Non-triviality of CM points in ring class field towers

Esther Aflalo, Jan Nekovář

Introduction

In [Co-Va 1], Cornut and Vatsal proved a generalisation of Mazur's conjecture on higher Heegner points, valid for CM points on Shimura curves with unramified central character over totally real number fields. In the present work, which grew out of the thesis of the first author at University Paris 6, we extend the results of [Co-Va 1] to more general families of CM points on slightly less general Shimura curves (those with trivial central character).

In more concrete terms, let X be such a Shimura curve over a totally real number field F and let A_0 be a simple quotient of the Jacobian of X. Fix a quadratic CM extension K of F and two relatively prime ideals c and $b \neq (1)$ of O_F. Let $K[cb^\infty]$ be the union of all ring class fields of K (in the sense of 1.1.2 below) of conductors cb^m $(m \geq 1)$; fix a character χ_0 of $\mathrm{Gal}(K[cb^\infty]/K)_{\mathrm{tors}}$. If $\{x_m\}_{m\in\mathbf{N}}$ is a sequence of CM points on X lying in the same b-isogeny class (in the sense of 2.2.4 below) and such that b^m divides the conductor $c(x_m)$ of x_m for all $m \geq 1$, then our main result (Theorem 4.3.1) states that, asuming a certain natural necessary compatibility condition for A_0 and χ_0, for each sufficiently large $m >> 0$ there exists a character χ_m of $\mathrm{Gal}(K[cb^\infty]/K)$ of finite order which extends χ_0, whose conductor differs from $c(x_m)$ by a divisor of c, and for which the $\overline{\chi}_m$-component of the image of x_m in A_0 is not torsion.

The proof is based on the techniques of Cornut and Vatsal, who proved in [Co-Va 1, Thm. 4.2] this result (for slightly more general Shimura curves) in the special case $c = (1)$, $b = P$ prime: the Galois action on the CM points in the given b-isogeny class is decomposed into a "chaotic" and a "geometric" part. The chaotic action is treated using a known case of the André-Oort conjecture proved by Edixhoven and Yafaev [Ed-Ya, Thm. 1.2]. The corresponding argument in [Co-Va 2, 3.21] was somewhat sketchy; the appendix [Co 2] to the present article provides the required translation of the general results of [Ed-Ya] to the special case considered here and in [Co-Va 1,2] (it is very likely that the chaotic action could have also been treated using a p-adic version of Ratner's theorem, as in [Co-Va 1,2], but we did not pursue this line of argument). The geometric Galois action is analysed - and this our principal innovation - using representation theory of $GL(2)$ and its inner forms over p-adic fields. This approach leads naturally to the questions considered by Gross and Prasad [Gr-Pr]. In particular, the compatibility condition for A_0 and χ_0 alluded to above has a natural formulation in terms of "test vectors" in the sense of [Gr-Pr]. The existence of test vectors is controlled by the values of certain local ε-factors ([Tu], [Wa], [Sa]); this is exploited in [Ne 3], where the second author deduces a fairly general result on the parity-induced growth of Selmer groups of Hilbert modular forms over ring class fields.

Let us describe the contents of the present article in more detail. In §1 we collect background material on ring class fields and local ε-factors. In §2, after preliminaries on Shimura curves (§2.1) and CM points (§2.2) we analyse first the chaotic (§2.3) and then the geometric Galois action (§2.4); this leads to a weak non-vanishing result (Theorem 2.5.1) which generalises [Co-Va 1, Thm. 4.10]. The purpose of the representation-theoretical interlude in §3 is two-fold: firstly, to exhibit explicit test vectors (and thus generalise [Gr-Pr, Prop. 2.3]), which implies an explicit variant (Theorem 2.5.2) of Theorem 2.5.1; secondly, to prepare ground for §4, in which we deduce our main result (Theorem 4.3.1) from Theorem 2.5.1 and the distribution relations for CM points that were studied in [Co-Va 1, §6].

The authors are grateful to C. Cornut for helpful discussions and for the appendix [Co 2].

1. Galois groups

1.1. Ring class fields

1.1.1. Throughout §1, F is a totally real number field of degree d and K a totally imaginary quadratic extension of F. Let $\eta = \eta_{K/F} : \mathbf{A}_F^*/F^* N_{K/F}(\mathbf{A}_K^*) \longrightarrow \{\pm 1\}$ be the quadratic character associated to the extension K/F. We denote by $\mathrm{rec}_K : \widehat{K}^* = (K \otimes \widehat{\mathbf{Z}})^* \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)$ the reciprocity map normalised by letting uniformisers correspond to *geometric* Frobenius elements. For a subextension L/K of K^{ab}/K we denote by $\mathrm{rec}_{L/K}$ the morphism $\widehat{K}^* \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K) \twoheadrightarrow \mathrm{Gal}(L/K)$.

1.1.2. For a non-zero ideal $c \subset O_F$, the **ring class field of K of conductor** c is the finite abelian extension $K[c]$ of K satisfying

$$\mathrm{rec}_K : \widehat{K}^*/K^*\widehat{F}^*\widehat{O}_c^* \xrightarrow{\sim} \mathrm{Gal}(K[c]/K),$$

where $O_c = O_F + cO_K$ is the O_F-order in O_K of conductor c, $\widehat{O}_c = O_c \otimes \widehat{\mathbf{Z}}$ and $\widehat{O}_c^* = (\widehat{O}_c)^*$. The group

$$U(c) = \widehat{O}_K^*/\widehat{O}_c^* \xrightarrow{\sim} (O_K/cO_K)^*/(O_F/cO_F)^*$$

maps naturally to $\widehat{K}^*/K^*\widehat{F}^*\widehat{O}_c^*$, giving rise to an exact sequence

$$0 \longrightarrow Z(c) \longrightarrow U(c) \longrightarrow \mathrm{Gal}(K[c]/K) \longrightarrow \mathrm{Gal}(K[1]/K) \longrightarrow 0, \tag{1.1.2.1}$$

in which $Z(c)$ is a subquotient of the finite abelian group $(K^* \cap \widehat{F}^*\widehat{O}_K^*)/F^*$. Put $K[\infty] = \bigcup_c K[c]$.

1.1.3. If $b \subset O_F$ is a non-zero ideal satisfying $bO_K \nmid \mathrm{lcm}\,\{(u-1) \mid u \in (O_K^*)_{\mathrm{tors}},\, u \neq 1\}$, then $Z(bc) = Z(b)$ by ([Ne 2, Cor. 2.11]), hence

$$\widehat{O}_b^*/\widehat{O}_{bc}^* \xrightarrow{\sim} \mathrm{Gal}(K[bc]/K[b]).$$

1.1.4. For any subgroup $X \subset \widehat{K}^*$, put $X^{(c)} = \{x \in X \mid (\forall w|c)\ x_w = 1\}$.

1.1.5. For any finite prime v of F, put $K_v = K \otimes_F F_v$ and $O_{K,v} = O_K \otimes_{O_F} O_{F,v}$. Let $(K_v^*)^\circ \subset K_v^*$ be the preimage of $(K_v^*/O_{K,v}^*)^{\mathrm{Gal}(K/F)} \subset K_v^*/O_{K,v}^*$. As

$$K_v^*/O_{K,v}^*F_v^* \xrightarrow{\sim} \begin{cases} 0, & v \text{ is inert in } K/F \\ \mathbf{Z}/2\mathbf{Z}, & v \text{ is ramified in } K/F \\ \mathbf{Z}, & v \text{ splits in } K/F, \end{cases}$$

it follows that the quotient $(K_v^*)^\circ/F_v^*$ is compact and

$$K_v^*/(K_v^*)^\circ \xrightarrow{\sim} \begin{cases} \mathbf{Z}, & v \text{ splits in } K/F \\ 0, & \text{otherwise,} \end{cases} \qquad (K_v^*)^\circ/O_{K,v}^*F_v^* \xrightarrow{\sim} \begin{cases} \mathbf{Z}/2\mathbf{Z}, & v \text{ is ramified in } K/F \\ 0, & \text{otherwise.} \end{cases}$$

Put $\widehat{K}^{*\circ} = \{x \in \widehat{K}^* \mid (\forall v)\ x_v \in (K_v^*)^\circ\}$.

1.1.6. Let $b, c \subset O_F$ be ideals satisfying $(b, c) = (1)$. Denote by $U(c, b)$ (resp., by $U'(c, b)$) the image of $\widehat{O}_K^{*(b)}$ (resp., of $(\widehat{K}^{*\circ})^{(b)}$) in $\widehat{K}^*/\widehat{F}^*\widehat{O}_{bc}^*$. There are canonical isomorphisms

$$U(c, b) \xrightarrow{\sim} \widehat{O}_K^{*(b)}/\widehat{O}_{bc}^{*(b)} \xrightarrow{\sim} \widehat{O}_K^*/\widehat{O}_c^* = U(c) = \prod_{v|c} U(c)_v, \qquad U'(c, b) \xrightarrow{\sim} \prod_{\substack{v|cd \\ v\nmid b}} U'(c, b)_v \qquad (d = d_{K/F}),$$

where $U(c)_v$ (resp., $U'(c, b)_v$) is the quotient of $O_{K,v}^*/O_{F,v}^*$ (resp., of $(K_v^*)^\circ/F_v^*$) by $(O_{F,v} + cO_{K,v})^*/O_{F,v}^*$. It follows from 1.1.5 that, for each prime $v \mid cd_{K/F}$ not dividing b,

$$U'(c, b)_v/U(c)_v \xrightarrow{\sim} \begin{cases} \mathbf{Z}/2\mathbf{Z}, & v \mid d_{K/F},\ v \nmid b \\ 0, & \text{otherwise,} \end{cases} \tag{1.1.6.1}$$

which implies that there is an exact sequence

$$0 \longrightarrow U(c) \longrightarrow U'(c,b) \longrightarrow (\mathbf{Z}/2\mathbf{Z})^{\{v \mid d_{K/F},\, v \nmid b\}} \longrightarrow 0. \tag{1.1.6.2}$$

1.1.7. For any subextension L/K of $K[\infty]/K$, denote by $\mathrm{Gal}(L/K)^{b-\mathrm{rat}} \supset \mathrm{Gal}(L/K)^{\circ, b-\mathrm{rat}}$ the respective images of $\widehat{K}^{*(b)} \supset (\widehat{K}^{*\circ})^{(b)}$ in $\mathrm{Gal}(L/K)$ under the map $\mathrm{rec}_{L/K}$. In particular, $\mathrm{rec}_{K[bc]/K}$ induces a surjective morphism

$$U'(c,b) \twoheadrightarrow \mathrm{Gal}(K[bc]/K)^{\circ, b-\mathrm{rat}}. \tag{1.1.7.1}$$

In the terminology of [Co-Va 2], the elements of $\mathrm{Gal}(L/K)^{b-\mathrm{rat}}$ are "b-rational" (note, however, that our ring class fields $K[c]$ are slightly smaller than the corresponding fields in [Co-Va 1,2]).

1.2. Ring class towers

1.2.1. Data. Throughout this section, $P_1, \ldots, P_s$ ($s \geq 1$) are distinct prime ideals of O_F and $c \subset O_F$ is an ideal relatively prime to $P_1 \cdots P_s$. We denote by p_i the rational prime below P_i.

1.2.2. Notation. Let $\mathbf{N}_\infty = \mathbf{N} \cup \{\infty\}$ and $\mathbf{N}_{>0} = \mathbf{N} - \{0\}$. For a multiindex $n = (n_1, \ldots, n_s) \in \mathbf{N}_\infty^s$, we abbreviate $P^n := P_1^{n_1} \cdots P_s^{n_s}$ and we use the following notation:

$$G^{(c)}(n) = \mathrm{Gal}(K[cP^n]/K), \qquad G(n) = G^{(1)}(n) \qquad (n \in \mathbf{N}^s),$$
$$G^{(c)} = G^{(c)}(\infty) = \mathrm{Gal}(K[cP^\infty]/K) = \varprojlim_n G^{(c)}(n), \qquad K[cP^\infty] = \bigcup_{n \in \mathbf{N}^s} K[cP^n], \qquad G = G^{(1)}.$$

1.2.3. Proposition-Definition. (1) *The kernels and cokernels of the maps*

$$U(cP^n) = U(c) \oplus \bigoplus_{i=1}^{s} U(P_i^{n_i}) \longrightarrow G^{(c)}(n) \qquad (n \in \mathbf{N}^s)$$

(defined in 1.1.2) are finite and their orders are bounded.
(2) *The torsion subgroups* $G_0^{(c)} := (G^{(c)})_{\mathrm{tors}}$ *and* $G_0 := G_0^{(1)} = G_{\mathrm{tors}}$ *are finite and*

$$G^{(c)}/G_0^{(c)} \xrightarrow{\sim} G/G_0 \xrightarrow{\sim} \bigoplus_{i=1}^{s} \mathbf{Z}_{p_i}^{[F_{P_i}:\mathbf{Q}_{p_i}]}.$$

(3) *We define* $G_1^{(c)} := G_0^{(c)} \cap (G^{(c)})^{P_1 \cdots P_s - \mathrm{rat}}$ *and* $G_1 := G_0 \cap G^{P_1 \cdots P_s - \mathrm{rat}}$.

Proof. (1) Apply (1.1.2.1). (2) This follows from (1) and the structure of $\varprojlim_m U(P_i^m)$.

1.2.4. Proposition. *For all but finitely many $n \in \mathbf{N}^s$, the map from 1.2.3(1) gives rise to an exact sequence*

$$0 \longrightarrow U(c) \longrightarrow G^{(c)}(n) \longrightarrow G(n) \longrightarrow 0.$$

Proof. Apply 1.1.3 with $b = P^n$.

1.2.5. Corollary. *There is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U(c) & \longrightarrow & G_0^{(c)} & \longrightarrow & G_0 & \longrightarrow & 0 \\ & & \| & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & U(c) & \longrightarrow & G^{(c)} & \longrightarrow & G & \longrightarrow & 0. \end{array}$$

1.2.6. Denote by d the prime-to-$(P_1 \cdots P_s)$ part of $d_{K/F}$. It follows from 1.1.6 that

$$\forall n \in \mathbf{N}^s_{>0} \qquad U'(c, P^n) = U'(c, P_1 \cdots P_s), \qquad U'(c, P^n)/U(c) \xrightarrow{\sim} (\mathbf{Z}/2\mathbf{Z})^{\{v|d\}} \xrightarrow{\sim} U'(1, P^n).$$

1.2.7. Proposition. (1) *The reciprocity map* rec_K *induces isomorphisms*

$$\widehat{K}^*/K^*\widehat{F}^*\widehat{O}_c^{*(P_1\cdots P_s)} \xrightarrow{\sim} G^{(c)}, \qquad \widehat{K}^{*(P_1\cdots P_s)}/\widehat{F}^{*(P_1\cdots P_s)}\widehat{O}_c^{*(P_1\cdots P_s)} \xrightarrow{\sim} (G^{(c)})^{P_1\cdots P_s-\mathrm{rat}}.$$

(2) *The surjective maps (1.1.7.1)*

$$U'(1, P_1 \cdots P_s) = U'(1, P^n) \twoheadrightarrow G(n)^{\circ, P_1\cdots P_s-\mathrm{rat}} \qquad (n \in \mathbf{N}^s_{>0})$$

give rise to an isomorphism

$$(\mathbf{Z}/2\mathbf{Z})^{\{v|d_{K/F},\, v\nmid P_1\cdots P_s\}} \xrightarrow{\sim} U'(1, P_1 \cdots P_s) \xrightarrow{\sim} G_1.$$

(3) *The surjective maps (1.1.7.1)*

$$U'(c, P_1 \cdots P_s) = U'(c, P^n) \twoheadrightarrow G^{(c)}(n)^{\circ, P_1\cdots P_s-\mathrm{rat}} \qquad (n \in \mathbf{N}^s_{>0})$$

give rise to an isomorphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U(c) & \longrightarrow & U'(c, P_1 \cdots P_s) & \longrightarrow & U'(1, P_1 \cdots P_s) & \longrightarrow & 0 \\ & & \| & & \big\downarrow \wr & & \big\downarrow \wr & & \\ 0 & \longrightarrow & U(c) & \longrightarrow & G_1^{(c)} & \longrightarrow & G_1 & \longrightarrow & 0. \end{array}$$

Proof. (1) The first isomorphism is obtained as in ([Co-Va 1], Lemma 2.1); the second follows from $\widehat{K}^{*(P_1\cdots P_s)} \cap K^*\widehat{F}^*\widehat{O}_c^{*(P_1\cdots P_s)} = \widehat{F}^{*(P_1\cdots P_s)}\widehat{O}_c^{*(P_1\cdots P_s)}$.
(2) (cf. [Co-Va 1, Lemma 2.3]) Applying (1) to $c = (1)$, we obtain

$$G^{P_1\cdots P_s-\mathrm{rat}} \xrightarrow{\sim} \widehat{K}^{*(P_1\cdots P_s)}/\widehat{F}^{*(P_1\cdots P_s)}\widehat{O}_K^{*(P_1\cdots P_s)} \xrightarrow{\sim} I_K^{(P_1\cdots P_s)}/I_F^{(P_1\cdots P_s)},$$

where we have denoted by $I_E^{(a)}$ ($E = F, K$; $a \subset O_F$) the group of fractional ideals of E prime to a. It follows that

$$G_1 = (G^{P_1\cdots P_s-\mathrm{rat}})_{\mathrm{tors}} \xrightarrow{\sim} (I_K^{(P_1\cdots P_s)}/I_F^{(P_1\cdots P_s)})_{\mathrm{tors}} = (\mathbf{Z}/2\mathbf{Z})^{\{v|d_{K/F},\, v\nmid P_1\cdots P_s\}},$$

and this isomorphism is induced by the map (1.1.7.1).
(2) It follows from Corollary 1.2.5 and 1.2.6 that the reciprocity map defines a map between exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U(c) & \longrightarrow & U'(c, P_1 \cdots P_s) & \longrightarrow & U'(1, P_1 \cdots P_s) & \longrightarrow & 0 \\ & & \| & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & U(c) & \longrightarrow & G_1^{(c)} & \longrightarrow & G_1 & & \end{array}$$

By (2), the third vertical arrow is an isomorphism; this implies that the map $G_1^{(c)} \longrightarrow G_1$ is surjective and that the middle vertical arrow is also an isomorphism (by the Snake Lemma).

1.2.8. We define the **conductor** $c(\chi)$ of a character $\chi : G_1^{(c)} \longrightarrow \mathbf{C}^*$ to be the smallest divisor c' of c (with respect to divisibility) such that χ factors through $G_1^{(c')}$. Equivalently, it is the smallest divisor c' of c such that the restriction of χ to $U(c)$ factors through $U(c')$ (by Proposition 1.2.7(3) applied to c and c').

1.3. Local ε-factors

1.3.1. Let v be a finite prime of F and B_v a quaternion algebra over F_v equipped with an F_v-embedding $t_v : K_v \hookrightarrow B_v$. Denote the invariant of B_v by

$$\mathrm{inv}_v(B_v) = \begin{cases} 1, & B_v = M_2(F_v) \\ -1, & B_v \neq M_2(F_v). \end{cases}$$

Let π_v be an irreducible smooth representation of B_v^*/F_v^* (on a $\mathbf{C}$-vector space) and $\chi_v : K_v^*/F_v^* \longrightarrow \mathbf{C}^*$ a continuous (quasi)-character. The ε-factor

$$\varepsilon(\pi_v \times \chi_v, \tfrac{1}{2}) = \varepsilon(\pi_v \times \chi_v, \psi_v, \tfrac{1}{2}) \in \{\pm 1\}$$

is independent of the choice of an additive character ψ_v of F_v and satisfies

$$\varepsilon(\pi_v \times \chi_v^{-1}, \tfrac{1}{2}) = \varepsilon(\pi_v \times \chi_v, \tfrac{1}{2})^{-1} = \varepsilon(\pi_v \times \chi_v, \tfrac{1}{2}). \tag{1.3.1.1}$$

Its value admits the following representation-theoretical characterisation.

1.3.2. Theorem ([Tu], [Wa, Thm. 2], [Sa]). *If $B_v = M_2(F_v)$, assume that $\dim(\pi_v) = \infty$.*
(1) *If $\eta_v(-1)\varepsilon(\pi_v \times \chi_v, \frac{1}{2}) = \mathrm{inv}_v(B_v)$, then there exists a non-zero $t_v(K_v^*)$-invariant linear form $\pi_v \otimes \chi_v \longrightarrow \mathbf{C}$ (unique up to a scalar multiple). Equivalently, there exists a non-zero $t_v(K_v^*)$-invariant linear map (unique up to a scalar multiple) $\ell_v : \pi_v \longrightarrow \mathbf{C}(\chi_v^{-1})$, i.e., a linear map $\ell_v : \pi_v \longrightarrow \mathbf{C}$ satisfying*

$$\forall a \in K_v^* \ \ \forall u \in \pi_v \qquad \ell_v\left(t_v(a)u\right) = \chi_v(a)^{-1}\, \ell_v(u).$$

A "test vector" (for ℓ_v) is a vector $u \in \pi_v$ such that $\ell_v(u) \neq 0$.
(2) *If $\eta_v(-1)\varepsilon(\pi_v \times \chi_v, \frac{1}{2}) \neq \mathrm{inv}_v(B_v)$, then no such ℓ_v exists.*

1.3.3. (1) If v splits in K/F or if ($B_v = M_2(F_v)$ and π_v is a principal series representation), then ([Ne 1, Prop. 12.6.2.4]):

$$\eta_v(-1)\varepsilon(\pi_v \times \chi_v, \tfrac{1}{2}) = 1 = \mathrm{inv}_v(B_v).$$

(2) If v does not split in K/F, then K_v^*/F_v^* is compact, which implies that π_v is a direct sum of finite-dimensional (smooth) representations of $t_v(K_v^*)$. This means that the existence of a non-zero $t_v(K_v^*)$-invariant linear form $\pi_v \otimes \chi_v \longrightarrow \mathbf{C}$ is equivalent to the existence of a non-zero $t_v(K_v^*)$-invariant vector in $\pi_v \otimes \chi_v$.
(3) If $B_v = M_2(F_v)$ and $\pi_v = \mathrm{St} \otimes \mu$ (where St is the Steinberg representation and $\mu : F_v^* \longrightarrow \{\pm 1\}$), then ([Ne 1, Prop. 12.6.2.4]):

$$\eta_v(-1)\varepsilon(\pi_v \times \chi_v, \tfrac{1}{2}) = \begin{cases} -1, & v \text{ does not split in } K/F \text{ and } \chi_v = \mu \circ N_{K_v/F_v} \\ 1, & \text{otherwise.} \end{cases}$$

(4) If $B_v \neq M_2(F_v)$, denote by $JL(\pi_v)$ the representation of $PGL_2(F_v)$ associated to π_v by the Jacquet-Langlands correspondence; then $\varepsilon(\pi_v \times \chi_v, \frac{1}{2}) = \varepsilon(JL(\pi_v) \times \chi_v, \frac{1}{2})$.

