Level raising and anticyclotomic Selmer groups for Hilbert modular forms of weight two

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0. Introduction

0.0. Convention. A 'character' always means a continuous character. For any perfect field k we denote by $G_k = \mathrm{Gal}(\overline{k}/k)$ the absolute Galois group of k.

0.1. Let F be a totally real number field; let $f \in S_2(\mathfrak{n}, \omega)$ be – in the notation of [N2, ch. 12] – a cuspidal Hilbert modular eigenform over F of parallel weight 2, (exact) level $\mathfrak{n}$ and character $\omega : \mathbf{A}_F^\times/F^\times \longrightarrow \mathbf{C}^\times$ (ω is a totally real character of finite order).

For each prime $v \nmid \mathfrak{n}\infty$ of F denote by $\lambda_f(v)$ the eigenvalue of the standard Hecke operator $T(v)$ acting on f: $T(v)f = \lambda_f(v)f$. The field $L_f \subset \mathbf{C}$ generated over $\mathbf{Q}$ by all Hecke eigenvalues $\lambda_f(v)$ and by the values of ω (in fact, L_f is generated by $\{\lambda_f(v)\}$ for v belonging to any set of primes of F of density 1) is a totally real (resp. a CM) number field if $\omega = 1$ (resp. if $\omega \neq 1$).

0.2. The (unitary) automorphic representation $\pi(f)$ of $GL_2(\mathbf{A}_F)$ attached to f has central character ω and its standard L-function is related to the classical L-function of f by the relation

$$L(\pi(f), s - \tfrac{1}{2}) = \Gamma_{\mathbf{C}}(s)^{[F:\mathbf{Q}]}\, L(f, s), \qquad \Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s),$$

which is valid Euler factor by Euler factor. In particular, the Euler factor at a prime $v \nmid \mathfrak{n}\infty$ is equal to

$$L(\pi(f)_v, s) = \left(1 - \lambda_f(v)(Nv)^{-s-1/2} + \omega(v)(Nv)^{-2s}\right)^{-1}.$$

As $\widetilde{\pi(f)} = \pi(f) \otimes \omega^{-1}$, the functional equation

$$L(\pi(f), s) = \varepsilon(\pi(f), s)\, L(\widetilde{\pi(f)}, 1-s), \qquad \varepsilon(\pi(f), s) = c(\pi)^{1/2-s}\,\varepsilon(\pi(f), \tfrac{1}{2})$$

becomes self-dual if $\omega = 1$, in which case $\varepsilon(\pi(f), \frac{1}{2}) = \pm 1$.

0.3. Let K be a totally imaginary quadratic extension of F and $\chi : \mathbf{A}_K^\times/K^\times \longrightarrow \mathbf{C}^\times$ a character of finite order satisfying

$$\chi|_{\mathbf{A}_F^\times} \cdot \omega = 1. \tag{0.3.1}$$

This condition implies that

$$^{c}\chi \cdot \chi \cdot (\omega \circ N_{K/F}) = 1, \tag{0.3.2}$$

where c is the non-trivial element of $\mathrm{Gal}(K/F)$ and $^{c}\chi : \mathbf{A}_K^\times/K^\times \longrightarrow \mathbf{C}^\times$ is the character

$$(^{c}\chi)(a) = \chi(c^{-1}(a)) = \chi(c(a)).$$

0.4. Fix a number field $L \subset \mathbf{C}$ containing L_f and the values of χ. Fix a prime $\mathfrak{p}$ of L above a rational prime p and denote by $V_{\mathfrak{p}}(f)$ the (cohomologically normalised) two-dimensional representation of G_F with coefficients in $L_{\mathfrak{p}}$ attached to f: if $v \nmid p\mathfrak{n}\infty$ is a prime of F, then $V_{\mathfrak{p}}(f)$ is unramified at v and

$$\det\left(1 - \mathrm{Fr}_{\mathrm{geom}}(v)X \mid V_{\mathfrak{p}}(f)\right) = 1 - \lambda_f(v)X + \omega(v)(Nv)X^2.$$

We identify ω (resp. χ) with a Galois character $\omega : G_F \longrightarrow O_L^\times$ (resp. $\chi : G_K \longrightarrow O_L^\times$) via the reciprocity map $\mathrm{rec}_F : \mathbf{A}_F^\times/F^\times \longrightarrow G_F^{\mathrm{ab}}$ (resp. rec_K) normalised by letting the uniformisers correspond to geometric Frobenius elements. The relation (0.3.2) then reads as follows:

$$^{c}\chi \cdot \chi \cdot \omega|_{G_K} = 1.$$

It is known (see [C1, Thm. A] and [T1, Thm. 2]) that, for each prime $v \nmid p\infty$ of F, the restriction $V_{\mathfrak{p}}(f)_v$ of $V_{\mathfrak{p}}(f)$ to the decomposition group G_{F_v} corresponds to $\pi(f)_v \otimes |\cdot|^{-1/2}$ via the local Langlands correspondence. This implies that $L(f,s)$ coincides with the L-function of the strongly compatible system of L-rational Galois representations $\{V_{\mathfrak{p}}(f)\}_{\mathfrak{p}}$ of G_F in the sense that

$$\forall v \nmid p\infty \qquad L_v(f,s) = \det\left(1 - \mathrm{Fr}_{\mathrm{geom}}(v)(Nv)^{-s} \mid V_{\mathfrak{p}}(f)^{I_v}\right)^{-1}.$$

0.5. Denote by θ_χ the automorphic representation of $GL_2(\mathbf{A}_F)$ generated by the theta series of χ (which is a weight one Hilbert eigenform over F); its central character is equal to $\chi|_{\mathbf{A}_F^\times} \cdot \eta$, where $\eta = \eta_{K/F} : \mathbf{A}_F^\times/F^\times N_{K/F}(\mathbf{A}_K^\times) \xrightarrow{\sim} \{\pm 1\}$ is the quadratic character corresponding to the extension K/F. As $\theta_\chi \otimes \eta = \theta_\chi$, the condition (0.3.1) implies that the Rankin-Selberg L-function $L(\pi(f) \times \theta_\chi, s)$ (which will be abusively denoted by $L(\pi(f) \times \chi, s)$) coincides with $L(\widetilde{\pi(f)} \times \widetilde{\theta_\chi}, s)$, hence admits a self-dual functional equation

$$\begin{aligned} &L(\pi(f) \times \chi, s) = \varepsilon(\pi(f) \times \chi, s)\, L(\pi(f) \times \chi, 1-s), \\ &\varepsilon(\pi(f) \times \chi, s) = c(\pi(f) \times \chi)^{1/2-s}\, \varepsilon(\pi(f) \times \chi, \tfrac{1}{2}), \qquad \varepsilon(\pi(f) \times \chi, \tfrac{1}{2}) = \pm 1. \end{aligned} \tag{0.5.1}$$

In more concrete terms,

$$L(\pi(f) \times \chi, s - \tfrac{1}{2}) = \Gamma_{\mathbf{C}}(s)^{[K:\mathbf{Q}]}\, L(f_K, \chi, s),$$

where $L(f_K, \chi, s)$ is the L-function of the strongly compatible system of L-rational Galois representations $\{V_{\mathfrak{p}}(f)|_{G_K} \otimes \chi\}_{\mathfrak{p}}$ of G_K:

$$\forall v \nmid p\infty \qquad L_v(f_K, \chi, s) = \prod_{w|v} \det\left(1 - \mathrm{Fr}_{\mathrm{geom}}(w)(Nw)^{-s} \mid (V_{\mathfrak{p}}(f) \otimes \chi)^{I_w}\right)^{-1}.$$

Above, v (resp. w) is a prime of F (resp. of K). Set

$$r_{\mathrm{an}}(f_K, \chi) := \mathrm{ord}_{s=1}\, L(f_K, \chi, s) = \mathrm{ord}_{s=1/2}\, L(\pi(f) \times \chi, s).$$

0.6. For any G_K-module M we denote by ${}^cM$ the abelian group M equipped with a new action of G_K given by $m \mapsto (\widetilde{c}^{-1} g \widetilde{c})m$ ($m \in M$, $g \in G_K$), where $\widetilde{c} \in G_F$ is any element of G_F that does not belong to G_K (the isomorphism class of the G_K-module ${}^cM$ does not depend on the choice of $\widetilde{c}$).

The map sending a non-homogeneous n-cochain $z \in C^n(G_K, M)$ to the cochain $z' \in C^n(G_K, {}^cM)$ given by $z'(g_1, \ldots, g_n) = z(\widetilde{c}^{-1} g_1 \widetilde{c}, \ldots, \widetilde{c}^{-1} g_n \widetilde{c})$ induces an isomorphism

$$H^n(G_K, M) \xrightarrow{\sim} H^n(G_K, {}^cM). \tag{0.6.1}$$

In the special case when M is a G_F-module the map $\widetilde{c} : M \xrightarrow{\sim} {}^cM$ ($m \mapsto \widetilde{c}m$) is an isomorphism of G_K-modules.

0.7. As $\det(V_{\mathfrak{p}}(f)) = L_{\mathfrak{p}}(-1) \otimes \omega$, the $L_{\mathfrak{p}}[G_F]$-module $V := V_{\mathfrak{p}}(f)(1)$ is equipped with a non-degenerate skew-symmetric G_F-equivariant pairing

$$V \times V \longrightarrow L_{\mathfrak{p}}(1) \otimes \omega, \tag{0.7.1}$$

which induces a non-degenerate G_K-equivariant pairing

$$(V|_{G_K} \otimes \chi) \times (V|_{G_K} \otimes {}^c\chi) \longrightarrow L_{\mathfrak{p}}(1).$$

As a result, there is an isomorphism of G_F-modules

$$V^*(1) := \mathrm{Hom}_{L_{\mathfrak{p}}}(V, L_{\mathfrak{p}})(1) \xrightarrow{\sim} V \otimes \omega^{-1}$$

and isomorphisms of G_K-modules

$$(V|_{G_K} \otimes \chi)^*(1) \xrightarrow{\sim} V|_{G_K} \otimes {}^c\chi \xrightarrow{\sim} {}^c(V|_{G_K} \otimes \chi).$$

The fields $K_\varphi = \overline{K}^{\mathrm{Ker}(\varphi)}$ $(\varphi = \chi, {}^c\chi)$ are finite abelian extensions of K satisfying

$$K_\chi K_{{}^c\chi} = K_\chi F_\omega = K_{{}^c\chi} F_\omega,$$

where $F_\omega = \overline{F}^{\mathrm{Ker}(\omega)}$ is the (totally real) finite abelian extension of F trivialising ω.

0.8. We are interested in the Bloch-Kato Selmer groups

$$H^1_f(K, V \otimes \varphi) = \left(H^1_f(K_\varphi, V) \otimes \varphi\right)^{\mathrm{Gal}(K_\varphi/K)} = H^1_f(K_\varphi, V)^{(\varphi^{-1})} \qquad (\varphi = \chi, {}^c\chi),$$

where

$$M^{(\varphi)} = \{m \in M \mid \forall g \in \mathrm{Gal}(K_\varphi/K) \quad g(m) = \varphi(g)m\}$$

for any $O_L[\mathrm{Gal}(K_\varphi/K)]$-module M. Set

$$h^1_f(K, V \otimes \varphi) := \dim_{L_\mathfrak{p}} H^1_f(K, V \otimes \varphi).$$

The discussion in 0.6–0.7 implies that there are isomorphisms

$$H^1_f(K, (V \otimes \chi)^*(1)) \xrightarrow{\sim} H^1_f(K, V \otimes {}^c\chi) \xrightarrow{\sim} H^1_f(K, {}^c(V \otimes \chi)) \xrightarrow{\sim} H^1_f(K, V \otimes \chi). \tag{0.8.1}$$

As the Galois representation V is pure of weight -1 by the generalised Ramanujan conjecture [Bl],

$$H^0(K, (V \otimes \chi)^*(1)) = H^0(K, V \otimes {}^c\chi) = H^0(K, {}^c(V \otimes \chi)) = H^0(K, V \otimes \chi) = 0. \tag{0.8.2}$$

0.9. If we take into account (0.8.1-2), the conjectures of Bloch and Kato [BK] predict that

$$h^1_f(K, V \otimes \chi) \stackrel{?}{=} r_{\mathrm{an}}(f_K, \chi). \tag{0.9.1}$$

In the present article we concentrate only on the implication

$$r_{\mathrm{an}}(f_K, \chi) = 0 \stackrel{?}{\Longrightarrow} H^1_f(K, V \otimes \chi) = 0. \tag{0.9.2}$$

Results of this kind were first proved by Bertolini and Darmon [BD 1-2]; their method was further developed in [L1-3], [LV 1-2], [LRV], [H], [PW], [Cd] and [TZ]. Our aim is to eliminate, whenever possible, the restrictive assumptions imposed in [*loc. cit.*]. Our main result is the following.

Theorem A. *Let $f \in S_2(\mathfrak{n}, \omega)$, $\chi : \mathbf{A}_K^\times/K^\times \longrightarrow \mathbf{C}^\times$ $(\chi|_{\mathbf{A}_F^\times} \cdot \omega = 1)$ and $V = V_\mathfrak{p}(f)(1)$ be as in 0.1–0.7. If f has CM by a totally imaginary quadratic extension $K(f)$ of F (i.e., if f is the θ-series attached to a Hecke character of $K(f)$), assume that $K(f) \not\subset K_\chi F_\omega$. Assume that there exists an element $g_\mathfrak{p} \in G_F$ satisfying the following conditions (A1)–(A3).*
(A1) $g_\mathfrak{p}$ acts trivially on F_ω ($\iff \omega(g_\mathfrak{p}) = 1$);
(A2) $\det(1 - g_\mathfrak{p}X \mid V) = (1 - \lambda_1 X)(1 - \lambda_2 X)$, where $\lambda_1^2 = 1 \neq \lambda_2^2$; if f has CM, then $\forall n \geq 1$ $\lambda_2^n \neq 1$;
(A3) $g_\mathfrak{p}$ does not act trivially on K;
(such an element exists for $\mathfrak{p}$ belonging to a set of primes of L of positive density).
If $L(f_K, \chi, 1) \neq 0$, then $H^1_f(K, V \otimes \chi) = H^1_f(K, V \otimes {}^c\chi) = 0$; if, in addition, $\mathfrak{p}$ does not belong to a finite set of primes of L depending only on f, K and χ, then $H^1_f(K, (V/T) \otimes \chi) = H^1_f(K, (V/T) \otimes {}^c\chi) = 0$ for any G_F-stable $O_{L,\mathfrak{p}}$-lattice $T \subset V$.

0.10. Conjecturally, for each f in Theorem A there exists an abelian variety A_f defined over F (unique up to isogeny) such that

$$\dim(A_f) = [L_f : \mathbf{Q}], \qquad \mathrm{End}_F(A_f) = O_{L_f}, \qquad L(\iota A_f/F, s) = L(f, s) \tag{0.10.1}$$

(Euler factor by Euler factor), where ι denotes the inclusion $L_f \subset \mathbf{C}$. In this case we have, for any prime $\mathfrak{p}$ of L above a prime $\mathfrak{p}_f$ of L_f,

$$V = V_{\mathfrak{p}}(f)(1) = V_{\mathfrak{p}_f}(A_f) \otimes_{L_{f,\mathfrak{p}_f}} L_{\mathfrak{p}}, \qquad V_{\mathfrak{p}_f}(A_f) = T_p(A_f) \otimes_{(O_{L_f} \otimes \mathbf{Z}_p)} L_{f,\mathfrak{p}_f}.$$

Moreover, the Bloch-Kato Selmer group of $A_f[p^\infty]$ over any finite extension F' of F coincides with the classical Selmer group for the p-power descent on A_f. In view of the standard descent sequence

$$0 \longrightarrow (A_f(F') \otimes \mathbf{Q}_p/\mathbf{Z}_p) \otimes_{(O_{L_f} \otimes \mathbf{Z}_p)} O_{L,\mathfrak{p}} \longrightarrow H^1_f(F', V/T) \longrightarrow Ш(A_f/F')[p^\infty] \otimes_{(O_{L_f} \otimes \mathbf{Z}_p)} O_{L,\mathfrak{p}} \longrightarrow 0,$$
$$T = T_p(A_f) \otimes_{(O_{L_f} \otimes \mathbf{Z}_p)} O_{L,\mathfrak{p}},$$

Theorem A can be rephrased in this context as follows.

Theorem A'. *Assume that $f \in S_2(\mathfrak{n}, \omega)$ in Theorem A is attached to an abelian variety A_f satisfying (0.10.1). If A_f acquires CM over a totally imaginary quadratic extension $K(f)$ of F, assume that $K(f) \not\subset K_\chi F_\omega$. Assume that there exists an element $g_{\mathfrak{p}_f} \in G_F$ satisfying the following conditions (A1')–(A3').*
(A1') $g_{\mathfrak{p}_f}$ acts trivially on F_ω ($\iff \omega(g_{\mathfrak{p}_f}) = 1$);
(A2') $\det(1 - g_{\mathfrak{p}_f} X \mid V_{\mathfrak{p}_f}(A_f)) = (1-\lambda_1 X)(1-\lambda_2 X)$, where $\lambda_1^2 = 1 \neq \lambda_2^2$; if A_f has CM, then $\forall n \geq 1\ \ \lambda_2^n \neq 1$;
(A3') $g_{\mathfrak{p}_f}$ does not act trivially on K;
(such an element exists for $\mathfrak{p}_f$ belonging to a set of primes of L_f of positive density).
If $L(\iota A_f/K, \iota \circ \chi, 1) \neq 0$, then $A_f(K_\varphi)^{(\varphi^{-1})}$ is finite ($\varphi = \chi, {}^c\chi$); if, in addition, $\mathfrak{p}_f$ does not belong to a finite set of primes of L_f depending only on f, K and χ, then $\Big(Ш(A_f/K_\varphi)[p^\infty] \otimes_{(O_{L_f} \otimes \mathbf{Z}_p)} O_{L,\mathfrak{p}}\Big)^{(\varphi^{-1})}$ is finite ($\varphi = \chi, {}^c\chi$).

0.11. Applying Theorem A to $\chi = 1$ for variable K we obtain the case (c) of the following result. The cases (a) and (b) are well-known consequences (cf. [Zh 2, Thm. 4.3.2]) of [N1, Thm. 3.2], [YZZ, Thm. 1.3.1] and [FH, Thm. B.2], which generalise, respectively, the Euler system argument, the Gross-Zagier formula and the non-vanishing results for quadratic twists used by Kolyvagin and Logachev [KoLo] in their proof of the corresponding result for $F = \mathbf{Q}$.

Theorem B. *Assume that $f \in S_2(\mathfrak{n}, 1)$ from 0.1 has trivial character, $V = V_{\mathfrak{p}}(f)(1)$, $L(f,1) \neq 0$ and that at least one of the following three conditions holds:*
(a) $2 \nmid [F : \mathbf{Q}]$;
(b) *there exists a finite prime v of F for which $\pi(f)_v$ is not a principal series representation;*
(c) *there exists $g_{\mathfrak{p}} \in G_F$ satisfying the conditions (A1) and (A2) from Theorem A (if f has no CM this is equivalent to V not being quaternionic in the sense of B.4.7, which holds for all but finitely many $\mathfrak{p}$).*
Then $H^1_f(F, V) = 0$ and, if $\mathfrak{p}$ does not belong to a certain finite set of primes of L, then $H^1_f(F, V/T) = 0$ for any G_F-stable $O_{L,\mathfrak{p}}$-lattice $T \subset V$.

There is also an analogue of Theorem A' in this situation (the field L_f being totally real in this case):

Theorem B'. *If $f \in S_2(\mathfrak{n}, 1)$ in Theorem B is attached to an abelian variety A_f satisfying (0.10.1) and $L(\iota A_f/F, 1) \neq 0$, then $A_f(F)$ is finite. Moreover, if at least one of the following five condition holds, then $Ш(A_f/F)[\mathfrak{p}_f^\infty]$ is finite (and equal to zero if $\mathfrak{p}_f$ does not belong to a certain finite set of primes of L_f):*
(a) $2 \nmid [F : \mathbf{Q}]$;
(b1) *A_f does not have potentially good reduction everywhere;*
(b2) *A_f does not acquire semistable reduction everywhere over any cyclic extension of F;*
(c1) *A_f does not have CM and the localisation $C \otimes_{Z(C)} Z(C)_{\mathfrak{p}_C}$ of the simple algebra $C := \mathrm{End}_{\overline{\mathbf{Q}}}(A_f) \otimes \mathbf{Q}$ at the prime $\mathfrak{p}_C$ of $Z(C) \subset L_f$ below $\mathfrak{p}_f$ is isomorphic to $M_n(Z(C)_{\mathfrak{p}_C})$;*
(c2) *A_f has CM by a totally imaginary quadratic extension L' of L_f, the prime $\mathfrak{p}_f$ splits in L'/L_f and $V_{\mathfrak{p}_f}(A_f)|_{G_{K(f)}} = \psi_1 \oplus \psi_2$, where $\psi_i : G_{K(f)} \longrightarrow L_{f,\mathfrak{p}_f}^\times$ are characters for which $\psi_2(\mathrm{Ker}(\psi_1))$ is infinite.*
[In particular, if $2 \nmid [F : \mathbf{Q}]$ or if f does not have CM then $Ш(A_f/F)[\mathfrak{p}_f^\infty] = 0$ for all but finitely many $\mathfrak{p}_f$.]

Corollary. *If E is a modular elliptic curve over F satisfying $L(E/F, 1) \neq 0$, then:*
(1) *(cf. [L2, Thm. A]) $E(F)$ is finite.*

(2) *If $2 \nmid [F:\mathbf{Q}]$ or if E has no CM, then $Ш(E/F)$ is finite.*
(3) *If $2 \mid [F:\mathbf{Q}]$ and E has CM by an imaginary quadratic field K', then the following group is finite:*

$$Ш(E/F)_{\mathrm{split}} := \bigoplus_{p \text{ splits in } K'/\mathbf{Q}} Ш(E/F)[p^\infty].$$

0.12. The proof of Theorem A is based on the method of Bertolini and Darmon [BD 2], with the following improvements:

• assumptions such as $(\mathfrak{n}, d_{K/F} N_{K/F}(\mathrm{cond}(\chi))) = 1$, which had been used to transfer f to an explicit definite quaternion algebra, can be eliminated by an appeal to results of Tunnell [Tu] and Saito [Sa] on local toric linear forms (such linear forms were used in the context of a generalised Gross-Zagier formula by Zhang et al. [Z1], [YZZ]);
• it is not necessary to assume that there exists a level-raising congruence $f \equiv f' \pmod{\mathfrak{p}^m}$, where f' is an eigenform of level $\mathfrak{n}\ell$, new at a well-chosen prime ℓ of F. As in [T1], it is sufficient to work with an eigenform with coefficients in $O_L/\mathfrak{p}^m O_L$, which always exists (for a suitable ℓ). A variant of the arguments of Boston-Lenstra-Ribet [BLR] then allows us to realise the reduction modulo $\mathfrak{p}^{m-C}$ of (a certain lattice T in a Tate twist of) the $\mathfrak{p}$-adic Galois representation attached to f as a quotient of the Tate module of a suitable Shimura curve;
• assumptions such as $(\mathfrak{n}, p) = 1$ or $p \nmid D_F$, which had been used to control the local behaviour at p of the cohomology class $c(\ell) \in H^1(K_\chi, T/\mathfrak{p}^{m-C}T)$ constructed from a certain CM point on the Shimura curve alluded to above, can be avoided by a consistent use of Raynaud extensions and their flat cohomology, combined with a uniformity result A.1.8 for Barsotti-Tate groups.
• the assumptions about the image of the Galois representation V can be formulated in an abstract – and probably optimal – form (the conditions (A1)–(A3) in Theorem A).

The contents of the present article are as follows. In §1 we sum up the geometric machinery behind level raising from a weight two form on a definite quaternion algebra over F to a (weight two) form arising from a suitable Shimura curve over F. There are no original results in this part of the article; we simply translate back relevant parts of [T1] to a natural geometric situation similar to [Ri 7] (see also [J] and [R]). In §2 we construct the cohomology classes $c(\ell) \in H^1(K_\chi, T/\mathfrak{p}^n T)$ using "weak level raising modulo $\mathfrak{p}^n$"; we study their local properties and prove the main result (and its corollaries) by combining the annihilation relation arising from the reciprocity law

$$\forall s \in H^1(K_\chi, (T/\mathfrak{p}^n T) \otimes \omega^{-1}) \qquad \sum_v \mathrm{inv}_v(c(\ell)_v \cup s_v) = 0 \in O_L/\mathfrak{p}^n O_L$$

with the Čebotarev density theorem, as in Kolyvagin's method. In Appendix A (resp. Appendix B) we collect useful results about flat cohomology of finite group schemes and Raynaud extensions (resp. about images of Galois representations attached to Hilbert modular forms of regular weight).

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Notation: Throughout this article, F is a totally real number field of degree d and $S_\infty = \{\tau_1, \ldots, \tau_d\}$ (resp. $S_p = \{v \mid p\}$) the set of archimedean primes (resp. of primes above a rational prime p) in F. For a quaternion algebra D over F we denote by $\mathrm{Ram}(D) = \{v \mid \mathrm{inv}_v(D_v) = -1\}$ the set of primes of F at which D is ramified (above, $D_v = D \otimes_F F_v$). We also write $\otimes$ for $\otimes_{\mathbf{Z}}$ and $\widehat{A}$ for $A \otimes \widehat{\mathbf{Z}}$ ($\widehat{\mathbf{Z}} = \varprojlim_n \mathbf{Z}/n\mathbf{Z}$), for any abelian group A.

1. Level raising

Throughout §1, B is a totally definite quaternion algebra over F (i.e., such that $\mathrm{Ram}(B) \supset S_\infty$).

1.1. Automorphic forms of weight two on definite quaternion algebras

1.1.1. The space $\mathbb{S}(U; A)$. For any open compact subgroup $U \subset \widehat{B}^\times$ and any abelian group A, denote by

$$\mathbb{S}(U; A) = \left\{ f : \widehat{B}^\times \longrightarrow A \mid \forall b \in B^\times \ \ \forall g \in \widehat{B}^\times \ \ \forall u \in U \quad f(bgu) = f(g) \right\} = A[B^\times \backslash \widehat{B}^\times / U]$$

the abelian group of automorphic forms "of weight two" and level U on $B_{\mathbf{A}}^\times$ with values in A (note that the set $B^\times \backslash \widehat{B}^\times / U$ is finite).

1.1.2. Action of $\widehat{B}^\times$. The action of $\widehat{B}^\times$ on $\mathbb{S}(B_{\mathbf{A}}^\times; A) = \bigcup_U \mathbb{S}(U; A)$ by right translations $(g \cdot f)(g') = f(g'g)$ is smooth (the stabiliser of each element is open in $\widehat{B}^\times$) and, for each open compact subgroup U, the space of U-invariants is equal to $\mathbb{S}(B_{\mathbf{A}}^\times; A)^U = \mathbb{S}(U; A)$.

The restriction of this action to $Z(\widehat{B}^\times) = \widehat{F}^\times$ leaves each $\mathbb{S}(U; A)$ stable. More precisely, $\widehat{F}^\times$ acts on $\mathbb{S}(U; A)$ through the finite abelian group $F^\times \backslash \widehat{F}^\times / (\widehat{F}^\times \cap U)$.

1.1.3. Some subspaces of $\mathbb{S}(U; A)$. It will be useful to consider the following subgroups of $\mathbb{S}(U; A)$:

$$\mathbb{S}(U; A)_{\mathrm{triv}} = \left\{ f \in \mathbb{S}(U; A) \mid f \text{ factors through } \mathrm{Nrd} : B^\times \backslash \widehat{B}^\times / U \longrightarrow F_+^\times \backslash \widehat{F}^\times / \mathrm{Nrd}(U) \right\};$$

$$\mathbb{S}(UY; A) := \mathbb{S}(U; A)^Y = \mathbb{S}(U; A)^{Y(\widehat{F}^\times \cap U)} = A[B^\times \backslash \widehat{B}^\times / UY],$$

where Y is an open – but not necessarily compact – subgroup of $\widehat{F}^\times = Z(\widehat{B}^\times)$.

1.1.4. Action of double cosets. If $U, U' \subset \widehat{B}^\times$ are compact open subgroups and $x \in \widehat{B}^\times$, then ${}^xU = xUx^{-1}$ is commensurable with U'. If we write $U' = \coprod_i \alpha_i (U' \cap {}^xU)$ as a disjoint union of cosets, then $U'xU = \coprod_i x_i U$, where $x_i = \alpha_i x$. The linear map

$$[U'xU] : \mathbb{S}(U; A) \longrightarrow \mathbb{S}(U'; A), \qquad ([U'xU]f)(g) = \sum_i f(gx_i)$$

has the following properties.

1.1.4.1. $[U'xU]$ commutes with the action of $\widehat{F}^\times$. In particular, it maps $\mathbb{S}(UY; A)$ to $\mathbb{S}(U'Y; A)$, for any open subgroup $Y \subset \widehat{F}^\times$.

1.1.4.2. The endomorphisms $[UxU]$ define a left action of the double coset algebra $\mathbf{Z}[U \backslash \widehat{B}^\times / U]$ on $\mathbb{S}(U; A)$. If $x \in Z(\widehat{B}^\times) = \widehat{F}^\times$, then $[UxU]$ coincides with the action of x.

1.1.4.3. If $x = 1$ and $U' \subset U$, then $[U'1U] : \mathbb{S}(U; A) \longrightarrow \mathbb{S}(U'; A)$ is the inclusion and $[U1U'] : \mathbb{S}(U'; A) \longrightarrow \mathbb{S}(U; A)$ is the trace map.

Recall that $\mathbf{Z}[U \backslash \widehat{B}^\times / U]$ embeds into the Hecke algebra $C_c^\infty(\widehat{B}^\times)$ of locally constant functions on $\widehat{B}^\times$ with compact support, equipped with the convolution product

$$(\alpha * \beta)(g) = \int_{\widehat{B}^\times} \alpha(h) \beta(h^{-1}g) \, dh$$

(for a fixed Haar measure dh on $\widehat{B}^\times$), via the map which sends $\mathrm{vol}(U)[UgU]$ to the characteristic function of UgU. Denote by $x \mapsto x^\vee$ the (anti)-involution of $\mathbf{Z}[U \backslash \widehat{B}^\times / U]$ which sends $[UgU]$ to $[Ug^{-1}U]$.

1.1.5. Action of the spherical Hecke algebra. Fix an open compact subgroup $U \subset \widehat{B}^\times$. There exists a finite set $S \supset \mathrm{Ram}(B)$ of primes of F such that $U = U_S U^S$, where U_S is an open compact subgroup of $\prod_{v \in S_f} B_v^\times$ ($S_f = S - S_\infty$) and $U^S = \prod_{v \notin S} U_v$, where each U_v is a maximal compact subgroup of $B_v^\times$.

Fix such a set S and denote by $\mathbb{T}^S(U)$ the (commutative) subring of $\mathbf{Z}[U \backslash \widehat{B}^\times / U]$ generated by the double cosets $[UxU]$ for all $x \in (\widehat{B}^S)^\times = \{x \in \widehat{B}^\times \mid \forall v \in S_f \quad x_v = 1\}$. As a ring, it is isomorphic to $\mathbf{Z}[T(v), S(v), S(v)^{-1} \mid v \notin S]$, where $T(v)$ and $S(v)$ are the standard Hecke operators

$$T(v) = [U\xi_v U], \qquad S(v) = [U\varpi_v U] \qquad\qquad (v \notin S).$$

Above, ϖ_v is a uniformiser of F_v and ξ_v is as in 1.2.1 below. The involution $x \mapsto x^\vee$ acts on $\mathbb{T}^S(U)$ by $T(v)^\vee = T(v)S(v)^{-1}$, $S(v)^\vee = S(v)^{-1}$ $(v \notin S)$.

For any open subgroup $Y \subset \widehat{F}^\times$, the image $\mathbb{T}^S(UY)$ of $\mathbb{T}^S(U)$ in $\mathbf{Z}[UY\backslash\widehat{B}^\times/UY]$ is the quotient of $\mathbb{T}^S(U)$ by the ideal generated by $[UyU] - 1$ $(y \in Y)$.

The image of $\mathbb{T}^S(UY)$ in $\mathrm{End}_{\mathbf{Z}}(\mathbb{S}(UY;\mathbf{Z}))$ is an order in a finite product of number fields. It acts on $\mathbb{S}(UY;A) = \mathbb{S}(UY;\mathbf{Z}) \otimes_{\mathbf{Z}} A$, for any abelian group A.

For any $\mathbb{T}^S(UY)$-module N set

$${}^hN = N \otimes_{\mathbb{T}^S(UY),\vee} \mathbb{T}^S(UY).$$

The map $n \mapsto {}^hn := n \otimes 1$ is an isomorphism of abelian groups $N \longrightarrow {}^hN$ satisfying

$$\forall t \in \mathbb{T}^S(UY)\ \forall n \in N \qquad t({}^hn) = {}^h(t^\vee(n)).$$

1.1.6. The Eisenstein part of $\mathbb{S}(U;\mathbf{C})$. As a $\widehat{B}^\times$-module, $\mathbb{S}(B_{\mathbf{A}}^\times;\mathbf{C})_{\mathrm{triv}} = \bigcup_U \mathbb{S}(U;\mathbf{C})_{\mathrm{triv}}$ decomposes into a direct sum of one-dimensional eigenspaces $\mathbf{C}f_\varphi$, one for each character of finite order $\varphi : B^\times\backslash\widehat{B}^\times \longrightarrow \mathbf{C}^\times$ factoring as $B^\times\backslash\widehat{B}^\times \xrightarrow{\mathrm{Nrd}} F_+^\times\backslash\widehat{F}^\times \xrightarrow{\varphi'} \mathbf{C}^\times$. If U satisfies $\varphi'(\mathrm{Nrd}(U)) = 1$, then

$$\forall v \notin S \qquad T(v)f_\varphi = (N(v)+1)\varphi'(\varpi_v)f_\varphi, \qquad S(v)f_\varphi = \varphi'(\varpi_v)^2 f_\varphi.$$

1.1.7. The non-Eisenstein part of $\mathbb{S}(U;\mathbf{C})$. As a $\mathbb{T}^S(U)$-module, the space $\mathbb{S}(U;\mathbf{C}) = \oplus_\lambda \mathbb{S}(U;\mathbf{C})_\lambda$ is a direct sum of its isotypic components, for certain ring morphisms $\lambda : \mathbb{T}^S(U) \longrightarrow \mathbf{C}$.

If $f \in \mathbb{S}(U;\mathbf{C})_\lambda$ but $f \notin \mathbb{S}(U;\mathbf{C})_{\mathrm{triv}}$, it follows from the Jacquet-Langlands correspondence that there exists a cuspidal Hilbert eigenform of parallel weight 2 on F with the same Hecke eigenvalues under $T(v)$ and $S(v)$ $(\forall v \notin S)$ as f. The generalised Ramanujan conjecture for Hilbert modular forms, whose proof was completed in [Bl], implies that

$$\forall v \notin S \qquad T(v)f = \lambda(T(v))f, \qquad |\lambda(T(v))| \leq 2N(v)^{1/2}. \tag{1.1.7.1}$$

As a result, for each $v \notin S$, $T(v)$ has no common eigenvalues on $\mathbb{S}(U;\mathbf{C})_{\mathrm{triv}}$ and $\mathbb{S}(U;\mathbf{C})/\mathbb{S}(U;\mathbf{C})_{\mathrm{triv}}$. It follows that there is a unique $\mathbb{T}^S(U)$-submodule $\mathbb{S}(U;\mathbf{C})_0 \subset \mathbb{S}(U;\mathbf{C})$ such that

$$\mathbb{S}(U;\mathbf{C}) = \mathbb{S}(U;\mathbf{C})_{\mathrm{triv}} \oplus \mathbb{S}(U;\mathbf{C})_0.$$

Moreover, for each $v \notin S$, any eigenvector of $T(v)$ in $\mathbb{S}(U;\mathbf{C})$ lies either in $\mathbb{S}(U;\mathbf{C})_{\mathrm{triv}}$ or in $\mathbb{S}(U;\mathbf{C})_0$.

For a subring $A \subset \mathbf{C}$ and an open subgroup $Y \subset \widehat{F}^\times$ set

$$\mathbb{S}(UY;A)_0 := \mathbb{S}(UY;A) \cap \mathbb{S}(U;\mathbf{C})_0.$$

If f is as in (1.1.7.1), then $\sigma \circ f \in \mathbb{S}(U;\mathbf{C})_{\sigma\circ\lambda}$ and $\sigma \circ f \notin \mathbb{S}(U;\mathbf{C})_{\mathrm{triv}}$, for each $\sigma \in \mathrm{Aut}(\mathbf{C})$. This implies that $\mathbb{S}(U;\mathbf{C})_0$ is stable under $\mathrm{Aut}(\mathbf{C})$; as the same is true for $\mathbb{S}(U;\mathbf{C})_{\mathrm{triv}}$, it follows that

$$\mathbb{S}(UY;A) = \mathbb{S}(UY;A)_{\mathrm{triv}} \oplus \mathbb{S}(UY;A)_0, \tag{1.1.7.2}$$

for any subring $A \subset \mathbf{C}$ containing $\mathbf{Q}$ and any open subgroup $Y \subset \widehat{F}^\times$.

1.2. Oldforms and newforms

Let U and S be as in 1.1.5.

1.2.1. Degeneracy and trace maps. Fix a prime $\ell \notin S$ of F, a uniformiser ϖ_ℓ of F_ℓ and an isomorphism $M_2(F_\ell) \xrightarrow{\sim} B_\ell$ sending $GL_2(O_\ell)$ onto U_ℓ (where $O_\ell = O_{F,\ell}$ is the ring of integers of F_ℓ). Set

$$\xi_\ell = \begin{pmatrix} 1 & 0 \\ 0 & \varpi_\ell \end{pmatrix} \in GL_2(F_l) \xrightarrow{\sim} B_\ell^\times \subset \widehat{B}^\times, \qquad U(\ell) = U \cap {}^{\xi_\ell}U = \left\{ u \in U \mid u_\ell \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{\varpi_\ell} \right\}.$$

For any abelian group A we have the standard injective degeneracy maps

$$\begin{aligned} \alpha^* &= [U(\ell)1U] : \mathbb{S}(U;A) \longrightarrow \mathbb{S}(U(\ell);A), \qquad (\alpha^* f)(g) = f(g), \\ \beta^* &= [U(\ell)\xi_\ell U] : \mathbb{S}(U;A) \longrightarrow \mathbb{S}(U(\ell);A), \qquad (\beta^* f)(g) = f(g\xi_\ell) \end{aligned}$$

and the corresponding trace maps

$$\begin{aligned} \alpha_* &= [U1U(\ell)] : \mathbb{S}(U(\ell);A) \longrightarrow \mathbb{S}(U;A), \\ \beta_* &= [U\xi_\ell^{-1}U(\ell)] : \mathbb{S}(U(\ell);A) \longrightarrow \mathbb{S}(U;A). \end{aligned}$$

All these maps commute with the respective actions of $Z(\widehat{B}^\times) = \widehat{F}^\times$ and $T(v)$ $(v \notin S \cup \{\ell\})$ on $\mathbb{S}(U;A)$ and $\mathbb{S}(U(\ell);A)$.

1.2.2. Definition. *The* **ℓ-new subspace** *of* $\mathbb{S}(U(\ell);A)$ *is defined as*

$$\mathbb{S}(U(\ell);A)^{\ell-\mathrm{new}} := \mathrm{Ker}\left(\mathbb{S}(U(\ell);A) \xrightarrow{(\alpha_*,\beta_*)} \mathbb{S}(U;A)^{\oplus 2}\right).$$

It is stable by the action of $Z(\widehat{B}^\times)$ and $T(v)$ $(v \notin S \cup \{\ell\})$ and satisfies

$$\begin{aligned} \mathbb{S}(U(\ell);A)^{\ell-\mathrm{new}} &= \mathbb{S}(U(\ell);A) \cap \mathbb{S}(U(\ell);A')^{\ell-\mathrm{new}} \qquad (A \subset A'), \\ \mathbb{S}(U(\ell)Y;A)^{\ell-\mathrm{new}} &:= \mathbb{S}(U(\ell)Y;A) \cap \mathbb{S}(U(\ell);A)^{\ell-\mathrm{new}} = \left(\mathbb{S}(U(\ell);A)^{\ell-\mathrm{new}}\right)^Y, \end{aligned}$$

for any open subgroup $Y \subset \widehat{F}^\times$.

