# MASLOV INDEX AND CLIFFORD ALGEBRAS

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The Maslov index of triples of Lagrangean subspaces of a symplectic space over a local field $F$ is considered in the book of G. Lion and M. Vergne [4]. The authors use the Maslov index to construct a central extension of $Sp(2n, F)$ by the Witt group $WF$ of the field $F$. The cocycle of this extension determines the cocycle of the Weil representation of $Sp(2n, F)$.

In the present work we introduce a generalised Maslov index associated to triples of non-zero vectors in a plane. It takes values in a certain non-commutative group $\widetilde{N}$, which contains $K_2(F)$ as a subgroup. The generalised Maslov index can be reduced to this subgroup, giving rise to a central extension of $SL(2, F)$ by $K_2(F)$ (the field $F$ is arbitrary). The cocycle of this extension coincides with Matsumoto's cocycle ([1], [6]). The Witt group (more precisely, its quotient) should be regarded as a "reduction of the group $\widetilde{N} \pmod 2$". In this work we give an analogous interpretation of the "reduction of $\widetilde{N} \pmod n$": it is related to a certain class of $\mathbf{Z}/n$-graded algebras, which generalise Clifford algebras.

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## 1. Maslov Index (after [4])

The Maslov index arises naturally in the study of invariants of systems of one-dimensional subspaces of a two-dimensional space. Fix the following notation: let $F$ be a field of characteristic $char(F) \neq 2$, $V = F^2$ a two-dimensional vector space over $F$, $\mathcal{B}$ a symplectic form on $V$, $G = SL(2, F) = \mathrm{Aut}(V, \mathcal{B})$. Denote by

$$T = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}, \qquad B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}, \qquad U = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\},$$

respectively, the standard maximal torus in $G$, the standard Borel subgroup and its unipotent radical.

As $\{\text{lines in } V\} = \mathbf{P}(V) = G/B$, we have

$$G\backslash\{\text{pairs of lines in } V\} = G\backslash(G/B \times G/B) = B\backslash G/B = W,$$

where $W$ is the Weyl group of $G$. The non-trivial element of $W$ corresponds to pairs of transversal lines.

Let us consider invariants of triples of lines. We first restrict our attention to the case of general position, when $l_0, l_1, l_2$ are pair-wise transversal lines. After applying suitable $g \in G$, we have

$$l_0 = \left\{ \begin{bmatrix} * \\ 0 \end{bmatrix} \right\}, \qquad l_1 = \left\{ \begin{bmatrix} 0 \\ * \end{bmatrix} \right\}, \qquad l_2 = \left\{ \begin{bmatrix} x \\ ax \end{bmatrix} \middle| \, x \in F \right\},$$

for some $a \in F^*$. The stabiliser of the pair $(l_0, l_1)$ is the torus $T$ and the action of an element $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in T$ changes the parameter $a$ to $at^2$. This yields an invariant

$$m : G\backslash\{\text{triples of lines in general position in } V\} \xrightarrow{\sim} F^*/F^{*2}$$
$$\left( \left\{ \begin{bmatrix} * \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ * \end{bmatrix} \right\}, \left\{ \begin{bmatrix} x \\ ax \end{bmatrix} \right\} \right) \mapsto a.$$

Let $l_0, l_1, l_2, l_3$ be four lines in general position. We can again assume that $l_0$ and $l_1$ are the axes of coordinates and that $l_2 = \left\{ \begin{bmatrix} x \\ ax \end{bmatrix} \right\}$, $l_3 = \left\{ \begin{bmatrix} x \\ bx \end{bmatrix} \right\}$, in which case

$$m_{012} = m(l_0, l_1, l_2) = a, \qquad m_{013} = b, \qquad m_{023} = \frac{ab}{a-b}, \qquad m_{123} = a - b,$$
$$m_{012}\, m_{013}^{-1}\, m_{023}\, m_{123}^{-1} = 1 \in F^*/F^{*2},$$

hence $m$ is a 2-cocycle. In fact, a stronger statement holds: let $\langle a \rangle$ (where $a \in F^*/F^{*2}$) be the class of the one-dimensional quadratic form $x \mapsto ax^2$ in the Witt ring $WF$ of the field $F$. We have the equality in $WF$

$$\langle a - b \rangle - \left\langle \frac{ab}{a-b} \right\rangle = \langle a \rangle - \langle b \rangle\,,$$

as

$$(a-b)X^2 - \frac{ab}{a-b}\,Y^2 = a\left(X - \frac{b}{a-b}\,Y\right)^2 - b\left(X - \frac{a}{a-b}\,Y\right)^2.$$

This identity implies that $m$ is a 2-cocycle, when considered as a function with values in $WF$. We extend the domain of definition of $m$ as follows: set $m(l_0, l_1, l_2) = 0 \in WF$ for any triple of lines $l_0, l_1, l_2$ not in a general position.

