BEILINSON'S CONJECTURES

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Abstract. We give a survey of Beilinson's conjectures about special values of L-functions, with emphasis on the underlying philosophy of mixed motives and motivic cohomology.

Introduction.

In his seminal paper [1], A.A. Beilinson formulated far reaching conjectures about values of motivic L-functions at integers, and produced a compelling body of evidence in their favour by making ingenious calculations in several special cases. The main gist of [1] was a construction of "higher regulators", expected to explain these L-values in the same spirit as the (slightly modified) classical Dirichlet regulator

$$r : \mathcal{O}_F^* \oplus \mathbf{Z} \longrightarrow \mathbf{R}^{r_1+r_2}$$

does for the zeta function of a number field F at $s = 0$ (resp. $s = 1$). In this case, $\zeta_F(s)$ satisfies a functional equation relating its values at s and $1-s$. It has a simple pole at $s = 1$ and a zero of order $r_1 + r_2 - 1$ at $s = 0$. Its leading Taylor coefficient at $s = 0$ is equal to

$$\lim_{s \longrightarrow 0} \zeta_F(s) s^{-(r_1+r_2-1)} = -\frac{\sharp \mathrm{Pic}(\mathcal{O}_F) \cdot R}{\sharp (\mathcal{O}_F^*)_{tors}}, \tag{0.1}$$

where R denotes the covolume of the lattice $\mathrm{Im}(r)$ in $\mathbf{R}^{r_1+r_2}$.

The quest for higher regulators, extending (0.1) to other values of $\zeta_F(s)$, has been initiated by Lichtenbaum [47]. He observed that for $m > 1$, the order d_m of vanishing of $\zeta_F(s)$ at $s = 1 - m$ is equal to the dimension of the higher K-group $K_{2m-1}(F) \otimes \mathbf{Q}$. This led to a conjecture that the leading coefficient

$$\lim_{s \longrightarrow 1-m} \zeta_F(s)(s + m - 1)^{-d_m}$$

should be equal, up to a rational factor, to the covolume of $\mathrm{Im}(r_m)$ for a certain map

$$r_m : K_{2m-1}(F) \longrightarrow \mathbf{R}^{d_m}$$

1991 *Mathematics Subject Classification.* Primary 19F27, 11G40, 14A20; Secondary 14C35, 14G10.

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This conjecture has been proved by Borel [14],[15], for a slightly modified version of the regulator map r_m.

The next step was taken by Bloch [6],[7], who defined a regulator

$$r : K_2(E) \longrightarrow H^1(E(\mathbf{C}), \mathbf{R})$$

for elliptic curves $E_{/\mathbf{C}}$ and verified that r computes the value $L(E, 2)$ for curves with complex multiplication defined over $\mathbf{Q}$. Later, Bloch [8] defined a regulator

$$K_2(X) \longrightarrow H^1(X(\mathbf{C}), \mathbf{C}^*)$$

for any curve $X_{/\mathbf{C}}$.

Beilinson defines, for a given quasi-projective variety $X_{/\mathbf{Q}}$, its motivic cohomology $H^{\cdot}_{\mathcal{M}}(X, \mathbf{Q}(*))$ as a suitable piece of K-theory of X. Let X be a smooth projective variety over $\mathbf{Q}$. The L-function $L(h^i(X), s)$, associated to the i^{th} cohomology of X, is expected to satisfy a functional equation relating its values at s and $i + 1 - s$. Suppose that n is an integer greater than $1 + i/2$. Beilinson defines a regulator map

$$r : H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(n)) \longrightarrow H^{i+1}_{\mathcal{D}}(X_{/\mathbf{R}}, \mathbf{R}(n)) \tag{0.2}$$

from the motivic cohomology of X into Deligne cohomology. The significance of the latter lies in the fact that its dimension is equal to the order of vanishing of $L(h^i(X), s)$ at $i + 1 - n$. Deligne cohomology also admits a natural $\mathbf{Q}$-structure and Beilinson conjectures that $\det(r) \in \mathbf{R}^*/\mathbf{Q}^*$ with respect to this $\mathbf{Q}$-structure is equal to the leading coefficient of the L-function at $i + 1 - n$. At the central point $n = (i+1)/2$ resp. near central point $n = i/2+1$ the conjecture has to be modified.

In a letter to Soulé [22], Deligne suggested a motivic formulation of the regulator 0.2, writing it as a 'Hodge realization'

$$r : \mathrm{Ext}^1(\mathbf{Q}(0), M) \longrightarrow \mathrm{Ext}^1(\mathbf{R}(0), M_{Hodge}),$$

where the first Ext is the group of "motivic extensions" of the trivial motive by $M = h^i(X)(n)$ and the second one is the group of extensions in a suitable category of Hodge structures. Deligne also introduced another $\mathbf{Q}$-structure on the target of r, related to the value of $L(h^i(X), s)$ at $s = n$.

In [3],[4], Beilinson followed this suggestion and put his conjectures about special values of L-functions into a general perspective of mixed motives and motivic sheaves. In this context, $H^i_{\mathcal{M}}(X, \mathbf{Q}(j))$ are expected to form a universal 'absolute', or 'arithmetic', cohomology theory of X, as opposed to Grothendieck's $h^i(X)$, which should provide only 'geometric' information about $X \times_{\mathbf{Q}} \overline{\mathbf{Q}}$.

Following this motivic thread, Scholl [63] proposed a unified formulation of Beilinson's conjectures at all integers (including the central and near central points) as Deligne's conjecture [21] for critical mixed motives.

In a separate development, Bloch and Kato [12] formulated a conjecture about the precise value of $L(h^i(X), n)$, eliminating the undetermined rational factor in Beilinson's approach. Recently, Fontaine and Perrin-Riou [32], [33] found a common generalization of the conjectures of Bloch-Kato and Scholl.

Apart from Beilinson's original papers [1]–[4], there exist excellent surveys of his conjectures [29],[53],[56],[61], [67]. For geometric aspects of the conjectures, [43] is indispensable.

This survey attempts to explain not only the K-theoretic formulation of the conjectures, but also the underlying motivic intuition. The reader will have no trouble in distinguishing real mathematical statements from a mere wishful thinking (which prevails) simply by counting frequency of expressions "should", "is expected" and the like.

1. Pure motives and realizations.

(1.1) We first recall basic notions of (pure) motives. To a smooth projective variety X, defined over a number field K, and integers $i \geq 0$, $n \in \mathbf{Z}$, one hopes to associate a 'motive' $M = h^i(X)(n)$ (pure of weight $w = i - 2n$), being a universal cohomology group of X. In Sec. 1–2, M will intervene only through its realizations, namely

- étale realizations (for each prime number ℓ)

$$M_\ell = H^i((X \times_K \overline{K})_{et}, \mathbf{Q}_\ell)(n)$$

— a finite dimensional ℓ-adic (continuous) representation of $G(\overline{K}/K)$, pure of weight w. The last condition has the following meaning: there is a finite set S of places at which X has bad reduction. If $v \notin S$ is nonarchimedean and prime to ℓ, then M_ℓ is unramified at v and all eigenvalues of the geometric Frobenius Fr_v on M_ℓ are algebraic numbers with absolute value $(Nv)^{w/2}$ (by [20]).

- Betti realizations (for each embedding $\sigma : K \hookrightarrow \mathbf{C}$)

$$M_{\sigma,B} = H^i((X \times_{K,\sigma} \mathbf{C})(\mathbf{C}), \mathbf{Q}(n))$$

— a pure $\mathbf{Q}$-Hodge structure of weight w. If σ is a real embedding, then the action of the complex conjugation $c \in G(\mathbf{C}/\mathbf{R})$ on both $(X \times_{K,\sigma} \mathbf{C})(\mathbf{C})$ and $\mathbf{Q}(n) = (2\pi i)^n \mathbf{Q}$ induces an involution ϕ_σ on $M_{\sigma,B}$ such that $\phi_\sigma \otimes c$ preserves the Hodge decomposition

$$M_{\sigma,B} \otimes_{\mathbf{Q}} \mathbf{C} = \bigoplus_{p+q=w} H^{p,q}.$$

- de Rham realization

$$M_{dR} = H^i(X_{Zar}, \Omega^{\cdot}_{X/K})(n)$$

— a finite dimensional K-vector space with a decreasing filtration

$$F^k M_{dR} = H^i(X_{Zar}, \Omega^{\geq k+n}_{X/K}).$$

(1.2) There are standard comparison isomorphisms between different realizations:

- $I_\sigma : M_{\sigma,B} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} M_{dR} \otimes_{K,\sigma} \mathbf{C}$

("de Rham theorem"), under which $(F^k M_{dR}) \otimes \mathbf{C}$ corresponds to $\bigoplus_{p \geq k} H^{p,q}$. If σ is a real embedding, then $\phi_\sigma \otimes c$ corresponds to the action of $1 \otimes c$ on the R.H.S.

- $I_{\ell,\overline{\sigma}} : M_{\sigma,B} \otimes_{\mathbf{Q}} \mathbf{Q}_\ell \xrightarrow{\sim} M_\ell$

(depending on the choice of an extension $\overline{\sigma} : \overline{K} \hookrightarrow \mathbf{C}$ of σ). For a real embedding σ, the action of $\phi_\sigma \otimes 1$ corresponds to the automorphism $\overline{\sigma}^*(c) \in G(\overline{K}/K)$ induced on $\overline{K}$ by c (via $\overline{\sigma}$).

The common dimension

$$\mathrm{rk}(M) := \dim_{\mathbf{Q}}(M_{\sigma,B}) = \dim_K(M_{dR}) = \dim_{\mathbf{Q}_\ell}(M_\ell)$$

is called the rank of M (over K).

(1.3) All constructions of linear algebra (duals, tensor products, ...) apply to M, at least on the level of realizations. In particular,

$$M = h^i(X) \otimes \mathbf{Q}(1)^{\otimes n},$$

where the Tate motive $\mathbf{Q}(1) = h^2(\mathbf{P}^1_K)^\vee$ has realizations

$$\begin{aligned} \mathbf{Q}(1)_\ell &= \varprojlim_k (\mu_{\ell^k}(\overline{K})) \otimes \mathbf{Q} \\ \mathbf{Q}(1)_B &= (2\pi i)\mathbf{Q}, \quad \text{Hodge type } (-1,-1),\ \phi_\sigma = -1 \\ \mathbf{Q}(1)_{dR} &= K, \quad F^0 = 0,\ F^{-1} = K \end{aligned}$$

By Poincaré duality and hard Lefschetz theorem (true in all realizations), we have

$$\begin{aligned} M^\vee(1) &= h^i(X)^\vee(1-n) = h^{2d-i}(X)(d+1-n) \\ &= h^i(X)(i+1-n) = M(w+1), \end{aligned} \tag{1.3.1}$$

where d is the dimension of X and $w = i - 2n$ the weight of M.

There is also an operation of restriction of scalars for motives:

$$R_{K/\mathbf{Q}}(M) = h^i(X_{/\mathbf{Q}})(n),$$

where $X_{/\mathbf{Q}}$ is X viewed as a $\mathrm{Spec}(\mathbf{Q})$-scheme.

(1.4) The local L-factor of M at a non-archimedean place v of K is defined as

$$L_v(M,s) = \det(1 - Fr_v(Nv)^{-s} \mid M_\ell^{I_v})^{-1} \qquad (\ell \nmid Nv)$$

(where I_v is the inertia group of v), conjecturally independent of ℓ. For $v \notin S$, $L_v(M,s)$ is indeed independent of ℓ by [20] and all its poles have real part equal to $\mathrm{Re}(s) = w/2$. For $v \in S$, independence of ℓ is not known; if true, purity conjecture for monodromy filtration then predicts that all poles of $L_v(M,s)$ have real part $\mathrm{Re}(s) = w/2, (w-1)/2, \dots, (w-i)/2$ (see [40]).

Local L-factors satisfy the following relations:

$$\begin{aligned} L_v(M(m),s) &= L_v(M,s+m) \\ L_p(R_{K/\mathbf{Q}}(M),s) &= \prod_{v \mid p} L_v(M,s) \\ L_v(M_1 \oplus M_2) &= L_v(M_1,s) L_v(M_2,s) \end{aligned} \tag{1.4.1}$$

For an archimedean place v, corresponding to an embedding $\sigma : K \hookrightarrow \mathbf{C}$, $L_v(M,s)$ depends only on the real Hodge structure $H = M_{\sigma,B} \otimes_{\mathbf{Q}} \mathbf{R}$. The relations (1.4.1) are satisfied for archimedean places as well, and they determine $L_v(M,s)$ for all M, once they are known for three basic Hodge structures (for a real place v):

For M of rank 2 with H of Hodge type $(k,0)+(0,k)$ with $k > 0$, $L_v(M,s) = \Gamma_{\mathbf{C}}(s)$; for M of rank 1 with H of Hodge type $(0,0)$, $L_v(M,s) = \Gamma_{\mathbf{R}}(s)$ (resp. $\Gamma_{\mathbf{R}}(s+1)$), if ϕ_σ acts on H by $+1$ (resp. -1). Here we use the standard notation

$$\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2), \quad \Gamma_{\mathbf{C}}(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s).$$

The total L-factor at infinity will be denoted by

$$L_\infty(M,s) = \prod_{v|\infty} L_v(M,s).$$

See [26],[27],[28] for a unified definition of L-factors at all places.

(1.5) Assuming that L-factors at places of bad reduction are defined and behave as expected (i.e. the location of their poles is that predicted by the purity conjecture), then the L-function

$$L(M,s) = \prod_{v\not|\infty} L_v(M,s),$$

apriori only a formal Dirichlet series with rational coefficients, is absolutely convergent (and without zeroes) for $\mathrm{Re}(s) > 1 + w/2$.
The full L-function

$$\Lambda(M,s) = L_\infty(M,s)L(M,s)$$

conjecturally satisfies a functional equation

$$\Lambda(M,s) = \varepsilon(M,s)\Lambda(M^\vee(1),-s)$$

with an ε-factor of the form $\varepsilon(M,s) = a \cdot b^s$ (see [28]). We have, of course,

$$\Lambda(M^\vee(1),-s) = \Lambda(M,w+1-s)$$

by (1.3.1) and (1.4.1), so the functional equation becomes

$$\Lambda(M,s) = \varepsilon(M,s)\Lambda(w+1-s) \tag{1.4.2}$$

Suppose that we are interested in the behaviour of $L(h^i(X),s)$ at an integer $s=n$. Restricting the scalars to $\mathbf{Q}$, applying a Tate twist by $\mathbf{Q}(n)$ and using (1.4.1), we may assume that $K = \mathbf{Q}$ and that the point of our interest is $s = 0$. Using the functional equation (1.4.2) (which we *assume* to hold), it is sufficient to treat only the case when $s = 0$ lies to the right of the central point $(w+1)/2$ (or coincides with it), which happens iff $w \leq -1$. In fact,

$$w = -1 \iff s = 0 \text{ is the central value } (w+1)/2$$
$$w = -2 \iff s = 0 \text{ is the near central value } w/2+1$$
$$w \leq -3 \iff s = 0 \text{ is in the convergence region}$$

From now on, we shall assume that X is a smooth projective variety over $\mathbf{Q}$ and that $M = h^i(X)(n)$ is of weight $w = i - 2n \leq -1$. Since $\mathbf{Q}$ admits only one embedding $\infty : \mathbf{Q} \hookrightarrow \mathbf{C}$, we shall often drop it from the notation.