1.2.3. Proposition. (1) *For any open subgroup $Y \subset \widehat{F}^\times$ containing $F_\ell^\times$, the composite map*

$$\mu : \mathbb{S}(UY;A)^{\oplus 2} \xrightarrow{\alpha^* - \beta^*} \mathbb{S}(U(\ell)Y;A) \xrightarrow{(-\alpha_*,\beta_*)} \mathbb{S}(UY;A)^{\oplus 2}$$

is given by the matrix

$$\begin{pmatrix} -\alpha_*\alpha^* & \alpha_*\beta^* \\ \beta_*\alpha^* & -\beta_*\beta^* \end{pmatrix} = \begin{pmatrix} -N(\ell)-1 & T(\ell) \\ T(\ell) & -N(\ell)-1 \end{pmatrix}.$$

(2) $\mathrm{Ker}\left(\mathbb{S}(U;A)^{\oplus 2} \xrightarrow{\alpha^* - \beta^*} \mathbb{S}(U(\ell);A)\right) = \{(\xi_\ell \cdot f, f) \mid f \in \mathbb{S}(U;A)_{\mathrm{triv}}\}$.

Proof. (1) [Ri 7, Proof of Thm. 3.22], [T1, Lemma 2]. (2) If $f, f' \in \mathbb{S}(U;A)$ satisfy $\alpha^*(f) = \beta^*(f')$, then f is invariant by both $U_\ell = GL_2(O_\ell)$ and $\xi_\ell^{-1} GL_2(O_\ell)\xi_\ell$, hence by $\{g \in GL_2(F_\ell) \mid \det(g) \in O_\ell^*\}$. As

$$(\widehat{B}^\times)^{\mathrm{Nrd}=1} = (B^\times)^{\mathrm{Nrd}=1} SL_2(F_\ell)(U^{(\ell)})^{\mathrm{Nrd}=1}$$

by the strong approximation theorem [Vi, Thm. III.4.3], it follows that f factors through Nrd, hence $f \in \mathbb{S}(U;A)_{\mathrm{triv}}$. The relation $\alpha^*(f) = \beta^*(f')$ implies that $f = \xi_\ell \cdot f'$.

1.3. Shimura curves

Let U, S and ℓ be as in 1.2.1.

1.3.1. New quaternion algebra. Let B' be the quaternion algebra over F obtained from B by "switching invariants" at ℓ and a fixed infinite prime $\tau_1 \in S_\infty$:

$$\mathrm{Ram}(B') = (\mathrm{Ram}(B) \setminus \{\tau_1\}) \cup \{\ell\}.$$

Denote by $\widehat{F}^{(\ell)}$ (resp. $\widehat{D}^{(\ell)}$, for $D = B, B'$) the restricted product of F_v (resp. of D_v) over all finite primes $v \neq \ell$ of F (in other words, we have $\widehat{F} = F_\ell \times \widehat{F}^{(\ell)}$ and $\widehat{D} = D_\ell \times \widehat{D}^{(\ell)}$). Fix an isomorphism of $\widehat{F}^{(\ell)}$-algebras

$$\varphi : \widehat{B}^{(\ell)} \xrightarrow{\sim} \widehat{B}'^{(\ell)}$$

and set

$$U^{(\ell)} = U_S \prod_{v \notin S \cup \{\ell\}} U_v \subset \widehat{B}^{(\ell)\times}, \qquad U' = \varphi(U^{(\ell)}) O_{B'_\ell}^\times \subset \widehat{B}'^\times,$$

where $O_{B'_\ell}$ is the maximal order of the division algebra B'_ℓ.

1.3.2. Shimura curve (complex uniformisation). Fix an isomorphism $B'_{\tau_1} \xrightarrow{\sim} M_2(\mathbf{R})$ and consider the Shimura curve $M_{U'}$ corresponding to the open compact subgroup $U' \subset \widehat{B}'^\times$, using the notation and conventions of [CV 1, §3] and [CV 2, §3] with $\epsilon = 1$ (see also [C2] and [N1, §1]). In concrete terms, $M_{U'}$ is a smooth and projective curve over F whose associated Riemann surface $M_{U'}^{\mathrm{an}}$ is naturally identified with

$$M_{U'}^{\mathrm{an}} = (M_{U'} \otimes_{F,\tau_1} \mathbf{C})(\mathbf{C}) = B'^\times \backslash (\mathbf{C} - \mathbf{R}) \times \widehat{B}'^\times / U',$$

where $B'^\times \subset B'^\times_{\tau_1} \xrightarrow{\sim} GL_2(\mathbf{R})$ acts on $\mathbf{C} - \mathbf{R}$ by the standard action $z \mapsto \frac{az+b}{cz+d}$.

The curve $M_{U'}$ is irreducible, but not necessarily geometrically irreducible. Its field of constants is isomorphic to the finite abelian extension $F_{U'}$ of F characterised by the isomorphism

$$\mathrm{rec}_F : F_+^\times \backslash \widehat{F}^\times / \mathrm{Nrd}(U') = F_+^\times \backslash \widehat{F}^\times / \mathrm{Nrd}(U^{(\ell)}) O_\ell^\times \xrightarrow{\sim} \mathrm{Gal}(F_{U'}/F).$$

This is consistent with the fact that the reduced norm for B' induces a bijection

$$\pi_0(M_{U'}^{\mathrm{an}}) = B'^\times \backslash \pi_0(\mathbf{C} - \mathbf{R}) \times \widehat{B}'^\times / U' \xrightarrow{\sim} \mathrm{Nrd}(B'^\times) \backslash \{\pm 1\} \times \widehat{F}^\times / \mathrm{Nrd}(U') = F_+^\times \backslash \widehat{F}^\times / \mathrm{Nrd}(U'),$$

by strong approximation [Vi, Thm. III.4.3] and Eichler's norm theorem [Vi, Thm. III.4.1].

1.3.3. Quotient Shimura curve. For $z \in \mathbf{C} - \mathbf{R}$ and $b' \in \widehat{B}'^\times$, denote by $[z, b']_{U'}$ the complex point of $M_{U'}$ represented by the pair (z, b'). The centre $Z(\widehat{B}'^\times) = \widehat{F}^\times$ acts on $M_{U'}$ (by morphisms defined over F) according to the formula

$$g([z, b']_{U'}) = [z, b'g]_{U'} = [z, gb']_{U'}. \tag{1.3.3.1}$$

This action factors through the finite abelian group $F^\times \backslash \widehat{F}^\times / (\widehat{F}^\times \cap U')$.

For any open subgroup $Y \subset \widehat{F}^\times$, denote by $M_{U'Y}$ the quotient of $M_{U'}$ by (the image in $F^\times \backslash \widehat{F}^\times / (\widehat{F}^\times \cap U')$) of Y; it is a smooth projective curve over F satisfying

$$M_{U'Y}^{\mathrm{an}} = (M_{U'Y} \otimes_{F,\tau_1} \mathbf{C})(\mathbf{C}) = B'^\times \backslash (\mathbf{C} - \mathbf{R}) \times \widehat{B}'^\times / U'Y.$$

Denote by $[z, b']_{U'Y} \in M_{U'Y}(\mathbf{C})$ the image of $[z, b']_{U'}$.

In the case $Y = \widehat{F}^\times \cap U'$ (resp. $Y = \widehat{F}^\times$) the curve $M_{U'Y}$ coincides with $M_{U'}$ (resp. with the curve which was denoted by X in [Z1, 1.5.1] and by $N_{U'}$ in [N1]).

1.3.4. Hecke correspondences. For each $g \in B'^\times$, the right multiplication $[\cdot g] : [z, b'] \mapsto [z, b'g]$ and the diagram

$$\begin{array}{ccc} M_{(U' \cap gU'g^{-1})Y} & \xrightarrow{[\cdot g]} & M_{(g^{-1}U'g \cap U')Y} \\ \Big\downarrow{\scriptstyle \mathrm{pr}} & & \Big\downarrow{\scriptstyle \mathrm{pr}'} \\ M_{U'Y} & \overset{[U'YgU'Y]}{\dashrightarrow} & M_{U'Y} \end{array} \tag{1.3.4.1}$$

define a multivalued map (a "Hecke correspondence")

$$[U'YgU'Y] : M_{U'Y} - - - \dashrightarrow M_{U'Y}. \tag{1.3.4.2}$$

1.4. ℓ-adic uniformisation of Shimura curves

Let U, S and ℓ be as in 1.2.1. Denote by O_ℓ^{ur} (resp. by O_{ℓ^2}) the ring of integers in the maximal unramified extension F_ℓ^{ur} (resp. in the unramified quadratic extension $F_{\ell^2} \subset F_\ell^{ur}$) of F_ℓ and by $\widehat{O}_\ell^{ur}$ (resp. $\mathbf{C}_\ell$) the completion of O_ℓ^{ur} (resp. of $\overline{F}_\ell$) with respect to the ℓ-adic topology.

1.4.1. Theorem of Čerednik and Drinfeld. Fix an isomorphism $B_\ell \xrightarrow{\sim} M_2(F_\ell)$. According to [Če] and [Dr] (see also [BC, Thm. 5.2], [BZ, Thm. 3.1] and [V, Thm. 5.3]) there is a natural integral model $\mathbf{M}_{U'}$ of $M_{U'} \otimes_F F_\ell$ over O_ℓ, whose completion $\mathcal{M}_{U'}$ along the special fibre is canonically identified with

$$B^\times \backslash (\widehat{\mathscr{H}_\ell} \widehat{\otimes}_{O_\ell} \widehat{O}_\ell^{ur}) \times \widehat{B}^{(\ell)\times}/U^{(\ell)},$$

where $\widehat{\mathscr{H}_\ell}$ is the formal scheme over $\mathrm{Spf}(O_\ell)$ [BC, I.3] which is a natural formal model of Drinfeld's ℓ-adic half plane $\mathbf{P}^1(\mathbf{C}_\ell) - \mathbf{P}^1(F_\ell)$ and $b \in B^\times$ acts on $\widehat{\mathscr{H}_\ell}$ (resp. on $\widehat{O}_\ell^{ur}$) via the natural action of $B^\times \subset B_\ell^\times \xrightarrow{\sim} GL_2(F_\ell)$ on $\mathbf{P}^1_{F_\ell}$ (resp. by $\mathrm{Fr}_{\mathrm{geom}}(\ell)^{\mathrm{ord}_\ell(\mathrm{Nrd}(b))}$).

Denote by $\mathcal{M}_{U'}^{\mathrm{an}}$ the corresponding rigid analytic space over F_ℓ. As in 1.3.2, the components of

$$\mathcal{M}_{U'}^{\mathrm{an}}(\mathbf{C}_\ell) = B^\times \backslash (\mathbf{P}^1(\mathbf{C}_\ell) - \mathbf{P}^1(F_\ell)) \times \widehat{B}^{(\ell)\times}/U^{(\ell)}$$

are in bijection with

$$B^\times \backslash \mathbf{Z} \times \widehat{B}^{(\ell)\times}/U^{(\ell)} = B^\times \backslash \mathrm{Nrd}(B_\ell^\times/U_\ell) \times \widehat{B}^{(\ell)\times}/U^{(\ell)} \xrightarrow{\sim} F_+^\times \backslash \widehat{F}^\times / O_\ell^\times \mathrm{Nrd}(U^{(\ell)}) = F_+^\times \backslash \widehat{F}^\times/\mathrm{Nrd}(U'),$$

where the middle bijection is again induced by the reduced norm, this time for B.

For $z \in \mathbf{P}^1(\mathbf{C}_\ell) - \mathbf{P}^1(F_\ell)$ and $b \in \widehat{B}^{(\ell)\times}$, denote by $[z, b]_U \in M_{U'}(\mathbf{C}_\ell)$ the point represented by (z, b).

1.4.2. ℓ-adic uniformisation of the quotient Shimura curve. The action (1.3.3.1) of $g \in \widehat{F}^\times = Z(\widehat{B}'^\times)$ on $M_{U'}$ extends to an action on $\mathbf{M}_{U'}$. The corresponding action on $\mathcal{M}_{U'}$ is given by the following formula: write $g = g_\ell g^{(\ell)}$ with $g_\ell \in F_\ell^\times$ and $g^{(\ell)} \in \widehat{F}^{(\ell)\times}$; then g acts on $\widehat{\mathscr{H}_\ell} \widehat{\otimes}_{O_\ell} \widehat{O}_\ell^{ur}$ (resp. on $\widehat{B}^{(\ell)\times}$) by $\mathrm{id} \times \mathrm{Fr}_{\mathrm{geom}}(\ell)^{\mathrm{ord}_\ell(\mathrm{Nrd}(g_\ell))}$ (resp. by multiplication by $g^{(\ell)}$).

Fix an open subgroup $Y \subset \widehat{F}^\times$ **containing** $F_\ell^\times$ (i.e., such that $Y = Y_\ell \times Y^{(\ell)}$, where $Y_\ell = F_\ell^\times$ and $Y^{(\ell)}$ is an open subgroup of $\widehat{F}^{(\ell)\times}$) and consider the quotient curve

$$M = M_{U'Y} = M_{U'}/Y$$

from 1.3.3. Taking this quotient (by a finite abelian group) makes sense for $\mathbf{M}_{U'}$ and $\mathcal{M}_{U'}$; this yields an integral model $\mathbf{M} = \mathbf{M}_{U'}/Y$ of M over O_ℓ and its completion $\mathcal{M} = \mathcal{M}_{U'}/Y$ along the special fibre. The field of constants $F_{U'Y}$ of M satisfies

$$\mathrm{rec}_F : F_+^\times \backslash \widehat{F}^\times/\mathrm{Nrd}(U')Y^2 \xrightarrow{\sim} \mathrm{Gal}(F_{U'Y}/F) \qquad (Y^2 = \{a^2 \mid a \in Y\}).$$

In the notation of 1.4.1 we denote by $[z, b]_{UY} \in M_{U'Y}(\mathbf{C}_\ell)$ the image of $[z, b]_U$.

The assumption $Y = F_\ell^\times \times Y^{(\ell)}$ implies that

$$\mathcal{M} = B^\times \backslash (\widehat{\mathscr{H}_\ell} \widehat{\otimes}_{O_\ell} \widehat{O}_\ell^{ur}) \times \widehat{B}^{(\ell)\times}/U^{(\ell)}Y = B^\times \backslash (\widehat{\mathscr{H}_\ell} \otimes_{O_\ell} O_{\ell^2}) \times \widehat{B}^{(\ell)\times}/U^{(\ell)}Y^{(\ell)}.$$

Write

$$\widehat{B}^{(\ell)\times} = \coprod_i B^\times \alpha_i U^{(\ell)} Y^{(\ell)}$$

as a disjoint union of double cosets and set, for each i,

$$\Gamma_i = B^\times \cap \alpha_i U^{(\ell)} Y^{(\ell)} \alpha_i^{-1};$$

there is an isomorphism

$$\coprod_i \Gamma_i \backslash (\widehat{\mathscr{H}_\ell} \otimes_{O_\ell} O_{\ell^2}) \xrightarrow{\sim} \mathcal{M},$$

sending $\Gamma_i z$ to the class represented by the pair (z, α_i). As the subgroup

$$\Gamma_{i,+} := \{\gamma \in \Gamma_i \mid \mathrm{ord}_\ell(\mathrm{Nrd}(\gamma)) \equiv 0 \pmod 2\} \subset \Gamma_i$$

acts trivially on O_{ℓ^2}, we obtain (cf. [BC, 5.3.3]) an isomorphism

$$\mathcal{M} \xrightarrow{\sim} \coprod_i W_i \backslash \left((\overline{\Gamma}_{i,+} \backslash \widehat{\mathscr{H}_\ell}) \otimes_{O_\ell} O_{\ell^2} \right), \tag{1.4.2.1}$$

where $W_i = \Gamma_i/\Gamma_{i,+}$ is a group of order 1 or 2 and where we have denoted by $\overline{\Gamma}_{i,+}$ the image of $\Gamma_{i,+}$ in $B_\ell^\times/F_\ell^\times \xrightarrow{\sim} PGL_2(F_\ell)$.

1.5. Bad reduction of the (quotient) Shimura curve at ℓ

Let U, S and ℓ be as in 1.2.1. In addition, let $Y = Y_\ell \times Y^{(\ell)} = F_\ell^\times \times Y^{(\ell)}$ be as in 1.4.2.

1.5.1. Bruhat-Tits tree. Denote by $\mathcal{T}_\ell$ the Bruhat-Tits tree attached to $B_\ell^\times \xrightarrow{\sim} GL_2(F_\ell)$. Its set of vertices is equal to $\mathcal{V}(\mathcal{T}_\ell) = B_\ell^\times/U_\ell F_\ell^\times$, the set of oriented edges (= of ordered pairs of adjacent vertices) to $\overrightarrow{\mathcal{E}}(\mathcal{T}_\ell) = B_\ell^\times/U(\ell)_\ell F_\ell^\times$, the incidence relation is given by the maps

$$s, t : \overrightarrow{\mathcal{E}}(\mathcal{T}_\ell) \longrightarrow \mathcal{V}(\mathcal{T}_\ell), \qquad s(gU(\ell)_\ell F_\ell^\times) = gU_\ell F_\ell^\times, \qquad t(gU(\ell)_\ell F_\ell^\times) = g\xi_\ell U_\ell F_\ell^\times$$

(s = source, t = target) and the inversion of an edge by

$$\iota : \overrightarrow{\mathcal{E}}(\mathcal{T}_\ell) \longrightarrow \overrightarrow{\mathcal{E}}(\mathcal{T}_\ell), \qquad gU(\ell)_\ell F_\ell^\times \mapsto g \begin{pmatrix} 0 & 1 \\ \varpi_\ell & 0 \end{pmatrix} U(\ell)_\ell F_\ell^\times = g\xi_\ell \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U(\ell)_\ell F_\ell^\times.$$

Denote by $\mathcal{E}(\mathcal{T}_\ell) = \overrightarrow{\mathcal{E}}(\mathcal{T}_\ell)/\{\mathrm{id}, \iota\}$ the set of non-oriented edges of $\mathcal{T}_\ell$.

1.5.2. Special fibre of $\widehat{\mathscr{H}_\ell}$. The group $B_\ell^\times \xrightarrow{\sim} GL_2(F_\ell)$ acts on $\widehat{\mathscr{H}_\ell}$. The special fibre of $\widehat{\mathscr{H}_\ell}$ is identified, in a $B_\ell^\times$-equivariant way, with a collection of projective lines $\mathbf{P}^1_{k(\ell)}$ glued together according to the incidence relation given by $\mathcal{T}_\ell$; in other words, with the coequaliser of the pair of morphisms

$$s, t : \coprod_{\overrightarrow{\mathcal{E}}(\mathcal{T}_\ell)} \mathrm{Spec}(k(\ell)) \longrightarrow \coprod_{\mathcal{V}(\mathcal{T}_\ell)} \mathbf{P}^1_{k(\ell)},$$

where s (resp. t) maps the point $\mathrm{Spec}(k(\ell))$ corresponding to an oriented edge e to ∞ (resp. to 0) on the copy of $\mathbf{P}^1_{k(\ell)}$ corresponding to the vertex $s(e)$ (resp. to $t(e)$).

1.5.3. Special fibre of C_i. According to Kurihara [Ku, Prop. 3.2], who extended earlier results of Mumford [Mu], each quotient $C_i = \overline{\Gamma}_{i,+} \backslash \widehat{\mathscr{H}_\ell}$ from (1.4.2.1) is an admissible curve over O_ℓ, in the terminology of [JL, § 3]:

1.5.3.1. C_i is a proper and flat curve over O_ℓ with a smooth generic fibre.

1.5.3.2. The special fibre $C_i \otimes_{O_\ell} k(\ell)$ is reduced; the normalisation of each of its irreducible components is isomorphic to $\mathbf{P}^1_{k(\ell)}$; its only singular points are ordinary double points, rational over $k(\ell)$.

1.5.3.3. The completion of the local ring of C_i at each of its singular points x is isomorphic, as an O_ℓ-algebra, to $O_\ell[[X, Y]]/(XY - \varpi_\ell^w)$, where $w = w(x) \in \{1, 2, 3, \ldots\}$.

In addition, the combinatorics of the special fibre is described as follows.

1.5.3.4. The set of irreducible components of $C_i \otimes_{O_\ell} k(\ell)$ is naturally identified with $\overline{\Gamma}_{i,+}\backslash \mathcal{V}(\mathcal{T}_\ell)$.
1.5.3.5. The set of singular points of $C_i \otimes_{O_\ell} k(\ell)$ is naturally identified with $(\overline{\Gamma}_{i,+}\backslash \mathcal{E}(\mathcal{T}_\ell))^*$, where the star in the superscript refers to the fact that we remove from $\overline{\Gamma}_{i,+}\backslash \mathcal{E}(\mathcal{T}_\ell)$ the images of those oriented edges $e \in \overrightarrow{\mathcal{E}}(\mathcal{T}_\ell)$ for which there exits $\gamma \in \overline{\Gamma}_{i,+}$ such that $\gamma(e) = \iota(e)$. As we are going to see in 1.5.4 below, no such edges exist in our case; as a result, $(\overline{\Gamma}_{i,+}\backslash \mathcal{E}(\mathcal{T}_\ell))^* = \overline{\Gamma}_{i,+}\backslash \mathcal{E}(\mathcal{T}_\ell)$.
1.5.3.6. The incidence relation between the irreducible components and the singular points is inherited from $\mathcal{T}_\ell$.
1.5.3.7. If x is a singular point of $C_i \otimes_{O_\ell} k(\ell)$ represented by an oriented edge $e \in \overrightarrow{\mathcal{E}}(\mathcal{T}_\ell)$, then the integer $w(x)$ from 1.5.3.3 is equal to the order of the stabiliser $(\overline{\Gamma}_{i,+})_e$ of e in $\overline{\Gamma}_{i,+}$.

Kurihara rephrased 1.5.3.4–1.5.3.7 by saying that the dual graph $\mathcal{G}(C_i)$ of the special fibre of C_i is equal to

$$\mathcal{V}(\mathcal{G}(C_i)) = \overline{\Gamma}_{i,+}\backslash \mathcal{V}(\mathcal{T}_\ell), \qquad \overrightarrow{\mathcal{E}}(\mathcal{G}(C_i)) = (\overline{\Gamma}_{i,+}\backslash \overrightarrow{\mathcal{E}}(\mathcal{T}_\ell))^*,$$

and is equipped with the function

$$w : \overrightarrow{\mathcal{E}}(\mathcal{G}(C_i)) \longrightarrow \{1, 2, 3, \ldots\}$$

given by 1.5.3.7.

1.5.4. Special fibre of $\mathcal{M} \otimes_{O_\ell} O_{\ell^2}$. The formula (1.4.2.1) and the discussion in the previous paragraph imply that $\mathcal{M} \otimes_{O_\ell} O_{\ell^2}$ is an admissible curve over O_{ℓ^2}. Moreover, the dual graph $\mathcal{G}$ of the special fibre $\mathcal{M} \otimes_{O_\ell} k(\ell^2)$ (whose vertices correspond to irreducible components and edges to singular points) is given by

$$\mathcal{V}(\mathcal{G}) = \coprod_i \Gamma_i\backslash\left(\mathcal{V}(\mathcal{T}_\ell) \times \mathbf{Z}/2\mathbf{Z}\right), \qquad \overrightarrow{\mathcal{E}}(\mathcal{G}) = \coprod_i \left(\Gamma_i\backslash(\overrightarrow{\mathcal{E}}(\mathcal{T}_\ell) \times \mathbf{Z}/2\mathbf{Z})\right)^*,$$

where $\gamma \in \Gamma_i \subset B^\times$ acts on $\mathbf{Z}/2\mathbf{Z}$ by translation by $\mathrm{ord}_\ell(\mathrm{Nrd}(\gamma)) \pmod 2$. In adelic terms,

$$\mathcal{V}(\mathcal{G}) = B^\times\backslash\left(\mathcal{V}(\mathcal{T}_\ell) \times \mathbf{Z}/2\mathbf{Z} \times \widehat{B}^{(\ell)\times}/U^{(\ell)}Y^{(\ell)}\right) = B^\times\backslash\left(B_\ell^\times/U_\ell Y_\ell \times \mathbf{Z}/2\mathbf{Z} \times \widehat{B}^{(\ell)\times}/U^{(\ell)}Y^{(\ell)}\right) \xrightarrow{\sim}$$
$$\xrightarrow{\sim} \left(B^\times\backslash\widehat{B}^\times/UY\right) \times \mathbf{Z}/2\mathbf{Z},$$

where the last bijection is given by

$$B^\times\left(b_\ell U_\ell Y_\ell,\, j,\, b^{(\ell)}U^{(\ell)}Y^{(\ell)}\right) \mapsto \left(B^\times b_\ell b^{(\ell)} UY,\, j + \mathrm{ord}_\ell(\mathrm{Nrd}(b_\ell))\right).$$

The same formula for $U(\ell)$ instead of U induces a bijection

$$\coprod_i \Gamma_i\backslash\left(\overrightarrow{\mathcal{E}}(\mathcal{T}_\ell) \times \mathbf{Z}/2\mathbf{Z}\right) \xrightarrow{\sim} B^\times\backslash\left(B_\ell^\times/U(\ell)_\ell Y_\ell \times \mathbf{Z}/2\mathbf{Z} \times \widehat{B}^{(\ell)\times}/U^{(\ell)}Y^{(\ell)}\right) \xrightarrow{\sim} \left(B^\times\backslash\widehat{B}^\times/U(\ell)Y\right) \times \mathbf{Z}/2\mathbf{Z}.$$

The maps

$$s, t : \left(B^\times\backslash\widehat{B}^\times/U(\ell)Y\right) \times \mathbf{Z}/2\mathbf{Z} \longrightarrow \left(B^\times\backslash\widehat{B}^\times/UY\right) \times \mathbf{Z}/2\mathbf{Z}$$

induced by the incidence relation on $\mathcal{T}_\ell$ are given by

$$s(B^\times bU(\ell)Y,\, j) = (B^\times bUY,\, j), \qquad t(B^\times bU(\ell)Y,\, j) = (B^\times b\xi_\ell UY,\, j+1).$$

In particular, each quotient $\Gamma_i\backslash(\mathcal{T}_\ell \times \mathbf{Z}/2\mathbf{Z})$ is a bipartite graph, hence

$$\left(\Gamma_i\backslash(\overrightarrow{\mathcal{E}}(\mathcal{T}_\ell) \times \mathbf{Z}/2\mathbf{Z})\right)^* = \Gamma_i\backslash(\overrightarrow{\mathcal{E}}(\mathcal{T}_\ell) \times \mathbf{Z}/2\mathbf{Z}), \qquad \overrightarrow{\mathcal{E}}(\mathcal{G}) \xrightarrow{\sim} \left(B^\times\backslash\widehat{B}^\times/U(\ell)Y\right) \times \mathbf{Z}/2\mathbf{Z}.$$

1.5.5. Proposition. (1) *$\mathcal{M} \otimes_{O_\ell} O_{\ell^2}$ is an admissible curve over O_{ℓ^2}.*
(2) *The dual graph $\mathcal{G}$ of the special fibre of $\mathcal{M} \otimes_{O_\ell} O_{\ell^2}$ is bipartite, with*

$$\mathcal{V}(\mathcal{G}) = \left(B^\times\backslash\widehat{B}^\times/UY\right) \times \mathbf{Z}/2\mathbf{Z}, \qquad \overrightarrow{\mathcal{E}}(\mathcal{G}) = \left(B^\times\backslash\widehat{B}^\times/U(\ell)Y\right) \times \mathbf{Z}/2\mathbf{Z},$$
$$s(B^\times bU(\ell)Y,\, j) = (B^\times bUY,\, j), \qquad t(B^\times bU(\ell)Y,\, j) = (B^\times b\xi_\ell UY,\, j+1),$$
$$\iota(B^\times bU(\ell)Y,\, j) = (B^\times b \begin{pmatrix} 0 & 1 \\ \varpi_\ell & 0 \end{pmatrix}_\ell U(\ell)Y, j+1).$$

(3) *If there exists an integer $N > 2$ such that $\{v \mid N\} \cap \mathrm{Ram}(B) = \emptyset$ and $\forall v \mid N \quad U_v \subseteq 1 + NR(v)$ for some maximal order $R(v) \subset B_v \xrightarrow{\sim} M_2(F_v)$, then the curve $\mathcal{M} \otimes_{O_\ell} O_{\ell^2}$ is semi-stable ($\Longleftrightarrow$ the integers w attached to the singular points of $\mathcal{M} \otimes_{O_\ell} k(\ell^2)$ are all equal to 1).*

Proof. The statements (1) and (2) follow from the discussion in 1.5.4. The assertion (3) is a consequence of 1.5.3.7 and the following Lemma.

1.5.6. Lemma. *If U satisfies the assumptions of 1.5.5(3), then, for any $t \in \widehat{B}^\times$, the group $\Gamma_t := B^\times \cap tU\widehat{F}^\times t^{-1}$ ($\supseteq F^\times$) is equal to $F^\times$.*

Proof. Suppose first that $\gamma \in \Gamma_t^{\mathrm{Nrd}=1}$, $\gamma \notin F^\times$. In this case $\gamma \in B^\times \cap tU(\widehat{O}_F)^\times t^{-1}$ is integral over O_F, $K := F(\gamma) \subset B$ is a totally imaginary quadratic extension of F and $N_{K/F}(\gamma) = 1$, which implies that the roots of the characteristic polynomial $X^2 - \mathrm{Trd}(\gamma)X + \mathrm{Nrd}(\gamma) = X^2 - \mathrm{Tr}_{K/F}X + 1$ are ζ and ζ^{-1}, where $\zeta \neq \zeta^{-1}$ is a root of unity such that $\zeta + \zeta^{-1} \in O_F$. Writing $\gamma = tuu't^{-1}$ with $u \in U$ and $u' \in (\widehat{O}_F)^\times$, we have $\zeta + \zeta^{-1} = u' \,\mathrm{Trd}(u)$, $u'^2 \,\mathrm{Nrd}(u) = 1$ and $\mathrm{Nrd}(u) \in 1 + N\widehat{O}_F$, which implies that, for each $v \mid N$ and a suitable choice of a sign, $\pm\gamma \in I + NM_2(O_{F,v})$ is a matrix of finite order, which implies that $\pm\gamma = I$, since $N > 2$. This shows that $\Gamma_t^{\mathrm{Nrd}=1} = (F^\times)^{\mathrm{Nrd}=1} = \{\pm 1\}$.

If $\gamma \in \Gamma_t$, then $\mathrm{Nrd}(\gamma)\gamma^{-2} = \overline{\gamma}\gamma^{-1} \in \Gamma_t^{\mathrm{Nrd}=1} = \{\pm 1\}$ (where $\gamma \mapsto \overline{\gamma}$ is the standard involution on B). If $\overline{\gamma}\gamma^{-1} = 1$, then $\gamma \in F^\times$. If $\overline{\gamma}\gamma^{-1} = -1$, then $\mathrm{Trd}(\gamma) = \gamma + \overline{\gamma} = 0$. However, $\gamma = tuu't^{-1}$ with $u \in U$ and $u' \in \widehat{F}^\times$, which implies that $\mathrm{Trd}(\gamma) = u' \,\mathrm{Trd}(u) \neq 0$, since $\mathrm{Trd}(u) \in 2 + N\widehat{O}_F$ and $N > 2$. This contradiction shows that $\Gamma_t = F^\times$, as claimed.

1.5.7. (Co)homology of $\mathcal{G}$. As the graph $\mathcal{G}$ is bipartite, it has two natural orientations (by an orientation of $\mathcal{G}$ we mean a section of the canonical projection $\overrightarrow{\mathcal{E}}(\mathcal{G}) \longrightarrow \overrightarrow{\mathcal{E}}(\mathcal{G})/\{\mathrm{id}, \iota\} = \mathcal{E}(\mathcal{G})$). Fix one of them, say, the following one:

$$\mathcal{E}(\mathcal{G}) = B^\times\backslash\widehat{B}^\times/U(\ell)Y \xrightarrow{\sim} \left(B^\times\backslash\widehat{B}^\times/U(\ell)Y\right) \times \{0\} \subset \left(B^\times\backslash\widehat{B}^\times/U(\ell)Y\right) \times \mathbf{Z}/2\mathbf{Z} = \overrightarrow{\mathcal{E}}(\mathcal{G}),$$

for which

$$s : \mathcal{E}(\mathcal{G}) \longrightarrow \left(B^\times\backslash\widehat{B}^\times/UY\right) \times \{0\}, \qquad s(B^\times bU(\ell)Y) = (B^\times bUY, 0),$$
$$t : \mathcal{E}(\mathcal{G}) \longrightarrow \left(B^\times\backslash\widehat{B}^\times/UY\right) \times \{1\}, \qquad t(B^\times bU(\ell)Y) = (B^\times b\xi_\ell UY, 1).$$

The chain and cochain complexes of $\mathcal{G}$

$$\mathbf{Z}[\mathcal{E}(\mathcal{G})] \xrightarrow{d_* = -s_* + t_*} \mathbf{Z}[\mathcal{V}(\mathcal{G})], \qquad \mathbf{Z}[\mathcal{V}(\mathcal{G})] \xrightarrow{d^* = -s^* + t^*} \mathbf{Z}[\mathcal{E}(\mathcal{G})]$$

are then identified, respectively, with

$$\mathbb{S}(U(\ell)Y; \mathbf{Z}) \xrightarrow{(-\alpha_*, \beta_*)} \mathbb{S}(UY; \mathbf{Z})^{\oplus 2}, \qquad \mathbb{S}(UY; \mathbf{Z})^{\oplus 2} \xrightarrow{-\alpha^* + \beta^*} \mathbb{S}(U(\ell)Y; \mathbf{Z}),$$

in the notation of 1.2.1 (in particular, both maps d_* and d^* are $\mathbb{T}^{S\cup\{\ell\}}(U(\ell))$-linear). As a result,

$$H_1(\mathcal{G}) = \mathrm{Ker}(d_*) = \mathbb{S}(U(\ell)Y; \mathbf{Z})^{\ell-\mathrm{new}}$$

and

$$H^0(\mathcal{G}) = \{(\xi_\ell \cdot f, f) \mid f \in \mathbb{S}(UY;\mathbf{Z})_{\mathrm{triv}}\},$$

by Proposition 1.2.3(2). Set

$$\mathbf{Z}[\mathcal{V}(\mathcal{G})]_0 := \mathrm{Ker}(\mathbf{Z}[\mathcal{V}(\mathcal{G})] \longrightarrow H_0(\mathcal{G})) = \mathrm{Im}(d_*).$$

1.5.8. Scalar products. Under the assumptions of 1.5.5(3), the formulas

$$u : \mathbf{Z}[\mathcal{E}(G)] \times \mathbf{Z}[\mathcal{E}(G)] = \mathbb{S}(U(\ell)Y;\mathbf{Z}) \times \mathbb{S}(U(\ell)Y;\mathbf{Z}) \longrightarrow \mathbf{Z}, \qquad u(f,f') = \sum_{e\in\mathcal{E}(G)} f(e)f'(e)$$

and

$$u' : \mathbf{Z}[\mathcal{V}(G)] \times \mathbf{Z}[\mathcal{V}(G)] = \mathbb{S}(UY;\mathbf{Z})^{\oplus 2} \times \mathbb{S}(UY;\mathbf{Z})^{\oplus 2} \longrightarrow \mathbf{Z}, \qquad u'(f,f') = \sum_{a\in\mathcal{V}(G)} f(a)f'(a)$$

define non-degenerate symmetric bilinear pairings satisfying

$$\forall T \in \mathbf{Z}[U(\ell)\backslash \widehat{B}^\times/U(\ell)] \qquad u(Tf,f') = u(f,T^\vee f'),$$
$$\forall T \in \mathbf{Z}[U\backslash \widehat{B}^\times/U] \qquad u'(Tf,f') = u'(f,T^\vee f')$$

and

$$\forall f \in \mathbf{Z}[\mathcal{E}(G)] \ \ \forall f' \in \mathbf{Z}[\mathcal{V}(G)] \qquad u(f,s^*f') = u'(s_*f,f'), \qquad u(f,t^*f') = u'(t_*f,f'). \tag{1.5.8.1}$$

In particular, for each finite prime $v \notin S\cup\{\ell\}$ of F, the adjoint of $T(v)$ (resp. of $S(v)$) with respect to u or u' is equal to $T(v)^\vee = S(v)^{-1}T(v)$ (resp. to $S(v)^\vee = S(v)^{-1}$).

1.5.9. Proposition. *Let $A \subset \mathbf{C}$ be a subring. Let $v_0 \notin S\cup\{\ell\}$ be a finite prime of F with trivial class in $F_+^\times\backslash \widehat{F}^\times/\mathrm{Nrd}(U)$.*
(1) We have $(T(v_0) - N(v_0) - 1)\,\mathbb{S}(UY;A)^{\oplus 2} \subseteq A[\mathcal{V}(\mathcal{G})]_0$, where $A[\mathcal{V}(\mathcal{G})]_0 := \mathbf{Z}[\mathcal{V}(\mathcal{G})]_0 \otimes A$.
(2) If $\widetilde{f} \in \mathbb{S}(UY;A)_0^{\oplus 2}$ satisfies $T(v_0)\widetilde{f} = \lambda\widetilde{f}$ for some $\lambda \in A$, then $\widetilde{f} \in A[\mathcal{V}(\mathcal{G})]_0$. In particular, if $\widetilde{f} \in \mathbb{S}(UY;A)_0^{\oplus 2}$ is an eigenform for the action of $\mathbb{T}^{S\cup\{\ell\}}(U)$, then $\widetilde{f} \in A[\mathcal{V}(\mathcal{G})]_0$.

Proof. (1) Under the non-degenerate pairing $A[\mathcal{V}(G)] \times A[\mathcal{V}(G)] \longrightarrow A$ obtained from u' by extending the scalars, the orthogonal complement of $A[\mathcal{V}(\mathcal{G})]_0$ is equal to

$$(A[\mathcal{V}(\mathcal{G})]_0)^\perp = (\mathrm{Im}(-s_*+t_*))^\perp = \mathrm{Ker}(-s^*+t^*) = \{(\xi_\ell\cdot f, f) \mid f \in \mathbb{S}(UY;A)_{\mathrm{triv}}\},$$

where the second (resp. the third) equality follows from (1.5.8.1) (resp. from Proposition 1.2.3(2)). The elements $S(v_0)-1$ and $T(v_0)-N(v_0)-1$ annihilate $\mathbb{S}(UY;A)_{\mathrm{triv}}$, which implies that both $S(v_0)^\vee - 1 = S(v_0)^{-1} - 1$ and $T(v_0)^\vee - N(v_0) - 1 = S(v_0)^{-1}T(v_0) - N(v_0) - 1$ (hence also $T(v_0)-N(v_0)-1$) annihilate $(A[\mathcal{V}(\mathcal{G})]/A[\mathcal{V}(\mathcal{G})]_0)\otimes\mathbf{Q} \supset A[\mathcal{V}(\mathcal{G})]/A[\mathcal{V}(\mathcal{G})]_0$ (recall that $A[\mathcal{V}(\mathcal{G})]/A[\mathcal{V}(\mathcal{G})]_0 = H_0(\mathcal{G})\otimes A$ is a free A-module). It follows that

$$(T(v_0)-N(v_0)-1)\,\mathbb{S}(UY;A)^{\oplus 2} = (T(v_0)-N(v_0)-1)\,A[\mathcal{V}(\mathcal{G})] \subseteq A[\mathcal{V}(\mathcal{G})]_0.$$

(2) We know by (1) that the image of $(\lambda - N(v_0)-1)\widetilde{f}$ in the free A-module $A[\mathcal{V}(\mathcal{G})]/A[\mathcal{V}(\mathcal{G})]_0$ is trivial. As $\lambda - N(v_0) - 1 \in A - \{0\}$ by (1.1.7.1), the image of $\widetilde{f}$ is trivial, too.

1.6. Bad reduction of the Jacobian of the Shimura curve at ℓ

Let U, S and ℓ be as in 1.2.1. Let $Y = Y_\ell \times Y^{(\ell)} = F_\ell^\times \times Y^{(\ell)}$ be as in 1.4.2. In addition, assume that U satisfies the condition from 1.5.5(3).

1.6.1. Components and geometric components. Recall that $M = M_{U'Y}$ is an irreducible smooth projective curve over F, whose field of constants $F' := F_{U'Y} \subset F^{\mathrm{ab}}$ satisfies

$$\mathrm{rec}_F : F_+^\times \backslash \widehat{F}^\times / \mathrm{Nrd}(U')Y^2 \xrightarrow{\sim} \mathrm{Gal}(F'/F).$$

For any field $L \supset F$, the set of irreducible (= connected) components of $M \otimes_F L$ is in bijection with $\mathrm{Spec}(F' \otimes_F L)$. As $\mathrm{Nrd}(U')Y^2 \supset O_\ell^\times F_\ell^{\times 2}$, the completion of F' at any prime above ℓ is isomorphic to F_ℓ or F_{ℓ^2}, which implies that $F' \otimes_F F_{\ell^2} \xrightarrow{\sim} F_{\ell^2}^{[F':F]}$. As a result, each irreducible component of $M \otimes_F F_{\ell^2}$ is geometrically irreducible.

