Eichler-Shimura relations and semisimplicity of étale cohomology of quaternionic Shimura varieties

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Abstract: we show that the non CM part of ℓ-adic étale cohomology of any compact quaternionic Shimura variety with coefficients in any automorphic local system is a semisimple Galois representation. If the local system has weight $k = (k_1, \ldots, k_d)$ with all k_i of the same parity, the full ℓ-adic étale cohomology is semisimple. For Hilbert modular varieties, analogous results are proved for ℓ-adic intersection cohomology of the Baily-Borel compactification. The proof combines a representation-theoretical criterion of semisimplicity with Eichler-Shimura relations for partial Frobenius morphisms.

Résumé : on montre que l'action galoisienne sur la partie sans multiplication complexe de la cohomologie étale d'un faisceau ℓ-adique lisse automorphe sur une variété de Shimura quaternionique compacte est semi-simple. Si le poids du faisceau s'écrit $k = (k_1, \ldots, k_d)$, où les k_i ont la même parité, toute la cohomologie étale est semi-simple. Les mêmes résultats sont montrés pour la cohomologie d'intersection ℓ-adique de la compactification de Baily-Borel des variétés modulaires de Hilbert. La preuve utilise un critère abstrait de semi-simplicité et les relations d'Eichler-Shimura pour les morphismes de Frobenius partiels.

0. Introduction

(0.1) General conventions and notation. The characteristic polynomial of an endomorphism u of a finite-dimensional vector space over a field k will be denoted by $P_u(X) = \det(X \cdot \mathrm{id} - u) \in k[X]$. If $k \subset K$ are fields and X is a k-vector subspace of a K-vector space Y, we denote by $K \cdot X$ the K-vector subspace of Y generated by X. We abbreviate $\otimes_{\mathbf{Z}}$ as $\otimes$. For an abelian group A we let $\widehat{A} = A \otimes \widehat{\mathbf{Z}}$. We denote by $\mathbf{A}$ and $\mathbf{A}_k$, respectively, the ring of adeles of $\mathbf{Q}$ and of a number field k.

Throughout the article we fix an isomorphism $\mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_\ell$. For any algebraic object $(-)$ defined over a subfield of $\mathbf{C}$ we denote by $(-)_\ell$ its base change to $\overline{\mathbf{Q}}_\ell$. Let $\overline{\mathbf{Q}}$ be the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$. The reciprocity map of class field theory is normalised by letting uniformisers correspond to geometric Frobenius elements $\mathrm{Fr}(P)$. All representations and characters are assumed to be continuous with respect to the natural topologies involved. A representation of a profinite group is called strongly irreducible if its restriction to every open subgroup is irreducible. By an automorphic representation we mean an irreducible automorphic representation.

(0.2) Let us recall basic facts about decomposition of singular and étale cohomology of compact Shimura varieties. As in the classical case of cuspidal cohomology of modular curves, everything boils down to the fact that the Hecke operators act on the space of cuspidal automorphic forms in a selfadjoint way (up to a twist).

Let $(G, \mathscr{X})$ be a (pure) Shimura datum. A rational representation $\xi : G_{\mathbf{C}} \longrightarrow GL(N)_{\mathbf{C}}$ (whose restriction to the centre $Z_{\mathbf{C}}$ satisfies an appropriate condition) gives rise, for each sufficiently small open compact subgroup $K \subset G(\widehat{\mathbf{Q}})$, to a locally constant sheaf of complex vector spaces $\mathscr{L}_\xi$ on the complex manifold $Sh_K(G, \mathscr{X})^{an} = G(\mathbf{Q})\backslash(\mathscr{X} \times G(\widehat{\mathbf{Q}})/K)$.

(0.3) If, in addition, the derived group G^{der} is anisotropic, then $Sh_K(G, \mathscr{X})^{an}$ is compact and its cohomology $H^*(Sh_K(G, \mathscr{X})^{an}, \mathscr{L}_\xi)$ is described in terms of relative Lie algebra cohomology.

Write $\mathscr{X} = G(\mathbf{R})/K_\infty$, where K_∞ is the stabiliser of a fixed base point in $\mathscr{X}$, and denote, for any $G(\mathbf{R})$-module V, by V_0 the subspace of K_∞-finite vectors in V. There is a canonical isomorphism

$$H^i(Sh_K(G, \mathscr{X})^{an}, \mathscr{L}_\xi) = H^i(\mathfrak{g}, K_\infty; C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K) \otimes \xi) = H^i(\mathfrak{g}, K_\infty; C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K)_0 \otimes \xi),$$

where $\mathfrak{g} = \mathrm{Lie}(G(\mathbf{R}))$ [BW, VII.2.7]. It gives rise to a $G(\widehat{\mathbf{Q}})$-equivariant isomorphism

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$$H^i(Sh(G,\mathscr{X})^{an},\mathscr{L}_\xi)=\varinjlim_{K} H^i(Sh_K(G,\mathscr{X})^{an},\mathscr{L}_\xi)=H^i(\mathfrak{g},K_\infty;C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A}))_0\otimes\xi). \qquad (0.3.1)$$

For every character $\omega : Z(\mathbf{Q})\backslash Z(\mathbf{A}) \longrightarrow \mathbf{C}^\times$ fix a character $\omega' : G(\mathbf{Q})\backslash G(\mathbf{A}) \longrightarrow \mathbf{R}_+^\times$ such that $\omega'|_{Z(\mathbf{A})} = |\omega|$. The space $G(\mathbf{Q})Z(\mathbf{A})\backslash G(\mathbf{A})$ is compact (since G^{der} is anisotropic) and the completion $L^2(G,\omega)$ of

$$C^\infty(G,\omega)=\{f\in C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A}))\mid f(gz)=\omega(z)f(g)\ \ \forall z\in Z(\mathbf{A})\}$$

with respect to the norm

$$|f|^2=\int_{G(\mathbf{Q})Z(\mathbf{A})\backslash G(\mathbf{A})}(\omega'(g)^{-1}|f(g)|)^2\,dg$$

is a unitary representation of $G(\mathbf{A})$ under the action $(g*f)(h)=\omega'(g)^{-1}f(hg)$. This representation decomposes as a discrete Hilbert sum $L^2(G,\omega)=\widehat{\bigoplus} m(\pi')\,\pi'$ of unitary automorphic representations π' of $G(\mathbf{A})$ with finite multiplicities $m(\pi')$.