(1.6) An easy calculation, based on basic properties of the Gamma function, shows that

$$
\begin{aligned}
-\operatorname{ord}_{s=0} L_\infty(M^\vee(1),s) &= \sum_{0>p>q} \dim(H^{p,q})\big(+\dim(H^{w/2,w/2})^-\big)\\
&= \dim_K(M_{dR}/F^0) - \dim_{\mathbf{Q}}(M_B^+) \qquad (1.6.1)\\
\operatorname{ord}_{s=0} L_\infty(M,s) &= 0
\end{aligned}
$$

(where the $\pm$ superscript denotes the (± 1)-eigenspace for ϕ_∞)
and that leading terms of the Taylor expansions of L_∞ at $s=0$ satisfy

$$
\frac{L_\infty^*(M,0)}{L_\infty^*(M^\vee(1),0)} \in (2\pi)^{w\cdot \mathrm{rk}(M)/2+\dim(M_B^-)}\mathbf{Q}^* \qquad (1.6.2)
$$

Here we define $f^*(a) = \lim_{z\longrightarrow a}(z-a)^{-r}f(z) = b$, where $r = \operatorname{ord}_{z=a} f(z)$ (for a function f meromorphic in a neighbourhood of $a \in \mathbf{C}$).

2. Deligne's period map.

(2.1) Under the comparison map $I_\infty : M_B \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} M_{dR} \otimes_{\mathbf{Q}} \mathbf{C}$, $M_{dR} \otimes \mathbf{R}$ corresponds to $(M_B^+ \otimes \mathbf{R}) \oplus (M_B^- \otimes \mathbf{R}(-1))$. Deligne's period map is the induced map

$$
\alpha_M : M_B^+ \otimes \mathbf{R} \longrightarrow (M_{dR}/F^0) \otimes \mathbf{R},
$$

first introduced in [21]. As M has negative weight by assumption, we have

$$
\operatorname{Ker}(\alpha_M) \subseteq (F^0 \cap \overline{F}^0)(M_B \otimes \mathbf{C}) = 0.
$$

We may, therefore, reformulate (1.6.1) as

$$
\begin{aligned}
\operatorname{ord}_{s=0} L_\infty(M,s) &= \dim_{\mathbf{R}} \operatorname{Ker}(\alpha_M) = 0\\
-\operatorname{ord}_{s=0} L_\infty(M^\vee(1),s) &= \dim_{\mathbf{R}} \operatorname{Coker}(\alpha_M)
\end{aligned}
\qquad (2.1.1)
$$

The isomorphism $M(-1)_B^+ \otimes \mathbf{R} = M_B^- \otimes \mathbf{R}(-1) \xrightarrow{\sim} M_{dR} \otimes \mathbf{R}/I_\infty(M_B^+ \otimes \mathbf{R})$ also induces a map

$$
\beta_M : \operatorname{Ker}(\alpha_{M(-1)}) \longrightarrow \operatorname{Coker}(\alpha_M).
$$

For $w \neq -2$, the domain of β_M vanishes; for $w=-2$, β_M is injective for the same reason as α_M is.

(2.2) The $\mathbf{Q}$-structures M_B^+ and M_{dR}/F^0, on the domain and target of α_M respectively, define a natural $\mathbf{Q}$-structure $\mathcal{D}(M)$ on the real vector space $\det(\operatorname{Coker}(\alpha_M))$ (where by $\det(V)$ we denote the highest exterior power of a vector space V).
Deligne [21] calls the motive M *critical*, if the period map α_M is an isomorphism. If this is the case, he defines the period of M as

$$
c^+(M) = \det(\alpha_M) \in \mathbf{R}^*/\mathbf{Q}^*,
$$

the determinant being taken with respect to the $\mathbf{Q}$-structures M_B^+, M_{dR}/F^0. Of course, $\operatorname{Coker}(\alpha_M)$ vanishes for such M, so its determinant is canonically isomorphic to $\mathbf{R}$, and the $\mathbf{Q}$-structure $\mathcal{D}(M)$ is equal to $c^+(M)^{-1}\cdot \mathbf{Q}$.

In general, there is a natural commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 \longrightarrow & (F^0 M_{dR} \oplus M_B^+) \otimes \mathbf{R} & \longrightarrow & (M_B^-(-1) \oplus M_B^+) \otimes \mathbf{R} & \longrightarrow & \operatorname{Ker}(\alpha_{M^\vee(1)})^\vee & \longrightarrow 0 \\ & \| & & \downarrow \wr I_\infty & & \downarrow \wr & \\ 0 \longrightarrow & (F^0 M_{dR} \oplus M_B^+) \otimes \mathbf{R} & \longrightarrow & M_{dR} \otimes \mathbf{R} & \longrightarrow & \operatorname{Coker}(\alpha_M) & \longrightarrow 0, \end{array}$$

which defines a canonical isomorphism

$$\operatorname{Coker}(\alpha_M)^\vee \xrightarrow{\sim} \operatorname{Ker}(\alpha_{M^\vee(1)}) \tag{2.2.1}$$

Similarly, $\operatorname{Coker}(\alpha_{M^\vee(1)}) \xrightarrow{\sim} \operatorname{Ker}(\alpha_M)^\vee = 0$, and thus $\det(\operatorname{Ker}(\alpha_{M^\vee(1)}))$ also has a natural $\mathbf{Q}$-structure, inherited from the $\mathbf{Q}$-structures $M^\vee(1)_B^+$ and $M^\vee(1)_{dR}/F^0$ on the domain and target of $\alpha_{M^\vee(1)}$. The above diagram shows that this $\mathbf{Q}$-structure on $\det(\operatorname{Ker}(\alpha_{M^\vee(1)}))$ corresponds, via (2.2.1), to the $\mathbf{Q}$-structure

$$\mathcal{B}(M) = (2\pi i)^{-\dim(M_B^-)} \delta(M) \mathcal{D}(M)$$

on $\det(\operatorname{Coker}(\alpha_M))$. Here $\delta(M) \in \mathbf{C}^*/\mathbf{Q}^*$ denotes the determinant of I_∞ with respect to the $\mathbf{Q}$-structures M_B, M_{dR}.

The calculation in [21, 5.6], based on a conjectural description of motives of rank 1 over $\mathbf{Q}$, shows that

$$(2\pi i)^{-\dim(M_B^-)} \delta(M) \in (2\pi)^{-\dim(M_B^-) - w\cdot \operatorname{rk}(M)/2} \varepsilon(M, 0) \mathbf{Q}^*,$$

hence, in view of (1.6.2),

$$L^*(M^\vee(1), 0) \mathcal{B}(M) = L^*(M, 0) \mathcal{D}(M) \tag{2.2.2}$$

(2.3) We have seen that (by 1.4.2 and 2.1.1)

$$\dim_{\mathbf{R}}(\operatorname{Coker}(\alpha_M)) = \operatorname{ord}_{s=0} L(M^\vee(1), s) - \operatorname{ord}_{s=0} L(M, s)$$

(where the last term vanishes if $w \le -3$). This makes $\operatorname{Coker}(\alpha_M)$ a natural candidate for the target of a regulator map, which should "explain" the values $L^*(M, 0)$ and $L^*(M^\vee(1), 0)$. What we need is a $\mathbf{Q}$-vector space A_M 'of arithmetic nature' (a generalization of $\mathbf{Q} \oplus \mathcal{O}_F^* \otimes \mathbf{Q}$ for $\zeta_F(s)$ at $s = 0$) and regulator map $r : A_M \longrightarrow \operatorname{Coker}(\alpha_M)$, inducing an isomorphism after tensoring with $\mathbf{R}$. The value $L^*(M, 0) \in \mathbf{R}^*/\mathbf{Q}^*$ (resp. $L^*(M^\vee(1), 0)$) should be then equal to the determinant of r with respect to the $\mathbf{Q}$-structure $\mathcal{D}(M)$ (resp. $\mathcal{B}(M)$) on $\det(\operatorname{Coker}(\alpha_M))$.

(2.4) As a first step towards the construction of a regulator map we give an interpretation of $\operatorname{Coker}(\alpha_M)$ in terms of Hodge theory.

For a subring $A \subset \mathbf{R}$, denote by $\mathcal{MH}_A$ (resp. $\mathcal{MH}_A^+$) the category of mixed A-Hodge structures (resp. mixed A-Hodge structures with infinite Frobenius, i.e. an involution ϕ_∞ compatible with the weight filtration and such that $\phi_\infty \otimes c$ preserves the Hodge filtration). Both $\mathcal{MH}_A$ and $\mathcal{MH}_A^+$ are tensor categories with a unit object $\mathbf{1} = A(0)$.

For $H \in \operatorname{Ob} \mathcal{MH}_{\mathbf{R}}$, the complex (in degrees 0 and 1)

$$W_0 H \oplus F^0 (W_0 H)_{\mathbf{C}} \xrightarrow{i_W - i_F} (W_0 H)_{\mathbf{C}} \tag{2.4.1}$$

(where i_W and i_F denote the obvious inclusions) represents $R\operatorname{Hom}(\mathbf{R}(0), H)$, i.e. there are (natural) isomorphisms

$$\operatorname{Hom}_{\mathcal{MH}_{\mathbf{R}}}(\mathbf{R}(0), H) \xrightarrow{\sim} W_0 H \cap F^0 H_{\mathbf{C}} \tag{2.4.2}$$
$$\operatorname{Ext}^1_{\mathcal{MH}_{\mathbf{R}}}(\mathbf{R}(0), H) \xrightarrow{\sim} W_0 H \backslash (W_0 H)_{\mathbf{C}} / F^0 (W_0 H)_{\mathbf{C}} \tag{2.4.3}$$

and higher Ext^i vanish for $i > 1$. For a proof of this basic fact of life in $\mathcal{MH}_{\mathbf{R}}$ we refer the reader to [3],[16],[17]. Note that (2.4.2) is obvious: we associate to a morphism $f : \mathbf{R}(0) \longrightarrow H$ in $\mathcal{MH}_{\mathbf{R}}$ the value $f(1)$. Let us indicate how the morphism in (2.4.3) is defined: given an extension

$$0 \longrightarrow H \longrightarrow E \longrightarrow \mathbf{R}(0) \longrightarrow 0$$

in $\mathcal{MH}_{\mathbf{R}}$, we can lift $1 \in \mathbf{R}(0)$ to $1_W \in W_0 E$ (defined modulo $W_0 H$) and to $1_F \in F^0(W_0 E)_{\mathbf{C}}$ (defined modulo $F^0(W_0 H)_{\mathbf{C}}$). The class of $1_F - 1_W$ in

$$W_0 H \backslash (W_0 H)_{\mathbf{C}} / F^0 (W_0 H)_{\mathbf{C}}$$

is then well defined, depends only on the extension class of E and is additive in E.

(2.5) Similarly, for $H \in \operatorname{Ob} \mathcal{MH}^+_{\mathbf{R}}$, we just take $\phi_\infty \otimes c$-invariants: $R\operatorname{Hom}(\mathbf{R}(0), H)$ is represented by

$$W_0 H^+ \oplus F^0(W_0 H_{dR}) \longrightarrow W_0 H_{dR}, \tag{2.5.1}$$

hence

$$\operatorname{Hom}_{\mathcal{MH}^+_{\mathbf{R}}}(\mathbf{R}(0), H) \xrightarrow{\sim} W_0 H^+ \cap F^0 H_{\mathbf{C}} \tag{2.5.2}$$
$$\operatorname{Ext}^1_{\mathcal{MH}^+_{\mathbf{R}}}(\mathbf{R}(0), H) \xrightarrow{\sim} W_0 H^+ \backslash W_0 H_{dR} / F^0 (W_0 H_{dR}), \tag{2.5.3}$$

where $H_{dR} = H_{\mathbf{C}}^{\phi_\infty \otimes c = 1}$, $H^{\pm} = H^{\phi_\infty = \pm 1}$.

In particular, for $H = M_B \otimes \mathbf{R} \in \operatorname{Ob} \mathcal{MH}^+_{\mathbf{R}}$, we have $H = W_0 H$ and $H_{dR} = M_{dR} \otimes \mathbf{R}$, hence

$$\operatorname{Ext}^1_{\mathcal{MH}^+_{\mathbf{R}}}(\mathbf{R}(0), H) \xrightarrow{\sim} M_B^+ \otimes \mathbf{R} \backslash M_{dR} \otimes \mathbf{R} / F^0 M_{dR} \otimes \mathbf{R} = \operatorname{Coker}(\alpha_M)$$

(2.6) Having interpreted $\operatorname{Coker}(\alpha_M)$ as an Ext-group in the category of mixed Hodge structures, it is quite tempting to make a guess as to what the regulator map should be: simply the canonical map ('Hodge realization')

$$\operatorname{Ext}^1_{\mathcal{MM}_{\mathbf{Q}}}(\mathbf{Q}(0), M) \longrightarrow \operatorname{Ext}^1_{\mathcal{MH}^+_{\mathbf{R}}}(\mathbf{R}(0), M_B \otimes \mathbf{R}), \tag{2.6.1}$$

where the first Ext-group is computed in a suitable category of "mixed motives", which extends Grothendieck's category of pure motives over $\mathbf{Q}$. In the following two sections we shall try to make this idea more precise.