1.6.2. Jacobian. The Jacobian of M

$$J(M) := \mathrm{Pic}^\circ_{M/F} \xrightarrow{\sim} \mathrm{Res}_{F'/F}\,\mathrm{Pic}^\circ_{M/F'}$$

is an abelian variety defined over F. If $L \supset F$ is a field and D is a divisor on $M \otimes_F L$ which has degree zero on each connected component of $M \otimes_F \overline{L}$, then D represents a point $cl(D) \in J(M)(L)$.

1.6.3. Actions of Hecke correspondences. For any cohomology theory $H(-)$ which admits trace maps for finite flat morphisms between curves we let the Hecke correspondence $[U'YgU'Y]$ from (1.3.4.1-2) ($g \in \widehat{B}'^\times$) act on $H(M) = H(M_{U'Y})$ as follows (by "Picard functoriality"):

$$H(M_{U'Y}) \xrightarrow{\mathrm{pr}'^*} H(M_{(g^{-1}U'g\cap U')Y}) \xrightarrow{[\cdot g]^*} H(M_{(U'\cap gU'g^{-1})Y}) \xrightarrow{\mathrm{pr}_*} H(M_{U'Y}). \tag{1.6.3.1}$$

This formula (which also applies to functors such as $H(-) = \mathrm{Pic}^\circ_{-/F} = J(-)$ or $H(-) = \Gamma(-, \Omega_{-/F})$) defines a ring homomorphism

$$\mathbf{Z}[U'Y\backslash \widehat{B}'^\times/U'Y] \longrightarrow \mathrm{End}_{\mathbf{Z}}(H(M_{U'Y})). \tag{1.6.3.2}$$

As in 1.1.5, denote by $\mathbb{T}'^{S\cup\{\ell\}}(U'Y)$ the (commutative) subring of $\mathbf{Z}[U'Y\backslash \widehat{B}'^\times/U'Y]$ generated by the double cosets $[U'YxU'Y]$ for all $x \in (\widehat{B}'^{S\cup\{\ell\}})^\times$. The isomorphism $\varphi : \widehat{B}^{(\ell)} \xrightarrow{\sim} \widehat{B}'^{(\ell)}$ induces a ring isomorphism

$$\varphi_* : \mathbb{T}^{S\cup\{\ell\}}(UY) \xrightarrow{\sim} \mathbb{T}'^{S\cup\{\ell\}}(U'Y), \qquad \varphi_*([UYgUY]) = [U'Y\varphi(g)U'Y] \qquad (g \in (\widehat{B}^{S\cup(\ell)})^\times). \tag{1.6.3.3}$$

For each $v \notin S \cup \{\ell\}$, the elements

$$T'(v) := \varphi_*(T(v)), \qquad S'(v) := \varphi_*(S(v)) \in \mathbb{T}'^{S\cup\{\ell\}}(U'Y)$$

are independent of the choice of φ. Combining (1.6.3.2) with (1.6.3.3) we obtain a ring homomorphism

$$\mathbb{T}^{S\cup\{\ell\}}(UY) \longrightarrow \mathrm{End}_{\mathbf{Z}}(H(M_{U'Y})). \tag{1.6.3.4}$$

1.6.4. Néron model. Denote by J the Néron model of $J(M) \otimes_F F_{\ell^2}$ over O_{ℓ^2}, by $J_s = J \otimes_{O_{\ell^2}} k(\ell^2)$ its special fibre and by $\Phi = J_s/J_s^\circ$ the étale group scheme over $k(\ell^2)$ of connected components of J_s.

As $M \otimes_F F_{\ell^2}$ has a semi-stable model $\mathbf{M} \otimes_{O_\ell} O_{\ell^2}$ whose special fibre consists of several copies of $\mathbf{P}^1_{k(\ell^2)}$ intersecting at ordinary double points defined over $k(\ell^2)$, the general theory [NM, ch. 9] tells us that J_s° is a split torus over $k(\ell^2)$ and Φ is a constant group scheme. By abuse of language we identify Φ with the finite abelian group $\Phi(\overline{k(\ell)})$.

1.6.5. Connected components of the Néron model. There are two equivalent descriptions of Φ, due to Raynaud [Ra 1, Prop. 8.1.2] (see also [NM, Thm. 1 in 9.6]) and Grothendieck [G2, Thm. 11.5, Thm. 12.5], respectively. They can be summed up by the following commutative diagram with exact rows and columns, which we have borrowed from [Ed].

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \big\downarrow & & \big\downarrow & & \\
 & & X & = & X & & \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle i} & & \\
\mathbf{Z}[\mathcal{V}(\mathcal{G})] & \xrightarrow{d^*} & \mathbf{Z}[\mathcal{E}(\mathcal{G})] & \longrightarrow & X^\vee & \longrightarrow & 0 \\
\big\downarrow{\scriptstyle -\mathrm{id}} & & \big\downarrow{\scriptstyle d_*} & & \big\downarrow & & \\
\mathbf{Z}[\mathcal{V}(\mathcal{G})] & \xrightarrow{\mu_0} & \mathbf{Z}[\mathcal{V}(\mathcal{G})]_0 & \longrightarrow & \Phi & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \\
 & & 0 & & 0 & &
\end{array}
\tag{1.6.5.1}
$$

In this diagram, the free abelian group

$$X := \mathrm{Ker}(d_*) = H_1(\mathcal{G}) = \mathbb{S}(U(\ell)Y;\mathbf{Z})^{\ell-\mathrm{new}} \xrightarrow{\sim} X^*(J_s^\circ)$$

is canonically isomorphic to the character group of the torus J_s° ([Ed, p. 140]). The map

$$i : X \longrightarrow X^\vee = \mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{Z}) \tag{1.6.5.2}$$

is induced by the monodromy pairing

$$X \times X \hookrightarrow \mathbf{Z}[\mathcal{E}(\mathcal{G})] \times \mathbf{Z}[\mathcal{E}(\mathcal{G})] \xrightarrow{u} \mathbf{Z}, \tag{1.6.5.3}$$

where u is the scalar product from 1.5.8. The composition of μ_0 with the inclusion $\mathbf{Z}[\mathcal{V}(\mathcal{G})]_0 \hookrightarrow \mathbf{Z}[\mathcal{V}(\mathcal{G})]$ is equal to the map

$$\mu : \mathbf{Z}[\mathcal{V}(\mathcal{G})] \longrightarrow \mathbf{Z}[\mathcal{V}(\mathcal{G})], \qquad \mu(C) = \sum_{C'} (C \cdot C')\, C',$$

where $C, C' \in \mathcal{V}(\mathcal{G})$ are irreducible components of the special fibre of $\mathbf{M} \otimes_{O_\ell} O_{\ell^2}$ and $(C \cdot C') \in \mathbf{Z}$ is their intersection product on the regular scheme $\mathbf{M} \otimes_{O_\ell} O_{\ell^2}$.

1.6.6. Specialisation of divisors. Let $\mathcal{K}$ be a finite unramified extension of F_{ℓ^2}. Each divisor D on $M \otimes_F \mathcal{K}$ naturally extends to a Cartier divisor $\widetilde{D}$ on $\mathbf{M} \otimes_{O_\ell} O_{\mathcal{K}}$ (write $D = D_1 - D_2$ with effective divisors D_i and let $\widetilde{D} := \widetilde{D}_1 - \widetilde{D}_2$, where $\widetilde{D}_i$ is the closure of D_i in $\mathbf{M} \otimes_{O_\ell} O_{\mathcal{K}}$).

If D has degree zero on each connected component of $M \otimes_F \overline{\mathcal{K}}$, then $cl(D) \in J(M)(\mathcal{K}) = J(O_{\mathcal{K}})$ and its image in Φ is represented by

$$cl(D)_{\mathcal{V}} := \sum_{C \in \mathcal{V}(\mathcal{G})} (C_{\mathcal{K}} \cdot \widetilde{D})\, C \in \mathbf{Z}[\mathcal{V}(\mathcal{G})]_0, \qquad C_{\mathcal{K}} = C \otimes_{O_{\ell^2}} O_{\mathcal{K}}.$$

In the special case when $D = \sum n_P P$ is a linear combination of $\mathcal{K}$-rational points $P \in M(\mathcal{K})$, each $\widetilde{P}$ intersects $C_{\mathcal{K}}$ for exactly one irreducible component $C = C(P) \in \mathcal{V}(\mathcal{G})$ of the special fibre of $\mathbf{M} \otimes_{O_\ell} O_{\ell^2}$ (we say that P specialises to $C(P)$). Consequently,

$$cl(\sum n_P P)_{\mathcal{V}} = \sum n_P\, C(P).$$

1.6.7. Compatibility of Hecke actions. The recipe from [Ed, p. 140] defines, for any $k(\ell^2)$-algebra A, a morphism of abelian groups $\mathbf{Z}[\mathcal{E}(\mathcal{G})] \otimes A^\times \longrightarrow J_s^\circ(A)$ which is functorial in A and which sits in an exact sequence

$$\mathbf{Z}[\mathcal{V}(\mathcal{G})] \otimes A^\times \xrightarrow{d^* \otimes \mathrm{id}} \mathbf{Z}[\mathcal{E}(\mathcal{G})] \otimes A^\times \longrightarrow J_s^\circ(A) \longrightarrow 0. \tag{1.6.7.1}$$

The resulting isomorphism between $H^1(\mathcal{G}) = \mathrm{Coker}(d^*)$ and the group of cocharacters $X^\vee = X_*(J_s^\circ) = \mathrm{Hom}_{\mathbf{Z}}(X^*(J_s^\circ), \mathbf{Z})$ of the torus J_s° does not depend on the choice of orientation of $\mathcal{G}$. Moreover, for each $v \notin S \cup \{\ell\}$, the action of $T'(v)$ (resp. $S'(v)$) on $J_s^\circ(A)$ given by (1.6.3.1) for $H(-) = \mathrm{Pic}^\circ_{-/F}$ and the functoriality of the Néron model is induced by the action of $T(v)$ (resp. $S(v)$) on $\mathbf{Z}[\mathcal{E}(\mathcal{G})] = \mathbb{S}(U(\ell)Y, \mathbf{Z})$. In other words, (1.6.7.1) becomes an exact sequence of $\mathbb{T}^{S\cup\{\ell\}}(UY)$-modules (with the action on the third term via (1.6.3.4)).

We equip $X = H_1(\mathcal{G})$ (resp. $X^\vee = H^1(\mathcal{G})$) with the structure of a $\mathbb{T}^{S\cup\{\ell\}}(UY)$-module induced by the inclusion $H_1(\mathcal{G}) \subset \mathbf{Z}[\mathcal{E}(\mathcal{G})] = \mathbb{S}(U(\ell)Y, \mathbf{Z})$ (resp. by the surjection $\mathbf{Z}[\mathcal{E}(\mathcal{G})] = \mathbb{S}(U(\ell)Y, \mathbf{Z}) \longrightarrow H^1(\mathcal{G})$). The map (1.6.5.2) induces a morphism of $\mathbb{T}^{S\cup\{\ell\}}(UY)$-modules

$$i : {}^hX \longrightarrow X^\vee$$

and $\Phi = \mathrm{Coker}(i)$ inherits a $\mathbb{T}^{S\cup\{\ell\}}(UY)$-module structure as a quotient of $X^\vee$.

For any commutative ring $A \supset \mathbf{Z}$ of characteristic zero and $0 \neq m \in A$ there is an isomorphism of $\mathbb{T}^{S\cup\{\ell\}}(UY) \otimes A$-modules (which depends on m)

$$(\Phi \otimes A)[m] \xrightarrow{\sim} \mathrm{Ker}\left({}^hX \otimes A/mA \xrightarrow{i\otimes \mathrm{id}} X^\vee \otimes A/mA\right)$$

arising from snake lemma.

1.6.8. The Eichler-Shimura relation. For each integer $m \geq 1$, the canonical isomorphism

$$H^1_{et}(M \otimes_F \overline{F}, \mu_m) \xrightarrow{\sim} J[m] \qquad (1.6.8.1)$$

is $\mathbb{T}^{S\cup\{\ell\}}(UY)$-equivariant. The Eichler-Shimura congruence relation [C2, §10] states that, for every prime $v \notin S \cup \{\ell\}$ of F, the special fibre of (the flat extension to a proper smooth model of M over $O_{F,v}$) the Hecke correspondence $T(v)$ is equal to

$$T(v) \pmod v = \Gamma_{\mathrm{Fr}_v \circ [\cdot \varpi_v]} + {}^t\Gamma_{\mathrm{Fr}_v} = \Gamma_{\mathrm{Fr}_v} \circ S(v) + {}^t\Gamma_{\mathrm{Fr}_v}. \qquad (1.6.8.2)$$

The proper and smooth base change theorems for étale cohomology imply that, if v does not divide m, the G_F-module $H^1_{et}(M \otimes_F \overline{F}, \mathbf{Z}/m\mathbf{Z})$ is unramified. Letting both sides of (1.6.8.2) act on $H^1_{et}(M \otimes_F \overline{F}, \mathbf{Z}/m\mathbf{Z})$ contravariantly (as in (1.6.3.1)), we obtain the following relation:

$$\mathrm{Fr}_{\mathrm{geom}}(v)^2 - T(v)S(v)^{-1}\,\mathrm{Fr}_{\mathrm{geom}}(v) + N(v)S(v)^{-1} = 0 \in \mathrm{End}_{\mathbf{Z}}(H^1_{et}(M \otimes_F \overline{F}, \mathbf{Z}/m\mathbf{Z})). \qquad (1.6.8.3)$$

Applying the Tate twist and the involution $t \mapsto t^\vee$ yields, respectively,

$$\mathrm{Fr}_{\mathrm{geom}}(v)^2 - T(v)S(v)^{-1}N(v)^{-1}\,\mathrm{Fr}_{\mathrm{geom}}(v) + N(v)^{-1}S(v)^{-1} = 0 \in \mathrm{End}_{\mathbf{Z}}(J[m])$$

and

$$\mathrm{Fr}_{\mathrm{geom}}(v)^2 - T(v)N(v)^{-1}\,\mathrm{Fr}_{\mathrm{geom}}(v) + S(v)N(v)^{-1} = 0 \in \mathrm{End}_{\mathbf{Z}}({}^hJ[m]). \qquad (1.6.8.4)$$

1.6.9. Erratum for [N1]. The congruence relation (1.6.8.2) (which can be checked, for example, on the classical modular curve $Y_1(N)$ parameterising elliptic curves E equipped with a level structure $\mu_N \hookrightarrow E$) was stated incorrectly in [N1, (1.14.1)]. As a result, the decomposition of $H^1_{et}(M \otimes_F \overline{F}, \overline{\mathbf{Q}}_\ell)$ in [N1, Prop. 1.18(ii)] should involve the Galois representations $V_\ell(\widetilde{\pi}) = V_\ell(\pi) \otimes \omega_\pi^{-1}$ rather than $V_\ell(\pi)$.

1.6.10. Self-duality of Φ. The monodromy pairing (1.6.5.3) gives rise to a non-degenerate symmetric pairing [G2, (11.4.1)]

$$(\ ,\ ) : \Phi \times \Phi \longrightarrow \mathbf{Q}/\mathbf{Z},$$

defined as follows. According to Grothendieck's description,

$$\Phi = \mathrm{Coker}\left(X \xrightarrow{i} X^\vee = \mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{Z})\right), \qquad (i(x))(y) = u(x, y).$$

Fix an integer $m \geq 1$ such that $m\Phi = 0$. Given $x^\vee, y^\vee \in X^\vee$ there is a unique $x \in X$ such that $i(x) = mx^\vee$; the value

$$([x^\vee], [y^\vee]) := \frac{1}{m}\, y^\vee(x) + \mathbf{Z} = \frac{1}{m}\, u(x, \widetilde{y}^\vee) + \mathbf{Z} \in \frac{1}{m}\mathbf{Z}/\mathbf{Z} \subset \mathbf{Q}/\mathbf{Z} \tag{1.6.10.1}$$

is independent of the choice of m and depends only on the respective classes $[x^\vee], [y^\vee] \in \Phi$ of $x^\vee$ and $y^\vee$. Above, $\widetilde{y}^\vee \in \mathbf{Z}[\mathcal{E}(\mathcal{G})]$ is any representative of $y^\vee \in X^\vee \in \mathrm{Coker}(d^*)$.

If we do not assume that $m\Phi = 0$, the same formula defines a non-degenerate pairing

$$(\ ,\ )_m : \Phi[m] \times \Phi/m\Phi \longrightarrow \frac{1}{m}\mathbf{Z}/\mathbf{Z}$$

and, for any commutative ring $A \supset \mathbf{Z}$ of characteristic zero and $0 \neq m \in A$, a pairing

$$(\ ,\ )_m : (\Phi \otimes A)[m] \times (\Phi/m\Phi) \otimes A \longrightarrow \frac{1}{m}A/A$$

satisfying

$$\forall t \in \mathbb{T}^{S\cup\{\ell\}}(UY) \qquad (t(x), y)_m = (x, t^\vee(y))_m.$$

1.6.11. Proposition. *Let $A \supset \mathbf{Z}$ be a commutative ring and $m \geq 1$ an integer. Assume that $P, Q \in A[\mathcal{V}(\mathcal{G})]_0$ and that $mP = \mu_0(P')$ for some $P' \in A[\mathcal{V}(\mathcal{G})]$. If we denote by $[P]$ and $[Q]$ the respective images of P and Q in $(\Phi \otimes A)[m]$ and $(\Phi/m\Phi) \otimes A$, then*

$$([P], [Q])_m = \frac{1}{m}u'(P', Q) + A \in \frac{1}{m}A/A.$$

Proof. This follows easily from the definitions and the diagram (1.6.5.1): fix $\widetilde{P} \in A[\mathcal{E}(\mathcal{G})]$ such that $P = d_*\widetilde{P}$; then $[P] = [x^\vee]$, where $x^\vee$ is the image of $\widetilde{P}$ in $X^\vee \otimes A$. As $d_*(m\widetilde{P} + d^*P') = mP - \mu_0(P') = 0$, the element $x := m\widetilde{P} + d^*P'$ lies in $X \otimes A$ and satisfies $i(x) = mx^\vee$. Similarly, if we fix $\widetilde{Q} \in A[\mathcal{E}(\mathcal{G})]$ such that $Q = d_*\widetilde{Q}$ and denote by $y^\vee$ its image in $\Phi \otimes A$, then we can take $\widetilde{y}^\vee = \widetilde{Q}$ in (1.6.10.1), hence

$$([P], [Q])_m = \frac{1}{m}\, y^\vee(x) + A = \frac{1}{m}\, u(m\widetilde{P} + d^*P', \widetilde{Q}) + A = u(d^*P', \widetilde{Q}) + A = u'(P', d_*\widetilde{Q}) + A = u'(P', Q) + A.$$

1.7. ℓ-adic uniformisation of the Jacobian

The assumptions of §1.6 are in force.

1.7.1. ℓ-adic uniformisation. As J has a split totally toric reduction over F_{ℓ^2} and a canonical principal polarisation, there is a commutative diagram of $\mathbb{T}^{S\cup\{\ell\}}(UY)[G_{F_{\ell^2}}]$-modules with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}^hX & \longrightarrow & X^\vee \otimes \overline{F}_\ell^\times & \longrightarrow & J(\overline{F}_\ell) & \longrightarrow & 0 \\ & & \| & & \big\downarrow{\scriptstyle \mathrm{ord}_\ell} & & & & \\ 0 & \longrightarrow & {}^hX & \xrightarrow{\ i\ } & X^\vee \otimes \mathbf{Q} & & & & \end{array} \tag{1.7.1.1}$$

in which $G_{F_{\ell^2}} = \mathrm{Gal}(\overline{F}_\ell/F_{\ell^2})$ acts trivially on X and $X^\vee$. This yields, for every integer $m \geq 1$, an exact sequence of $\mathbb{T}^{S\cup\{\ell\}}(UY)[G_{F_{\ell^2}}]$-modules (in which $J[m] := J(\overline{F}_\ell)[m]$)

$$0 \longrightarrow X^\vee \otimes \mu_m \longrightarrow J[m] \longrightarrow {}^hX \otimes \mathbf{Z}/m\mathbf{Z} \longrightarrow 0, \tag{1.7.1.2}$$

which is self-dual (up to a sign and the involution $t \mapsto t^\vee$) with respect to the Weil pairing

$$J[m] \times J[m] \longrightarrow \mu_m \tag{1.7.1.3}$$

corresponding to the canonical principal polarisation of J.

1.7.2. Connected components. Let $\mathcal{K}$ be a finite unramified extension of F_{ℓ^2}. The canonical map $J(\mathcal{K}) = J(O_{\mathcal{K}}) \longrightarrow \Phi$ sits in a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}^hX & \longrightarrow & X^\vee \otimes \mathcal{K}^\times & \longrightarrow & J(\mathcal{K}) & \longrightarrow & 0 \\ & & \| & & \big\downarrow \mathrm{ord}_\ell & & \big\downarrow & & \\ 0 & \longrightarrow & {}^hX & \xrightarrow{i} & X^\vee & \longrightarrow & \Phi & \longrightarrow & 0, \end{array}$$

whose first row is obtained from (1.7.1.1) by taking invariants under $\mathrm{Gal}(\overline{F}_\ell/\mathcal{K})$.

1.7.3. Kummer maps. Let $I = \mathrm{Gal}(\overline{F}_\ell/F_\ell^{ur})$ be the common inertia group of F_{ℓ^2} and the field $\mathcal{K}$ from 1.7.2. Fix an integer $m \geq 1$ prime to $N(\ell)$; denote by

$$\partial : J(\mathcal{K}) \otimes \mathbf{Z}/m\mathbf{Z} \hookrightarrow H^1(\mathcal{K}, J[m])$$

the Kummer map arising from the standard descent sequence

$$0 \longrightarrow J(\mathcal{K})[m] \longrightarrow J(\mathcal{K}) \xrightarrow{m} J(\mathcal{K}) \longrightarrow H^1(\mathcal{K}, J[m]) \longrightarrow H^1(\mathcal{K}, J) \longrightarrow \cdots$$

and by ∂_{ram} ("the ramified part of ∂") the composite map

$$\partial_{\mathrm{ram}} : J(\mathcal{K}) \otimes \mathbf{Z}/m\mathbf{Z} \longrightarrow H^1(\mathcal{K}, J[m]) \longrightarrow H^1(I, J[m]).$$

The key point of the construction of Bertolini and Darmon [BD 2] is the fact that, unlike in the case of good reduction, the map ∂_{ram} can be far from being zero.

The corresponding Kummer maps for the torus $X^\vee \otimes \mathbf{G}_m$ over $\mathcal{K}$ and F_ℓ^{ur} are isomorphisms related by a commutative diagram

$$\begin{array}{ccc} X^\vee \otimes \mathcal{K}^\times \otimes \mathbf{Z}/m\mathbf{Z} & \xrightarrow{\sim} & H^1(\mathcal{K}, X^\vee \otimes \mu_m) \\ \big\downarrow \mathrm{ord}_\ell & & \big\downarrow \\ X^\vee \otimes \mathbf{Z}/m\mathbf{Z} & \xrightarrow{\sim} & H^1(I, X^\vee \otimes \mu_m) \end{array}$$

The three Kummer maps can be combined into the following commutative diagram whose first two rows are exact.

$$\begin{array}{ccccccc} {}^hX \otimes \mathbf{Z}/m\mathbf{Z} & \longrightarrow & X^\vee \otimes \mathcal{K}^\times \otimes \mathbf{Z}/m\mathbf{Z} & \longrightarrow & J(\mathcal{K}) \otimes \mathbf{Z}/m\mathbf{Z} & \longrightarrow & 0 \\ \| & & \| & & \big\downarrow \partial & & \\ {}^hX \otimes \mathbf{Z}/m\mathbf{Z} & \longrightarrow & H^1(\mathcal{K}, X^\vee \otimes \mu_m) & \longrightarrow & H^1(\mathcal{K}, J[m]) & & \\ \big\downarrow i\otimes\mathrm{id} & & \big\downarrow & & \big\downarrow & & \\ X^\vee \otimes \mathbf{Z}/m\mathbf{Z} & = & H^1(I, X^\vee \otimes \mu_m) & \longrightarrow & H^1(I, J[m]) & & \end{array}$$

This implies that the map ∂_{ram} factors as

$$J(\mathcal{K}) \otimes \mathbf{Z}/m\mathbf{Z} \longrightarrow \Phi/m\Phi = \mathrm{Coker}(X/mX \xrightarrow{i\otimes\mathrm{id}} X^\vee/mX^\vee) \longrightarrow H^1(I, J[m]).$$

1.7.4. Unramified cohomology. The cohomology sequence of (1.7.1.2) over F_ℓ^{ur}

$$0 \longrightarrow X^\vee \otimes \mu_m \longrightarrow J[m]^I \longrightarrow {}^hX \otimes \mathbf{Z}/m\mathbf{Z} \xrightarrow{i\otimes\mathrm{id}} H^1(I, X^\vee \otimes \mu_m) = X^\vee/mX^\vee$$

yields

$$0 \longrightarrow X^\vee \otimes \mu_m \longrightarrow J[m]^I \longrightarrow \Phi[m] \longrightarrow 0,$$

which in turn implies another exact sequence

$$H^1_{ur}(\mathcal{K}, X^\vee \otimes \mu_m) \longrightarrow H^1_{ur}(\mathcal{K}, J[m]) \longrightarrow H^1_{ur}(\mathcal{K}, \Phi[m]) \longrightarrow 0.$$

Denote by r the composite surjection

$$r : H^1_{ur}(\mathcal{K}, J[m]) \longrightarrow H^1_{ur}(\mathcal{K}, \Phi[m]) = \mathrm{Hom}(\mathrm{Gal}(F_\ell^{ur}/\mathcal{K}), \Phi[m]) \xrightarrow{\sim} \Phi[m],$$

where the last isomorphism is the evaluation map at the geometric Frobenius over $\mathcal{K}$.

According to local duality, the cup product

$$\cup : H^1(\mathcal{K}, J[m]) \times H^1(\mathcal{K}, J[m]) \longrightarrow H^2(\mathcal{K}, \mu_m) \xrightarrow{\sim} \mathbf{Z}/m\mathbf{Z}$$

induced by the Weil pairing (1.7.1.3) gives rise to a perfect pairing

$$\cup : H^1_{ur}(\mathcal{K}, J[m]) \times \mathrm{Im}\left(H^1(\mathcal{K}, J[m]) \longrightarrow H^1(I, J[m])\right) \longrightarrow \mathbf{Z}/m\mathbf{Z}. \qquad (1.7.4.1)$$

1.7.5. Proposition (Explicit reciprocity law). *Let $\mathcal{K}$ be a finite unramified extension of F_{ℓ^2} and $m \geq 1$ an integer prime to $N(\ell)$. If $c \in H^1_{ur}(\mathcal{K}, J[m])$ and $R \in J(\mathcal{K})$, then the cup product (1.7.4.1) satisfies*

$$c \cup \partial_{\mathrm{ram}}(R) = \pm m(r(c), \text{the image of } R \text{ in } \Phi/m\Phi)_m.$$

Proof. This follows from the definitions and [Ru, Lemma 1.4.7(ii)].

1.8. CM points (unramified at ℓ)

The assumptions of §1.6 are in force.

1.8.1. Embeddings. Let K be a totally imaginary quadratic extension of F in which none of the primes from $\mathrm{Ram}(B)$ splits. Under this assumption there exists an F-embedding $t : K \hookrightarrow B$; fix such a t. It induces embeddings $t_v : K_v \hookrightarrow B_v$ and $\widehat{t} : \widehat{K} \hookrightarrow \widehat{B}$ (for each prime v of F we use the slightly ambiguous notation $K_v := K \otimes_F F_v$ and $O_{K,v} := O_K \otimes_{O_F} O_{F,v}$).

Assume, in addition, that the prime ℓ satisfies the following conditions.

(1.8.1.1) ℓ is inert in K/F; denote by λ the unique prime of K above ℓ.

(1.8.1.2) $t_\ell^{-1}(U_\ell) = O^\times_{K,\ell}$.

The existence of t together with (1.8.1.1) imply that there exists an F-embedding $t' : K \hookrightarrow B'$; any such embedding automatically satisfies the following analogue of (1.8.1.2): $t'^{-1}_\ell(O_{B'_\ell}) = O_{K,\ell}$ $(= O_{K,\lambda})$.

According to the Skolem-Noether theorem, two F_v-embeddings $K_v \hookrightarrow B'_v$ are conjugate by an element of $B'^\times_v$. This implies that, for fixed t and t', after replacing $\varphi : \widehat{B}^{(\ell)} \xrightarrow{\sim} \widehat{B}'^{(\ell)}$ by a conjugate isomorphism we can – and will – assume that φ, t and t' are compatible outside ℓ in the sense that the composite map

$$\widehat{K}^{(\ell)} \xrightarrow{t^{(\ell)}} \widehat{B}^{(\ell)} \xrightarrow{\varphi} \widehat{B}'^{(\ell)}$$

is equal to $t'^{(\ell)}$.

1.8.2. CM points in the complex uniformisation. There are exactly two points of $\mathbf{C} - \mathbf{R}$ which are fixed under the action of $t_{\tau_1}(K^\times) \subset t_{\tau_1}(K^\times_{\tau_1}) \subset B^\times_{\tau_1} \xrightarrow{\sim} GL_2(\mathbf{R})$; fix one of them and denote it by z'.

The set of CM points by K unramified at ℓ on the curve $M = M_{U'Y}$ is defined as

$$CM(M, K)_{\ell-ur} := \{[z', b']_{U'Y} \mid b' \in \widehat{B}'^\times, \, b'_\ell = 1\} \subset (M \otimes_{F,\tau_1} \mathbf{C})(\mathbf{C}).$$

Shimura's reciprocity law states that

$$CM(M, K)_{\ell-ur} \subset M(K^{\mathrm{ab}})$$

and that G_K^{ab} acts on this set as follows:

$$\forall a \in \widehat{K}^\times \qquad \mathrm{rec}_K(a)\,[z', b']_{U'Y} = [z', \widehat{t'}(a)b']_{U'Y}$$

(in order to make sense of this statement one must appropriately choose an embedding $K \hookrightarrow \mathbf{C}$ extending $\tau_1 : F \hookrightarrow \mathbf{R}$; see [Mi 3, II.5]). This implies that a CM point $x = [z', b']_{U'Y} \in CM(M, K)_{\ell-ur}$ is defined over a finite abelian extension $K(x)$ of K satisfying

$$\mathrm{rec}_K : \mathrm{Gal}(K(x)/K) \xrightarrow{\sim} K^\times \backslash \widehat{K}^\times / \widehat{t}'^{-1}(b'U'Yb'^{-1}).$$

In particular, λ splits completely in $K(x)/K$, since $t'_\ell(K_\ell^\times) = t'_\ell(O_{K,\ell}^\times F_\ell^\times) \subset U'_\ell Y_\ell = b'_\ell U'_\ell Y_\ell b_\ell'^{-1}$.

1.8.3. CM points in the ℓ-adic uniformisation. Fix one of the two F_ℓ-embeddings $K_\ell \hookrightarrow F_\ell^{ur}$ (i.e., fix an isomorphism $K_\ell \xrightarrow{\sim} F_{\ell^2}$ over F_ℓ). Under the ℓ-adic uniformisation 1.4.2 of M, the set $CM(M, K)_{\ell-ur}$ corresponds to

$$CM(M, K)_{\ell-ur} = \{[z, b]_{UY} \mid b \in \widehat{B}^{(\ell)\times}\} \subset M(K_\ell),$$

where z is one of the two fixed points of $t_\ell(K^\times) \subset t_\ell(K_\ell^\times) \subset GL_2(F_\ell)$ acting on $\mathbf{P}^1(K_\ell) - \mathbf{P}^1(F_\ell)$ (z is determined by the choice of $K_\ell \hookrightarrow F_\ell^{ur}$). Moreover, the action of G_K^{ab} is given by the formula

$$\forall a \in \widehat{K}^\times \qquad \mathrm{rec}_K(a)\,[z, b]_{UY} = [z, \widehat{t}^{(\ell)}(a^{(\ell)})b]_{UY},$$

where $a^{(\ell)}$ denotes the projection of a to $\widehat{K}^{(\ell)\times}$.

2. Proof of Theorem A and its corollaries

From now on until the end of §2.9 we assume that $f \in S_2(\mathfrak{n}, \omega)$, $\chi : \mathbf{A}_K^\times / K^\times \longrightarrow \mathbf{C}^\times$ ($\chi|_{\mathbf{A}_F^\times} \cdot \omega = 1$) and $V = V_\mathfrak{p}(f)(1)$ are as in Theorem A. In particular, $L(f_K, \chi, 1) \neq 0$ and there exists $g_\mathfrak{p} \in G_F$ satisfying (A1)–(A3). In addition, if f has CM by a totally imaginary quadratic extension $K(f)$ of F, we assume in 2.9 that $K(f) \not\subset K_\chi F_\omega$.

2.1. Local linear forms and transfer to a definite quaternion algebra

We begin by recalling results of Tunnell, H. Saito and D. Prasad on local invariant linear forms.

2.1.1. Proposition [Tu], [Sa]. *Let v be a finite prime of F, let D be a quaternion algebra over F_v, let π_D be an irreducible smooth complex representation of $D^\times$ (of infinite dimension if $D = M_2(F_v)$) with central character $\omega_D : F_v^\times \longrightarrow \mathbf{C}^\times$. Let $\varphi : K_v^\times \longrightarrow \mathbf{C}^\times$ be a smooth character satisfying $\varphi|_{F_v^\times} \cdot \omega_D = 1$.*
(1) *If there exists an F_v-embedding $K_v \hookrightarrow D$ (which we fix), then*

$$\dim_{\mathbf{C}} \mathrm{Hom}_{K_v^\times}(\pi_D \otimes \varphi, \mathbf{C}) = \begin{cases} 1, & \varepsilon(\pi_D \times \varphi, \frac{1}{2}) = \mathrm{inv}_v(D)\, (\eta_v \omega_D)(-1) \\ 0, & \varepsilon(\pi_D \times \varphi, \frac{1}{2}) = -\mathrm{inv}_v(D)\, (\eta_v \omega_D)(-1). \end{cases}$$

(2) *If $D = M_2(F_v)$ and π_D is a principal series representation, then $\mathrm{Hom}_{K_v^\times}(\pi_D \otimes \varphi, \mathbf{C}) \neq 0$.*
(3) *If there is no F_v-embedding $K_v \hookrightarrow D$ ($\iff$ $D \neq M_2(F_v)$ and v splits in K/F), then $\varepsilon(\pi_D \times \varphi, \frac{1}{2}) = \omega_D(-1)$.*

2.1.2. Proposition [Pr, Thm. 4]. *Assume that, in the situation of Proposition 2.1.1, $\varphi = 1$. Denote by N_v the normaliser of $K_v^\times$ in $D^\times$. If $\mathrm{Hom}_{K_v^\times}(\pi_D, \mathbf{C}) \neq 0$, then the non-trivial element of $N_v / K_v^\times \xrightarrow{\sim} \{\pm 1\}$ acts on $\mathrm{Hom}_{K_v^\times}(\pi_D, \mathbf{C})$ by multiplication by $\mathrm{inv}_v(D)\, \varepsilon(\pi_D, \frac{1}{2})$.*

2.1.3. The archimedean factors $\Gamma_{\mathbf{C}}(s + 1/2)$ of $L(\pi(f) \times \chi, s)$ have no pole (and no zero) at the central point $s = 1/2$. The assumption $L(f_K, \chi, 1) \neq 0$ is, therefore, equivalent to $L(\pi(f) \times \chi, \frac{1}{2}) \neq 0$. The functional equation (0.5.1) implies that

$$\varepsilon(\pi(f) \times \chi, \tfrac{1}{2}) = \prod_v \varepsilon(\pi(f)_v \times \chi_v, \tfrac{1}{2}) = 1. \tag{2.1.3.1}$$

Set

$$S_B = \{v \text{ a prime of } F \mid \varepsilon(\pi(f)_v \times \chi_v, \tfrac{1}{2}) \neq (\eta_v \omega_v)(-1)\},$$

where $\omega_v : F_v^\times \longrightarrow \mathbf{C}^\times$ and $\chi_v : K_v^\times = (K \otimes_F F_v)^\times \longrightarrow \mathbf{C}^\times$ denote the local components of ω and χ, respectively.

2.1.4. Proposition. (1) *S_B is a finite set of even cardinality containing S_∞. If $v \in S_B - S_\infty$, then v does not split in K/F and $\pi(f)_v$ is not a principal series representation.*
(2) *Denote by B the unique quaternion algebra over F such that* $\mathrm{Ram}(B) = S_B$. *The algebra is totally definite and there exists an F-embedding $t : K \hookrightarrow B$ (which we fix).*
(3) *There exists a unique irreducible automorphic representation π of $B_{\mathbf{A}}^\times$ which corresponds to $\pi(f)$ via the Jacquet-Langlands correspondence. The central character of π is equal to ω and the archimedean component of π is trivial: $\pi_\infty = 1$.*
(4) $\forall v \nmid \infty \quad \dim_{\mathbf{C}} \mathrm{Hom}_{K_v^\times}(\pi_v \otimes \chi_v, \mathbf{C}) = 1$.

Proof. The cardinality of S_B is even, by (2.1.3.1) and the product formula $\prod_v (\eta_v \omega_v)(-1) = 1$. Each $v \in S_\infty$ belongs to S_B, since $\varepsilon(\pi(f)_v \times \chi_v, \frac{1}{2}) = \omega_v(-1) = 1$ and $\eta_v(-1) = -1$. The remaining statements follow from Proposition 2.1.1 and basic properties of the Jacquet-Langlands correspondence.

2.1.5. As $\pi_\infty = 1$, we can – and will – consider π as a representation of $\widehat{B}^\times = B_{\mathbf{A}}^\times / (B \otimes \mathbf{R})^\times$. It occurs with multiplicity 1 as a subrepresentation of $\mathbb{S} = \bigcup_U \mathbb{S}(U; \mathbf{C})$ and the embedding $\pi \hookrightarrow \mathbb{S}$ is unique up to a scalar multiple. Moreover, the generalised Ramanujan conjecture [Bl] for f implies, as in 1.1.7, that the image of π is contained in $\mathbb{S}_0 = \bigcup_U \mathbb{S}(U;\mathbf{C})_0$.

2.2. Global linear forms and global test vectors

The fixed embedding $t : K \hookrightarrow B$ (which is unique up to conjugation by an element of $B^\times$) induces local embeddings $t_v : K_v \hookrightarrow B_v$ and an adelic embedding $\widehat{t} : \widehat{K} \hookrightarrow \widehat{B}$.

2.2.1. The property 2.1.4(4) and the fact that, for all but finitely many v, the image of a spherical vector in π_v by a non-trivial element of $\mathrm{Hom}_{K_v^\times}(\pi_v \otimes \chi_v, \mathbf{C})$ is non-zero, imply that

$$\dim_{\mathbf{C}} \mathrm{Hom}_{\widehat{K}^\times}(\pi \otimes \chi, \mathbf{C}) = 1.$$

2.2.2. A suitably regularised integral

$$f_B \mapsto \ell_\chi(f_B) := \int_{\widehat{K}^\times/\widehat{F}^\times K^\times} \chi(x) f_B(\widehat{t}(x))\, dx \qquad (f_B \in \pi \subset \mathbb{S}_0)$$

(for a fixed Haar measure dx on $\widehat{K}^\times$) defines an element $\ell_\chi \in \mathrm{Hom}_{\widehat{K}^\times}(\pi\otimes\chi, \mathbf{C})$. According to a fundamental result of Waldspurger [W2, Thm. 2], our assumption about the non-vanishing of $L(\pi(f)\times\chi, \frac{1}{2}) = L(\pi\times\chi, \frac{1}{2})$ is equivalent to $\ell_\chi \neq 0$.

In concrete terms, there exist an open compact subgroup $U \subset \widehat{B}^\times$ (sufficiently small in the sense that $\chi(\widehat{t}^{-1}(U)) = 1$) and a function $f_B \in \pi^U \subset \mathbb{S}(U;\mathbf{C})_0$ satisfying

$$\mathscr{L}_\chi(f_B) := \sum_{\widehat{K}^\times/\widehat{F}^\times K^\times \widehat{t}^{-1}(U)} \chi(x) f_B(\widehat{t}(x)) \neq 0 \in \mathbf{C}$$

(f_B is a "test vector" for the global linear form ℓ_χ).