2. Shimura curves and CM points

2.1. Shimura curves

We recall the basic definitions in the language of [Co-Va 1] (see also [Ne 2]).

2.1.1. Let F be as in 1.1.1. Let B be a quaternion algebra over F such that $B_{\tau_j} := B \otimes_{F,\tau_j} \mathbf{R}$ is isomorphic to $M_2(\mathbf{R})$ (resp., to the algebra of Hamilton quaternions) for one infinite prime τ_1 of F (resp., for the remaining infinite primes $\tau_2, \ldots, \tau_d$ of F). If $F = \mathbf{Q}$, we also allow $B = M_2(\mathbf{Q})$.

2.1.2. Put $\widehat{B} = B \otimes \widehat{\mathbf{Z}}$ and, for each prime v of B, $B_v = B \otimes_F F_v$. For any open compact subgroup $H \subset \widehat{B}^*$ (for example, if $R \subset B$ is an O_F-order, then $\widehat{R}^*$ is an open compact subgroup of $\widehat{B}^*$), let N_H be the (non-singular) Shimura curve over F whose associated Riemann surface is given by

$$N_H^{\mathrm{an}} = (N_H \otimes_{F,\tau_1} \mathbf{C})(\mathbf{C}) = B^* \backslash ((\mathbf{C} - \mathbf{R}) \times \widehat{B}^* / H \widehat{F}^*),$$

where $B^* \subset B_{\tau_1}^*$ acts on $\mathbf{C} - \mathbf{R}$ via a fixed isomorphism $B_{\tau_1} \xrightarrow{\sim} M_2(\mathbf{R})$ and the standard action of $GL_2(\mathbf{R})$ by Möbius transformations. The point of N_H^{an} represented by $z \in \mathbf{C} - \mathbf{R}$ and $b \in \widehat{B}^*$ will be denoted by $[z, b]_H = [z, b]$. If $H' \subset H$ is an open subgroup of H, then there is a natural projection map

$$\text{(2.1.2.1)} \qquad \mathrm{pr} : N_{H'} \longrightarrow N_H, \qquad [z, b]_{H'} \mapsto [z, b]_H,$$

which is finite and flat.

2.1.3. If $B \neq M_2(\mathbf{Q})$, then the curve N_H is projective; set $N_H^* = N_H$. If $B = M_2(\mathbf{Q})$, denote by N_H^* the standard non-singular compactification of N_H obtained by adjoining the cusps. In either case, the smooth projective curve N_H^* over F is connected, but not necessarily geometrically connected. Its field of constants is (non-canonically) isomorphic to the subfield $F_H \subset F^{\mathrm{ab}}$ satisfying

$$\mathrm{rec}_F : \widehat{F}^* / F_+^* \widehat{F}^{*2} \mathrm{nr}(H) \xrightarrow{\sim} \mathrm{Gal}(F_H/F),$$

where F_+^* is the group of totally positive elements of F^* and nr is the reduced norm on B. As in [Co-Va 1, 3.3], we define the Jacobian of N_H^* as

$$J(N_H^*) = \mathrm{Pic}^\circ(N_H^*/F) = R_{F_H/F}(\mathrm{Pic}^\circ(N_H^*/F_H))$$

and we let

$$\iota_H : N_H^* \longrightarrow J(N_H^*)$$

be the F-morphism given by a suitable multiple of the Hodge class (resp., of the cusp at infinity), as in [Co-Va 1, 3.5] (see also [Ne 2, 1.19]).

2.1.4. The curves N_H^* (for variable H) form a projective system with respect to the projections (2.1.2.1) (resp., their natural extensions to N_H^*, if $B = M_2(\mathbf{Q})$). Note that N_H^* depends only on $H\widehat{O}_F^*$, which always contains a subgroup of the form $\widehat{R}^*$, for a suitable O_F-order $R \subset B$. As a result, the set of curves $\{N_{\widehat{R}^*}^*\}_R$ is cofinal in the projective system $\{N_H^*\}_H$.

2.1.5. For each $g \in \widehat{B}^*$, right multiplication by g induces a holomorphic isomorphism

$$[\cdot g] : N_H^{\mathrm{an}} \xrightarrow{\sim} N_{g^{-1}Hg}^{\mathrm{an}}, \qquad [z, b]_H \mapsto [z, bg]_{g^{-1}Hg},$$

which comes from an F-isomorphism

$$[\cdot g] : N_H^* \xrightarrow{\sim} N_{g^{-1}Hg}^*.$$

2.1.6. As explained in [Co-Va 1, 3.6] (see also [Ne 2, 1.6-1.7]), the space of holomorphic 1-forms

$$\varinjlim_{H, \mathrm{pr}^*} \Gamma(N_H^{*\mathrm{an}}, \Omega^{\mathrm{an}})$$

is naturally identified with a certain space $\mathcal{S}_2^{\widehat{F}^*}$ of automorphic forms of weight $(2,0,\ldots,0)$ on $B_{\mathbf{A}}^*$ with trivial central character. All we need to know about this space are the following three facts.

2.1.6.1. For each $g \in \widehat{B}^*$, the left action

$$[\cdot g]^* : \Gamma(N_{g^{-1}Hg}^*, \Omega^{\mathrm{an}}) \longrightarrow \Gamma(N_H^{*\mathrm{an}}, \Omega^{\mathrm{an}})$$

corresponds, in the limit, to the standard action of g on $\mathcal{S}_2$.

2.1.6.2. As a (smooth) representation of $\widehat{B}^*$, the space $\mathcal{S}_2^{\widehat{F}^*}$ decomposes into a direct sum

$$\mathcal{S}_2^{\widehat{F}^*} = \bigoplus_{\pi} \pi^{\infty},$$

where $\pi = \pi_\infty \otimes \pi^\infty = \bigotimes_v' \pi_v$ runs through all irreducible cuspidal representations of $B_{\mathbf{A}}^*$ with trivial central character whose archimedean component π_∞ is isomorphic to the weight two holomorphic discrete series at τ_1 and to the trivial representation at the remaining infinite primes $\tau_2, \ldots, \tau_d$. Moreover, each representation $\pi^\infty = \otimes_{v\nmid\infty}' \pi_v$ occurs in $\mathcal{S}_2^{\widehat{F}^*}$ with multiplicity one.

2.1.6.3. For each open compact subgroup H of $\widehat{B}^*$, $\Gamma(N_H^{*\mathrm{an}}, \Omega^{\mathrm{an}})$ is identified with the subspace $\mathcal{S}_2^{\widehat{F}^* H}$ of H-fixed vectors in $\mathcal{S}_2^{\widehat{F}^*}$.

2.1.7. As recalled in ([Ne 2, 1.18]), for each open compact subgroup H of $\widehat{B}^*$ there exists an isogeny (defined over F)

$$J(N_H^*) \longrightarrow \prod_{j \in J} A_j^{a_j} \qquad (a_j \geq 1), \tag{2.1.7.1}$$

where each A_j is an F-simple abelian variety over F for which $\mathrm{End}_F(A_j) = O_{L_j}$ is the ring of integers in a totally real number field L_j of degree $[L_j : \mathbf{Q}] = \dim(A_j)$. In particular, $\Gamma(A_j, \Omega^1_{A_j/F})$ is a free $(L_j \otimes_{\mathbf{Q}} F)$-module of rank one, which implies that the term corresponding to A_j in

$$\mathcal{S}_2^{\widehat{F}^* H} = \Gamma(J(N_H^*)^{\mathrm{an}}, \Omega^{\mathrm{an}}) = \Gamma(J(N_H^*), \Omega^1_{J(N_H^*)/F}) \otimes_{F,\tau_1} \mathbf{C} \xrightarrow{\sim} \bigoplus_{j \in J} \Gamma(A_j, \Omega^1_{A_j/F})^{a_j} \otimes_{F,\tau_1} \mathbf{C} \tag{2.1.7.2}$$

is a free module of rank a_j over

$$L_j \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} \prod_{\sigma : L_j \hookrightarrow \mathbf{R}} \mathbf{C}, \qquad a \otimes b \mapsto (\sigma \mapsto \sigma(a)b).$$

More precisely, there exists a finite set S of finite primes of F such that $H = H_S H^S$, where H_S is an open compact subgroup of $\prod_{v \in S} B_v^*$ and $H^S = \prod_{v \notin S} H_v$, where each H_v is a maximal compact subgroup of B_v^*. As recalled in [Ne 2, 1.17], the spherical Hecke algebra $\mathbf{T}_H^S = \mathbf{Z}[H^S \backslash (\widehat{B}^*)^S / H^S]$ acts on the j-th factor of (2.1.7.2) via a morphism $\varphi_j : \mathbf{T}_H^S \longrightarrow O_{L_j}$ and the natural action of O_{L_j} on A_j. Strong multiplicity one theorem for $B_{\mathbf{A}}^*$ implies that, for each embedding $\sigma : L_j \hookrightarrow \mathbf{R}$, there exists a unique automorphic representation $\pi = \pi(\sigma)$ of the type described in 2.1.6.2 for which $\mathbf{T}_H^S$ acts on $\pi(\sigma)^H \neq 0$ by $\sigma \circ \varphi_j$. In particular,

$$\forall \sigma : L_j \hookrightarrow \mathbf{Q} \qquad \dim_{\mathbf{C}} (\pi(\sigma)^\infty)^H = a_j.$$

In more arithmetic terms, the L-function of A_j is automorphic in the following sense ([Ne 2, 1.18]):

$$\forall \sigma : L_j \hookrightarrow \mathbf{Q} \qquad L(\sigma A_j/F, s) = L(\pi(\sigma), s - \tfrac{1}{2})$$

(Euler factor by Euler factor).

2.2. CM points

2.2.1. Let F and B be as in 2.1.1. Let K be a totally imaginary quadratic extension of F such that all finite primes of F at which B ramifies are either inert or ramified in K/F. This implies that there exists an embedding $t : K \hookrightarrow B$ over F, which is unique up to B^*-conjugacy, by the Skolem-Noether theorem. Fix such an embedding t and denote by

$$t_v : K_v \hookrightarrow B_v, \qquad \widehat{t} : \widehat{K} \hookrightarrow \widehat{B}$$

the corresponding local and finite-adelic embeddings.

2.2.2. There exists a unique point $z \in \mathbf{C}$ with $\mathrm{Im}(z) > 0$ which is fixed by the action of $t(K^*) \subset B^* \subset B^*_{\tau_1} \xrightarrow{\sim} GL_2(\mathbf{R})$. The set of CM-points by K on the curve N_H is defined as

$$CM(N_H, K) = \{[z, b]_H \in N_H(\mathbf{C}) \mid b \in \widehat{B}^*\}.$$

This set does not depend on the choice of t and is contained in $N_H(K[\infty])$, thanks to "Shimura's reciprocity law"

$$\text{(2.2.2.1)} \qquad \forall a \in \widehat{K}^* \qquad \mathrm{rec}_K(a)\, [z, b]_H = [z, \widehat{t}(a)b]_H.$$

In particular, a CM point $x = [z, b]_H$ is defined over $K[c]$, provided $\widehat{t}(\widehat{O}_c^*) \subseteq bHb^{-1}$.

The map $b \mapsto [z, b]_H$ defines a bijection $t(K^*)\backslash \widehat{B}^*/H\widehat{F}^* \xrightarrow{\sim} CM(N_H, K)$.

2.2.3. From now on (with the exception of §2.6) we assume, for simplicity, that $H = \widehat{R}^*$ for some O_F-order $R \subset B$ (in view of 2.1.4, this is not a serious restriction). For each CM point $x = [z, b]_{\widehat{R}^*} \in CM(N_{\widehat{R}^*}, K)$, there is a unique non-zero ideal $c(x) \subset O_F$ (called the conductor of x) such that $t^{-1}(bRb^{-1}) = O_{c(x)} = O_F + c(x)O_K$. As $\widehat{t}^{-1}(b\widehat{R}^* b^{-1}) = \widehat{O}^*_{c(x)}$, it follows from (2.2.2.1) that x is defined over $K[c(x)]$.

2.2.4. Given distinct prime ideals $P_1, \ldots, P_s$ ($s \geq 1$) of O_F, a $(P_1 \cdots P_s)$-isogeny class of CM points is a subset of $CM(N_H, K)$ of the form

$$\{[z, g_1 \cdots g_s b]_H \mid g_i \in B^*_{P_i}\}$$

for fixed $b \in \widehat{B}^*$. Note that, if $x, x' \in CM(N_H, K)$ lie in the same $(P_1 \cdots P_s)$-isogeny class, then

$$\text{(2.2.4.1)} \qquad \forall v \nmid P_1 \cdots P_s \qquad \mathrm{ord}_v(c(x)) = \mathrm{ord}_v(c(x')).$$

2.3. CM points in ring class towers: the chaotic Galois action

2.3.1. Proposition ([Co-Va 2, Prop. 3. 18] if $s = 1$). *Let $P_1, \ldots, P_s$ be distinct prime ideals of O_F such that $B_{P_i} = M_2(F_{P_i})$ for all $i = 1, \ldots, s$. Fix $\sigma \in \mathrm{Gal}(K[\infty]/K)$ and put, for any $x \in CM(N_H, K)$, $\delta_\sigma(x) = (x, \sigma x) \in (N_H \otimes_{F,\tau_1} \mathbf{C})(\mathbf{C})^2$. The following properties of σ are equivalent:*
(1) $\sigma \in \mathrm{Gal}(K[\infty]/K)^{P_1 \cdots P_s-\mathrm{rat}}$ (= the image of $\widehat{K}^{*(P_1 \cdots P_s)}$ under $\mathrm{rec}_{K[\infty]/K}$).
(2) For any $(P_1 \cdots P_s)$-isogeny class $\mathcal{C} \subset CM(N_H, K)$, the Zariski closure of $\delta_\sigma(\mathcal{C})$ in $(N_H \otimes_{F,\tau_1} \mathbf{C})^2$ has dimension 1.
(3) For any sequence of CM points $\{x_n\}_{n \in \mathbf{N}}$ contained in some $(P_1 \cdots P_s)$-isogeny class $\mathcal{C} \subset CM(N_H, K)$ and satisfying $(P_1 \cdots P_s)^n \mid c(x_n)$ for all $n \in \mathbf{N}$, the Zariski closure of $\{\delta_\sigma(x_n)\}_{n \in \mathbf{N}}$ in $(N_H \otimes_{F,\tau_1} \mathbf{C})^2$ has dimension 1.

Proof. The proof of ([Co-Va 2, Prop. 3. 18]) applies with trivial modifications (one has to consider the distances on the trees associated to $B^*_{P_i}/F^*_{P_i}$ for all $i = 1, \ldots, s$).

2.3.2. Corollary ([Co-Va 2, Rmk. 3. 21] if $s = 1$). *Given elements $\sigma_1, \dots, \sigma_r \in \mathrm{Gal}(K[\infty]/K)$ that are distinct modulo $\mathrm{Gal}(K[\infty]/K)^{P_1 \cdots P_s-\mathrm{rat}}$, then for any sequence $\{x_n\}_{n\in\mathbf{N}}$ satisfying the assumptions of 2.3.1(3), the Zariski closure of $\{(\sigma_1(x_n), \dots, \sigma_r(x_n))\}_{n\in\mathbf{N}}$ in $(N_H \otimes_{F,\tau_1} \mathbf{C})^r$ contains a connected component of $(N_H \otimes_{F,\tau_1} \mathbf{C})^r$.*

Proof. The statement being trivial for $r = 1$, assume $r > 1$. Let Y be a connected component of the Zariski closure of $\{(\sigma_1(x_n), \dots, \sigma_r(x_n))\}_{n\in\mathbf{N}}$. According to a special case of the André-Oort conjecture proved in ([Ed-Ya, Thm. 1.2]), Y is a subvariety of Hodge type of $(N_H \otimes_{F,\tau_1} \mathbf{C})^r$. For $1 \le i < j \le r$, denote by $p_{ij} : (N_H \otimes_{F,\tau_1} \mathbf{C})^r \longrightarrow (N_H \otimes_{F,\tau_1} \mathbf{C})^2$ the projection on the i-th and the j-th factors. Proposition 2.3.1 implies that $p_{ij}(Y)$ contains a connected component of $(N_H \otimes_{F,\tau_1} \mathbf{C})^2$. The description of the subvarieties of Hodge type of $(N_H \otimes_{F,\tau_1} \mathbf{C})^r$ given in [Co 2, Prop. 2.1] (which generalises the special case $B = M_2(\mathbf{Q})$ treated in [Ed, Prop. 2.1]) then yields the desired result.

2.4. CM points in ring class towers: the geometric Galois action

2.4.1. We continue to assume that $H = \widehat{R}^*$ for some O_F-order $R \subset B$ (which implies that $H = \prod_v H_v$, $H_v = R_v^*$). Assume that we are given the following data:

(D1) Distinct prime ideals $P_1, \dots, P_s$ ($s \ge 1$) of O_F such that $B_{P_i} = M_2(F_{P_i})$ for all $i = 1, \dots, s$. We denote by p_i the rational prime below P_i.

(D2) An ideal $c \subset O_F$ relatively prime to $P_1 \cdots P_s$.

(D3) A $(P_1 \cdots P_s)$-isogeny class $\mathcal{C} \subset CM(N_H, K)$ such that for at least one $x \in \mathcal{C}$ ($\Longleftrightarrow$ for all $x \in \mathcal{C}$) $c(x) \mid c(P_1 \cdots P_s)^\infty$.

(D4) A simple quotient $J(N_H^*) \twoheadrightarrow A_0$ defined over F. As recalled in 2.1.7, there exists a totally real number field L_0 such that $[L_0 : \mathbf{Q}] = \dim(A_0)$ and (possibly after replacing A_0 by an isogeneous abelian variety) $\mathrm{End}_F(A_0) = O_{L_0}$. Denote by

$$\alpha : N_H^* \xrightarrow{\iota_H} J(N_H^*) \twoheadrightarrow A_0$$

the composition with the map ι_H from 2.1.3 (α is not constant and is defined over F).

(D5) A character $\chi_0 : G_0^{(c)} \longrightarrow O_L^*$, where L is a number field containing L_0. Put

$$e_{\overline{\chi}_0} = \sum_{g\in G_0^{(c)}} \chi_0(g)\, g \in O_L[G_0^{(c)}].$$

2.4.2. Our goal is to show (under suitable assumptions) that, for sufficiently many CM points $x \in \mathcal{C}$,

$$e_{\overline{\chi}_0}(1 \otimes \alpha(x)) = \sum_{g\in G_0^{(c)}} \chi_0(g) \otimes \alpha(g(x)) \in O_L \otimes_{O_{L_0}} A_0(K[cP^\infty])$$

is not torsion. As in [Co-Va 1, 4.5-4.6], we decompose the action of $G_0^{(c)}$ into a "geometric" action of $G_1^{(c)}$ and a "chaotic" action of a set of representatives of $G_0^{(c)}/G_1^{(c)}$.

2.4.3. Put $S = \{v \mid cd_{K/F},\, v \nmid P_1 \cdots P_s\}$ (which should not be confused with the set that was denoted by the same letter in 2.1.7). As observed in Proposition 1.2.7, the reciprocity map induces an isomorphism $U' \xrightarrow{\sim} G_1^{(c)}$, where

$$U' = U'(c, P_1 \cdots P_s) = \bigoplus_{v\in S} U'_v, \qquad U'_v = U'(c, P_1 \cdots P_s)_v = (K_v^*)^\circ / F_v^*(O_{F,v} + cO_{K,v})^*.$$

This means that we can consider the restriction of χ_0 to $G_1^{(c)}$ as a character $\chi_1 : U' \longrightarrow O_L^*$, and then write it as a product of its local components $\chi_{1,v} : U'_v \longrightarrow O_L^*$ ($v \in S$). We shall abuse the notation and denote by $\chi_{1,v}$ also the corresponding character

$$\chi_{1,v} : (K_v^*)^\circ / F_v^* \twoheadrightarrow U'_v \longrightarrow O_L^* \qquad (v \in S).$$

2.4.4. The decomposition $\chi_1 = \prod_{v\in S} \chi_{1,v}$ implies that

$$e_{\overline{\chi}_1} := \sum_{a\in U'} \chi_1(a)\, a = \prod_{v\in S} \Big(\sum_{a_v\in U'_v} \chi_{1,v}(a_v)\, a_v \Big) = \prod_{v\in S} e_{\overline{\chi}_{1,v}} \in O_L[U']. \tag{2.4.4.1}$$

We are going to write the action of $e_{\overline{\chi}_1}$ on all CM points in $\mathcal{C}$ in geometric terms. Fix $g \in \widehat{B}^*$ such that $\mathcal{C} = \{[z, b_1 \cdots b_s g]_H \mid b_i \in B^*_{P_i}\}$ and define new embeddings

$$j_v = \mathrm{Ad}(g_v)^{-1} \circ t_v : K_v \hookrightarrow B_v, \qquad j_v(a) = g_v^{-1} t_v(a) g_v \qquad (v \in S)$$
$$j = (j_v)_{v\in S} : \prod_{v\in S} K_v \hookrightarrow \prod_{v\in S} B_v.$$

Shimura's reciprocity law (2.2.2.1) implies that

(2.4.4.2)
$$\forall x = [z, b]_H \in \mathcal{C} \ \ \forall a \in \prod_{v\in S} K_v^* \ \ \forall a' \in \widehat{K}^* \qquad \mathrm{rec}_K(a)\mathrm{rec}_K(a')(x) = [z, \widehat{t}(a)\widehat{t}(a')b]_H = [z, \widehat{t}(a')bj(a)]_H,$$

and that, for each $v \in S$, the order $j_v^{-1}(R_v) = O_{F_v} + c(x)O_{K,v}$ (with $\mathrm{ord}_v(c(x)) \leq \mathrm{ord}_v(c)$) is independent of $x \in \mathcal{C}$. In particular, for each $a \in (K_v^*)^\circ$ $(v \in S)$, the order $\mathrm{Ad}j_v(a)R_v = j_v(a)R_v j_v(a)^{-1} \subset B_v$ depends only on the image of a in $(K_v^*)^\circ/F_v^*(O_{F_v} + cO_{K,v})^* = U'_v$.