3. Arithmetic vs. geometric cohomology.

(3.1) The group $\mathrm{Coker}(\alpha_M)$ (where, as before, $M = h^i(X)(n)$ for a smooth projective variety $X_{/\mathbf{Q}}$ and $w = i - 2n \leq -1$) can be obtained as a composition of two cohomological functors: $H^i_B(X(\mathbf{C}), -)$ (applied to $\mathbf{R}(n)$) and $\mathrm{Ext}^1_{\mathcal{MH}^+_{\mathbf{R}}}(\mathbf{R}(0), -)$. This suggests that $\mathrm{Coker}(\alpha_M)$ is, in fact, equal to $H^{i+1}_?(X, \mathbf{R}(n))$ in some fancy cohomology theory $H^{\cdot}_?$, and that the isomorphism

$$H^{i+1}_?(X, \mathbf{R}(n)) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{MH}^+_{\mathbf{R}}}(\mathbf{R}(0), H^i_B(X(\mathbf{C}), \mathbf{R}(n)))$$

comes from the standard spectral sequence for composition of derived functors.

(3.2) This is indeed the case and the corresponding cohomology theory fits into the following general framework:

• Let $\mathcal{T}$ be a tensor category with a unit object $\mathbf{1}$ and Tate twists $- \mapsto -(j)$. Set $A = \mathrm{End}_{\mathcal{T}}(\mathbf{1})$,

$$\Gamma(\mathcal{T}, -) = \mathrm{Hom}_{\mathcal{T}}(\mathbf{1}, -) : \mathcal{T} \longrightarrow (A - mod).$$

• Let $\mathcal{V}$ be a sufficiently large subcategory of the category of schemes of finite type over a given field F (e.g. containing all smooth quasi-projective varieties over F).

• Suppose that for each $X \in \mathrm{Ob}\,\mathcal{V}$ and $j \in \mathbf{Z}$, there is a complex (contravariant in X)

$$\underline{R\Gamma}(X, j) = \underline{R\Gamma}(X, 0)(j) \in \mathrm{Ob}\, D^b(\mathcal{T}),$$

whose cohomology

$$\underline{H}^p(X, j) = \underline{H}^p(X, 0)(j) \in \mathrm{Ob}\,\mathcal{T}$$

are 'geometric cohomology groups' of X.

Applying the functor $R\Gamma(\mathcal{T}, -) : D^b(\mathcal{T}) \longrightarrow D^+(A - mod)$, we get a complex

$$R\Gamma_{\mathcal{T}}(X, j) = R\Gamma(\mathcal{T}, \underline{R\Gamma}(X, j)) \in \mathrm{Ob}\, D^+(A - mod)$$

with cohomology groups

$$H^p_{\mathcal{T}}(X, j) \in \mathrm{Ob}\,(A - mod)$$

— 'arithmetic', or 'absolute' cohomology groups of X.

The spectral sequence, referred to in 3.1, is then

$$\mathrm{Ext}^p_{\mathcal{T}}(\mathbf{1}, \underline{H}^q(X, j)) \Longrightarrow H^{p+q}_{\mathcal{T}}(X, j) \tag{3.2.1}$$

Reasonable geometric cohomology theories are usually equipped with additional structure: cohomology with supports, cup products, dual homology theory (in the sense of [13]). See 3.5 below for more details.

(3.3) As a basic example, consider étale cohomology. For a fixed integer n prime to the characteristic of F, let $\mathcal{T}$ be the category of finite $\mathbf{Z}/n\mathbf{Z}[G]$-modules, where $G = G(F^{sep}/F)$.Then $\mathbf{1} = \mathbf{Z}/n\mathbf{Z}(0)$, $A = \mathbf{Z}/n\mathbf{Z}$, $R^p\Gamma_{\mathcal{T}}(-) = H^p(G, -)$ and the geometric resp. arithmetic cohomology groups are étale cohomology of X over F^{sep} resp. F:

$$\underline{H}^p(X, j) = H^p((X \times_F F^{sep})_{et}, \mathbf{Z}/n\mathbf{Z})(j)$$
$$H^p_{\mathcal{T}}(X, j) = H^p(X_{et}, (\mathbf{Z}/n\mathbf{Z})(j))$$

They are related by the Hochschild-Serre spectral sequence.

If F is a finitely generated extension of $\mathbf{Q}$, then $\underline{R\Gamma}(X, j)$ exist also for ℓ-adic cohomology. In this case, $\mathcal{T}$ is the category of $\mathbf{Q}_\ell$-vector spaces of finite dimension equipped with a continuous action of G, $\mathbf{1} = \mathbf{Q}_\ell(0)$, $A = \mathbf{Q}_\ell$,

$$\underline{H}^p(X, j) = H^p((X \times_F \overline{F})_{et}, \mathbf{Q}_\ell)(j)$$
$$H^p_{\mathcal{T}}(X, j) = H^p(X_{et}, \mathbf{Q}_\ell(j)),$$

where the ℓ-adic cohomology over F is the *continuous* étale cohomology in the sense of [41]. Similarly, (3.2.1) becomes the Hochschild-Serre spectral sequence

$$H^p(G, H^q((X \times_F \overline{F})_{et}, \mathbf{Q}_\ell(j)) \Longrightarrow H^{p+q}(X_{et}, \mathbf{Q}_\ell(j))$$

for continuous Galois cohomology ([41]).

(3.4) Another example is provided by what Beilinson calls 'absolute Hodge cohomology'. For any separated scheme X of finite type over $\mathbf{C}$, Beilinson constructs in [3] a complex $\underline{R\Gamma}(X, 0) \in \mathrm{Ob}\, D^b(\mathcal{MH}_\mathbf{R})$, whose cohomology objects are $H^i_B(X(\mathbf{C}), \mathbf{R})$ with Deligne's Hodge structures [18],[19]. The formalism of 3.2 (with $\mathcal{T} = \mathcal{MH}_\mathbf{R}$) then produces 'absolute Hodge cohomology' of X, sitting in an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{MH}_\mathbf{R}}(\mathbf{R}(0), H^i_B(X(\mathbf{C}), \mathbf{R}(n))) \longrightarrow H^{i+1}_{\mathcal{MH}_\mathbf{R}}(X, n) \longrightarrow$$
$$\longrightarrow \mathrm{Hom}_{\mathcal{MH}_\mathbf{R}}(\mathbf{R}(0), H^{i+1}_B(X(\mathbf{C}), \mathbf{R}(n))) \longrightarrow 0.$$

For X over $\mathbf{R}$, we just replace $\mathcal{MH}_\mathbf{R}$ by $\mathcal{MH}^+_\mathbf{R}$, and get a similar sequence for $H^{i+1}_{\mathcal{MH}^+_\mathbf{R}}(X, n)$. If $X_{/\mathbf{R}}$ is proper and smooth, then (writing $w = i - 2n$, as usual) we get from (2.5.2–3)

$$H^{i+1}_{\mathcal{MH}^+_\mathbf{R}}(X, n) = \begin{cases} \mathrm{Ext}^1_{\mathcal{MH}^+_\mathbf{R}}(\mathbf{R}(0), H^i_B(X(\mathbf{C}), \mathbf{R}(n))) & w \le -2 \\ H^{2n}_B(X(\mathbf{C}), \mathbf{R}(n))^+ \cap F^0 & w = -1 \\ 0 & w \ge 0 \end{cases} \tag{3.4.1}$$

(3.5) Grothendieck's philosophy of motives stipulates existence of functors $X \mapsto h^i(X)$, which are universal among cohomology groups for *smooth projective* varieties over a given field F. The point of view advocated by Beilinson and Deligne is that the category $\mathcal{M}_F$ of motives with respect to homological equivalence embeds into a larger category of 'mixed motives' $\mathcal{MM}_F$, which should be a universal target for 'geometric cohomology theories' in the sense of 3.2. A fundamental object associated to a quasi-projective variety $X_{/F}$ should then be the complex

$$\underline{R\Gamma}(X, 0) \in \mathrm{Ob}\, D^b(\mathcal{MM}_F),$$

rather then its cohomology groups

$$h^i(X) = H^i(\underline{R\Gamma}(X,0)) \in \mathrm{Ob}\,\mathcal{MM}_F$$

(which would coincide with Grothendieck's $h^i(X)$ for smooth projective X).

There should be versions with support $\underline{R\Gamma}_Y(X,j)$ (for $Y \subseteq X$ closed), cup product $\underline{R\Gamma}(X,i)\otimes^{\mathbf{L}}\underline{R\Gamma}(X,j) \longrightarrow \underline{R\Gamma}(X,i+j)$) and homology complexes $\underline{R\Gamma}'(X,j)$ satisfying several axioms (see [1, 2.3.2]), which ensure that

$$h^i_Y(X)(j) = H^i(\underline{R\Gamma}_Y(X,j)) \quad \text{and}$$
$$h_i(X,j) = H^{-i}(\underline{R\Gamma}'(X,-j))$$

form a twisted Poincaré duality theory in the sense of Bloch and Ogus [13]. The most important axiom is the duality isomorphism

$$\underline{R\Gamma}'(Y,j) \xrightarrow{\sim} \underline{R\Gamma}_Y(X,j+d)[2d],$$

valid for $Y \subseteq X$ closed in a smooth X of dimension d. If Y has pure codimension i, then its cycle class is a map

$$cl(Y) : \mathbf{1} \longrightarrow h_{2d-2i}(Y)(d-i) \xrightarrow{\sim} h^{2i}_Y(X)(i)$$

Motivic cohomology and homology are then defined by

$$H^i_{\mathcal{MM}_F}(X,j) = \mathrm{Ext}^i_{\mathcal{MM}_F}(\mathbf{1},\underline{R\Gamma}(X,j)) = \mathrm{Hom}_{D^b(\mathcal{MM}_F)}(\mathbf{1},\underline{R\Gamma}(X,j)[i])$$
$$H_i^{\mathcal{MM}_F}(X,j) = \mathrm{Ext}^{-i}_{\mathcal{MM}_F}(\mathbf{1},\underline{R\Gamma}'(X,-j)) = \mathrm{Hom}_{D^b(\mathcal{MM}_F)}(\mathbf{1},\underline{R\Gamma}'(X,-j)[-i]).$$

Of course, $\mathbf{1} = \mathbf{Q}(0) = h^0(\mathrm{Spec}(F))$.

(3.6) Beilinson conjectures that the spectral sequence (3.2.1)

$$\mathrm{Ext}^i_{\mathcal{MM}_F}(\mathbf{1}, h^j(X)(n)) \Longrightarrow H^{i+j}_{\mathcal{MM}_F}(X,n) \tag{3.6.1}$$

degenerates for smooth projective X and that $\mathrm{Ext}^i_{\mathcal{MM}_F}$ vanishes for i greater than the Kronecker dimension of F (equal to the transcendence degree $tr.deg(F/\mathbf{F}_p)$ in characteristic p, resp. to $1 + tr.deg(F/\mathbf{Q})$ in characteristic zero). In particular, if X is a smooth projective variety over $\mathbf{Q}$, then the spectral sequence (3.6.1) should degenerate into exact sequences

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{MM}_\mathbf{Q}}(\mathbf{1}, h^i(X)(n)) \longrightarrow H^{i+1}_{\mathcal{MM}_\mathbf{Q}}(X,n) \longrightarrow \mathrm{Hom}_{\mathcal{MM}_\mathbf{Q}}(\mathbf{1}, h^{i+1}(X)(n)) \longrightarrow 0,$$

with the third group vanishing, unless $w = i - 2n = -1$ (for weight reasons). This suggests another description of the regulator map (2.6.1) for $w < -1$: the Hodge realization

$$H^{i+1}_{\mathcal{MM}_\mathbf{Q}}(X,n) \longrightarrow H^{i+1}_{\mathcal{MH}^+_\mathbf{R}}(X,n) \tag{3.6.2}$$

For $i + 1 = 2n$, the above sequence should be isomorphic to

$$0 \longrightarrow CH^n(X)_0 \otimes \mathbf{Q} \longrightarrow CH^n(X) \otimes \mathbf{Q} \xrightarrow{cl} \mathrm{Hom}_{\mathcal{MM}_\mathbf{Q}}(\mathbf{Q}(0), h^{2n}(X)(n)) \longrightarrow 0,$$

where $CH^n(X)$ is the Chow group of codimension n cycles on X modulo rational equivalence, and $CH^n(X)_0$ the subgroup of homologically trivial cycles (cf. 4.2).

(3.7) Furthemore, there should be a relative version of motivic cohomology for morphisms $f : X \longrightarrow Y$; there should be a notion of "motivic sheaves" on every variety $X_{/F}$, together with the standard formalism of Grothendieck's six functors $(f^*, Rf_*, Rf_!, Rf^!, R\,\mathrm{Hom}, \otimes^{\mathbf{L}})$ between corresponding derived categories. Denoting the category of motivic sheaves on X by $\mathcal{M}(X)$, then $\mathcal{MM}_F$ should be identified with $\mathcal{M}(\mathrm{Spec}(F))$. Writing $a : X \longrightarrow \mathrm{Spec}(F)$ for the structural morphism, we should have

$$\underline{R\Gamma}(X, j) = Ra_* a^* \mathbf{1}(j)$$
$$H^i_{\mathcal{MM}_F}(X, j) = \mathrm{Ext}^i_{\mathcal{M}(\mathrm{Spec}(F))}(\mathbf{1}, Ra_* a^* \mathbf{1}(j)) = \mathrm{Ext}^i_{\mathcal{M}(X)}(a^*\mathbf{1}, a^*\mathbf{1}(j)).$$

Note that such a relative theory exists in both étale cohomology (3.3) and Hodge theory (3.4) (see [58]).

4. Mixed motives.

(4.1) The category $\mathcal{MM}_{\mathbf{Q}}$ of mixed motives over $\mathbf{Q}$ is expected to enjoy (at least) the following four properties:

- The category of semisimple objects of $\mathcal{MM}_{\mathbf{Q}}$ is equivalent to the category $\mathcal{M}_{\mathbf{Q}}$ of motives with respect to homological equivalence (this makes sense only if homological and numerical equivalences of cycles coincide, which is one of Grothendieck's Standard Conjectures [39]; otherwise $\mathcal{M}_{\mathbf{Q}}$ itself would not be semisimple, by [44]).
- Each mixed motive $E \in \mathrm{Ob}\,\mathcal{MM}_{\mathbf{Q}}$ admits a functorial weight filtration $W_{\bullet}E$ (increasing) with graded factors $Gr^W_i(E) \in \mathrm{Ob}\,\mathcal{M}_{\mathbf{Q}}$ pure of weight i.
- Each quasi-projective variety (more generally, simplicial variety) X has cohomology $h^i(X) \in \mathrm{Ob}\,\mathcal{MM}_{\mathbf{Q}}$.
- $\mathrm{Ext}^i_{\mathcal{MM}_{\mathbf{Q}}} = 0$ for $i > 1$.

