ERRATUM: On the parity of ranks of Selmer groups III

Jan Nekovář

Remark 4.1.2(4) and the treatment of archimedean ε-factors in 4.1.3 are incorrect. Contrary to what is stated in 0.3, the individual archimedean ε-factors $\varepsilon_u(M)$ $(u \mid \infty)$ cannot be expressed, in general, in terms of $M_{\mathfrak{p}}$, but their product can.

To motivate the corrections below, consider a motive M (pure of weight w) over F with coefficients in L. Set $\widetilde{S}_\infty = \{\tau : F \hookrightarrow \mathbf{C}\}$, $\widetilde{S}_p = \{\sigma : F \hookrightarrow \overline{\mathbf{Q}}_p\}$ and denote by $r_\infty : \widetilde{S}_\infty \longrightarrow S_\infty$, $r_p : \widetilde{S}_p \longrightarrow S_p$ the canonical surjections. Fix an embedding $\iota : L \hookrightarrow \mathbf{C}$ and an isomorphism $\lambda : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$ such that $\mathfrak{p}$ is induced by $\iota_p = \lambda^{-1} \circ \iota : L \hookrightarrow \overline{\mathbf{Q}}_p$. To each $v \in S_p$ then corresponds a subset

$$S_\infty(v) = \{r_\infty(\lambda \circ \sigma) \mid r_p(\sigma) = v\} \subset S_\infty$$

such that

$$\sum_{w \in S_\infty(v)} [L_w : \mathbf{R}] = [F_v : \mathbf{Q}_p].$$

For each $\tau \in \widetilde{S}_\infty$, the Betti realization $M_{B,\tau}$ is an L-vector space and there is a Hodge decomposition

$$M_{B,\tau} \otimes_{L,\iota} \mathbf{C} = \bigoplus_{i \in \mathbf{Z}} (\iota M_\tau)^{i,w-i}.$$

The corresponding Hodge numbers

$$h^{i,w-i}(\iota M_u) := h^{i,w-i}(\iota M_\tau) = \dim_{\mathbf{C}} (\iota M_\tau)^{i,w-i}$$

depend only on $u = r_\infty(\tau) \in S_\infty$. The de Rham realization M_{dR} is a free $L \otimes_{\mathbf{Q}} F$-module; its Hodge filtration is given by submodules $F^r M_{dR}$ (not necessarily free) which correspond, under the de Rham comparison isomorphism

$$M_{dR} \otimes_{L \otimes_{\mathbf{Q}} F, \iota \otimes \tau} \mathbf{C} \xrightarrow{\sim} M_{B,\tau} \otimes_{L,\iota} \mathbf{C},$$

to

$$(F^r M_{dR}) \otimes_{L \otimes_{\mathbf{Q}} F, \iota \otimes \tau} \mathbf{C} \xrightarrow{\sim} \bigoplus_{i \geq r} (\iota M_\tau)^{i,w-i},$$

hence

$$\dim_{\mathbf{C}} \left((gr^i_F M_{dR}) \otimes_{L \otimes_{\mathbf{Q}} F, \iota \otimes \tau} \mathbf{C} \right) = h^{i,w-i}(\iota M_\tau).$$

The $\mathfrak{p}$-adic realization $M_{\mathfrak{p}}$ of M is isomorphic, as an $L_{\mathfrak{p}}$-vector space, to $M_{B,\tau} \otimes_L L_{\mathfrak{p}}$ (for any $\tau \in \widetilde{S}_\infty$). For each $v \in S_p$, $D_{dR}(M_{\mathfrak{p},v})$ is a free $L_{\mathfrak{p}} \otimes_{\mathbf{Q}_p} F_v$-module equipped with a filtration satisfying

$$D^r_{dR}(M_{\mathfrak{p},v}) \xrightarrow{\sim} F^r M_{dR} \otimes_{L \otimes_{\mathbf{Q}} F} (L_{\mathfrak{p}} \otimes_{\mathbf{Q}_p} F_v).$$