2.2.3. Note that π has a model over L in the sense that

$$\pi^{U_0} = \left(\pi^{U_0} \cap \mathbb{S}(U_0; L)\right) \otimes_L \mathbf{C},$$

for any open compact subgroup $U_0 \subset \widehat{B}^\times$. Indeed, the multiplicity one theorem for automorphic forms on $B_{\mathbf{A}}^\times$ implies that π^{U_0} coincides with the space of complex-valued functions f_0 on the finite set $B^\times\backslash\widehat{B}^\times/U_0$ which satisfy a system of linear equations with coefficients in L

$$T(v) f_0 = \lambda_f(v) f_0,$$

for v belonging to a sufficiently large finite set of primes of F.

2.2.4. Combining 2.2.2 with 2.2.3 we deduce that there exists $f_B \in \mathbb{S}(U;L)_0 \cap \pi$ satisfying $\mathscr{L}_\chi(f_B) \neq 0$. After replacing f_B by a suitable scalar multiple, we can assume that

$$f_B \in \mathbb{S}(U;O_L)_0 \cap \pi, \qquad \chi(\widehat{t}^{-1}(U)) = 1, \qquad \mathscr{L}_\chi(f_B) \in O_L - \{0\}. \tag{2.2.4.1}$$

We define

$$C_1 = C_1(\mathfrak{p}) := \mathrm{ord}_{\mathfrak{p}}(\mathscr{L}_\chi(f_B)) \geq 0. \tag{2.2.4.2}$$

2.2.5. We fix U and f_B satisfying (2.2.4.1). We also fix a finite set S as in 1.1.5 and a prime $v_0 \notin S$ of F which has trivial class in $F_+^\times\backslash\widehat{F}^\times/\mathrm{Nrd}(U)$ and for which $\omega(v_0) = 1$. Furthermore, we set $Y := \mathrm{Ker}(\omega : \widehat{F}^\times \longrightarrow \mathbf{C}^\times)$.

2.2.6. By definition of f_B, the spherical Hecke algebra $\mathbb{T}^S(U)$ acts on f_B by the character

$$\lambda_B : \mathbb{T}^S(U) \longrightarrow O_L, \qquad \lambda_B(T(v)) = \lambda_f(v), \qquad \lambda_B(S(v)) = \omega(v) \qquad (v \notin S).$$

Denote by ${}^h\lambda_B$ the ring homomorphism

$$^h\lambda_B : \mathbb{T}^S(U) \xrightarrow{\vee} \mathbb{T}^S(U) \xrightarrow{\lambda_B} O_L.$$

For any $O_L[\mathbb{T}^S(U)]$-module N we set

$$\begin{aligned} N^{(\lambda_B)} &:= \{n \in N \mid \forall t \in \mathbb{T}^S(U) \quad t(n) = \lambda_B(t)n\} = N[\mathrm{Ker}(\lambda_B)] \\ N_{(\lambda_B)} &:= N \otimes_{\mathbb{T}^S(U),\lambda_B} O_L = N/\mathrm{Ker}(\lambda_B)N \end{aligned}$$

(and similarly for $^h\lambda_B$ instead of λ_B).

2.3. Choosing a Kolyvagin prime ℓ

Denote by $e(\mathfrak{p}) := \mathrm{ord}_{\mathfrak{p}}(p)$ the absolute ramification index of $L_{\mathfrak{p}}$ and fix a G_F-stable $O_{L,\mathfrak{p}}$-lattice $T \subset V$. For a suitable scalar multiple of the pairing (0.7.1) (which we fix) the lattice T will be self-dual in the sense that (0.7.1) will induce an isomorphism of $O_{L,\mathfrak{p}}[G_F]$-modules $T^*(1) := \mathrm{Hom}_{O_{L,\mathfrak{p}}}(T, O_{L,\mathfrak{p}})(1) \xrightarrow{\sim} T \otimes \omega^{-1}$.

2.3.1. The assumptions (A1)–(A3) tell us that there exists $g = g_{\mathfrak{p}} \in G_{F_\omega}$, $g|_K \neq \mathrm{id}$, which acts on T (in a suitable $O_{L,\mathfrak{p}}$-basis e_1, e_2, which we fix) by a matrix $\begin{pmatrix} \varepsilon & 0 \\ 0 & u_g \end{pmatrix}$, where $\varepsilon = \pm 1$ and $u_g \in O_{L,\mathfrak{p}}^\times$, $u_g^2 \neq 1$. Set

$$C_2 = C_2(\mathfrak{p}) := \mathrm{ord}_{\mathfrak{p}}(u_g^2 - 1).$$

It follows from Appendix B.5.2 and B.5.5(7) (resp. B.6.5(5) and B.6.5(7)) in the case when f does not have (resp. has) complex multiplication that for all but finitely many $\mathfrak{p}$ satisfying (A1)–(A3) one can choose $g_{\mathfrak{p}}$ in such a way that $C_2(\mathfrak{p}) = 0$.

2.3.2. Fix a large integer $n >> 0$ and set

$$F(T/\mathfrak{p}^nT) = \overline{F}^{\mathrm{Ker}(G_F \longrightarrow \mathrm{Aut}(T/\mathfrak{p}^nT))}.$$

Using the basis e_1, e_2 of T from 2.3.1 we consider the Galois group $\mathrm{Gal}(F(T/\mathfrak{p}^nT)/F)$ as a subgroup of the group $\mathrm{Aut}_{O_{L,\mathfrak{p}}}(T/\mathfrak{p}^nT) \xrightarrow{\sim} GL_2(O_L/\mathfrak{p}^nO_L)$.

Let ℓ be a prime of F satisfying the following properties.

(2.3.2.1) $\ell \notin S \cup \{v_0\}$;
(2.3.2.2) ℓ does not divide p;
(2.3.2.3) ℓ is unramified in K_χ/F;
(2.3.2.4) $t_\ell^{-1}(U_\ell) = O_{K,\ell}^\times$;
(2.3.2.5) $\mathrm{Fr}_{\mathrm{geom}}(\ell)$ with respect to the extension $KF_\omega F(T/\mathfrak{p}^nT)/F$ is equal to the conjugacy class of the restriction of g (this makes sense, since $\ell \nmid \mathfrak{n}$ by (2.3.2.1), hence is unramified in this extension, by (2.3.2.2-3)).

The set of ℓ satisfying (2.3.2.1-5) has positive density.

2.3.3. Proposition. *The prime ℓ and the Hecke eigenvalue $a_\ell := \lambda_f(\ell)$ of $T(\ell)$ acting on f have the following properties.*

$$\omega(\ell) = 1, \qquad \ell \text{ is inert in } K/F, \qquad N(\ell)^{-1} - \varepsilon u_g \in \mathfrak{p}^nO_{L,\mathfrak{p}}, \qquad N(\ell)^{-1}a_\ell - (\varepsilon + u_g) \in \mathfrak{p}^nO_{L,\mathfrak{p}},$$
$$N(\ell) + 1 - \varepsilon a_\ell \in \mathfrak{p}^nO_{L,\mathfrak{p}}, \qquad \mathrm{ord}_{\mathfrak{p}}(N(\ell)^2 - 1) = \mathrm{ord}_{\mathfrak{p}}(u_g^2 - 1) = C_2.$$

Proof. As g acts trivially (resp. non-trivially) on F_ω (resp. on K), $\omega(\ell) = 1$ (resp. ℓ is inert in K/F). The remaining properties follow from the fact that

$$1 - a_\ell N(\ell)^{-1}X + \omega(\ell)N(\ell)^{-1}X^2 = \det(1 - \mathrm{Fr}_{\mathrm{geom}}(\ell)X \mid V) \equiv 1 - (\varepsilon + u_g)X + \varepsilon u_g X^2 \pmod{\mathfrak{p}^nO_{L,\mathfrak{p}}}.$$

2.3.4. Definition. *Denote by $n' \geq n$ the integer*

$$n' := \mathrm{ord}_{\mathfrak{p}}(N(\ell) + 1 - \varepsilon a_\ell)$$

(note that $N(\ell) + 1 \pm a_\ell \neq 0$, by the generalised Ramanujan conjecture [Bl]).

2.4. Variations on a theme of Boston-Lenstra-Ribet

In this section we prove a weak version of the results of [BLR] for slightly more general coefficient rings.

Proposition 2.4.1. *Let A be a quotient of a discrete valuation ring by a non-zero ideal, M a free A-module of finite rank $r \geq 1$, $\rho' : R \longrightarrow \mathrm{End}_A(M)$ a morphism of A-algebras and $I_0, I_1 \subset A$ ideals such that $I_0 \cdot \mathrm{Ker}(\rho') = 0$ and $I_1 \cdot \mathrm{Coker}(\rho') = 0$. Then, for each left R-module N and an element $x \in N$ there exists $j \in \mathrm{Hom}_R(N, M)$ such that $I_0 I_1^2 \,\mathrm{Ann}_A(u(x)) \subseteq \mathrm{Ann}_A(x)$.*

Proof. After replacing R (resp. N) by $R/\mathrm{Ker}(\rho')$ (resp. $N/\mathrm{Ker}(\rho')N$) and x by its image in $N/\mathrm{Ker}(\rho')N$ we can assume that $I_0 = 0$. In other words, there is an exact sequence of A-modules

$$0 \longrightarrow R \longrightarrow \mathrm{End}_A(M) \longrightarrow C \longrightarrow 0, \qquad I_1 C = 0.$$

The map

$$\lambda : \mathrm{End}_A(M) \longrightarrow \mathrm{Hom}_A(\mathrm{End}_A(M), A), \qquad (\lambda(X))(Y) = \mathrm{Tr}(YX)$$

is an isomorphism of left $\mathrm{End}_A(M)$-modules, the module structure on the R.H.S. being given by $(Zf)(Y) = f(YZ)$. Its composition with the restriction map to R gives rise to a morphism of left R-modules

$$\lambda' : M^{\oplus r} = \mathrm{End}_A(M) \xrightarrow{\lambda} \mathrm{Hom}_A(\mathrm{End}_A(M), A) \longrightarrow \mathrm{Hom}_A(R, A)$$

whose kernel and cokernel is killed by I_1. It follows that the induced morphism of A-modules

$$\lambda'_* : \mathrm{Hom}_R(N, M^{\oplus r}) \longrightarrow \mathrm{Hom}_R(N, \mathrm{Hom}_A(R, A)) \xrightarrow{\sim} \mathrm{Hom}_A(N, A)$$

satisfies $I_1^2 \cdot \mathrm{Coker}(\lambda'_*) = 0$. Fix $j' \in \mathrm{Hom}_A(N, A)$ such that $\mathrm{Ann}_A(j'(x)) = \mathrm{Ann}_A(x)$; as $I_1 = t^m A$ (where $t \in A$ is a generator of the maximal ideal of A), there exist $j_i \in \mathrm{Hom}_R(N, M)$ such that $t^{2m} j' = \lambda'(j_1, \ldots, j_r)$. This implies that

$$\bigcap_{i=1}^{r} \mathrm{Ann}_A(j_i(x)) \subseteq \mathrm{Ann}_A(t^{2m} j'(x)) = \mathrm{Ann}_A(t^{2m} x);$$

thus $\mathrm{Ann}_A(j_i(x)) \subseteq \mathrm{Ann}_A(t^{2m} x)$ for some $i = 1, \ldots, r$, as claimed.

2.4.2. Let A be a commutative ring, M a free A-module of rank 2, G a group and $\rho : G \longrightarrow \mathrm{Aut}_A(M)$ ($\xrightarrow{\sim} GL_2(A)$) a group homomorphism. Denote by $J \subset A[G]$ the bilateral ideal generated by the elements $g^2 - \mathrm{Tr}(\rho(g))g + \det(\rho(g))$ (for all $g \in G$) and set $R = A[G]/J$. The Cayley-Hamilton theorem implies that the morphism of A-algebras $A[G] \longrightarrow \mathrm{End}_A(M)$ ($\xrightarrow{\sim} M_2(A)$) induced by ρ – which will still be denoted by ρ – factors as

$$\rho : A[G] \longrightarrow R \xrightarrow{\rho'} \mathrm{End}_A(M).$$

Proposition 2.4.3. *If, under the assumptions of 2.4.2, $I \subset A$ is an ideal such that $\mathrm{Im}(\rho) \supseteq I \cdot \mathrm{End}_A(M)$, then $I^4 \cdot \mathrm{Ker}(\rho') = 0$.*

Proof. This statement is proved in [BLR, Prop. 2] in the case when $A = I$ is a field. In general, the arguments in the proof of [*loc. cit.*] show that the annihilator $\mathrm{Ann}_R(x) = \{r \in R \mid rx = 0\}$ of each element $x \in \mathrm{Ker}(\rho')$ is a bilateral ideal of R containing $\{yz - zy \mid y, z \in R\}$. It follows that $\rho'(\mathrm{Ann}_R(x))$ is a bilateral ideal of $\mathrm{Im}(\rho') = \mathrm{Im}(\rho)$ containing $\{y'z' - z'y' \mid y', z' \in I \cdot \mathrm{End}_A(M)\}$, which implies that $\rho'(\mathrm{Ann}_R(x))$ contains $\{t \in I^2 \cdot \mathrm{End}_A(M) \mid \mathrm{Tr}(t) = 0\}$. In particular, for any $a, b \in I^2$ there is $w \in R$ such that $wx = 0$ and $\rho'(w) = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$. As $w^2 - \mathrm{Tr}(\rho(w))w + \det(\rho'(w)) = 0 \in R$ by [BLR, Prop. 1], we have $abx = -\det(\rho'(w))x = 0$; thus $I^4 x = 0$.

Corollary 2.4.4. *In the situation of Proposition 2.4.3, assume that $A = \widetilde{A}/t^n\widetilde{A}$ and $M = \widetilde{M}/t^n\widetilde{M}$, where $\widetilde{A}$ is a discrete valuation ring with a uniformiser t, $\widetilde{M}$ is a free $\widetilde{A}$-module of rank 2 and $\widetilde{\rho} : G \longrightarrow \mathrm{Aut}_{\widetilde{A}}(\widetilde{M})$ a group homomorphism lifting ρ.*

(1) *If $\widetilde{M} \otimes_{\widetilde{A}} \mathrm{Frac}(\widetilde{A})$ is an absolutely irreducible representation of G with coefficients in the fraction field $\mathrm{Frac}(\widetilde{A})$ of $\widetilde{A}$, then there exists an integer $c \geq 0$ such that $\mathrm{Im}(\widetilde{\rho}) \supseteq t^c \,\mathrm{End}_{\widetilde{A}}(\widetilde{M})$ (hence $\mathrm{Im}(\rho) \supseteq t^c \,\mathrm{End}_A(M)$).*

(2) $t^{4c}\,\mathrm{Ker}(\rho') = 0$.
(3) *If the residual representation M/tM of G is an absolutely irreducible representation of G with coefficients in A/tA, then* $\mathrm{Im}(\rho) = \mathrm{End}_A(M)$ *and* $\mathrm{Ker}(\rho') = 0$.

Proof. (1) The existence of c is equivalent to $\mathrm{Im}(\widetilde{\rho}) \otimes_{\widetilde{A}} \mathrm{Frac}(\widetilde{A}) = \mathrm{End}_{\widetilde{A}}(\widetilde{M}) \otimes_{\widetilde{A}} \mathrm{Frac}(\widetilde{A})$, which is, in turn, equivalent to the absolute irreducibility of $\widetilde{M} \otimes_{\widetilde{A}} \mathrm{Frac}(\widetilde{A})$, by a theorem of Burnside [CR, Thm. 3.32] (and Schur's lemma [CR, Prop. 3.33]).
(2) By (1), Proposition 2.4.3 applies with $I = t^c A$.
(3) By the argument used in the proof of (1), the absolute irreducibility of $M/tM = \widetilde{M}/t\widetilde{M}$ is equivalent to $\mathrm{Im}(\widetilde{\rho}) + t\,\mathrm{End}_{\widetilde{A}}(\widetilde{M}) = \mathrm{End}_{\widetilde{A}}(\widetilde{M})$, which implies that $\mathrm{Im}(\widetilde{\rho}) = \mathrm{End}_{\widetilde{A}}(\widetilde{M})$, by Nakayama's lemma; thus $c = 0$.

Proposition 2.4.5. *Assume that, in the situation of 2.4.2, A is a quotient of a discrete valuation ring by a non-zero ideal and that we are given $h \in G$, $a \in A$ and a basis e_1, e_2 of M such that $\rho(h)e_1 = e_1$ and $\rho(h)e_2 = ae_2$. If N is a left R-module and $N_0 \subset N$ an A-submodule such that $h(n_0) = an_0$ for each $n_0 \in N_0$, then the cokernel of the restriction map*

$$\mathrm{Hom}_R(N, M) \longrightarrow \mathrm{Hom}_{A[h]}(N_0, M)$$

is killed by $(a-1)^2 I^{12}$.

Proof. Write $I = bA$ for some $b \in A$. We must show that, for each $j_0 \in \mathrm{Hom}_{A[h]}(N_0, M)$, there exists $j \in \mathrm{Hom}_R(N, M)$ such that $j|_{N_0} = (a-1)^2 b^{12} j_0$. Set

$$N_1 := \mathrm{Ker}(j_0) \subset N_0 \subset N^{h=a}.$$

As j_0 injects N_0/N_1 into $M^{h=a}$ and $(a-1)M^{h=a} = A(a-1)e_2$ is a cyclic A-module, $(a-1)N_0/N_1$ and its quotients, such as $(a-1)(N_0/(N_0 \cap RN_1)) \xrightarrow{\sim} (a-1)((N_0 + RN_1)/RN_1)$, are also cyclic.

Lemma 2.4.6 below implies that the map $(a-1)b^6 j_0 : N_0 \longrightarrow (a-1)b^6 M^{h=a} = A(a-1)b^6 e_2$ factors through $j_1 \in \mathrm{Hom}_A(N_0/(N_0 \cap RN_1), A(a-1)b^6 e_2)$. Fix an element $x \in N' := (N_0 + RN_1)/RN_1$ such that $(a-1)x$ is a generator of the cyclic A-module $(a-1)N'$. Applying Proposition 2.4.1 to the R-module N/RN_1 (with $I_1 = bA$ by definition and $I_0 = b^4 A$ by Proposition 2.4.2), we obtain a morphism $j_2 \in \mathrm{Hom}_R(N/RN_1, M)$ such that $b^6 \mathrm{Ann}_A(j_2(x)) \subseteq \mathrm{Ann}_A(x)$; thus

$$(a-1)b^6 \mathrm{Ann}_A((a-1)j_2(x)) \subseteq b^6 \mathrm{Ann}_A(j_2(x)) \subseteq \mathrm{Ann}_A(x) \subseteq \mathrm{Ann}_A(j_1(x)).$$

As both $j_1(x)$ and $(a-1)j_2(x)$ belong to the cyclic A-module $A(a-1)b^6 e_2$, there exists $a' \in A$ such that $(a-1)b^6 j_1(x) = a'(a-1)j_2(x)$. This implies, by definition of x, that the map $j := a'(a-1)j_2 \in \mathrm{Hom}_R(N, M)$ satisfies $j|_{RN_1} = 0$ and

$$j|_{N_0} = (a-1)b^6 j_1 = (a-1)^2 b^{12} j_0,$$

as required.

Lemma 2.4.6. $(a-1)b^6(N_0 \cap RN_1) = (a-1)b^6 N_1$.

Proof. It is enough to prove the non-trivial inclusion $(a-1)b^6(N_0 \cap RN_1) \subseteq (a-1)b^6 N_1$. If

$$n_0 = \sum_{i=1}^{k} r_i n_i \in N_0 \cap RN_1 \qquad (\forall i \geq 1 \quad r_i \in R,\ n_i \in N_1),$$

then

$$\forall j_0' \in \mathrm{Hom}_R(RN_0, M)\ \forall i \geq 0 \quad (a-1)j_0'(n_i) = (a-1)a_i e_2 \in (a-1)M^{h=a} = A(a-1)e_2 \qquad (a_i \in A).$$

Writing $r_i e_2 = \lambda_i e_1 + \mu_i e_2$, we obtain

$$0 = (a-1)j_0' \left(n_0 - \sum_{i=1}^{k} r_i n_i \right) = (a-1) \left(a_0 e_2 - \sum_{i=1}^{k} a_i (\lambda_i e_1 + \mu_i e_2) \right),$$

hence

$$0 = (a-1) \left(a_0 e_2 - \sum_{i=1}^{k} a_i \mu_i e_2 \right).$$

This implies that the element $n' := n_0 - \sum_{i=1}^{k} \mu_i n_i \in N_0$ satisfies

$$\forall j_0' \in \mathrm{Hom}_R(RN_0, M) \quad (a-1)j_0'(n') = 0.$$

Applying Proposition 2.4.1 to the R-module RN_0 and $n' \in RN_0$ (again with $I_1 = bA$ and $I_0 = b^4 A$) we deduce that $(a-1)b^6 n' = 0$ and $(a-1)b^6 n_0 = (a-1)b^6 \sum_{i=1}^{k} \mu_i n_i \in (a-1)b^6 N_1$, which finishes the proof.

2.5. Weak level raising modulo $\mathfrak{p}^n$

In this section we are going to work with the curve $M_{U'Y}$ and its Jacobian $J = J(M_{U'Y})$ attached to U and $Y = \mathrm{Ker}(\omega)$ from 2.2.5 and ℓ from 2.3.2 (note that $Y = F_\ell^\times \times Y^{(\ell)}$, since $\omega(\ell) = 1$).

2.5.1. The element

$$\widetilde{f} := \begin{pmatrix} f_B \\ \varepsilon f_B \end{pmatrix} \in \left(\mathbb{S}(UY; O_L)_0^{\oplus 2}\right)^{(\lambda_B)} \subset O_{L,\mathfrak{p}}[\mathcal{V}(\mathcal{G})]_0^{(\lambda_B)} \tag{2.5.1.1}$$

(see Prop. 1.5.9(2)) satisfies

$$(\mu_0 \otimes \mathrm{id})(\widetilde{f}) = \begin{pmatrix} -N(\ell) - 1 & T(\ell) \\ T(\ell) & -N(\ell) - 1 \end{pmatrix} \begin{pmatrix} f_B \\ \varepsilon f_B \end{pmatrix} = (\varepsilon a_\ell - N(\ell) - 1)\widetilde{f}.$$

Consequently, the image $[\widetilde{f}]$ of $\widetilde{f}$ in

$$\Phi \otimes O_{L,\mathfrak{p}} = \mathrm{Coker}\left(\mu_0 \otimes \mathrm{id} : \mathbb{S}(UY; O_{L,\mathfrak{p}})^{\oplus 2} \longrightarrow \mathbb{S}(UY; O_{L,\mathfrak{p}})_0^{\oplus 2}\right)$$

satisfies, by definition of n',

$$[\widetilde{f}] \in (\Phi \otimes O_{L,\mathfrak{p}})[\mathfrak{p}^{n'}]^{(\lambda_B)}. \tag{2.5.1.2}$$

2.5.2. Proposition-Definition. *The O_L-module*

$$I(f_B) := \{a \in L \mid a f_B \in \mathbb{S}(UY; O_L) + \mathbb{S}(UY; L)_{\mathrm{triv}}\}$$

is a fractional O_L-ideal containing O_L; set

$$C_3 = C_3(\mathfrak{p}) := -\mathrm{ord}_{\mathfrak{p}}(I(f_B)) \geq 0.$$

The element $[\widetilde{f}] \in (\Phi \otimes O_{L,\mathfrak{p}})[\mathfrak{p}^{n'}]$ satisfies $\mathfrak{p}^{n'-C_3-1}[\widetilde{f}] \neq 0 \in \Phi \otimes O_{L,\mathfrak{p}}$.

Proof. $I(f_B)$ is a fractional O_L-ideal, since $f_B \notin \mathbb{S}(UY; L)_{\mathrm{triv}}$, which means that C_3 is defined. We must show that any relation

$$c \begin{pmatrix} f_B \\ \varepsilon f_B \end{pmatrix} = \begin{pmatrix} -N(\ell) - 1 & T(\ell) \\ T(\ell) & -N(\ell) - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -(N(\ell)+1)x + T(\ell)y \\ T(\ell)x - (N(\ell)+1)y \end{pmatrix}$$

with $c \in O_{L,\mathfrak{p}}$ and $x, y \in \mathbb{S}(UY; O_{L,\mathfrak{p}})$ implies that $c \in \mathfrak{p}^{n'-C_3} O_{L,\mathfrak{p}}$. As in 1.1.7, it follows from the relation

$$(T(\ell)+\varepsilon(N(\ell)+1))(y-\varepsilon x)=0$$

that $y-\varepsilon x\in\mathbb{S}(UY;O_{L,\mathfrak{p}})_{\mathrm{triv}}$. Decomposing $x=x_0+x_{\mathrm{triv}}$ ($x_0\in\mathbb{S}(UY;L_{\mathfrak{p}})_0$, $x_{\mathrm{triv}}\in\mathbb{S}(UY;L_{\mathfrak{p}})_{\mathrm{triv}}$), we deduce from

$$2\varepsilon c f_B=(T(\ell)-\varepsilon(N(\ell)+1))(x+\varepsilon y)=2(T(\ell)-\varepsilon(N(\ell)+1))x-2(N(\ell)+1)(y-\varepsilon x)$$

that

$$(T(\ell)-\varepsilon(N(\ell)+1))x_0=\varepsilon c f_B,$$

hence

$$x_0=c'f_B,\qquad c'=\varepsilon c/(a_\ell-\varepsilon(N(\ell)+1))$$

(since $T(\ell)-\varepsilon(N(\ell)+1)$ is invertible on $\mathbb{S}(UY;L_{\mathfrak{p}})_0$ and $T(\ell)f_B=a_\ell f_B$). By definition of $I(f_B)$, the relation

$$y=((y-\varepsilon x)+\varepsilon x_{\mathrm{triv}})+\varepsilon c'f_B$$

implies that $c'\in I(f_B)O_{L,\mathfrak{p}}$, hence $\mathrm{ord}_{\mathfrak{p}}(c)=\mathrm{ord}_{\mathfrak{p}}(a_\ell-\varepsilon(N(\ell)+1))+\mathrm{ord}_{\mathfrak{p}}(c')\geq n'-C_3$, as claimed.

2.5.3. Set

$$T_{\mathfrak{p}}(J\otimes O_L):=T_p(J)\otimes_{\mathbf{Z}_p}O_{L,\mathfrak{p}},\qquad (J\otimes O_L)[\mathfrak{p}^m]:=T_p(J)\otimes_{\mathbf{Z}_p}O_L/\mathfrak{p}^mO_L\qquad (m\geq 0).$$

These are $\mathbb{T}^{S\cup\{\ell\}}(U(\ell)Y)\otimes O_{L,\mathfrak{p}}[G_F]$-modules (see 1.6.7) and there are exact sequences of $\mathbb{T}^{S\cup\{\ell\}}(U(\ell)Y)\otimes O_{L,\mathfrak{p}}[G_{F(\ell^2)}]$-modules (see 1.7.1)

$$\begin{gathered}0\longrightarrow X^\vee\otimes O_{L,\mathfrak{p}}(1)\longrightarrow T_{\mathfrak{p}}(J\otimes O_L)\longrightarrow {}^hX\otimes O_{L,\mathfrak{p}}\longrightarrow 0\\ 0\longrightarrow X^\vee\otimes O_L/\mathfrak{p}^mO_L(1)\longrightarrow (J\otimes O_L)[\mathfrak{p}^m]\longrightarrow {}^hX\otimes O_L/\mathfrak{p}^mO_L\longrightarrow 0.\end{gathered}\tag{2.5.3.1}$$

Dualising the maps

$$\begin{array}{rcccl}[\cdot\widetilde{f}]:\ O_L/\mathfrak{p}^{n'}O_L & \longrightarrow & (\Phi\otimes O_{L,\mathfrak{p}})[\mathfrak{p}^{n'}]^{(\lambda_B\otimes\mathrm{id})} & \hookrightarrow (\Phi\otimes O_{L,\mathfrak{p}})[\mathfrak{p}^{n'}]\hookrightarrow {}^hX\otimes O_L/\mathfrak{p}^{n'}O_L\\ 1 & \mapsto & [\widetilde{f}] & \end{array}$$

(using the pairings from 1.6.10 and fixing a generator of $\mathfrak{p}^{n'}O_{L,\mathfrak{p}}$) we obtain a morphism of $\mathbb{T}^{S\cup\{\ell\}}(U(\ell)Y)\otimes O_{L,\mathfrak{p}}[G_{F(\ell^2)}]$-modules

$$\begin{array}{rcccl}[\cdot\widetilde{f}]^\vee: X^\vee\otimes O_L/\mathfrak{p}^{n'}O_L\twoheadrightarrow\Phi\otimes O_L/\mathfrak{p}^{n'}O_L\twoheadrightarrow & (\Phi\otimes O_L/\mathfrak{p}^{n'}O_L)_{({}^h\lambda_B\otimes\mathrm{id})} & \longrightarrow & O_L/\mathfrak{p}^{n'}O_L & \twoheadrightarrow O_L/\mathfrak{p}^nO_L\\ & a & \mapsto & ([\widetilde{f}],a)_{\mathfrak{p}^{n'}} & \end{array}\tag{2.5.3.2}$$

which factors through $(X^\vee\otimes O_L/\mathfrak{p}^nO_L)_{({}^h\lambda_B\otimes\mathrm{id})}$ and whose cokernel is killed by $\mathfrak{p}^{C_3}$, by Proposition 2.5.2 and non-degeneracy of the pairing $\Phi\times\Phi\longrightarrow\mathbf{Q}/\mathbf{Z}$. Above,

$$\lambda_B\otimes\mathrm{id}:\mathbb{T}^{S\cup\{\ell\}}(U(\ell)Y)\otimes O_{L,\mathfrak{p}}\longrightarrow O_L/\mathfrak{p}^{n'}O_L$$

is obtained from λ_B by extension of scalars.

After tensoring (2.5.3.1) over $\mathbb{T}^{S\cup\{\ell\}}(U(\ell)Y)\otimes O_{L,\mathfrak{p}}$ via the morphism

$${}^h\lambda_B\otimes\mathrm{id}:\mathbb{T}^{S\cup\{\ell\}}(U(\ell)Y)\otimes O_{L,\mathfrak{p}}\longrightarrow O_L/\mathfrak{p}^nO_L,$$

we obtain an exact sequence of $O_L/\mathfrak{p}^nO_L[G_{F(\ell^2)}]$-modules

$$\begin{gathered}\mathrm{Tor}_1^{\mathbb{T}^{S\cup\{\ell\}}(U(\ell)Y)\otimes O_{L,\mathfrak{p}}}({}^hX\otimes O_L/\mathfrak{p}^nO_L,O_L/\mathfrak{p}^nO_L)\xrightarrow{\partial}(X^\vee\otimes O_L/\mathfrak{p}^nO_L)_{({}^h\lambda_B\otimes\mathrm{id})}(1)\longrightarrow(J\otimes O_L)[\mathfrak{p}^n]_{({}^h\lambda_B\otimes\mathrm{id})}\\ \longrightarrow{}^h\big((X\otimes O_L/\mathfrak{p}^nO_L)_{(\lambda_B\otimes\mathrm{id})}\big)\longrightarrow 0.\end{gathered}$$

The Frobenius element $\mathrm{Fr}_{\mathrm{geom}}(\ell^2)$ acts on the domain (resp. on the target) of ∂ trivially (resp. by multiplication by $N(\ell)^2$), which implies that

$$\mathfrak{p}^{C_2}\,\mathrm{Im}(\partial) = (N(\ell)^2 - 1)\,\mathrm{Im}(\partial) = 0.$$

Set

$$N := (J \otimes O_L)[\mathfrak{p}^n]_{({}^h\lambda_B \otimes \mathrm{id})} \supset N_0 := \mathrm{Coker}(\partial);$$

the previous discussion implies that the Tate twist of the map $(N(\ell)^2 - 1)[\cdot\widetilde{f}]^\vee$ factors through a morphism of $O_L/\mathfrak{p}^n O_L[G_{F(\ell^2)}]$-modules

$$j_0 : N_0 \longrightarrow O_L/\mathfrak{p}^n O_L(1), \qquad j_0(x) = (N(\ell)^2 - 1)([\widetilde{f}], x)_{\mathfrak{p}^{n'}} \pmod{\mathfrak{p}^n}.$$

As $\mathfrak{p}^{C_3}\mathrm{Coker}([\cdot\widetilde{f}]^\vee) = 0$, we have $\mathfrak{p}^{C_2+C_3}\mathrm{Coker}(j_0) = 0$. To sum up, we have proved the following statement.

2.5.4. Proposition. *There exists a diagram of morphisms of $O_L/\mathfrak{p}^n O_L[G_{F(\ell^2)}]$-modules*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_0 & \longrightarrow & N(= (J \otimes O_L)[\mathfrak{p}^n]_{({}^h\lambda_B \otimes \mathrm{id})}) & \longrightarrow & {}^h\big((X \otimes O_L/\mathfrak{p}^n O_L)_{(\lambda_B \otimes \mathrm{id})}\big) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle j_0} & & & & & & \\ & & O_L/\mathfrak{p}^n O_L(1) & & & & & & \end{array}$$

whose row is exact and in which $\mathfrak{p}^{C_2+C_3}\,\mathrm{Coker}(j_0) = 0$.

2.5.5. Definition. *Denote by $C_4 = C_4(\mathfrak{p})$ the smallest integer $C_4 \geq 0$ satisfying*

$$\mathrm{Im}\left(O_{L,\mathfrak{p}}[G_F] \longrightarrow \mathrm{End}_{O_{L,\mathfrak{p}}}(T)\right) \supseteq \mathfrak{p}^{C_4}\,\mathrm{End}_{O_{L,\mathfrak{p}}}(T)$$

(C_4 exists, since V is an absolutely irreducible representation of G_F [T2, Prop. 3.1]; see Corollary 2.4.4(1) above).

2.5.6. The following objects satisfy the assumptions of Proposition 2.4.3:

$$A = O_L/\mathfrak{p}^n O_L, \qquad G = G_F, \qquad M = T/\mathfrak{p}^n T, \qquad I = \mathfrak{p}^{C_4} A.$$

As in 2.4.2, let $R = A[G]/J$.

The construction of ℓ implies that $\omega(\ell) = 1$ and that $h = \mathrm{Fr}_{\mathrm{geom}}(\ell^2)$ acts on M in the basis e_1, e_2 from 2.3.1 by

$$\rho(h)e_1 = e_1, \qquad \rho(h)e_2 = u_g^2 e_2 = N(\ell)^{-2} e_2.$$

Furthermore, the Eichler-Shimura relation (1.6.8.4) (together with the Čebotarev density theorem) implies that N is an R-module and that

$$j_0 : N_0 \longrightarrow O_L/\mathfrak{p}^n O_L(1) = (O_L/\mathfrak{p}^n O_L)e_2 \subset M$$

is a morphism of $A[h]$-modules.

2.5.7. Proposition. *There exists a morphism of $O_L/\mathfrak{p}^n O_L[G_F]$-modules*

$$j : N = (J \otimes O_L)[\mathfrak{p}^n]_{({}^h\lambda_B \otimes \mathrm{id})} \longrightarrow T/\mathfrak{p}^n T = M$$

whose restriction to N_0 is equal to

$$(N(\ell)^{-2} - 1)^2 b^{12} j_0,$$

where b is a fixed generator of $\mathfrak{p}^{C_4}O_{L,\mathfrak{p}}$. Furthermore, $\mathfrak{p}^{3C_2+C_3+13C_4}\,\mathrm{Coker}(j)=0$.

Proof. The existence of j follows from Proposition 2.4.5, which applies with $a=N(\ell)^{-2}$, thanks to the discussion in 2.5.6. By construction, $j(N_0)$ contains $\mathfrak{p}^{3C_2+C_3+12C_4}e_2$, which implies that $j(N)$ contains $\mathfrak{p}^{3C_2+C_3+13C_4}e_1$, by definition of C_4.

2.5.8. The proof of Proposition 2.5.7 used only the existence of an element $g_\mathfrak{p}\in G_F$ satisfying (A1) and (A2). Neither the property (A3) nor the assumption $L(f_K,\chi,1)\neq 0$ were used in the proof.

2.6. The cohomology class $c(\ell)$

The assumptions and notation from 2.5 are in force.

2.6.1. The CM point $x(\ell):=[z,1]_{UY}\in CM(M,K)_{\ell-ur}$ is defined over the field $K_{UY}\subset F_\ell^{ur}$ which is abelian over K ($\subset K_\ell\subset F_\ell^{ur}$) and which satisfies

$$\mathrm{rec}_K:\mathrm{Gal}(K_{UY}/K)\xrightarrow{\sim}\widehat{K}^\times/K^\times\,\widehat{t}^{-1}(U)Y=\widehat{K}^\times/K^\times K_\ell^\times(\widehat{t}^{(\ell)})^{-1}(U^{(\ell)})Y^{(\ell)}=\widehat{K}^{(\ell)\times}/K^\times(\widehat{t}^{(\ell)})^{-1}(U^{(\ell)})Y^{(\ell)}$$

(this isomorphism does not depend on the choice of an embedding $K_{UY}\hookrightarrow K^{\mathrm{ab}}$). Note that the field K_{UY} (more precisely, its image in K^{ab}) does not depend on ℓ and contains K_χ, by (2.2.4.1).

2.6.2. The divisor

$$D:=e_{\overline{\chi}}\left(\mathrm{Tr}_{K_{UY}/K_\chi}(x(\ell))\right)\in\mathrm{Div}(M\otimes_F K_\chi)\otimes O_L,$$

where

$$e_{\overline{\chi}}:=\sum_{\sigma\in\mathrm{Gal}(K_\chi/K)}\chi(\sigma)\,\sigma\in O_L[\mathrm{Gal}(K_\chi/K)],$$

is equal to

$$D=\sum_{a\in\widehat{K}^{(\ell)\times}/K^\times(\widehat{t}^{(\ell)})^{-1}(U^{(\ell)})Y^{(\ell)}}\chi(a)\,[z,\widehat{t}^{(\ell)}(a)]_{UY},\tag{2.6.2.1}$$

by 1.8.3.

According to Proposition 1.5.9(1), the divisor $D'=(N(v_0)+1-T(v_0))D$ has degree zero on each connected component of $M\otimes_F\overline{F}$; its class defines an element

$$cl(D')\in(J(K_\chi)\otimes O_L)^{(\chi^{-1})}.$$

2.6.3. The Kummer map

$$J(K_\chi)\otimes\mathbf{Z}_p=\varprojlim_m J(K_\chi)\otimes\mathbf{Z}/p^m\mathbf{Z}\hookrightarrow\varprojlim_m H^1(K_\chi,J[p^m])=H^1(K_\chi,T_p(J))$$

gives rise to a map

$$\partial:J(K_\chi)\otimes O_{L,\mathfrak{p}}\hookrightarrow H^1(K_\chi,T_p(J)\otimes_{\mathbf{Z}_p}O_{L,\mathfrak{p}})\longrightarrow H^1(K_\chi,(J\otimes O_L)[\mathfrak{p}^n]).$$

Set

$$c'(\ell):=j_*(\partial(cl(D')))\in H^1(K_\chi,T/\mathfrak{p}^nT)^{(\chi^{-1})},$$

where

$$j_*:H^1(K_\chi,(J\otimes O_L)[\mathfrak{p}^n])\longrightarrow H^1(K_\chi,T/\mathfrak{p}^nT)$$

is induced by the map j from Proposition 2.5.7.

2.6.4. Definition. *Denote by $c(\ell)$ the image of $c'(\ell)$ in $H^1(K_\chi, T/\mathfrak{p}^{n-C_0}T)^{(\chi^{-1})}$, where C_0 is the constant defined in Definition 2.7.9 below.*

2.7. Localisation of $c(\ell)$ outside ℓ

2.7.1. Let E/F be a finite extension. For each finite prime w of E denote by

$$\begin{aligned} H^1_f(E_w, V) &= \mathrm{Ker}\left(H^1(E_w,V) \longrightarrow \begin{cases} H^1(E_w, V\otimes_{\mathbf{Q}_p} B_{\mathrm{cris}}), & w \mid p \\ H^1(I_{E_w}, V), & w\nmid p\end{cases}\right) \\ H^1_f(E_w,T) &= \mathrm{Ker}\left(H^1(E_w,T)\longrightarrow H^1(E_w,V)/H^1_f(E_w,V)\right) \\ H^1_f(E_w,V/T) &= \mathrm{Ker}\left(H^1(E_w,V/T)\longrightarrow \mathrm{Coker}(H^1_f(E_w,V)\longrightarrow H^1(E_w,V/T))\right) \\ H^1_f(E_w,T/\mathfrak{p}^mT) &= \mathrm{Im}\left(H^1_f(E_w,T)\longrightarrow H^1(E_w,T/\mathfrak{p}^mT)\right)\end{aligned}$$

the Bloch-Kato subspaces [BK] of local Galois cohomology and by

$$H^1_f(E,-) = \mathrm{Ker}(H^1(E,-)\longrightarrow \prod_{w\nmid\infty} H^1(E_w,-)/H^1_f(E_w,-)) \qquad (- = V, T, V/T, T/\mathfrak{p}^mT)$$

the corresponding Bloch-Kato Selmer groups (and similarly for $T^*(1) \xrightarrow{\sim} T\otimes\omega^{-1}$). Note that

$$H^1_f(E,T) = \varprojlim_m H^1_f(E,T/\mathfrak{p}^mT), \qquad H^1_f(E,V/T) = \varinjlim_m H^1_f(E,T/\mathfrak{p}^mT).$$

For each finite extension $E'_{w'}/E_w$ the restriction (resp. corestriction) map $res : H^1(E_w,-)\longrightarrow H^1(E'_{w'},-)$ (resp. $cor : H^1(E'_{w'},-)\longrightarrow H^1(E_w,-)$) maps the H^1_f-subspace into the H^1_f-subspace. The formula $cor\circ res = [E'_{w'} : E_w]$ implies that

$$[E'_{w'} : E_w]\cdot res^{-1}\left(H^1_f(E'_{w'},-)\right) \subseteq H^1_f(E_w,-) \qquad (- = V,T,V/T,T/\mathfrak{p}^mT). \tag{2.7.1.1}$$

2.7.2. The goal of §2.7 is to define integers $C_0, C_5, C_6 \geq 0$ depending on f and $\mathfrak{p}$ (but not on K, χ or ℓ) and prove Proposition 2.7.12 below.