There is no Grothendieck style definition of $\mathcal{MM}_{\mathbf{Q}}$ as yet. All definitions proposed so far ([23],[32],[43]) are based on the same principle: one constructs first a suitable Tannakian category of mixed realizations $\mathcal{MR}_{\mathbf{Q}}$ and then defines $\mathcal{MM}_{\mathbf{Q}}$ as a full subgategory of $\mathcal{MR}_{\mathbf{Q}}$ consisting of objects of 'geometric origin', e.g. the smallest Tannakian subcategory of $\mathcal{MR}_{\mathbf{Q}}$ containing cohomology realizations of all quasi-projective varieties. This is based on a tacit assumption that the realization functor $\mathcal{MM}_{\mathbf{Q}} \longrightarrow \mathcal{MR}_{\mathbf{Q}}$ is fully faithful.

All realizations discussed in (1.1) in the context of pure motives have analogues for mixed motives. For arbitrary quasi-projective variety $X_{/\mathbf{Q}}$, the mixed motive $E = h^i(X)$ has realizations

$$\begin{aligned} E_\ell &= H^i((X \times_{\mathbf{Q}} \overline{\mathbf{Q}})_{et}, \mathbf{Q}_\ell) \\ E_B &= H^i(X(\mathbf{C}), \mathbf{Q}) \\ E_{dR} &= H^i((Y_{\bullet})_{Zar}, \Omega^{\bullet}_{Y_{\bullet}/\mathbf{Q}}), \end{aligned}$$

where $Y_{\bullet} \longrightarrow X$ is a smooth hypercovering of X (for Zariski topology). In this situation E_B is a *mixed* Hodge structure with infinite Frobenius ϕ_∞ and the weight filtration on E_B corresponds, under comparison isomorphisms

$$I_{\ell,\overline{\sigma}} : E_B \otimes_{\mathbf{Q}} \mathbf{Q}_\ell \xrightarrow{\sim} E_\ell, \qquad I_\infty : E_B \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} E_{dR} \otimes_{\mathbf{Q}} \mathbf{C},$$

to natural filtrations $W_{\cdot}E_\ell$ (by subrepresentations of $G(\overline{\mathbf{Q}}/\mathbf{Q})$) and $W_{\cdot}E_{dR}$. The graded objects $Gr_j^W(E_\ell)$ and $Gr_j^W(E_B)$ are pure of weight j (as representations of $G(\overline{\mathbf{Q}}/\mathbf{Q})$ in the sense of (1.1) resp. as $\mathbf{Q}$-Hodge structures).

(4.2) Let us give a few examples of mixed motives over arbitrary (say, finitely generated) field F. We shall confidently use cohomology with supports $h^i_Y(X)$ resp. relative cohomology $h^i(X,Y)$ for $Y \subseteq X$ a closed subvariety of X (see [64] for a realization of the relative cohomology).
The first example gives a motivic interpretation of Abel-Jacobi maps. Let $X_{/F}$ be a smooth projective variety, $Y \subset X$ a cycle of (pure) codimension i, homologically trivial. Consider the long exact cohomology sequence

$$0 \longrightarrow h^{2i-1}(X)(i) \longrightarrow h^{2i-1}(X-Y)(i) \longrightarrow h^{2i}_Y(X)(i) \xrightarrow{\beta} h^{2i}(X)(i) \longrightarrow \dots$$

As Y is homologically trivial, the composition of β with the cycle class of Y

$$cl(Y) : \mathbf{Q}(0) \longrightarrow h^{2i}_Y(X)(i)$$

vanishes. Taking pullback of the above exact sequence via β, we get an extension of motives

$$0 \longrightarrow h^{2i-1}(X)(i) \longrightarrow E \longrightarrow \mathbf{Q}(0) \longrightarrow 0$$

Note that E is a motive with two weights, namely -1 and 0. The extension class of E depends, in fact, only on the rational equivalence class of Y and the map $Y \mapsto E$ induces a homomorphism

$$CH^i(X)_0 \longrightarrow \mathrm{Ext}^1_{\mathcal{MM}_F}(\mathbf{Q}(0), h^{2i-1}(X)(i))$$

This is a 'motivic' Abel-Jacobi map; if $F \subseteq \mathbf{C}$, we may do the same with singular cohomology with *integral* coefficients, getting the usual Abel-Jacobi map with values in Griffiths' Jacobian

$$\mathrm{Ext}^1_{\mathcal{MH}_{\mathbf{Z}}}(\mathbf{Z}(0), H_{\mathbf{Z}}) = H_{\mathbf{Z}} \backslash H_{\mathbf{C}} / F^0,$$

where $H_{\mathbf{Z}} = H^{2i-1}(X(\mathbf{C}), \mathbf{Z}(i))$ (cf. (2.4.3)).

(4.3) The second example is related to a motivic construction of height pairings (see [64]). Let $X_{/F}$ be a smooth projective variety, equidimensional of dimension d, let $Y, Z \subset X$ be homologically trivial cycles of pure codimensions $i, j = d+1-i$, with disjoint supports. The relative cohomology $H = h^{2i-1}(X-Y, Z)(i)$ appears in exact sequences

$$0 \longrightarrow h^{2i-1}(X,Z)(i) \longrightarrow h^{2i}(X-Y,Z)(i) \longrightarrow h^{2i}_Y(X)(i) \longrightarrow h^{2i}(X,Z)(i)$$

$$h^{2i-2}(X-Y)(i) \longrightarrow h^{2i-2}(Z)(i) \longrightarrow h^{2i-1}(X-Y,Z)(i) \longrightarrow h^{2i-1}(X-Y)(i) \longrightarrow 0$$

Taking pullback of H via the cycle class $cl(Y) : \mathbf{Q}(0) \longrightarrow h^{2i}_Y(X)(i)$ and pushout via the trace map $Tr(Z) : h^{2i-2}(Z)(i) \longrightarrow \mathbf{Q}(1)$, we get a mixed motive E sitting in a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Q}(1) & \longrightarrow & E_2 & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Q}(1) & \longrightarrow & E & \longrightarrow & E_1 & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathbf{Q}(0) & = & \mathbf{Q}(0) & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

where $M = h^{2i-1}(X)(i)$ and E_1 (resp. $E_2^{\vee}(1)$) is associated to the cycle Y (resp. Z) as in the previous example. The weight filtration of E is given by

$$W_{-3}E = 0,\ W_{-2}E = \mathbf{Q}(1),\ W_{-1}E = E_2,\ W_0E = E.$$

An important special case of this construction, when $X = \mathbf{P}^1$, $Y = (0) - (\infty)$, $Z = (1) - (u)$ ($u \in F^* - \{1\}$) gives a 'Kummer motive' $E = h^1(\mathbf{G}_m, \{1, u\})(1)$, which is an extension

$$0 \longrightarrow \mathbf{Q}(1) \longrightarrow E \longrightarrow \mathbf{Q}(0) \longrightarrow 0$$

It is believed that Kummer motives exhaust all motivic extensions of $\mathbf{Q}(0)$ by $\mathbf{Q}(1)$, in other words that

$$\mathrm{Ext}^1_{\mathcal{MM}_F}(\mathbf{Q}(0), \mathbf{Q}(1)) = F^* \otimes \mathbf{Q} \tag{4.3.1}$$

Note that the extension class of E in various realizations is given by

$$[E_\ell] = u \otimes 1 \in H^1(F, \mathbf{Q}_\ell(1)) = F^* \widehat{\otimes} \mathbf{Q}_\ell$$
$$[E_B] = \log(u) \in \mathrm{Ext}^1_{\mathcal{MH}^+_{\mathbf{R}}}(\mathbf{R}(0), \mathbf{R}(1)) \xrightarrow{\sim} \mathbf{R}$$

(for ℓ different from the characteristic of F, resp. for an embedding $F \hookrightarrow \mathbf{R}$), after a suitable normalization of signs [64].

In general, we get an extension of $\mathbf{Q}(0)$ by $\mathbf{Q}(1)$ whenever $M = 0$. It is by no means clear that extensions obtained in this way are Kummer (I am grateful to the referee for this remark).

(4.4) The last example is borrowed from the appendix to [29]. Let

$$\Delta_n = \mathrm{Spec}(F[T_0, \dots, T_n]/(\sum T_i - 1))$$

be a simplex of dimension n, let $\partial_i : \Delta_n \hookrightarrow \Delta_{n+1}$ (for $0 \le i \le n+1$) be the i^{th} face map (sending T_i to 0 and renumbering $T_j \mapsto T_{j-1}$ for $j > i$).

Let $X_{/F}$ be an equidimensional smooth projective variety. Fix $n > 0$ and suppose that Y is a cycle of codimension i on $X \times \Delta_n$, meeting all faces of $X \times \Delta_n$ properly. By a face we mean the image of $X \times \Delta_m$ by any composition of face maps ∂_i (for $m < n$). We also assume that $\partial_j^*(Y) = 0$ for all $0 \le j \le n$.

We shall write $\Delta^n_X = X \times \Delta_n$, $\partial\Delta^n_X$ for the union of all faces of codimension one $\partial_i(X \times \Delta_{n-1})$, $|\partial Y| = |Y| \cap \partial\Delta^n_X$, $U = \Delta^n_X - Y$, $\partial U = U \cap \partial\Delta^n_X$.
There is a natural exact sequence ([29, (A.3)])

$$0 \longrightarrow h^{2i-n-1}(X)(i) \longrightarrow h^{2i-1}(U, \partial U)(i) \longrightarrow \mathrm{Ker}(\beta) \longrightarrow h^{2i-n}(X)(i),$$

where

$$\beta : \mathrm{Ker}[h^{2i}_Y(\Delta^n_X)_0(i) \longrightarrow h^{2i}(\Delta^n_X)_0(i)] \longrightarrow \mathrm{Ker}[h^{2i}_{\partial Y}(\partial\Delta^n_X)_0(i) \longrightarrow h^{2i}(\partial\Delta^n_X)_0(i)]$$

is induced by taking intersection with $\partial\Delta^n_X$ (the subscript 0 denotes kernel of the map 'forget the supports').
The class of Y is a map $cl(Y) : \mathbf{Q}(0) \longrightarrow \mathrm{Ker}(\beta)$ and its composite with $\mathrm{Ker}(\beta) \longrightarrow h^{2i-n}(X)(i)$ vanishes. Taking pullback by $cl(Y)$, gives, finally, an extension

$$0 \longrightarrow h^{2i-n-1}(X)(i) \longrightarrow E \longrightarrow \mathbf{Q}(0) \longrightarrow 0$$

(4.5) The example of Kummer motives in 4.3 shows that the motivic regulator (2.6.1) still needs a minor adjustment: for $M = R_{F/\mathbf{Q}}\mathbf{Q}(1)$, we have

$$\mathrm{Ext}^1_{\mathcal{MM}_\mathbf{Q}}(\mathbf{Q}(0), M) = \mathrm{Ext}^1_{\mathcal{MM}_F}(\mathbf{Q}(0), \mathbf{Q}(1)) \stackrel{?}{=} F^* \otimes \mathbf{Q},$$

but the classical regulator is made up only of units $\mathcal{O}^*$ (where $\mathcal{O}$ is the ring of integers of F). We need, therefore, a motivic interpretation of $\mathcal{O}^* \otimes \mathbf{Q}$.

We say, after A.J. Scholl, that a mixed motive $E \in \mathrm{Ob}\,\mathcal{MM}_F$ is defined over $\mathcal{O}$, if the weight filtration of E_ℓ splits as a representation of the inertia group I_v, for all ℓ and $v \not| \ell$. Mixed motives defined over $\mathcal{O}$ form a full subcategory $\mathcal{MM}_\mathcal{O}$ of $\mathcal{MM}_F$, containing $\mathcal{M}_F$.
For $v \not| \ell$, the valuation v induces isomorphisms

$$H^1(F_v, \mathbf{Q}_\ell(1)) = F^*_v \widehat{\otimes} \mathbf{Q}_\ell \stackrel{\sim}{\longrightarrow} H^1(I_v, \mathbf{Q}_\ell(1)) = F^*_{v,ur} \widehat{\otimes} \mathbf{Q}_\ell \stackrel{\sim}{\longrightarrow} \mathbf{Q}_\ell,$$

which shows that the Kummer motive corresponding to $u \in F^*$ is defined over $\mathcal{O}$ iff $u \in \mathcal{O}^*$. Modulo (4.3.1), this gives the desired motivic formula for $\mathcal{O}^* \otimes \mathbf{Q}$:

$$\mathcal{O}^* \otimes \mathbf{Q} = \mathrm{Ext}^1_{\mathcal{MM}_\mathcal{O}}(\mathbf{Q}(0), \mathbf{Q}(1)) = \mathrm{Ext}^1_{\mathcal{MM}_\mathbf{Z}}(\mathbf{Q}(0), R_{F/\mathbf{Q}}\mathbf{Q}(1)).$$

Similarly, we denote by $H^{i+1}_{\mathcal{MM}_\mathbf{Z}}(X, n)$ (for smooth projective $X_{/\mathbf{Q}}$ and $w = i - 2n \leq -2$) the image of

$$\mathrm{Ext}^1_{\mathcal{MM}_\mathbf{Z}}(\mathbf{Q}(0), M) \hookrightarrow \mathrm{Ext}^1_{\mathcal{MM}_\mathbf{Q}}(\mathbf{Q}(0), M) \stackrel{\sim}{\longrightarrow} H^{i+1}_{\mathcal{MM}_\mathbf{Q}}(X, n).$$

The final version of the motivic regulator should then be given by restricting (3.6.1) to

$$H^{i+1}_{\mathcal{MM}_\mathbf{Z}}(X, n) \hookrightarrow H^{i+1}_{\mathcal{MM}_\mathbf{Q}}(X, n) \longrightarrow H^{i+1}_{\mathcal{MH}^+_\mathbf{R}}(X_{/\mathbf{R}}, n) \qquad (4.5.1)$$

5. Motivic cohomology.

(5.1) One of the most disturbing features of the motivic regulator map (4.5.1) is the way how the motivic cohomology $H^{i+1}_{\mathcal{MM}_{\mathbf{Z}}}$ is 'defined', in terms of homological algebra in $\mathcal{MM}_{\mathbf{Q}}$ and $\mathcal{MM}_{\mathbf{Z}}$, two Tannakian categories of rather dubious status. What we need is a direct description of motivic cohomology and the regulator map.