**Definition.** *The Maslov index is the above defined function* $\quad m : G\backslash\{\text{triples of lines in } V\} \longrightarrow WF.$

**Proposition 1.** *The Maslov index is a skew-symmetric 2-cocycle.*

*Proof.* The skew-symmetry follows from the definition. The cocycle relation has been verified in the case of general position; it is trivial in all other cases.

**Corollary.** *Fix a line* $l \in \mathbf{P}(V)$*. The formula* $\quad (g_1, g_2) \mapsto m(l, g_1 l, g_1 g_2 l) \quad$ *then defines a 2-cocycle on* $G$ *with values in* $WF$*, hence a central extension*

$$1 \longrightarrow WF \longrightarrow ? \longrightarrow G \longrightarrow 1.$$

## 2. Reduction of the Maslov Index

We shall reduce the Maslov index to a certain subgroup of $WF$. An **oriented line** is a pair $\widetilde{l} = (l, v)$, where $l \in \mathbf{P}(V)$ and $v \in (l - \{0\})/F^{*2}$. The space $\widetilde{\mathbf{P}}(V)$ of oriented lines is naturally identified with $G/\widetilde{B}$, where $\widetilde{B} = \left\{ \begin{pmatrix} a^2 & * \\ 0 & a^{-2} \end{pmatrix} \right\} \subset B$. As before, we have

$$G\backslash\{\text{pairs of oriented lines in } V\} = \widetilde{B}\backslash G/\widetilde{B}.$$

This set projects onto $B\backslash G/B = \mathbf{Z}/2$ with fibres $F^*/F^{*2}$ and admits a natural structure of an abelian group, which is closely related to the Witt ring; its definition follows.

Let us first recall basic facts about the structure of $WF$ (see [3]). Every quadratic space $Q$ has a well-defined dimension $\dim(Q) = n \in \mathbf{N}$ and discriminant $d(Q) \in F^*/F^{*2}$, but it is only $n \pmod 2$ and $d_\pm(Q) = (-1)^{n(n-1)/2} d(Q)$ that descend to the Witt ring. Denote by $IF$ the kernel of the homomorphism $\dim : WF \longrightarrow \mathbf{Z}/2$. As an abelian group, $IF$ is generated by the forms $\langle 1, -a \rangle$, that is by the norm forms attached to quadratic extensions. Set

$$QF = \{(e, a) \mid e \in \mathbf{Z}/2,\, a \in F^*/F^{*2}\}$$

with the abelian group law

$$(e, a) \cdot (e', a') = (e + e', (-1)^{ee'} aa').$$

**Proposition 2 (see [3]).** *The map $s = (\dim, d_\pm)$ induces an isomorphism of exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & IF/I^2F & \longrightarrow & WF/I^2F & \longrightarrow & WF/IF & \longrightarrow & 0 \\ & & \big\downarrow \wr\, d_\pm & & \big\downarrow \wr\, s & & \big\downarrow \wr\, \dim & & \\ 0 & \longrightarrow & F^*/F^{*2} & \longrightarrow & QF & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & 0. \end{array}$$

We define a lifting $\widetilde{\ } : QF \longrightarrow WF$ of the morphism $s$ by the formulas

$$\widetilde{(0,a)} = \langle 1, -a\rangle\,, \qquad \widetilde{(1,a)} = \langle a\rangle\,.$$

As a set, $QF$ has the same structure as $\widetilde{B}\backslash G/\widetilde{B}$, for the following reason: if we denote

$$\widetilde{T} = \left\{ \begin{pmatrix} a^2 & 0 \\ 0 & a^{-2} \end{pmatrix} \right\} \subset T, \qquad N = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\} \subset G,$$

then $\widetilde{B}\backslash G/\widetilde{B}$ is naturally identified with $N/\widetilde{T}$ and the formulas

$$\begin{pmatrix} 0 & A \\ -A^{-1} & 0 \end{pmatrix} \widetilde{T} \mapsto (1, A), \qquad \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \widetilde{T} \mapsto (0, A)$$

define an isomorphism $N/\widetilde{T} \xrightarrow{\sim} QF$. Putting everything together we obtain an identification

$$n : \{\text{pairs of oriented lines in } V\} = \widetilde{B}\backslash G/\widetilde{B} = N/\widetilde{T} \xrightarrow{\sim} QF$$

and, composing with the lifting $\widetilde{\ } : QF \longrightarrow WF$, a map

$$\widetilde{n} : \{\text{pairs of oriented lines in } V\} \longrightarrow WF.$$

An explicit formula for $n$: if $\widetilde{l}_i = (l_i, v_i)$, then

$$n_{01} = n(\widetilde{l}_0, \widetilde{l}_1) = \begin{cases} \left(0, \frac{v_1}{v_0}\right), & l_0 = l_1 \\ (1, -\mathcal{B}(v_0, v_1))\,, & l_0 \neq l_1. \end{cases}$$