2.7.3. Proposition. *For each finite prime $v\in S\cup S_p$ there exists a finite Galois extension F'_v/F_v which depends on U and ω but not on ℓ such that $J(M_{U'Y})\otimes_F F'_v$ has split semi-abelian reduction. In particular, the discussion in Appendix A.3 applies to $J(M_{U'Y})\otimes_F F'_v$ and $\mathcal{K} = F'_v$.*

Proof. If $v\in S_p - S$, then $M_{U'Y}$ and its Jacobian have good reduction at v, so we can – and will – take $F'_v = F_v$.

For each rational prime q,

$$H^1_{et}(J(M_{U'Y})\otimes_F \overline{F}, \overline{\mathbf{Q}}_q) = \bigoplus_{\pi'} (\pi')^{U'Y}\otimes V_q(\widetilde{\pi}'),$$

where π' runs through irreducible (cuspidal) representations of $B'^{\times}_{\mathbf{A}}$ such that $\pi'_\infty = \sigma_2$ ([N1, Prop. 1.18(ii)] and 1.6.9).

For each $v\in S$ fix an open compact subgroup $U_v\subset B'^{\times}_v$ such that $U_S\supset\prod_{v\in S}U_v$. If π' as above satisfies $(\pi')^{U'Y}\neq 0$, then its central character $\omega_{\pi'}$ is equal to a power of ω (since $Y = \mathrm{Ker}(\omega)\subset\mathrm{Ker}(\omega_{\pi'})$) and $(\pi'_v)^{U_v}\neq 0$ for each $v\notin S_\infty\cup\{\ell\}$.

It is known ([C1, Thm. A], [T1, Thm. 2]) that, for each $v\nmid q$, $V_q(\pi')|_{G_{F_v}}$ corresponds to $JL(\pi'_v)$ by the local Langlands correspondence. Above, $JL(\pi'_v)$ denotes π'_v if $v\notin\mathrm{Ram}(B')$ (resp. the smooth representation of $GL_2(\pi'_v)$ attached to π'_v by the Jacquet-Langlands correspondence if $v\in\mathrm{Ram}(B')$).

In concrete terms, for $v\in S$ and q diferent from the residue characteristic of v, the condition $(\pi'_v)^{U_v}\neq 0$ implies the following.

(1) If $JL(\pi'_v) = \pi(\mu_1, \mu_2)$ ($\mu_i : F_v^\times \longrightarrow \mathbf{C}^\times$) is a principal series representation, then the inertia group I_{F_v} acts on $V_q(\pi')$ – through a finite abelian quotient – by $(\mu_1 \oplus \mu_2)|_{O_{F,v}^\times}$. The conductor exponent $o(\mu_1)+o(\mu_2) = o(JL(\pi'_v))$ is bounded above by a constant depending on U_v, which implies that there is a finite abelian extension $F_{v,1}$ of F_v depending only on U_v whose inertia group acts trivially on $V_q(\pi')$.
(2) If $JL(\pi'_v)$ is supercuspidal, then the argument in (a) applies to the base change of $JL(\pi'_v)$ to a suitable extension of F_v of degree 2 (resp. degree dividing 12) if $v \nmid 2$ (resp. if $v \mid 2$). We obtain, again, a finite Galois extension $F_{v,2}$ of F_v depending only on U_v whose inertia group acts trivially on $V_q(\pi')$.
(3) If $JL(\pi'_v) = \mathrm{St} \otimes \mu$ ($\mu : F_v^\times \longrightarrow \mathbf{C}^\times$), then $\mu^2 = (\omega_{\pi'})_v$ is a power of ω_v, which implies that that the order of μ is bounded above by a constant depending on ω_v, hence there is a finite abelian extension $F_{v,3}$ of F_v depending only on ω_v containing $\overline{F}_v^{\mathrm{Ker}(\mu)}$. By Prop. 2.7.8(2) below, the inertia group of $F_{v,3}$ acts unipotently on $V_q(\pi')$ and its absolute Galois group acts trivially on the corresponding graded quotients of $V_q(\pi')$.

For each $v \in S$, let F'_v be any finite Galois extension of F_v containing the fields $F_{v,1}$, $F_{v,2}$ and $F_{v,3}$. By construction, if $\pi'_\infty = \sigma_2$ and $(\pi')^{U'Y} \neq 0$, then the base change of $JL(\pi'_v)$ to F'_v is an unramified principal series representation in the cases (1) and (2) (resp. the Steinberg representation St in the case (3)). The above description of the action of $G_{F'_v}$ on $V_q(\pi')$ implies that $J(M_{U'Y}) \otimes_F F'_v$ has split semi-abelian reduction, as required.

2.7.4. We can – and will – enlarge each F'_v so that the base change of $\pi(f)_v$ to F'_v be either an unramified principal series representation or the Steinberg representation of $GL_2(F'_v)$.

2.7.5. Definition. *For each finite prime $v \in S \cup S_p$ fix F'_v as in 2.7.3-4 and define $C_{5,v} = C_{5,v}(\mathfrak{p}) := \mathrm{ord}_{\mathfrak{p}}([F'_v : F_v]) \geq 0$, $C_5 = C_5(\mathfrak{p}) := \max_v C_{5,v}$.*

2.7.6. Proposition. *If $v \in S_p$, $v \nmid \mathfrak{n}$ (and if there exists $g_{\mathfrak{p}} \in G_F$ satisfying (A1)–(A2)), then there is a Barsotti-Tate group H over $O_{F,v}$ equipped with an action of $O_{L,\mathfrak{p}}$ such that $T_p(H) = T|_{G_{F_v}}$.*

Proof. This is known in general for an arbitrary Hilbert modular form of parallel weight two (even without the assumption about the existence of $g_{\mathfrak{p}}$):
- if f has CM, since $T \subset T_p(A)$, where A is an abelian variety with CM with good reduction at v;
- if $T/\mathfrak{p}T$ is an irreducible G_F-module [T2, Thm. 1.6];
- if $p \neq 2$ ($V|_{G_{F_v}}$ is crystalline with Hodge-Tate weights contained in $\{0, 1\}$ [Li3, Thm. 4.2.1], [Sk, Thm. 1]; the existence of H follows from [Br, Thm 1.4]);
- if p is arbitrary (replace the reference to [Br, Thm 1.4] by [Ki, Cor. 2.2.6]).

However, it may be of interest to note that in our case there is an argument along the lines of [T2, Thm. 1.6]: fix, for each $n >> 0$, a prime ℓ of F as in 2.3.2. The assumption $v \nmid \mathfrak{n}$ implies that $J(M_{U'Y})$ has good reduction at v, hence the G_{F_v}-module N in Proposition 2.4.6 comes from a finite flat group scheme over $O_{F,v}$ (equipped with an $O_L/\mathfrak{p}^n O_L$-action). This implies that $T/\mathfrak{p}^{n-3C_2-C_3-13C_4} T|_{G_{F_v}}$, which is a subquotient of N, also comes from such a finite flat group scheme. The existence of H then follows from [Ra 2, Prop. 2.3.1].

2.7.7. Corollary. *If $v \in S_p$ and $\pi(f)_v \neq \mathrm{St} \otimes \mu$ (and if there exists $g_{\mathfrak{p}} \in G_F$ satisfying (A1)–(A2)), then there is a Barsotti-Tate group H with an action of $O_{L,\mathfrak{p}}$ over $O_{F'_v}$ such that $T_p(H) = T|_{G_{F'_v}}$.*

Proof. Again, this is known if $p \neq 2$ even without assuming the existence of $g_{\mathfrak{p}}$. Under our assumptions there exists a totally real solvable extension F'/F and a prime $v' \mid v$ of F' such that $F'_{v'}/F_v$ is isomorphic to the extension F'_v/F_v from 2.7.3-4. If f has no CM, B.5.5(2) tells us that the existence of $g_{\mathfrak{p}}$ implies that there is $g'_{\mathfrak{p}} \in G_{F'}$ satisfying (A1)–(A2). The result follows by applying Proposition 2.7.6 to the base change of f to F'. If f has CM, then the base change of the corresponding abelian variety with CM to F' has good reduction at v', so we can conclude as in 2.7.6.

2.7.8. Proposition. (1) *Let E and w be as in 2.7.1. If $w \nmid p$, then $H^i(E_w, V) = 0$ fir each $i \geq 0$, $H^1_f(E_w, T) = H^1(E_w, T)$ is finite and $H^1_f(E_w, V/T) = 0$.*
(2) *If $\pi(f)_v = \mathrm{St} \otimes \mu$ ($\mu : F_v^\times \longrightarrow \mathbf{C}^\times$, $\mu^2 = \omega_v$), then $V_v := V|_{G_{F_v}}$ sits in an exact sequence of $L_{\mathfrak{p}}[G_{F_v}]$-modules*

$$0 \longrightarrow V_v^+ \longrightarrow V_v \longrightarrow V_v^- \longrightarrow 0,$$

where $V_v^+ = L_{\mathfrak{p}}(1) \otimes \mu$ *and* $V_v^- = L_{\mathfrak{p}} \otimes \mu$*. Set* $T_v^+ := T \cap V_v^+$ *and* $T_v^- := T/T_v^+$*. The extension class* $[T_v]$ *of the exact sequence* $0 \longrightarrow T_v^+ \longrightarrow T_v \longrightarrow T_v^- \longrightarrow 0$ *is an element of* $H^1(F_v, O_{L,\mathfrak{p}}(1)) \xrightarrow{\sim} F_v^\times \widehat{\otimes} O_{L,\mathfrak{p}}$ *whose image under* $\mathrm{ord}_v \otimes \mathrm{id} : F_v^\times \widehat{\otimes} O_{L,\mathfrak{p}} \longrightarrow O_{L,\mathfrak{p}}$ *is non-zero.*
(3) *If* $\pi(f)_v = \mathrm{St} \otimes \mu$ *and* $v \in S_p$*, then*

$$H^1_f(E_w, V) = \mathrm{Ker}\left(H^1(E_w, V) \longrightarrow H^1(E_w, V_v^-)\right),$$

for each finite extension E_w/F_v*.*

Proof. This is well-known (see, for example, [N2, (12.4.4.2)], [N2, Lemma 12.5.4(ii)] and [N3, Prop. 3.3.2]).

2.7.9. Definition. *Let* v *be as in 2.7.3.*
(1) *If* $v \notin S_p$*, set* $C_{0,v} := 0$*.*
(2) *If* $v \in S_p$ *and* $\pi(f)_v = \mathrm{St} \otimes \mu$*, set* $C_{0,v} := e(\mathfrak{p}) \cdot a$*, where* $a = a(O_{F'_v}, \mu_{p^\infty})$ *is the integer from A.1.7.*
(3) *If* $v \in S_p$ *and* $\pi(f)_v \neq \mathrm{St} \otimes \mu$*, set* $C_{0,v} := e(\mathfrak{p}) \cdot a$*, where* $a = a(O_{F'_v}, H)$ *is the integer from A.1.7 for the Barsotti-Tate group* H *from Corollary 2.7.7.*
Finally, set $C_0 = C_0(\mathfrak{p}) := \max_{v|p} C_{0,v}$*.*

2.7.10. Proposition-Definition. *Let* v *be as in 2.7.3.*
(1) *If* $v \nmid p\mathfrak{n}$*, set* $C_{6,v} := 0$*.*
(2) *If* $v \mid p\mathfrak{n}$ *and* $\pi(f)_v = \mathrm{St} \otimes \mu$*, let* $C_{6,v} \geq 0$ *be the biggest non-negative integer such that*

$$e(F'_v/F_v)(\mathrm{ord}_v \otimes \mathrm{id})[T_v] \in \mathfrak{p}^{C_{6,v}} O_{L,\mathfrak{p}}.$$

(3) *If* $v \mid p\mathfrak{n}$ *and* $\pi(f)_v \neq \mathrm{St} \otimes \mu$*, then* $H^0(F'_v, V(-1)) = 0$*. We let* $C_{6,v} \geq 0$ *be the smallest non-negative integer such that*

$$\mathfrak{p}^{C_{6,v}} H^0(F'_v, V/T(-1)) = 0.$$

Set $C_6 = C_6(\mathfrak{p}) := \max_{v|p} C_{6,v}$*.*

Proof. The existence of $C_{6,v}$ in the case (2) follows from Proposition 2.7.8(2). The existence of $C_{6,v}$ in the case (3) follows from the vanishing of $H^0(F'_v, V(-1))$, which is, in turn, a consequence of the fact that $V(-1)|_{G_{F'_v}}$ is an unramified (resp. a crystalline) representation of $G_{F'_v}$ if $v \nmid p$ (resp. if $v \mid p$) which is pure of weight 1, by the compatibility of $V_{\mathfrak{p}}(f)$ with the local Langlands correspondence [T1, Thm. 2] (resp. [Li3, Thm. 4.2.1] and [Sk, Thm. 1]) and the generalised Ramanujan conjecture for f [Bl].

2.7.11. Proposition. *Assume that* $\pi(f)_v = \mathrm{St} \otimes \mu$*.*
(1) *If* $v \nmid p$*, then*

$$\forall m \geq 0 \qquad \mathfrak{p}^{C_{6,v}}\, H^0(I_{F'_v}, T/\mathfrak{p}^m T) \subseteq T_v^+/\mathfrak{p}^m T_v^+.$$

(2) *If* $v \mid p$*, assume that* $H \in \mathrm{Gr}_{O_{F'_v}}$ *in the notation of Appendix A.1.1 and that* $j' : H(\overline{F}_v) \longrightarrow T/\mathfrak{p}^m T|_{G_{F'_v}}$ *is a morphism of* $O_{L,\mathfrak{p}}[G_{F'_v}]$*-modules for which* $\mathrm{Im}(j') \cap T_v^+/\mathfrak{p}^m T_v^+$ *has a sufficiently large exponent (= bigger than a suitable constant depending on* T_v *and* F'_v*). Then*

$$\mathfrak{p}^{C_{0,v}+C_{6,v}}\, \mathrm{Im}(j') \subseteq T_v^+/\mathfrak{p}^m T_v^+.$$

Proof. (1) This follows from the exact cohomology sequence of

$$0 \longrightarrow T_v^+/\mathfrak{p}^m T_v^+ \longrightarrow T/\mathfrak{p}^m T \longrightarrow T_v^-/\mathfrak{p}^m T_v^- \longrightarrow 0$$

over the maximal unramified extension of F'_v, which reads as follows:

$$0 \longrightarrow T_v^+/\mathfrak{p}^m T_v^+ \longrightarrow H^0(I_{F'_v}, T/\mathfrak{p}^m T) \longrightarrow O_L/\mathfrak{p}^m O_L \xrightarrow{\delta} H^1(I_{F'_v}, O_L/\mathfrak{p}^m O_L(1)) = O_L/\mathfrak{p}^m O_L,$$

where $\delta(1)$ is equal to the image of $e(F'_v/F_v)(\mathrm{ord}_v \otimes \mathrm{id})[T_v] \in O_{L,\mathfrak{p}} - \{0\}$ in $O_L/\mathfrak{p}^m O_L$.
(2) Let $H' \in \mathrm{Gr}_{O_{F'_v}}$ be the quotient of H by the scheme-theoretical closure of $\mathrm{Ker}(j')$. The induced map $j' : H'(\overline{F}_v) \hookrightarrow T/\mathfrak{p}^m T|_{G_{F'_v}}$ is injective and the statement (2) follows from A.2.9 (since $\mathfrak{p}^{C_{0,v}} O_{L,\mathfrak{p}} = p^a O_{L,\mathfrak{p}}$).

2.7.12. Proposition. *Let v be as in 2.7.3, let E_w/F_v be a finite extension. Denote by δ_{E_w} the map*

$$\delta_{E_w} : J(E_w) \otimes O_L/\mathfrak{p}^n O_L \xrightarrow{\partial} H^1(E_w, (J \otimes O_L)[\mathfrak{p}^n]) \xrightarrow{j_*} H^1(E_w, T/\mathfrak{p}^n T) \xrightarrow{(j_{C_{0,v}})_*} H^1(E_w, T/\mathfrak{p}^{n-C_{0,v}} T).$$

(1) *If $E_w \supseteq F'_v$, then*

$$\mathfrak{p}^{C_{6,v}} \operatorname{Im}(\delta_{E_w}) \subseteq H^1_f(E_w, T/\mathfrak{p}^{n-C_{0,v}} T).$$

(2) *In general,*

$$[E_w F'_v : E_w]\, \mathfrak{p}^{C_{6,v}} \operatorname{Im}(\delta_{E_w}) \subseteq H^1_f(E_w, T/\mathfrak{p}^{n-C_{0,v}} T),$$
$$\mathfrak{p}^{C_{5,v}+C_{6,v}} \operatorname{Im}(\delta_{E_w}) \subseteq H^1_f(E_w, T/\mathfrak{p}^{n-C_{0,v}} T).$$

Proof. Note that (2) follows from (1) for $E_w F'_v$ instead of E_w and (2.7.1.1) for $E_w F'_v/F'_v$. We can assume, therefore, that $E_w \supseteq F'_v$. We distinguish several cases.
(a) If $v \nmid p\mathfrak{n}$ ($C_{0,v} = C_{5,v} = C_{6,v} = 0$), then $w \nmid p$ and J has good reduction over E_w, which implies that $\operatorname{Im}(\partial) = H^1_{ur}(E_w, (J \otimes O_L)[\mathfrak{p}^n])$, hence $\operatorname{Im}(\delta_{E_w}) \subseteq H^1_{ur}(E_w, T/\mathfrak{p}^n T) = H^1_f(E_w, T/\mathfrak{p}^n T)$ (the last equality is a consequence of the fact that T is an unramified G_{E_w}-module).
(b) If $v \nmid p$ and $\pi(f)_v \neq \mathrm{St} \otimes \mu$ ($C_{0,v} = 0$), then T is again an unramified G_{E_w}-module, so we must show that $\operatorname{Im}(\delta_{E_w})$ is contained in $H^1_{ur}(E_w, T/\mathfrak{p}^n T)$. According to A.3.6 (applied to $J(M_{U'Y}) \otimes_F F'_v$, $\mathcal{K} = F'_v$ and $\mathcal{K}' = E_w$) combined with A.2.3(1),

$$\operatorname{Im}(\delta_{E_w}) \subseteq X_1 \oplus X_2, \qquad X_1 = \operatorname{Im}\left(H^1_{ur}(E_w, (\mathscr{G}^\circ \otimes O_L)[\mathfrak{p}^n](\overline{F}_v)) \xrightarrow{j_*} H^1(E_w, T/\mathfrak{p}^n T)\right),$$
$$X_2 = \operatorname{Im}\left(H^1(E_w, (\mathscr{T} \otimes O_L)[\mathfrak{p}^n](\overline{F}_v)) = H^1(E_w, (O_L/\mathfrak{p}^n O_L)(1))^{\operatorname{rk}(\mathscr{T})} \xrightarrow{j_*} H^1(E_w, T/\mathfrak{p}^n T)\right).$$

The statement follows from the fact that j_* maps unramified cohomology into unramified cohomology (hence $X_1 \subseteq H^1_{ur}(E_w, T/\mathfrak{p}^n T)$) and that $\mathfrak{p}^{C_{6,v}}\, j(\mathscr{T} \otimes O_L)[\mathfrak{p}^n] = 0$ by definition of $C_{6,v}$ (hence $\mathfrak{p}^{C_{6,v}} X_2 = 0$).
(c) If $v \mid p$ and $\pi(f)_v \neq \mathrm{St} \otimes \mu$, then $T|_{G_{E_w}} = T_p(H)$ for some Barsotti-Tate group H over O_{E_w} equipped with an $O_{L,\mathfrak{p}}$-action. It follows from A.2.6 that $H^1_f(E_w, T/\mathfrak{p}^n T) = X(H) \otimes_{O_L} O_L/\mathfrak{p}^n O_L = H^1_{fl}(O_{E_w}, H[\mathfrak{p}^n])$. Using A.3.6 again, we have

$$\operatorname{Im}(\delta_{E_w}) \subseteq Y_1 \oplus Y_2, \qquad Y_1 = \operatorname{Im}\left(H^1_{fl}(O_{E_w}, (\mathscr{G}^\circ \otimes O_L)[\mathfrak{p}^n]) \xrightarrow{(j_{C_{0,v}})_* \circ j_*} H^1(E_w, T/\mathfrak{p}^{n-C_{0,v}} T)\right),$$
$$Y_2 = \operatorname{Im}\left(H^1(E_w, (\mathscr{T} \otimes O_L)[\mathfrak{p}^n](\overline{F}_v)) = H^1(E_w, (O_L/\mathfrak{p}^n O_L)(1))^{\operatorname{rk}(\mathscr{T})} \xrightarrow{(j_{C_{0,v}})_* \circ j_*} H^1(E_w, T/\mathfrak{p}^{n-C_{0,v}} T)\right).$$

According to A.2.8, $Y_1 \subseteq H^1_{fl}(O_{E_w}, H[\mathfrak{p}^{n-C_{0,v}}])$, and the same argument as in (b) shows that $\mathfrak{p}^{C_{6,v}} Y_2 = 0$.
(d) If $\pi(f)_v = \mathrm{St} \otimes \mu$, then A.3.5(1) tells us that

$$\operatorname{Im}(\delta_{E_w}) \subseteq Z := \operatorname{Im}\left(H^1(E_w, (\mathscr{G}^\circ \otimes O_L)[\mathfrak{p}^n](\overline{F}_v)) \xrightarrow{(j_{C_{0,v}})_* \circ j_*} H^1(E_w, T/\mathfrak{p}^{n-C_{0,v}} T)\right).$$

Proposition 2.7.11 implies that

$$\mathfrak{p}^{C_{6,v}}\, j_{C_{0,v}} \circ j(\mathscr{G}^\circ \otimes O_L)[\mathfrak{p}^n](\overline{F}_v) \subseteq (T_v^+/\mathfrak{p}^{n-C_{0,v}} T_v^+)|_{G_{E_w}} = O_L/\mathfrak{p}^{n-C_{0,v}} O_L(1),$$

hence $\mathfrak{p}^{C_{6,v}} \operatorname{Im}(\delta_{E_w}) \subseteq \mathfrak{p}^{C_{6,v}} Z$ is contained (using Proposition 2.7.8(3) for the last inclusion) in

$$\operatorname{Im}\left(H^1(E_w, O_L/\mathfrak{p}^{n-C_{0,v}} O_L(1)) = H^1(E_w, O_{L,\mathfrak{p}}(1)) \otimes_{O_L} O_L/\mathfrak{p}^{n-C_{0,v}} O_L \longrightarrow H^1(E_w, T/\mathfrak{p}^{n-C_{0,v}} T)\right)$$
$$\subseteq H^1_f(E_w, T/\mathfrak{p}^{n-C_{0,v}} T),$$

Note that, in the case when $v \mid p$, the assumption of Proposition 2.7.11(2) is satisfied: as the G_{E_w}-module $J[p^m](\overline{\mathcal{K}})/\mathscr{G}^\circ[p^m](\overline{\mathcal{K}})$ is unramified for each $m \geq 0$, it follows from Proposition 2.5.7 that the intersection of $j((\mathscr{G}^\circ \otimes O_L)[\mathfrak{p}^n](\overline{\mathcal{K}}))$ with $T_v^+/\mathfrak{p}^n T_v^+|_{G_{E_w}} = O_L/\mathfrak{p}^n O_L(1)$ contains $\mathfrak{p}^C (O_L/\mathfrak{p}^n O_L)(1)$, for some constant C.

2.7.13. Corollary. *Let K'/K_χ be a finite extension and w a finite prime of K'.*
(1) *If $w \nmid \ell$, then*

$$\mathfrak{p}^{C_5+C_6}\,(res_{K'/K_\chi}(c(\ell)))_w \in H^1_f(K'_w, T/\mathfrak{p}^{n-C_0}T).$$

(2) *For each $s \in H^1_f(K', (T/\mathfrak{p}^{n-C_0}T)\otimes\omega^{-1})$ the local cup products*

$$\cup : H^1(K'_w, T/\mathfrak{p}^{n-C_0}T) \times H^1(K'_w, (T/\mathfrak{p}^{n-C_0}T)\otimes\omega^{-1}) \longrightarrow H^2(K'_w, O_L/\mathfrak{p}^{n-C_0}O_L(1)) \xrightarrow{\mathrm{inv}_w} O_L/\mathfrak{p}^{n-C_0}O_L$$

satisfy

$$\mathfrak{p}^{C_5+C_6}\sum_{w|\ell}\mathrm{inv}_w\left(s_w \cup (res_{K'/K_\chi}(c(\ell)))_w\right) = 0 \in O_L/\mathfrak{p}^{n-C_0}O_L.$$

Proof. (1) This follows from Proposition 2.7.12(2). The statment (2) is a consequence of the reciprocity law

$$\forall x \in H^1(K', (T/\mathfrak{p}^{n-C_0}T)\otimes\omega^{-1}) \qquad \sum_w \mathrm{inv}_w\left(x_w \cup (res_{K'/K_\chi}(c(\ell)))_w\right) = 0 \in O_L/\mathfrak{p}^{n-C_0}O_L$$

and the fact that $H^1_f(K'_w, T/\mathfrak{p}^{n-C_0}T) \cup H^1_f(K'_w, (T/\mathfrak{p}^{n-C_0}T)\otimes\omega^{-1}) = 0$ for each w.

2.8. Localisation of $c(\ell)$ at ℓ

2.8.1. Recall that we have denoted by λ the unique prime of K above ℓ. As $\mathrm{rec}_K(K_\ell^\times)$ acts trivially on K_{UY} and $\omega(\ell) = 1$, λ splits completely in $F_\omega K_{UY}/K$. As a result, the inclusions $K_\chi \subset K_{UY} \subset F_\ell^{ur}$ define a prime $\lambda_\chi \mid \lambda$ of K_χ such that $(K_\chi)_{\lambda_\chi} = K_\lambda = K_\ell$.
2.8.2. By construction of ℓ,

$$(T/\mathfrak{p}^nT)\big|_{G_{K_\ell}} = (O_L/\mathfrak{p}^nO_L)e_1 \oplus (O_L/\mathfrak{p}^nO_L)(1)e_2.$$

We are interested in the image $c'(\ell)_{\lambda_\chi,\mathrm{ram}}$ of

$$c'(\ell)_{\lambda_\chi} \in H^1((K_\chi)_{\lambda_\chi}, T/\mathfrak{p}^nT) = H^1(K_\ell, O_L/\mathfrak{p}^nO_L)e_1 \oplus H^1(K_\ell, O_L/\mathfrak{p}^nO_L(1))e_2$$

in

$$\begin{aligned} H^1(I_{K_\ell}, T/\mathfrak{p}^nT) &= H^1(I_{K_\ell}, O_L/\mathfrak{p}^nO_L)e_1 \oplus H^1(I_{K_\ell}, O_L/\mathfrak{p}^nO_L(1))e_2 \\ &\xrightarrow{\sim} (O_L/\mathfrak{p}^nO_L)(-1)e_1 \oplus (O_L/\mathfrak{p}^nO_L)e_2. \end{aligned}$$

2.8.3. Proposition. *We have $c'(\ell)_{\lambda_\chi,\mathrm{ram}} = c\,e_2$, where c is the image of*

$$(N(v_0)+1-a_{v_0})(N(\ell)^2-1)(N(\ell)^{-2}-1)^2 b^{12}(K^\times\widehat{t}^{-1}(U)\widehat{F}^\times : K^\times\widehat{t}^{-1}(U)Y)\mathscr{L}_\chi(f_B) \in O_{L,\mathfrak{p}} - \{0\}$$

in $O_L/\mathfrak{p}^nO_L$. Above, $\mathrm{ord}_\mathfrak{p}(b) = C_4$ and $(K^\times\widehat{t}^{-1}(U)\widehat{F}^\times : K^\times\widehat{t}^{-1}(U)Y)$ divides $(\widehat{F}^\times : Y) = [F_\omega : F]$.

Proof. We consider $cl(D')$ as an element of $J((K_\chi)_{\lambda_\chi})\otimes O_L = J(K_\ell)\otimes O_L$. Using the notation from 1.6.5 and 1.6.6, its image $cl(D')_\Phi$ in $\Phi\otimes O_L$ is represented by

$$\widetilde{cl}(D')_\Phi := (N(v_0)+1-a_{v_0})\begin{pmatrix}\sum_a \chi(a)\,[\widehat{t}^{(\ell)}(a)] \\ 0\end{pmatrix} \in O_L[\mathcal{V}(\mathcal{G})]_0 \subset \mathbb{S}(UY;O_L)^2,$$

thanks to (2.6.2.1). Above, a runs through $\widehat{K}^{(\ell)\times}/K^\times(\widehat{t}^{(\ell)})^{-1}(U^{(\ell)})Y^{(\ell)}$ and $[b]$ is the image of $b \in \widehat{B}^\times$ in $B^\times\backslash\widehat{B}^\times/UY$ (we embed $\widehat{B}^{(\ell)\times} \hookrightarrow \widehat{B}^\times$ by $b \mapsto b\times\{1\}$, $1 \in B_\ell^\times$).

The construction of the map j_0 together with the discussion in 1.7.3 (for $\mathcal{K} = (K_\chi)_{\lambda_\chi} = K_\ell$) imply that $c'(\ell)_{\lambda_\chi,\mathrm{ram}} = c\,e_2$, where c is the image in $O_L/\mathfrak{p}^nO_L$ of

$$(N(v_0)+1-a_{v_0})(N(\ell)^2-1)(N(\ell)^{-2}-1)^2 b^{12}\, u([\widetilde{f}],\widetilde{cl}(D')_\Phi) \in O_{L,\mathfrak{p}}.$$

The statement of the Proposition follows from the fact that

$$u([\widetilde{f}],\widetilde{cl}(D')_\Phi) = \sum_{a\in \widehat{K}^\times/K^\times \widehat{t}^{-1}(U)Y} \chi(a) f_B(\widehat{t}(a)) = (K^\times \widehat{t}^{-1}(U)\widehat{F}^\times : K^\times \widehat{t}^{-1}(U)Y)\mathscr{L}_\chi(f_B).$$

2.8.4. Corollary. *If $x = x_1 e_1 + x_2 e_2$ is an element of*

$$H^1_{ur}((K_\chi)_{\lambda_\chi}, (T/\mathfrak{p}^n T)\otimes\omega^{-1}) = H^1_{ur}(K_\ell, T/\mathfrak{p}^n T) \xrightarrow{\sim} (T/\mathfrak{p}^n T)/(\mathrm{Fr}_{\mathrm{geom}}(\ell)^2-1) =$$
$$= (O_L/\mathfrak{p}^n O_L)e_1 \oplus (O_{L,\mathfrak{p}}/(u_g^2-1)O_{L,\mathfrak{p}})e_2,$$

then

$$\mathrm{inv}_{\lambda_\chi}\left(x \cup c'(\ell)_{\lambda_\chi}\right) = \pm c\, x_1 \in O_L/\mathfrak{p}^n O_L,$$

where the sign depends on the choice of the sign of the isomorphism beween the p-primary part of I_{K_ℓ} and $\mathbf{Z}_p(1)$.

Proof. As in 1.7.5, this follows from Proposition 2.8.3 and [Ru, Lemma 1.4.7(ii)] (recall that the dual basis to $\{e_1, e_2\}$ with respect to the pairing (0.7.1) is equal to $\{e_2, e_1\}$).

2.9. The annihilation relation and the completion of the proof of Theorem A

The assumptions listed at the beginning of §2 are in force. In addition, if f has CM by a totally imaginary quadratic extension $K(f)$ of F, we assume that $K(f) \not\subset F_\omega K_\chi$.

2.9.1. Let $m >> 0$ be a large enough integer and consider the field extensions

$$F \subset K \subset F_\omega K \subset H := F_\omega K_\chi \subset H_n := F(T/\mathfrak{p}^n T)H$$

for $n = m + C_0$; set $U_n := \mathrm{Gal}(H_n/H)$ and denote by $g_n \in \mathrm{Gal}(H_n/F)$ the restriction of $g = g_\mathfrak{p}$ to H_n.

The restriction $g_n|_H$ is of the form $g_n = c h_n$, where $c \in \mathrm{Gal}(K^{\mathrm{ab}}/F)$ is the complex conjugation for a fixed embedding $K^{\mathrm{ab}} \subset \mathbf{C}$ and $h_n \in \mathrm{Gal}(H/F_\omega K)$. If we consider $g_n^2 = c h_n c h_n = {}^c h_n h_n$ as an element of $\mathrm{Gal}(H/F_\omega K) \subset \mathrm{Gal}(K_\chi/K)$, then

$$\chi(g_n^2) = ({}^c\chi \cdot \chi)(h_n) = \omega^{-1}\left(h_n|_{F_\omega}\right) = 1;$$

thus $g_n^2 \in U_n$. Denote by

$$H'_n := H_n^{\langle g_n^2\rangle} \subset H_n$$

the fixed field of the subgroup $\langle g_n^2\rangle \subset U_n$ generated by g_n^2.

2.9.2. Proposition-Definition. *There exists an integer $C_7 = C_7(\mathfrak{p}) \geq 0$ (equal to 0 for all but finitely many $\mathfrak{p}$) such that the restriction maps*

$$res' : H^1(H, (T/\mathfrak{p}^m T)\otimes\omega^{-1}) \longrightarrow H^1(H'_n, (T/\mathfrak{p}^m T)\otimes\omega^{-1}),$$
$$res : H^1(H, (T/\mathfrak{p}^m T)\otimes\omega^{-1}) \longrightarrow H^1(H_n, (T/\mathfrak{p}^m T)\otimes\omega^{-1}) = \mathrm{Hom}_{U_n}(G_{H_n}^{\mathrm{ab}} \otimes \mathbf{Z}/p^m\mathbf{Z}, (T/\mathfrak{p}^m T)\otimes\omega^{-1})$$

satisfy $\mathfrak{p}^{C_7}\,\mathrm{Ker}(res) = \mathfrak{p}^{C_7}\,\mathrm{Ker}(res') = 0.$

Proof. According to B.5.2 and B.6.5(2), $Z := \{a \in \mathbf{Z}_p^\times \mid a\cdot \mathrm{id} \in \mathrm{Im}(G_H \longrightarrow \mathrm{Aut}(T\otimes\omega^{-1}))\}$ is an open subgroup of $\mathbf{Z}_p^\times$. Fix $a \in Z - \{1\}$; then $a \cdot \mathrm{id}$ is an element of the centre of U_n which acts on $T\otimes\omega^{-1}$ by multiplication by a. Sah's Lemma [S, Proof of Prop. 2.7(b)] implies that $a-1$ acts trivially on

$$H^1(U_n, (T/\mathfrak{p}^m T)\otimes\omega^{-1}) = \mathrm{Ker}(res) \supseteq \mathrm{Ker}(res'),$$

which means that we can take $C_7 := \mathrm{ord}_\mathfrak{p}(a-1)$. The fact that we can choose a so that $C_7 = 0$ for all but finitely many $\mathfrak{p}$ follows from B.5.2 and B.6.5(5),(7).

2.9.3. Proposition-Definition. *There exists an integer $C_8 = C_8(\mathfrak{p}) \geq 0$ (equal to 0 for all but finitely many $\mathfrak{p}$) such that*

$$\mathrm{Im}\left(O_{L,\mathfrak{p}}[G_H] \longrightarrow \mathrm{End}_{O_{L,\mathfrak{p}}}(T \otimes \omega^{-1})\right) \supseteq \mathfrak{p}^{C_8}\, \mathrm{End}_{O_{L,\mathfrak{p}}}(T \otimes \omega^{-1}).$$

Proof. The existence of C_8 is equivalent to $V \otimes \omega^{-1}|_{G_H} = V|_{G_H}$ being (absolutely) irreducible, which is, in turn, equivalent to our assumption "if f has CM by $K(f)$, then $K(f) \not\subset H$", by a variant of [N1, Prop. 6.2.1]. The fact that C_8 is equal to 0 for all but finitely many $\mathfrak{p}$ follows from [N1, Prop. 6.2.2].

2.9.4. Proposition. *Let $\widetilde{s} \in Z^1(G_H, (T/\mathfrak{p}^m T) \otimes \omega^{-1})$ be a 1-cocycle representing an element $s \in H^1_f(H, (T/\mathfrak{p}^m T) \otimes \omega^{-1})^{(\chi)}$. Then:*
(1) $\forall h' \in G_{H'_n} \subset G_H$ *such that* $h'|_{H_n} = g_n^2$

$$\mathfrak{p}^C\, \widetilde{s}(h) \in (O_L/\mathfrak{p}^m O_L)e_2 \subset (T/\mathfrak{p}^m T) \otimes \omega^{-1},$$

where $C = C_1 + 3C_2 + 12C_4 + C_5 + C_6 + \mathrm{ord}_{\mathfrak{p}}((N(v_0) + 1 - a_{v_0})[F_\omega : F][H : K])$ (recall that e_1, e_2 is the $O_L/\mathfrak{p}^n O_L$-basis of $T/\mathfrak{p}^n T$ from 2.3.1).
(2) $\forall h' \in G_{H_n}$ $\quad \mathfrak{p}^C\, (res(s))(h) \in (O_L/\mathfrak{p}^m O_L)e_2 \subset (T/\mathfrak{p}^m T) \otimes \omega^{-1}$.

Proof. If $h \in G_{H_n}$ and $h'|_{H_n} = g_n^2$, then $hh'|_{H_n} = g_n^2$. As

$$(res(s))(h) = \widetilde{s}(h) = \widetilde{s}(hh') - \widetilde{s}(h')$$

by the cocycle relation, it is sufficient to prove (1).

Let $\lambda_n \mid \ell$ be the prime of H_n whose (geometric) Frobenius element with respect to the extension H_n/F_ω is equal to g_n. Denote by λ'_n (resp. λ_H) the prime of H'_n (resp. of H) below λ_n. As g_n^2 acts trivially on H'_n, the unique prime λ of K above ℓ splits completely in H'_n/K.

Corollary 2.7.13(2) for $K' = H$ says that

$$\mathfrak{p}^{C_5+C_6} \sum_{w \mid \ell} \mathrm{inv}_w \left(s_w \cup (res_{H/K_\chi}(c(\ell)))_w\right) = 0 \in O_L/\mathfrak{p}^m O_L,$$

where w runs through the set of primes of H above ℓ. All terms on the L.H.S. are equal to each other, since s (resp. $res_{H/K_\chi}(c(\ell))$) lies in the χ-eigenspace (resp. χ^{-1}-eigenspace) for the action of $\Delta = \mathrm{Gal}(H/K) \twoheadrightarrow \mathrm{Gal}(K_\chi/K)$ and the action of Δ on the set $\{w \mid \ell\}$ is transitive. As a result, we obtain the following **annihilation relation**:

$$[H : K]\, \mathfrak{p}^{C_5+C_6}\, \mathrm{inv}_{\lambda_H} \left(s_{\lambda_H} \cup (res_{H/K_\chi}(c(\ell)))_{\lambda_H}\right) = 0 \in O_L/\mathfrak{p}^m O_L. \qquad (\star)$$

The localisation

$$\begin{aligned} s_{\lambda_H} \in H^1_{ur}(H_{\lambda_H}, (T/\mathfrak{p}^m T) \otimes \omega^{-1}) = H^1_{ur}((H'_n)_{\lambda'_n}, T/\mathfrak{p}^m T) = (T/\mathfrak{p}^m T)/(g_n^2 - 1) = \\ = (O_L/\mathfrak{p}^m O_L)e_1 \oplus (O_L/\mathfrak{p}^{C_2} O_L)e_2 \end{aligned}$$

is represented by

$$\widetilde{(g_n^2)} = \widetilde{s}(h') = \widetilde{s}(h')_1\, e_1 \oplus \widetilde{s}(h')_2\, e_2, \qquad \widetilde{s}(h')_i \in O_L/\mathfrak{p}^m O_L.$$

Combining $(\star)$ with Corollary 2.8.4, we obtain

$$(N(v_0) + 1 - a_{v_0})[F_\omega : F][H : K]\, \mathfrak{p}^{C_1+3C_2+12C_4+C_5+C_6}\, s(h')_1 = 0 \in O_L/\mathfrak{p}^m O_L,$$

as claimed.