2.4.5. Change of the Shimura curve. Define a new O_F-order $R_1 \subset B$ by its localisations

$$R_{1,v} = \begin{cases} R_v, & v \notin S \\ \bigcap_{a\in U'_v} \mathrm{Ad}j_v(a)R_v, & v \in S \end{cases}$$

and consider the Shimura curve $N^*_{H_1}$ for $H_1 = \widehat{R}_1^*$.

2.4.6. Degeneracy maps. For each $a \in U'$, choose its lift $\widetilde{a} \in \prod_{v\in S}(K_v^*)^\circ$; the morphism

$$f_a : N^*_{H_1} \xrightarrow{[\cdot j(\widetilde{a})]} N^*_{j(\widetilde{a})^{-1}H_1 j(\widetilde{a})} \xrightarrow{\mathrm{pr}} N^*_H$$

is defined over F and depends only on a, not on the chosen lift $\widetilde{a}$. Put

$$\beta_a = \alpha \circ f_a : N^*_{H_1} \xrightarrow{f_a} N^*_H \xrightarrow{\alpha} A_0. \tag{2.4.6.1}$$

The formula (2.4.4.2) implies that

$$\forall x = [z, b]_H \in \mathcal{C} \ \ \forall g \in \mathrm{Gal}(K[\infty]/K) \qquad e_{\overline{\chi}_1}(1 \otimes \alpha(g(x))) = \sum_{a\in U'} \chi_1(a) \otimes \beta_a(g(x_1)) \in O_L \otimes_{O_{L_0}} A_0(K[cP^\infty]),$$

where $x_1 = [z, b]_{H_1} \in CM(N_{H_1}, K)$. Equivalently, consider the abelian variety $A = O_L \otimes_{O_{L_0}} A_0$ (see [Con, §7]) and define a morphism (over F)

$$\alpha_1 : N^*_{H_1} \longrightarrow A, \qquad \alpha_1 = \sum_{a\in U'} \chi_1(a)\, \beta_a; \tag{2.4.6.2}$$

then

$$\forall x = [z, b]_H \in \mathcal{C} \ \ \forall g \in \mathrm{Gal}(K[\infty]/K) \qquad e_{\overline{\chi}_1}(1 \otimes \alpha(g(x))) = \alpha_1(g([z, b]_{H_1})). \tag{2.4.6.3}$$

Functoriality of the Hodge class [Co-Va 1, §3.5] implies that, if we multiply the Hodge classes of H and H_1 by the same integer in order to define ι_H and ι_{H_1}, then α_1 is also given by

$$(2.4.6.4)\qquad \alpha_1 : N^*_{H_1} \xrightarrow{\iota_{H_1}} J(N^*_{H_1}) \xrightarrow{((f_a)_*)} J(N^*_H)^{U'} \xrightarrow{\sum \chi_1(a)(-)} O_L \otimes_{O_{L_0}} J(N^*_H) \xrightarrow{\mathrm{id}\otimes\alpha} A.$$

2.4.7. Proposition ([Co-Va 1, §4.6] if $s = 1$). *Assume that the morphism α_1 from (2.4.6.2) is not constant. If $\{x_n\}_{n\in\mathbf{N}} \subseteq \mathcal{C}$ is a sequence of CM points contained in the $(P_1 \cdots P_s)$-isogeny class $\mathcal{C}$ such that $(P_1 \cdots P_s)^n \mid c(x_n)$ for all $n \in \mathbf{N}$, then the point*

$$e_{\overline{\chi}_0}(1 \otimes \alpha(x_n)) = \sum_{g\in G_0^{(c)}} \chi_0(g) \otimes \alpha(g(x_n)) \in O_L \otimes_{O_{L_0}} A_0(K[cP^\infty]) = A(K[cP^\infty])$$

is non-torsion for each sufficiently large $n >> 0$.

Proof. We briefly repeat the argument from [Co-Va 1]. Fix $b \in \widehat{B}^*$ such that $x_0 = [z, b]_H$. For each $n \in \mathbf{N}$, choose an element $u_n \in \prod_{i=1}^s B_{P_i}$ such that $x_n = [z, u_n b]_H$ and put $x_{n,1} = [z, u_n b]_{H_1}$; the sequence $\{x_{n,1}\}_{n\in\mathbf{N}}$ is then contained in the $(P_1 \cdots P_s)$-isogeny class $\mathcal{C}_1 \subset CM(N_{H_1}, K)$ of $x_{0,1}$. Fix a set of representatives $\mathcal{R} \subset G_0^{(c)}$ of $G_0^{(c)}/G_1^{(c)}$ and define

$$\begin{array}{rcccc} \alpha' : & (N^*_{H_1})^{\mathcal{R}} & \xrightarrow{\alpha_1^{\mathcal{R}}} & A^{\mathcal{R}} & \xrightarrow{\Sigma} & A \\ & & & (a_g)_{g\in\mathcal{R}} & \mapsto & \sum_{g\in\mathcal{R}} \chi_0(g) a_g, \end{array}$$

which is an algebraic morphism defined over F. We also define a map

$$\begin{array}{rccc} \Delta : & \mathcal{C}_1 & \longrightarrow & (N^*_{H_1} \otimes_{F,\tau_1} \mathbf{C})(\mathbf{C})^{\mathcal{R}} \\ & x_1 & \mapsto & (g(x_1))_{g\in\mathcal{R}}. \end{array}$$

The formula (2.4.6.3) implies that

$$\forall n \in \mathbf{N} \qquad e_{\overline{\chi}_0}(1 \otimes \alpha(x_n)) = \alpha' \circ \Delta(x_{n,1}).$$

Assume that $\alpha' \circ \Delta(x_{n,1}) \in A_{\mathrm{tors}}$ for infinitely many n. As $A(K[\infty])_{\mathrm{tors}}$ is finite [Co-Va 1, 4.2], it follows that there is a subsequence $\{x_{n(k),1}\}$ of $\{x_{n,1}\}$ for which $\alpha' \circ \Delta(\{x_{n(k),1}\}) = \{y\}$ is a single point ($y \in A(K[\infty])$), which implies that the Zariski closure Z of $\{\Delta(x_{n(k),1})\}$ in $(N^*_{H_1} \otimes_{F,\tau_1} \mathbf{C})(\mathbf{C})^{\mathcal{R}}$ satisfies $\alpha'(Z) = \{y\}$. On the other hand, Corollary 2.3.2 applies to $\mathcal{R} = \{\sigma_1, \ldots, \sigma_r\}$ and $\{x_{n(k),1}\}$, hence for each $g \in \mathcal{R}$ there exists a connected component Y_g of $(N^*_{H_1} \otimes_{F,\tau_1} \mathbf{C})(\mathbf{C})$ such that $\prod_{g\in\mathcal{R}} Y_g \subseteq Z$; thus $\alpha' \otimes \mathrm{id}$ is constant on $\prod_{g\in\mathcal{R}} Y_g$, which implies that $\alpha_1 \otimes \mathrm{id}$ is constant on each Y_g. As α_1 is defined over F and $N^*_{H_1}$ is connected, α_1 is constant, which contradicts our assumptions.

2.4.8. Action on differentials. The morphism α_1 is non-constant if and only if the induced map on differentials

$$\alpha_1^* : \Gamma(A, \Omega^1_{A/F}) \longrightarrow \Gamma(N^*_{H_1}, \Omega^1_{N^*_{H_1}/F})$$

is non-zero. Let us express this map in terms of the representation-theoretical description of differentials given in 2.1.6-2.1.7. Firstly, applying $(-) \otimes_{F,\tau_1} \mathbf{C}$ to the algebraic differentials in the sequence

$$\beta_a^* : \Gamma(A_0, \Omega^1_{A_0/F}) \xrightarrow{\alpha^*} \Gamma(N^*_H, \Omega^1_{N^*_H/F}) \xrightarrow{f_a^*} \Gamma(N^*_{H_1}, \Omega^1_{N^*_{H_1}/F}),$$

we obtain the following commutative diagram:

$$\begin{array}{ccccccc} \beta_a^* \otimes \mathrm{id} : & \Gamma(A_0^{\mathrm{an}}, \Omega^{\mathrm{an}}) & \hookrightarrow & \Gamma(N_H^{*\mathrm{an}}, \Omega^{\mathrm{an}}) & \longrightarrow & \Gamma(N_{H_1}^{*\mathrm{an}}, \Omega^{\mathrm{an}}) \\ & \big\downarrow & & \| & & \| \\ & \bigoplus_{\sigma_0 : L_0 \hookrightarrow \mathbf{R}} (\pi(\sigma_0)^\infty)^H & \hookrightarrow & \mathcal{S}_2^{\widehat{F}^* H} & \xrightarrow{j(a)} & \mathcal{S}_2^{\widehat{F}^* H_1}, \end{array}$$

from which we deduce another commutative diagram

$$\begin{array}{ccc}
\Gamma(A^{\mathrm{an}},\Omega^{\mathrm{an}}) = L\otimes_{L_0}\Gamma(A_0^{\mathrm{an}},\Omega^{\mathrm{an}}) & \xrightarrow{(\alpha_1\otimes\mathrm{id})^*} & \Gamma(N_{H_1}^{*\mathrm{an}},\Omega^{\mathrm{an}}) \\
\big\downarrow & & \| \\
\bigoplus_{\sigma:L\hookrightarrow\mathbf{C}}(\pi(\sigma|_{L_0})^\infty)^H & \longrightarrow & \mathcal{S}_2^{\widehat{F}^*H_1} \\
\omega\in(\pi(\sigma|_{L_0})^\infty)^H = ({}^\sigma\pi^\infty)^H & \mapsto & \left(\sum_{a\in U'}{}^\sigma\chi_1(a)j(a)\right)\omega,
\end{array}$$

where we have put (for each embedding $\sigma : L\hookrightarrow\mathbf{C}$) ${}^\sigma\chi_1 = \sigma\circ\chi_1 : G_1^{(c)}\longrightarrow\mathbf{C}^*$ and ${}^\sigma\pi = \pi(\sigma|_{L_0})$; the left vertical arrow is given on $L\otimes_{L_0}(\Gamma(A_0^{\mathrm{an}},\Omega^{\mathrm{an}})\cap\pi(\sigma_0))$ by

$$\lambda\otimes\omega_0\mapsto\left(\sigma\mapsto\begin{cases}\sigma(\lambda)\omega_0, & \text{if } \sigma|_{L_0}=\sigma_0\\ 0, & \text{otherwise}\end{cases}\right).$$

In other words, we have

(2.4.8.1)
$$\forall\sigma : L\hookrightarrow\mathbf{C}\ \ \forall\omega\in\Gamma(A^{\mathrm{an}},\Omega^{\mathrm{an}})\cap({}^\sigma\pi^\infty)^H\quad(\alpha_1\otimes\mathrm{id})^*(\omega) = \Big(\sum_{a\in U'}{}^\sigma\chi_1(a)j(a)\Big)\omega = \prod_{v\in S}\Big(\sum_{a\in U_v'}{}^\sigma\chi_{1,v}(a)j_v(a)\Big)\omega.$$

This formula can be reinterpreted as follows: for each $\sigma : L\hookrightarrow\mathbf{C}$, the space ${}^\sigma\pi$ decomposes under the action of the compact abelian group $j(\prod_{v\in S}(K_v^*)^\circ/F_v^*)$ into a direct sum of one-dimensional representations. The action on the subspace $({}^\sigma\pi^\infty)^H = \otimes'_{v\nmid\infty}{}^\sigma\pi_v^{H_v}$ (recall that $H=\widehat{R}^*=\prod_v R_v^*=\prod_v H_v$) factors through U', and (2.4.8.1) simply says that

(2.4.8.2)
$$|U'|^{-1}(\alpha_1\otimes\mathrm{id})^*(\omega) = |U'|^{-1}\,{}^\sigma e_{\overline{\chi}_1}(\omega) = |U'|^{-1}\Big(\prod_{v\in S}{}^\sigma e_{\overline{\chi}_{1,v}}\Big)(\omega)$$

is equal to the projection of ω on the ${}^\sigma\chi_1^{-1}$-eigenspace with respect to this action. A fancy reformulation of this simple observation is given in Proposition 2.4.10 below.

2.4.9. Proposition. *If* $\forall v\in S\ \ \dim({}^\sigma\pi_v^{H_v})=1$ *(for one, hence for all* $\sigma : L\hookrightarrow\mathbf{C}$*), then: the map* α_1 *is not constant* $\iff$ *there exists* $\sigma : L\hookrightarrow\mathbf{C}$ *such that* $\forall v\in S\ \ {}^\sigma e_{\overline{\chi}_{1,v}}({}^\sigma\pi_v^{H_v})\neq 0$.

Proof. The assumptions imply that, for each $\sigma : L\hookrightarrow\mathbf{C}$, any non-zero element $\omega\in({}^\sigma\pi^\infty)^H = \bigotimes'_{v\nmid\infty}{}^\sigma\pi_v^{H_v}$ is of the form $\omega=\bigotimes_{v\in S}\omega_v\otimes\omega^S$ ($\mathbf{C}\omega_v={}^\sigma\pi_v^{H_v}$, $\omega^S\neq 0$); then

$$(\alpha_1\otimes\mathrm{id})^*(\omega) = \bigotimes_{v\in S}{}^\sigma e_{\overline{\chi}_{1,v}}(\omega_v)\otimes\omega^S$$

is non-zero $\iff$ $\forall v\in S\ \ {}^\sigma e_{\overline{\chi}_{1,v}}({}^\sigma\pi_v^{H_v})\neq 0$.

2.4.10. Proposition. (1) *The map* α_1 *is not constant* $\iff$ *there exist* $\sigma : L\hookrightarrow\mathbf{C}$, $\omega\in\Gamma(A^{\mathrm{an}},\Omega^{\mathrm{an}})\cap{}^\sigma\pi^H$ *and a* $j(\prod_{v\in S}(K_v^*)^\circ/F_v^*)$*-invariant linear map* ${}^\sigma\ell : {}^\sigma\pi\longrightarrow\mathbf{C}({}^\sigma\chi_1^{-1})$ *(i.e., such that* ${}^\sigma\ell(j(a)u)={}^\sigma\chi_1(a)^{-1}\,{}^\sigma\ell(u)$*) satisfying* ${}^\sigma\ell(\omega)\neq 0$.
(2) *Assume that there exist* $\sigma : L\hookrightarrow\mathbf{C}$ *and* $0\neq\omega=\bigotimes_{v\in S}\omega_v\otimes\omega^S\in\Gamma(A^{\mathrm{an}},\Omega^{\mathrm{an}})\cap{}^\sigma\pi^H$ *(*$\omega_v\in{}^\sigma\pi_v^{H_v}$*) and, for each* $v\in S$*, a* $j_v((K_v^*)^\circ/F_v^*)$*-invariant linear map* ${}^\sigma\ell_v : {}^\sigma\pi_v\longrightarrow\mathbf{C}({}^\sigma\chi_{1,v}^{-1})$ *such that* ${}^\sigma\ell_v(\omega_v)\neq 0$*. Then* α_1 *is not constant.*
(3) *Assume that, for each* $v\in S$, $\dim({}^\sigma\pi_v^{H_v})=1$ *(for one, hence for all* $\sigma : L\hookrightarrow\mathbf{C}$*). Then: the map* α_1 *is not constant* $\iff$ *there exist* $\sigma : L\hookrightarrow\mathbf{C}$ *and, for each* $v\in S$*, a* $j_v((K_v^*)^\circ/F_v^*)$*-invariant linear map* ${}^\sigma\ell_v : {}^\sigma\pi_v\longrightarrow\mathbf{C}({}^\sigma\chi_{1,v}^{-1})$ *such that* ${}^\sigma\ell_v({}^\sigma\pi_v^{H_v})\neq 0$.

Proof. (1) If ${}^\sigma\ell$ exists, then $\omega' := (\alpha_1\otimes\mathrm{id})^*(\omega)$ is non-zero (and hence α_1 is not constant), since

$$ {}^\sigma\ell(\omega') = \sum_{a\in U'} {}^\sigma\chi_1(a)\, {}^\sigma\ell(j(a)\omega) = |U'|\cdot {}^\sigma\ell(\omega) \neq 0. $$

Conversely, if α_1 is not constant, then there exist σ and ω as above such that $\omega' := (\alpha_1\otimes \mathrm{id})^*(\omega) \neq 0$; we let ${}^\sigma\ell$ be the $j(\prod_{v\in S}(K_v^*)^\circ/F_v^*)$-equivariant projection of ${}^\sigma\pi$ onto the line $\mathbf{C}\omega'$ (on which $j(\prod_{v\in S}(K_v^*)^\circ/F_v^*)$ acts by the character ${}^\sigma\chi_1^{-1}$).
(2) Fix a linear form $\ell^S : \bigotimes'_{v\notin S, v\nmid\infty} {}^\sigma\pi_v \longrightarrow \mathbf{C}$ such that $\ell^S(\omega^S)\neq 0$ and apply (1) to the linear map ${}^\sigma\ell = \bigotimes_{v\in S} {}^\sigma\ell_v \otimes \ell^S$.
(3) The assumptions imply that, for each $\sigma : L \hookrightarrow \mathbf{C}$, any element $\omega \in ({}^\sigma\pi^\infty)^H$ is of the form $\omega = \bigotimes_{v\in S}\omega_v\otimes\omega^S$ ($\omega_v \in {}^\sigma\pi_v^{H_v}$); the two implications then follow from (1) and (2), respectively.

2.4.11. Invariant linear maps. Let $v\in S$. If v does not split in K/F, then $(K_v^*)^\circ = K_v^*$ and $\chi_{1,v}$ is a character of K_v^*/F_v^*. The value of $\varepsilon({}^\sigma\pi_v\times{}^\sigma\chi_{1,v},\frac12)\in\{\pm1\}$ does not depend on σ; denote it by $\varepsilon(\pi_v\times\chi_v,\frac12)$. As recalled in 1.3.2, a non-zero $j_v(K_v^*/F_v^*)$-invariant linear map ${}^\sigma\ell_v : {}^\sigma\pi_v \longrightarrow \mathbf{C}({}^\sigma\chi_{1,v}^{-1})$ exists if and only if the following condition is satisfied:

$$ (\star_v) \qquad \eta_v(-1)\varepsilon(\pi_v\times\chi_v,\tfrac12) = \mathrm{inv}_v(B_v). $$

If this is the case, then ${}^\sigma\ell_v$ is unique up to a scalar multiple.

If, on the other hand, v splits in K/F, then $B = M_2(F_v)$ and there are many different $j_v((K_v^*)^\circ/F_v^*)$-invariant linear maps ${}^\sigma\pi_v \longrightarrow \mathbf{C}({}^\sigma\chi_{1,v}^{-1})$. For example, if $\widetilde{\chi}_{1,v} : K_v^*/F_v^* \longrightarrow O_L^*$ is any character extending $\chi_{1,v}$ (there are plenty of them, since $K_v^*/(K_v^*)^\circ F_v^* \xrightarrow{\sim} \mathbf{Z}$), then $\varepsilon(\pi_v\times\chi_v,\frac12) := \varepsilon({}^\sigma\pi_v\times{}^\sigma\widetilde{\chi}_{1,v},\frac12)\in\{\pm1\}$ (where ${}^\sigma\widetilde{\chi}_{1,v} = \sigma\circ\widetilde{\chi}_{1,v} : K_v^*/F_v^*\longrightarrow \mathbf{C}^*$) does not depend on σ, nor on the chosen extension $\widetilde{\chi}_{1,v}$ of $\chi_{1,v}$, and the condition $(\star_v)$ automatically holds (by 1.3.3(1)). As a result, there exists a non-zero $j_v(K_v^*/F_v^*)$-invariant linear map (unique up to a scalar multiple) ${}^\sigma\ell_v : {}^\sigma\pi_v \longrightarrow \mathbf{C}({}^\sigma\widetilde{\chi}_{1,v}^{-1})$, which is then $j_v((K_v^*)^\circ/F_v^*)$-invariant as a map ${}^\sigma\pi_v \longrightarrow \mathbf{C}({}^\sigma\chi_{1,v}^{-1})$. However, there are many $j_v((K_v^*)^\circ/F_v^*)$-invariant maps ${}^\sigma\pi_v \longrightarrow \mathbf{C}({}^\sigma\chi_{1,v}^{-1})$ which cannot be obtained in this way.