**Proposition 3.** *For every triple of oriented lines $(\widetilde{l}_0, \widetilde{l}_1, \widetilde{l}_2)$ we have*

$$m_{012} = \widetilde{n}_{01} - \widetilde{n}_{02} + \widetilde{n}_{12} \pmod{I^2F}.$$

*Proof.* If the three lines are not in general position, then both sides vanish. In the case of general position we can assume that $v_0 = \begin{bmatrix} x \\ 0 \end{bmatrix}$, $v_1 = \begin{bmatrix} 0 \\ y \end{bmatrix}$, $v_2 = \begin{bmatrix} z \\ az \end{bmatrix}$. In this case we have

$$m_{012} - \widetilde{n}_{01} + \widetilde{n}_{02} - \widetilde{n}_{12} = \langle a, xy, -axz, -yz\rangle \in \operatorname{Ker}(\dim, d_\pm) = I^2F. \qquad \square$$

We define the **reduced Maslov index** of a triple $(\widetilde{l}_0, \widetilde{l}_1, \widetilde{l}_2)$ of oriented lines by the formula

$$\widetilde{m}_{012} = \widetilde{m}(\widetilde{l}_0, \widetilde{l}_1, \widetilde{l}_2) = m_{012} - \widetilde{n}_{01} + \widetilde{n}_{02} - \widetilde{n}_{12}.$$

**Corollary.** *$\widetilde{m} : G\backslash\{\text{triples of oriented lines in } V\} \longrightarrow I^2F$ is a 2-cocycle cohomologous to $m$ (more precisely, to the lift of $m$ via the projection $\widetilde{\mathbf{P}}(V) \longrightarrow \mathbf{P}(V)$).*

## 3. Relation to $K$-theory

Let us recall the structure of $I^2F$. As an abelian group, $I^2F$ is generated by the classes of forms $\langle 1, -a\rangle \otimes \langle 1, -b\rangle = \langle 1, -a, -b, ab\rangle$. Such forms are the reduced norms on quaternion algebras. Denote by $\left(\frac{a,b}{F}\right)_2$ the quaternion algebra

$$X^2 = a, \qquad Y^2 = b, \qquad YX = -XY$$

over $F$, and by $\mathrm{Quat}(F)$ the subgroup of the Brauer group of $F$ generated by the classes of quaternion algebras (of course, $\mathrm{Quat}(F) \subset \mathrm{Br}(F)_2$ is an abelian group of exponent 2). Let $\mathrm{BW}(F)$ be the Brauer-Wall group of the field $F$, i.e. the group of similitude classes of $\mathbf{Z}/2$-graded central simple algebras over $F$ (in the graded sense, see [3]). The abelian group law in $\mathrm{BW}(F)$ is given by the graded tensor product of algebras:

$$(a \widehat{\otimes} b)(a' \widehat{\otimes} b') = (-1)^{\deg(b)\deg(a')} aa' \otimes bb'.$$

**Proposition 4 (see [3]).** *The functor $C : Q \mapsto C(Q)$ which associates to each quadratic space its Clifford algebra induces a homomorphism of abelian groups $C : WF \longrightarrow \mathrm{BW}(F)$ with kernel $\mathrm{Ker}(C) = I^3F$ and a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I^2F/I^3F & \longrightarrow & WF/I^3F & \longrightarrow & WF/I^2F & \longrightarrow & 0 \\
 & & \big\downarrow \wr & & \big\downarrow \wr\, C & & \big\downarrow \wr\, s & & \\
0 & \longrightarrow & \mathrm{Quat}(F) & \longrightarrow & \mathrm{Clif}(F) & \longrightarrow & QF & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Br}(F) & \longrightarrow & \mathrm{BW}(F) & \longrightarrow & QF & \longrightarrow & 0,
\end{array}$$

*in which $\mathrm{Clif}(F)$ denotes the image of $C$ and the isomorphism $I^2F/I^3F \xrightarrow{\sim} \mathrm{Quat}(F)$ is given by the formula $\langle 1, -a\rangle \otimes \langle 1, -b\rangle \mapsto \left(\frac{a,b}{F}\right)_2$.*

According to Proposition 4, $\widetilde{m}$ induces a 2-cocycle

$$G\backslash\{\text{triples of oriented lines in } V\} \longrightarrow I^2F/I^3F \xrightarrow{\sim} \mathrm{Quat}(F).$$

Recall that the Milnor groups $K_n^M(F)$ are generated by the symbols $\{a_1, \ldots, a_n\}$ ($a_i \in F^*$) which are multiplicative in each argument and which satify the relation $\{\cdots, a, 1-a, \cdots\} = 1$.