2.9.5. Corollary. *For each sufficiently large $m >> 0$ we have $\mathfrak{p}^{C+C_7+C_8}\, H^1_f(H, (T/\mathfrak{p}^m T)\otimes\omega^{-1})^{(\chi)} = 0$.*

Proof. Proposition 2.9.4(2) tells us that, for each $s \in H^1_f(H, (T/\mathfrak{p}^m T)\otimes\omega^{-1})^{(\chi)}$, the U_n-stable subgroup

$$\mathfrak{p}^C\,(res(s))(G_{H_n}\otimes \mathbf{Z}/p^m\mathbf{Z}) \subset (T/\mathfrak{p}^m T)\otimes\omega^{-1}$$

is contained in $(O_L/\mathfrak{p}^m O_L)e_2$. By definition of C_8, it is killed by $\mathfrak{p}^{C_8}$, hence $\mathfrak{p}^{C+C_8}\,res(s) = 0$. As $\mathfrak{p}^{C_7}\,\mathrm{Ker}(res) = 0$, the statement follows.

2.9.6. Proof of Theorem A. Corollary 2.9.5 implies that

$$\mathfrak{p}^{C+C_7+C_8}\, H^1_f(H, (V/T)\otimes\omega^{-1})^{(\chi)} = 0.$$

The kernel of the restriction map

$$H^1_f(K, (V/T)\otimes {}^c\chi) \longrightarrow \left(H^1_f(H, (V/T)\otimes\omega^{-1})\otimes\chi^{-1}\right)^{\mathrm{Gal}(H/K)}$$

is killed by $[H : K]$; thus

$$[H : K]\,\mathfrak{p}^{C+C_7+C_8}\, H^1_f(K, (V/T)\otimes {}^c\chi) = 0,$$

which proves that both groups $H^1_f(K, (V/T)\otimes\chi) \xrightarrow{\sim} H^1_f(K, (V/T)\otimes {}^c\chi)$ (see (0.6.1)) are finite (and equal to zero if $\mathfrak{p}$ does not belong to a certain finite set of primes of F, since the constants $C_i = C_i(\mathfrak{p})$ are equal to zero for all but finitely many $\mathfrak{p}$).

Finally, the conditions (A1)–(A3) do not depend on χ and $L_\mathfrak{p}$. It is legitimate, therefore, to analyse them by considering the minimal coefficient field $L = L_f$ (the field generated over $\mathbf{Q}$ by the Hecke eigenvalues $\lambda_f(v)$ ($v \nmid \mathfrak{n}$) of f). The corresponding analysis is carried out in Appendix B.5-6 in a more general context. The Galois representation $V = V_\mathfrak{p}(f)(1)$ corresponds to $V_\mathfrak{p}(\pi)$ from B.2.1 for $\pi = \pi(f)\otimes|\cdot|$; thus $\forall v \in S_\infty\quad k_v = m = 2$ in the notation of B.2.1. In particular, B.5.5(6) (resp. B.6.5(4),(6),(7)) implies that an element $g_\mathfrak{p} \in G_F$ satisfying (A1)–(A3) exists for $\mathfrak{p}$ belonging to a set of density 1 (resp. for $\mathfrak{p}_+ = L^+\cap\mathfrak{p}$ belonging to a set of density 1/2, where L^+ is the maximal totally real subfield of L) if f does not have (resp. has) complex multiplication. This completes the proof of Theorem A (and of Theorem A', which is its immediate consequence).

2.10. Proof of Theorem B

In this section we assume that $f \in S_2(\mathfrak{n}, 1)$ (hence the corresponding automorphic representation $\pi(f)$ of $GL_2(\mathbf{A}_F)$ as well) has trivial character. We take $L = L_f$.

2.10.1. For any totally imaginary quadratic extension K/F we have

$$L(f_K, 1_K, s) = L(f, s)\,L(f\otimes\eta, s), \qquad L(\pi(f)\times 1_K, s) = L(\pi(f), s)\,L(\pi(f)\otimes\eta, s),$$

using the notation of 0.5 for the trivial character $1_K : \mathbf{A}_K^\times/K^\times\mathbf{A}_F^\times \longrightarrow \mathbf{C}^\times$ (and writing $\eta = \eta_{K/F}$).

2.10.2. Proposition. *The following conditions are equivalent.*
(1) $\pi(f) = JL(\pi')$ is associated by the Jacquet-Langlands correspondence to an irreducible (cuspidal) automorphic representation π' of $B'^\times_\mathbf{A}/F^\times_\mathbf{A}$, where B' is a quaternion algebra over F such that $B'\otimes\mathbf{R} \xrightarrow{\sim} M_2(\mathbf{R})\times\mathbf{H}^{[F:\mathbf{Q}]-1}$ ($\iff$ $V_\mathfrak{p}(f)$ occurs in $H^1_{et}(M_H\otimes_F\overline{F}, L_\mathfrak{p})$ for a suitable (compactified) Shimura curve M_H over F arising from B' $\iff$ there exists a simple quotient A_f of the Jacobian $J(M_H)$ satisfying (0.10.1));
(2) $2\nmid[F:\mathbf{Q}]$ or there exists a finite prime v of F such that $\pi(f)_v$ is not a principal series representation;
(3) there exists a totally imaginary quadratic extension K/F such that $\varepsilon(\pi(f)\times 1_K, \frac{1}{2}) = -1$;
(4) there exists a finite set Σ of finite primes of F and for each $v\in\Sigma$ a character $\mu_v : F_v^\times \longrightarrow \{\pm1\}$ with the following property: for each totally imaginary quadratic extension K/F satisfying $\forall v\in\Sigma\quad (\eta_{K/F})_v = \mu_v$ we have $\varepsilon(\pi(f)\times 1_K, \frac{1}{2}) = -1$.

Proof. The equivalence (1) $\iff$ (2) is standard. If $\pi(f)_v$ is a principal series representation for each $v\notin S_\infty$, then $\varepsilon(\pi(f)_v\times 1_K, \frac{1}{2}) = \eta_v(-1)$ for such v (Proposition 2.1.1), hence $\varepsilon(\pi(f)\times 1_K, \frac{1}{2}) = \prod_{v|\infty}\eta_v(-1) =$

$(-1)^{[F:\mathbf{Q}]}$, which proves the implication (3) $\Longrightarrow$ (2). It remains to show that (2) $\Longrightarrow$ (4). If $2 \nmid [F:\mathbf{Q}]$, then we can take $\Sigma = \{w \mid \mathfrak{n}\}$ and $\mu_w = 1$ for each $w \in \Sigma$. If $2 \mid [F:\mathbf{Q}]$ and $\pi(f)_v$ is a twist of the Steinberg representation by an unramified character ($\Longleftrightarrow$ $\mathrm{ord}_v(\mathfrak{n}) = 1$), then we can take $\Sigma = \{w \mid \mathfrak{n}\}$, $\mu_w = 1$ if $w \in \Sigma - \{v\}$ and $\mu_v =$ the unramified quadratic character of $F_v^\times$. More generally, if $2 \mid [F:\mathbf{Q}]$ and $\pi(f)_v$ is not a principal series representation, let D be the quaternion division algebra over F_v and let π'_v be the (finite-dimensional) smooth representation of $D^\times/F_v^\times$ corresponding to $\pi(f)_v$ by the Jacquet-Langlands correspondence. Take any non-zero vector x in the representation space of π'_v. The stabiliser of x is open in $D^\times$; it contains an element $d \in D^\times$ which does not belong to $F_v^\times$. The commutative F_v-subalgebra $E := F_v[d] \subset D$ is a field of degree two over F_v and, by construction, $\mathrm{Hom}_{E^\times}(1_E, \pi'_v) \neq 0$. As π'_v decomposes under the action of $E^\times$ into a direct sum of finitely many one-dimensional representations, we also have $\mathrm{Hom}_{E^\times}(\pi'_v, 1_E) \neq 0$, hence $\varepsilon(\pi(f)_v \times 1_E, \frac{1}{2}) = -\eta_{E/F_v}(-1)$, by Proposition 2.1.1(1). If we take $\Sigma = \{w \mid \mathfrak{n}\}$, $\mu_w = 1$ if $w \in \Sigma - \{v\}$ and $\mu_v = \eta_{E/F_v}$, then

$$\varepsilon(\pi(f) \times 1_K, \tfrac{1}{2}) = \prod_{w \notin \Sigma} 1 \left(\prod_{w \in \Sigma - \{v\}} \eta_w(-1) \right) (-\eta_v(-1)) = -\prod_{w \mid \infty} \eta_w(-1) = -(-1)^{[F:\mathbf{Q}]} = -1,$$

for every totally imaginary quadratic extension K/F satisfying $\eta_w = \mu_w$ for all $w \in \Sigma$, where $\eta = \eta_{K/F}$.

2.10.3. Proposition. *If K from 2.10.1 satisfies $\mathrm{ord}_{s=1} L(f_K, 1_K, s) = 1$, then there exists a (compactified) Shimura curve $M_{H\widehat{F}^\times}$ (for an open compact subgroup $H \subset \widehat{B}'^\times$), a simple quotient A_f of the Jacobian $J(M_{H\widehat{F}^\times})$ satisfying (0.10.1), a finite subextension K_0/K of $(K^{\mathrm{ab}})^{\mathrm{rec}_K(\mathbf{A}_F^\times)}/K$ and a CM point $x = [z', 1]_{H\widehat{F}^\times} \in M_{H\widehat{F}^\times}(K_0)$ whose image $y \in A_f(K_0)$ under $\alpha : M_{H\widehat{F}^\times} \longrightarrow J(M_{H\widehat{F}^\times}) \longrightarrow A_f$ (where the first map is given by a suitable integral multiple of the Hodge class [Z1, p. 30], [CV 1, 3.5], [N1, 1.19]) has the following properties:*

(1) *the point $y_K := \mathrm{Tr}_{K_0/K}(y) \in A_f(K)$ is not torsion;*

(2) *the point $c(y_K) + \varepsilon(\pi(f), \frac{1}{2})\, y_K \in A_f(K)$ is torsion, where c is the non-trivial element of $\mathrm{Gal}(K/F)$.*

Proof. The proof of Proposition 2.1.4 implies that there exists π' as in Proposition 2.10.2(1) for the quaternion algebra B' satisfying $\forall v \nmid \infty \quad \varepsilon(\pi(f)_v \times 1_K, \frac{1}{2}) = \mathrm{inv}_v(B')\, \eta_v(-1)$.

The generalised Gross-Zagier formula [Z2, Thm. 4.2.1] proved in [YZZ] states that the Néron-Tate distribution from [Z2, 4.2]

$$NT_{1,\pi'} \in \mathrm{Hom}_{\widehat{K}^\times}(\pi'^\infty, \mathbf{C}) \otimes \mathrm{Hom}_{\widehat{K}^\times}(\pi'^\infty, \mathbf{C})$$

is a non-zero multiple of the distribution $\beta_{\pi',1}$ defined in [Z2, 4.2], which implies that, for suitable H and A_f, the point y_K has non-zero height, hence is non-torsion. The action of c on y_K is given by $c(P) = \mathrm{Tr}_{K_0/K}(\alpha([z', n]_{H\widehat{F}^\times}))$, where $n \in N_{B^\times}(K^\times)$ is an element of the normaliser of $K^\times$ in $B^\times$ ($n \notin K^\times$). It follows from Proposition 2.1.2 that n acts on $\mathrm{Hom}_{\widehat{K}^\times}(\pi'^\infty, \mathbf{C})$ by multiplication by

$$\prod_{v \nmid \infty} \mathrm{inv}_v(B')\, \varepsilon(\pi(f)_v, \tfrac{1}{2}) = \varepsilon(\pi(f), \tfrac{1}{2}) \prod_{v \mid \infty} \mathrm{inv}_v(B')\, \varepsilon(\pi(f)_v, \tfrac{1}{2}) = -\varepsilon(\pi(f), \tfrac{1}{2}),$$

since $\varepsilon(\pi(f)_v, \frac{1}{2}) = -1$ for each $v \mid \infty$. As a result, the point $c(y_K) + \varepsilon(\pi(f), \frac{1}{2})\, y_K$ has trivial height pairing with y_K, hence also with $c(y_K)$ and itself; it must be torsion, as claimed.

2.10.4. Corollary. *If $L(f, 1) \neq 0$ and if K/F is a totally imaginary quadratic extension such that f does not have CM by K and $\mathrm{ord}_{s=1} L(f_K, 1_K, s) = 1$, then the groups $A_f(F)$ and $Ш(A_f/F)$ are finite.*

Proof. It follows from Proposition 2.10.3(1) and [N1, Thm. 3.2] that the groups $A_f(K)/O_L \cdot y_K$ and $Ш(A_f/K)$ are finite. As $c(y_K) + y_K$ is torsion by Proposition 2.10.3(2), the group $A_f(F) = A_f(K)^{c=1}$ is finite, and so is $Ш(A_f/F)$ (since $2 \cdot \mathrm{Ker}(Ш(A_f/F) \longrightarrow Ш(A_f/K)) = 0$).

2.10.5. Proof of Theorem B. (a), (b) Combining Proposition 2.10.2(4) with [FH, Thm. B.2] we obtain an extension K/F to which Corollary 2.10.4 applies. The exact sequence

$$0 \longrightarrow A_f(F) \otimes L_{\mathfrak{p}}/O_{L,\mathfrak{p}} \longrightarrow H^1_f(F, V/T) \longrightarrow Ш(A_f/F)[\mathfrak{p}^\infty] \longrightarrow 0$$

(where $T = T_{\mathfrak{p}}(A_f) \subset V = V_{\mathfrak{p}}(A_f) = V_{\mathfrak{p}}(f)(1)$) then implies that $H^1_f(F, V/T)$ is finite (and equal to zero for all but finitely many $\mathfrak{p}$).

(c) Thanks to (a) we can assume that $2 \mid [F : \mathbf{Q}]$. If K/F is a totally imaginary quadratic extension in which all primes dividing $\mathfrak{n}$ split, then $2 \mid \mathrm{ord}_{s=1} L(f_K, 1_K, s)$. According to [FH, Thm. B.1], there exists such a K for which K/F is ramified at two primes $\mathfrak{q}_1, \mathfrak{q}_2$ not dividing $\mathfrak{n}$ lying above two distinct rational primes and $L(f_K, 1_K, 1) = L(f, 1)\, L(f \otimes \eta_{K/F}, 1) \neq 0$. The ramification assumption at $\mathfrak{q}_i$ implies that $K \not\subset F_{\Gamma_{\mathfrak{p}}} F(V_{\mathfrak{p}})$ in the notation of B.5.1 (resp. $K \not\subset K(f)F(V_{\mathfrak{p}})$) if f does not have (resp. has) CM. It follows from B.5.5(2),(4) (resp. B.6.5(2),(4)) that there exists $g_{\mathfrak{p}} \in G_F$ satisfying (A1)–(A3). Applying Theorem A for f, K and $\chi = 1$ we obtain that the group $H^1_f(K, V/T)$ is finite (and equal to zero if $\mathfrak{p}$ does not belong to a certain finite set). As $2 \cdot \mathrm{Ker}\left(H^1_f(F, V/T) \longrightarrow H^1_f(K, V/T)\right) = 0$, the same finiteness result holds over F.

2.10.6. Proof of Theorem B'. The statement (a) is an immediate consequence of Theorem B(a). In the case (b1) (resp. (b2)) there exists a finite prime v of F at which A_f does not have potentially good reduction (resp. such that $A_f \otimes_F F_v$ does not acquire semistable reduction over any cyclic extension of F_v, by [AT, ch. 10, Thm. 5]); thus $\pi(f)_v$ is a twist of a Steinberg representation (resp. is a supercuspidal representation), so Theorem B(b) applies. Finally, in the case (c1) (resp. (c2)) Theorem B(c) applies, thanks to B.4.12 (resp. B.6.5(2)).

Appendix A: Finite flat group schemes and their cohomology

In this appendix we recall basic facts about flat cohomology of finite flat group schemes over the ring of integers in a finite extension of $\mathbf{Q}_p$. The main references are [Ma] and [Mi 2, §III.1].

A.1 Finite flat group schemes

Let R be a complete DVR of mixed characteristic $(0,p)$, let $\mathcal{K}$ (resp. k) be its fraction field (resp. its residue field).

A.1.1. Denote by Gr_R the (exact) category of commutative finite flat group schemes over R. The generic fibre $H_{\mathcal{K}}$ of any $H \in \mathrm{Gr}_R$ is a finite étale group scheme over $\mathcal{K}$, which is determined by the $G_{\mathcal{K}}$-module $H_{\mathcal{K}}(\overline{\mathcal{K}})$.

A.1.2. The functor

$$H \mapsto H(\overline{\mathcal{K}}) = H_{\mathcal{K}}(\overline{\mathcal{K}}) \qquad (A.1.2.1)$$

from Gr_R into the category of discrete $G_{\mathcal{K}}$-modules is faithful. In other words, the map

$$\alpha_{H,H'} : \mathrm{Hom}_{Gr_R}(H,H') \longrightarrow \mathrm{Hom}_{G_{\mathcal{K}}}(H(\overline{\mathcal{K}}), H'(\overline{\mathcal{K}}))$$

is injective for any $H, H' \in Gr_R$. Moreover,

$$\mathrm{Coker}(\alpha_{H,H'}) \otimes \mathbf{Z}[1/p] = 0.$$

If the absolute ramification index $e = v_R(p)$ of $\mathcal{K}$ satisfies $e < p-1$, then the functor (A.1.2.1) is fully faithful [Ra 2, Cor. 3.3.6] (i.e., $\alpha_{H,H'}$ is bijective).

A.1.3. Question. *In general, is there a constsnt s depending only on e and p such that*

$$\forall H, H' \in Gr_R \qquad p^s\,\mathrm{Coker}(\alpha_{H,H'}) = 0 \quad ?$$

A.1.4. According to [Bo, Thm. A,B] (see also [Li1, Thm. 1.0.5], [Li2, Thm. 2.4.2] and [VZ, Thm. 1]), the answer is "yes". However, a much weaker statement (Corollary A.1.8-9) is sufficient for our purposes.

A.1.5. Recall [Ra 2, Prop. 2.2.2] that the generic fibre $H_{\mathcal{K}}$ of any $H \in \mathrm{Gr}_R$ admits a maximal (resp. a minimal) prolongation $H_{\max} \in \mathrm{Gr}_R$ (resp. $H_{\min} \in \mathrm{Gr}_R$). By definition, there are canonical morphisms in Gr_R

$$H_{\max} \longrightarrow H \longrightarrow H_{\min}$$

inducing the identity on the generic fibre. Moreover,

$$\mathrm{Coker}(\alpha_{H_{\max},H'_{\max}}) = \mathrm{Coker}(\alpha_{H_{\min},H'_{\min}}) = 0.$$

A.1.6. For any Barsotti-Tate group $H = (H_n)_{n\geq 1}$ over R denote by

$$i_n : H_m \longrightarrow H_{m+n}, \qquad j_n : H_{m+n} \longrightarrow H_m$$

the standard transition morphisms (the composite maps $i_n j_n$ and $j_n i_n$ are given by multiplication by p^n).

A.1.7. Proposition. *For each Barsotti-Tate group $H' = (H'_n)$ over R there is an integer $a = a(R,H') \geq 0$ such that, for each $n \geq 0$, the minimal prolongation $(H'_{n+a})_{\min} \longrightarrow (H'_n)_{\min}$ of $(j_a)_{\mathcal{K}} : (H'_{n+a})_{\mathcal{K}} \longrightarrow (H'_n)_{\mathcal{K}}$ factors through a morphism*

$$(H'_{n+a})_{\min} \xrightarrow{\phi} H'_n \longrightarrow (H'_n)_{\min}.$$

Proof. We follow the proof of [Ra 2, Prop. 2.3.1] for $\mathcal{G}(n) = (H'_n)_{\min}$. The construction in [*loc. cit.* (b) - (d)] gives an integer $i_0 \geq 0$ and prolongations $\mathcal{G}(n)_{i_0} \subset \mathcal{G}(n)$ of $(H'_{i_0})_{\mathcal{K}}$ ($\forall n \geq i_0$) such that the quotients $\mathcal{H}(n) := \mathcal{G}(n+i_0)/\mathcal{G}(n+i_0)_{i_0}$ ($n \geq 0$) have the following property: for fixed n, the inductive system of the scheme-theoretical closure of $\mathcal{H}(n)_{\mathcal{K}} = \mathcal{G}(n)_{\mathcal{K}}$ in $\mathcal{H}(n+j)$ (indexed by j) becomes stationary; its stationary values $\mathcal{H}_n$ form a Barsotti-Tate group $(\mathcal{H}_n)$ over R extending $((H'_n)_{\mathcal{K}})$. The functor $(H_n) \mapsto (H_n)_{\mathcal{K}}$ from Barsotti-Tate groups over R to Barsotti-Tate groups over $\mathcal{K}$ being fully faithful [Ta, p. 180], there exists a compatible system of isomorphisms in Gr_R

$$u_n : \mathcal{H}_n \xrightarrow{\sim} H'_n.$$

Set $a := i_0$. The composite morphism

$$\phi : (H'_{n+a})_{\min} = \mathcal{G}(n+a) \longrightarrow \mathcal{H}(n) \longrightarrow \mathcal{H}_n \xrightarrow{u_n} H'_n$$

has the required property, as it is a prolongation of the morphism $(j_a)_{\mathcal{K}} : (H'_{n+a})_{\mathcal{K}} \longrightarrow (H'_n)_{\mathcal{K}}$.

A.1.8. Corollary. *For each $H \in \mathrm{Gr}_R$ and each $f \in \mathrm{Hom}_{G_{\mathcal{K}}}(H(\overline{\mathcal{K}}), H'_{n+a}(\overline{\mathcal{K}}))$ $(n \geq 0)$ the composite morphism*

$$H(\overline{\mathcal{K}}) \xrightarrow{f} H'_{n+a}(\overline{\mathcal{K}}) \xrightarrow{j_a} H'_n(\overline{\mathcal{K}})$$

is of the form $\alpha_{H,H'_n}(\widetilde{f})$ for some $\widetilde{f} \in \mathrm{Hom}_{\mathrm{Gr}_R}(H, H'_n)$.

A.1.9. Corollary. *For each $H \in \mathrm{Gr}_R$ and each $m \geq 0$ $p^a \,\mathrm{Coker}(\alpha_{H,H'_m}) = 0$.*

A.2 Flat cohomology

A.2.1. For any $H \in \mathrm{Gr}_R$ we denote by $H^i_{fl}(R, H)$ the cohomology of the sheaf on the flat site of $\mathrm{Spec}(R)$ represented by H [Mi 1, II.1.7]. We can consider either the small or the big site, equipped with any of the following topologies: fpqf, fppf, syntomic; the cohomology groups $H^i_{fl}(R, H)$ remain the same [Mi 1, III.3.4].

A.2.2. The functor (A.1.2.1) induces maps into Galois cohomology

$$H^i_{fl}(R, H) \longrightarrow H^i(K, H_{\mathcal{K}}) := H^i(G_{\mathcal{K}}, H_K(\overline{\mathcal{K}})).$$

A.2.3. Local flat duality [Ma], [Mi 2, III.1]. *Let $H \in \mathrm{Gr}_R$, let $H^D \in \mathrm{Gr}_R$ be its Cartier dual. Assume that $\mathcal{K}$ is a finite extension of $\mathbf{Q}_p$.*
(1) *If H is étale over R (e.g., if the order of H is prime to p), then $H_K(\overline{\mathcal{K}})$ is an unramified $G_{\mathcal{K}}$-module and $H^i_{fl}(R, H) = H^i_{et}(R, H) = H^i(G_k, H_K(\overline{\mathcal{K}})) = H^1_{ur}(\mathcal{K}, H_K(\overline{\mathcal{K}}))$.*
(2) *The map $H^0_{fl}(R, H) \xrightarrow{\sim} H^0(K, H_{\mathcal{K}})$ is an isomorphism.*
(3) $\forall i > 1 \quad H^i_{fl}(R, H) = 0$.
(4) *The map $H^1_{fl}(R, H) \hookrightarrow H^1(K, H_{\mathcal{K}})$ is injective.*
(5) *Under the (non-degenerate) Tate pairing*

$$H^1(K, H_{\mathcal{K}}) \times H^1(K, H^D_{\mathcal{K}}) \xrightarrow{\cup} H^2(\mathcal{K}, \mathbf{G}_{m,\mathcal{K}}) \xrightarrow{\mathrm{inv}_{\mathcal{K}}} \mathbf{Q}/\mathbf{Z},$$

the orthogonal complement of $H^1_{fl}(R, H)$ is equal to $H^1_{fl}(R, H^D)$.
(6) *For each $i \geq 0$ $H^i_{fl}(R, H)$ is a finite abelian group killed by the order of H (= the cardinality of $H(\overline{\mathcal{K}})$).*

A.2.4. Restriction, corestriction, conjugation. Assume that $\mathcal{K}$ is a finite extension of $\mathbf{Q}_p$ and $\mathcal{K}'/\mathcal{K}$ is a finite extension; let R' be the ring of integers in $\mathcal{K}'$. For each $H \in \mathrm{Gr}_R$ the standard functoriality of the flat site gives rise to a "restriction map" res_{fl} which sits, thanks to A.2.3(5), in a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_{fl}(R', H_{R'}) & \longrightarrow & H^1(K', H_{\mathcal{K}'}) & \longrightarrow & \mathrm{Hom}(H^1_{fl}(R', H^D_{R'}), \mathbf{Q}/\mathbf{Z}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle res_{fl}} & & \uparrow{\scriptstyle res} & & \uparrow & & \\
0 & \longrightarrow & H^1_{fl}(R, H) & \longrightarrow & H^1(K, H_{\mathcal{K}}) & \longrightarrow & \mathrm{Hom}(H^1_{fl}(R, H^D), \mathbf{Q}/\mathbf{Z}) & \longrightarrow & 0.
\end{array} \qquad (A.2.4.1)$$

Applying A.2.3(5) to this diagram, we obtain a "corestriction map" cor_{fl} sitting in a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_{fl}(R', H^D_{R'}) & \longrightarrow & H^1(K', H^D_{\mathcal{K}'}) & \longrightarrow & \mathrm{Hom}(H^1_{fl}(R', H_{R'}), \mathbf{Q}/\mathbf{Z}) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle cor_{fl}} & & \Big\downarrow{\scriptstyle cor} & & \Big\downarrow & & \\ 0 & \longrightarrow & H^1_{fl}(R, H^D) & \longrightarrow & H^1(K, H^D_{\mathcal{K}}) & \longrightarrow & \mathrm{Hom}(H^1_{fl}(R, H), \mathbf{Q}/\mathbf{Z}) & \longrightarrow & 0. \end{array} \tag{A.2.4.2}$$

Similarly, any field automorphism $\sigma : \mathcal{K}' \longrightarrow \mathcal{K}'$ fixing $\mathcal{K}$ gives rise, by functoriality, to a commutative diagram

$$\begin{array}{ccc} H^1_{fl}(R', H_{R'}) & \hookrightarrow & H^1(K', H_{\mathcal{K}'}) \\ \Big\downarrow{\scriptstyle \sigma_{fl}} & & \Big\downarrow{\scriptstyle \sigma} \\ H^1_{fl}(R', H_{R'}) & \hookrightarrow & H^1(K', H_{\mathcal{K}'}). \end{array} \tag{A.2.4.3}$$

Moreover, $\sigma_{fl} \circ res_{fl} = res_{fl}$, since $\sigma \circ res = res$.

A.2.5. Proposition. *The maps from A.2.4 have the following properties.*
(1) $cor_{fl} \circ res_{fl} = [\mathcal{K}' : \mathcal{K}] \cdot \mathrm{id}$.
(2) *If $[\mathcal{K}' : \mathcal{K}]$ is prime to p, then res_{fl} is injective, cor_{fl} is surjective and the left square in (A.2.4.1) is cartesian.*
(3) *More generally, $res^{-1}(H^1_{fl}(R', H_{R'}))/H^1_{fl}(R, H)$ is killed by the greatest common divisor d of $[\mathcal{K}' : \mathcal{K}]$ and the order of H.*
(4) *If $\mathcal{K}'/\mathcal{K}$ is a Galois extension with Galois group Δ, then $res_{fl} \circ cor_{fl} = \sum_{\sigma \in \Delta} \sigma_{fl}$. The kernel and cokernel of the map*

$$res_{fl} : H^1_{fl}(R, H) \longrightarrow H^1_{fl}(R', H_{R'})^{\Delta}$$

is killed by the integer d from (3).

Proof. This follows from the corresponding statements for Galois cohomology and from A.2.3(4),(6).

A.2.6. Proposition. *Assume that $\mathcal{K}$ is a finite extension of $\mathbf{Q}_p$. Let $H = (H_n)$ be a Barsotti-Tate group over R, let $H^t = (H_n^D)$ be the dual Barsotti-Tate group. Denote by $T_p(H) := \varprojlim_n H_n(\overline{\mathcal{K}})$ the Tate module of H and set $V_p(H) = T_p(H) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.*
(1) *$V_p(H)$ is a crystalline representation of $G_{\mathcal{K}}$.*
(2) *For each $m, n \geq 1$ the map $j_m : H_{m+n} \longrightarrow H_n$ induces an isomorphism*

$$H^1_{fl}(R, H_{m+n}) \otimes \mathbf{Z}/p^n\mathbf{Z} \xrightarrow{\sim} H^1_{fl}(R, H_n).$$

(3) *The $\mathbf{Z}_p$-module $X(H) := \varprojlim_n H^1_{fl}(R, H_n)$ is of finite type and, for each $n \geq 1$, there is a canonical isomorphism $X(H)/p^n H(H) \xrightarrow{\sim} H^1_{fl}(R, H_n)$.*
(4) *The orthogonal complement of $X(H)$ under the (non-degenerate) pairing*

$$H^1(\mathcal{K}, T_p(H)) \times H^1(\mathcal{K}, T_p(H^t)) \xrightarrow{\cup} H^2(\mathcal{K}, T_p(\mathbf{G}_m)) = H^2(\mathcal{K}, \mathbf{Z}_p(1)) \xrightarrow{\sim} \mathbf{Z}_p$$

is equal to $X(H^t)$.
(5) *The subgroup $X(H) \hookrightarrow \varprojlim_n H^1(\mathcal{K}, (H_n)_{\mathcal{K}}) = H^1(\mathcal{K}, T_p(H))$ is equal to the Bloch-Kato subspace*

$$H^1_f(\mathcal{K}, T_p(H)) = \mathrm{Ker}\left(H^1(\mathcal{K}, T_p(H)) \longrightarrow H^1(\mathcal{K}, V_p(H) \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}})\right).$$

Proof. (1) This is a theorem of Fontaine [Fo, Thm. 6.2]. The statements (2) and (3) follow from the exact cohomology sequences attached to

$$0 \longrightarrow H_m \xrightarrow{i_n} H_{m+n} \xrightarrow{j_m} H_n \longrightarrow 0$$
$$0 \longrightarrow H_n \xrightarrow{i_m} H_{m+n} \xrightarrow{j_n} H_m \longrightarrow 0$$

and the vanishing of $H^2_{fl}(R, H_j)$, while (4) is a consequence of A.2.3(5).
(5) We first prove the inclusion $X(H) \subseteq H^1_f(\mathcal{K}, T_p(H))$. Let $x = (x_n) \in X(H) \subset H^1(\mathcal{K}, T_p(H))$. Each element $x_n \in H^1_{fl}(R, H_n)$ is represented by $\widetilde{H}_n \in \mathrm{Gr}_R$ sitting in an exact sequence (of flat sheaves)

$$0 \longrightarrow H_n \longrightarrow \widetilde{H}_n \longrightarrow \mathbf{Z}/p^n\mathbf{Z} \longrightarrow 0.$$

The generic fibres $(\widetilde{H}_n)_{\mathcal{K}}$ form a Barsotti-Tate group over $\mathcal{K}$; it extends, thanks to [Ra 2, Prop. 2.3.1], to a Barsotti-Tate group $H' = (H'_n)$ over R. By construction, x is the extension class of the exact sequence of $G_{\mathcal{K}}$-modules

$$0 \longrightarrow T_p(H) \longrightarrow T_p(H') \longrightarrow \mathbf{Z}_p \longrightarrow 0.$$

As $V_p(H')$ is a crystalline representation of $G_{\mathcal{K}}$ (by (1)), $x \in H^1_f(\mathcal{K}, T_p(H))$, proving $X(H) \subseteq H^1_f(\mathcal{K}, T_p(H))$.

Combining the inclusion $X(H^t) \subseteq H^1_f(\mathcal{K}, T_p(H^t))$ with (4) and the equality [BK, Prop. 3.8]

$$H^1_f(\mathcal{K}, T_p(H)) = H^1_f(\mathcal{K}, T_p(H^t))^\perp$$

we obtain the converse inclusion $X(H) \supseteq H^1_f(\mathcal{K}, T_p(H))$.

A.2.7. Proposition. *Let $\mathscr{B}$ be a semi-abelian variety over R siting in an exact sequence*

$$0 \longrightarrow \mathscr{T} \longrightarrow \mathscr{B} \longrightarrow \mathscr{A} \longrightarrow 0,$$

where $\mathscr{T}$ (resp. $\mathscr{A}$) is a torus (resp. an abelian scheme) over R.
(1) *For each integer $m \geq 1$ there is an exact sequence of flat sheaves on* $\mathrm{Spec}(R)$

$$0 \longrightarrow \mathscr{B}[m] \longrightarrow \mathscr{B} \xrightarrow{m} \mathscr{B} \longrightarrow 0.$$

(2) *If $\mathcal{K}$ is a finite extension of $\mathbf{Q}_p$, then $H^1_{fl}(R, \mathscr{B}) = 0$ and the coboundary map in the exact cohomology sequence of (1) induces an isomorphism*

$$\partial_R : \mathscr{B}(R) \otimes \mathbf{Z}/m\mathbf{Z} \xrightarrow{\sim} H^1_{fl}(R, \mathscr{B}[m]).$$

Proof. (1) [NM, Lemma 7.3.2(a)]. The vanishing statement in (2) is a consequence of the isomorphisms

$$H^1_{fl}(R, \mathscr{B}) \xrightarrow{\sim} H^1_{et}(R, \mathscr{B}) \xrightarrow{\sim} H^1(G_k, \mathscr{B}(\overline{k})) \xrightarrow{\sim} 0$$

(the first one is a theorem of Grothendieck [G1, Thm. 11.7], the second one an elementary property of étale cohomology of $\mathrm{Spec}(R)$ [Mi 2, Prop. II.1.1(b)] and the third one a theorem of Lang [La, Prop. 3]). The exact cohomology sequence

$$0 \longrightarrow \mathscr{B}(R)[m] \longrightarrow \mathscr{B}(R) \xrightarrow{m} \mathscr{B}(R) \longrightarrow H^1_{fl}(R, \mathscr{B}[m]) \longrightarrow H^1_{fl}(R, \mathscr{B}) = 0$$

implies that ∂_R is an isomorphism.

A.2.8. Proposition. *In the situation of A.1.7, for each $H \in \mathrm{Gr}_R$, $f \in \mathrm{Hom}_{G_{\mathcal{K}}}(H(\overline{\mathcal{K}}), H'_{n+a}(\overline{\mathcal{K}}))$ $(n \geq 0)$ and a finite extension $\mathcal{K}'/\mathcal{K}$ with ring of integers R',*

$$\mathrm{Im}\left(H^1_{fl}(R', H_{R'}) \longrightarrow H^1(\mathcal{K}', H_{\mathcal{K}'}) \xrightarrow{f_*} H^1(\mathcal{K}', (H'_{n+a})_{\mathcal{K}'}) \xrightarrow{j_{a*}} H^1(\mathcal{K}', (H'_n)_{\mathcal{K}'})\right) \subseteq H^1_{fl}(R', (H'_n)_{R'}).$$

Proof. This follows from Corollary A.1.8 and the fact that the maps A.2.2 are functorial in H.

A.2.9. Proposition. *There exists an integer $a \geq 0$ depending on $\mathcal{K}$ with the following property. For each $m \geq 0$ and for each $H \in \mathrm{Gr}_R$ such that there is an exact sequence of $G_{\mathcal{K}}$-modules*

$$\mathcal{E} : 0 \longrightarrow \mu_{p^m}(\overline{\mathcal{K}}) \longrightarrow H(\overline{\mathcal{K}}) \longrightarrow \mathbf{Z}/p^m\mathbf{Z} \longrightarrow 0,$$

the image of the extension class $[\mathcal{E}] \in H^1(\mathcal{K}, \mu_{p^m}) = \mathcal{K}^\times \otimes \mathbf{Z}/p^m\mathbf{Z}$ in $\mathcal{K}^\times \otimes \mathbf{Z}/p^{m-a}\mathbf{Z}$ is contained in $R^\times \otimes \mathbf{Z}/p^{m-a}\mathbf{Z}$.

Proof. There is an exact sequence in Gr_R

$$0 \longrightarrow H_1 \longrightarrow H \longrightarrow H_2 \longrightarrow 0,$$

where H_1 is the scheme-theoretical closure of $(\mu_{p^m})_{\mathcal{K}}$ in H. Applying Proposition A.1.7 to the Barsotti-Tate group μ_{p^∞} over R we obtain an integer $a \geq 0$ such that the surjection

$$H_1(\overline{\mathcal{K}}) \xrightarrow{\sim} \mu_{p^m}(\overline{\mathcal{K}}) \xrightarrow{j_a} \mu_{p^{m-a}}(\overline{\mathcal{K}})$$

extends to a morphism $H_1 \longrightarrow \mu_{p^{m-a}}$ in Gr_R. Applying the same argument to H_2^D and dualising, we see that the inclusion

$$\mathbf{Z}/p^{m-a}\mathbf{Z} \xrightarrow{i_a} \mathbf{Z}/p^m\mathbf{Z} \xrightarrow{\sim} H_2(\overline{\mathcal{K}})$$

extends to a morphism $\mathbf{Z}/p^{m-a}\mathbf{Z} \longrightarrow H_2$ in Gr_R. The fibre product $H' := H \times_{H_2} \mathbf{Z}/p^{m-a}\mathbf{Z} \in Gr_R$ sits in an exact sequence in Gr_R

$$0 \longrightarrow H_1 \longrightarrow H' \longrightarrow \mathbf{Z}/p^{m-a}\mathbf{Z} \longrightarrow 0.$$

Applying the same argument to H'^D and dualising we obtain an exact sequence in in Gr_R

$$0 \longrightarrow \mu_{p^{m-a}} \longrightarrow H' \longrightarrow \mathbf{Z}/p^{m-a}\mathbf{Z} \longrightarrow 0,$$

whose $\overline{\mathcal{K}}$-valued points coincide with $\mathcal{E} \otimes \mathbf{Z}/p^{m-a}\mathbf{Z}$. It follows that the image of $[\mathcal{E}]$ in $H^1(\mathcal{K}, \mu_{p^{m-a}}) = \mathcal{K}^\times \otimes \mathbf{Z}/p^{m-a}\mathbf{Z}$ is contained in the image of $H^1_{fl}(R, \mu_{p^{m-a}}) = R^\times \otimes \mathbf{Z}/p^{m-a}\mathbf{Z} \longrightarrow H^1(\mathcal{K}, \mu_{p^{m-a}})$, as claimed.

A.3 Raynaud extensions

A.3.1. Let J be an abelian variety over $\mathcal{K}$ with semi-abelian reduction, let $\mathscr{J}$ be its Néron model over R. By definition, the connected component of the identity $\mathscr{J}_s^\circ$ of the special fibre $\mathscr{J}_s$ of $\mathscr{J}$ is a semi-abelian variety over k.

A.3.2. The *Raynaud extension* [SGA 7/I, §7] attached to J is a smooth commutative group scheme $\mathscr{G}$ over R with the following properties [FC, ch. II,III].