In particular, if $(\star_v)$ holds for all $v\in S$, then there is a non-zero $j(\prod_{v\in S}(K_v^*)^\circ/F_v^*)$-invariant linear map

$$ \bigotimes_{v\in S} {}^\sigma\ell_v : {}^\sigma\pi_S = \bigotimes_{v\in S} {}^\sigma\pi_v \longrightarrow \mathbf{C}({}^\sigma\chi_1^{-1}). $$

Tensoring it with any non-zero linear form $\ell^S : \bigotimes'_{v\notin S, v\nmid\infty} {}^\sigma\pi_v \longrightarrow \mathbf{C}$, we obtain a non-zero $j(\prod_{v\in S}(K_v^*)^\circ/F_v^*)$-invariant linear map

$$ {}^\sigma\ell = \bigotimes_{v\in S} {}^\sigma\ell_v\otimes\ell^S : {}^\sigma\pi \longrightarrow \mathbf{C}({}^\sigma\chi_1^{-1}). $$

2.4.12. Proposition. (1) *If α_1 is not constant, then $(\star_v)$ holds for all $v\in S$.*
(2) *If $\chi : G^{(c)} \longrightarrow O_L^*$ is a character of finite order whose restriction to $G_1^{(c)}$ is equal to χ_1, then $(\star_v)$ holds for all finite primes v of F not dividing $cd_{K/F}P_1\cdots P_s$.*
(3) *If $(\star_v)$ holds for all $v\in S$ and $\chi : G^{(c)}\longrightarrow O_L^*$ is as in (2), then the global ε-factor $\varepsilon(\pi\times\chi,\frac12) = \prod_v\varepsilon(\pi_v\times\chi_v,\frac12)$ is equal to*

$$ \varepsilon(\pi\times\chi,\tfrac12) = -\prod_{v\mid P_1\cdots P_s}\eta_v(-1)\varepsilon(\pi_v\times\chi_v,\tfrac12). $$

If all primes $P_1,\ldots,P_s$ split in K/F or if $(P_1\cdots P_s)^n \mid c(\chi)$ for large enough n, then $\varepsilon(\pi\times\chi,\frac12) = -1$.

Proof. (1) As α_1 is not constant, there exist $\sigma : L\hookrightarrow\mathbf{C}$ and $\omega\in\Gamma(A^{\mathrm{an}},\Omega^{\mathrm{an}})\cap{}^\sigma\pi$ such that $\omega' := (\alpha_1\otimes\mathrm{id})^*(\omega)\neq 0$. As

$$ \forall a\in\prod_{v\in S}(K_v^*)^\circ/F_v^* \qquad {}^\sigma\pi(j(a))\,\omega' = {}^\sigma\chi_1(a)^{-1}\,\omega', $$

it follows that, for each $v \in S$, there exists a non-zero element of ${}^\sigma\pi_v \otimes {}^\sigma\chi_{1,v}$ invariant under $j_v((K_v^*)^\circ/F_v^*)$. If v splits in K/F, then the condition $(\star_v)$ is automatically satisfied, by 1.3.3(1). If v does not split, then $(K_v^*)^\circ = K_v^*$ and the statement follows from 1.3.3(2) and 1.3.2.
(2) As before, there is nothing to prove if v splits in K/F. If v does not split, then our assumptions imply that v is inert and χ_v is unramified, hence χ_v is trivial on $F_v^* O_{K,v}^* = K_v^*$. Similarly, the fact that $\forall x \in \mathcal{C} \quad v \nmid c(x)$ implies that $t_v(O_{K,v}^*) \subseteq R_v^*$; thus $t_v(K_v^*) = t_v(F_v^* O_{K,v}^*)$ acts trivially on ${}^\sigma\pi_v^{R_v^*} \neq 0$, hence there exists a $t_v(K_v^*)$-invariant vector in ${}^\sigma\pi_v \otimes {}^\sigma\chi_v$; we conclude again by 1.3.3(2) and 1.3.2.
(3) For each prime v of F, put $a_v = \mathrm{inv}_v(B_v)\eta_v(-1)\varepsilon_v \in \{\pm 1\}$, where $\varepsilon_v = \varepsilon(\pi_v \times \chi_v, \frac{1}{2})$; then the global ε-factor is equal to $\varepsilon = \prod_v a_v$. If $v \nmid \infty P_1 \cdots P_s$, then $a_v = 1$, by (2) (resp., by assumption) if $v \notin S$ (resp., if $v \in S$). If $v = \tau_j \mid \infty$, then $\varepsilon_v = 1$, $\eta_v(-1) = -1$ and $\mathrm{inv}_v(B_v) = -1$ (resp., $\mathrm{inv}_v(B_v) = 1$) if $j \neq 1$ (resp., if $j = 1$); thus

$$\varepsilon = \prod_{v \mid \infty P_1 \cdots P_s} a_v = -\prod_{v \mid P_1 \cdots P_s} a_v = -\prod_{v \mid P_1 \cdots P_s} \eta_v(-1)\varepsilon_v$$

(the last equality follows from the assumption (D1)). If $v = P_i$ splits in K/F, then $\eta_v(-1)\varepsilon_v = 1$, by 1.3.3(1). Finally, if $v = P_i$ does not split and χ_v is sufficiently ramified, then $\eta_v(-1)\varepsilon_v = 1$, by [J-L, Prop. 3.8] and [J, Thm. 20.6].

2.4.13. We are now going to describe several special cases in which Proposition 2.4.10(3) (equivalently, Proposition 2.4.9) applies. We drop σ from the notation and consider the following, purely local, situation: let v be a finite prime of F, B_v a quaternion algebra over F_v, K_v a quadratic extension of F_v (the split case $K_v = F_v \times F_v$ is allowed), $j_v : K_v \hookrightarrow B_v$ an F_v-embedding, $\chi_v : (K_v^*)^\circ/F_v^* \longrightarrow \mathbf{C}^*$ a character (of finite order), π an irreducible smooth representation of B_v^*/F_v^* (of infinite dimension if $B_v = M_2(F_v)$) and $R_v \subset B_v$ an $O_{F,v}$-order such that $\pi_v^{R_v^*} \neq 0$.

We are interested in the existence of a non-zero $j_v((K_v^*)^\circ/F_v^*)$-invariant linear map $\ell_v : \pi_v \longrightarrow \mathbf{C}(\chi_v^{-1})$, i.e., a linear form $\ell_v : \pi_v \longrightarrow \mathbf{C}$ satisfying

$$\forall a \in (K_v^*)^\circ/F_v^* \ \ \forall u \in \pi_v \qquad \ell_v(j_v(a)u) = \chi_v(a)^{-1}\, u, \tag{2.4.13.1}$$

for which

$$\ell_v(\pi_v^{R_v^*}) \neq 0, \tag{2.4.13.2}$$

particularly in the following case (studied in [Gr-Pr], for example):

$$\dim(\pi_v^{R_v^*}) = 1. \tag{2.4.13.3}$$

The requirements (2.4.13.1-2) are compatible only if $\chi_v(j_v^{-1}(R_v^*)) = 1$, which is equivalent to

$$c(\chi_v) \mid c(j_v^{-1}(R_v)), \tag{2.4.13.4}$$

where the conductor of any $O_{F,v}$-order O of K_v is the ideal $c(O) \subset O_{F,v}$ given by $O = O_{F,v} + c(O)O_{K,v}$, and the conductor $c(\chi_v)$ of χ_v is the conductor of the biggest $O_{F,v}$-order $O \subset K_v$ such that $O^* \subseteq \mathrm{Ker}(\chi_v)$.

In the situation considered in 2.4.4 we have, for each $v \in S$, $\chi_v = \chi_{1,v}$, hence $c(\chi_v)$ is equal to the v-part of $c(\chi_1)$, and $c(j_v^{-1}(R_v))$ is equal to the common v-part of $c(x)$ for all $x \in \mathcal{C}$. The divisibility (2.4.13.4) then simply says that

$$\forall v \in S \ \ \forall x \in \mathcal{C} \qquad \mathrm{ord}_v(c(\chi_1)) \leq \mathrm{ord}_v(c(x)). \tag{2.4.13.5}$$

We can reformulate (2.4.13.1-2) in more down-to-earth terms, by returning to the formula (2.4.8.1): the assumption (2.4.13.4) implies that χ_v factors through the finite abelian group

$$U'_v = (K_v^*)^\circ/F_v^*(O_{F,v} + c(j_v^{-1}(R_v))O_{K,v})^*;$$

similarly, the action of $j_v(K_v^*)^\circ/F_v^*)$ on $\pi_v^{R_v^*}$ factors through U'_v. Denoting

$$e_{\overline{\chi}_v} = \sum_{a\in U'_v} \chi_v(a)\, a \in \mathbf{C}[U'_v],$$

then the two conditions (2.4.13.1-2) are equivalent to the condition

$$e_{\overline{\chi}_v}(\pi_v^{R_v^*}) \neq 0 \qquad (2.4.13.6)$$

from Proposition 2.4.9, as shown in the proof of Proposition 2.4.10.

2.4.14. We retain the notations of 2.4.13. We list a few cases in which the properties (2.4.13.1-3) are known to be satisfied.

2.4.14.1. If $B_v = M_2(F_v)$ and $\dim(\pi_v) = \infty$, denote by $o(\pi_v)$ the conductor exponent of π_v (i.e., the valuation of the conductor of π_v). If $R_v \subset B_v$ is an Eichler order of B_v of level $n \geq 0$, then

$$\dim(\pi_v^{R_v^*}) = \max(n - o(\pi_v) + 1, 0),$$

by a representation-theoretical reformulation of the Atkin-Lehner theory, due to Casselman [Ca, §1] (cf. 3.1.7 below). In particular, (2.4.13.3) holds if $n = o(\pi_v)$.

2.4.14.2. If $B_v \neq M_2(F_v)$, then $\dim(\pi_v) < \infty$; put $o(\pi_v) := o(JL(\pi_v))$. We have

$$\dim(\pi_v) = 1 \iff \pi_v = \mu \circ \mathrm{nr} : B_v^* \xrightarrow{\mathrm{nr}} F_v^* \xrightarrow{\mu} \{\pm 1\} \iff JL(\pi_v) = \mathrm{St} \otimes \mu.$$

2.4.14.3. Assume that $o(\pi_v) = 0$ and that R_v is a maximal order. In this case $B_v = M_2(F_v)$, π_v is an unramified principal series representation and (2.4.13.3) holds.

A result of Gross and Prasad ([Gr-Pr, Prop. 2.3]) implies that a linear form satisfying (2.4.13.1-2) exists, provided $c(\chi_v) = c(j_v^{-1}(R_v))$.

In Proposition 3.4.3 below we generalise this result as follows: write the eigenvalue of the Hecke operator T_v on $\pi_v^{R_v^*}$ as $(Nv)^{1/2}(\lambda_v + \lambda_v^{-1})$ ($\lambda_v \in \mathbf{C}$). Assume that $c(\chi_v) = v^m$, $c(j_v^{-1}(R_v)) = v^n$, $1 \leq m \leq n$. If $\lambda_v^2 = 1$ or $\lambda_v^{2(n-m+1)} \neq 1$ (which is automatic if $m = n$, the case covered by [Gr-Pr]), then (2.4.13.6) holds.

2.4.14.4. Assume that $o(\pi_v) = 1$, $B_v \neq M_2(F_v)$ and that R_v is any order in B_v. In this case v does not split in K/F, $\dim(\pi_v) = 1$, $\pi_v = \mu \circ \mathrm{nr}$, $\mu : F_v^* \longrightarrow \{\pm 1\}$ is unramified and (2.4.13.3) holds. It follows from 1.3.2-3 that a linear form ℓ_v satisfying (2.4.13.1) exists (and is then trivially unique up to a scalar multiple) if and only if

$$\chi_v = \mu \circ N_{K_v/F_v} \iff \begin{cases} v \text{ is inert in } K/F,\ \chi_v = 1 \\ v \text{ is ramified in } K/F,\ \chi_v \text{ is unramified},\ \chi_v(w) = \mu(v) \quad (vO_K = w^2). \end{cases} \qquad (2.4.14.4.1)$$

If (2.4.14.4.1) holds, then the form ℓ_v satisfies (2.4.13.2), for trivial reasons, and (2.4.13.6) holds.

2.4.14.5. Assume that $o(\pi_v) = 1$, $B_v = M_2(F_v)$ and that R_v is an Eichler order of level one. In this case $\pi_v = \mathrm{St} \otimes \mu$, $\mu : F_v^* \longrightarrow \{\pm 1\}$ is unramified and (2.4.13.3) holds. It follows again from 1.3.2-3 that a linear form ℓ_v satisfying (2.4.13.1) exists (and is unique up to a scalar multiple if v does not split in K/F) if and only if either

$$v \text{ splits in } K/F, \qquad (2.4.14.5.1)$$

or

(2.4.14.5.2)

$$v \text{ does not split in } K/F,\ \chi_v \neq \mu \circ N_{K_v/F_v} \iff \begin{cases} v \text{ is inert in } K/F,\ \chi_v \neq 1 \\ v \text{ is ramified in } K/F,\ \chi_v \text{ is ramified} \\ v \text{ is ramified in } K/F,\ \chi_v \text{ is unramified},\ \chi_v(w) = -\mu(v). \end{cases}$$

($vO_K = w^2$). It is shown in Proposition 3.4.5 below that if (2.4.14.5.1) or (2.4.14.5.2) holds, then (2.4.13.6) holds in this case, too.

2.5. Weak non-vanishing results

2.5.1. Theorem. *Assume that we are given the data (D1) - (D5) from 2.4.1. Assume also that there exist* $\sigma : L \hookrightarrow \mathbf{C}$, $\omega \in \Gamma(A^{\mathrm{an}}, \Omega^{\mathrm{an}}) \cap {}^{\sigma}\pi^{H}$ *and a* $j(\prod_{v\in S}(K_v^*)^\circ/F_v^*)$*-invariant linear map* ${}^{\sigma}\ell : {}^{\sigma}\pi \longrightarrow \mathbf{C}({}^{\sigma}\chi_1^{-1})$ *(i.e., such that* ${}^{\sigma}\ell(j(a)u) = {}^{\sigma}\chi_1(a)^{-1}\,{}^{\sigma}\ell(u))$ *satisfying* ${}^{\sigma}\ell(\omega) \neq 0$*. If* $\{x_n\}_{n\in\mathbf{N}} \subseteq \mathcal{C}$ *is a sequence of CM points contained in the* $(P_1\cdots P_s)$*-isogeny class* $\mathcal{C}$ *such that* $(P_1\cdots P_s)^n \mid c(x_n)$ *for all* $n \in \mathbf{N}$*, then*

$$e_{\overline{\chi}_0}(1\otimes\alpha(x_n)) = \sum_{g\in G_0^{(c)}} \chi_0(g)\otimes\alpha(g(x_n)) \in O_L \otimes_{O_{L_0}} A_0(K[cP^\infty]) = A(K[cP^\infty])$$

is non-torsion for all sufficiently large $n >> 0$*.*

Proof. Combine Proposition 2.4.7 and 2.4.10(1).

2.5.2. Theorem (an explicit special case of 2.5.1). *Assume that we are given the data (D1) - (D5) from 2.4.1. Assume also that, for each* $v \in S$*, the following conditions (i)–(iv) hold:*
(i) $o(\pi_v) \leq 1$.
(ii) *If* $o(\pi_v) = 0$*, then* R_v *is a maximal order in* $B_v = M_2(F_v)$*; if* $o(\pi_v) = 1$ *and* $B_v = M_2(F_v)$*, then* R_v *is an Eichler order of level one in* B_v*.*
(iii) $(\star_v)$ *holds.*
(iv) *If* $o(\pi_v) = 0$*, write the eigenvalue of* T_v *on* A_0 *as* $(Nv)^{1/2}(\lambda_v + \lambda_v^{-1})$*. Put* $n_v = \mathrm{ord}_v(c(x))$ *(for any* $x \in \mathcal{C}$*) and* $m_v = \mathrm{ord}_v(c(\chi_0))$*. Assume that* $1 \leq m_v \leq n_v$ *and* $[\lambda_v^2 = 1$ *or* $\lambda_v^{2(n_v-m_v+1)} \neq 1]$ *(which is automatic if* $m_v = n_v \geq 1$*).*
Then: if $\{x_n\}_{n\in\mathbf{N}} \subseteq \mathcal{C}$ *is a sequence of CM points contained in the* $(P_1\cdots P_s)$*-isogeny class* $\mathcal{C}$ *such that* $(P_1\cdots P_s)^n \mid c(x_n)$ *for all* $n \in \mathbf{N}$*, then*

$$e_{\overline{\chi}_0}(1\otimes\alpha(x_n)) = \sum_{g\in G_0^{(c)}} \chi_0(g)\otimes\alpha(g(x_n)) \in O_L \otimes_{O_{L_0}} A_0(K[cP^\infty]) = A(K[cP^\infty])$$

is non-torsion for all sufficiently large $n >> 0$*.*

Proof. Combine Proposition 2.4.7, 2.4.9 and 2.4.14.3-5.

2.6. Non-vanishing and ε-factors

In this section $H \subset \widehat{B}^*$ denotes any open compact subgroup (not necessarily of the form $\widehat{R}^*$).

2.6.1. Let $\alpha : N_H^* \xrightarrow{\iota_H} J(N_H^*) \twoheadrightarrow A_0$ be as in (D4). Assume that we are given a CM point $x = [z,b]_H \in CM(N_H, K)$ and a character $\chi : G_x = \mathrm{Gal}(K(x)/K) \longrightarrow O_L^*$ ($L \supset L_0$), where $K(x) = (K^{\mathrm{ab}})^{\mathrm{rec}_K(\widehat{t}^{-1}(bH\widehat{F}^*b^{-1}))} \subset K[\infty]$ is the field of definition of x. As in 2.4.1-6, set $e_{\overline{\chi}} = \sum_{g\in G_x}\chi(g)\,g \in O_L[G_x]$ and $A = O_L \otimes_{O_{L_0}} A_0$. The character

$$\widehat{K}^* \xrightarrow{\mathrm{rec}_K} \mathrm{Gal}(K^{\mathrm{ab}}/K) \twoheadrightarrow G_x \xrightarrow{\chi} O_L^*$$

will also be denoted by χ. For each embedding $\sigma : L \hookrightarrow \mathbf{C}$, let ${}^{\sigma}\pi$ and ${}^{\sigma}\chi = \sigma\circ\chi : G_x \longrightarrow \mathbf{C}^*$ be as in 2.4.8. For each prime v of F, the local ε-factor $\varepsilon({}^{\sigma}\pi_v \times {}^{\sigma}\chi_v, \frac{1}{2}) \in \{\pm 1\}$ does not depend on σ; denote it by $\varepsilon(\pi_v \times \chi_v, \frac{1}{2})$ (cf. 2.4.11).

The following Proposition seems to be well-known to the experts ([Gr], [Zh]), but we have not been able to find a precise reference.

2.6.2. Proposition. *If the point* $e_{\overline{\chi}}(1\otimes\alpha(x)) \in A(K(x))$ *is not torsion, then:*
(1) $\forall v \nmid \infty \quad \eta_v(-1)\,\varepsilon(\pi_v\times\chi_v,\frac{1}{2}) = \mathrm{inv}_v(B_v)$. *In particular, B is uniquely determined by K, χ and the isogeny class of A_0 over F.*
(2) *The global ε-factor is equal to* $\varepsilon(\pi\times\chi,\frac{1}{2}) = -1$.