This implies that, for each $i \in \mathbf{Z}$, the dimension

$$d^i_v(M_{\mathfrak{p}}) := \dim_{L_{\mathfrak{p}}} \left(D^i_{dR}(M_{\mathfrak{p},v}) / D^{i+1}_{dR}(M_{\mathfrak{p},v}) \right)$$

is equal to

$$\dim_{L_{\mathfrak{p}}} \left(gr^i_F M_{dR} \right) \otimes_{L \otimes_{\mathbf{Q}} F} (L_{\mathfrak{p}} \otimes_{\mathbf{Q}_p} F_v) = \dim_{\overline{\mathbf{Q}}_p} \left(gr^i_F M_{dR} \right) \otimes_{L \otimes_{\mathbf{Q}} F, \iota_p \otimes \mathrm{incl}} (\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F_v) =$$
$$= \sum_{\sigma : F_v \hookrightarrow \overline{\mathbf{Q}}_p} \dim_{\overline{\mathbf{Q}}_p} \left(gr^i_F M_{dR} \right) \otimes_{L \otimes_{\mathbf{Q}} F, \iota_p \otimes \sigma} \overline{\mathbf{Q}}_p = \sum_{u \in S_\infty(v)} [F_u : \mathbf{R}] \, h^{i,w-i}(\iota M_u),$$

hence

$$d_v^-(M_{\mathfrak p}) := \sum_{i<0} i\, d_v^i(M_{\mathfrak p}) = \sum_{u\in S_\infty(v)} [F_u:\mathbf{R}]\, d^-(\iota M_u), \qquad d^-(\iota M_u) := \sum_{i<0} i\, h^{i,w-i}(\iota M_u). \tag{$\star$}$$

Corrections to §4.1 and §5.1: firstly, 4.1.2(4) and 5.1.2(9) should be deleted. Secondly, §4.1.3 should be reformulated as follows: we assume that V satisfies 4.1.2(1)-(3). For each $v \in S_p$ we define

$$d_v^-(V) := \sum_{i<0} i\, d_v^i(V), \qquad d_v^i(V) = \dim_{L_{\mathfrak p}} \left(D^i_{dR}(V_v)/D^{i+1}_{dR}(V_v)\right) \tag{4.1.3.1$'$}$$

and

$$\prod_{u\in S_\infty(v)} \varepsilon(V_u) := (-1)^{d_v^-(V)} \prod_{u\in S_\infty(v), F_u=\mathbf{C}} (-1)^{\dim_{L_{\mathfrak p}}(V)/2} \tag{4.1.3.2$'$}$$

(even though we are unable to define the individual $\varepsilon(V_u)$). If $V = M_{\mathfrak p}$, where $M \xrightarrow{\sim} M^*(1)$ is pure (of weight -1), it follows from $(\star)$ and (2.3.1) that this definition gives the correct product of archimedean ε-factors.

The formula (4.1.3.6) should be replaced by

$$\forall v \in S_p \qquad \widetilde{\varepsilon}(V_v) = (-1)^{d_v^-(V)} (\det V_v^+)(-1) = \varepsilon(WD(V_v)^{N-ss}), \tag{4.1.3.6$'$}$$

which implies that

$$\widetilde{\varepsilon}(V_v) \prod_{u\in S_\infty(v)} \varepsilon(V_u) = (\det V_v^+)(-1) \prod_{u\in S_\infty(v), F_u=\mathbf{C}} (-1)^{\dim_{L_{\mathfrak p}}(V)/2},$$

hence

$$\prod_{v\in S_p\cup S_\infty} \widetilde{\varepsilon}(V_v) = (-1)^{r_2(F)\, \dim_{L_{\mathfrak p}}(V)/2} \prod_{v\in S_p} (\det V_v^+)(-1), \tag{4.1.3.7$'$}$$

where $r_2(F)$ denotes the number of complex places of F.

Corrections to Theorem 5.3.1 and its proof: the statement should say that, under the assumptions 5.1.2(1)-(8), the quantity

$$\begin{aligned} &(-1)^{h^1_f(F,V)}/\varepsilon(V) = (-1)^{\widetilde{h}^1_f(F,V)}/\widetilde{\varepsilon}(V) = \\ &= (-1)^{\widetilde{h}^1_f(F,V)} (-1)^{r_2(F)\, \dim_{\mathscr{L}}(\mathcal{V})/2} \prod_{v\in S_p} (\det \mathcal{V}_v^+)(-1) \prod_{v\notin S_p\cup S_\infty} \varepsilon(\mathcal{V}_v) \end{aligned}$$

depends only on $\mathcal{V}$ and $\mathcal{V}_v^+$ $(v \in S_p)$.