Each π' has central character $\omega_{\pi'}=\omega/|\omega|$ and gives rise to an automorphic representation $\pi=\omega'\pi'=\pi_\infty\otimes\pi^\infty$ of $G(\mathbf{A})=G(\mathbf{R})\times G(\widehat{\mathbf{Q}})$ with central character $\omega_\pi=\omega$.

Matsushima's formula [BW, Thm. VII.5.2] yields a $G(\widehat{\mathbf{Q}})$-equivariant isomorphism

$$H^i(\mathfrak{g},K_\infty;C^\infty(G,\omega)_0\otimes\xi)=\bigoplus_{\pi=\pi_\infty\otimes\pi^\infty} m(\pi)\,H^i(\mathfrak{g},K_\infty;\pi_\infty\otimes\xi)\otimes\pi^\infty, \qquad m(\pi)=m((\omega')^{-1}\pi),$$

hence, after putting the contributions of all ω together, a $G(\widehat{\mathbf{Q}})$-equivariant isomorphism

$$H^i(Sh(G,\mathscr{X})^{an},\mathscr{L}_\xi)=\bigoplus_{\pi=\pi_\infty\otimes\pi^\infty} m(\pi)\,H^i(\mathfrak{g},K_\infty;\pi_\infty\otimes\xi)\otimes\pi^\infty, \qquad (0.3.2)$$

where π runs through automorphic representations of $G(\mathbf{A})$ and $m(\pi)$ is the multiplicity of the unitary representation $\pi'=(\omega'_\pi)^{-1}\pi$ in $L^2(G,\omega_\pi)$.

(0.4) The Shimura variety $Sh(G,\mathscr{X})$ is defined over its reflex field $E=E(G,\mathscr{X})\subset\overline{\mathbf{Q}}\subset\mathbf{C}$. For sufficiently small K, the representation $\xi_\ell : G_{\overline{\mathbf{Q}}_\ell}\longrightarrow GL(N)_{\overline{\mathbf{Q}}_\ell}$ gives rise to a lisse $\overline{\mathbf{Q}}_\ell$-sheaf $\mathscr{L}_{\xi,\ell}$ on $Sh_K(G,\mathscr{X})$ and the comparison theorem between analytic and étale cohomology defines a $G(\widehat{\mathbf{Q}})$-equivariant isomorphism

$$H^i_{et}(Sh(G,\mathscr{X})\otimes_E\overline{\mathbf{Q}},\mathscr{L}_{\xi,\ell})=\varinjlim_{K} H^i_{et}(Sh_K(G,\mathscr{X})\otimes_E\overline{\mathbf{Q}},\mathscr{L}_{\xi,\ell})\simeq H^i(Sh(G,\mathscr{X})^{an},\mathscr{L}_\xi)_\ell=\bigoplus_{\pi^\infty} V^i(\pi^\infty)\otimes\pi^\infty, \qquad (0.4.1)$$

where $V^i(\pi^\infty)=\mathrm{Hom}_{G(\widehat{\mathbf{Q}})}(\pi^\infty,H^i_{et})$, π^∞ runs through irreducible unitarisable smooth representations of $G(\widehat{\mathbf{Q}})$ for which there exists a unitarisable irreducible $(\mathfrak{g},K_\infty)$-module π_∞ such that $H^i(\mathfrak{g},K_\infty;\pi_\infty\otimes\xi)\neq 0$ and $\pi_\infty\otimes\pi^\infty$ is an automorphic representation of $G(\mathbf{A})$. If ξ is irreducible (more generally, if it admits a central character ω_ξ), then the condition $H^i(\mathfrak{g},K_\infty;\pi_\infty\otimes\xi)\neq 0$ ("π_∞ being cohomological in degree i for ξ") implies a compatibility of central characters $\omega_\xi|_{Z(\mathbf{R})}=\omega_{\pi_\infty}^{-1}$.

The $\overline{\mathbf{Q}}_\ell$-vector space $V^i(\pi^\infty)$ has finite dimension

$$\dim V^i(\pi^\infty)=\sum_{\pi=\pi_\infty\otimes\pi^\infty} m(\pi)\,\dim H^i(\mathfrak{g},K_\infty;\pi_\infty\otimes\xi)$$

and the natural Galois action of $\Gamma_E=\mathrm{Gal}(\overline{\mathbf{Q}}/E)$ on étale cohomology (which commutes with the action of $G(\widehat{\mathbf{Q}})$) gives rise to a representation

$$\Gamma_E\longrightarrow \mathrm{Aut}_{G(\widehat{\mathbf{Q}})}(V^i(\pi^\infty)\otimes\pi^\infty)=\mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V^i(\pi^\infty))$$

(the last equality follows from $\dim \operatorname{End}_{G(\widehat{\mathbf{Q}})}(\pi^\infty) = 1$, which holds by a variant of Schur's Lemma).

(0.5) There is a huge industry based on pioneering work of Langlands and Kottwitz (with first steps due to Ihara in the case of Shimura curves) whose aim is to determine the isomorphism class of the semisimplification of the Galois representation $V^i(\pi^\infty)$. The main steps in this approach – which is still far from being completed in full generality – are the following:

(0.5.1) a construction of a canonical integral model $S_{K,\mathfrak{p}}$ of $Sh_K(G,\mathscr{X})$ over $O_{E,\mathfrak{p}}$, for (almost) all finite primes $\mathfrak{p}$ of E at which $Sh_K(G,\mathscr{X})$ has good reduction;

(0.5.2) a group-theoretical description of the set of $\overline{k(\mathfrak{p})}$-rational points of the special fibre of $S_{K,\mathfrak{p}}$;

(0.5.3) a comparison of the expression for

$$\sum_{i=0}^{2\dim} (-1)^i \operatorname{Tr}\left(\operatorname{Fr}(\mathfrak{p})^n \mid H^i_{et}(Sh_K(G,\mathscr{X}) \otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell})\right)$$

obtained from the previous two steps (via the Lefschetz formula) with terms occurring in the stable trace formula.

(0.6) In the present article we use a much more elementary method, based on Eichler-Shimura relations, to obtain information about the Galois representations $V^i(\pi^\infty)$. The study of Eichler-Shimura relations in this context has a long history. What is relevant to us is the approach of Faltings and Chai [FC, ch. VII], generalised by Wedhorn [W]. Our results are, naturally, weaker than those obtained by the Langlands-Kottwitz method, with one notable exception: in favourable cases we are able to show that $V^i(\pi^\infty)$ is a semisimple representation of Γ_E.