Proof. (1) We slightly modify the arguments from 2.4.4-12. Let S' be a finite set of non-archimedean primes of F containing all primes at which B, π or $K(x)/F$ ramify and such that the composite morphism

$$r = (r_v)_{v\in S'} : \prod_{v\in S'} K_v^* \longrightarrow \widehat{K}^* \xrightarrow{\mathrm{rec}_K} \mathrm{Gal}(K^{\mathrm{ab}}/K) \twoheadrightarrow G_x \qquad\qquad (r_v : K_v^* \longrightarrow G_x)$$

is surjective. As in 2.4.4, if we define

$$j_v = \mathrm{Ad}(b_v)^{-1}\circ t_v : K_v \hookrightarrow B_v \quad (v\in S'), \qquad j = (j_v)_{v\in S'} : \prod_{v\in S'} K_v \hookrightarrow \prod_{v\in S'} B_v,$$

then

$$\forall a\in\prod_{v\in S'} K_v^* \qquad \mathrm{rec}_K(a)[z,b]_H = [z,\widehat{t}(a)b]_H = [z,bj(a)]_H.$$

Denote by S_1' (resp., S_2') the set of those primes $v\in S'$ that do (resp., do not) split in K/F. For each $v\in S'$, $C_v = O_{K,v}^*\cap\mathrm{Ker}(r_v)$ is an open subgroup of $O_{K,v}^*$ and the quotient $V_v^\circ := (K_v^*)^\circ/F_v^*C_v$ is finite. If $v\in S_2'$, then $V_v^\circ = V_v := K_v^*/F_v^*C_v$. If $v\in S_1'$, fix a splitting of the surjection $K_v^*\twoheadrightarrow K_v^*/(K_v^*)^\circ = \Delta_v \xrightarrow{\sim} \mathbf{Z}$; then $K_v^* = (K_v^*)^\circ\times\Delta_v$ and $\mathrm{Ker}(r_v|_{\Delta_v}) = n_v\Delta_v$ for some $n_v\geq 1$. Fix a set of representatives $\Delta_v'\subset\Delta_v$ of $\Delta_v/n_v\Delta_v$ and set $V_v = V_v^\circ\Delta_v'\subset K_v^*/F_v^*C_v$. This defines a finite subset $V = \prod_{v\in S'} V_v \subset \prod_{v\in S'} K_v^*/F_v^*C_v$ stable under multiplication by the abelian group $V^\circ = \prod_{v\in S'} V_v^\circ$ and such that the composite map $V\hookrightarrow \prod_{v\in S'} K_v^*/F_v^*C_v \xrightarrow{r} G_x$ is surjective, with each fibre of cardinality $|V|/|G_x|$. It follows that

$$\frac{|V|}{|G_x|}\, e_{\overline{\chi}}(1\otimes\alpha(x)) = \sum_{a\in V}\chi(a)\,\alpha([z,bj(a)]_H).$$

As in 2.4.5-6, fix an open compact subgroup $H_1\subset\widehat{B}^*$ such that

$$H_1\widehat{F}^* \subseteq \bigcap_{a\in V}\mathrm{Ad}(j(a))H\widehat{F}^*$$

and define, for $a\in V$,

$$\beta_a : N_{H_1}^* \xrightarrow{[\cdot j(a)]} N_{j(a)^{-1}H_1 j(a)}^* \xrightarrow{\mathrm{pr}} N_H^* \xrightarrow{\alpha} A_0, \qquad \alpha_1 = \sum_{a\in V}\chi(a)\,\beta_a : N_{H_1}^* \longrightarrow A;$$

then

$$\frac{|V|}{|G_x|}\, e_{\overline{\chi}}(1\otimes\alpha(x)) = \alpha_1(x_1), \qquad x_1 = [z,b]_{H_1}.$$

We claim that the map

$$\alpha_1^* : \Gamma(A,\Omega_{A/F}) \longrightarrow \Gamma(N_{H_1}^*,\Omega_{N_{H_1}^*/F})$$

is non-zero. If this were not the case, the morhism α_1 (which is defined over F) would be constant, as the curve $N_{H_1}^*$ is irreducible. However, α_1 commutes with the Hecke operator $T(v)$ for any non-archimedean prime v of F at which B and ${}^\sigma\pi$ are unramified, which implies that the closed point $\alpha_1(N_{H_1}^*)$ of A is killed by $\deg(T(v))-\lambda(v) = Nv+1-\lambda(v)\in O_{L_0}$, where $\sigma(\lambda(v))\in\mathbf{C}$ is the eigenvalue of $T(v)$ on ${}^\sigma\pi$. As $Nv+1-\lambda(v)$ is non-zero by Weil's bounds, it follows that $\alpha_1(N_{H_1}^*)$ is torsion. This contradiction with our assumption that $\alpha_1(x_1)$ is not torsion implies that $\alpha_1^*\neq 0$, as claimed. As in 2.4.8, we obtain that there is an embedding $\sigma : L\hookrightarrow\mathbf{C}$ and a differential form $\omega\in\Gamma(A^{\mathrm{an}},\Omega^{\mathrm{an}})\cap({}^\sigma\pi^\infty)^H$ for which

$$\omega_1 := (\alpha_1 \otimes \mathrm{id})^*(\omega) = \sum_{a \in V} {}^\sigma\chi(a)\, j(a)\omega = \prod_{v \in S'} \left(\sum_{a_v \in V_v} {}^\sigma\chi_v(a_v)\, j_v(a_v) \right) \omega \neq 0$$

(above, $\chi_v = \chi \circ r_v : K_v^* \longrightarrow O_L^*$). As V is stable by multiplication by V°, it follows that

$$\forall a \in \prod_{v \in S'} (K_v^*)^\circ \qquad j(a)\omega_1 = {}^\sigma\chi^{-1}(a)\, \omega_1.$$

Writing

$$\omega_1 = \omega_{1,S'} \otimes \bigotimes_{v \notin S'} (\text{a spherical vector in } {}^\sigma\pi_v), \qquad \omega_{1,S'} \in \bigotimes_{v \in S'} {}^\sigma\pi_v,$$

we deduce that $\omega_{1,S'}$ is a non-zero $j\left(\prod_{v \in S'} (K_v^*)^\circ\right)$-invariant vector in $\bigotimes_{v \in S'} ({}^\sigma\pi_v \otimes {}^\sigma\chi_v)$. As each ${}^\sigma\pi_v$ is a direct sum of one-dimensional representations of $(K_v^*)^\circ$ (on each of which $(K_v^*)^\circ$ acts through a finite quotient), it follows (as in the proof of Proposition 2.4.10(1)) that there exists a non-zero $j\left(\prod_{v \in S'} (K_v^*)^\circ\right)$-invariant linear map ${}^\sigma\ell_{S'} : \bigotimes_{v \in S'} {}^\sigma\pi_v \longrightarrow \mathbf{C}({}^\sigma\chi^{-1})$ for which ${}^\sigma\ell_{S'}(\omega_{1,S'}) \neq 0$. In particular, for each $v \in S'_2$ there exists a non-zero $j_v(K_v^*)$-invariant linear map ${}^\sigma\pi_v \longrightarrow \mathbf{C}({}^\sigma\chi_v^{-1})$, hence

$$\eta_v(-1)\, \varepsilon(\pi_v \times \chi_v, \tfrac{1}{2}) = \mathrm{inv}_v(B_v),$$

by 1.3.2. For each non-archimedean prime $v \notin S'_2$ it follows from 1.3.3(1) and the definition of S' that

$$\varepsilon(\pi_v \times \chi_v, \tfrac{1}{2}) = \eta_v(-1) = \mathrm{inv}_v(B_v) = 1.$$

(2) If v is an archimedean prime of F, then

$$\varepsilon(\pi_v \times \chi_v, \tfrac{1}{2}) = 1, \qquad \eta_v(-1) = -1, \qquad \mathrm{inv}_v(B_v) = \begin{cases} -1, & v \neq \tau_1 \\ 1, & v = \tau_1. \end{cases}$$

Combined with (1), this implies that

$$\varepsilon(\pi \times \chi, \tfrac{1}{2}) = -\prod_v \eta_v(-1)\, \mathrm{inv}_v(B_v) = -1.$$

2.6.3. (1) In particular, the assumptions of ([Ne 2], Thm. 3.2) imply that the corresponding global ε-factor is equal to -1 (note that, if $A_j = A_0$ acquires CM over K, then this ε-factor is very often equal to $+1$).
(2) With minor modifications, Proposition 2.6.2 also holds for a simple quotient $J(M_H^*) \twoheadrightarrow A_0$ of the Jacobian of the Shimura curve M_H^* ([Co-Va 1], [Ne 2]) which does not factor through $J(N_H^*)$. In this case L_0 is a CM field, ${}^\sigma\pi$ has a non-trivial central character (of finite order, with trivial archimedean part) ${}^\sigma\varphi = \sigma \circ \varphi$, where $\varphi : \mathbf{A}_F^*/F^* \longrightarrow O_{L_0}^*$. A CM point $x = [z, b]_H \in CM(M_H, K)$ is defined over $K(x) = (K^{\mathrm{ab}})^{\mathrm{rec}_K(\widehat{t}^{-1}(bH\widehat{b}^{-1}))}$, which is not necessarily contained in $K[\infty]$. Fix $\chi : \widehat{K}^* \longrightarrow G_x = \mathrm{Gal}(K(x)/K) \longrightarrow O_L^*$ ($L \supset L_0$) and assume that $e_{\overline{\chi}}(1 \otimes \alpha(x)) \in A(K(x))$ is not torsion. The proof of Proposition 2.6.2(1) yields a non-zero $j\left(\prod_{v \in S'} (K_v^*)^\circ\right)$-invariant vector in $\bigotimes_{v \in S'} ({}^\sigma\pi_v \otimes {}^\sigma\chi_v)$, which implies that $\varphi \cdot \chi|_{\widehat{F}^*} = 1$. Appealing to the results of ([Tu], [Wa, Thm. 2], [Sa]) in the case of a non-trivial central character, we deduce, as in the proof of Proposition 2.6.2, that

$$\forall v \nmid \infty \quad \varepsilon(\pi_v \times \chi_v, \tfrac{1}{2}) = (\varphi\eta)_v(-1)\, \mathrm{inv}_v(B_v), \qquad \varepsilon(\pi \times \chi, \tfrac{1}{2}) = -1.$$

3. Representation theory

3.1. The Bruhat-Tits tree

3.1.1. For any set X, denote by $\mathbf{C}[X] = \{\sum_{x\in X} n_x x \mid n_x \in \mathbf{C}, \text{the sum is finite}\}$ the $\mathbf{C}$-vector space on X. If G is a group acting on X (on the left), then $\mathbf{C}[X]$ is a left G-module. A map $f : X \longrightarrow Y$ with finite fibres induces linear maps

$$\begin{array}{rclcrcl} f^* : \mathbf{C}[Y] & \longrightarrow & \mathbf{C}[X] & \qquad & f_* : \mathbf{C}[X] & \longrightarrow & \mathbf{C}[Y] \\ \sum\limits_{y\in Y} n_y y & \mapsto & \sum\limits_{x\in X} n_{f(x)} x, & & \sum\limits_{x\in X} n_x x & \mapsto & \sum\limits_{y\in Y} \Big(\sum\limits_{f(x)=y} n_x \Big) y \end{array}$$

satisfying

$$\mathbf{C}[X] = \mathrm{Im}(f^*) \oplus \mathrm{Ker}(f_*). \tag{3.1.1.1}$$

3.1.2. Let O be a complete discrete valuation ring with a uniformiser ϖ and a finite residue field $O/\varpi O$ with q elements; denote by E the fraction field of O and by $v : E^* \longrightarrow \mathbf{Z}$ the valuation. Let $\underline{G} = GL(2) \supset \underline{B} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \supset \underline{Z} = \left\{ * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ (as group schemes over O) and put, for $n \in \mathbf{N}$,

$$G = \underline{G}(E), \qquad B = \underline{B}(E), \qquad Z = \underline{Z}(E), \qquad K = \underline{G}(O), \qquad K(n) = \mathrm{Ker}(\underline{G}(O) \longrightarrow \underline{G}(O/\varpi^n O)),$$
$$K_0(n) = \text{ the inverse image of } \underline{B}(O/\varpi^n O) \subset \underline{G}(O/\varpi^n O) \text{ in } \underline{G}(O).$$

3.1.3. Let $\mathcal{T}$ be the Bruhat-Tits tree associated to $PGL_2(E)$; denote by $\mathcal{V}$ (resp., by $\mathcal{E}$) the set of its vertices (resp., oriented edges). A vertex $x \in \mathcal{V}$ is a homothety class $[L]$ of an O-lattice $L \subset E^2$; an oriented edge $\overrightarrow{x_0x_1} \in \mathcal{E}$ is a homothety class of ordered pairs of O-lattices $L_1 \subset L_0 \subset E^2$ satisfying $L_0/L_1 \xrightarrow{\sim} O/\varpi O$. We put $s(e) = x_0$, $t(e) = x_1$ and $\iota(e) = -e = \overrightarrow{x_1x_0}$ (= the homothety class of $\varpi L_0 \subset L_1 \subset E^2$). The degree of each vertex is equal to $q+1$. Let $o = [O^2] \in \mathcal{V}$ be the origin of $\mathcal{T}$. Denote by $d : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbf{N}$ the distance function on the vertices of the tree (if $L/L' \xrightarrow{\sim} O/\varpi^n O$, then $d([L],[L']) = n$).

3.1.4. There is a natural transitive left action (by isometries) of G on $\mathcal{V}$, hence also on $\mathcal{E}$. For each $n \geq 0$, put $g_n = \begin{pmatrix} 1 & 0 \\ 0 & \varpi^n \end{pmatrix}$ and $o_n = g_n o \in \mathcal{V}$; let $e_0 \in \mathcal{E}$ be the edge satisfying $s(e_0) = o$, $t(e_0) = o_1$. As

$$\mathrm{Stab}_G(o) = KZ, \qquad \mathrm{Stab}_G(e_0) = K_0(1)Z,$$

there are natural G-equivariant bijections

$$\begin{array}{rclcrcl} G/KZ & \xrightarrow{\sim} & \mathcal{V} & \qquad & G/K_0(1)Z & \xrightarrow{\sim} & \mathcal{E} \\ gKZ & \mapsto & gO, & & gK_0(1)Z & \mapsto & ge_0, \end{array} \tag{3.1.4.1}$$

which can be interpreted in a coordinate-free form as G-equivariant bijections

$$\begin{array}{rclcrcl} \{\text{maximal orders in } M_2(E)\} & \xrightarrow{\sim} & \mathcal{V} & \qquad & \{\text{Eichler orders of level one in } M_2(E)\} & \xrightarrow{\sim} & \mathcal{E}/\{\mathrm{id},\iota\} \\ R & \mapsto & \mathcal{V}^{R^*}, & & R_1 & \mapsto & \mathcal{E}^{R_1^*} \end{array} \tag{3.1.4.2}$$

(with G acting on the respective sets of orders by conjugation).

3.1.5. For $n \in \mathbf{N}$ (resp., $n \in \mathbf{N}_{>0}$), put

$$\mathcal{V}_{\leq n} = \{x \in \mathcal{V} \mid d(o,x) \leq n\}, \qquad \mathcal{V}_n = \{x \in \mathcal{V} \mid d(o,x) = n\},$$

resp.,

$$\mathcal{E}_n = \{e \in \mathcal{E} \mid s(e) \in \mathcal{V}_{n-1},\, t(e) \in \mathcal{V}_n\}.$$

For $n \geq m \geq 0$, denote by $r_{n,m} : \mathcal{V}_n \longrightarrow \mathcal{V}_m$ the geodesic projection: $r_{n,m}(x)$ is the unique point $x' \in \mathcal{V}_m$ satisfying $d(x, x') = n - m$. Similarly, let $r_{n,m} : \mathcal{E}_n \longrightarrow \mathcal{E}_m$ be the map satisfying $t \circ r_{n,m} = r_{n,m} \circ t$ (for $n \geq m \geq 1$).

If $x \in \mathcal{V}_{\leq n}$, then $x = [L]$ for some lattice $O^2 \supset L \supset \varpi^n O^2$, which implies that $K(n)x = x$. As the fibres of $r_{n,m}$ coincide with the orbits of $K(m)$ on $\mathcal{V}_n$, we obtain

$$\mathcal{V}^{K(n)} = \mathcal{V}_{\leq n} \quad (n \geq 0), \qquad \mathcal{E}^{K(n)} = \bigcup_{k=1}^{n} (\mathcal{E}_k \cup \iota\mathcal{E}_k) \quad (n \geq 1). \tag{3.1.5.1}$$

Moreover,

$$\mathrm{Stab}_K(o_n) = K \cap g_n K g_n^{-1} = K_0(n) \quad (n \geq 0),$$

which yields K-equivariant bijections

$$K/K_0(n) \xrightarrow{\sim} \mathcal{V}_n, \qquad kK_0(n) \mapsto k\, o_n = k g_n o \quad (n \geq 0). \tag{3.1.5.2}$$

3.1.6. More generally, for each $n \geq 0$ the group G acts transitively on the set $\mathcal{P}_n(\mathcal{T})$ of oriented paths of length n (without backtracking) in $\mathcal{T}$ (above, $\mathcal{V} = \mathcal{P}_0(\mathcal{T})$, $\mathcal{E} = \mathcal{P}_1(\mathcal{T})$). As

$$\mathrm{Stab}_G(o \to o_1 \to \cdots \to o_n) = K_0(n)Z,$$

there are natural G-equivariant bijections

$$\begin{aligned} G/K_0(n)Z &\xrightarrow{\sim} \mathcal{P}_n(\mathcal{T}) \\ gK_0(n)Z &\mapsto g(o \to o_1 \to \cdots \to o_n), \end{aligned} \tag{3.1.6.1}$$

$$\begin{aligned} \{\text{Eichler orders of level } n \text{ in } M_2(E)\} &\xrightarrow{\sim} \mathcal{P}_n(\mathcal{T})/\{\mathrm{id}, \iota\} \\ R_n &\mapsto \mathcal{P}_n(\mathcal{T})^{R_n^*}, \end{aligned} \tag{3.1.6.2}$$

where ι inverses the orientation of every path.

3.1.7. Degeneracy maps. For $0 \leq m \leq n$ and $0 \leq j \leq n-m$ there are natural G-equivariant "degeneracy maps"

$$\begin{aligned} d_j : \mathcal{P}_n(\mathcal{T}) &\longrightarrow \mathcal{P}_m(\mathcal{T}) \\ (x_0 \to \cdots \to x_n) &\mapsto (x_j \to \cdots \to x_{j+m}), \end{aligned} \tag{3.1.7.1}$$

which can also be written using bijections (3.1.6.1) as

$$[\cdot g_j] : G/K_0(n)Z \longrightarrow G/K_0(m)Z, \qquad gK_0(n)Z \mapsto g g_j K_0(m)Z,$$

since

$$d_j(o \to o_1 \to \cdots \to o_n) = g_j(o \to o_1 \to \cdots \to o_m).$$

If π is an infinite-dimensional irreducible admissible representation of G/Z and $m = o(\pi)$, then $\dim \pi^{K_0(m)} = 1$. Moreover, if $\pi^{K_0(m)} = \mathbf{C}f$ and $n \geq m$, then the elements $g_j \cdot f$ $(0 \leq j \leq n-m)$ form a basis of $\pi^{K_0(n)}$ ([Ca, §1]).

3.1.8. Unramified induced representations. Given $\alpha_1, \alpha_2 \in \mathbf{C}^*$, let $\mu_1, \mu_2 : E^* \longrightarrow \mathbf{C}^*$ be the unramified characters given by $\mu_j(a) = \alpha_j^{v(a)}$; define

$$\mu : B \longrightarrow \mathbf{C}^*, \qquad \mu\Big(\begin{pmatrix} a_1 & t \\ 0 & a_2 \end{pmatrix}\Big) \mapsto \mu_1(a_1)\mu_2(a_2),$$

$$\rho = \rho(\mu_1, \mu_2) = \mathrm{Ind}_B^G(\mu) = \{f : G \longrightarrow \mathbf{C} \text{ locally constant} \mid f(bg) = \mu(b)f(g) \ (b \in B,\, g \in G)\}$$

with the G-action $(g \cdot f)(g') = f(g'g)$. This is a smooth representation of G with central character $\mu_1\mu_2$, which is irreducible $\iff$ $\alpha_1/\alpha_2 \neq 1, q^{-2}$. Moreover, the Iwasawa decomposition $G = BK$ together with $B \cap K \subseteq \mathrm{Ker}(\mu)$ imply that

$$\rho^K = \mathbf{C} f_o, \qquad f_o(bk) = \mu(b) \qquad (b \in B,\, k \in K). \tag{3.1.8.1}$$

More generally, for $n \geq 1$, the Iwasawa decomposition implies that

$$B\backslash G/K(n) = B\backslash BK/K(n) = (B \cap K)\backslash K/K(n) = \underline{B}(O/\varpi^n O)\backslash \underline{G}(O/\varpi^n O) = K_0(n)\backslash K, \tag{3.1.8.2}$$

hence

$$\dim \rho^{K(n)} = |K_0(n)\backslash K| = |\mathcal{V}_n| = q^n + q^{n-1}. \tag{3.1.8.3}$$

More precisely, putting together the identifications (3.1.5.2) and (3.1.8.2), we obtain an isomorphism of $\mathbf{C}$-vector spaces

$$\begin{aligned} \mathrm{Iw}_n : \rho^{K(n)} &\xrightarrow{\sim} \mathbf{C}[\mathcal{V}_n] \\ f &\mapsto \sum_{k \in K_0(n)\backslash K} f(k)\,(k^{-1} g_n o) \end{aligned} \tag{3.1.8.4}$$

(note that, for any $k \in K$, both $f(k) \in \mathbf{C}$ and $k^{-1}g_n o \in \mathcal{V}_n$ depend only on the class $K_0(n)k \in K_0(n)\backslash K$).

3.2. Unramified principal series with trivial central character

3.2.1. Assume that, in the notation of 3.1.8, $\alpha_1\alpha_2 = 1$ and $\alpha_1^2 \neq 1, q^{-2}$. The representation ρ is then irreducible and Z acts trivially. The orbit map

$$\begin{array}{ccccc} \mathcal{V} & \xrightarrow{\sim} & G/KZ & \longrightarrow & \rho \\ go & \mapsto & gKZ & \mapsto & g \cdot f_o \end{array}$$

gives rise to a G-equivariant map

$$\begin{array}{rccc} F : & \mathbf{C}[\mathcal{V}] & \longrightarrow & \rho \\ & \sum n_g\, go & \mapsto & \sum n_g (g \cdot f_o) \end{array}$$

between two smooth representations of G. The Hecke operator

$$\begin{aligned} T : \mathbf{C}[\mathcal{V}] &\longrightarrow \mathbf{C}[\mathcal{V}] \\ x &\mapsto \sum_{\overrightarrow{xy} \in \mathcal{E}} y \end{aligned} \tag{3.2.1.1}$$

commutes with the G-action and

(3.2.1.2) $$F \circ T = (q\alpha_1 + \alpha_2)F.$$

3.2.2. Proposition (cf. [Gr-Pr, Lemma 3.1]). (1) $F : \mathbf{C}[\mathcal{V}_0] \xrightarrow{\sim} \rho^K$.
(2) $\forall m \geq n \geq 0 \qquad F \circ r^*_{m,n}(\mathbf{C}[\mathcal{V}_{\leq n}]) \subseteq F(\mathbf{C}[\mathcal{V}_{\leq n}])$.
(3) $\forall n \geq 0$ *the map* $\mathbf{C}[\mathcal{V}_{\leq n}] = \mathbf{C}[\mathcal{V}^{K(n)}] \xrightarrow{F} \rho^{K(n)}$ *is surjective.*
(4) $\forall n \geq 1$ *the map* $\mathbf{C}[\mathcal{V}_n] = \mathbf{C}[\mathcal{V}_{\leq n}]/\mathbf{C}[\mathcal{V}_{\leq n-1}] \xrightarrow{F} \rho^{K(n)}/\rho^{K(n-1)}$ *induces an isomorphism*

$$\mathbf{C}[\mathcal{V}_n]/r^*_{n,n-1}(\mathbf{C}[\mathcal{V}_{n-1}]) \xrightarrow{\sim} \rho^{K(n)}/\rho^{K(n-1)}.$$

(5) $\forall n \geq 1$ F *induces an isomorphism*

$$\mathrm{Ker}\,((r_{n,n-1})_* : \mathbf{C}[\mathcal{V}_n] \longrightarrow \mathbf{C}[\mathcal{V}_{n-1}]) \xrightarrow{\sim} \mathrm{Ker}\left(\mathrm{Tr}_{K(n-1)/K(n)} : \rho^{K(n)} \longrightarrow \rho^{K(n-1)}\right),$$

where $\mathrm{Tr}_{H/H'} := \sum_{h \in H/H'} h$.