Beilinson [4] and Lichtenbaum [48] conjectured that motivic cohomology of X (even its version with integral coefficients) can be computed as hypercohomology of suitable complexes on Zariski resp. étale site of X. We shall not discuss this approach to motivic cohomology and instead refer the reader to the articles of S. Bloch, A.B. Gončarov and S. Lichtenbaum in these Proceedings.

There exist two candidates for motivic cohomology. Beilinson [1] defines, for a quasi-projective variety $X_{/F}$,

$$H^i_{\mathcal{M}}(X, \mathbf{Q}(j)) = (K_{2j-i}(X) \otimes \mathbf{Q})^{(j)} \tag{5.1.1}$$

as the subspace of weight j for Adams operations of a suitable K-group of X. Here K-theory enters the picture for two reasons. First, Beilinson was guided by the relationship between K-theory and singular cohomology in the topological situation (see the discussion of the Atiyah-Hirzebruch spectral sequence in [37]). The second hint came from arithmetic, through the works of Lichtenbaum, Borel and Bloch discussed in the introduction. In fact, in [1], Beilinson states that his work "owes its origin to an attempt to understand Bloch's ideas and computations".

The second construction is due to Bloch [9]. In the notation of 4.4, let $Z^q(X)_n$ be the group of cycles of codimension q on $X \times \Delta_n$, meeting all faces transversally. Then $\underline{\underline{n}} \mapsto Z^q(X)_n$ is a simplicial abelian group $Z^q(X)$. One defines the higher Chow group $CH^q(X, p)$ as the homotopy group $\pi_p(\mid Z^q(X) \mid)$ of the geometric realization of $Z^q(X)$ (or as homology of the corresponding normalized chain complex; see [37]). Then

$$H^i(X, \mathbf{Z}(j)) = CH^j(X, 2j-i)$$

is a candidate for motivic cohomology with integral coefficients.

At present, it is not even known if both recipes give the same result, i.e. if there exist canonical isomorphisms

$$(K_p(X) \otimes \mathbf{Q})^{(j)} \xrightarrow{\sim} CH^j(X, p) \otimes \mathbf{Q} \tag{5.1.2}$$

for all smooth varieties $X_{/F}$. This is certainly true for $p = 0$. In this case, $CH^j(X, 0) = CH^j(X)$ is the standard Chow group of codimension j cycles on X modulo rational equivalence, and the isomorphism $CH^j(X) \otimes \mathbf{Q} \xrightarrow{\sim} (K_0(X) \otimes \mathbf{Q})^{(j)}$ is a classical result of Grothendieck [66]. For $p > 0$, (5.1.2) still remains open (the argument in [9] runs into difficulties when applying various moving lemmas).
An account of Beilinson's conjectures in terms of higher Chow groups is presented in [29].

(5.2) It is possible to extend (5.1.1) and define K-theoretic cohomology with supports, homology theory (using $K'(X)$), cup products

$$\cup : H^i_{\mathcal{M}}(X, \mathbf{Q}(m)) \otimes H^j_{\mathcal{M}}(X, \mathbf{Q}(n)) \longrightarrow H^{i+j}_{\mathcal{M}}(X, \mathbf{Q}(m+n))$$

and show that they satisfy Galois descent and almost all axioms of Bloch-Ogus [13] (see [68],[69]).

Next we need a regulator map

$$r_{\mathcal{H}} : H^i_{\mathcal{M}}(X, \mathbf{Q}(j)) \longrightarrow H^i_{\mathcal{MH}^+_{\mathbf{R}}}(X, j) \tag{5.2.1}$$

to replace (3.6.2) (say, for a quasi-projective variety $X_{/\mathbf{Q}}$). Beilinson constructs $r_{\mathcal{H}}$ as a Chern class on higher K-theory, using the general machinery of characteristic classes due to Gillet [34]. We shall present a more direct construction of $r_{\mathcal{H}}$, which works for X smooth and quasi-projective.

Fix an integer N and denote by $B_{\textstyle .}GL_{N/\mathbf{R}}$ the classifying space of the algebraic group $GL_{N/\mathbf{R}}$. It is a simplicial scheme (cf. [37]) and there is a universal simplicial bundle $\mathcal{E}$ of rank N over $B_{\textstyle .}GL_{N/\mathbf{R}}$. The Betti cohomology of $B_{\textstyle .}GL_{N/\mathbf{R}}$ is well known:

$$H^{2n-1}(B_{\textstyle .}GL_{N/\mathbf{R}}(\mathbf{C}), \mathbf{Q}) = 0$$
$$\bigoplus_{n \geq 0} H^{2n}(B_{\textstyle .}GL_{N/\mathbf{R}}(\mathbf{C}), \mathbf{Q}(n)) = \mathbf{Q}[c_1, \dots, c_N],$$

where

$$c_i = c_i(\mathcal{E}) \in H^{2i}(B_{\textstyle .}GL_{N/\mathbf{R}}(\mathbf{C}), \mathbf{Q}(i))$$

are Chern classes of the universal bundle $\mathcal{E}$. As $\mathcal{E}$ is defined over $\mathbf{R}$, all c_i are fixed by ϕ_∞. According to [19, 9.1.1], all cohomology groups $H^{2n}(B_{\textstyle .}GL_{N/\mathbf{R}}(\mathbf{C}), \mathbf{Q}(n))$ have pure Hodge type $(0,0)$. From (3.4.1), we get isomorphisms

$$H^{2n}_{\mathcal{MH}^+_{\mathbf{R}}}(B_{\textstyle .}GL_{N/\mathbf{R}}, n) \xrightarrow{\sim} H^{2n}(B_{\textstyle .}GL_{N/\mathbf{R}}(\mathbf{C}), \mathbf{Q}(n)) \otimes \mathbf{R} \qquad (n \geq 0)$$

and we may view, therefore, c_i as elements of the first group.

Let A be an $\mathbf{R}$-algebra of finite type. We have the evaluation map

$$ev : \mathrm{Spec}(A) \times B_{\textstyle .}GL_N(A) \longrightarrow B_{\textstyle .}GL_{N/\mathbf{R}},$$

which is a morphism of (simplicial) $\mathbf{R}$-schemes. Let us compute in general $H^p_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}} \times Y_{\textstyle .}, q)$ for $X_{/\mathbf{R}}$ a separated scheme and $Y_{\textstyle .}$ a simplicial set. If we denote by $C^{\cdot}(Y_{\textstyle .}, \mathbf{R})$ the cochain complex of $Y_{\textstyle .}$ (with real coefficients), then, in the notation of 3.4,

$$\underline{R\Gamma}(X_{/\mathbf{R}} \times Y_{\textstyle .}, q) = s(\underline{R\Gamma}(X_{/\mathbf{R}}, q) \otimes_{\mathbf{R}} C^{\cdot}(Y_{\textstyle .}, \mathbf{R})),$$

hence the Künneth formula and the cap product

$$H^p_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}} \times Y_{\textstyle .}, q) \xrightarrow{\sim} \bigoplus_j H^{p-j}_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, q) \otimes_{\mathbf{R}} H^j(Y_{\textstyle .}, \mathbf{R})$$
$$\cap : H^{2n}_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}} \times Y_{\textstyle .}, n) \otimes H_i(Y_{\textstyle .}, \mathbf{R}) \longrightarrow H^{2n-i}_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n)$$

For $i > 0$, the Hurewicz map $K_i(A) \longrightarrow H_i(GL(A), \mathbf{R}) = \varinjlim_N H_i(B_{\textstyle .}GL_N(A), \mathbf{R})$ and cap product with $ev^*(c_n)$ induce a homomorphism

$$c_{i,n} : K_i(A) \longrightarrow H^{2n-i}_{\mathcal{MH}^+_{\mathbf{R}}}(\mathrm{Spec}(A), n)$$

Suppose now that $X_{/\mathbf{R}}$ is a smooth quasi-projective variety. By Jouanolou's trick [45], there is an affine variety $\mathrm{Spec}(A)$ and a morphism $\pi : \mathrm{Spec}(A) \longrightarrow X_{/\mathbf{R}}$ which makes $\mathrm{Spec}(A)$ a vector bundle over $X_{/\mathbf{R}}$. By the homotopy property, π induces isomorphisms both on K_i and $H^i_{\mathcal{MH}^+_{\mathbf{R}}}$, and we define

$$c_{i,n} : K_i(X_{/\mathbf{R}}) \longrightarrow H^{2n-i}_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n)$$

by transport of structure.

Finally, one defines Chern *character* maps

$$ch_i = \sum_{n \geq 0} \frac{(-1)^{n-1}}{(n-1)!} c_{i,n} : K_i(X_{/\mathbf{R}}) \otimes \mathbf{Q} \longrightarrow \bigoplus_n H^{2n-i}_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n)$$

for $i > 0$ and ch_0 as the usual Chern character

$$K_0(X_{/\mathbf{R}}) \otimes \mathbf{Q} \longrightarrow \bigoplus_{n \geq 0} H^{2n}_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n).$$

The weight properties of Chern classes imply that $(K_i(X_{/\mathbf{R}}) \otimes \mathbf{Q})^{(n)}$ is mapped by ch_i into $H^{2n-i}_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n)$.

For a smooth quasi-projective variety $X_{/\mathbf{Q}}$, Beilinson's regulator $r_{\mathcal{H}}$ is defined as

$$r_{\mathcal{H}} : H^i_{\mathcal{M}}(X, \mathbf{Q}(j)) = (K_{2j-i}(X) \otimes \mathbf{Q})^{(j)} \longrightarrow (K_{2j-i}(X_{/\mathbf{R}}) \otimes \mathbf{Q})^{(j)} \xrightarrow{ch_i} H^i_{\mathcal{MH}^+_{\mathbf{R}}}(X, j)$$

By multiplicativity of the Chern character, it satisfies

$$r_{\mathcal{H}}(x \cup y) = r_{\mathcal{H}}(x) \cup r_{\mathcal{H}}(y)$$

The basic example of Beilinson's regulator is the usual logarithm:

$$H^1_{\mathcal{M}}(\mathrm{Spec}(\mathbf{R}), \mathbf{Q}(1)) = \mathbf{R}^* \otimes \mathbf{Q} \longrightarrow H^1_{\mathcal{MH}^+_{\mathbf{R}}}(\mathrm{Spec}(\mathbf{R}), 1) = \mathrm{Ext}^1_{\mathcal{MH}^+_{\mathbf{R}}}(\mathbf{R}(0), \mathbf{R}(1)) \xrightarrow{\sim} \mathbf{R}$$

maps $x \otimes 1$ to $\log |x|$.

(5.3) Let X be, as before, a smooth projective variety over $\mathbf{Q}$. Chose a proper flat model $\mathcal{X}_{\mathbf{Z}}$ of X (it always exists) and put

$$H^i_{\mathcal{M}}(X, \mathbf{Q}(j))_{\mathbf{Z}} = \mathrm{Im}(K'_{2j-i}(\mathcal{X}) \otimes \mathbf{Q} \longrightarrow (K_{2j-i}(X) \otimes \mathbf{Q})^{(j)})$$

Beilinson conjectures that this subgroup is independent of the choice of $\mathcal{X}$. This is true if we consider only proper and *regular* models of X; unfortunately, such models are rarely known to exist.

The localization sequence

$$\cdots \longrightarrow K'_j(\mathcal{X}) \longrightarrow K_j(X) \longrightarrow \bigoplus_p K'_{j-1}(\mathcal{X} \times_{\mathbf{Z}} \mathbf{F}_p) \longrightarrow \cdots$$

and certain conjectures about K-theory of schemes over finite fields would imply that $H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(n))/H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}}$ depends only on the bad fibres of $\mathcal{X}$ for $w = i - 2n < -1$ and vanishes for $n > \max(i, \dim(X)) + 1$ (cf. [1, 2.4.2.2]).

(5.4) To sum up, Beilinson defines

$$H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}} \hookrightarrow H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(n)) \xrightarrow{r_{\mathcal{H}}} H^{i+1}_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n)$$

as a K-theoretic substitute for 4.5.1, which still remains only a wishful thinking.

6. Values of L-functions.

We are now ready to formulate Beilinson's conjectures concerning the special values of L-functions. Let X be a smooth projective variety over $\mathbf{Q}$, $i \geq 0$ and $n \in \mathbf{Z}$ integers satisfying $w = i - 2n < 0$.

(6.1) Conjecture. *Assume $w \leq -3$. Then:*

(1) $$r_{\mathcal{H}} \otimes \mathbf{R} : H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}} \otimes \mathbf{R} \longrightarrow H^{i+1}_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n)$$

is an isomorphism.

(2) $r_{\mathcal{H}}(\det\ H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}}) = L(h^i(X), n)\mathcal{D}(M) = L^*(h^i(X), i+1-n)\mathcal{B}(M)$ *in* $\det\ H^{i+1}_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n)$, *with $\mathcal{D}(M)$ and $\mathcal{B}(M)$ defined in 2.2 for $M = h^i(n)$.*

According to 2.3.1, the order of vanishing of $L(h^i(X), s)$ at $s = i + 1 - n$ is equal to $\dim_{\mathbf{R}} H^{i+1}_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n)$. If $M = h^i(X)(n)$ is critical (see 2.2), then 6.1 reduces to Deligne's conjecture [21]

$$L(h^i(X), n) = L(M, 0) \in c^+(M)\mathbf{Q}^*$$

(6.2) For $w = -2$, $L(h^i(X), s)$ can have a pole at $s = n = 1 + i/2$. The order of the pole is predicted by Tate's conjecture:

$$-\operatorname{ord}_{s=n} L(h^{2n-2}(X), s) = \dim_{\mathbf{Q}} N^{n-1}(X),$$

where

$$N^{n-1}(X) = (CH^{n-1}(X)/CH^{n-1}(X)_0) \otimes \mathbf{Q} \stackrel{?}{=} \operatorname{Hom}_{\mathcal{MM}_{\mathbf{Q}}}(\mathbf{Q}(0), h^{2n-2}(n-1))$$

The cycle class in Betti cohomology $cl_B : N^{n-1} \longrightarrow \operatorname{Ker}(\alpha_{M(-1)})$ (for $M = h^{2n-2}(X)(n)$) and the map $\beta_M : \operatorname{Ker}(\alpha_{M(-1)}) \longrightarrow \operatorname{Coker}(\alpha_M)$ of 2.1 define

$$r_B : N^{n-1}(X) \longrightarrow \operatorname{Coker}(\alpha_M) = H^{2n-1}_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n).$$

(6.3) Conjecture. *Assume $w = -2$. Then*

(1) $(r_{\mathcal{H}} \oplus r_B) \otimes \mathbf{R} : H^{2n-1}_{\mathcal{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}} \otimes \mathbf{R} \oplus N^{n-1}(X) \otimes \mathbf{R} \longrightarrow H^{2n-1}_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, n)$

is an isomorphism.