In the proof, a reference to (4.1.3.7) should be replaced by that to (4.1.3.7'), which yields

$$\widetilde{\varepsilon}(V) = \prod_{v\in S_p\cup S_\infty} \widetilde{\varepsilon}(V_v) \prod_{v\notin S_p\cup S_\infty} \varepsilon(V_v) = (-1)^{r_2(F)\, \dim_{L_{\mathfrak p}}(V)/2} \prod_{v\in S_p} (\det V_v^+)(-1) \prod_{v\notin S_p\cup S_\infty} \varepsilon(V_v).$$

Corrections to §5.3.3: the first question should ask whether

$$(-1)^{d_v^-(V)} \varepsilon(WD(V_v)^{N-ss}) \qquad\qquad (v \in S_p)$$

depends only on $\mathcal{V}_v$?

Corrections to §5.3.4-5: it is often useful to use a slightly more general version of Example 5.3.4 with $\Gamma = \Gamma_0 \times \Delta$, where Γ_0 is isomorphic to $\mathbf{Z}_p$ and Δ is finite (abelian). Given a character $\alpha : \Delta \longrightarrow \mathcal{O}_{\mathfrak p}^*$, set

$$R = \mathcal{O}_{\mathfrak{p}}[[\Gamma_0]], \qquad \mathcal{T} = \left(T \otimes_{\mathcal{O}_{\mathfrak{p}}} \mathcal{O}_{\mathfrak{p}}[[\Gamma^+]]\right) \otimes_{\mathcal{O}_{\mathfrak{p}}[\Delta],\alpha} \mathcal{O}_{\mathfrak{p}}, \qquad \mathcal{T}_v^+ = \left(T_v^+ \otimes_{\mathcal{O}_{\mathfrak{p}}} \mathcal{O}_{\mathfrak{p}}[[\Gamma^+]]\right) \otimes_{\mathcal{O}_{\mathfrak{p}}[\Delta],\alpha} \mathcal{O}_{\mathfrak{p}} \quad (v \in S_p).$$

As in 5.3.4(2)-(3), $\mathcal{T}$ is an $R[G_{F,S}]$-module equipped with a skew-symmetric R-bilinear pairing $(\ ,\ ) : \mathcal{T} \times \mathcal{T} \longrightarrow R(1)$ inducing an isomorphism

$$\mathcal{T} \otimes \mathbf{Q} \xrightarrow{\sim} \mathrm{Hom}_R(\mathcal{T}, R(1)) \otimes \mathbf{Q}.$$

In 5.3.4(5) we have to replace $\beta : \Gamma \longrightarrow L_{\mathfrak{p}}(\beta)$ by $\beta : \Gamma_0 \longrightarrow L_{\mathfrak{p}}(\beta)$; then

$$\mathcal{T}_P/\varpi_P\mathcal{T}_P = \mathrm{Ind}_{G_{F_0,S}}^{G_{F,S}}(V \otimes (\beta \times \alpha)).$$

In 5.3.5, we set, for any $L_{\mathfrak{p}}[\Gamma]$-module M,

$$M^{(\beta\times\alpha)} = \{x \in M \otimes_{L_{\mathfrak{p}}} L_{\mathfrak{p}}(\beta) \mid \forall \sigma \in \Gamma \quad \sigma(x) = (\beta \times \alpha)(x)\};$$

then

$$H^1_f(F, \mathcal{T}_P/\varpi_P\mathcal{T}_P) = H^1_f(F_0, V \otimes (\beta \times \alpha)) = \left(H^1_f(F_\beta, V) \otimes (\beta \times \alpha)\right)^{\mathrm{Gal}(F_\beta/F_0)} = H^1_f(F_\beta, V)^{(\beta^{-1}\times\alpha^{-1})}$$

and

$$\tau : H^1_f(F_\beta, V)^{(\beta^{-1}\times\alpha^{-1})} \xrightarrow{\sim} H^1_f(F_\beta, V)^{(\beta\times\alpha)}.$$

Applying Corollary 5.3.2, we obtain, for any pair of characters of finite order $\beta, \beta' : \Gamma_0 \longrightarrow \overline{L}_{\mathfrak{p}}^*$, that

$$(-1)^{h^1_f(F_0, V\otimes(\beta\times\alpha))}/\,\varepsilon(F_0, V \otimes (\beta \times \alpha)) = (-1)^{h^1_f(F_0, V\otimes(\beta'\times\alpha))}/\,\varepsilon(F_0, V \otimes (\beta' \times \alpha)). \qquad (5.3.5.1')$$