(0.7) An Eichler-Shimura relation (a "congruence relation") is a statement about compatibility of the commuting actions of Γ_E and the spherical part of the Hecke algebra $\overline{\mathbf{Q}}[G(\widehat{\mathbf{Q}})//K]$ on $H^i_{et}(Sh_K(G,\mathscr{X}) \otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell})$. In vague terms, the relation states that the (geometric) Frobenius $\operatorname{Fr}(\mathfrak{p})$ is a root of a certain Hecke polynomial.

(0.8) In the classical case when $G = GL(2)_{\mathbf{Q}}$, $K \supseteq K(N)$ $(N > 2)$ and ξ is the (k-2)-th symmetric power of the standard representation ($k \geq 2$), the Shimura variety $Y = Sh_K(G,\mathscr{X})$ is an open modular curve of level dividing N and the classical Eichler-Shimura relation states that

$$\forall p \nmid \ell N \qquad \operatorname{Fr}(p)^2 - T_p \operatorname{Fr}(p) + pS_p = 0 \tag{0.8.1}$$

on $H^i = H^i_{et}(Y \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell})$, where T_p and S_p denote the Albanese (= covariant) action of the double cosets $[K \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K]$ and $[K \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} K]$, respectively.

The spherical Hecke algebra $\mathbf{T} = \mathbf{Q}[T_p, pS_p]_{p\nmid N}$ acts semisimply on cuspidal cohomology

$$H^1_! = H^1_{et}(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_*\mathscr{L}_{\xi,\ell}),$$

where $j : Y \hookrightarrow X = Y \cup \{\text{cusps}\}$. For any cuspidal Hecke eigenform $\phi \in S_k(N,\chi)$ of weight k for $\mathbf{T}$

$$\phi|_{T_p} = \lambda_p \phi, \qquad \phi|_{pS_p} = p^{k-1}\chi(p)\phi,$$

the analogue of $V^i(\pi^\infty)$ from (0.4.1)

$$\rho := H^1_![\phi - \text{eigenspace for } \mathbf{T}]$$

is a non-zero representation of $\Gamma_{\mathbf{Q}}$. Let $\rho_\phi : \Gamma_{\mathbf{Q}} \longrightarrow GL_2(\overline{\mathbf{Q}}_\ell)$ be the Galois representation attached to ϕ. Its characteristic polynomial is characterised by the fact that

$$\forall p \nmid \ell N \qquad P_{\rho_\phi(\operatorname{Fr}(p))}(X) = X^2 - \lambda_p X + p^{k-1}\chi(p). \tag{0.8.2}$$

The Eichler-Shimura relation (0.8.1) implies that

$$\forall p \nmid \ell N \qquad P_{\rho_\phi(\mathrm{Fr}(p))}(\rho(\mathrm{Fr}(p))) = 0, \tag{0.8.3}$$

hence

$$\forall g \in \Gamma_{\mathbf{Q}} \qquad P_{\rho_\phi(g)}(\rho(g)) = 0, \qquad [\text{“}P_{\rho_\phi}(\rho) = 0\text{”}] \tag{0.8.4}$$

by the Čebotarev density theorem. Of course, $\rho \simeq \rho_\phi^{\oplus m}$ for some $m \geq 1$ (by a variant of (0.4.1) or by Theorem 3.7 below) and ρ_ϕ is constructed as ρ for a suitable choice of K (the formula (0.8.2) is deduced from (0.8.1) and Poincaré duality).

(0.9) We are interested in those Shimura varieties for which $G = R_{F/\mathbf{Q}}(H)$ is the restriction of scalars of a (connected reductive) algebraic group H defined over a totally real number field F. In this case the corresponding analytic objects decompose according to the decomposition of $F \otimes \mathbf{R} \xrightarrow{\sim} \prod_{v|\infty} F_v = \prod_{v|\infty} \mathbf{R}$:

$$G(\mathbf{R}) = \prod_{v|\infty} H_v(\mathbf{R}), \qquad \mathscr{X} = \prod_{v|\infty} \mathscr{X}_v, \qquad K_\infty = \prod_{v|\infty} K_{\infty,v}, \qquad \mathfrak{g} = \prod_{v|\infty} \mathfrak{h}_v, \qquad \pi_\infty = \bigotimes_{v|\infty} \pi_v.$$

If, in addition, $\xi = \bigotimes_{v|\infty} \xi_v$ with $\xi_v : (H_v)_{\mathbf{C}} \longrightarrow GL(N_v)_{\mathbf{C}}$, the Künneth formula

$$H^*(\mathfrak{g}, K_\infty; \pi_\infty \otimes \xi) = \bigotimes_{v|\infty} H^*(\mathfrak{h}_v, K_{\infty,v}; \pi_v \otimes \xi_v)$$

combined with (0.3.1) implies that the analytic cohomology $H^*(Sh_K(G,\mathscr{X})^{an}, \mathscr{L}_\xi)$ admits a natural decomposition as a finite direct sum of tensor products $\bigotimes_{v|\infty}$.

(0.10) Such a “weak Künneth decomposition” is expected to be of a motivic origin (see the discussion in [NS,§6]). In particular, there should be a finite extension E'/E depending only on $(G,\mathscr{X})$ and F for which the Galois representation $H^*_{et}(Sh_K(G,\mathscr{X}) \otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell})$ restricted to $\Gamma_{E'}$ is isomorphic to a direct sum of tensor products $\bigotimes_{v|\infty}$ of representations of $\Gamma_{E'}$. Equivalently, each $V(\pi^\infty) = \bigoplus_i V^i(\pi^\infty)$ in (0.4.1) should be of this form.

This behaviour is expected to reflect geometry of the integral models $S_{K,\mathfrak{p}}$ from (0.5.1) at those primes $\mathfrak{p}$ of E (of good reduction) which split completely in E'/E. For such primes the Frobenius morphism on the special fibre should be a product of partial Frobenius morphisms, for which a refinement of (0.5.2) and (0.5.3) should be valid. See 5.16 and the Appendix for a discussion of this phenomenon in a special case.