**Proposition 5 (see [7]).** *The map*

$$\{a_1, \ldots, a_n\} \pmod 2 \mapsto \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle \pmod{I^{n+1}F}$$

*induces a surjective homomorphism of abelian groups $s_n : K_n^M(F)/2 \longrightarrow I^nF/I^{n+1}F$.*

Milnor conjectured that the maps $s_n$ are all isomorphisms. For $n = 0, 1$ this follows from Proposition 2. The Merkurjev-Suslin theorem [5] implies that the composite homomorphism

$$K_2(F)/2 = K_2^M(F)/2 \xrightarrow{s_2} I^2F/I^3F \hookrightarrow \mathrm{Br}(F)_2$$

is an isomorphism, i.e. $\mathrm{Quat}(F) = \mathrm{Br}(F)_2$.

## 4. Generalised Maslov Index

We are going to modify the above construction to obtain a 2-cocycle with values in $K_2(F)$. We replace $\widetilde{B}$ by the unipotent group $U$ and we consider the basic affine space $G/U = V - \{0\}$ of non-zero vectors in $V$. There is a canonical identification

$$n : G\backslash\{\text{pairs of non} - \text{zero vectors in } V\} = U\backslash G/U = N,$$

where $N$ is the normaliser of the torus $T$. This means that the group $QF$ should be replaced by its extension $N$. We must construct, however, an analogue of the group $\mathrm{BW}(F)$. The latter abelian group has the following structure.

**Proposition 6 (see [3]).** *The extension*

$$0 \longrightarrow \mathrm{Br}(F) \longrightarrow \mathrm{BW}(F) \longrightarrow QF \longrightarrow 0$$

*is given by the cocycle*

$$c'((0,a),(0,b)) = c'((1,a),(1,b)) = \left(\frac{a,b}{F}\right)_2, \qquad c'((0,a),(1,b)) = c'((1,b),(0,a)) = \left(\frac{a,-b}{F}\right)_2$$

**Definition of the group $\widetilde{N}$:** set

$$(0,A) = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}, \qquad (1,A) = \begin{pmatrix} 0 & A \\ -A^{-1} & 0 \end{pmatrix} \in N.$$

The formulas

$$c((0,A),(0,B)) = -c((1,A),(1,B)) = -\{A,B\}$$
$$c((0,A),(1,B)) = c((1,B),(0,A)) = -\{A,-B\}$$

define a 2-cocycle, which gives rise to a central extension

$$1 \longrightarrow K_2(F) \longrightarrow \widetilde{N} \longrightarrow N \longrightarrow 1$$

together with a section $\widetilde{\ } : N \longrightarrow \widetilde{N}$ satisfying $\widetilde{n} \cdot \widetilde{n}' = c(n,n')\, nn'$. According to Proposition 6, the subgroup $\mathrm{Clif}(F)$ of $\mathrm{BW}(F)$ is equal to "the reduction of $\widetilde{N} \pmod 2$", i.e. to the image of $\widetilde{N}$ via the homomorphisms $s_2 : K_2(F) \longrightarrow \mathrm{Quat}(F)$, $N \longrightarrow N/\widetilde{T} = QF$.
**Definition of the generalised Maslov index:** for $v_0, v_1, v_2 \in G/U$ set

$$m_{012} = m(v_0,v_1,v_2) = \left(\widetilde{n_{12} n_{02}^{-1} n_{01}}\right)^{-1} \in \widetilde{N} \qquad \text{(Maslov index)}$$
$$\widetilde{m}_{012} = \widetilde{n}_{12}(\widetilde{n}_{02})^{-1}\widetilde{n}_{01}\, m_{012} \in K_2(F) \qquad \text{(reduced Maslov index)}$$

Note that, for $n, n' \in N$, we have

$$m(U, nU, nn'U) = 1 \in \widetilde{N}, \qquad \widetilde{m}(U, nU, nn'U) = c(n,n'),$$

since $n_{01} = n$, $n_{02} = nn'$, $n_{12} = n'$ and $\widetilde{n} \cdot \widetilde{n}' = c(n,n')\, nn'$ (where $c$ is the cocycle for the extension $\widetilde{N}$).
**Formulas for the Maslov index:**
(1) $u, v$ linearly independent, $A, B \in F^*$: $\quad \widetilde{m}(u, v, Au + Bv) = \{A, B\}$.
(2) $u, v$ linearly independent, $A \in F^*$, $B = \mathcal{B}(u,v)$: $\quad \widetilde{m}(u, Au, v) = -\widetilde{m}(u, v, Au) = \widetilde{m}(v, u, Au) = \{A, -B\}$.
(3) $A, B \in F^*$: $\quad \widetilde{m}(u, Au, Bu) = \{-B, A\}$.