Proof. The statement (1) is just (3.1.8.1), while (2) follows by induction on $m - n$ from (3.2.1.2) and the geometry of $\mathcal{T}$. In order to show (3), assume that $f \in \rho^{K(n)}$; then $f = F(\varphi)$ for some $\varphi \in \mathcal{V}_{\leq m}$, $m \geq n$ (as ρ is irreducible, F is surjective). As f is $K(n)$-invariant, we have

$$(K(n) : K(m))\, F(\varphi) = \mathrm{Tr}_{K(n)/K(m)} F(\varphi) = F(\mathrm{Tr}_{K(n)/K(m)}\varphi).$$

Writing $\varphi = \sum_{i=0}^m \varphi_i$, $\varphi_i \in \mathbf{C}[\mathcal{V}_i]$, the claim follows from the fact that, for each $i > n$, $F(\mathrm{Tr}_{K(n)/K(m)}\varphi_i)$ is a rational multiple of $F(r^*_{i,n} \circ (r_{i,n})_*\varphi)$ (as the orbits of $K(n)/K(m)$ on $\mathcal{V}_m$ coincide with the fibres of $r_{m,n}$), and the last term is contained in $F(\mathbf{C}[\mathcal{V}_{\leq n}])$, by (2).
(4) The map in question is surjective (by (3)) and its kernel contains $r^*_{n,n-1}(\mathbf{C}[\mathcal{V}_{n-1}])$ (by (2)); we conclude by dimension count, using the equality (3.1.8.3) for n and $n-1$.
(5) This follows from (4), the decomposition (3.1.1.1) for $f = r_{n,n-1}$ and the analogous decomposition

$$\rho^{K(n)} = \rho^{K(n-1)} \oplus \mathrm{Ker}\left(\mathrm{Tr}_{K(n-1)/K(n)}\right)$$

(since the fibres of $r_{n,n-1}$ coincide with the orbits of $K(n-1)/K(n)$ on $\mathcal{V}_n$).

3.2.3. Proposition. *For each* $m \geq 0$, $\mathbf{C}[\mathcal{V}_m] = r^*_{m,0}(\mathbf{C}[\mathcal{V}_0]) \oplus \bigoplus_{k=1}^m r^*_{m,k}\,(\mathrm{Ker}(r_{k,k-1})_*)$.

Proof. Apply successively (3.1.1.1) to $f = r_{1,0}, \ldots, r_{m,m-1}$.

3.2.4. Iwasawa decomposition revisited. Our next goal is to describe, for $0 \leq m \leq n$, the image of each term in the decomposition 3.2.3 under the map

$$\mathbf{C}[\mathcal{V}_m] \xrightarrow{F} \rho^{K(m)} \hookrightarrow \rho^{K(n)}.$$

Composing it with the isomorphism Iw_n from (3.1.8.4), we obtain a linear map

(3.2.4.1)
$$\begin{array}{ccccc} \mathbf{C}[\mathcal{V}_m] & \xrightarrow{F} & \rho^{K(m)} \hookrightarrow \rho^{K(n)} & \xrightarrow{\mathrm{Iw}_n} & \mathbf{C}[\mathcal{V}_n] \\ {[k_1^{-1} g_m o]} & \mapsto & k_1^{-1} g_m \cdot f_o & \mapsto & \sum_{k_2 \in K_0(n)\backslash K} (k_1^{-1} g_m \cdot f_o)(k_2)\, [k_2^{-1} g_n o] \end{array}$$

(where $k_1 \in K_0(m)\backslash K$) with matrix elements

(3.2.4.2) $$M_{x_2,x_1} = (k_1^{-1} g_m \cdot f_o)(k_2) = f_o(k_2 k_1^{-1} g_m), \qquad x_1 = k_1^{-1} g_m o \in \mathcal{V}_m, \quad x_2 = k_2^{-1} g_n o \in \mathcal{V}_n.$$

Note that the composite map (3.2.4.1) makes sense for principal series representations with arbitrary central characters. We compute its matrix coefficients (3.2.4.2) in the general context of 3.1.8.

3.2.5. Proposition. *For all* $0 \le m \le n$,

$$M_{x,y} = \alpha_1^{d(x',y)/2} \alpha_2^{m-d(x',y)/2} \qquad (x \in \mathcal{V}_n,\ y \in \mathcal{V}_m,\ x' = r_{n,m}(x) \in \mathcal{V}_m)$$

[which implies that the eigenvalue of T *acting on* ρ^K *is, indeed, equal to* $\sum_{x \in \mathcal{V}_1} M_{x,o_1} = q\alpha_1 + \alpha_2$*].*

Proof. As $f_o(k_2 k_1^{-1} g_m)$ does not change if we replace k_i by $k_i k$ ($k \in K$), we have $M_{kx,ky} = M_{x,y}$ ($k \in K$). In particular, we can suppose that x corresponds to $K_0(n)k_2 = K_0(n)$. Identifying $K_0(m)\backslash K$ with $\mathbf{P}^1(O/\varpi^m O)$ via $K_0(m)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [c:d]$, we obtain the following set of representatives k_1 of $K_0(m)\backslash K$:

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & b \end{pmatrix},$$

where a runs through a set of representatives of $O/\varpi^m O$ in O (say, with $\varpi^{m+1} \nmid a$) and b runs through a set of representatives of $\varpi O/\varpi^m O$ in O (if $m \ge 1$). Expressing $k_2 k_1^{-1} g_m = k_1^{-1} g_m$ in terms of the Iwasawa decomposition

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \varpi^m \end{pmatrix} = \begin{pmatrix} \varpi^m/a & -1 \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & \varpi^m/a \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 \\ 1 & b \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \varpi^m \end{pmatrix} = \begin{pmatrix} \varpi^m & -b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we obtain, respectively,

$$M_{x,y} = \begin{cases} \alpha_1^{m-v(a)} \alpha_2^{v(a)}, \\ \alpha_1^m, \end{cases} \qquad x' = [0:1], \quad y = \begin{cases} [a:1] \\ [1:b] \end{cases} \in \mathbf{P}^1(O/\varpi^m O).$$

As

$$r_{m,k}(x') = r_{m,k}(y) \iff \begin{cases} v(a) \ge k, \\ k = 0, \end{cases}$$

it follows that

$$d(x',y)/2 = \begin{cases} m - v(a) \\ m, \end{cases}$$

proving the claim.

3.2.6. Proposition. *For all* $1 \le k \le m \le n$, *the restriction of the map*

$$F_{m,n} : \mathbf{C}[\mathcal{V}_m] \xrightarrow{F} \rho^{K(m)} \hookrightarrow \rho^{K(n)} \xrightarrow{\mathrm{Iw}_n} \mathbf{C}[\mathcal{V}_n]$$

to the subspace $r_{m,k}^* \left(\mathrm{Ker}(r_{k,k-1})_*\right)$ *is equal to* $C_{k,m} r_{n,m}^*$, *where*

$$C_{k,m} = \alpha_2^{k-1}(\alpha_2 - \alpha_1) \prod_{\substack{\zeta^{m-k+1}=1 \\ \zeta \ne 1}} (\alpha_2 - q\zeta\alpha_1).$$

Proof. The space $r_{m,k}^* \left(\mathrm{Ker}(r_{k,k-1})_*\right)$ is generated by elements of the form $u(y,y') = r_{m,k}^*([y]-[y'])$, where $y, y' \in \mathcal{V}_k$ and $r_{k,k-1}(y) = r_{k,k-1}(y')$. It follows from Proposition 3.2.5 that the coefficient of $x \in \mathcal{V}_n$ in $F_{m,n}(u(y,y')) \in \mathbf{C}[\mathcal{V}_n]$ depends only on $z = r_{n,k}(x) \in \mathcal{V}_k$, and is equal to

$$\begin{cases} 0, & z \neq y, y' \\ C, & z = y \\ -C, & z = y', \end{cases}$$

where

$$C = \alpha_2^m + \alpha_1\alpha_2^{m-1}(q-1) + \cdots + \alpha_1^{m-k}\alpha_2^k(q^{m-k} - q^{m-k-1}) - \alpha_1^{m-k+1}\alpha_2^{k-1}q^{m-k} =$$
$$= \alpha_2^{k-1}(\alpha_2 - \alpha_1)\sum_{j=0}^{m-k} q^j\alpha_1^j\alpha_2^{m-k-j} = \alpha_2^{k-1}(\alpha_2 - \alpha_1)\prod_{\substack{\zeta^{m-k+1}=1 \\ \zeta\neq 1}}(\alpha_2 - q\zeta\alpha_1),$$

hence $F_{m,n}(u(y,y')) = C\, r_{n,m}^*(u(y,y'))$, proving the claim.

3.2.7. Corollary. *For all $1 \leq k \leq m \leq n$, the image of the map*

$$r_{m,k}^*\left(\mathrm{Ker}(r_{k,k-1})_*\right) \hookrightarrow \mathbf{C}[\mathcal{V}_m] \xrightarrow{F} \rho^{K(m)} \hookrightarrow \rho^{K(n)} \xrightarrow{\mathrm{Iw}_n} \mathbf{C}[\mathcal{V}_n]$$

is equal to $r_{n,k}^*\left(\mathrm{Ker}(r_{k,k-1})_*\right)$ *(resp., to zero) if* $[(q\alpha_1/\alpha_2)^{m-k+1} \neq 1$ *or* $q\alpha_1 = \alpha_2]$ *(resp., if* $[q\alpha_1 \neq \alpha_2$ *and* $(q\alpha_1/\alpha_2)^{m-k+1} = 1]$*.*

Proof. According to Proposition 3.2.6, the two cases are distinguished by the (non-)vanishing of $C_{k,m}$; we recall that $\alpha_1 \neq \alpha_2$, as ρ is irreducible.

3.2.8. Corollary (the case of the trivial central character). *Assume that $\alpha_1\alpha_2 = 1$ and put $\lambda = q^{1/2}\alpha_1$. The eigenvalue of the Hecke operator T acting on ρ^K is equal to $q^{1/2}(\lambda+\lambda^{-1})$ and, for $1 \leq k \leq m \leq n$, the image of the map*

$$r_{m,k}^*\left(\mathrm{Ker}(r_{k,k-1})_*\right) \hookrightarrow \mathbf{C}[\mathcal{V}_m] \xrightarrow{F} \rho^{K(m)} \hookrightarrow \rho^{K(n)} \xrightarrow{\mathrm{Iw}_n} \mathbf{C}[\mathcal{V}_n]$$

is equal to $r_{n,k}^*\left(\mathrm{Ker}(r_{k,k-1})_*\right)$ *(resp., to zero) if* $[\lambda^{2(m-k+1)} \neq 1$ *or* $\lambda^2 = 1]$ *(resp., if* $[\lambda^2 \neq 1$ *and* $\lambda^{2(m-k+1)} = 1]$*.*

3.3. Unramified special representations with trivial central character

3.3.1. Unramified special representations. Assume that, in the notation of 3.1.8, $\alpha_1 = \alpha_2 = \alpha$. The function f_o is then given by $f_o(g) = \alpha^{v(\det(g))}$ and the subspace $\mathbf{C}f_o \subset \rho$ is G-stable. Put $\overline{\rho} = \rho/\mathbf{C}f_o$ and denote by $\overline{f}$ the image in $\overline{\rho}$ of any $f \in \rho$. The (irreducible) representation $\overline{\rho} = \mathrm{St} \otimes \mu$ of G is a twist of the Steinberg representation by the unramified character $\mu = \mu_1 = \mu_2$ satisfying $\mu(\varpi) = \alpha$.

The double coset space

$$B\backslash G/K_0(1) = B\backslash BK/K_0(1) = (B \cap K)\backslash K/K_0(1) = \underline{B}(O/\varpi O)\backslash\underline{G}(O/\varpi O)/\underline{B}(O/\varpi O),$$

contains two elements, represented by the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It follows that the space

$$\rho^{K_0(1)} = \mathbf{C}f_I \oplus \mathbf{C}f_J,$$

is two-dimensional, with basis given by the functions

$$f_X(bYk_0) = \begin{cases} \mu(b), & X = Y \\ 0, & X \neq Y \end{cases} \qquad (X, Y \in \{I, J\},\, b \in B,\, k_0 \in K_0(1)).$$

As $f_o = f_I + f_J$, we have $\overline{f}_I = -\overline{f}_J$ in $\overline{\rho}$ and

$$\overline{\rho}^{K_0(1)} = \mathbf{C}\overline{f}_I = \mathbf{C}\overline{f}_J.$$

An easy calculation shows that the matrix $w = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}$ acts on $\rho^{K_0(1)}$ by

$$w \cdot f_I = \alpha f_J, \qquad w \cdot f_J = \alpha f_I,$$

hence

(3.3.1.1) $$w \text{ acts on } \overline{\rho}^{K_0(1)} \text{ by multiplication by } -\alpha = -\mu(\varpi).$$

3.3.2. The case of the trivial central character. Assume that $\alpha = \pm 1$ ($\iff \mu^2 = 1 \iff$ the central character of $\overline{\rho} = \mathrm{St} \otimes \mu$ is trivial). As in 3.2.1, the orbit map

$$\begin{array}{ccccc} \mathcal{E} & \xrightarrow{\sim} & G/K_0(1)Z & \longrightarrow & \overline{\rho} \\ ge_0 & \mapsto & gK_0(1)Z & \mapsto & g \cdot f_I \end{array}$$

gives rise to a G-equivariant map

$$\begin{array}{cccc} \overline{F}: & \mathbf{C}[\mathcal{E}] & \longrightarrow & \overline{\rho} \\ & \sum n_g\, ge_0 & \mapsto & \sum n_g (g \cdot f_I). \end{array}$$

It follows from (3.3.1.1) that

(3.3.2.1) $$\overline{F} \circ \iota = -\alpha \overline{F} = -\mu(\varpi)\overline{F} \qquad (\iff \forall e \in \mathcal{E} \qquad \overline{F}(-e) = -\alpha \overline{F}(e)).$$

Moreover, $\overline{F}$ satisfies the "harmonicity condition"

(3.3.2.2) $$\forall x \in \mathcal{V} \qquad \overline{F}\Big(\sum_{s(e)=x} e\Big) = \overline{F}\Big(\sum_{t(e)=x} e\Big) = 0$$

(for $x = o$, the two terms on the L.H.S. are contained in $\overline{\rho}^K = 0$; the case of general x follows from the G-equivariance of $\overline{F}$).

3.3.3. Proposition. (1) $\forall n \geq 1 \qquad \overline{F}(\mathbf{C}[\mathcal{E}_n]) \subseteq \overline{F}(\mathbf{C}[\mathcal{E}_{n+1}])$.
(2) $\forall n \geq 1$ *the map* $\overline{F} : \mathbf{C}[\mathcal{E}_n] \longrightarrow \overline{\rho}^{K(n)}$ *is surjective.*
(3) $\forall n \geq 1$ *the sequence*

$$0 \longrightarrow \mathbf{C} \sum_{e \in \mathcal{E}_n} e \longrightarrow \mathbf{C}[\mathcal{E}_n] \xrightarrow{\overline{F}} \overline{\rho}^{K(n)} \longrightarrow 0$$

is exact.

Proof. (1) This follows from the harmonicity condition (3.3.2.2).
(2) The argument used in the proof of Proposition 3.2.2(3) shows that the map

$$\overline{F} : \mathbf{C}[\bigcup_{k=1}^{n} \mathcal{E}_k] \longrightarrow \overline{\rho}^{K(n)}$$

is surjective; we conclude by (1).
(3) The element $\sum_{e\in\mathcal{E}_n} e$ lies in the kernel of $\overline{F}$, by a repeated application of (3.3.2.2). On the other hand, $\dim \mathbf{C}[\mathcal{E}_n] = |\mathcal{E}_n| = |\mathcal{V}_n|$ and $\dim \overline{\rho}^{K(n)} = \dim \rho^{K(n)} - 1 = |\mathcal{V}_n| - 1$ (by (3.1.8.3)), hence $\dim(\mathrm{Ker}(\overline{F} : \mathbf{C}[\mathcal{E}_n] \longrightarrow \overline{\rho}^{K(n)})) = 1$, by (2).

3.4. New vectors as test vectors

In this section we apply the results of 3.1-3.3 to the question formulated in 2.4.13.

3.4.1. Orders and conductors. Return to the abstract situation of 2.4.13 in the case $B_v = M_2(F_v)$. We are going to use the results of 3.1-3.3 over $E = F_v$ (hence $q = Nv$). The given embedding $j_v : K_v \hookrightarrow M_2(F_v)$ induces a map

$$\mathscr{L} : \mathcal{V} \xrightarrow{\sim} \{\text{maximal orders in } M_2(E)\} \xrightarrow{j_v^*} \{O_{F,v} - \text{orders in } K_v\} \xrightarrow{\mathrm{ord}_v(c(-))} \mathbf{N}$$

(where $j^*(R) = j_v^{-1}(R)$); for $x \in \mathcal{V}$, we say that "x has level $\mathscr{L}(x)$". The $j_v(K_v^*/F_v^*)$-orbits on $\mathcal{V}$ are precisely the fibres of the level map $\mathscr{L}$. They can be easily visualised on the tree $\mathcal{T}$, if we take into account the following rules ([Co 1, Cor. 2.2.2], [Co-Va 1, Lemma 6.2,6.5]):

If $x \in \mathcal{V}$ has level $n > 0$, then x has 1 neighbour of level $n-1$ and q neighbours of level $n+1$.
If $x \in \mathcal{V}$ has level 0, then x has $1+a$ neighbours of level 0 and $q-a$ neighbours of level 1, where

$$a = \begin{cases} -1, & \text{if } v \text{ is inert in } K/F \\ 0, & \text{if } v \text{ is ramified in } K/F \\ 1, & \text{if } v \text{ splits in } K/F. \end{cases}$$

In particular, $\mathscr{L}^{-1}(0)$ consists of a single point if v is inert, resp., of the two endpoints of an edge if v is ramified, resp., of the vertices of an infinite path (infinite in both directions) if v splits.

3.4.2. Unramified principal series. Assume that we are in the situation of 2.4.14.3, and that $c(\chi_v) = v^m$, $c(j_v^{-1}(R_v)) = v^n$, $m \le n$ (hence (2.4.13.4) holds). We are going to investigate the validity of (2.4.13.6) using the results and the notation of 3.2. Firstly, $\pi_v = \rho$ and R_v corresponds to the vertex $x \in \mathcal{V}$ fixed by R_v^* (by (3.1.4.2)), hence $n = \mathscr{L}(x)$ and $\rho^{R_v^*} = F(\mathbf{C}x)$. We need to decide whether the subspace (of dimension at most one)

$$e_{\overline{\chi}_v}(\rho^{R_v^*}) = e_{\overline{\chi}_v}(F(\mathbf{C}x)) = \mathbf{C}F(e_{\overline{\chi}_v}x) = \mathbf{C}F\Big(\sum_{a\in U_v'} \chi_v(a)(j_v(a)x)\Big) \subset \rho$$

is non-zero.

In order to conform to the notation of 3.2 we must change the origin of $\mathcal{T}$ (by conjugating everything in sight by suitable $g \in G$) so that $\mathscr{L}(o) = 0$ and $x = o_n \in \mathcal{V}_n$ (equivalently, we could have stated everything in 3.2 in a coordinate-free form).

Denote by $X \subseteq \mathscr{L}^{-1}(n) \subseteq \mathcal{V}_n$ the $j_v^*((K_v^*)^\circ/F_v^*)$-orbit of x. The description of the function $\mathscr{L}$ recalled in 3.4.1 implies the following:

$$(3.4.2.1) \qquad X = \begin{cases} \mathcal{V}_n, & \text{if } v \text{ is inert in } K/F \\ (\mathcal{V}_n - r_{n,1}^{-1}(o')) \cup r_{n+1,1}^{-1}(o'), & \text{if } v \text{ is ramified in } K/F \\ \mathcal{V}_n - r_{n,1}^{-1}(o', o''), & \text{if } v \text{ splits in } K/F, \end{cases}$$

where $o' \ne r_{n,1}(x)$ (resp., $o', o'' \ne r_{n,1}(x)$) is (resp., are) the unique neighbour(s) of o of level 0 if v is ramified (resp., if v splits).

3.4.3. Proposition. *In the situation of 3.4.2, write the eigenvalue of the Hecke operator $T = T_v$ acting on ρ^K as $(Nv)^{1/2}(\lambda + \lambda^{-1})$ ($\lambda \in \mathbf{C}$). Assume that $1 \leq m = \mathrm{ord}_v(c(\chi_v)) \leq n = \mathrm{ord}_v(c(j_v^{-1}(R_v)))$ and [$\lambda^{2(n-m+1)} \neq 1$ or $\lambda^2 = 1$]. Then*

$$e_{\overline{\chi}_v}(\rho^{R_v^*}) = \mathbf{C}F(e_{\overline{\chi}_v}x) \neq 0.$$

Proof. If v is unramified in K/F, it follows from (3.4.2.1) that $0 \neq e_{\overline{\chi}_v}x \in r^*_{n,m}(\mathrm{Ker}(r_{m,m-1})_*)$, hence

$$\mathrm{Iw}_n \circ F(e_{\overline{\chi}_v}x) = C_{m,n}e_{\overline{\chi}_v}x \neq 0,$$

by Proposition 3.2.6 (as $C_{m,n} \neq 0$ by the assumptions on λ).

If v is ramified in K/F, then (3.4.2.1) implies that $e_{\overline{\chi}_v}x = \varphi_0 + \varphi_1$, where

$$0 \neq \varphi_0 \in r^*_{n,m}(\mathrm{Ker}(r_{m,m-1})_*), \qquad 0 \neq \varphi_1 \in r^*_{n+1,m+1}(\mathrm{Ker}(r_{m+1,m})_*)$$

and $r^*_{n+1,n}(\varphi_0), \varphi_1 \in \mathbf{C}[\mathcal{V}_{n+1}]$ have disjoint supports. Proposition 3.2.6 then implies that

$$\mathrm{Iw}_{n+1} \circ F(e_{\overline{\chi}_v}x) = C_{m,n}r^*_{n+1,n}(\varphi_0) + C_{m+1,n+1}\varphi_1 \neq 0,$$

(as $C_{m,n}C_{m+1,n+1} \neq 0$ by the assumptions on λ).