(2) $$(r_{\mathcal{H}} \oplus r_B)(\det\ (H^{2n-1}_{\mathcal{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}} \oplus N^{n-1}(X)) = L^*(h^{2n-2}(X), n-1)\mathcal{B}(M) = \\ = L^*(h^{2n-2}(X), n)\mathcal{D}(M)$$

(6.4) In the remaining case of $w = -1$, the conjecture has to be modified. A new ingredient is the height pairing

$$h : CH^n(X)_0 \otimes \mathbf{Q} \ \otimes \ CH^{\dim X+1-n}(X)_0 \otimes \mathbf{Q} \longrightarrow \mathbf{R} \tag{6.4.1}$$

We shall not discuss possible definitions of h (unfortunately, all definitions proposed so far are conditional, except for $n = 1$) and refer to [64] for more details.

(6.5) Conjecture. *Assume $w = -1$. Then:*
(1) *The height pairing h in 6.4.1 is non-degenerate.*
(2) $\mathrm{ord}_{s=n} L(h^{2n-1}(X), s) = \dim_{\mathbf{Q}} CH^n(X)_0 \otimes \mathbf{Q}$
(3) $L^*(h^{2n-1}(X), n) \in c^+(h^{2n-1}(X)(n)) \det(h)\mathbf{Q}^*$,
where $c^+(M)$ is Deligne's period defined in 2.2.

(6.6) The only case when 6.1 is known is, essentially, if $X = \mathrm{Spec}(F)$ for a finite extension $F/\mathbf{Q}$. In many cases, however, one can verify the following

Weak Conjecture. *Assume $w = i - 2n < 0$. Then 6.1 (or 6.3), resp. 6.5 holds if $H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}}$ $(\oplus N^{n-1})$, resp. $CH^{\cdot}(X)_0 \otimes \mathbf{Q}$ is replaced by a suitable $\mathbf{Q}$-subspace.*

(6.7) For quite a few interesting L-series one needs a refined version of the above conjectures, when $L(h^i(X), s)$ is replaced by $L(M, s)$ for a suitable 'submotive' $M \subset h^i(X)$. This is indeed possible for Chow motives, which include, among others, motives of Dirichlet characters. Fix a number field E – the field of coefficients. Let $\mathcal{V}_k$ be the category of smooth projective varieties over a field k. The category $\mathcal{MC}(k, E)$ of Chow motives over k with coefficients in E has as objects triples

$$M = (X, p, m),$$

where $X \in \mathrm{Ob}\,\mathcal{V}_k$, $p \in CH^{\dim X}(X \times X) \otimes E$ a projector ($p^2 = p$) and $m \in \mathbf{Z}$. For $N = (Y, q, n)$ one has

$$\mathrm{Hom}(M, N) = q\, CH^{\dim Y + m - n}(X \times Y) \otimes E\, p,$$

with composition of $f : M_1 \longrightarrow M_2$ and $g : M_2 \longrightarrow M_3$ defined by the intersection product: $g \circ f = p_{13*}(p_{12}^* f \cdot p_{23}^* g)$, where $p_{ij} : X_1 \times X_2 \times X_3$ are the projections. There is a natural covariant functor $\mathcal{V}_k \longrightarrow \mathcal{MC}(k, E)$, sending X to the triple $(X, id, 0)$ and $f : X \longrightarrow Y$ to the graph of f. This definition is due to U. Jannsen [42]; a more traditional construction works in two steps, first adjoining images of projectors and then adding Tate twists.

The functors $X \mapsto H^{\cdot}_{\mathcal{M}}(X, \mathbf{Q}(*))$ extend to $\mathcal{MC}(k, E)$: we set

$$H^i_{\mathcal{M}}((X, p, m), \mathbf{Q}(j)) = p^* H^{i+2m}_{\mathcal{M}}(X, \mathbf{Q}(j + m)) \otimes E$$

and, for $f = q \circ g \circ p \in \mathrm{Hom}((X, p, m), (Y, q, n))$,

$$f^* : H^i_{\mathcal{M}}((Y, q, n), \mathbf{Q}(j)) \longrightarrow H^i_{\mathcal{M}}((X, p, m), \mathbf{Q}(j))$$

by $f^*(\alpha) = p^*(\pi_X)_*(f \cup \pi_Y^*(q^*\alpha))$, where

$$(\pi_X)_* : H^{i+2\dim Y}_{\mathcal{M}}((X \times Y, id, m - n), \mathbf{Q}(j + \dim Y)) \longrightarrow H^i_{\mathcal{M}}((X, id, m), \mathbf{Q}(j))$$

is the Gysin map.

Similarly, for $k = \mathbf{Q}$, one can extend to $\mathcal{MC}(\mathbf{Q}, E)$ functors $X \mapsto H^{\cdot}_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, *)$ and the regulator

$$r_{\mathcal{H}} : H^i_{\mathcal{M}}(M, \mathbf{Q}(j)) \longrightarrow H^i_{\mathcal{MH}^+_{\mathbf{R}}}(M_{/\mathbf{R}}, j).$$

The L-series $L(M, s)$ has values in $E \otimes \mathbf{C}$ and the above regulator is expected to determine its special value modulo E^*. See [1], [24],[42] for more details.

Beilinson conjectures [3, 8.5.1] that $H^i_{\mathcal{M}}(-, \mathbf{Q}(j))$ and the regulator in fact extend to the category of motives modulo *homological* equivalence.

7. Some computations.

(7.1) In this section we present some explicit formulas for Beilinson's regulator. First, we slightly modify the target $H^i_{\mathcal{MH}^+_{\mathbf{R}}}$ and use instead a weaker cohomology theory, known as Deligne cohomology (see [31], [35]).

Let us forget the weight filtration in the complex 2.4.1, which represents $R\,\mathrm{Hom}(\mathbf{R}(0), H)$ in $\mathcal{MH}_{\mathbf{R}}$. The functor which associates to $H \in \mathrm{Ob}\,\mathcal{MH}_{\mathbf{R}}$ the complex

$$H \oplus F^0 H_{\mathbf{C}} \xrightarrow{i_W - i_F} H_{\mathbf{C}} \tag{7.1.1}$$

extends to a functor $R\Gamma^w(\mathcal{MH}_{\mathbf{R}}, -) : D^b(\mathcal{MH}_{\mathbf{R}}) \longrightarrow D^b(\mathbf{R} - \mathit{mod})$. For a separated scheme X of finite type over $\mathbf{C}$, its Deligne cohomology is defined as

$$H^i_{\mathcal{D}}(X_{/\mathbf{C}}, \mathbf{R}(j)) = H^i(R\Gamma^w(\mathcal{MH}_{\mathbf{R}}, \underline{R\Gamma}(X, 0)(j))),$$

where $\underline{R\Gamma}(X, 0) \in \mathrm{Ob}\, D^b(\mathcal{MH}_{\mathbf{R}})$ is Beilinson's complex from 3.4.

Almost by definition, Deligne cohomology sits in an exact sequence

$$\begin{aligned} \cdots \longrightarrow H^i(X(\mathbf{C}), \mathbf{R}(n)) \longrightarrow H^i_{dR}(X_{/\mathbf{C}})/F^n \longrightarrow H^{i+1}_{\mathcal{D}}(X_{/\mathbf{C}}, \mathbf{R}(n)) \longrightarrow \\ \longrightarrow H^{i+1}(X(\mathbf{C}), \mathbf{R}(n)) \longrightarrow H^{i+1}_{dR}(X_{/\mathbf{C}})/F^n \longrightarrow \cdots \end{aligned} \tag{7.1.2}$$

In particular, for $n > i+1$, $H^{i+1}_{\mathcal{D}}(X_{/\mathbf{C}}, \mathbf{R}(n)) \xrightarrow{\sim} H^i(X(\mathbf{C}), \mathbf{C}/\mathbf{R}(n))$.

In a similar fashion, for $H \in \mathrm{Ob}\,\mathcal{MH}^+_{\mathbf{R}}$, one replaces 2.5.1 by

$$H^+ \oplus F^0 H_{dR} \longrightarrow H_{dR}$$

to get $R\Gamma^w(\mathcal{MH}^+_{\mathbf{R}}, -) : D^b(\mathcal{MH}^+_{\mathbf{R}}) \longrightarrow D^b(\mathbf{R} - \mathit{mod})$ and Deligne cohomology

$$H^i_{\mathcal{D}}(X_{/\mathbf{R}}, \mathbf{R}(j)) = H^i_{\mathcal{D}}(X_{/\mathbf{C}}, \mathbf{R}(j))^{\phi_\infty \otimes c = 1}$$

for $X_{/\mathbf{R}}$ separated of finite type.

There is a canonical map $H^i_{\mathcal{MH}^+_{\mathbf{R}}}(X_{/\mathbf{R}}, j) \longrightarrow H^i_{\mathcal{D}}(X_{/\mathbf{R}}, \mathbf{R}(j))$. It is an isomorphism for $i \leq j$, or even for $i \leq 2j$ if X is proper. The composition with the regulator $r_{\mathcal{H}}$ defines a regulator with values in Deligne cohomology (for any quasi-projective variety X over $\mathbf{Q}$)

$$r_{\mathcal{D}} : H^i_{\mathcal{M}}(X, \mathbf{Q}(j)) \longrightarrow H^i_{\mathcal{D}}(X_{/\mathbf{R}}, \mathbf{R}(j))$$

(7.2) For smooth varieties, Deligne cohomology can be computed as hypercohomology of quite explicit complexes of sheaves.
Suppose first that $X_{/\mathbf{C}}$ is proper and smooth. The complex

$$\mathbf{R}(n)_{\mathcal{D}} = [\mathbf{R}(n) \longrightarrow \mathcal{O}_X \longrightarrow \cdots \longrightarrow \Omega_X^{n-1}] \tag{7.2.1}$$

in degrees 0 to n is part of an exact triangle

$$\Omega_X^{<n}[-1] \longrightarrow \mathbf{R}(n)_{\mathcal{D}} \longrightarrow \mathbf{R}(n) \tag{7.2.2}$$

By GAGA and the degeneration of the Hodge spectral sequence we have

$$H^i(X(\mathbf{C}), \Omega_X^{<n}) \xrightarrow{\sim} H^i(X_{Zar}, \Omega_{X/\mathbf{C}}^{<n}) = H^i_{dR}(X_{/\mathbf{C}})/F^n$$

Comparing 7.1.2 and the cohomology sequence of 7.2.2, we see that

$$H^i_{\mathcal{D}}(X_{/\mathbf{C}}, \mathbf{R}(n)) \xrightarrow{\sim} H^i(X(\mathbf{C}), \mathbf{R}(n)_{\mathcal{D}})$$

If $X_{/\mathbf{C}}$ is smooth and separated, then there exists an open immersion $j : X \hookrightarrow \overline{X}$ of X into a proper smooth variety $\overline{X}$ such that the complement $D = \overline{X} - X$ is a divisor with normal crossings. There are natural maps

$$Rj_*\mathbf{R}(n) \longrightarrow Rj_*\Omega^{\cdot}_X, \qquad \Omega_{\overline{X}}^{\geq n}(\log D) \longrightarrow Rj_*\Omega^{\cdot}_X$$

and using their difference we define

$$\mathbf{R}(n)_{\mathcal{D}} = \mathrm{Cone}(\Omega_{\overline{X}}^{\geq n}(\log D) \oplus Rj_*\mathbf{R}(n) \longrightarrow Rj_*\Omega^{\cdot}_X)[-1]$$

Again, the degeneration of the logarithmic Hodge spectral sequence

$$H^i(\overline{X}(\mathbf{C}), \Omega_{\overline{X}}^{\geq n}(\log D)) \xrightarrow{\sim} F^n H^i_{dR}(X_{/\mathbf{C}})$$

and 7.1.2 imply that

$$H^i_{\mathcal{D}}(X_{/\mathbf{C}}, \mathbf{R}(n)) \xrightarrow{\sim} H^i(\overline{X}(\mathbf{C}), \mathbf{R}(n)_{\mathcal{D}})$$

(7.3) A version of the complex $\mathbf{R}(n)_{\mathcal{D}}$ more suitable for calculations is given by

$$\widetilde{\mathbf{R}(n)}_{\mathcal{D}} = \mathrm{Cone}(\Omega_{\overline{X}}^{\geq n}(\log D) \longrightarrow j_*\mathcal{A}^{\cdot}_X \otimes \mathbf{R}(n-1))[-1],$$

where $\mathcal{A}^{\cdot}_X$ is the de Rham complex of smooth real valued differential forms on $X(\mathbf{C})$. There is a canonical quasi-isomorphism between $\mathbf{R}(n)_{\mathcal{D}}$ and $\widetilde{\mathbf{R}(n)}_{\mathcal{D}}$ induced by quasi-isomorphisms

$$Rj_*\mathbf{C} \longrightarrow Rj_*\Omega^{\cdot}_X \longrightarrow j_*\mathcal{A}^{\cdot}_X \otimes \mathbf{C}$$

and the projection $\pi_{n-1} : \mathbf{C} \longrightarrow \mathbf{R}(n-1)$ along $\mathbf{R}(n)$.
This gives an explicit description of $H^n_{\mathcal{D}}(X_{/\mathbf{C}}, \mathbf{R}(n)) \xrightarrow{\sim} H^n(\overline{X}(\mathbf{C}), \widetilde{\mathbf{R}(n)}_{\mathcal{D}})$ as the $\mathbf{R}$-vector space

$$\frac{\{\varphi \in H^0(X(\mathbf{C}), \mathcal{A}^{n-1} \otimes \mathbf{R}(n-1)) \mid d\varphi = \pi_{n-1}(\omega), \omega \in H^0(\overline{X}(\mathbf{C}), \Omega^n_{\overline{X}}(\log D))\}}{dH^0(X(\mathbf{C}), \mathcal{A}^{n-2} \otimes \mathbf{R}(n-1))} \tag{7.3.1}$$

The cup product of the classes of φ_m, φ_n in $H^m_{\mathcal{D}}(X_{/\mathbf{C}}, \mathbf{R}(m))$ resp. $H^n_{\mathcal{D}}(X_{/\mathbf{C}}, \mathbf{R}(n))$ is represented by

$$\varphi_m \cup \varphi_n = \varphi_m \wedge \pi_n \omega_n + (-1)^m \pi_m \omega_m \wedge \varphi_n \tag{7.3.2}$$

See [31] for general formulas for the cup product in Deligne cohomology.