These formulas follow directly from the definitions. Recall that the function

$$n : G \backslash \{\text{pairs of non} - \text{zero vectors in } V\} \longrightarrow N$$

is given by the formulas

$$n(u, Au) = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = (0,A) \in N, \qquad n(u,v) = \begin{pmatrix} 0 & -B^{-1} \\ B & 0 \end{pmatrix} = (1, -B^{-1}) \in N.$$

We set, using the previous notation, $x(u, Au) = A^{-1}$, $x(u,v) = B$.

**Proposition 7.** *For any triple of non-zero vectors $v_0, v_1, v_2$ in $V$ we have*

$$\widetilde{m}_{012} = \left\{ \frac{x(v_0, v_1)}{x(v_0, v_2)}, \frac{x(v_1, v_2)}{x(v_0, v_2)} \right\} = \left\{ \frac{x_{01}}{x_{02}}, \frac{x_{12}}{x_{02}} \right\}.$$

*Proof.* In the case of general position we use the identity

$$\widetilde{(1,A)}\widetilde{(1,B)}^{-1}\widetilde{(1,C)} = \widetilde{\left(1, \frac{AC}{B}\right)} \left\{ \frac{C}{B}, \frac{A}{B} \right\}$$

for $A = -x_{12}^{-1}$, $B = -x_{02}^{-1}$, $C = -x_{01}^{-1}$. The remaining cases are similar.

**Theorem 1.** *The map* $\widetilde{m} : G\backslash\{\text{triples of non} - \text{zero vectors in } V\} \longrightarrow K_2(F)$ *is a 2-cocycle.*

*Proof.* The function $\widetilde{m}$ is almost skew-symmetric: it changes sign if we exchange two arguments in general position; in the case of exchanging linearly dependent vectors $u, Au$ one has to add an additional term $\{A, -1\}$. It follows that the coboundary

$$(\delta\widetilde{m})_{0123} = \widetilde{m}_{012}\, \widetilde{m}_{013}^{-1}\, \widetilde{m}_{023}\, \widetilde{m}_{123}^{-1}$$

is fully skew-symmetric and that it is enough to consider only the case of linearly dependent $v_0, v_1$ and the case of four vectors in general position.

If $v_0 = Av_1$, then $x_{01} = x_{02}x_{12}^{-1} = x_{03}x_{13}^{-1} = A$ and

$$\widetilde{m}_{012}\, \widetilde{m}_{013}^{-1} = \{A, x_{03}x_{02}^{-1}\}, \qquad \widetilde{m}_{012}\, \widetilde{m}_{013}^{-1}\, \widetilde{m}_{023} = \{x_{23}x_{03}^{-1}A, x_{03}x_{02}^{-1}\} = \{x_{23}x_{13}^{-1}, x_{13}x_{12}^{-1}\} = \widetilde{m}_{123}.$$

In the case of general position the function $\widetilde{m}$ is $GL(2)$-invariant, so we can assume that $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} A \\ B \end{bmatrix}$, $v_3 = \begin{bmatrix} C \\ D \end{bmatrix}$. Consider a new quadruple of vectors $v_0' = v_0$, $v_1' = v_1$, $v_2' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_3' = \begin{bmatrix} AC^{-1} \\ BD^{-1} \end{bmatrix}$. Multiplicativity of $\{\ ,\ \}$ implies that $(\delta\widetilde{m})(v_0, v_1, v_2, v_3) = (\delta\widetilde{m})(v_0', v_1', v_2', v_3')$. However, for the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} X \\ Y \end{bmatrix}$ we have

$$\widetilde{m}_{012} = 1, \qquad \widetilde{m}_{013} = \{X, Y\}, \qquad \widetilde{m}_{023} = \{X - Y, Y\}, \qquad \widetilde{m}_{123} = \{Y - X, X\},$$

$$(\delta\widetilde{m})_{0123} = \left\{ 1 - \frac{Y}{X}, \frac{Y}{X} \right\} = 1. \qquad \square$$

Note that we had not used the Steinberg relation $\{A, 1 - A\} = 1$ before in its full force, only its consequence $\{A, -A\} = 1$. The relation itself can be reconstructed during the construction: one requires that $\widetilde{m}(v_0, v_1, v_2) = 1$ for all triples of vectors whose endpoints are collinear.

The above construction then has to be modified as follows.

(1) Suppose that we are given a bimultiplicative function $[\ ,\ ] : F^* \otimes F^* \longrightarrow A$ with values in an abelian group $A$, which satisfies the relation $[X, -X] = 0$.

(2) Using the function $[\ ,\ ]$ instead of $\{\ ,\ \}$, define a central extension $1 \longrightarrow A \longrightarrow \widetilde{N} \longrightarrow N \longrightarrow 1$ and a lifting $\widetilde{\ } : N \longrightarrow \widetilde{N}$ using the same formulas as before.