A.3.2.1. The connected component $\mathscr{G}^\circ$ is a semi-abelian scheme over R: as in A.2.7 there is an exact sequence

$$0 \longrightarrow \mathscr{T} \longrightarrow \mathscr{G}^\circ \longrightarrow \mathscr{A} \longrightarrow 0.$$

A.3.2.2. The quotient $\mathscr{G}/\mathscr{G}^\circ$ is a finite étale group scheme over R whose special fibre is isomorphic to $\mathscr{J}/\mathscr{J}_s^\circ$.

A.3.2.3. There exists a monomorphism $Z \longrightarrow \mathscr{G}^\circ_{\mathcal{K}}$ of étale group schemes over $\mathcal{K}$, where $Z_{\mathcal{K}^{ur}} \xrightarrow{\sim} \mathbf{Z}^{\mathrm{rk}(\mathscr{T})}$ and $\mathscr{G}^\circ(O_{\overline{\mathcal{K}}}) \cap Z(\overline{\mathcal{K}}) = 0$, and an isomorphism of $G_{\mathcal{K}}$-modules

$$\mathscr{G}^\circ_{\mathcal{K}}(\overline{\mathcal{K}})/Z(\overline{\mathcal{K}}) \xrightarrow{\sim} J(\overline{\mathcal{K}}).$$

A.3.2.4. For any finite extension $\mathcal{K}'/\mathcal{K}$ with ring of integers R', the following properties are equivalent:

$Z_{\mathcal{K}'}$ is a constant group scheme $\iff$ $\mathscr{T}_{R'}$ is a split torus $\iff$ $J_{\mathcal{K}'}$ has split semiabelian reduction.

A.3.3. The exact Galois cohomology sequence of

$$0 \longrightarrow Z(\overline{\mathcal{K}}) \longrightarrow \mathscr{G}^\circ_{\mathcal{K}}(\overline{\mathcal{K}}) \longrightarrow J(\overline{\mathcal{K}}) \longrightarrow 0$$

over a finite extension $\mathcal{K}'$ of $\mathcal{K}$ reads as

$$0 \longrightarrow Z(\mathcal{K}') \longrightarrow \mathscr{G}^\circ_{\mathcal{K}}(\mathcal{K}') \longrightarrow J(\mathcal{K}') \longrightarrow H^1(G_{\mathcal{K}'}, Z(\overline{\mathcal{K}})). \tag{A.3.3.1}$$

If $Z_{\mathcal{K}'}$ is a constant group scheme, then $G_{\mathcal{K}'}$ acts trivially on $Z(\overline{\mathcal{K}})$ and (A.3.3.1) reduces to

$$\mathscr{G}^\circ_{\mathcal{K}}(\mathcal{K}')/Z(\overline{\mathcal{K}}) \xrightarrow{\sim} J(\mathcal{K}'). \tag{A.3.3.2}$$

A.3.4. For each integer $m \geq 1$ there is an exact sequence of flat sheaves over $\mathrm{Spec}(R)$

$$0 \longrightarrow \mathscr{T}[m] \longrightarrow \mathscr{G}^\circ[m] \longrightarrow \mathscr{A}[m] \longrightarrow 0$$

and an exact sequence of $G_{\mathcal{K}}$-modules

$$0 \longrightarrow \mathscr{G}^\circ[m](\overline{\mathcal{K}}) \longrightarrow J(\overline{\mathcal{K}})[m] \longrightarrow Z(\overline{\mathcal{K}}) \otimes \mathbf{Z}/m\mathbf{Z} \longrightarrow 0.$$

For any finite extension $\mathcal{K}'$ of $\mathcal{K}$ denote by

$$\partial_{\mathcal{K}'} : J(\mathcal{K}') \otimes \mathbf{Z}/m\mathbf{Z} \hookrightarrow H^1(\mathcal{K}', J[m]) := H^1(G_{\mathcal{K}'}, J(\overline{\mathcal{K}})[m])$$

(as in 1.7.3) the injective map induced by the cohomology sequence of

$$0 \longrightarrow J[m] \longrightarrow J(\overline{\mathcal{K}}) \xrightarrow{m} J(\overline{\mathcal{K}}) \longrightarrow 0.$$

Note that there is a commutative diagram (where R' is the ring of integers in $\mathcal{K}'$)

$$\begin{array}{ccccccc}
\mathscr{G}^\circ(R') \otimes \mathbf{Z}/m\mathbf{Z} & \longrightarrow & \mathscr{G}^\circ_{\mathcal{K}}(\mathcal{K}') \otimes \mathbf{Z}/m\mathbf{Z} & \longrightarrow & J(\mathcal{K}') \otimes \mathbf{Z}/m\mathbf{Z} & \longleftarrow & \mathscr{T}_{\mathcal{K}}(\mathcal{K}') \otimes \mathbf{Z}/m\mathbf{Z} \\
\downarrow{\scriptstyle \partial_{R'}} & & \downarrow & & \downarrow{\scriptstyle \partial_{\mathcal{K}'}} & & \downarrow \\
H^1_{fl}(R', \mathscr{G}^\circ[m]) & \longrightarrow & H^1(\mathcal{K}', \mathscr{G}^\circ[m]) & \longrightarrow & H^1(\mathcal{K}', J[m]) & \longleftarrow & H^1(\mathcal{K}', \mathscr{T}[m]).
\end{array} \tag{A.3.4.1}$$

If $Z_{\mathcal{K}'}$ is a constant group scheme, then the second map in the top row is surjective.

A.3.5. Proposition. *Assume that $\mathcal{K}$ is a finite extension of $\mathbf{Q}_p$. Let $\mathcal{K}'$ be a finite extension of $\mathcal{K}$, let R' be its ring of integers.*
(1) $\partial_{\mathcal{K}'}\left(\mathrm{Im}\left(\mathscr{G}^\circ_{\mathcal{K}}(\mathcal{K}') \otimes \mathbf{Z}/m\mathbf{Z} \longrightarrow J(\mathcal{K}') \otimes \mathbf{Z}/m\mathbf{Z}\right)\right) \subseteq \mathrm{Im}\left(H^1(\mathcal{K}', \mathscr{G}^\circ[m]) \longrightarrow H^1(\mathcal{K}', J[m])\right)$.
(2) $\partial_{\mathcal{K}'}\left(\mathrm{Im}\left(\mathscr{G}^\circ(R') \otimes \mathbf{Z}/m\mathbf{Z} \longrightarrow J(\mathcal{K}') \otimes \mathbf{Z}/m\mathbf{Z}\right)\right) = \mathrm{Im}\left(H^1_{fl}(R', \mathscr{G}^\circ[m]) \longrightarrow H^1(\mathcal{K}', J[m])\right)$.
(3) *The L.H.S. of (1) is contained in*

$$\partial_{\mathcal{K}'}\left(\mathrm{Im}\left(\mathscr{G}^\circ(R') \otimes \mathbf{Z}/m\mathbf{Z} \longrightarrow J(\mathcal{K}') \otimes \mathbf{Z}/m\mathbf{Z}\right)\right) + \mathrm{Im}\left(H^1(\mathcal{K}', \mathscr{T}[m]) \longrightarrow H^1(\mathcal{K}', J[m])\right).$$

Proof. The statements (1) and (2) follow from the commutative diagram (A.3.4.1) and A.2.7(2). The inclusion (3) is a consequence of the fact that $\mathscr{G}^\circ_{\mathcal{K}}(\mathcal{K}') = \mathscr{T}_{\mathcal{K}}(\mathcal{K}') + \mathscr{G}^\circ(R')$.

A.3.6. Corollary. *If $J_{\mathcal{K}'}$ has split semi-abelian reduction, then:*
(1) $\mathrm{Im}(\partial_{\mathcal{K}'}) \subseteq \mathrm{Im}\left(H^1(\mathcal{K}', \mathscr{G}^\circ[m]) \longrightarrow H^1(\mathcal{K}', J[m])\right)$.
(2) $\mathrm{Im}(\partial_{\mathcal{K}'}) \subseteq \mathrm{Im}\left(H^1_{fl}(R', \mathscr{G}^\circ[m]) \longrightarrow H^1(\mathcal{K}', J[m])\right) + \mathrm{Im}\left(H^1(\mathcal{K}', \mathscr{T}[m]) \longrightarrow H^1(\mathcal{K}', J[m])\right)$.

Appendix B: Galois images for cohomological Hilbert modular forms

In this appendix we collect basic statements about the images of Galois representations attached to Hilbert modular forms of regular weight. For elliptic modular forms, these results were proved by Ribet [Ri 1-4, 6] and Momose [Mo]. The case of Hilbert modular forms seems to be well-known, but we have not been able to find a good reference.

B.1 Twisted endomorphisms and the Brauer group

B.1.1. Assume that we are given the following data: a field L, a group G, an $L[G]$-module V (of finite dimension over L) satisfying $\mathrm{End}_{L[G]}(V) = L$, and a finite group $\Gamma \subset \mathrm{Aut}(L)$ of field automorphisms of L.

We assume, in addition, that the isomorphism class of V is Γ-invariant in the sense that, for each $\sigma \in \Gamma$, there is an isomorphism of $L[G]$-modules

$$\alpha_\sigma : {}^\sigma V \xrightarrow{\sim} V. \qquad (B.1.1.1)$$

B.1.2. In concrete terms, after choosing an L-basis of V the action of G on V will be given by a group homomorphism

$$\rho : G \longrightarrow \mathrm{Aut}_L(V) \xrightarrow{\sim} GL_n(L) \qquad (n = \dim_L(V))$$

and the isomorphisms (B.1.1.1) by matrices $\alpha_\sigma \in GL_n(L)$ satisfying

$$\forall g \in G \qquad \alpha_\sigma \cdot {}^\sigma\rho(g) = \rho(g) \cdot \alpha_\sigma \qquad (\iff \mathrm{Ad}(\alpha_\sigma)({}^\sigma\rho(g)) = \rho(g)). \qquad (B.1.2.1)$$

For each couple $\sigma, \tau \in \Gamma$ the composite map

$$\alpha_\sigma \circ {}^\sigma\alpha_\tau \circ \alpha_{\sigma\tau}^{-1} \in \mathrm{Aut}_{L[G]}(V)$$

is given by multiplication by a scalar $\beta_{\sigma,\tau} \in L^\times$ and the function $\beta = (\beta_{\sigma,\tau}) \in Z^2(\Gamma, L^\times)$ is a 2-cocycle with cohomology class

$$[\beta] = \delta([P(\alpha)]) \in H^2(\Gamma, L^\times) = H^2(L/L^\Gamma, L^\times) \subset \mathrm{Br}(L^\Gamma),$$

where

$$\delta : H^1(\Gamma, PGL_n(L)) \longrightarrow H^2(\Gamma, L^\times)$$

is the coboundary arising from the exact sequence

$$1 \longrightarrow L^\times \longrightarrow GL_n(L) \xrightarrow{P} PGL_n(L) \longrightarrow 1$$

and $P(\alpha) = (P(\alpha_\sigma)) \in Z^1(\Gamma, PGL_n(L))$. If we choose another basis of V, then $P(\alpha)$ will be replaced by a cohomologous cocycle. Moreover, $n[\beta] = 0 \in H^2(\Gamma, L^\times)$.

B.1.3. Twisted action, twisted endomorphisms. The formula

$${}^{tw(\sigma)}f := \mathrm{Ad}(\alpha_\sigma)({}^\sigma f) = \alpha_\sigma \circ {}^\sigma f \circ \alpha_\sigma^{-1} \qquad (f \in \mathrm{End}_L(V) \xrightarrow{\sim} M_n(L))$$

defines a twisted action of Γ on $W := \mathrm{End}_L(V)$ (by morphisms of L^Γ-algebras). Let us call the L^Γ-subalgebra of endomorphisms invariant by the twisted action of Γ

$$\mathrm{End}_L(V)^{tw(\Gamma)} \subset \mathrm{End}_L(V)$$

the algebra of "twisted endomorphisms". According to (B.1.2.1), the morphism of L-algebras

$$L[G] \longrightarrow \mathrm{End}_L(V) \qquad (B.1.3.1)$$

given by the G-action is obtained from a morphism of L^Γ-algebras

$$L^\Gamma[G] \longrightarrow \mathrm{End}_L(V)^{tw(\Gamma)} \tag{B.1.3.2}$$

by extension of scalars via the multiplication map

$$m : L \otimes_{L^\Gamma} \mathrm{End}_L(V)^{tw(\Gamma)} \longrightarrow \mathrm{End}_L(V), \qquad m(c \otimes f) = cf. \tag{B.1.3.3}$$

B.1.4. Proposition. *The map (B.1.3.3) is an isomorphism of L-algebras. The ring $\mathrm{End}_L(V)^{tw(\Gamma)}$ is a central simple algebra over L^Γ whose class in the Brauer group of L^Γ is equal to*

$$[\mathrm{End}_L(V)^{tw(\Gamma)}] = [\beta] \in H^2(\Gamma, L^\times) \subset \mathrm{Br}(L^\Gamma)$$

(in particular, this class is killed by n). If, in addition, V is a simple $L[G]$-module, then the maps (B.1.3.1) and (B.1.3.2) are surjective.

Proof. As $H^1(\Gamma, \mathrm{Aut}_L(W)) = 1$, the 1-cocycle $(\mathrm{Ad}(\alpha_\sigma)) \in Z^1(\Gamma, \mathrm{Aut}_L(W))$ is a coboundary: there exists an L-linear automorphism φ of $W = \mathrm{End}_L(V)$, necessarily of the form $\varphi(f) = a \circ f \circ b^{-1}$ for some $a, b \in \mathrm{Aut}_L(V)$, such that

$$\forall \sigma \in \Gamma \ \ \forall f \in W \qquad \mathrm{Ad}(\alpha_\sigma) f = \varphi^{-1} \circ {}^\sigma\varphi(f).$$

As a result,

$$\forall \sigma \in \Gamma \ \ \forall f \in W \qquad \varphi({}^{tw(\sigma)}f) = {}^\sigma(\varphi(f)),$$

hence

$$\mathrm{End}_L(V)^{tw(\Gamma)} = \varphi^{-1}(M_n(L^\Gamma)) = \{a^{-1} \circ f \circ b \mid M_n(L^\Gamma)\}$$

(note that φ is a morphism of L^Γ-algebras iff $a = b$ iff $[P(\alpha)] = 0 \in H^1(\Gamma, PGL_n(L))$ iff $[\beta] = 0 \in H^2(\Gamma, L^\times)$).

The map (B.1.3.3) is a surjective morphism of L-algebras, since its image contains $a^{-1} \circ L \otimes_{L^\Gamma} M_n(L^\Gamma) \circ b = M_n(L)$. A dimension count implies that m is bijective. In other words, $\mathrm{End}_L(V)^{tw(\Gamma)}$ is an L/L^Γ-form of the matrix algebra $M_n(L^\Gamma)$, hence it is a central simple algebra over L^Γ. Its Brauer class in $H^2(\Gamma, L^\times)$ is represented by the 2-cocycle $b = (b_{\sigma,\tau})$ defined as follows: for each $\sigma \in \Gamma$ there exists $a_\sigma \in \mathrm{Aut}(V) \xrightarrow{\sim} GL_n(L)$ such that $m \circ (\sigma \otimes \mathrm{id}) = \mathrm{Ad}(a_\sigma) \circ (\sigma \circ m)$; then $b_{\sigma,\tau} = a_\sigma {}^\sigma a_\tau a_{\sigma\tau}^{-1} \in L^\times$. By definition of the twisted action, we can take $a_\sigma = \alpha_\sigma$, hence $b = \beta$.

If V is a simple $L[G]$-module, then (B.1.3.1) is surjective by Burnside's theorem [CR, Thm. 3.32]. As (B.1.3.1) is obtained from (B.1.3.2) by extension of scalars and m is an isomorphism, the map (B.1.3.2) must also be surjective.

B.1.5. Proposition. *The centraliser A of the L^Γ-subalgebra $\mathrm{End}_L(V)^{tw(\Gamma)}$ of $\mathrm{End}_{L^\Gamma}(V) \xrightarrow{\sim} M_{n|\Gamma|}(L^\Gamma)$ is a central simple algebra over L^Γ satisfying $M_n(A)^{\mathrm{op}} \xrightarrow{\sim} M_{|\Gamma|}(\mathrm{End}_L(V)^{tw(\Gamma)})$. In particular, $\dim_{L^\Gamma}(A) = |\Gamma|^2$ and $[A] = -[\mathrm{End}_L(V)^{tw(\Gamma)}] \in \mathrm{Br}(L^\Gamma)$.*

Proof. According to [Sc, Thm. 4.5], A is a central simple algebra over L^Γ. The remaining statements follow from the isomorphism of L^Γ-algebras [Sc, Thm. 4.5]

$$A \otimes_{L^\Gamma} \mathrm{End}_L(V)^{tw(\Gamma)} \xrightarrow{\sim} \mathrm{End}_{L^\Gamma}(V), \qquad a \otimes b \mapsto ab.$$

B.1.6. Proposition. *The normaliser $\{g \in \mathrm{End}_L(V)^\times \mid g \circ \mathrm{End}_L(V)^{tw(\Gamma)} \circ g^{-1} = \mathrm{End}_L(V)^{tw(\Gamma)}\}$ of the subring $\mathrm{End}_L(V)^{tw(\Gamma)} \subset \mathrm{End}_L(V)$ is equal to $(\mathrm{End}_L(V)^{tw(\Gamma)})^\times L^\times = L^\times (\mathrm{End}_L(V)^{tw(\Gamma)})^\times$.*

Proof. One inclusion is obvious. To prove the opposite one, let g be an element of the normaliser. By definition,

$$\forall \sigma \in \Gamma \ \ \forall f \in \mathrm{End}_L(V)^{tw(\Gamma)} \qquad g \circ f \circ g^{-1} = {}^{tw(\sigma)}(g \circ f \circ g^{-1}) = {}^{tw(\sigma)}g \circ f \circ {}^{tw(\sigma)}g^{-1},$$

which implies that $g^{-1} \circ {}^{tw(\sigma)}g$ centralises $L \otimes_{L^\Gamma} \mathrm{End}_L(V)^{tw(\Gamma)} = \mathrm{End}_L(V)$, hence ${}^{tw(\sigma)}g = c_\sigma g$ for some $c_\sigma \in L^\times$. As ${}^{tw(\sigma)}a = {}^\sigma a$ for all $a \in L$, the function $\{c_\sigma\} \in Z^1(\Gamma, L^\times)$ is a 1-cocycle. It follows that $c_\sigma = {}^\sigma b b^{-1}$ for some $b \in L^\times$; thus $g = (gb^{-1})b$ with $gb^{-1} \in (\mathrm{End}_L(V)^{tw(\Gamma)})^\times$.

B.2 Automorphic representations and Galois representations

Let F be a totally real number field. As in the main body of this article, denote by S_∞ (resp. by S_p) the set of all infinite primes (resp. all primes above a rational prime p) of F.

B.2.1. Automorphic representations. Fix an irreducible cuspidal automorphic representation π of $GL_2(\mathbf{A}_F)$ whose infinity type is of the form $\pi_\infty = \otimes_{v|\infty}\pi_v$, where π_v is a discrete series representation of weight k_v and algebraic central character, with all $k_v \geq 2$ of the same parity.

Denote by $\omega : \mathbf{A}_F^\times/F^\times \longrightarrow \mathbf{C}^\times$ the central character of π. Unlike in [N2, ch. 12] or in 0.2 we do not insist on ω being unitary ($\iff$ of finite order). Our assumptions imply that $\omega = |\cdot|^m\varphi$, where $m \in \mathbf{Z}$ and $\varphi : \mathbf{A}_F^\times/F^\times \longrightarrow \mathbf{C}^\times$ is a character of finite order satisfying $\varphi_v(-1) = (-1)^{k_v} = (-1)^m$ for all $v \in S_\infty$. Denote by

$$S = \{v \mid \pi_v \neq \text{unramified principal series}\} \supset S_\infty$$

the (finite) ramification set of π.

B.2.2. Fields of moduli. As in [W1], set

$$\begin{aligned} \mathscr{S}(\pi_v) &= \{\sigma \in \mathrm{Aut}(\mathbf{C}) \mid {}^\sigma\pi_v \xrightarrow{\sim} \pi_v\} \qquad (v \notin S_\infty) \\ \mathscr{S}(\pi) &= \{\sigma \in \mathrm{Aut}(\mathbf{C}) \mid {}^\sigma\pi \xrightarrow{\sim} \pi\} \end{aligned}$$

and define the field of moduli of π_v and π, respectively, to be

$$\mathbf{Q}(\pi_v) := \mathbf{C}^{\mathscr{S}(\pi_v)}, \qquad \mathbf{Q}(\pi) := \mathbf{C}^{\mathscr{S}(\pi)}.$$

Note that, for each $\sigma \in \mathrm{Aut}(\mathbf{C})$, the representation ${}^\sigma\pi$ is also of the form considered in B.2.1, with infinity type $({}^\sigma\pi)_\infty = \pi_\infty$ and central character ${}^\sigma\omega = |\cdot|^m\,{}^\sigma\varphi$.

Let $c \in \mathrm{Aut}(\mathbf{C})$ be the complex conjugation. The existence of a hermitian scalar product on the space of cusp forms implies that, for each $\sigma \in \mathrm{Aut}(\mathbf{C})$,

$${}^c({}^\sigma\pi \otimes |\cdot|^{-m/2}) \xrightarrow{\sim} (\widetilde{{}^\sigma\pi \otimes |\cdot|^{-m/2}}) \xrightarrow{\sim} {}^\sigma\pi \otimes |\cdot|^{-m/2} \otimes \varphi^{-1},$$

hence

$${}^{c\sigma}\pi \xrightarrow{\sim} {}^\sigma\pi \otimes {}^\sigma\varphi^{-1} \xrightarrow{\sim} {}^{\sigma c}\pi. \tag{B.2.2.1}$$

B.2.3. Proposition [W1]. (1) *$\mathbf{Q}(\pi)$ is a number field which is either totally real or a CM field. If $\varphi = 1$, then $\mathbf{Q}(\pi)$ is totally real.*

(2) $\mathbf{Q}(\pi) \supset \mathbf{Q}(\omega) := \mathbf{Q}(\mathrm{Im}(\omega)) = \mathbf{Q}(\mathrm{Im}(\varphi))$.

(3) *$\mathbf{Q}(\pi) \supset \mathbf{Q}(\pi_v)$ for all $v \notin S_\infty$.*

(4) *$\mathbf{Q}(\pi)$ is the compositum of $\{\mathbf{Q}(\pi_v)\}_{v\notin\Sigma}$, for any finite set of primes $\Sigma \supset S$ of F.*

(5) *For each $v \notin S_\infty$, π_v has a model over $\mathbf{Q}(\pi_v)$.*

(6) *π has a model over $\mathbf{Q}(\pi)$.*

(7) *$\forall v \notin S \quad \mathbf{Q}(\pi_v) = \mathbf{Q}(\lambda_\pi(v), \omega(v))$, where $\lambda_\pi(v)$ is the eigenvalue of the Hecke operator $T(v)$ on the spherical line in π_v. Explicitly, $\pi_v = \pi(\mu_1, \mu_2)$ acts by right translations on the space*

$$\mathcal{B}(\mu_1,\mu_2) = \left\{f : GL_2(F_v) \longrightarrow \mathbf{C} \,\middle|\, f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \mu_1(a)\mu_2(d)|ad^{-1}|_v^{1/2} f(g)\right\},$$

where $\mu_i : F^\times \longrightarrow \mathbf{C}^\times$ are unramified characters, and $\lambda_\pi(v) = (\mu_1(v) + \mu_2(v))(Nv)^{1/2}$.

B.2.4. In the situation of B.2.3(7), $\mathbf{Q}(\pi_v)$ is equal to the field of definition of the local Hecke polynomial

$$P^H(\pi_v, X) = 1 - \lambda_\pi(v)X + \omega(v)(Nv)X^2,$$

which is related to the local L-factor

$$L(\pi_v, s) = [(1 - \mu_1(v)(Nv)^{-s})(1 - \mu_2(v)(Nv)^{-s})]^{-1}$$

by the following relation:

$$L(\pi_v, s - \tfrac{1}{2}) = P^H(\pi_v, (Nv)^{-s})^{-1}.$$

B.2.5. Galois representations. From now on (until the end of Appendix B) we are going to denote the number field $\mathbf{Q}(\pi)$ by L. It is known, thanks to the work of many people culminating in [T1] (see also [C1]), that for each finite prime $\mathfrak{p}$ of L above a rational prime p there exists a representation $V_\mathfrak{p}(\pi)$ of G_F which is two-dimensional over $L_\mathfrak{p}$, unramified outside $S \cup S_p$ and which satisfies

$$\forall v \notin S \cup S_p \qquad \det(1 - X\,\mathrm{Fr}_{\mathrm{geom}}(v) \mid V_\mathfrak{p}(\pi)) = P^H(\pi_v, X).$$

In other words, $\{V_\mathfrak{p}(\pi)\}_\mathfrak{p}$ is a strongly compatible system of L-rational representations of G_F whose L-function is equal to

$$L^{S_\infty}(\pi, s - \tfrac{1}{2}) = \prod_{v \nmid \infty} L(\pi_v, s - \tfrac{1}{2}).$$

It follows from B.2.4 that

$$\forall \sigma \in \mathrm{Aut}(\mathbf{C}) \qquad V_{\sigma(\mathfrak{p})}({}^\sigma\pi) \xrightarrow{\sim} V_\mathfrak{p}(\pi) \otimes_{L_\mathfrak{p}, \sigma} \sigma(L)_{\sigma(\mathfrak{p})}.$$

Let us identify characters of finite order of $\mathbf{A}_F^\times / F^\times$ with characters of G_F as in 0.4; then

$$\Lambda^2 V_\mathfrak{p}(\pi) \xrightarrow{\sim} L_\mathfrak{p}(-1) \otimes L_\mathfrak{p}(m) \otimes \varphi = L_\mathfrak{p}(m-1) \otimes \varphi. \qquad (B.2.5.1)$$

If $f \in S_k(\mathfrak{n}, \varphi)$ is a Hilbert modular newform of parallel even weight k as in [N2, 12.3] and $\pi = \pi(f)$ is normalised as in [*loc. cit.*] by $\omega = \varphi$, then

$$V_\mathfrak{p}(\pi) = V_\mathfrak{p}(f)(k/2 - 1).$$

In particular, $V_\mathfrak{p}(\pi) = V_\mathfrak{p}(f)$ for newforms $f \in S_2(\mathfrak{n}, \varphi)$ considered in the main body of this article.

We define the (semi-simplified) residual representation $\overline{V_\mathfrak{p}(\pi)}^{ss}$ of $V_\mathfrak{p}(\pi)$ to be the semi-simplification of $T_\mathfrak{p}(\pi)/\mathfrak{p}T_\mathfrak{p}(\pi)$, for any G_F-stable $O_{L,\mathfrak{p}}$-lattice $T_\mathfrak{p}(\pi) \subset V_\mathfrak{p}(\pi)$ (up to isomorphism, $\overline{V_\mathfrak{p}(\pi)}^{ss}$ does not depend on the choice of the lattice).

B.2.6. Proposition [T2]. (1) *$V_\mathfrak{p}(\pi)$ is an absolutely irreducible $L_\mathfrak{p}[G_F]$-module.*
(2) *For all but finitely many $\mathfrak{p}$ the residual representation $\overline{V_\mathfrak{p}(\pi)}^{ss}$ is an absolutely irreducible $O_L/\mathfrak{p}[G_F]$-module.*

B.2.7. Proposition. *The following properties are equivalent:*
(1) *There exists a finite extension E/F such that $V_\mathfrak{p}(\pi)|_{G_E}$ is not an absolutely irreducible $L_\mathfrak{p}[G_E]$-module.*
(2) *There exists a totally imaginary quadratic extension $K(\pi)/F$ such that $\pi \otimes \eta \xrightarrow{\sim} \pi$, where $\eta = \eta_{K(\pi)/F}$: $\mathrm{Gal}(K(\pi)/F) \xrightarrow{\sim} \{\pm 1\}$ is the quadratic character corresponding to $K(\pi)/F$.*
(3) *There exists $K(\pi)/F$ as in (2) such that $\pi = I_{K(\pi)/F}(\psi)$ is obtained by automorphic induction from a Hecke character $\psi : \mathbf{A}_{K(\pi)}^\times / K(\pi)^\times \longrightarrow \mathbf{C}^\times$ of infinity type $\psi_w(z) = z^{a_w}\overline{z}^{b_w}$ ($a_w, b_w \in \mathbf{Z}$, $|a_w - b_w| = k_v - 1$ and $a_w + b_w = m - 1$ for each $w \mid v \mid \infty$).*
[If they are satisfied, we say that π has complex multiplication by $K(\pi)$.]

Proof. Ribet's argument [Ri 3, Thm. 4.5] for $F = \mathbf{Q}$ combined with [He, Thm. 2] works in general (see [N1, Prop. 6.2.1] for a special case).

B.2.8. Proposition-Definition [Ri 2, p. 788]. *Let E be a finite extension of F contained in $\overline{F}$. The* **Frobenius field** *M_E attached to π over E is the subfield of L generated over $\mathbf{Q}$ by the traces $\mathrm{Tr}(\mathrm{Fr}_{\mathrm{geom}}(w) \mid V_{\mathfrak{p}}(\pi)) \in L$ (which do not depend on $\mathfrak{p}$), for w running through all primes of E not dividing S. We have $M_F = L$.*

B.3 Inner twists

Let π, $L = \mathbf{Q}(\pi)$ and $V_{\mathfrak{p}}(\pi)$ be as in B.2.1 and B.2.5. Denote by L^+ the maximal real subfield of L.

B.3.1. In order to simplify the notation we are going to write $V_{\mathfrak{p}} := V_{\mathfrak{p}}(\pi)$. The direct sum $V_p := \oplus_{\mathfrak{p}|p} V_{\mathfrak{p}}$ is a free module of rank 2 over $L \otimes \mathbf{Q}_p = \prod_{\mathfrak{p}|p} L_{\mathfrak{p}}$. The action of G_F defines a morphism of $(L \otimes \mathbf{Q}_p)$-algebras

$$(L \otimes \mathbf{Q}_p)[G_F] \longrightarrow \mathrm{End}_{L\otimes\mathbf{Q}_p}(V_p) = \prod_{\mathfrak{p}|p} \mathrm{End}_{L_{\mathfrak{p}}}(V_{\mathfrak{p}}) \xrightarrow{\sim} M_2(L \otimes \mathbf{Q}_p) = \prod_{\mathfrak{p}|p} M_2(L_{\mathfrak{p}}).$$

B.3.2. Proposition-Definition. *An* **inner twist** *of π is a pair (σ, χ), where $\sigma : L \hookrightarrow \mathbf{C}$ and $\chi : \mathbf{A}_F^\times / F^\times \longrightarrow \mathbf{C}^\times$ is a character such that there exists an isomorphism ${}^\sigma\pi \xrightarrow{\sim} \pi \otimes \chi$ (more precisely, such that there exists an isomorphism ${}^{\sigma'}\pi \xrightarrow{\sim} \pi \otimes \chi$ for some – hence for each – field automorphism $\sigma' \in \mathrm{Aut}(\mathbf{C})$ extending σ). Each such a pair has the following properties.*
(1) *The character χ is unramified outside S.*
(2) $\forall v \notin S \quad {}^\sigma\lambda_\pi(v) = \chi(v)\,\lambda_\pi(v)$.
(3) $\chi^2 = {}^\sigma\omega/\omega = {}^\sigma\varphi/\varphi$.
(4) *$\chi = \varphi^i\mu$, where $i \in \mathbf{Z}$ and $\mu^2 = 1$ ($\Longrightarrow \mathrm{Im}(\chi) \subseteq L$).*
(5) *$\sigma(L) \subseteq L$, hence $\sigma \in \mathrm{Aut}(L/\mathbf{Q})$.*
(6) *For each rational prime p there is an isomorphism of $(L \otimes \mathbf{Q}_p)[G_F]$-modules ${}^\sigma V_p \xrightarrow{\sim} V_p \otimes \chi$ (recall from B.2.5 that we identify χ with a character $\chi : G_F \longrightarrow L^\times \subset \mathbf{C}^\times$).*

Proof. The arguments from [Mo, §1] and [Ri 4, §3] apply.

B.3.3. Proposition. (1) *The inner twists of π form a group Γ under $(\sigma, \chi) \cdot (\sigma', \chi') = (\sigma \circ \sigma', \chi \cdot {}^\sigma\chi')$.*
(2) *The map "forget χ" is a group homomorphism $\Gamma \longrightarrow \mathrm{Aut}(L/\mathbf{Q})$, $(\sigma, \chi) \mapsto \sigma$, whose kernel is trivial (resp. is cyclic of order 2 generated by $(\mathrm{id}, \eta_{K(\pi)/F})$) if π has no CM (resp. if π has CM by $K(\pi)$ as in B.2.7). Consequently, Γ is a finite group.*
(3) *The image of Γ in $\mathrm{Aut}(L/\mathbf{Q})$ is an abelian group.*
(4) *If $\varphi \neq 1$, then $(c, \varphi^{-1}) \in \Gamma$ (if, in addition, π has no CM, then L is not totally real).*
(5) *For each $\mathfrak{p} \mid p$ in L, the subgroup $\Gamma_{\mathfrak{p}} := \{(\sigma, \chi) \in \Gamma \mid \sigma(\mathfrak{p}) = \mathfrak{p}\} \subset \Gamma$ is equal to*

$$\{(\sigma, \chi) \mid \sigma \in \mathrm{Aut}(L_{\mathfrak{p}}/\mathbf{Q}_p),\ \chi : G_F \longrightarrow L_{\mathfrak{p}}^\times,\ {}^\sigma V_{\mathfrak{p}} \xrightarrow{\sim} V_{\mathfrak{p}} \otimes \chi\}.$$

Proof. Again, everything works as in [Mo, §1] and [Ri 4, §3].

B.3.4. Proposition-Definition. *From now on, until the end of B.5, we assume that π* **has no CM**. *Under this assumption we can identify Γ with a (commutative) subgroup of $\mathrm{Aut}(L/\mathbf{Q})$ and write its elements as (σ, χ_σ), since χ is determined by σ. For any subgroup $\Delta \subset \Gamma$ denote by F_Δ the fixed field of the open subgroup $\bigcap_{\sigma\in\Delta} \mathrm{Ker}(\chi_\sigma) \subseteq G_F$.*
(1) *F_Δ is a finite abelian extension of F unramified outside S.*
(2) $\Delta \subset \Delta' \iff F_\Delta \subset F_{\Delta'}$.
(3) $F_\Delta F_{\Delta'} = F_{\Delta\Delta'}$.
(4) $F_{\{1\}} = F$, $F_{\{1,c\}} = F_\varphi := \overline{F}^{\mathrm{Ker}(\varphi)}$.
(5) *$F_\Gamma = F_\varphi(\sqrt{a_1}, \ldots, \sqrt{a_r})$ for some $a_j \in F_\varphi^\times$.*
(6) *F_φ is totally real (resp. totally complex) if $2 \mid m$ (resp. if $2 \nmid m$).*
(7) *If $2 \mid m$, then an intermediate field $F \subset F' \subset F_\Gamma$ is totally real $\iff$ it is not totally complex.*
(8) *If the prime $\mathfrak{p}_\Gamma := \mathfrak{p} \cap L^\Gamma$ of L^Γ splits completely in L^+/L^Γ, then $\Gamma_{\mathfrak{p}} \subset \{1, c\}$ and $F_{\Gamma_{\mathfrak{p}}} \subset F_\varphi$.*

Proof. Easy exercise.

B.3.5. Definition. *Set $\Gamma_{\rm triv} = \{1\}$ (resp. $\{1, c\}$) if $\varphi = 1$ (resp. if $\varphi \neq 1$). We say that π has no non-trivial inner twist if $\Gamma = \Gamma_{\rm triv}$ ($\iff$ $F_\Gamma = F_\varphi$). Similarly, we say that $V_{\mathfrak{p}}$ has no non-trivial inner twist if $\Gamma_{\mathfrak{p}} \subset \Gamma_{\rm triv}$ ($\iff$ $F_{\Gamma_{\mathfrak{p}}} \subset F_\varphi$).*

B.3.6. Proposition-Definition [Ri 2]. *For any field embedding $\sigma : L \hookrightarrow \overline{\mathbf{Q}}_p$ let V_σ be the $\overline{\mathbf{Q}}_p[G_F]$-module $V_p \otimes_{L\otimes\mathbf{Q}_p,\sigma\otimes{\rm id}} \overline{\mathbf{Q}}_p = V_{\mathfrak{p}} \otimes_{L_{\mathfrak{p}},\sigma} \overline{\mathbf{Q}}_p$, where $\mathfrak{p} \mid p$ is the prime of L induced by σ. Let E be a finite extension of F contained in $\overline{F}$, let $\sigma, \tau : L \hookrightarrow \overline{\mathbf{Q}}_p$ be field embeddings. The following conditions are equivalent.*
(1) $\sigma|_{M_E} = \tau|_{M_E}$.
(2) The $\overline{\mathbf{Q}}_p[G_E]$-modules $V_\sigma|_{G_E}$ and $V_\tau|_{G_E}$ are isomorphic.
(3) [In the case when E/F is a Galois extension.] There exists a character $\chi : {\rm Gal}(E/F) \longrightarrow \overline{\mathbf{Q}}_p^\times$ and an isomorphism of $\overline{\mathbf{Q}}_p[G_F]$-modules $V_\sigma \xrightarrow{\sim} V_\tau \otimes \chi$.

Proof. [Ri 2, Lemma 4.4.5], [Ri 4, Proof of Thm. 4.7], [Ch, Prop. 5.4].

B.3.7. Corollary. $\bigcap_E M_E = M_{F_\Gamma} = L^\Gamma$.

B.4 Image of the Galois representation V_p (the non CM case)

The assumptions of B.3 are in force. In particular, π does not have CM.

B.4.1. Fix a rational prime p and denote by

$$\rho_p : G_F \longrightarrow {\rm End}_{L\otimes\mathbf{Q}_p}(V_p) \xrightarrow{\sim} GL_2(L\otimes\mathbf{Q}_p)$$

the morphism given by the action of G_F on V_p.

The restriction of ρ_p to G_{F_Γ} gives rise to a semi-local version of the situation considered in B.1 (for $G = G_{F_\Gamma}$): for each $\sigma \in \Gamma$ there is an isomorphism of $(L\otimes\mathbf{Q}_p)[G_{F_\Gamma}]$-modules $\alpha_\sigma : {}^\sigma V \xrightarrow{\sim} V$. The corresponding subalgebra of twisted endomorphisms

$$D(p) := {\rm End}_{L\otimes\mathbf{Q}_p}(V_p)^{tw(\Gamma)} \subset {\rm End}_{L\otimes\mathbf{Q}_p}(V_p) \xrightarrow{\sim} \prod_{\mathfrak{p}|p} M_2(L_{\mathfrak{p}})$$

is an Azumaya algebra with centre $(L\otimes\mathbf{Q}_p)^\Gamma = L^\Gamma \otimes \mathbf{Q}_p \xrightarrow{\sim} \prod_{\mathfrak{p}_\Gamma|p} (L^\Gamma)_{\mathfrak{p}_\Gamma}$. It satisfies

$$(L\otimes\mathbf{Q}_p) \otimes_{(L^\Gamma\otimes\mathbf{Q}_p)} D(p) \xrightarrow{\sim} {\rm End}_{L\otimes\mathbf{Q}_p}(V_p)$$

and its class in

$$H^2(\Gamma, (L\otimes\mathbf{Q}_p)^\times) = \bigoplus_{\mathfrak{p}_\Gamma|p} H^2(\Gamma_{\mathfrak{p}}, L_{\mathfrak{p}}^\times) \subset \bigoplus_{\mathfrak{p}_\Gamma|p} {\rm Br}((L^\Gamma)_{\mathfrak{p}_\Gamma})$$

is killed by 2 (above, $\mathfrak{p}_\Gamma$ runs through all primes of L^Γ above p, $\mathfrak{p} \mid \mathfrak{p}_\Gamma$ is any prime of L above $\mathfrak{p}_\Gamma$ and $\Gamma_{\mathfrak{p}} = {\rm Gal}(L_{\mathfrak{p}}/(L^\Gamma)_{\mathfrak{p}_\Gamma}) \subset \Gamma$ is – as in B.3.3(5) – the decomposition group of $\mathfrak{p}$ in L/L^Γ). In other words,

$$D(p) = \bigoplus_{\mathfrak{p}_\Gamma|p} D(\mathfrak{p}_\Gamma), \qquad D(\mathfrak{p}_\Gamma) \text{ is a quaternion algebra over } (L^\Gamma)_{\mathfrak{p}_\Gamma}.$$

B.4.2. Proposition. *Let p be any rational prime.*
(1) The restriction of ρ_p to G_{F_Γ} defines a surjective morphism of $L\otimes\mathbf{Q}_p$-algebras $(L\otimes\mathbf{Q}_p)[G_{F_\Gamma}] \longrightarrow {\rm End}_{L\otimes\mathbf{Q}_p}(V_p)$, which is obtained by extension of scalars from a surjective morphism of $L^\Gamma\otimes\mathbf{Q}_p$-algebras $(L^\Gamma\otimes\mathbf{Q}_p)[G_{F_\Gamma}] \longrightarrow D(p)$.
(2) If $2 \mid m$, then $\rho_p(G_{F_\varphi}) \subseteq GL_2(L^+\otimes\mathbf{Q}_p)$, in a suitable basis of V_p (this is also true if π has complex multiplication).