(0.11) Eichler-Shimura relations in this context lead to identities of the form

$$P_{(\rho_1\otimes\cdots\otimes\rho_r)(\mathrm{Fr}(\mathfrak{p}))}(\rho(\mathrm{Fr}(\mathfrak{p}))) = 0$$

and

$$\forall g \in \Gamma_{E'} \qquad P_{(\rho_1\otimes\cdots\otimes\rho_r)(g)}(\rho(g)) = 0 \qquad [\text{“}P_{\rho_1\otimes\cdots\otimes\rho_r}(\rho) = 0\text{”}] \tag{0.11.1}$$

which generalise (0.8.3-4). Above, $\rho = V^i(\pi^\infty) \otimes (\pi^\infty)^K|_{\Gamma_{E'}}$ and $\rho_1, \ldots, \rho_r$ are certain representations of $\Gamma_{E'}$ attached to π.

Note that the formulation (0.11.1) of Eichler-Shimura relations requires the knowledge of the existence of the Galois representations ρ_i.

(0.12) It is natural to consider the relation (0.11.1) in the following abstract context. Let $\rho_1, \ldots, \rho_r$ be irreducible finite-dimensional representations of Γ (which can be a group, a profinite group, a Lie algebra, an algebraic group etc.).

(Q1) If a finite-dimensional representation ρ of Γ satisfies $P_{\rho_1\otimes\cdots\otimes\rho_r}(\rho) = 0$, is it true that

$$\rho^{\mathrm{ss}} \subseteq \left((\rho_1 \otimes \cdots \otimes \rho_r)^{\oplus m}\right)^{\mathrm{ss}} \tag{0.12.1}$$

for some $m \geq 1$? Note that $\rho_1 \otimes \cdots \otimes \rho_r$ is automatically semisimple if the field of coefficients is of characteristic zero.

(Q2) If (0.12.1) holds, under what additional assumptions is ρ semisimple (i.e., when is $\rho = \rho^{\mathrm{ss}}$)?

In fact, the discussion in 0.10 leads naturally to the following, more precise version of (Q2).

(Q2') Let Γ be a profinite group containing a dense subset Σ such that, for each $g \in \Sigma$, $\rho(g) = u_1 \cdots u_r$, where u_i commute with each other and $P_{\rho_i(g)}(u_i) = 0$ for all $i = 1, \ldots, r$ (u_i corresponds to the action of a partial Frobenius morphism). Under what additional assumptions is ρ semisimple?

If the polynomials $P_{\rho_i(g)}(X)$ ($g \in \Sigma$) have distinct roots, then each u_i is semisimple, and so is their product $\rho(g) = u_1 \cdots u_r$, for all $g \in \Sigma$. It is then natural to ask the following question.

(Q3) If a finite-dimensional representation ρ of Γ satisfies $P_{\rho_1 \otimes \cdots \otimes \rho_r}(\rho) = 0$ and if $\rho(\Gamma)$ contains many semisimple elements, under what additional assumptions is ρ semisimple?

(0.13) Boston, Lenstra and Ribet [BLR] showed that both questions (Q1) and (Q2) have a positive answer if $r = 1$ and ρ_1 is a two-dimensional absolutely irreducible representation of a group Γ. Their result and its applications were inspired by [Mz, Prop. 14.2]. Dimitrov [Di, Lemma 6.5] considered a variant of question (Q1) for certain two-dimensional representations $\rho_1, \ldots, \rho_r : \Gamma \longrightarrow GL_2(\mathbf{F}_q)$.

For certain higher-dimensional representations $\rho_1 : \Gamma \longrightarrow GL_n(\overline{\mathbf{F}}_p)$ of a finite group Γ which have a sufficiently large image, Emerton and Gee [EG, Sect. 4] showed that (Q1) for $r = 1$ has a positive answer.

(0.14) It may be helpful to keep in mind the following two toy models.

(0.15) A toy model for (Q1). If Γ is the Lie algebra $sl(2)$ (over a field of characteristic zero), every finite-dimensional representation of Γ is semisimple and the irreducible representations $V_n = Sym^n(V_1)$ ($\dim(V_n) = n + 1$) are indexed by their highest weights $n \in \mathbf{N}$. As

$$\{\text{weights of } V_n\} = \{n, n-2, \ldots, 2-n, -n\}$$

(all weights occurring with multiplicity one), we have

$$P_{V_m}(V_n) = 0 \iff \{\text{weights of } V_n\} \subset \{\text{weights of } V_m\} \iff n \le m, \quad n \equiv m \pmod 2.$$

In particular, question (Q1) has a negative answer even for $r = 1$ (namely, for $\rho_1 = V_n$, $n > 1$). On the other hand, the Clebsch-Gordan formula

$$V_n \otimes V_1 = V_{n+1} \oplus V_{n-1}$$

implies that

$$P_{V_1^{\otimes r}}(V_n) = 0 \iff \{\text{weights of } V_n\} \subset \{\text{weights of } V_1^{\otimes r}\} \iff n \le r, \quad n \equiv r \pmod 2 \iff V_n \subset V_1^{\otimes r}.$$

In other words, question (Q1) has a positive answer if $\rho_1 = \cdots = \rho_r = V_1$, for any $r \ge 1$.

The key point is that V_1 is a **minuscule representation** of $sl(2)$. This is generalised in Proposition 1.6 to representations of arbitrary split reductive Lie algebras. In Proposition 3.10 we deduce from this result an affirmative answer to question (Q1) for representations $\rho_1, \ldots, \rho_r$ of a profinite group which have a large image.

(0.16) A toy model for (Q3). Assume that Γ is a profinite group, $r = 2$, $\rho_1 = \rho_2$, $\dim(\rho_1) = 2$, $\rho_1(\Gamma) \subset GL_2(\overline{\mathbf{Q}}_\ell)$ is big (for example, contains a conjugate of an open subgroup of $SL_2(\mathbf{Z}_\ell)$) and the semisimplification of ρ is of the form $\rho^{\mathrm{ss}} \simeq \rho_1^{\otimes 2} = Sym^2(\rho_1) \oplus \wedge^2(\rho_1)$.

We have (possibly after dualising if necessary) an exact sequence of representations of Γ

$$0 \longrightarrow Sym^2(\rho_1) \longrightarrow \rho \longrightarrow \wedge^2(\rho_1) \longrightarrow 0. \tag{0.16.1}$$

After choosing a splitting of (0.16.1) we can write ρ in a matrix form as

$$\rho(g) = \begin{pmatrix} Sym^2(\rho_1(g)) & * \\ 0 & \det(\rho_1(g)) \end{pmatrix}, \qquad * = \begin{pmatrix} * \\ * \\ * \end{pmatrix} = c(g)\det(\rho_1(g)),$$

where $c \in Z^1(\Gamma, \mathrm{Hom}(\wedge^2(\rho_1), Sym^2(\rho_1)))$ is a 1-cocycle describing the extension (0.16.1).