(3) For $v_0, v_1, v_2 \in G/U$ define

$$\widetilde{m}_{012} = \widetilde{n}_{12}(\widetilde{n}_{02})^{-1}\widetilde{n}_{01} \left( \widetilde{n_{12}n_{02}^{-1}n_{01}} \right)^{-1} \in A.$$

(4) Let $X = \{(v_0, v_1, v_2) \in (G/U)^3 \mid \exists g \in G, \exists u_{ij} \in gUg^{-1} \;\; v_i = u_{ij} v_j\}$ be the set of the triples of vectors whose endpoints are collinear. We divide $A$ by the Steinberg relation: let $\overline{A} = A/$(subgroup generated by $\widetilde{m}_{012}(X)$).

(5) The induced function $\overline{m} : G\backslash(G/U \times G/U \times G/U) \longrightarrow \overline{A}$ then turns out to be a 2-cocycle.

It is natural to ask whether the construction (1)–(5) can be applied to the groups $SL(n, F)$ as an alternative to Matsumoto's construction ([1], [6]). According to Proposition 7, the cocycle $\widetilde{m}$ coincides with Matsumoto's cocycle for $SL(2)$ in Kubota's form ([1], [2]). For $n > 2$, however, the function $\widetilde{m}$ defined according to (1)–(4) will coincide with Matsumoto's cocycle only on those triples of elements of $G/U$ which are "not in a very general position".

## 5. $\mathbf{Z}/n$-graded Clifford Algebras

The construction of the generalized Maslov index $\widetilde{m}$ with values in $K_2(F)$ implies that the corresponding central extension

$$1 \longrightarrow K_2(F) \longrightarrow ? \longrightarrow SL(2, F) \longrightarrow 1$$

is mapped via the morphism $K_2(F) \longrightarrow K_2(F)/2 \xrightarrow{s_2} \mathrm{Quat}(F)$ to the extension with kernel $\mathrm{Quat}(F) \xrightarrow{\sim} I^2F/I^3F$, which is given by the usual (reduced) Maslov index.

We are now going to give an interpretation of the objects related to $K_2(F)/n$ for $n > 2$. Even though the definition of "the Witt ring of forms of degree $n$" is not known in this case, there exists a natural analogue of the group $WF/I^3F$, namely the group of generalised Clifford algebras defined below.

Let $F$ be a field of characteristic prime to $n$, which contains the group $\mu_n$ of the $n$-th roots of unity. Fix once for all a primitive root of unity $\zeta \in \mu_n$. Any ordered set of elements $a_1, \ldots, a_N \in F^*$ determines an $F$-algebra $A = \langle a_1, \ldots, a_N\rangle$ with generators $X_1, \ldots, X_N$ and relations

$$X_i^n = a_i, \qquad X_j X_i = \zeta X_i X_j \qquad\qquad (i < j).$$

The algebra $A$ has a natural $\mathbf{Z}/n$-grading, for which $\deg(X_i) = 1$ for all $i$.

We define the graded tensor product $A \widehat{\otimes} B$ of two $\mathbf{Z}/n$-graded algebras $A, B$ as follows: as a vector space it coincides with $A \otimes B$ and the multiplication is defined by the formula

$$(a \otimes b)\,(a' \otimes b') = \zeta^{\deg(b)\deg(a')} aa' \otimes bb'.$$

We have $\langle a_1, \ldots, a_N\rangle = \langle a_1\rangle \,\widehat{\otimes} \cdots \widehat{\otimes}\, \langle a_N\rangle$; in particular, $\dim \langle a_1, \ldots, a_N\rangle = n^N$.

The algebra $\langle a, b\rangle$ is the standard cyclic central simple algebra over $F$. Its class $(a, b)_{n,\zeta}$ in the Brauer group $\mathrm{Br}(F)$ depends on the choice of $\zeta$, but

$$(a, b)_n = (a, b)_{n,\zeta} \otimes \zeta \in \mathrm{Br}(F) \otimes \mu_n$$

does not depend on $\zeta$. The map $\{a, b\} \mapsto (a, b)_n$ defines a homomorphism $K_2(F) \longrightarrow \mathrm{Br}(F) \otimes \mu_n$; denote by $\mathrm{Cyc}_n(F)$ its image.

Before we start developing structure theory of the algebras $\langle a_1, \ldots, a_N\rangle$, recall an elementary lemma:

**Lemma.** *Let $A$ be a central simple algebra over $F$. If $A = \oplus A_i$ admits a $\mathbf{Z}/n$-grading, then there exists an invertible element $z \in A_0$ such that*

$$A_i = \{a \in A \mid az = \zeta^i za\}.$$

*The element $z$ is determined uniquely up to a scalar multiple and $z^n = F^*$.*

*Proof.* The automorphism $f(a) = \zeta^{-\deg(a)} a$ of $A$ is inner, by the Skolem-Noether theorem: $f(a) = zaz^{-1}$. As $f^n = \mathrm{id}$, $z^n$ is contained in the centre $F$ of $A$, which also contains the ambiguity of the choice of $z$. □

Denote by $d(A)$ the image of $z^n$ in $F^*/F^{*n}$; it is uniquely determined by the $\mathbf{Z}/n$-grading. In other words, the centraliser $Z_A(A_0)$ of the subalgebra $A_0$ in $A$ is isomorphic to $\langle d(A)\rangle$.