(7.4) Any successful attack on the Weak Conjecture 6.6 usually proceeds in three steps:
(1) Construction of elements in $H^i_{\mathcal{M}}(X, \mathbf{Q}(j))$.
(2) Calculation of their image under the regulator map.
(3) Comparison of the result with the value of the L-function.

Let us give a simple example. If X is a smooth quasi-projective variety over $\mathbf{Q}$, then the regulator

$$r_{\mathcal{D}} : \mathcal{O}(X)^* \otimes \mathbf{Q} = H^1_{\mathcal{M}}(X, \mathbf{Q}(1)) \longrightarrow H^1_{\mathcal{D}}(X_{/\mathbf{C}}, \mathbf{R}(1)) \tag{7.4.1}$$

maps $f \in \mathcal{O}(X)^*$ into $\varphi = \log|f|$ (with corresponding $\omega = d\log(f)$). For $f_1, \dots, f_n \in H^1_{\mathcal{M}}(X, \mathbf{Q}(1))$, denote by $\{f_1, \dots, f_n\}$ their cup product in $H^n_{\mathcal{M}}(X, \mathbf{Q}(n))$. Both regulators $r_{\mathcal{H}}, r_{\mathcal{D}}$ are multiplicative, hence

$$r_{\mathcal{D}}(\{f_1, \dots, f_n\}) = r_{\mathcal{D}}(f_1) \cup \dots \cup r_{\mathcal{D}}(f_n) \in H^n_{\mathcal{D}}(X_{/\mathbf{C}}, \mathbf{R}(n))$$

In particular, if $f_1, \dots, f_n$ are any rational functions on X, write $U \subseteq X$ for the complement of their divisors. The symbol $\{f_1, \dots, f_n\}$ lies in $H^n_{\mathcal{M}}(U, \mathbf{Q}(n))$ and the problem is how to extend it to X. It is possible, sometimes, to exploit an inherent symmetry of the situation in question and construct a natural projector $\pi_{\mathcal{M}}$ from $H^n_{\mathcal{M}}(U, \mathbf{Q}(n))$ to $H^n_{\mathcal{M}}(X, \mathbf{Q}(n))$, completing thus Step 1.

As a next step, one copies the construction of $\pi_{\mathcal{M}}$ to get a similar projector $\pi_{\mathcal{D}}$ in Deligne cohomology. The regulator is then equal to

$$r_{\mathcal{D}}(\pi_{\mathcal{M}}(\{f_1, \dots, f_n\})) = \pi_{\mathcal{D}}(r_{\mathcal{D}}(f_1) \cup \dots \cup r_{\mathcal{D}}(f_n))$$

Consider the simplest case of $H^2_{\mathcal{M}}(X, \mathbf{Q}(2))$. Suppose that X is a smooth scheme over a field k. According to [68, Th.4], there is a spectral sequence

$$E_1^{pq} = \coprod_{x \in X^{(p)}} (K_{-p-q}(k(x)) \otimes \mathbf{Q})^{(j-p)} \Longrightarrow (K_{-p-q}(X) \otimes \mathbf{Q})^{(j)},$$

where $X^{(p)}$ denotes the set of points of codimension p on X. For a field F and $i = 1, 2$, $(K_i(F) \otimes \mathbf{Q})^{(j)}$ vanishes unless $j = i$, so the spectral sequence reduces to

$$H^2_{\mathcal{M}}(X, \mathbf{Q}(2)) = \mathrm{Ker}(K_2(k(X)) \otimes \mathbf{Q} \xrightarrow{T} \coprod_{x \in X^{(1)}} k(x)^* \otimes \mathbf{Q})$$

Here the value of $T(\{f, g\})$ at $x \in X^{(1)}$ is equal to the tame symbol

$$(-1)^{\mathrm{ord}_x(f)\cdot \mathrm{ord}_x(g)}\cdot\left(\frac{f^{\mathrm{ord}_x(g)}}{g^{\mathrm{ord}_x(f)}}\right)(x).$$

The following construction is due to Bloch. Suppose that X is a smooth projective curve over a number field k and $f, g \in k(X)^*$ two rational functions on X. Write D for the union of supports of the divisors of f and g and k_D for the splitting field of D. We assume that, for a fixed integer N, a difference of any two geometric points $P, Q \in D(k_D)$ in the Jacobian of X is torsion of order dividing N. Fix $O \in D(k_D)$. For any $P \in D(k_D) - \{O\}$ there is a rational function $f_P \in k_D(X)^*$ with divisor $N(P) - N(O)$. Write $c_P \in k_D^*$ for the value of the tame symbol $T(\{f, g\})$ at $P \in D(k_D)$. Then

$$\pi(\{f,g\}) = \{f,g\} + \sum_{O\neq P\in D(k_D)} \{f_P, c_P\}\otimes\frac{1}{N}$$

lies in $K_2(k_D(X))\otimes\mathbf{Q}$, is $G(k_D/k)$-invariant, does not depend on the choice of f_P (as K_2 is torsion for number fields) and lies in the kernel of the tame symbol T (this is clear outside of O; vanishing of T at O then follows from Weil's reciprocity law), hence represents an element of

$$H^2_{\mathcal{M}}(X\times_k k_D, \mathbf{Q}(2))^{G(k_D/k)} \xrightarrow{\sim} H^2_{\mathcal{M}}(X, \mathbf{Q}(2))$$

Interestingly, there exist examples of families of elliptic curves $E_{/\mathbf{Q}}$ with elements in $H^2_{\mathcal{M}}(E, \mathbf{Q}(2))$ coming from functions with divisors supported at non-torsion points.

(7.5) Fix an embedding $k \hookrightarrow \mathbf{C}$. Let us compute the regulator $r_{\mathcal{D}}$ on $\pi(\{f, g\})$. Set $U = X - D$. According to 7.3.1, we have

$$H^2_{\mathcal{D}}(U_{/\mathbf{C}}, \mathbf{R}(2)) \xrightarrow{\sim} H^1(U(\mathbf{C}), \mathbf{R}(1)), \quad H^2_{\mathcal{D}}(X_{/\mathbf{C}}, \mathbf{R}(2)) \xrightarrow{\sim} H^1(X(\mathbf{C}), \mathbf{R}(1)).$$

By 7.4.1, $r_{\mathcal{D}}(f) \in H^1_{\mathcal{D}}(U_{/\mathbf{C}}, \mathbf{R}(1))$ is represented by $\varphi_f = \log|f|$ with $df = \pi_0(\omega_f)$ for $\omega_f = \partial f/f$.

According to 7.3.2, the cup product $r_{\mathcal{D}}(\{f, g\}) \in H^1(U(\mathbf{C}), \mathbf{R}(2))$ is represented by

$$\psi(f,g) = \varphi_f(\pi_1\omega_g) - (\pi_1\omega_f)\varphi_g = \log|f|d\arg(g) - \log|g|d\arg(f)$$

The pairing

$$\langle\ ,\ \rangle:\ \psi,\omega \mapsto \frac{1}{2\pi i}\int_{U(\mathbf{C})} \psi\wedge\omega$$

defines a map $H^1(U(\mathbf{C}), \mathbf{R}(1)) \longrightarrow H^0(X(\mathbf{C}), \Omega^1_X)^*$, whose restriction to $H^1(X(\mathbf{C}), \mathbf{R}(1))$ is an isomorphism (of $\mathbf{R}$-vector spaces).

An elementary calculation shows that, modulo an exact form,

$$\psi(f,g)\wedge\omega \sim \log|g|d\log(\overline{g})\wedge\omega,$$

which vanishes if g is a constant function. Consequently, for $\omega \in H^0(X(\mathbf{C}), \Omega^1_X)$,

$$\langle r_{\mathcal{D}}(\pi(\{f,g\})), \omega\rangle = \frac{1}{2\pi i}\int_{X(\mathbf{C})} \log|g|d\log(\overline{g})\wedge\omega, \qquad (7.5.1)$$

completing Step 2 of the program formulated in 7.4.

For an elliptic curve, this integral can be computed explicitly in terms of certain Kronecker (– Eisenstein – Lerch) series. More precisely, suppose that X is an elliptic curve defined over $\mathbf{R}$ with complex points $X(\mathbf{C}) = \mathbf{C}/\Gamma$ for $\Gamma = \mathbf{Z} + \mathbf{Z}\tau$ ($\mathrm{Im}(\tau) > 0$) and f, g two rational functions on X with divisors supported at torsion points. Writing dz for the canonical differential on $\mathbf{C}/\Gamma$ and

$$(z,\gamma) = exp\left(\frac{2\pi i}{\tau - \overline{\tau}}(z\overline{\gamma} - \overline{z}\gamma)\right), \qquad z \in \mathbf{C}/\Gamma,\ \gamma \in \Gamma = \mathbf{Z} + \mathbf{Z}\tau$$

for the duality between $\mathbf{C}/\Gamma$ and Γ, the regulator is given by the formula

$$\langle r_{\mathcal{D}}(\pi(\{f,g\})), dz\rangle = -\frac{1}{2}\sqrt{\frac{\tau-\overline{\tau}}{2\pi i}} \sum_{x,y \in X(\mathbf{C})} \mathrm{ord}_x(f)\,\mathrm{ord}_y(g) \sum_{0 \neq \gamma \in \Gamma} \frac{\overline{\gamma}}{|\gamma|^4}(y-x,\gamma) \tag{7.5.2}$$

The reader may wish to consult [1],[30],[57] for the details of the computation. If the curve has complex multiplication, then its L-function at $s = 2$ is a series of the same type and it is relatively easy to compare the regulator in 7.5.2 with $L(X, 2)$. Historically, this computation was performed first by Bloch [6],[7], using another definition of the regulator.

If $X_{/\mathbf{Q}}$ is an elliptic curve *without* complex multiplication but with a nontrivial torsion over $\mathbf{Q}$, one can repeat the construction of 7.4 and get an element of $H^2_{\mathcal{M}}(X, \mathbf{Q}(2))$, but not necessarily of $H^2(X, \mathbf{Q}(2))_{\mathbf{Z}}$. Amusingly, the obstruction to integrality is related to the third Bernoulli polynomial in [11],[60]. In fact, it was only after the calculations of [11] that the cohomology of the integral model was incorporated into the conjectures.

(7.6) Note that the above construction works in a family: if S is the (open) modular curve over $\mathbf{C}$ classifying elliptic curves with a full level N structure (for a fixed $N \geq 3$), consider the universal elliptic curve $p : X \longrightarrow S$ and choose two sections u, v which generate the subgroup of N-torsion of X. There exist rational functions f, g on X with divisors equal to $N(u) - N(0)$ resp. $N(v) - N(0)$ and they can be normalized in such a way that $f_{|v} = 1$, $g_{|u} = 1$. The symbol $\{f, g\}$ then represents an element of $H^2_{\mathcal{M}}(X, \mathbf{Q}(2))$. Its restriction to each fiber X_s has the same field of definition as $s \in S(\mathbf{C})$.

In general, if $p : X \longrightarrow S$ is a proper smooth map between two varieties over $\mathbf{C}$ and $a \in H^i_{\mathcal{M}}(X, \mathbf{Q}(n))$ a global element with $i < n$, then the regulator of its restriction to the fibre X_s

$$r_{\mathcal{D}}(a_s) \in H^i_{\mathcal{D}}(X_s, \mathbf{R}(n))$$

is "locally constant" as a function of s. The intuition is quite clear: for $i < n$, the groups $H^i_{\mathcal{D}}(X_s, \mathbf{R}(n)) \xrightarrow{\sim} H^{i-1}(X_s(\mathbf{C}), \mathbf{C}/\mathbf{R}(n))$ form a locally constant sheaf on $S(\mathbf{C})$. The formal argument goes as as follows: $r_{\mathcal{D}}(a) \in H^i(\overline{X}(\mathbf{C}), \mathbf{R}(n)_{\mathcal{D}})$ defines a global section of the sheaf $R^i p_* j^* \mathbf{R}(n)_{\mathcal{D}}$, where $\mathbf{R}(n)_{\mathcal{D}}$ is the complex defined in 7.3, living on a suitable compactification $j : X \hookrightarrow \overline{X}$. The complex $j^*\mathbf{R}(n)_{\mathcal{D}}$ is quasi isomorphic to

$$\mathcal{F}_X = [\mathbf{R}(n) \longrightarrow \mathcal{O}_X \longrightarrow \cdots \Omega^{n-1}_X]$$

Denote by $\mathcal{F}_{X/S}$ the analogous complex in which we replace differentials on X by relative differentials.
We may assume that S is a curve. Then the exact triangle

$$p^*\Omega^1_S \otimes \Omega^{\cdot}_{X/S}/F^{n-1}[-2] \longrightarrow \mathcal{F}_X \longrightarrow \mathcal{F}_{X/S}$$

induces an exact sequence

$$R^i p_* \mathcal{F}_X \longrightarrow R^i p_* \mathcal{F}_{X/S} \xrightarrow{D} \mathcal{H}^{i-1}_{dR}(X/S)/\mathcal{F}^{n-1} \otimes_{\mathcal{O}_S} \Omega^1_S.$$

Here the map D comes from an exact sequence

$$H^{i-1}_B(X/S, \mathbf{R}(n)) \longrightarrow \mathcal{H}^{i-1}_{dR}(X/S)/\mathcal{F}^n \longrightarrow R^i p_* \mathcal{F}_{X/S} \longrightarrow 0$$

and the Gauss-Manin connection

$$\mathcal{H}^{i-1}_{dR}(X/S)/\mathcal{F}^n \xrightarrow{\nabla} \mathcal{H}^{i-1}_{dR}(X/S)/\mathcal{F}^{n-1},$$

with respect to which the Betti cohomology of the fibres is horizontal.
Finally, $r_{\mathcal{D}}(a_s) \in H^0(S, R^i p_* \mathcal{F}_{X/S})$ must be a section of $\mathrm{Ker}(D)$, which is (assuming $i < n$), nothing else than the locally constant sheaf of $H^{i-1}(X_s(\mathbf{C}), \mathbf{C}/\mathbf{R}(n))$.

This digression explains why, for example, there is no "uniform" construction of elements in $H^2_{\mathcal{M}}(E, \mathbf{Q}(n))$, for elliptic curves E and $n > 2$ (corresponding to the values $L(E, n) = L(h^1(E), n)$, conjecturally). Such a uniform construction works, however, for $H^{n+1}_{\mathcal{M}}(\mathrm{Sym}^n(h^1(E)), n+1)$ (Eisenstein symbols, constructed by Beilinson [2]; see also [24],[29]). For curves with complex multiplication, Deninger [24] defines a map

$$H^{2m}_{\mathcal{M}}(\mathrm{Sym}^{2m-1}(h^1(E)), 2m) \longrightarrow H^2_{\mathcal{M}}(h^1(E), m+1)$$

into cohomology responsible for the value $L(E, m+1)$.