If the matrix $\rho_1(g) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is diagonal, then

$$\rho(g) = \begin{pmatrix} a^2 & & & x \\ & ab & & y \\ & & b^2 & z \\ & & & ab \end{pmatrix},$$

which means that

$$\rho(g) \text{ is semisimple } \iff y = 0.$$

In particular, the condition "$\rho(g)$ is semisimple for many $g \in \Gamma$" imposes many non-trivial constraints on the cocycle c. A natural guess would be that these constraints force c to be a coboundary (which implies that ρ is isomorphic to $\rho_1^{\otimes 2}$, and therefore is semisimple). This is indeed the case. A general version of this argument (which uses in a crucial way the fact that $Sym^2(\rho_1(g))$ and $\wedge^2(\rho_1(g))$ have a common eigenvalue) is given in Theorem 2.4 below (note that our toy model corresponds to Example 2.5 for $sl(2)$ and the adjoint representation).

(0.17) In 0.16 we considered a special case of (Q3) when we knew in advance that (Q1) had a positive answer (a fairly general result in this direction is proved in Theorem 3.3 below). In his thesis at Université Pierre et Marie Curie, K. Fayad [Fa] shows that (Q2') has a positive answer in many cases when (Q1) does not.

(0.18) In this article we consider questions (Q1), (Q2) and (Q3) for representations of a profinite group Γ with coefficients in $\overline{\mathbf{Q}}_\ell$. Our main abstract results proved in §3 involve a passage to Lie algebras and an application of the general results on Lie algebra representations proved in §1 and §2. This means that the assumptions are far from being optimal. In §4 we consider the simplest possible case of induced representations, when the methods of §3 do not apply.

(0.19) In §5 we combine the results of §3 and §4 with Eichler-Shimura relations on quaternionic Shimura varieties $Sh(G, \mathscr{X})$, for which $G = D^\times$ is the multiplicative group of a quaternion algebra D over a totally real number field F. As mentioned in 0.6, we recover only a weak form of the results obtained by the Langlands-Kottwitz method (Theorem 5.18, Theorem 5.20(1),(2)).

However, we are able to show (Theorem 5.20(3)) that the Galois action on the full (cuspidal) étale cohomology with coefficients in $\mathscr{L}_{\xi,\ell}$ of these Shimura varieties is semisimple, using the Eichler-Shimura relations proved in §5 and §6 of the Appendix. This result is new already in the special case of cuspidal cohomology of Hilbert modular varieties.

In §6 we study étale cohomology of closely related quaternionic Shimura varieties $Sh(G^*, \mathscr{X}^*)$, where G^* is the subgroup of $D^\times$ consisting of elements whose reduced norm lies in $\mathbf{Q}^\times$. For local systems $\mathscr{L}_{\xi^*,\ell}$ of non-motivic weight (i.e., for those whose Tate twists do not extend to $Sh(G, \mathscr{X})$) we show an analogous semisimplicity result, but only for the non CM part of the cohomology.

Partial results on semisimplicity of the Galois action on certain subspaces of non-endoscopic étale cohomology of unitary Shimura varieties are proved in K. Fayad's thesis [Fa]. These results are further generalised in [FaN].

(0.20) The general formalism of Eichler-Shimura relations on essentially PEL Shimura varieties at split primes is discussed in the Appendix.

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1. Lie algebras

(1.1) Notation and conventions (see [LIE, Ch. VI, VIII]). Let $\mathfrak{g}$ be a split semisimple Lie algebra over a field $k \supset \mathbf{Q}$. A choice of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ determines the following objects: the set of roots $R \subset \mathfrak{h}^*$, the root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^\alpha, \qquad \mathfrak{g}^\alpha = k \cdot X_\alpha,$$

the root lattice $Q = \Sigma_{\alpha \in R} \mathbf{Z}\alpha \subset \mathfrak{h}^*$, the dual root system $R^\vee = \{\alpha^\vee \mid \alpha \in R\} \subset \mathfrak{h}$ satisfying $\langle \alpha^\vee, \alpha \rangle = 2$ for all $\alpha \in R$, the coroot lattice $Q^\vee = \Sigma_{\alpha^\vee \in R^\vee} \mathbf{Z}\alpha^\vee \subset \mathfrak{h}$ (= the root lattice of $R^\vee$), the weight lattice $P \supset Q$ (the $\mathbf{Z}$-dual of $Q^\vee$) and the coweight lattice $P^\vee \supset Q^\vee$.

A choice of a Weyl chamber $C \subset \mathfrak{h}^*$ is equivalent to a decomposition $R = R_+ \cup R_-$, where $R_+ = \{\alpha \in R \mid \langle \alpha^\vee, C \rangle \geq 0\}$ (resp. $R_- = -R_+$) is the set of positive (resp. negative roots). Such a decomposition also determines the set $\Delta \subset R$ of simple roots (which forms a $\mathbf{Z}$-basis of Q), a decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \qquad \mathfrak{n}_\pm = \bigoplus_{\alpha \in R_\pm} \mathfrak{g}^\alpha,$$

the monoid of dominant weights $P_{++} = P \cap \overline{C} \subset P_+ = P \cap \sum_{\alpha \in R_+} \mathbf{Q}_{\geq 0}\, \alpha = P \cap \sum_{\alpha \in \Delta} \mathbf{Q}_{\geq 0}\, \alpha \subset P$ [LIE, Ch. VI, §1, no. 6], submonoids $Q_{++} = P_{++} \cap Q \subset Q_+ = P_+ \cap Q = \sum_{\alpha \in R_+} \mathbf{Z}_{\geq 0}\, \alpha = \sum_{\alpha \in \Delta} \mathbf{Z}_{\geq 0}\, \alpha \subset Q$ and a partial order on $x + Q$ (for any $x \in P$) given by $\lambda \leq \mu \iff \mu - \lambda \in Q_+$. Set $Q_- = -Q_+ \subset Q$.