For any $F$-algebra $A$ let $(A)$ be the $\mathbf{Z}/n$-graded algebra with $(A)_0 = A$ and $(A)_i = 0$ $(i \neq 0)$.

**Proposition 8.** *Let* $A = \langle a_1, \dots, a_N \rangle$.
(1) *If* $N \equiv 1 \pmod 2$, *then* $A \xrightarrow{\sim} (A_0)\widehat{\otimes}\langle d\rangle = (A_0) \otimes \langle d \rangle$, $A_0$ *is a central simple algebra over* $F$, $[A_0] \otimes \zeta \in \mathrm{Cyc}_n(F)$ *and* $Z_A(A_0) = \langle d \rangle$.
(2) *If* $N \equiv 0 \pmod 2$, *then* $A$ *is a central simple algebra over* $F$, $[A] \otimes \zeta \in \mathrm{Cyc}_n(F)$ *and* $Z_A(A_0) = \langle d(A) \rangle$.

**Remarks.** The usual tensor product $A \otimes B$ denotes the algebra with the usual multiplication

$$(a \otimes b)\,(a' \otimes b') = aa' \otimes bb'.$$

The symbol $[A]$ denotes the class of a central simple algebra $A$ in the Brauer group $\mathrm{Br}(F)$.

*Proof.* We use induction on $N$. The case $N = 1$ is trivial. We perform the induction step $N \mapsto N + 1$.
Case (1): $N \equiv 1 \pmod 2$. In this case

$$A \widehat{\otimes} \langle b \rangle = (A_0) \otimes (\langle d \rangle \,\widehat{\otimes}\, \langle b \rangle) = (A_0) \otimes \langle d, b \rangle$$

is indeed a central simple algebra over $F$ whose class lies in $\mathrm{Cyc}_n(F)$.
Case (2): $N \equiv 0 \pmod 2$. Let $x \in \langle b \rangle_1$ be a element satisfying $x^n = b$ and let $z \in A_0$ be an element defining the grading, as in Lemma. Set $B = A\widehat{\otimes}\langle b\rangle$. Then $y = z \otimes x$ generates the centre $Z(B) \xrightarrow{\sim} \langle d \rangle$ of the algebra $B$, as $y^n = z^n x^n = d(A)b$. It follows that $B = (B_0) \otimes \langle d \rangle$ and $Z_B(B_0) = \langle d \rangle$. By the induction hypothesis, we have $A = C \otimes \langle u, v \rangle$, where $C \subset A_0$ and $[C] \otimes \zeta \in \mathrm{Cyc}_n(F)$, so it is enough to consider the case $A = \langle u, v \rangle$. The algebra $A$ is generated by two elements $X, Y$ satisfying $X^n = u$, $Y^n = v$, $YX = \zeta XY$; then $\overline{X} = -X \otimes x^{-1}$ and $\overline{Y} = -Y \otimes x^{-1}$ generate $B_0$, $\overline{YX} = \zeta\overline{XY}$, $\overline{X}^n = -ub^{-1}$ and $\overline{Y}^n = -vb^{-1}$, hence $B_0 = \langle -ub^{-1}, -vb^{-1} \rangle$ and $[B_0] \otimes \zeta \in \mathrm{Cyc}_n(F)$, as claimed. □

As a consequence, we obtain that each algebra $A = \langle a_1, \dots, a_N \rangle$ determines a triple of invariants

$$C(A) = (e(A), d(A), D) \in \mathbf{Z}/2 \times F^*/F^{*n} \times \mathrm{Cyc}_n(F)$$
$$e(A) = N \pmod 2, \qquad \langle d(A) \rangle = Z_A(A_0), \qquad D = \begin{cases} [A] \otimes \zeta, & N \equiv 0 \pmod 2 \\ [A_0] \otimes \zeta, & N \equiv 1 \pmod 2 \end{cases}$$

The set $S$ of all algebras $\langle a_1, \dots, a_N \rangle$ forms a semi-group with respect to the graded tensor product $\widehat{\otimes}$. The set of triples $(e, d, D)$ inherits the group structure from $\widetilde{N}$ via the homomorphisms $(\ ,\ )_n : K_2(F) \longrightarrow \mathrm{Cyc}_n(F)$ and $F^* \longrightarrow F^*/F^{*n}$:

$$\begin{aligned}
(0, d, D) \cdot (0, d', D') &= \big(0, dd', DD'(d, d')_n^{-1}\big)\\
(0, d, D) \cdot (1, d', D') &= \big(1, dd', DD'(d, -d')_n^{-1}\big)\\
(1, d, D) \cdot (0, d', D') &= \big(1, dd'^{-1}, DD'(-d, d')_n\big)\\
(1, d, D) \cdot (1, d', D') &= \big(0, -dd'^{-1}, DD'(d, d')_n\big)\,.
\end{aligned}$$

The signs in the definition of the cocycle $c$ have been chosen in order to make the following statement hold.