8. Evidence for the conjectures.

In this section we make a survey of the progress towards the conjectures in various special cases; results on Deligne's conjecture in the critical case will not be mentioned.

(1) Zeta functions of number fields.
For $\zeta_F(s)$, the strong conjecture has been proved by Borel [14],[15] at all integers (with $s = 0, 1$ being classical), using his own definition of a regulator

$$K_{2n-1}(\mathbf{C}) \longrightarrow \mathbf{R}(n-1)$$

It is proved in [1] (cf. [55]) that Borel's and Beilinson's regulator coincide.

In recent years, there has been a resurgence of activity centered around classical polylogarithm functions, values of $\zeta_F(s)$ and their relation to Borel's regulator. See the articles [5],[10],[36] in these proceedings.

(2) Dirichlet L-series.
For every Dirichlet character χ there is a Chow motive M_χ over $\mathbf{Q}$ such that $L(s, \chi) = L(M_\chi, s)$. Beilinson [1] proved the weak conjecture at all integers; the results of Borel (1) then imply the strong conjecture.

Remark. These are the only cases when the strong conjecture has been proved, except for (9) below, dealing with the central point.

(3) Eliptic curves with complex multiplication.
For $E_{/\mathbf{Q}}$ an elliptic curve with complex multiplication, the weak conjecture for $L(h^1(E), s)$ has been proved at $s = 2$ by Bloch [6], [7], Beilinson [1]; for elliptic curves of Shimura type, proved at all integers $s \geq 2$ by Deninger [24].

(4) Motives of Hecke characters of an imaginary quadratic field.
The weak conjecture is proved by Deninger [25], where he also reproves (2).

(5) Modular forms.
If f is a newform of weight $k + 2$ on some congruence subgroup of $SL_2(\mathbf{Z})$, then there is a Grothendieck motive $M(f)$ associated to f ([62]) such that $L(f, s) = L(M(f), s)$. The motive $M(f)$ is constructed from a Chow motive M corresponding to all cusp forms of given type using a projector Π_f in Hecke algebra. For every integer $n \geq k + 2$, Beilinson [2] for $k = 0$ and Scholl [65] in general construct a subspace $P_n \subseteq H^{k+2}_{\mathcal{M}}(M, \mathbf{Q}(n))$ such that $\det(r_{\mathcal{D}}(\Pi_f(P_n)))$ gives the value of $L(f, n)$. However, for $k > 0$, it is not known if P_n lies in $H^{k+2}_{\mathcal{M}}(M, \mathbf{Q}(n))_{\mathbf{Z}}$. See also [29],[59] for more details.

(6) Shimura curves over Q.
For any Shimura curve X coming from an automorphic form on an indefinite quaternion algebra B over $\mathbf{Q}$, Ramakrishnan [52] proves the weak conjecture for $L(h^1(X), s)$ at all integers $s \geq 2$. Ramakrishnan uses Jacquet-Langlands correspondence between automorphic forms on B and GL_2, together with Faltings isogeny theorem to deduce this result from the corresponding statement for modular forms on GL_2, proved by Beilinson (see (5) above).

(7) Product of two modular curves.
For two modular curves C_1, C_2 defined over $\mathbf{Q}$, Beilinson [1] proves the weak conjecture for $L(h^2(C_1 \times C_2), s)$ at $s = 2$, but he makes an incorrect argument for integrality of elements in $H^3_{\mathcal{M}}(C_1 \times C_2, \mathbf{Q}(2))$ he constructs. A revised version of [54] is supposed to fill this gap.

(8) Hilbert-Blumenthal surfaces.
Let X be a Hilbert-Blumenthal surface over a real quadratic field F for some congruence subgroup of $GL_2(\mathcal{O}_F)$. There is a smooth toroidal compactification $\overline{X}$ of X. Ramakrishnan [54] proves the weak conjecture for an incomplete L-series $L_S(h^2(\overline{X}), s)$ (with bad Euler factors removed) at $s = 2$. The integrality of relevant elements in the motivic cohomology is not known, however.

(9) Elliptic curves at the central point.
If E is an elliptic curve over $\mathbf{Q}$ which is modular (i.e. admits a nontrivial map $X_0(N) \longrightarrow E$) and the order of vanishing of $L(E, s)$ at $s = 1$ is equal to 0 or 1, then the conjecture of Birch and Swinnerton-Dyer is true for E, up to a controlled rational factor. This follows from the work of Kolyvagin [46], combined with [38] and nonvanishing theorems about L-functions of modular forms (see [51]).

(10) Numerical evidence.
In [11], Bloch and Grayson give results of computations of the regulator on $H^2_{\mathcal{M}}(E, \mathbf{Q}(2))_{\mathbf{Z}}$ for certain elliptic curves without complex multiplication. The result compares favorably with the value $L(E, 2)$, as expected.
Mestre and Schappacher [50] report on similar computations for the symmetric square of an elliptic curve without complex multiplication.

9. Mixed motives revisited.

(9.1) This section contains a reformulation of Beilinson's conjectures in terms of mixed motives, due to Scholl [63]. We have to assume that the formalism of motivic cohomology over $\mathbf{Q}$, described in Sec. 3–4, makes sense and that the categories $\mathcal{MM}_{\mathbf{Z}}$, $\mathcal{MM}_{\mathbf{Q}}$ exist and behave as expected.

For the reader's convenience, we summarize a few relevant formulas. If X is a smooth projective variety over $\mathbf{Q}$ and $M = h^i(X)(n)$, then we should have

$$\mathrm{Ext}^0_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M) = \begin{cases} 0 & \text{if } i \neq 2n \\ CH^n(X)/CH^n(X)_0 \otimes \mathbf{Q} & \text{if } i = 2n \end{cases}$$
$$\mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M) = \begin{cases} H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}} & \text{if } i+1 \neq 2n \\ CH^n(X)_0 \otimes \mathbf{Q} & \text{if } i+1 = 2n \end{cases}$$

The Ext group in $\mathcal{MM}_{\mathbf{Q}}$ should be given by the same formula with $H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(n))_{\mathbf{Z}}$ replaced by $H^{i+1}_{\mathcal{M}}(X, \mathbf{Q}(n))$.
A special case of the above formulas (for $M = h^2(\mathbf{P}^1)(2)$) is

$$\mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Q}}}(\mathbf{Q}(0), \mathbf{Q}(1)) = \mathbf{Q}^* \otimes \mathbf{Q}$$
$$\mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), \mathbf{Q}(1)) = \mathbf{Z}^* \otimes \mathbf{Q} = 0$$

Finally, for $w = i - 2n < -1$, the regulator map should be none else than the Hodge realization

$$\mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), h^i(X)(n)) \longrightarrow \mathrm{Ext}^1_{\mathcal{MH}^+_{\mathbf{R}}}(\mathbf{R}(0), H^i(X(\mathbf{C}), \mathbf{R}(n)))$$

(9.2) Deligne's period map

$$\alpha_E : E_B^+ \otimes \mathbf{R} \longrightarrow (E_{dR}/F^0) \otimes \mathbf{R}$$

makes sense for any mixed motive $E \in \mathrm{Ob}\, \mathcal{MM}_{\mathbf{Q}}$. Scholl [63] calls E critical if α_E is an isomorphism.
For a critical mixed motive E, Deligne's period $c^+(E) \in \mathbf{R}^*/\mathbf{Q}^*$ is defined as the determinant of α_E with respect to the $\mathbf{Q}$-structures E_B^+ and E_{dR}/F^0.
One defines the L-function $L(E, s)$ of a mixed motive in the same way as $L(M, s)$ is defined in 1.4 (assuming the independence of ℓ of the local L-factors). If E is a mixed motive over $\mathbf{Z}$, then

$$L(E, s) = \prod_j L(Gr_j^W(E), s)$$

(9.3) Scholl [63] makes the following three conjectures:
(A) If $E \in \mathrm{Ob}\, \mathcal{MM}_{\mathbf{Q}}$ is critical, then $L(E, 0) \cdot c^+(E)^{-1} \in \mathbf{Q}$.
(B) For any mixed motive E over $\mathbf{Z}$,

$$\mathrm{ord}_{s=0}\, L(E, s) = \dim \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), E^\vee(1)) - \dim \mathrm{Ext}^0_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), E^\vee(1)).$$

(C) If E is a mixed motive over $\mathbf{Z}$ satisfying

$$\mathrm{Ext}^i_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), E) = \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), E^\vee(1)) \quad \text{for } i = 0, 1, \qquad (9.3.1)$$

then E is critical.

(9.4) There is a natural construction which transforms any mixed motive M over $\mathbf{Z}$ into a new mixed motive E over $\mathbf{Z}$ satisfying 9.3.1, by taking successive universal extensions and killing inconvenient subgroups. It proceeds in four steps

$$0 \longrightarrow \mathrm{Hom}(\mathbf{Q}(0), M) \otimes \mathbf{Q}(0) \longrightarrow M \longrightarrow M_1 \longrightarrow 0$$
$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M_1) \otimes \mathbf{Q}(0) \longrightarrow 0$$
$$0 \longrightarrow M_3 \longrightarrow M_2 \longrightarrow \mathrm{Hom}(\mathbf{Q}(0), M_2^\vee(1))^\vee \otimes \mathbf{Q}(1) \longrightarrow 0$$
$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M_3^\vee(1))^\vee \otimes \mathbf{Q}(1) \longrightarrow E \longrightarrow M_3 \longrightarrow 0$$

which do not require special comment. In verifying that E indeed satisfies 9.3.1 one has to remember that $\mathrm{Ext}^1(\mathbf{Q}(0), \mathbf{Q}(1))$ is supposed to vanish in $\mathcal{MM}_{\mathbf{Z}}$.
The L-function of E is equal to

$$L(E, s) = L(M, s)\zeta(s)^a\zeta(s+1)^b,$$

where

$$a = \dim \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M) - \dim \mathrm{Ext}^0_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M)$$
$$b = \dim \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M^\vee(1)) - \dim \mathrm{Ext}^0_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M^\vee(1))$$

The conjectures (A)–(C) predict that this new motive E is critical, $L(E, 0) \neq 0$ and

$$L^*(M, 0) \in L(E, 0) \cdot \mathbf{Q}^* = c^+(E) \cdot \mathbf{Q}^*. \tag{9.4.1}$$

(9.5) We shall now consider the case of a pure motive $M = h^i(X)(n)$ of weight $w = i - 2n$.
Suppose first that $w \leq -2$ and that $\mathrm{Hom}(\mathbf{Q}(0), M^\vee(1)) = 0$. For weight reasons and by semisimplicity of $\mathcal{M}_{\mathbf{Q}}$,

$$\mathrm{Ext}^0(\mathbf{Q}(0), M) = \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M^\vee(1)) = 0$$

This means that E is simply the universal extension

$$0 \longrightarrow M \longrightarrow E \longrightarrow \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M) \otimes \mathbf{Q}(0) \longrightarrow 0$$

as $\alpha_{\mathbf{Q}(0)} = 0$ and $\mathrm{Ker}(\alpha_M) = 0$, snake lemma implies that E is critical if and only if the connecting homomorphism

$$\mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M) \xrightarrow{\partial_M} \mathrm{Coker}(\alpha_M)$$

becomes an isomorphism after tensoring with $\mathbf{R}$. This map, however, is nothing else than the regulator and the canonical $\mathbf{Q}$-structure on its target is $\mathcal{D}(M)$, so 9.4.1 is equivalent to the first half of the conjecture 6.1.2

$$\det(\partial_M) = L(h^i(X), n)\mathbf{Q}^* \in \mathbf{R}^*/\mathbf{Q}^*$$

Continuing with the same notation, set $N = M^\vee(1)$. This is a pure motive of weight $-2 - w \geq 0$ with $\mathrm{Hom}(\mathbf{Q}(0), N) = 0$ and its 'universal extension' is

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M)^\vee \otimes \mathbf{Q}(1) \longrightarrow E^\vee(1) \longrightarrow N \longrightarrow 0$$

The period map for N is given by the connecting homomorphism

$$\partial_N : \mathrm{Ker}(\alpha_N) \longrightarrow \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M)^\vee \otimes \mathbf{R}.$$

Using 2.2.1 and the fact that ∂_M, ∂_N are adjoint maps, we see that 9.4.1 for N becomes

$$\det(\partial_N^\vee) = L^*(N,0)\mathbf{Q}^* = L^*(h^i(X), i+1-n)\mathbf{Q}^* \in \mathbf{R}^*/\mathbf{Q}^*,$$

which is the second half of the conjecture 6.1.2.

If M is pure of weight -2 with no other restrictions, then E has to be constructed in two steps. Its period map accounts for Beilinson's 'thickened regulator' $r_{\mathcal{H}} \oplus r_B$ in 6.2, but is more canonical; the reader is invited to verify that (A)–(C) predict that the following sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{MM}_{\mathbf{Z}}}(\mathbf{Q}(0), M) \otimes \mathbf{R} \longrightarrow \mathrm{Coker}(\alpha_M) \longrightarrow \mathrm{Hom}(\mathbf{Q}(0), M^\vee(1))^\vee \otimes \mathbf{R} \longrightarrow 0$$

is exact. The natural $\mathbf{Q}$-structures, say, $\mathcal{E}(M), \mathcal{D}(M), \mathcal{H}(M)$ on determinants of all three terms define the period

$$c^+(E) = \mathcal{E}(M) \cdot \mathcal{D}(M)^{-1} \cdot \mathcal{H}(M) \in \mathbf{R}^*/\mathbf{Q}^*,$$

equal to $L^*(M,0)$ by 9.4.1. This is equivalent to the conjecture 6.3.

The case of $w = -1$, related to height pairings, is discussed in detail in [64].

Scholl [63] shows that, modulo some other (reasonable) hypotheses about the structure of $\mathcal{MM}_{\mathbf{Q}}$, his conjectures (A)–(C) are in fact *equivalent* to Beilinson's conjectures 6.1, 6.3, 6.5.

Finally, let us mention that Fontaine and Perrin-Riou [32],[33] introduced a beautiful six-term exact sequence, which is effectively a translation of (A)–(C), but works for general mixed motives over $\mathbf{Q}$.

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