The universal envelopping algebra

$$U(\mathfrak{n}_\pm) = \bigoplus_{\mu \in Q_\pm} U(\mathfrak{n}_\pm)_\mu$$

has a natural grading $\deg(X_{\alpha_1} \cdots X_{\alpha_r}) = \alpha_1 + \cdots + \alpha_r$ by $Q_\pm$.

(1.2) More generally, if $\mathfrak{g}$ is a split reductive Lie algebra over k (of finite dimension), then $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathscr{D}\mathfrak{g}$ is a direct sum of the centre $\mathfrak{z}(\mathfrak{g})$ with the derived Lie algebra $\mathscr{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, which is split semisimple. Any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is of the form $\mathfrak{h} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{h}'$, where $\mathfrak{h}'$ is a Cartan subalgebra of $\mathscr{D}\mathfrak{g}$. The "root lattice" of $(\mathfrak{g}, \mathfrak{h})$ is the subgroup of $\mathfrak{h}^*$ generated by the weights of $\mathfrak{h}$ occurring in the adjoint representation of $\mathfrak{g}$. It is a lattice in $\mathrm{Ker}(\mathfrak{h}^* \longrightarrow \mathfrak{z}(\mathfrak{g})^*)$ and it corresponds to the root lattice of $\mathscr{D}\mathfrak{g}$ under the natural isomorphism between the latter space and $\mathfrak{h}'^*$.

A (non-zero) finite-dimensional simple $\mathfrak{g}$-module V will be called *minuscule* if its highest weight – considered as a weight of $\mathfrak{h}'$ – is a minuscule weight of the semisimple algebra $\mathscr{D}\mathfrak{g}$ (in other words, the weights of $\mathfrak{h}'$ occurring in V form a single orbit under the action of the Weyl group of $(\mathscr{D}\mathfrak{g}, \mathfrak{h}')$; see [LIE, Ch. VIII, §7, no. 3]).

(1.3) Proposition. *Let $\mathfrak{g}$ be as in 1.1, let V be a non-zero simple $\mathfrak{g}$-module of finite dimension. Write $V = \oplus_\mu V(\mu)$, where $V(\mu)$ denotes the subspace of weight $\mu \in \mathfrak{h}^*$ with respect to the action of $\mathfrak{h}$. Let $M \subset \mathrm{End}(V)$ be a non-zero $\mathfrak{g}$-submodule. Then:*
(1) *The weight zero subspace $M(0) = M^{\mathfrak{h}} \subset \mathrm{End}(V)(0) = \mathrm{End}(V)^{\mathfrak{h}} = \oplus_\mu \mathrm{End}(V(\mu))$ is non-zero.*
(2) *More precisely, if $\lambda \in P_{++}$ is the highest weight of V, then*

$$\mathrm{Im}\,(M(0) \longrightarrow \mathrm{End}(V(\lambda)) \neq 0.$$

Proof. (1) We can suppose that M is simple. All the weights of V are contained in $\lambda - Q_+$, which implies that the weights of $\mathrm{End}(V)$ (in particular, the highest weight λ_M of M) belong to Q. As $\lambda_M \in Q_{++} \subset Q_+$, it follows from [LIE, Ch. VIII, §7, Prop. 5] that $0 \in P_{++} \cap (\lambda_M - Q_+)$ occurs as a weight in M.
(2) Each element $m \in M(0)$ is of the form $m = (m_\mu)$, where $m_\mu \in \mathrm{End}(V(\mu))$. According to (1), the set $S = \{\mu \mid \exists\ m \in M(0) \quad m_\mu \neq 0\} \subset \lambda - Q_+$ is not empty. Let μ be a maximal element of S (i.e., $\forall \nu \in Q_+ \smallsetminus \{0\} \quad \mu + \nu \notin S$). If $\mu = \lambda$, then we are done. Assume that $\mu \neq \lambda$ and fix $m \in M(0)$ such that $m_\mu \neq 0$. Recall that $X \in \mathfrak{g}$ acts on $f \in \mathrm{End}(V)$ by $X * f = X \circ f - f \circ X$. This implies that, for any positive

roots $\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_s\in R_+$ ($r,s\geq 1$) such that $\alpha_1+\cdots+\alpha_r=\beta_1+\cdots+\beta_s=\nu$ ($\nu\in Q_+$), the element $m'=X_{\alpha_1}*\cdots*X_{\alpha_r}*Y_{\beta_1}*\cdots*Y_{\beta_s}*m\in M(0)$ satisfies

$$m'_{\mu+\nu}=\pm X_{\alpha_1}\circ\cdots\circ X_{\alpha_r}\circ m_\mu\circ Y_{\beta_s}\circ\cdots\circ Y_{\beta_1},$$

by the maximality of $\mu\in S$. As $m'_{\mu+\nu}=0$ (again by the maximality of μ), we deduce that

$$\forall\nu\in Q_+\smallsetminus\{0\}\quad U(\mathfrak{n}_+)_\nu\circ m_\mu\circ U(\mathfrak{n}_-)_{-\nu}=0\in\mathrm{End}(V(\mu+\nu)).$$

Taking $\nu=\lambda-\mu$ and using the equality $U(\mathfrak{n}_-)_{\mu-\lambda}V(\lambda)=V(\mu)$ we obtain that

$$\mathrm{Im}(m_\mu)\subset N(\mu):=\bigcap_{z\in U(\mathfrak{n}_+)_{\lambda-\mu}}\mathrm{Ker}\,(z:V(\mu)\longrightarrow V(\lambda))\,.$$

An easy induction shows that $N(\mu')=0$ for all $\mu'\in\lambda-Q_+$: indeed, $N(\lambda)=0$ by definition, and if $\mu'\neq\lambda$ but $N(\mu'+\alpha)=0$ for all $\alpha\in\Delta$, then $X_\alpha\,N(\mu')\subset N(\mu'+\alpha)=0$ for all such α, which means that $N(\mu')\subset V(\mu')\cap\{\text{highest weight vectors}\}=\mathrm{V}(\mu')\cap\mathrm{V}(\lambda)=0$. In particular, $N(\mu)=0$, which implies that $m_\mu=0$, contrary to our assumption $\mu\in S$. This contradiction shows that $\mu=\lambda$, as claimed.