3.4.4. Unramified special representations. We now turn to the situation considered in 2.4.14.5. In this case $\pi_v = \overline{\rho}$ and R_v is an Eichler order of level one. Let $e \in \mathcal{E}$ be the unique (up to ι) edge stabilised by R_v^*. As in 3.4.2, write $c(\chi_v) = v^m$, $c(j_v^{-1}(R_v)) = v^n$ (hence $n = \max(\mathscr{L}(s(e)), \mathscr{L}(t(e)))$) and assume that $m \leq n$. As $\overline{\rho}^{R_v^*} = \overline{F}(\mathbf{C}e)$, our goal is to decide whether the subspace

$$e_{\overline{\chi}_v}(\overline{\rho}^{R_v^*}) = e_{\overline{\chi}_v}(\overline{F}(\mathbf{C}e)) = \mathbf{C}\overline{F}(e_{\overline{\chi}_v}e) = \mathbf{C}\overline{F}\Big(\sum_{a \in U_v'} \chi_v(a)(j_v(a)e)\Big) \subset \overline{\rho}$$

is non-zero. As in 3.4.2, we change the origin of $\mathcal{T}$ so that $\mathscr{L}(o) = 0$ and (possibly after replacing e by $\iota(e)$) $e \in \mathcal{E}_n$ in the case $n > 0$. Let $Y \subset \mathcal{E}$ be the $j_v^*((K_v^*)^\circ/F_v^*)$-orbit of e. As in 3.4.2, let $\{o'\} = \mathcal{V}_1 \cap \mathscr{L}^{-1}(0)$ (resp., $\{o', o''\} = \mathcal{V}_1 \cap \mathscr{L}^{-1}(0)$) if v is ramified (resp., if v splits) in K/F.

If $n = 0$, then v is not inert and

$$Y = \begin{cases} \{e\}, & \text{if } v \text{ splits in } K/F \\ \{e, \iota e\} = \{\overrightarrow{oo'}, \overrightarrow{o'o}\}, & \text{if } v \text{ is ramified in } K/F. \end{cases} \tag{3.4.4.1}$$

If $n > 0$, then

$$Y = \begin{cases} \mathcal{E}_n, & \text{if } v \text{ is inert in } K/F \\ (\mathcal{E}_n - r_{n,1}^{-1}(\overrightarrow{oo'})) \cup r_{n+1,1}^{-1}(\overrightarrow{oo'}) = Y_0 \cup Y_1, & \text{if } v \text{ is ramified in } K/F \\ \mathcal{E}_n - r_{n,1}^{-1}(\overrightarrow{oo'}, \overrightarrow{oo''}), & \text{if } v \text{ splits in } K/F, \end{cases} \tag{3.4.4.2}$$

3.4.5. Proposition. *If, in the situation of 3.4.4, either (2.4.14.5.1) or (2.4.14.5.2) holds, then*

$$e_{\overline{\chi}_v}(\overline{\rho}^{R_v^*}) = \mathbf{C}\overline{F}(e_{\overline{\chi}_v}e) \neq 0.$$

Proof. Assume first $n = 0$. We have to consider only the case of v ramified in K/F, when $vO_K = w^2$, $m = 0$ and $\chi_v^2 = 1$. By (3.4.4.1) and (3.3.1.1),

$$\overline{F}(e_{\overline{\chi}_v}e) = \overline{F}(e + \chi_v(w)\iota(e)) = (1 - \mu(v)\chi_v(w))\overline{F}(e),$$

which is non-zero precisely when $\chi_v(w) = -\mu(v)$ (as $\chi_v(w), \mu(v) \in \{\pm 1\}$), which is guaranteed by (2.4.14.5.2). We now turn to the case $n > 0$. If v is inert in K/F, then $\chi_v \neq 1$ by the assumption (2.4.14.5.2), which implies that $m \geq 1$, hence $e_{\overline{\chi}_v}e \notin \mathbf{C}\sum_{e' \in \mathcal{E}_n} e'$ by (3.4.4.2); thus $\overline{F}(e_{\overline{\chi}_v}e) \neq 0$, by Proposition 3.3.3(3). If v

splits in K/F, then Y is a proper subset of $\mathcal{E}_n$ by (3.4.4.2), hence $e_{\overline{\chi}_v} e \notin \mathbf{C} \sum_{e' \in \mathcal{E}_n} e'$ for any χ_v; we conclude again by Proposition 3.3.3(3). Assume, finally, that v is ramified in K/F, $vO_K = w^2$. It follows from (3.4.4.2) that $e_{\overline{\chi}_v} e = \varphi_0 + \varphi_1$, with $\varphi_0 \in \mathbf{C}[\mathcal{E}_n]$, $\varphi_1 \in \mathbf{C}[\mathcal{E}_{n+1}]$ such that the supports of $r^*_{n+1,n}(\varphi_0), \varphi_1 \in \mathbf{C}[\mathcal{E}_{n+1}]$ are disjoint. If χ_v is ramified, then the coefficients of φ_1 are not all equal to each other, which implies that $e_{\overline{\chi}_v} e \notin \mathbf{C} \sum_{e' \in \mathcal{E}_n} e'$ in this case, too. If χ_v is unramified, then

$$\varphi_0 = \sum_{e' \in Y_0} e', \qquad \varphi_1 = \chi_v(w) \sum_{e' \in Y_1} e',$$

hence

$$\overline{F}(\varphi_0 + \varphi_1) = (-1)^n \overline{F}(\overrightarrow{oo'}) + (-1)^n \chi_v(w) F(\overrightarrow{o'o}) = (-1)^n (1 - \chi_v(w)\mu(v)) F(\overrightarrow{oo'}),$$

which is again non-zero, thanks to the assumption (2.4.14.5.2).

4. Strong non-vanishing results

In §4 we generalise the main result of [Co-Va 1] in the "indefinite case" ([Co-Va 1, Thm. 4.2]) in our setting. Assume that we are given the data (D1) - (D5) from 2.4.1.

4.1. Primitive characters

In this section we generalise [Co-Va 1, Lemma 2.8].

4.1.1. For $m \leq n \in \mathbf{N}^s$, put

$$G_0^{(c)}(n) = \mathrm{Im}(G_0^{(c)} \longrightarrow G^{(c)}(n)), \quad Z(n,m) = \mathrm{Ker}(G^{(c)}(n) \longrightarrow G^{(c)}(n-m)), \quad H(n) = G^{(c)}(n)/G_0^{(c)}(n)$$

(by 1.2.4-5, $H(n)$ does not depend on c, as the notation suggests, for all but finitely many $n \in \mathbf{N}^s$). For any subset $I \subseteq \{1,\ldots,s\}$ denote by $\delta_I \in \mathbf{N}^s$ the characteristic function of I: $\delta_I = (m_1,\ldots,m_s)$ with $m_i = 1$ (resp., $m_i = 0$) if $i \in I$ (resp., if $i \notin I$). If $n \in \mathbf{N}^s$ is large enough, then the following commutative diagram has exact rows and columns (for any $I \subseteq \{1,\ldots,s\}$):

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G_0^{(c)} & = & G_0^{(c)} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z(n,\delta_I) & \longrightarrow & G^{(c)}(n) & \longrightarrow & G^{(c)}(n-\delta_I) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z(n,\delta_I) & \longrightarrow & H(n) & \longrightarrow & H(n-\delta_I) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{4.1.1.1}$$

In particular, if n is large enough, then there is an exact sequence

$$0 \longrightarrow Z(n,\delta_I) \oplus G_0^{(c)} \longrightarrow G^{(c)}(n) \longrightarrow H(n-\delta_I) \longrightarrow 0,$$

in which

$$Z(n,\delta_I) \xrightarrow{\sim} \bigoplus_{i\in I} Z(n,\delta_{\{i\}}), \qquad Z(n,\delta_{\{i\}}) \xrightarrow{\sim} O_F/P_i.$$

4.1.2. Fix an embedding $L \hookrightarrow \mathbf{C}$ and consider χ_0 as a character $\chi_0 : G_0^{(c)} \longrightarrow \mathbf{C}^*$. For any $n \in \mathbf{N}^s$, put (following [Co-Va 1, 2.3])

$$\begin{aligned} P(\chi_0, n) &= \{\chi : G^{(c)}(n) \longrightarrow \mathbf{C}^* \mid \chi \text{ induces } \chi_0 \text{ on } G_0^{(c)} \text{ and } P^n \mid c(\chi)\} \\ &= \{\chi : G^{(c)}(n) \longrightarrow \mathbf{C}^* \mid \chi \text{ induces } \chi_0 \text{ on } G_0^{(c)} \text{ and } c(\chi) = c(\chi_0)P^n\}. \\ e(\chi_0, n) &= \sum_{\chi\in P(\chi_0,n)} e_{\overline{\chi}}, \qquad e_{\overline{\chi}} = \sum_{g\in G^{(c)}(n)} \chi(g)\, g \in \mathbf{C}[G^{(c)}(n)] \end{aligned}$$

(where $c(\chi_0) = c(\chi_0|_{G_1^{(c)}})$, in the notation of 1.2.8).

4.1.3. Lemma ([Co-Va 1, Lemma 2.8] if $s = 1$). *If $n \in \mathbf{N}^s$ is sufficiently large, then*

$$e(\chi_0, n) = |H(n)|\, e_{\overline{\chi}_0} \prod_{i=1}^{s} \left(1 - (NP_i)^{-1}\mathrm{Tr}_{Z(n,\delta_{\{i\}})}\right),$$

where

$$\mathrm{Tr}_{Z(n,\delta_I)} = \sum_{h\in Z(n,\delta_I)} h \in \mathbf{C}[G^{(c)}(n)], \qquad e_{\overline{\chi}_0} = \sum_{g\in G_0^{(c)}} \chi(g)\, g \in \mathbf{C}[G^{(c)}(n)].$$

Proof (following [Co-Va 1]). For any finite abelian group Δ denote by $\Delta^\vee = \mathrm{Hom}(\Delta, \mathbf{C}^*)$ its group of characters. Let $n \in \mathbf{N}^s$ be sufficiently large. It follows from (4.1.1.1) that there exists $\chi_0' \in \mathrm{Im}(G^{(c)}(n - \delta_{\{1,\dots,s\}})^\vee \longrightarrow G^{(c)}(n)^\vee)$ which induces χ_0 on $G_0^{(c)}$; then $\chi_0' = 1$ on each $Z(n,\delta_I)$ and

$$\{\chi : G^{(c)}(n) \longrightarrow \mathbf{C}^* \mid \chi \text{ induces } \chi_0 \text{ on } G_0^{(c)}\} = H(n)^\vee \chi_0'.$$

The inclusion-exclusion principle implies that

$$P(\chi_0, n) = \sum_{I\subseteq\{1,\dots,s\}} (-1)^{|I|}\, H(n-\delta_I)^\vee \chi_0'$$

(as sets with multiplicities), hence

$$\begin{aligned} e(\chi_0, n) &= \sum_{I\subseteq\{1,\dots,s\}} (-1)^{|I|} \sum_{\chi\in H(n-\delta_I)^\vee} \sum_{g\in G^{(c)}(n)} (\chi\chi_0')(g)\, g = \\ &= \sum_{I\subseteq\{1,\dots,s\}} (-1)^{|I|}\, \mathrm{Tr}_{Z(n,\delta_I)} \sum_{g\in G^{(c)}(n-\delta_I)} \sum_{\chi\in H(n-\delta_I)^\vee} (\chi\chi_0')(g)\, g = \\ &= \sum_{I\subseteq\{1,\dots,s\}} (-1)^{|I|}\, \mathrm{Tr}_{Z(n,\delta_I)} |H(n-\delta_I)| \sum_{g\in G_0^{(c)}} \chi_0(g)\, g = |H(n)| \prod_{i=1}^{s} \left(1 - (NP_i)^{-1}\mathrm{Tr}_{Z(n,\delta_{\{i\}})}\right) e_{\overline{\chi}_0}. \end{aligned}$$

4.1.4. Corollary. *If $n \in \mathbf{N}^s$ is sufficiently large, then each CM point $x \in CM(N_H^*, K)$ with $c(x) \mid cP^n$ satisfies*

$$\sum_{\chi\in P(\chi_0,n)} e_{\overline{\chi}}(\alpha(x)) = \frac{|H(n)|}{N(P_1\cdots P_s)}\, e_{\overline{\chi}_0}(y) \in \mathbf{C} \otimes_{O_L} A(K[cP^n]),$$

where

$$y = \prod_{i=1}^{s} \left(NP_i - \mathrm{Tr}_{Z(n,\delta_{\{i\}})}\right) \alpha(x) \in A_0(K[cP^n]).$$

4.1.5. The next step in the argument from [Co-Va 1, 4.4] is to rewrite the point $y \in A_0(K[cP^n])$ in terms of another Shimura curve N_{H^+}, a new CM point $x^+ \in CM(N^*_{H^+}, K)$ and a new map $\alpha_+ : N^*_{H^+} \longrightarrow J(N^*_{H^+}) \twoheadrightarrow A_0$ as $y = \alpha_+(x^+)$ (provided that $c(x) = c_0 P^n$, $c_0 \mid c$). This will be done in §4.2.

4.2. Reparameterisation

In this section we generalise [Co-Va 1, 4.4].

4.2.1. As in [Co-Va 1], we impose the following additional assumptions.

(A1) For each $i = 1, \ldots, s$, R_{P_i} is an Eichler order in B_{P_i} ($B_{P_i} \xrightarrow{\sim} M_2(F_{P_i})$, by (D1)); let $a_i \geq 0$ be its level.

(A2) For each $i = 1, \ldots, s$, the quotient $J(N^*_H) \twoheadrightarrow A_0$ is P_i-new in the sense of [Co-Va 1, Def. 3.13], which amounts to saying that $a_i = o({}^\sigma\pi_{P_i})$ (for one, hence for all $\sigma : L \hookrightarrow \mathbf{C}$).

4.2.2. We are going to describe the elements of the $(P_1 \cdots P_s)$-isogeny class $\mathcal{C}$ together with a large part of the Galois action on them in terms of the Bruhat-Tits trees $\mathcal{T}_i$ associated to $PGL_2(F_{P_i})$ ($i = 1, \ldots, s$).

Fix a CM point $x_0 = [z, b]_H \in \mathcal{C}$. The map

$$\prod_{i=1}^{s} B^*_{P_i}/F^*_{P_i}(b_{P_i} R_{P_i} b_{P_i}^{-1})^* \twoheadrightarrow \mathcal{C}, \qquad (g_1, \ldots, g_s) \mapsto [z, g_1 \cdots g_s b]_H \tag{4.2.2.1}$$

is surjective and $\prod_{i=1}^s K^*_{P_i}$-equivariant, with $K^*_{P_i}$ acting on the L.H.S. (resp., on the R.H.S.) via $t_{P_i} : K_{P_i} \hookrightarrow B_{P_i}$ and left multiplication (resp., via the reciprocity map).

For each $i = 1, \ldots, s$, fix an isomorphism $\varphi_i : B_{P_i} \xrightarrow{\sim} M_2(F_v)$ which maps $b_{P_i} B_{P_i} b_{P_i}^{-1}$ to the standard Eichler order

$$R_0(P_i^{a_i}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(O_{F,P_i}) \,\middle|\, c \in P_i^{a_i} \right\}; \tag{4.2.2.2}$$

together with (3.1.6.1), this yields identifications

$$\prod_{i=1}^{s} B^*_{P_i}/F^*_{P_i}(b_{P_i} R_{P_i} b_{P_i}^{-1})^* \xrightarrow{\sim} \prod_{i=1}^{s} GL_2(F_{P_i})/F^*_{P_i} R_0(P_i^{a_i})^* \xrightarrow{\sim} \prod_{i=1}^{s} \mathcal{P}_{a_i}(\mathcal{T}_i),$$

which allow us to rewrite (4.2.2.1) as a $\prod_{i=1}^s K^*_{P_i}$-equivariant surjection

$$X : \prod_{i=1}^{s} \mathcal{P}_{a_i}(\mathcal{T}_i) \twoheadrightarrow \mathcal{C}, \tag{4.2.2.3}$$

with $K^*_{P_i}$ acting on $\mathcal{T}_i$ and $\mathcal{P}_{a_i}(\mathcal{T}_i)$ via $\varphi_i \circ t_{P_i} : K_{P_i} \hookrightarrow M_2(F_v)$.

As in 3.4.1, the embedding $\varphi_i \circ t_{P_i}$ gives rise to a "level function"

$$\mathscr{L}_i : \mathcal{V}(\mathcal{T}_i) \longrightarrow \mathbf{N};$$

define

$$\begin{aligned} \overline{\mathscr{L}}_i : \mathcal{P}_{a_i}(\mathcal{T}_i) &\longrightarrow \mathbf{N} \\ (x_0 \to \cdots \to x_{a_i}) &\mapsto \max\nolimits_{0 \leq j \leq a_i} \mathscr{L}_i(x_j); \end{aligned}$$

then

$$(4.2.2.4) \qquad \forall i = 1, \ldots, s \ \ \forall \gamma = (\gamma_1, \ldots, \gamma_s) \in \prod_{i=1}^{s} \mathcal{P}_{a_i}(\mathcal{T}_i) \qquad \mathrm{ord}_{P_i}\left(c\left(X\left(\gamma\right)\right)\right) = \overline{\mathscr{L}}_i(\gamma).$$

For each $i = 1, \ldots, s$ there is a natural $K^*_{P_i}$-equivariant map

$$\mathscr{L}_i^{-1}\left(\mathbf{N}_{>0}\right) \longrightarrow \mathcal{V}(\mathcal{T}_i), \qquad x \mapsto x_l,$$

which associates to a vertex $x \in \mathcal{V}(\mathcal{T}_i)$ with $\mathscr{L}_i(x) > 0$ its unique "lower neighbour" x_l satisfying

$$d(x, x_l) = 1, \qquad \mathscr{L}_i(x_l) = \mathscr{L}_i(x) - 1$$

(this makes sense, thanks to the discussion in 3.4.1).

4.2.3. Definition ([Co-Va 1, Def. 1.6]). *A CM point $x \in \mathcal{C}$ is P_i-**good** if either $a_i = 0$, or if $x = X(\gamma)$, $\gamma = (\gamma_1, \ldots, \gamma_s)$, $\gamma_i = (x_0 \to \cdots \to x_{a_i})$, $\mathscr{L}_i(x_0) \neq \mathscr{L}_i(x_{a_i})$ (for one, hence for all γ such that $x = X(\gamma)$). [This is automatic if $a_i = 1$ and $P_i \mid c(x)$.] The point x is **good** if it is P_i-good for all $i = 1, \ldots, s$.*

4.2.4. Assume that $x \in \mathcal{C}$, $c(x) = c_0 P^n$, $c_0 \mid c$ and $n \in \mathbf{N}^s$ is sufficiently large. Following [Co-Va 1, §6], we are going to describe the Galois orbits $Z(n, \delta_{\{i\}})\, x \subset \mathcal{C}$. Writing $x = X(\gamma)$, $\gamma = (\gamma_1, \ldots, \gamma_s)$, $\gamma_i \in \mathcal{P}_{a_i}(\mathcal{T}_i)$, it is enough to describe the orbits

$$Z_i(n_i)\, \gamma_i \subset \mathcal{P}_{a_i}(\mathcal{T}_i),$$

where

$$Z_i(n_i) = \mathrm{Ker}\left(U(\mathcal{P}_i^{a_i}) \longrightarrow U(\mathcal{P}_i^{a_i - 1})\right) \subseteq K^*_{P_i}/F^*_{P_i}(O_{F,P_i} + P_i^{a_i} O_{K,P_i})^*$$

maps isomorphically onto $Z(n, \delta_{\{i\}})$ by the reciprocity map.

4.2.4.1. [Co-Va 1, Cor. 6.6]. If $a_i = 0$, then $\gamma_i = y \in \mathcal{V}(\mathcal{T}_i)$; we can assume that $n_i = \mathscr{L}_i(y) \geq 2$. In this case

$$Z_i(n_i)\, y = \{x \in \mathcal{V}(\mathcal{T}_i) \mid d(x, y_l) = 1,\, x \neq (y_l)_l\}.$$

4.2.4.2. [Co-Va 1, Lemma 6.11]. If $a_i = 1$, then $\gamma_i = (y_0 \to y_1) \in \mathcal{E}(\mathcal{T}_i)$; we can again assume that $n_i = \max(\mathscr{L}_i(y_0), \mathscr{L}_i(y_1)) \geq 2$. There are two possible cases:
If γ_i has **positive orientation** (notation: $\gamma_i > 0$), i.e., $n_i = \mathscr{L}_i(y_1) = \mathscr{L}_i(y_0) + 1$, then

$$Z_i(n_i)\,(y_0 \to y_1) = \{(y_0 \to y) \mid y \neq (y_0)_l\}.$$

If γ_i has **negative orientation** (notation: $\gamma_i < 0$), i.e., $n_i = \mathscr{L}_i(y_0) = \mathscr{L}_i(y_1) + 1$, then

$$Z_i(n_i)\,(y_0 \to y_1) = \{(y \to y_1) \mid y \neq (y_1)_l\}.$$

4.2.4.3. [Co-Va 1, §6.4]. If $a_i \geq 2$, then $\gamma_i = (y_0 \to \cdots \to y_{a_i})$. If x is P_i-**good**, then there are two possibilities, as in 4.2.4.2:
If $n_i = \mathscr{L}_i(y_{a_i}) > \mathscr{L}_i(y_0)$, then

$$Z_i(n_i)\, \gamma_i = \{(y_0 \to \cdots \to y_{a_i - 1} \to y)\} = d_0^{-1}(y_0 \to \cdots \to y_{a_i - 1}).$$

If $n_i = \mathscr{L}_i(y_0) > \mathscr{L}_i(y_{a_i})$, then

$$Z_i(n_i)\, \gamma_i = \{(y \to y_1 \to \cdots \to y_{a_i})\} = d_1^{-1}(y_1 \to \cdots \to y_{a_i}),$$

where

$$d_0, d_1 : \mathcal{P}_{a_i}(\mathcal{T}_i) \longrightarrow \mathcal{P}_{a_i - 1}(\mathcal{T}_i)$$

are the degeneracy maps from (3.1.7.1).