Proof. (1) Combine B.1.4 with B.2.6(1).
(2) The argument of [Ri 3, Cor. 5.2] applies.

B.4.3. Corollary. *For each rational prime p the G_{F_Γ}-action on V_p factors through*

$$\rho_p : G_{F_\Gamma} \longrightarrow \{x \in D(p)^\times \mid \mathrm{Nrd}(x) \in \chi_{\mathrm{cycl},p}(G_{F_\Gamma})^{m-1}\},$$

where $m \in \mathbf{Z}$ is as in B.2.1 and $\chi_{\mathrm{cycl},p} : G_{\mathbf{Q}} \longrightarrow \mathbf{Z}_p^\times$ is the cyclotomic character.

Proof. As $F_\Gamma \supset F_\varphi$, the formula (B.2.5.1) implies that $(\Lambda^2 V_p)|_{G_{F_\Gamma}} = \chi_{\mathrm{cycl},p}^{m-1}|_{G_{F_\Gamma}}$.

B.4.4. Corollary. *For each rational prime p the Lie algebra (over $\mathbf{Q}_p$) $\mathfrak{g}_p := Lie(\rho_p(G_{F_\Gamma})) \subset D(p)$ is contained in $\mathfrak{h}_p := \{x \in D(p) \mid \mathrm{Trd}(x) \in (m-1)\mathbf{Q}_p\}$.*

B.4.5. Theorem. *For each rational prime p we have $\mathfrak{g}_p = \mathfrak{h}_p$.*

Proof. The arguments in [Mo, Thm. 4.1] and [Ri 4, Prop. 4.5] apply almost word by word. For the reader's convenience we repeat the main points. There is an isomorphism of $\overline{\mathbf{Q}}_p$-algebras

$$D(p) \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p \xrightarrow{\sim} \prod_{\sigma : L^\Gamma \hookrightarrow \overline{\mathbf{Q}}_p} M_2(\overline{\mathbf{Q}}_p)$$

such that, for each field embedding $\sigma : L^\Gamma \hookrightarrow \overline{\mathbf{Q}}_p$, the composite morphism

$$\rho_\sigma : G_{F_\Gamma} \xrightarrow{\rho_p} D(p)^\times \hookrightarrow (D(p) \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p)^\times \xrightarrow{\sim} \prod_{\sigma : L^\Gamma \hookrightarrow \overline{\mathbf{Q}}_p} GL_2(\overline{\mathbf{Q}}_p) \xrightarrow{pr_\sigma} GL_2(\overline{\mathbf{Q}}_p)$$

corresponds to the action of G_{F_Γ} on $V_{\sigma'}|_{G_{F_\Gamma}}$, where $\sigma' : L \hookrightarrow \overline{\mathbf{Q}}_p$ is any embedding extending σ (see B.3.6 above). As $V_{\sigma'}|_{G_E}$ is an irreducible representation of G_E, for any finite extension E/F, the Lie algebra (over $\overline{\mathbf{Q}}_p$) $\mathfrak{g}_\sigma := Lie(\rho_\sigma(G_{F_\Gamma})) \subset \mathfrak{gl}_2(\overline{\mathbf{Q}}_p)$ is a reductive Lie subalgebra acting irreducibly on $\overline{\mathbf{Q}}_p^2$; thus $\mathfrak{g}_\sigma$ contains $\mathfrak{sl}_2(\overline{\mathbf{Q}}_p)$.

The Lie subalgebra $\mathfrak{g}_p \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p \subset \prod_{\sigma : L^\Gamma \hookrightarrow \overline{\mathbf{Q}}_p} \mathfrak{g}_\sigma$ has the following property: for two distinct embeddings $\sigma, \tau : L^\Gamma \hookrightarrow \overline{\mathbf{Q}}_p$ the $\overline{\mathbf{Q}}_p[G_{F_\Gamma}]$-modules $V_{\sigma'}|_{G_{F_\Gamma}}$ and $V_{\tau'}|_{G_{F_\Gamma}}$ are not isomorphic, by B.3.6. This implies that $(pr_\sigma \times pr_\tau)(\mathfrak{g}_p \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p)$ contains $\mathfrak{sl}_2(\overline{\mathbf{Q}}_p) \times \mathfrak{sl}_2(\overline{\mathbf{Q}}_p)$; this is enough to conclude that $\mathfrak{g}_p \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$ contains $\prod_{\sigma : L^\Gamma \hookrightarrow \overline{\mathbf{Q}}_p} \mathfrak{sl}_2(\overline{\mathbf{Q}}_p)$, by [Ri 4, Lemma 4.6]. Finally, the image of $\mathfrak{g}_p \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$ in $\prod_{\sigma : L^\Gamma \hookrightarrow \overline{\mathbf{Q}}_p} \mathfrak{gl}_2(\overline{\mathbf{Q}}_p)/\mathfrak{sl}_2(\overline{\mathbf{Q}}_p)$ is given by $\det(V_p)|_{G_{F_\Gamma}} = \chi_{\mathrm{cycl},p}^{m-1}|_{G_{F_\Gamma}}$.

B.4.6. Corollary. (1) *For each rational prime p the G_F-action on V_p factors through*

$$\rho_p : G_F \longrightarrow \{x \in D(p)^\times (L \otimes \mathbf{Q}_p)^\times \mid \mathrm{Nrd}(x) \in \chi_{\mathrm{cycl},p}(G_F)^{m-1}\mathrm{Im}(\varphi)\}.$$

(2) *For each rational prime p and each finite extension E/F_Γ we have*

$$\mathrm{Im}\left(\mathbf{Q}_p[G_E] \longrightarrow \mathrm{End}_{\mathbf{Q}_p}(V_p)\right) = D(p).$$

Proof. (1) As G_{F_Γ} is a normal subgroup of G_F, the image of G_F under ρ_p normalises $\mathfrak{g}_p + (L \otimes \mathbf{Q}_p)\mathrm{id} = D(p)$. The statement follows from a semi-local version of B.1.6.
(2) $D(p)$ contains the L.H.S., which in turn contains the $\mathbf{Q}_p$-algebra generated by the Lie algebra $\mathfrak{g}_p = \mathfrak{h}_p$, namely $D(p)$.

B.4.7. Definition. *let $\mathfrak{p}$ be a finite prime of L. We say that the representation $V_\mathfrak{p}$ is* **quaternionic** *if the quaternion algebra $D(\mathfrak{p}_\Gamma)$ (where $\mathfrak{p}_\Gamma = \mathfrak{p} \cap L^\Gamma$) is a division algebra (this depends only on the Γ-orbit of $\mathfrak{p}$).*

B.4.8. Proposition. (1) *If $\mathfrak{p}$ is unramified in L/L^Γ and $V_\mathfrak{p}$ is quaternionic, then the residual representation $\overline{V_\mathfrak{p}(\pi)}^{ss}$ is reducible.*
(2) *The set of $\mathfrak{p}$ for which $V_\mathfrak{p}$ is quaternionic is finite.*

(3) *There exists a quaternion algebra D over L^Γ such that $D \otimes \mathbf{Q}_p \xrightarrow{\sim} D(p)$ for each rational prime p (hence $D(\mathfrak{p}_\Gamma) \xrightarrow{\sim} D \otimes_{L^\Gamma} (L^\Gamma)_{\mathfrak{p}_\Gamma}$ for each finite prime $\mathfrak{p}_\Gamma$ of L^Γ).*

Proof. Thanks to B.2.6(2) it is enough to prove (1). Denote by O the (unique) maximal order of the quaternion division algebra $D(\mathfrak{p}_\Gamma)$. As $L_\mathfrak{p}/(L^\Gamma)_{\mathfrak{p}_\Gamma}$ is unramified, the maximal (bilateral) ideal of O is of the form $aO = Oa$, where a^2 is a uniformiser of $(L^\Gamma)_{\mathfrak{p}_\Gamma}$, hence of $L_\mathfrak{p}$. According to B.4.6, the image of the compact group G_F in $\mathrm{Aut}_{L_\mathfrak{p}}(V_\mathfrak{p})$ is contained in the maximal compact subgroup $O^\times O_{L,\mathfrak{p}}^\times$ of $D(\mathfrak{p}_\Gamma)^\times L_\mathfrak{p}^\times$. The image of O under that map $O \subset M_2(O_{L,\mathfrak{p}}) \longrightarrow M_2(O_L/\mathfrak{p})$ is isomorphic to $O/a^2O \xrightarrow{\sim} k[\varepsilon]/(\varepsilon^2)$, where $k = O/aO$. This implies that, possibly after conjugation by an element of $GL_2(O_L/\mathfrak{p})$, the image of O is contained in

$$R = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \,\middle|\, x, y \in O_L/\mathfrak{p} \right\} \subset M_2(O_L/\mathfrak{p}).$$

Consequently, the image of $O^\times O_{L,\mathfrak{p}}^\times$, hence of G_F, is contained in $R^\times$, which proves (1).

B.4.9. It is likely that there exists a (unique) quaternion algebra D as in B.4.8(3) which is totally indefinite (resp. totally definite) if $2 \mid m$ (resp. if $2 \nmid m$). In the case $F = \mathbf{Q}$ this was proved in [Mo, Thm. 3.1.2].

B.4.10. Theorem. *There exists a quaternion algebra D over L^Γ such that:*
(1) *For each rational prime p the representation ρ_p factors through*

$$\rho_p : G_{F_\Gamma} \longrightarrow (D \otimes \mathbf{Q}_p)^\times \subset ((D \otimes_{L^\Gamma} L) \otimes \mathbf{Q}_p)^\times \xrightarrow{\sim} GL_2(L \otimes \mathbf{Q}_p),$$
$$\rho_p : G_F \longrightarrow (D \otimes \mathbf{Q}_p)^\times (L \otimes \mathbf{Q}_p)^\times \subset GL_2(L \otimes \mathbf{Q}_p).$$

(2) *For each rational prime p the image $\rho_p(G_{F_\Gamma})$ is an open subgroup of*

$$\{x \in (D \otimes \mathbf{Q}_p)^\times \mid \mathrm{Nrd}(x) \in \chi_{\mathrm{cycl},p}(G_{F_\Gamma})^{m-1}\}.$$

(3) *For all but finitely many rational primes p there is a basis of V_p over $L \otimes \mathbf{Q}_p$ in which*

$$\rho_p(G_{F_\Gamma}) = \{x \in GL_2(O_{L^\Gamma} \otimes \mathbf{Z}_p) \mid \det(x) \in (\mathbf{Z}_p^\times)^{m-1}\}.$$

Proof. (1) This is a combination of B.4.2(1), B.4.6 and B.4.8(3). The statement (2) is equivalent to B.4.5. Finally, (3) is proved exactly as in [Ri 6, Thm. 3.1].

B.4.11. Modular abelian varieties over F. Assume that $m = 2$ and $k_v = 2$ for each $v \in S_\infty$. It is expected that there exists an abelian variety A over F with the following properties:

$$\dim(A) = [L : \mathbf{Q}], \qquad \mathrm{End}_F(A) = O_L, \qquad \forall p \quad V_p = H^1_{et}(A \otimes_F \overline{F}, \mathbf{Q}_p)(1).$$

The existence of A is known if $2 \nmid [F : \mathbf{Q}]$ or if there exists a finite prime v of F for which π_v is not a principal series representation (in this case A can be constructed as a quotient of the Jacobian of a suitable Shimura curve over F).

B.4.12. Proposition. (1) *There exists a quaternion algebra D over L^Γ such that*

$$\mathrm{End}_{\overline{\mathbf{Q}}}(A) \otimes \mathbf{Q} = \mathrm{End}_{F_\Gamma}(A) \otimes \mathbf{Q} \xrightarrow{\sim} \begin{cases} M_{|\Gamma|}(L^\Gamma), & \text{if } D \xrightarrow{\sim} M_2(L^\Gamma) \\ M_{|\Gamma|/2}(D), & \text{if } D \text{ is a division algebra} \end{cases}$$

and $D \otimes \mathbf{Q}_p \xrightarrow{\sim} D(p)$ for each rational prime p.
[Recall that π does not have CM, by assumption; this means that A does not have CM, either.]
(2) *$V_\mathfrak{p} = H^1_{et}(A \otimes_F \overline{F}, \mathbf{Q}_p)(1) \otimes_{L \otimes \mathbf{Q}_p} L_\mathfrak{p}$ is quaternionic in the sense of B.4.7 iff the central simple algebra $(\mathrm{End}_{\overline{\mathbf{Q}}}(A) \otimes \mathbf{Q}) \otimes_{L^\Gamma} (L^\Gamma)_{\mathfrak{p}_\Gamma}$ (where $\mathfrak{p}_\Gamma = \mathfrak{p} \cap L^\Gamma$) has a non-zero class in $\mathrm{Br}((L^\Gamma)_{\mathfrak{p}_\Gamma})$.*

Proof. It is enough to prove (1). For any finite extension E/F_Γ the Faltings isogeny theorem tells us that $\mathrm{End}_E(A) \otimes \mathbf{Q}_p$ is equal to the centraliser of $\mathrm{Im}\left(\mathbf{Q}_p[G_E] \longrightarrow \mathrm{End}_{\mathbf{Q}_p}(V_p)\right) = D(p)$ in $\mathrm{End}_{\mathbf{Q}_p}(V_p)$. As

$L^\Gamma \otimes \mathbf{Q}_p = Z(D(p)) \subseteq Z(\mathrm{End}_E(A) \otimes \mathbf{Q}_p)$, we deduce that $\mathrm{End}_E(A) \otimes \mathbf{Q}_p$ coicides with the centraliser of $D(p)$ in $\mathrm{End}_{L^\Gamma \otimes \mathbf{Q}_p}(V_p)$, hence is isomorphic to

$$\mathrm{End}_E(A) \otimes \mathbf{Q}_p \xrightarrow{\sim} \begin{cases} M_{|\Gamma|}(L^\Gamma \otimes \mathbf{Q}_p), & \text{if } D(p) \xrightarrow{\sim} M_2(L^\Gamma \otimes \mathbf{Q}_p) \\ M_{|\Gamma|/2}(D(p)), & \text{if not,} \end{cases}$$

thanks to a semi-local version of B.1.5. As $L \subset \mathrm{End}_E(A) \otimes \mathbf{Q}$ and $Z(\mathrm{End}_E(A) \otimes \mathbf{Q}_p) = L^\Gamma \otimes \mathbf{Q}_p$, it follows that $\mathrm{End}_E(A) \otimes \mathbf{Q}$ is a central simple algebra over L^Γ which does not depend on $E \supset F_\Gamma$, whose class in $\mathrm{Br}(L^\Gamma)$ is killed by 2 and whose localisations at finite primes are given by the above formula. The statement (1) is implied by these properties.

B.5 Image of the Galois representation $V_\mathfrak{p}$ (the non CM case)

The assumptions of B.3 are in force. In particular, π does not have CM. Let K be a totally imaginary quadratic extension of F.

B.5.1. For a prime $\mathfrak{p}$ of L above a rational prime p we denote by

$$\rho_\mathfrak{p} : G_F \longrightarrow \mathrm{Aut}_{L_\mathfrak{p}}(V_\mathfrak{p}) \xrightarrow{\sim} GL_2(L_\mathfrak{p})$$

the morphism defining the action of G_F on $V_\mathfrak{p}$ and by $F(V_\mathfrak{p}) := \overline{F}^{\mathrm{Ker}(\rho_\mathfrak{p})}$ the extension of F trivialising $\rho_\mathfrak{p}$. As before, we denote by $\mathfrak{p}_\Gamma = \mathfrak{p} \cap L^\Gamma$ the prime of L^Γ below $\mathfrak{p}$ and by $\Gamma_\mathfrak{p} \subset \Gamma$ the decomposition group of $\mathfrak{p}$ in the extension L/L^Γ.

B.5.2. Theorem. *$\rho_\mathfrak{p}(G_{F_\varphi F_{\Gamma_\mathfrak{p}}})$ is an open subgroup of $\{x \in D(\mathfrak{p}_\Gamma)^\times \mid \mathrm{Nrd}(x) \in (\mathbf{Z}_p^\times)^{m-1}\}$. For all but finitely many $\mathfrak{p}$ there exists a basis of $V_\mathfrak{p}$ in which*

$$\rho_\mathfrak{p}(G_{F_\varphi F_{\Gamma_\mathfrak{p}}}) = \{x \in GL_2(O_{L^\Gamma,\mathfrak{p}_\Gamma}) \mid \det(x) \in (\mathbf{Z}_p^\times)^{m-1}\} \qquad (\mathfrak{p}_\Gamma = \mathfrak{p} \cap L^\Gamma).$$

Proof. The proof of B.4.5 (resp. B.4.10(3)) applies with trivial modifications (taking into account B.3.3(5)).

B.5.3. Proposition. *If the field $F_\varphi F_{\Gamma_\mathfrak{p}}$ is not totally complex ($\iff$ $2 \mid m$ and $F_{\Gamma_\mathfrak{p}}$ is totally real), then $V_\mathfrak{p}$ is not quaternionic.*

Proof. Let $c \in G_{F_\varphi F_{\Gamma_\mathfrak{p}}}$ be the complex conjugation with respect to some real prime of $F_\varphi F_{\Gamma_\mathfrak{p}}$. The element $\rho_\mathfrak{p}(c) \in D(\mathfrak{p}_\Gamma)^\times \subset GL_2(L_\mathfrak{p})$ has two distinct eigenvalues $\pm 1 \in \mathbf{Q}_p \subset Z(D(\mathfrak{p}_\Gamma))$, which implies that $D(\mathfrak{p}_\Gamma)$ is not a division algebra.

B.5.4. As in Theorem A in the Introduction to this article, consider the following conditions on $g \in G_F$:

(A1) g acts trivially on F_φ ($\iff$ $\varphi(g) = 1$);
(A2) $\rho_\mathfrak{p}(g) \in GL_2(L_\mathfrak{p})$ has eigenvalues $\lambda_1, \lambda_2 \in L_\mathfrak{p}^\times$ satisfying $\lambda_1^2 = 1 \neq \lambda_2^2$;
(A3) g does not act trivially on K.

B.5.5. Proposition. (1) *Any $g \in G_F$ satisfying (A1) and (A2) acts trivially on $F_\varphi F_{\Gamma_\mathfrak{p}}$.*
(2) *There exists $g \in G_F$ satisfying (A1) and (A2) $\iff$ $V_\mathfrak{p}$ is not quaternionic.*
(3) *If there exists $g \in G_F$ satisfying (A1)–(A3), then $V_\mathfrak{p}$ is not quaternionic and $K \not\subset F_\varphi F_{\Gamma_\mathfrak{p}}$.*
(4) *If $V_\mathfrak{p}$ is not quaternionic and $K \not\subset F_\varphi F_{\Gamma_\mathfrak{p}} F(V_\mathfrak{p})$, then there exists $g \in G_F$ satisfying (A1)–(A3).*
(5) *If $2 \mid m$ and $F_{\Gamma_\mathfrak{p}}$ is totally real, then there exists $g \in G_F$ satisfying (A1)–(A3).*
(6) *If $2 \mid m$ and if $\mathfrak{p}_\Gamma$ splits completely in L^+/L^Γ, then there exists $g \in G_F$ satisfying (A1)–(A3).*
(7) *For all but finitely many $\mathfrak{p}$ satisfying $K \not\subset F_\varphi F_{\Gamma_\mathfrak{p}}$ there exists $g \in G_F$ satisfying (A1)–(A3).*
(8) *If $K \not\subset F_\Gamma$, then for all but finitely many $\mathfrak{p}$ there exists $g \in G_F$ satisfying (A1)–(A3).*
(9) *If $K \subset F_\Gamma$, then the set of primes $\mathfrak{p}_\Gamma$ of L^Γ for which there exists (for each $\mathfrak{p} \mid \mathfrak{p}_\Gamma$ in L) an element $g \in G_F$ satisfying (A1)–(A3) has density equal to at least $1 - [F_\varphi : F]/|\Gamma| \geq 1/2$.*
(10) *If $2 \mid m$ and if $V_\mathfrak{p}$ has no non-trivial inner twist in the sense of B.3.5, then there exists $g \in G_F$ satisfying (A1)–(A3).*

Proof. (1) As g satisfies (A1) and (A2), we have $\pm\lambda_2 = \det(\rho_\mathfrak{p}(g)) = \chi_{\mathrm{cycl},p}(g)^{m-1}\varphi(g) = \chi_{\mathrm{cycl},p}(g)^{m-1} \in \mathbf{Q}_p$. If $(\sigma, \chi_\sigma) \in \Gamma_\mathfrak{p}$, then ${}^\sigma V_\mathfrak{p} \xrightarrow{\sim} V_\mathfrak{p} \otimes \chi_\sigma$, which implies that $0 \neq \lambda_1 + \lambda_2 = {}^\sigma(\lambda_1 + \lambda_2) = \chi_\sigma(g)(\lambda_1 + \lambda_2)$, hence $\chi_\sigma(g) = 1$, as claimed.

(2) If $V_{\mathfrak{p}}$ is not quaternionic, then $D(\mathfrak{p}_\Gamma) \xrightarrow{\sim} M_2((L^\Gamma)_{\mathfrak{p}_\Gamma})$ and B.5.2 implies that there exists $g \in G_F$ which acts trivially on $F_\varphi F_{\Gamma_{\mathfrak{p}}}$ and for which $\rho_{\mathfrak{p}}(g) = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$ with $u \in \mathbf{Z}_p^\times$, $u^2 \neq 1$. Conversely, if there exists $g \in G_F$ satisfying (A1) and (A2), then g acts trivially on $F_\varphi F_{\Gamma_{\mathfrak{p}}}$ (by (1)), hence $\rho_{\mathfrak{p}}(g)$ is an element of $D(\mathfrak{p}_\Gamma)^\times \subset GL_2(L_{\mathfrak{p}})$ with two distinct eigenvalues lying in $\mathbf{Q}_p \subset Z(D(\mathfrak{p}_\Gamma))$; thus $D(\mathfrak{p}_\Gamma)$ cannot be a division algebra.
(3), (4) Both statements follow immediately from (1) and (2).
(5) Firstly, the assumptions imply that $V_{\mathfrak{p}}$ is not quaternionic, thanks to B.3.4(6-7) and B.5.3. Secondly, we can choose an isomorphism $D(\mathfrak{p}_\Gamma) \xrightarrow{\sim} M_2((L^\Gamma)_{\mathfrak{p}_\Gamma})$ in such a way that $\rho_{\mathfrak{p}} : G_{F_\varphi F_{\Gamma_{\mathfrak{p}}}} \longrightarrow GL_2((L^\Gamma)_{\mathfrak{p}_\Gamma})$ will map the complex conjugation c with respect to some real place of $F_\varphi F_{\Gamma_{\mathfrak{p}}}$ to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. According to B.5.2 there exists $h \in G_F$ which acts trivially on $F_\varphi F_{\Gamma_{\mathfrak{p}}} K$ and for which $\rho_{\mathfrak{p}}(h) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$, $a \in \mathbf{Z}_p^\times$. The element $g = ch \in G_F$ then satisfies (A1)–(A3).
(6) Combine (5) with B.3.4(6) and B.3.4(8).
(7) Thanks to (4) it is enough to show that there are only finitely many $\mathfrak{p} \nmid 6$ satisfying $K \not\subset F_\varphi F_{\Gamma_{\mathfrak{p}}}$ and $K \subset F_\varphi F_{\Gamma_{\mathfrak{p}}} F(V_{\mathfrak{p}})$. Fix a G_F-stable $O_{L,\mathfrak{p}}$-lattice $T \subset V_{\mathfrak{p}}$; then $F(V_{\mathfrak{p}}) = \bigcup_{n\geq 1} F(T/\mathfrak{p}^n T)$, where $F(T/\mathfrak{p}^n T)$ is the fixed field of $\mathrm{Ker}(G_F \longrightarrow \mathrm{Aut}(T/\mathfrak{p}^n T))$. As $F(V_{\mathfrak{p}})/F(T/\mathfrak{p}T)$ is a pro-p-extension and $p \neq 2$, the field K satisfies $F_\varphi F_{\Gamma_{\mathfrak{p}}} \subsetneq F_\varphi F_{\Gamma_{\mathfrak{p}}} K \subseteq F_\varphi F_{\Gamma_{\mathfrak{p}}} F(T/\mathfrak{p}T)$. According to B.5.2 (see also [Di, Prop. 0.1(ii)]), the Galois group $G := \mathrm{Gal}(F_\varphi F_{\Gamma_{\mathfrak{p}}} F(T/\mathfrak{p}T)/F_\varphi F_{\Gamma_{\mathfrak{p}}})$ is equal to $\{x \in GL_2(k(\mathfrak{p}_\Gamma)) \mid \det(x) \in (\mathbf{F}_p^\times)^{m-1}\}$, for all but finitely many $\mathfrak{p}$. As $p > 3$, the commutator of G contains $SL_2(k(\mathfrak{p}_\Gamma))$ [Gr, Thm. 1.9], which implies that $F_\varphi F_{\Gamma_{\mathfrak{p}}} \subsetneq F_\varphi F_{\Gamma_{\mathfrak{p}}} K \subseteq F_\varphi F_{\Gamma_{\mathfrak{p}}}(\mu_p)$, which can happen only for finitely many p (since there are only finitely many possible values of $F_{\Gamma_{\mathfrak{p}}}$).
(8) This is an immediate consequence of (7).
(9) If $\mathfrak{p}$ is unramified in L/L^Γ, then $\Gamma_{\mathfrak{p}} \subset \Gamma$ is cyclic, generated by the Frobenius element $\sigma(\mathfrak{p}_\Gamma) = \mathrm{Fr}_{\mathrm{geom}}(\mathfrak{p}_\Gamma)$. If $K \subset F_\varphi F_{\Gamma_{\mathfrak{p}}}$, then the restriction of $\chi_{\sigma(\mathfrak{p}_\Gamma)}$ to $\mathrm{Gal}(F_\Gamma/F_\varphi) \xrightarrow{\sim} (\mathbf{Z}/2\mathbf{Z})^r$ must be equal to the quadratic character $\eta_{F_\varphi K/F_\varphi} : \mathrm{Gal}(F_\Gamma/F_\varphi) \longrightarrow \{\pm 1\}$ associated to the quadratic extension $F_\varphi K/F_\varphi$, which implies that $\chi_{\sigma(\mathfrak{p}_\Gamma)} = \chi_0 \varphi^i$ for a fixed character χ_0 of $\mathrm{Gal}(F_\Gamma/F)$ extending $\eta_{F_\varphi K/F_\varphi}$ and some integer $i \in \mathbf{Z}$. It follows that $\chi_{\sigma(\mathfrak{p}_\Gamma)}$, hence $\sigma(\mathfrak{p}_\Gamma) \in \Gamma$, has only $[F_\varphi : F]$ possible values for which $K \subset F_\varphi F_{\Gamma_{\mathfrak{p}}}$. The remaining $|\Gamma| - [F_\varphi : F]$ values correspond to a set of primes $\mathfrak{p}_\Gamma$ of L^Γ of density $1 - [F_\varphi : F]/|\Gamma| \geq 1/2$ for which $K \not\subset F_\varphi F_{\Gamma_{\mathfrak{p}}}$, for each $\mathfrak{p} \mid \mathfrak{p}_\Gamma$; we conclude by (7).
(10) Combine (5) with B.3.4(6).

B.6 The case of complex multiplication

Assume that $\pi = I_{K(\pi)/F}(\psi)$ is as in B.2.7 and that $2 \mid m$. Let K be a totally imaginary quadratic extension of F.

B.6.1. The Hecke character $\psi : \mathbf{A}_{K(\pi)}^\times / K(\pi)^\times \longrightarrow \mathbf{C}^\times$ can be written as

$$\psi(x) = \psi_{\mathrm{alg}}(x)\, r(x_\infty)^{-1},$$

where

$$\psi_{\mathrm{alg}} : \mathbf{A}_{K(\pi)}^\times \longrightarrow \widetilde{L}^\times$$

is an algebraic Hecke character with values in a finite extension $\widetilde{L} \subset \mathbf{C}$ of L and

$$r : R_{K(\pi)/\mathbf{Q}} \mathbf{G}_m \longrightarrow R_{L'/\mathbf{Q}} \mathbf{G}_m$$

(for a suitable totally imaginary quadratic extension $L' \subset \widetilde{L}$ of L^+; $L' = L$ if $\varphi \neq 1$) is a morphism of algebraic tori satisfying

$$\forall x \in K(\pi)^\times \qquad \psi_{\rm alg}(x) = r(x).$$

The infinity type of ψ is related to r by the formula

$$\forall x_\infty \in (K(\pi)\otimes \mathbf{R})^\times \qquad \psi_\infty(x_\infty) = r(x_\infty)^{-1}.$$

B.6.2. For each rational prime p the action of $G_{K(\pi)F_\varphi} \subset G_F$ on V_p is given (via the reciprocity map) by the character

$$\psi^{(p)} : \mathbf{A}^\times_{K(\pi)F_\varphi}/(K(\pi)F_\varphi)^\times \longrightarrow (L'\otimes \mathbf{Q}_p)^\times = \prod_{\mathfrak{p}'|p} L'^\times_{\mathfrak{p}'} \subset GL_2(L^+\otimes \mathbf{Q}_p) = \prod_{\mathfrak{p}_+|p} GL_2(L^+_{\mathfrak{p}_+}),$$
$$\psi^{(p)}(x) = \psi_{\rm alg}(N(x))\, r(N(x_p))^{-1}, \qquad N = N_{K(\pi)F_\varphi/K(\pi)}.$$

Consequently, the action of $G_{K(\pi)F_\varphi}$ on $V_\mathfrak{p}$ is given by the projection $\psi^{(\mathfrak{p})}$ of $\psi^{(p)}$ to $\prod_{\mathfrak{p}'|\mathfrak{p}_+}$, where $\mathfrak{p}_+ = \mathfrak{p}\cap L^+$:

$$\psi^{(\mathfrak{p})} : \mathbf{A}^\times_{K(\pi)F_\varphi}/(K(\pi)F_\varphi)^\times \longrightarrow (L'\otimes_{L^+} L^+_{\mathfrak{p}_+})^\times = \prod_{\mathfrak{p}'|\mathfrak{p}_+} L'^\times_{\mathfrak{p}'} \subset GL_2(L^+_{\mathfrak{p}_+}).$$

In particular, if $\mathfrak{p}_+$ splits in L'/L^+, then ${\rm Im}(\psi^{(\mathfrak{p})})$ is contained (after conjugation) in the split Cartan subgroup $(L^+_{\mathfrak{p}_+})^\times \times (L^+_{\mathfrak{p}_+})^\times \subset GL_2(L^+_{\mathfrak{p}_+})$. If $\mathfrak{p}_+$ does not split in L'/L^+, then ${\rm Im}(\psi^{(\mathfrak{p})})$ is contained in a non-split Cartan subgroup $L'^\times_{\mathfrak{p}'} \subset GL_2(L^+_{\mathfrak{p}_+})$, where $\mathfrak{p}'$ is the unique prime of L' above $\mathfrak{p}_+$.

B.6.3. Proposition. (1) *For each rational prime p the Galois image $\rho_p(G_{K(\pi)F_\varphi}) = {\rm Im}(\psi^{(p)}) \subset (L'\otimes \mathbf{Q}_p)^\times \subset GL_2(L\otimes\mathbf{Q}_p)$ contains* r (an open subgroup of $(K(\pi)\otimes\mathbf{Q}_p)^\times$).
(2) *For all but finitely many rational primes p the image* ${\rm Im}(\psi^{(p)})$ *contains* $r\left((O_{K(\pi)}\otimes \mathbf{Z}_p)^\times\right)$.

Proof. (1) As ${\rm Ker}(\psi_{\rm alg})$ is open, there exists an open subgroup of $(K(\pi)\otimes\mathbf{Q}_p)^\times$ on which $\psi^{(p)} = r^{-1}$.
(2) For all but finitely many p the norm map $N : (O_{K(\pi)F_\varphi}\otimes\mathbf{Z}_p)^\times \longrightarrow (O_{K(\pi)}\otimes\mathbf{Z}_p)^\times$ is surjective and $(O_{K(\pi)}\otimes\mathbf{Z}_p)^\times \subset {\rm Ker}(\psi_{\rm alg})$.

B.6.4. Consider the following conditions on $g\in G_F$ corresponding to the CM case of Theorem A:

(A1) g acts trivially on F_φ ($\iff \varphi(g)=1$);
(A2) $\rho_\mathfrak{p}(g)\in GL_2(L_\mathfrak{p})$ has eigenvalues $\lambda_1,\lambda_2\in L^\times_\mathfrak{p}$ satisfying $\lambda_1^2 = 1 \neq \lambda_2^n$ ($\forall n\geq 1$);
(A3) g does not act trivially on K.

B.6.5. Proposition. (1) *Any $g\in G_F$ satisfying (A1) and (A2) acts trivially on $F_\varphi K(\pi)$.*
(2) *There exists $g\in G_F$ satisfying (A1) and (A2) $\iff$ $\mathfrak{p}_+$ splits in L'/L^+ and ${\rm Im}(\psi^{(\mathfrak{p})}) \subset (L^+_{\mathfrak{p}_+})^\times\times(L^+_{\mathfrak{p}_+})^\times$ contains an open subgroup of $\mathbf{Z}_p^\times\times\mathbf{Z}_p^\times$ $\iff$ $\mathfrak{p}_+$ splits in L'/L^+ and $\psi^{(\mathfrak{p})} = \psi_1\oplus\psi_2$, where the characters $\psi_i : \mathbf{A}^\times_{K(\pi)F_\varphi}/(K(\pi)F_\varphi)^\times \longrightarrow (L^+_{\mathfrak{p}_+})^\times$ are such that $\psi_2({\rm Ker}(\psi_1))$ is infinite.*
(3) *If there exists $g\in G_F$ satisfying (A1)–(A3), then $K\not\subset F_\varphi K(\pi)$ ($\iff$ $K(\pi)\not\subset F_\varphi K$), $\mathfrak{p}_+$ splits in L'/L^+ and ${\rm Im}(\psi^{(\mathfrak{p})})$ contains an open subgroup of $\mathbf{Z}_p^\times\times\mathbf{Z}_p^\times$.*
(4) *If $K\not\subset F_\varphi K(\pi)F(V_\mathfrak{p})$ and if there exists $g'\in G_F$ satisfying (A1) and (A2), then there exists $g\in G_F$ satisfying (A1)–(A3).*
(5) *There exists a constant $b(\psi)$ depending only on ψ such that for each $\mathfrak{p}$ for which there exists $g\in G_F$ satisfying (A1) and (A2) the image ${\rm Im}(\psi^{(\mathfrak{p})})$ contains a subgroup of $\mathbf{Z}_p^\times\times\mathbf{Z}_p^\times$ of index $\leq b(\psi)$.*
(6) *If $\mathfrak{p}_+$ splits in L'/L^+ and $L^+_{\mathfrak{p}_+}\xrightarrow{\sim}\mathbf{Q}_p$, then there exists $g\in G_F$ satisfying (A1)–(A2).*
(7) *If $K(\pi)\not\subset F_\varphi K$, then $K\not\subset F_\varphi K(\pi)F(V_\mathfrak{p})$ holds for all but finitely many $\mathfrak{p}$.*
(8) *If $K(\pi)\not\subset F_\varphi K$, then for all but finitely many $\mathfrak{p}$ for which there is $g'\in G_F$ satisfying (A1) and (A2) there exists $g\in G_F$ satisfying (A1)–(A3).*

Proof. (1) If $g\in G_F$ satisfying (A1) and (A2) acts non-trivially on $F_\varphi K(\pi)$, then $\rho_\mathfrak{p}(g)$ lies in the normaliser of a Cartan subgroup $C\subset GL_2(L^+_{\mathfrak{p}_+})$ but not in C itself. This implies that ${\rm Tr}(\rho_\mathfrak{p}(g)) = 0$, in contradiction with (A2).

(2) If $g \in G_F$ satisfies (A1) and (A2), then both eigenvalues of $\rho_{\mathfrak{p}}(g)$ belong to $L^+_{\mathfrak{p}_+}$, since $\pm\lambda_2 = \det(\rho_{\mathfrak{p}}(g)) = \chi_{\text{cycl},p}(g)^{m-1} \in \mathbf{Z}_p$. As a result, any Cartan subgroup of $GL_2(L^+_{\mathfrak{p}_+})$ containg $\rho_{\mathfrak{p}}(g)$ must be split; thus $\mathfrak{p}_+$ splits in L'/L^+. Furthermore, $\mathrm{Im}(\psi^{(\mathfrak{p})})$ contains $\rho_{\mathfrak{p}}(g^2)^{\mathbf{Z}_p} = \{1\} \times (\lambda_2^2)^{\mathbf{Z}_p}$ and $\rho_{\mathfrak{p}}(cg^2c^{-1})^{\mathbf{Z}_p} = (\lambda_2^2)^{\mathbf{Z}_p} \times \{1\}$ (where $c \in G_{F_\varphi}$, $c \notin G_{K(\pi)F_\varphi}$), hence it contains an open subgroup of $\mathbf{Z}_p^\times \times \mathbf{Z}_p^\times$. The remaining implications are easy (using the fact that $\psi_1\psi_2 = \chi^{m-1}_{\text{cycl},p}|_{G_{K(\pi)F_\varphi}}$ and $\psi_2(h) = \psi_1(chc^{-1})$ for all $h \in G_{K(\pi)F_\varphi}$).
(3) and (4) are immediate consequences of (1) and (2).
(5) This follows from [Ri 5, Thm. 2.4] combined with B.6.3(2).
(6) If not, then $\psi^{(\mathfrak{p})} = \psi_1 \oplus \psi_2$ with $\psi_i : G_{K(\pi)F_\varphi} \longrightarrow \mathbf{Z}_p^\times$ such that $\psi_2(\mathrm{Ker}(\psi_1))$ is finite. As $\psi_1\psi_2 = \chi^{m-1}_{\text{cycl},p}|_{G_{K(\pi)F_\varphi}}$, it follows that there exist integers a_1, a_2, b ($b \neq 0$) such that $\psi_i^b = \chi^{a_i}_{\text{cycl},p}|_{G_{K(\pi)F_\varphi}}$, $a_1 + a_2 = b(m-1)$. On the other hand, $\psi_2(h) = \psi_1(chc^{-1})$ as in (2) above, which implies that $a_1 = a_2$, hence $\psi_i^2/\chi^{m-1}_{\text{cycl},p}$ is a character of finite order of $G_{K(\pi)F_\varphi}$, which is impossible, since $2 \nmid (m-1)$ and ψ_i is a potentially crystalline representation at each prime of $K(\pi)F_\varphi$ above p.
(7) The equivalences $[K \subset F_\varphi K(\pi) \iff F_\varphi K = F_\varphi K(\pi) \iff F_\varphi K \supset K(\pi)]$ imply that $K \not\subset F_\varphi K(\pi)$. If $p \neq 2$ and $K \subset F_\varphi K(\pi) F(V_{\mathfrak{p}})$, then $F_\varphi K(\pi) \subsetneq F_\varphi K(\pi) K \subset F_\varphi K(\pi) F(T/\mathfrak{p}T)$, as in the proof of B.5.5(7). The Galois group $\mathrm{Gal}(F_\varphi K(\pi) F(T/\mathfrak{p}T)/F_\varphi K(\pi))$ injects into $k(\mathfrak{p}_+)^\times \times k(\mathfrak{p}_+)^\times$ ($k(\mathfrak{p}_+) = O_{L^+}/\mathfrak{p}_+$), with the non-trivial element of $\mathrm{Gal}(F_\varphi K(\pi)/F_\varphi)$ interchanging the two factors. It follows that $F_\varphi K(\pi) F(T/\mathfrak{p}T)/F_\varphi K(\pi)$ has at most one quadratic subextension which is a Galois extension of F_φ, namely the one contained in $F_\varphi(\mu_p)K(\pi)$. Consequently, $F_\varphi K(\pi) K \subset F_\varphi(\mu_p) K(\pi)$, which is possible for only finitely many p.
(8) Combine (4) and (7).

B.6.6. Question. *Is there an explicit criterion for deciding whether* $\mathrm{Im}(\psi^{(\mathfrak{p})})$ *contains an open subgroup of* $\mathbf{Z}_p^\times \times \mathbf{Z}_p^\times$*?*

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