(1.4) Both statements of Proposition 1.3 still hold if we merely assume that $\mathfrak{g}$ is a reductive Lie algebra (of finite dimension) and V is a (non-zero) simple $\mathfrak{g}$-module. In this case $\mathfrak{g}=\mathfrak{z}(\mathfrak{g})\oplus\mathscr{D}\mathfrak{g}$, each element of $\mathfrak{z}(\mathfrak{g})$ acts on V by a scalar, and the action of $\mathfrak{g}$ on $\mathrm{End}(V)$ factors through $\mathscr{D}\mathfrak{g}$.

(1.5) There is an analogue of Proposition 1.3 (in the form 1.4) in which $\mathfrak{g}$ is replaced by a split connected reductive group G over k, $\mathfrak{h}$ by a split maximal torus $T\subset G$ and V by a non-zero irreducible rational representation of G. The Lie algebra $\mathfrak{g}$ of G then acts on V and $\mathrm{End}(V)$, the weight subspaces for T and its Lie algebra $\mathfrak{h}$ (which is a Cartan subalgebra of $\mathfrak{g}$) coincide and G-submodules of $\mathrm{End}(V)$ are the same as $\mathfrak{g}$-submodules.

(1.6) Proposition. *Let $(\mathfrak{g},\mathfrak{h})$ be as in 1.2, let M be a finite direct sum of finite tensor products of one-dimensional or minuscule simple $\mathfrak{g}$-modules. If $N\neq 0$ is a simple $\mathfrak{g}$-module with the property that each weight of $\mathfrak{h}$ occurring in N occurs in M, then N is isomorphic to a submodule of M.*

Proof. Let ν be the highest weight of N. By assumption, ν occurs in M, hence in $M'=M_1\otimes\cdots\otimes M_r\subset M$, where each $\mathfrak{g}$-module M_i is one-dimensional or minuscule, which means that the weights occurring in M_i form a single orbit of the Weyl group W of $(\mathfrak{g},\mathfrak{h})$. If we denote by μ_i the highest weight of M_i, then $\nu=w_1(\mu_1)+\cdots+w_r(\mu_r)$ for some $w_i\in W$, since ν occurs in M'. The statement of the conjecture of Parthasarathy, Ranga Rao and Varadarajan (proved in [Ku] and [Ma]) then implies that there is an injective morphism of $\mathscr{D}\mathfrak{g}$-modules $f:N\hookrightarrow M'$. The centre $\mathfrak{z}(\mathfrak{g})$ acts on N (resp. on M') by a single weight equal to $\nu|_{\mathfrak{z}(\mathfrak{g})}$ (resp. to $\sum_{i=1}^r\mu_i|_{\mathfrak{z}(\mathfrak{g})}$). These two elements of $\mathfrak{z}(\mathfrak{g})^*$ coincide (again, since ν occurs in M'), which means that f is a morphism of $\mathfrak{g}$-modules.

(1.7) Let V be a finite-dimensional vector space over a field $k\supset\mathbf{Q}$ and $\mathfrak{g}\subset\mathrm{End}_k(V)$ a k-Lie subalgebra. As in [LIE, Ch. VII, §5, no. 3], denote by $\mathfrak{n}_V(\mathfrak{g})$ the set of all elements of the radical of $\mathfrak{g}$ that are nilpotent in $\mathrm{End}_k(V)$. It is a nilpotent ideal of $\mathfrak{g}$ containing the intersection of the radical with $\mathscr{D}\mathfrak{g}$.

Recall that $\mathfrak{g}$ is a **decomposable linear Lie algebra** [LIE, Ch. VII, §5, Def. 1] if both the semisimple and the nilpotent part of every element of $\mathfrak{g}$ belong to $\mathfrak{g}$. The following facts will be used in §2.

(1.8) Proposition. (1) *[LIE, Ch. VII, §5, Thm. 2] The Lie algebra $\mathfrak{g}\subset\mathrm{End}_k(V)$ is decomposable $\iff$ some ($\iff$ each) Cartan subalgebra of $\mathfrak{g}$ is decomposable $\iff$ the radical of $\mathfrak{g}$ is decomposable.*
(2) *[LIE, Ch. VII, §5, Thm. 1] If $\mathfrak{g}$ is generated as a k-Lie algebra by a subset $S\subset\mathfrak{g}$ such that every $X\in S$ is either semisimple or nilpotent in $\mathrm{End}_k(V)$, then $\mathfrak{g}$ is decomposable.*
(3) *[LIE, Ch. VII, §5, Prop. 7] If $\mathfrak{g}$ is decomposable, then there exists a Lie subalgebra $\mathfrak{m}\subset\mathfrak{g}$, reductive in $\mathrm{End}_k(V)$ (in particular, $\mathfrak{m}$ is a reductive Lie algebra acting semisimply on V), such that $\mathfrak{g}=\mathfrak{m}\ltimes\mathfrak{n}_V(\mathfrak{g})$.*
(4) *[LIE, Ch. VII, §3, Ex. 16] The set of elements of $\mathfrak{g}$ that are semisimple in $\mathrm{End}_k(V)$ is Zariski dense in $\mathfrak{g}$ $\iff$ some ($\iff$ each) Cartan subalgebra of $\mathfrak{g}$ is commutative and consists of elements that are semisimple in $\mathrm{End}_k(V)$.*

(1.9) Corollary. *If the set of elements of $\mathfrak{g}$ that are semisimple in $\mathrm{End}_k(V)$ is Zariski dense in $\mathfrak{g}$, then:*
(1) *$\mathfrak{g}$ is decomposable;*

(2) $\mathfrak{g} = \mathfrak{m} \ltimes \mathfrak{n}_V(\mathfrak{g})$ *for a suitable reductive Lie subalgebra* $\mathfrak{m} \subset \mathfrak{g}$ *acting semisimply on* V*;*
(3) *There exists a flag* $\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_s = V$ *of* $\mathfrak{g}$*-submodules such that* $\mathfrak{n}_V(\mathfrak{g})$ *acts trivially (and* $\mathfrak{m}$ *semisimply) on* $\mathrm{gr}(V) = \bigoplus_{i=1}^{s} V_i/V_{i-1}$ *(one such flag is* $V_0 = \{0\}$ *and* $V_{i+1} = \{v \in V \mid \mathfrak{n}_V(\mathfrak{g})\, v \in V_i\}$*). The isomorphism class of the semisimple* $\mathfrak{g}/\mathfrak{n}_V(\mathfrak{g})$*-module* $\mathrm{gr}(V)$ *does not depend on the choice of* $\{V_i\}$*.*