**Proposition 9.** *For* $A, B \in S$ *we have:*
(1) $C(\langle a, b \rangle) = (0, -ab^{-1}, (a, b)_n)$.
(2) $C(A\widehat{\otimes}B) = C(A) \cdot C(B)$.
(3) $C(A) = 1 \iff$ *there exists a* $\mathbf{Z}/n$*-graded vector space* $V$ *and an isomorphism* $A \xrightarrow{\sim} \mathrm{End}(V)$ *(the grading on* $\mathrm{End}(V)$ *is defined by the formula* $\mathrm{Hom}(V_i, V_j) \subset (\mathrm{End}(V))_{j-i}$*).*
(4) *For* $A = \langle a_1, \dots, a_N \rangle$ *put* $A^\circ = \langle -a_N, \dots, -a_1 \rangle$. *Then* $C(A) \cdot C(A^\circ) = C(A^\circ) \cdot C(A) = 1$.

*Proof.* (1) The grading is defined by the element $z = -XY^{-1}$, where $X^n = a$, $Y^n = b$, $YX = \zeta XY$, hence $z^n = -X^nY^{-n} = -ab^{-1}$.
(2) Thanks to the associativity of both products, it is enough to consider the case $A = \langle a_1, \dots, a_N \rangle$, $B = \langle b \rangle$.
Case (1): $N \equiv 1 \pmod 2$. In this case $C(A) = (1, d(A), [A_0] \otimes \zeta)$ and, by (1) and the proof of Proposition 8,

$$C(A\widehat{\otimes}\langle b \rangle) = (0, d(A'), ([A_0] \otimes \zeta)(d(A), b)_n),$$

where $A' = \langle d(A), b\rangle$.
Case (2): $N \equiv 0 \pmod 2$. Using again the proof of Proposition 8, it is enough to consider the case $N = 2$, $A = \langle u, v\rangle$, when we have $d(A\widehat{\otimes}B) = d(A)d(B)$ and

$$[(A\widehat{\otimes}B)_0] \otimes \zeta = (-ub^{-1}, -vb^{-1})_n = (u,v)_n(-uv^{-1}, -b)_n^{-1} = ([A] \otimes \zeta)\,(d(A), -d(B))_n^{-1}.$$

(3) If $A \xrightarrow{\sim} \mathrm{End}(V)$, then $Z(A) = F \Longrightarrow N \equiv 0 \pmod 2$ and $A$ is a central simple algebra over $F$ with a trivial class in $\mathrm{Br}(F)$. The grading is defined by the element $z = \sum \zeta^i p_i$, where $p_i : V \longrightarrow V_i$ is the natural projector, hence $z^n = 1$. Conversely, if $C(A) = 1$, then $N \equiv 0 \pmod 2$, $A \xrightarrow{\sim} \mathrm{End}(V)$ in the non-graded sense and the grading is defined by an element $z$ satisfying $z^n = 1$. As $\mu_n \subset F$, $z$ must be of the form $\sum \zeta^i p_i$, where $p_i$ are projectors.
(4) It is enough to consider the case $N = 1$; the statement then follows from (1) and (2). □

We say that two algebras $A, B \in S$ are **similar** if there exist $\mathbf{Z}/n$-graded vector spaces $V, W$ and an isomorphism $A\widehat{\otimes}\mathrm{End}(V) \xrightarrow{\sim} B\widehat{\otimes}\mathrm{End}(W)$. According to Proposition 9, the similitude classes of the algebras $\langle a_1, \ldots, a_N\rangle$ form a group with respect to $\widehat{\otimes}$, which will be denoted by $C_n(F)$ ("the Clifford algebras of degree $n$"). It is a natural generalisation of the group of Clifford algebras $\mathrm{Clif}(F) \subset \mathrm{BW}(F)$.

Putting everything together, we obtain the following statement.

**Theorem 2.** *There exists an exact sequence*

$$1 \longrightarrow \mathrm{Cyc}_n(F) \longrightarrow C_n(F) \longrightarrow Q_n(F) \longrightarrow 1,$$

*where $Q_n(F)$ is the group of pairs $(e, a)$, $e \in \mathbf{Z}/2$, $a \in F^*/F^{*n}$ with multiplication*

$$(e,a)\cdot(e',a') = \left(e + e', (-1)^{ee'} a(a')^{(-1)^a}\right).$$

As a final remark, we note that $\mathrm{Cyc}_n(F) = \mathrm{Br}(F)_n \otimes \mu_n$, according to [5].

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