4.2.5. Change of orientation. In the situation of 4.2.4.2, we have (for any $\sigma_0 : L_0 \hookrightarrow \mathbf{C}$) $\pi(\sigma_0)_{P_i} = \mathrm{St}\otimes\mu_i$, where $\mu_i : F_{P_i}^* \longrightarrow \{\pm 1\}$ is unramified. Put

$$\iota_i(\gamma_i) = (y_1 \to y_0), \qquad \iota_i(\gamma) = (\gamma_1, \ldots, \iota_i(\gamma_i), \ldots, \gamma_s);$$

then

$$X(\iota_i(\gamma)) = [\cdot g]\,(X(\gamma)), \qquad g = \varphi_i^{-1}\Big(\begin{pmatrix} 0 & 1 \\ \varpi_{P_i} & 0 \end{pmatrix}\Big) \in B_{P_i}^*,$$

hence

$$\alpha\,(X(\iota_i(\gamma))) = -\mu_i(P_i)\,\alpha(X(\gamma)), \tag{4.2.5.1}$$

by (3.3.2.1).

4.2.6. Change of the Shimura curve. For $i = 1, \ldots, s$, put

$$a_i^+ = \max(2, a_i), \qquad J_i = \{0, \ldots, a_i^+ - a_i\}, \qquad J = \prod_{i=1}^{s} J_i.$$

Let $R^+ \subset B$ be the order whose localisations are equal to

$$R_v^+ = \begin{cases} R_v, & v \nmid P_1 \cdots P_s \\ b_{P_i}^{-1}\Big(\varphi_i^{-1}\Big(R_0(P_i^{a_i^+})\Big)\Big)b_{P_i}, & v = P_i, \end{cases}$$

in the notation of (4.2.2.2); put $H^+ = (\widehat{R}^+)^*$. For $j = (j_1, \ldots, j_s) \in J$, put

$$g(j) = (g_1(j_1), \ldots, g_s(j_s)) \in \prod_{i=1}^{s} B_{P_i}^*, \qquad g_i(j_i) = b_{P_i}^{-1}\varphi_i^{-1}\Big(\begin{pmatrix} 1 & 0 \\ 0 & \varpi_{P_i}^{j_i} \end{pmatrix}\Big)b_{P_i} \in B_{P_i}^*.$$

As $g(j)Rg(j)^{-1} \supseteq R^+$, we obtain, for each $j \in J$, a degeneracy map

$$d(j) : N_{H^+}^* \xrightarrow{[\cdot g(j)]} N_{g(j)^{-1}H^+g(j)}^* \xrightarrow{\mathrm{pr}} N_H^*.$$

Let $\mathcal{C}^+ \subset CM(N_{H^+}, K)$ be the $(P_1 \cdots P_s)$-isogeny class of the CM point $[z, b]_{H^+}$. The construction from 4.2.2 (with the same isomorphisms φ_i) yields a $\prod_{i=1}^{s} K_{P_i}^*$-equivariant surjection

$$X^+ : \prod_{i=1}^{s} \mathcal{P}_{a_i^+}(\mathcal{T}_i) \twoheadrightarrow \mathcal{C}^+.$$

It follows from 3.1.7 that the following diagram is commutative, for any $j \in J$:

$$\begin{array}{ccc} \prod_{i=1}^{s} \mathcal{P}_{a_i^+}(\mathcal{T}_i) & \xrightarrow{X^+} & \mathcal{C}^+ \\ \big\downarrow{\scriptstyle (d_{j_1}, \ldots, d_{j_s})} & & \big\downarrow{\scriptstyle d(j)} \\ \prod_{i=1}^{s} \mathcal{P}_{a_i}(\mathcal{T}_i) & \xrightarrow{X} & \mathcal{C}. \end{array}$$

4.2.7. Lift of CM points. For each $i = 1, \ldots, s$ there is a natural $K_{P_i}^*$-equivariant map (commuting with $\overline{\mathscr{L}}_i$)

$$\overline{\mathscr{L}}_i^{-1}(\mathbf{N}_{>1}) \longrightarrow \mathcal{P}_{a_i^+}(\mathcal{T}_i), \qquad \gamma_i \mapsto \gamma_i^+, \tag{4.2.7.1}$$

defined as follows:
If $a_1 \geq 2$, then $a_i^+ = a_i$ and $\gamma_i^+ = \gamma_i$.
If $a_i = 1$ and $\gamma_i = (y_0 \to y_1)$, then

$$\gamma_i^+ = \begin{cases} ((y_0)_l \to y_0 \to y_1), & \text{if } \gamma_i > 0 \\ (y_0 \to y_1 \to (y_1)_l), & \text{if } \gamma_i < 0. \end{cases}$$

If $a_i = 0$ and $\gamma_i = y$, then $\gamma_i^+ = (y \to y_l \to (y_l)_l)$.
By construction,

$$(4.2.7.2) \qquad d_0(\gamma_i^+) = \gamma_i, \text{ unless } a_i = 1 \text{ and } \gamma_i > 0 \text{ (in which case } d_1(\gamma_i^+) = \gamma_i).$$

Putting the maps (4.2.7.1) together, we obtain $\prod_{i=1}^s K_{P_i}^*$-equivariant maps

$$\overline{\mathscr{L}}^{-1}(\mathbf{N}_{>1}^s) \longrightarrow \prod_{i=1}^s \mathcal{P}_{a_i^+}(\mathcal{T}_i), \qquad \gamma = (\gamma_1, \ldots, \gamma_s) \mapsto \gamma^+ = (\gamma_1^+, \ldots, \gamma_s^+)$$

and

$$\{x \in \mathcal{C}\,;\, (P_1 \cdots P_s)^2 \mid c(x)\} \longrightarrow \mathcal{C}^+, \qquad x = X(\gamma) \mapsto x^+ = X^+(\gamma^+).$$

4.2.8. P_i-new quotients. The assumption (A2) of 4.2.1 implies that the map

$$\mathbf{Z}[\prod_{i=1}^s \mathcal{P}_{a_i}(\mathcal{T}_i)] = \bigotimes_{i=1}^s \mathbf{Z}[\mathcal{P}_{a_i}(\mathcal{T}_i)] \xrightarrow{X_*} \mathbf{Z}[\mathcal{C}] \xrightarrow{\alpha_*} A_0(K[cP^\infty])$$

factors through the "new quotient"

$$\mathbf{Z}[\prod_{i=1}^s \mathcal{P}_{a_i}(\mathcal{T}_i)]_{\text{new}} = \bigotimes_{i=1}^s \mathbf{Z}[\mathcal{P}_{a_i}(\mathcal{T}_i)]_{\text{new}} = \bigotimes_{i=1}^s \mathbf{Z}[\mathcal{P}_{a_i}(\mathcal{T}_i)]/M_i,$$

where

$$M_i = \sum_{\substack{0 \leq k < a_i \\ 0 \leq j \leq a_i - k}} \text{Im}\Big(d_j^* : \mathbf{Z}[\mathcal{P}_k(\mathcal{T}_i)] \longrightarrow \mathbf{Z}[\mathcal{P}_{a_i}(\mathcal{T}_i)]\Big).$$

4.2.9. Lemma. *Assume that $\gamma_i = (y_0 \to \cdots \to y_{a_i}) \in \mathcal{P}_{a_i}(\mathcal{T}_i)$, $\overline{\mathscr{L}}_i(\gamma_i) = n_i \geq 2$ and γ_i is P_i-good (i.e., either $a_i = 0$, or $\mathscr{L}_i(y_0) \neq \mathscr{L}_i(y_{a_i})$). Then the image of $\text{Tr}_{Z_i(n_i)}(\gamma_i) = \sum_{a \in Z_i(n_i)} a \cdot \gamma_i$ in $\mathbf{Z}[\mathcal{P}_{a_i}(\mathcal{T}_i)]_{\text{new}}$ is equal to*

$$\begin{cases} T_{P_i}(d_1(\gamma_i^+)) - d_2(\gamma_i^+), & \text{if } a_i = 0 \\ -\iota_i(d_0(\gamma_i^+)), & \text{if } a_i = 1 \text{ and } \gamma_i > 0 \\ -\iota_i(d_1(\gamma_i^+)), & \text{if } a_i = 1 \text{ and } \gamma_i < 0 \\ 0, & \text{if } a_i \geq 2. \end{cases}$$

Above, T_{P_i} is the Hecke operator (3.2.1.1) on $\mathcal{T}_i$, which associates to each vertex the sum of its neighbours.

Proof. This follows from 4.2.4.1-3 and the definition of the new quotient in 4.2.8.

4.2.10. Change of the parameterisation α. For $j = (j_1, \ldots, j_s) \in J$, define

$$u(j) = \prod_{i=1}^s u_i(j_i) \in \text{End}_F(J(N_H^*)), \qquad u_i(j_i) = \begin{cases} NP_i, & \text{if } j_i = 0 \\ -T_{P_i}, & \text{if } j_i = 1,\, a_i = 0 \\ -\mu_i(P_i), & \text{if } j_i = 1,\, a_i = 1 \\ 1, & \text{if } j_i = 2,\, a_i = 0, \end{cases}$$

where this time T_{P_i} denotes the image of the eponymous Hecke operator in $\mathrm{End}_F(J(N_H^*))$, and μ_i is as in 4.2.5. The map

$$\alpha_+ : N_{H^+}^* \xrightarrow{(d(j))_{j\in J}} (N_H^*)^J \xrightarrow{\iota_H^J} J(N_H^*)^J \xrightarrow{\sum u(j)(-)} J(N_H^*) \twoheadrightarrow A_0$$

is an algebraic morphism defined over F. Functoriality of the Hodge class [Co-Va 1, §3.5] implies that, if we multiply the Hodge classes of H and H^+ by the same integer in order to define ι_H and ι_{H^+}, then α_+ is also given by

$$\text{(4.2.10.1)} \qquad \alpha_+ : N_{H^+}^* \xrightarrow{\iota_{H^+}} J(N_{H^+}^*) \xrightarrow{(d(j))_*} J(N_H^*)^J \xrightarrow{\sum u(j)(-)} J(N_H^*) \twoheadrightarrow A_0.$$

4.2.11. Proposition. *Assume that $x \in \mathcal{C}$ is a good CM point which is negatively oriented at each P_i with $a_i = 1$, $c(x) = c_0 P^n = c_0 P_1^{n_1} \cdots P_s^{n_s}$, $c_0 \mid c$, and $n \in \mathbf{N}^s$ is large enough. Then*

$$\prod_{i=1}^{s} \left(NP_i - \mathrm{Tr}_{Z(n,\delta_{\{i\}})} \right) \alpha(x) = \alpha_+(x^+) \in A_0(K[cP^\infty]).$$

Proof. This follows from (4.2.7.2), Lemma 4.2.9 and (3.3.2.1).

4.2.12. Proposition. *The morphism α_+ is not constant.*

Proof. As the map $\sum u(j)(-)$ in (4.2.10.1) is surjective, it is enough to show that the map

$$(d(j))^* : \Gamma(N_H^{*\mathrm{an}}, \Omega^{\mathrm{an}})^J \longrightarrow \Gamma(N_{H^+}^{*\mathrm{an}}, \Omega^{\mathrm{an}})$$

is injective. However, this map is given by

$$\bigoplus_{\sigma_0 : L_0 \hookrightarrow \mathbf{C}} \left(\bigotimes_{i=1}^{s} \left(\left(\pi(\sigma_0)_{P_i}^{R_i^*} \right)^{\oplus |J_i|} \longrightarrow \pi(\sigma_0)_{P_i}^{(R_i^+)^*} \right) \otimes \mathrm{id} \right),$$

where $|J_i| = a_i^+ - a_i + 1$, and each local map

$$(d_0, \ldots, d_{a_i^+ - a_i})^* : \left(\pi(\sigma_0)_{P_i}^{K_0(P_i^{a_i})} \right)^{\oplus |J_i|} \longrightarrow \pi(\sigma_0)_{P_i}^{K_0(P_i^{a_i^+})}$$

is injective, by Casselman's result recalled at the end of 3.1.7.

4.3. The main result

4.3.1. Theorem. *Assume that we are given the data (D1) - (D5) from 2.4.1 satisfying the assumptions (A1) - (A2) from 4.2.1. Assume also that there exist $\sigma : L \hookrightarrow \mathbf{C}$, $\omega \in \Gamma(A^{\mathrm{an}}, \Omega^{\mathrm{an}}) \cap {}^\sigma\pi^H$ and a $j(\prod_{v\in S}(K_v^*)^\circ / F_v^*)$-invariant linear map ${}^\sigma\ell : {}^\sigma\pi \longrightarrow \mathbf{C}({}^\sigma\chi_1^{-1})$ (i.e., such that ${}^\sigma\ell(j(a)u) = {}^\sigma\chi_1(a)^{-1}\, {}^\sigma\ell(u)$) satisfying ${}^\sigma\ell(\omega) \neq 0$. If $\{x_m\}_{m\in\mathbf{N}} \subseteq \mathcal{C}$ is a sequence of good CM points (in the sense of 4.2.3) contained in the $(P_1 \cdots P_s)$-isogeny class $\mathcal{C}$ such that $(P_1 \cdots P_s)^m \mid c(x_m)$ for all $m \in \mathbf{N}$, then, for each sufficiently large $m \in \mathbf{N}$, there exists a character $\chi_m : G^{(c)} \longrightarrow O_{L'_m}^*$ of finite order (where L'_m is a number field containing L) such that:*

(1) *the restriction of χ_m to $G_0^{(c)}$ is equal to χ_0;*
(2) *$\forall i = 1, \ldots, s \qquad \mathrm{ord}_{P_i}(c(\chi_m)) = \mathrm{ord}_{P_i}(c(x_m))$;*
(3) *the element*

$$e_{\overline{\chi}_m}(1 \otimes \alpha(x_m)) = \sum_{g \in G^{(c)}/\mathrm{Ker}(\chi_m)} \chi_m(g) \otimes \alpha(g(x_m)) \in O_{L'_m} \otimes_{O_{L_0}} A_0(K[cP^\infty])$$

is not torsion.

Proof. Applying, if necesary, the construction from 4.2.5 (which multiplies $\alpha(x_m)$ by ± 1), we can assume that each point x_m is negatively oriented at each P_i with $a_i = 1$. Combining Proposition 4.2.11 with Lemma 4.1.3, we obtain that, for $m \in \mathbf{N}$ large enough,

$$e(\chi_0, n(m))\,\alpha(x_m) = \frac{|H(n(m))|}{N(P_1 \cdots P_s)}\, e_{\overline{\chi}_0}\, \alpha_+(x_m^+),$$

where $c(x_m) = c_0 P^{n(m)}$, $c_0 \mid c$, $n(m) \in \mathbf{N}^s$. Applying Theorem 2.5.1 to the modified data $\{x_m^+\} \subseteq \mathcal{C}^+$ and $\alpha_+ : N_{H^+}^* \longrightarrow A_0$, we deduce that

$$e(\chi_0, n(m))\,\alpha(x_m) \neq 0 \in \mathbf{C} \otimes_{O_{L_0}} A_0(K[cP^\infty])$$

for all sufficiently large $m \in \mathbf{N}$, which proves the claim.

4.3.2. Corollary. *If A_0 does not acquire complex multiplication over any totally imaginary quadratic extension of F contained in $K[c(\chi_0)P^\infty]$, then, for all sufficiently large $m \in \mathbf{N}$, the following $O_{L'_m}$-modules are finite:*

$$e_{\overline{\chi}_m}\left(O_{L'_m} \otimes_{O_{L_0}} A_0(K[cP^{n(m)}])/O_{L'_m}(1 \otimes \alpha(x_m))\right), \qquad e_{\overline{\chi}_m}\left(O_{L'_m} \otimes_{O_{L_0}} Ш(A_0/K[cP^{n(m)}])\right),$$
$$e_{\chi_m}\left(O_{L'_m} \otimes_{O_{L_0}} A_0(K[cP^{n(m)}])/O_{L'_m}(1 \otimes \alpha(\rho(x_m)))\right), \qquad e_{\chi_m}\left(O_{L'_m} \otimes_{O_{L_0}} Ш(A_0/K[cP^{n(m)}])\right),$$

where $\rho \in \mathrm{Gal}(\overline{F}/F)$ is the complex conjugation w.r.t. some embedding $\overline{F} \hookrightarrow \mathbf{C}$ extending τ_1.

Proof. This follows from Theorem 4.3.1 and [Ne 2, Thm. 3.2].

4.3.3. Theorem (an explicit special case of 4.3.1). *Assume that we are given the data (D1) - (D5) from 2.4.1 satisfying the assumptions (A1) - (A2) from 4.2.1. Assume also that, for each $v \in S$, the following conditions (i)–(iv) hold:*

(i) $o(\pi_v) \leq 1$.

(ii) *If $o(\pi_v) = 0$, then R_v is a maximal order in $B_v = M_2(F_v)$; if $o(\pi_v) = 1$ and $B_v = M_2(F_v)$, then R_v is an Eichler order of level one in B_v.*

(iii) *$(\star_v)$ holds.*

(iv) *If $o(\pi_v) = 0$, write the eigenvalue of T_v on A_0 as $(Nv)^{1/2}(\lambda_v + \lambda_v^{-1})$. Put $n_v = \mathrm{ord}_v(c(x))$ (for any $x \in \mathcal{C}$) and $m_v = \mathrm{ord}_v(c(\chi_0))$. Assume that $1 \leq m_v \leq n_v$ and $[\lambda_v^2 = 1$ or $\lambda_v^{2(n_v - m_v + 1)} \neq 1]$ (which is automatic if $m_v = n_v \geq 1$).*

Then: if $\{x_m\}_{m \in \mathbf{N}} \subseteq \mathcal{C}$ is a sequence of good CM points (in the sense of 4.2.3) contained in the $(P_1 \cdots P_s)$-isogeny class $\mathcal{C}$ such that $(P_1 \cdots P_s)^m \mid c(x_m)$ for all $m \in \mathbf{N}$, then, for each sufficiently large $m \in \mathbf{N}$, there exists a character $\chi_m : G^{(c)} \longrightarrow O_{L'_m}^*$ of finite order (where L'_m is a number field containing L) such that:

(1) *the restriction of χ_m to $G_0^{(c)}$ is equal to χ_0;*

(2) $\forall i = 1, \ldots, s \qquad \mathrm{ord}_{P_i}(c(\chi_m)) = \mathrm{ord}_{P_i}(c(x_m))$;

(3) *the element*

$$\sum_{g \in G^{(c)}/\mathrm{Ker}(\chi_m)} \chi_m(g) \otimes \alpha(g(x_m)) \in O_{L'_m} \otimes_{O_{L_0}} A_0(K[cP^\infty])$$

is not torsion.

Proof. In the proof of Theorem 4.3.1, refer to Theorem 2.5.2 instead of Theorem 2.5.1.

References

[Ca] W. Casselman, *On some results of Atkin and Lehner*, Math. Ann. **201** (1973), 301–314.

[Con] B. Conrad, *Gross-Zagier revisited*, in: Heegner points and Rankin L-series, (H. Darmon, S.-W. Zhang, eds.), MSRI Publ. **49**, Cambridge Univ. Press, Cambridge, 2004, pp. 67–163.

[Co 1] C. Cornut, *Réduction de Familles de Points CMs*, Thesis, Strasbourg, 2000.

[Co 2] C. Cornut, Appendix to this paper.

[Co-Va 1] C. Cornut, V. Vatsal, *Nontriviality of Rankin-Selberg L-functions and CM points*, L-functions and Galois representations (Durham, July 2004), LMS Lecture Note Series **320**, Cambridge Univ. Press, 2007, pp. 121–186.

[Co-Va 2] C. Cornut, V. Vatsal, *CM points and quaternion algebras*, Documenta Math. **10** (2005), 263-309.

[Ed] B. Edixhoven, *Special points on products of modular curves*, Duke Math. J. **126** (2005), 325–348.

[Ed-Ya] B. Edixhoven, A. Yafaev *Subvarieties of Shimura varieties*, Ann. of Math. (2) **157** (2003), 621–645.

[Gr] B. Gross, *Heegner points and representation theory*, Heegner points and Rankin L-series, MSRI Publ. **49**, Cambridge Univ. Press, 2004, pp. 37–65.

[Gr-Pr] B. Gross, D. Prasad, *Test vectors for linear forms*, Math. Ann. **291** (1991), 343–355.

[Ja-La] H. Jacquet, R.P. Langlands, *Automorphic forms on* $GL(2)$, Lect. Notes in Math. **114**, Springer, Berlin, 1970.

[Ja] H. Jacquet, *Automorphic forms on* $GL(2)$ *II*, Lect. Notes in Math. **278**, Springer, Berlin, 1972.

[Ne 1] J. Nekovář, *Selmer complexes*, Astérisque **310**, S.M.F., Paris, 2006.

[Ne 2] J. Nekovář, *The Euler system method for CM points on Shimura curves*, L-functions and Galois representations (Durham, July 2004), LMS Lecture Note Series **320**, Cambridge Univ. Press, 2007, pp. 471–547.

[Ne 3] J. Nekovář, *Growth of Selmer groups of Hilbert modular forms over ring class fields*, Ann. E.N.S., sér. 4, **41** (2008), 1003–1022.

[Sa] H. Saito, *On Tunnell's formula for characters of* $GL(2)$, Compos. Math. **85** (1993), 99–108.

[Tu] J. Tunnell, *Local* ε*-factors and characters of* GL_2, Am. J. Math. **105** (1983), 1277–1308.

[Wa] J.-L. Waldspurger, *Correspondances de Shimura et quaternions*, Forum. Math. **3** (1991), 219–307.

[Zh] S.-W. Zhang, *Gross-Zagier formula for* GL_2, Asian J. Math. **5** (2001), 183–290.

Université Pierre et Marie Curie (Paris 6)
Institut de Mathématiques de Jussieu
Théorie des Nombres, Case 247
4, place Jussieu
F-75252 Paris cedex 05
FRANCE