2. Semisimplicity criteria for Lie algebra representations

(2.1) Proposition. *Let* $\mathfrak{g}_1, \ldots, \mathfrak{g}_m$ *be simple Lie algebras (of finite dimension) over a field* $k \supset \mathbf{Q}$*. Let* $\mathfrak{g} \subset \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_m$ *be a Lie subalgebra which projects surjectively on each factor* $\mathfrak{g}_i$*. Then there exist:*
- *a partition* $I = \{1, \ldots, m\} = I_1 \,\dot\cup\, \cdots \,\dot\cup\, I_n$ *(for non-empty subsets* $I_j \subset I$*),*
- *for each* $j \in J = \{1, \ldots, n\}$ *a Lie algebra* $\mathfrak{g}^{(j)}$*,*
- *for each* $j \in \{1, \ldots, n\}$ *and each* $i \in I_j$ *an isomorphism of Lie algebras* $f_{ji} : \mathfrak{g}^{(j)} \xrightarrow{\sim} \mathfrak{g}_i$*,*

such that

$$\mathfrak{g} = \mathrm{Im}\left(\prod_{j\in J} \mathfrak{g}^{(j)} \xrightarrow{\Delta} \prod_{j\in J} \left(\mathfrak{g}^{(j)}\right)^{I_j} \xrightarrow{f} \prod_{j\in J}\prod_{i\in I_j} \mathfrak{g}_i = \prod_{i\in I} \mathfrak{g}_i\right),$$

where $\Delta = (\Delta_j)_{j\in J}$*, each* $\Delta_j : \mathfrak{g}^{(j)} \longrightarrow \left(\mathfrak{g}^{(j)}\right)^{I_j}$ *is the diagonal map and* $f = (f_j)_{j\in J}$ *is a Lie algebra isomorphism with components* $f_j = (f_{ji})_{i\in I_j}$*.*

Proof. This is well-known. If we denote by p_i the projection map $\mathfrak{g} \hookrightarrow \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_m \longrightarrow \mathfrak{g}_i$, then the image $p_i(\mathfrak{n})$ of any abelian ideal $\mathfrak{n} \subset \mathfrak{g}$ is an abelian ideal of $\mathfrak{g}_i$, hence $p_i(\mathfrak{n}) = 0$ for all i, which implies that $\mathfrak{n} = 0$ and $\mathfrak{g}$ is semisimple, thus $\mathfrak{g} = \mathfrak{g}^{(1)} \times \cdots \times \mathfrak{g}^{(n)}$ for simple Lie algebras $\mathfrak{g}^{(j)}$. For each $j \in J = \{1, \ldots, n\}$ the set

$$I_j = \{i \in I \mid p_i(\mathfrak{g}^{(j)}) \neq 0\} = \{i \in I \mid p_i \text{ induces an isomorphism } \mathfrak{g}^{(j)} \xrightarrow{\sim} \mathfrak{g}_i\}$$

is non-empty. If $j \neq j'$ and $i \in I_j \cap I_{j'}$, then

$$\forall X \in \mathfrak{g}^{(j)} \quad \forall X' \in \mathfrak{g}^{(j')} \quad [f_{ji}(X), f_{j'i}(X')] = p_i([X, X']) = 0 \in \mathfrak{g}_i,$$

hence $[\mathfrak{g}_i, \mathfrak{g}_i] = 0$, which is not true. This contradiction implies that $I_j \cap I_{j'} = \emptyset$. As $\bigcup_j I_j = I$, the sets $I_1, \ldots, I_n$ form a partition of I. The rest of the proposition follows from the previous discussion.

(2.2) Proposition. *Let* $\mathfrak{g}_1, \ldots, \mathfrak{g}_m$ *be reductive Lie algebras (of finite dimension) over an algebraically closed field* $k \supset \mathbf{Q}$*. For each* $i \in I = \{1, \ldots, m\}$ *let* M_i *be a non-zero simple* $\mathfrak{g}_i$*-module of finite dimension. If* $\mathfrak{g} \subset \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_m$ *is a Lie subalgebra which projects surjectively on each factor* $\mathfrak{g}_i$*, then:*
- $\mathfrak{g}$ *is reductive,* $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathscr{D}\mathfrak{g}$*;*
- *each element of* $\mathfrak{z}(\mathfrak{g})$ *acts on* $M = M_1 \boxtimes \cdots \boxtimes M_m$ *by a scalar;*
- $\mathscr{D}\mathfrak{g}_i = \prod_t \mathfrak{g}_{i,t}$ *and* $M_i = \boxtimes_t M_{i,t}$*, where* $\mathfrak{g}_{i,t}$ *is a simple Lie algebra and* $M_{i,t}$ *is a simple* $\mathfrak{g}_{i,t}$*-module;*
- *applying Proposition 2.1 to* $\mathscr{D}\mathfrak{g} \subset \prod_{i,t} \mathfrak{g}_{i,t}$ *and replacing each* $\mathfrak{g}_{i,t}$ *(resp.* $M_{i,t}$*) for* $(i,t) \in I_j$ *by* $\mathfrak{g}^{(j)}$ *(resp. by the* $\mathfrak{g}^{(j)}$*-module* $N_{i,t} = f^*_{j,(i,t)}(M_{i,t})$*, where* $f_{j,(i,t)} : \mathfrak{g}^{(j)} \xrightarrow{\sim} \mathfrak{g}_{i,t}$ *is the isomorphism from Proposition 2.1), we have*

$$\mathscr{D}\mathfrak{g} = \mathrm{Im}\left(\prod_{j\in J} \mathfrak{g}^{(j)} \xrightarrow{\Delta=(\Delta_j)} \prod_{j\in J} \left(\mathfrak{g}^{(j)}\right)^{I_j} = \prod_{j\in J}\prod_{(i,t)\in I_j} \mathfrak{g}_{i,t} = \prod_{(i,t)} \mathfrak{g}_{i,t}\right);$$

- *if we identify* $\mathscr{D}\mathfrak{g}$ *with* $\prod_{j\in J} \mathfrak{g}^{(j)}$ *via* Δ*, then the* $\mathscr{D}\mathfrak{g}$*-module* M *is isomorphic to* $\boxtimes_{j\in J} M^{(j)}$*, where* $M^{(j)}$ *is the* $\mathfrak{g}^{(j)}$*-module*

$$M^{(j)} = \bigotimes_{(i,t)\in I_j} N_{i,t};$$

- *if* $\mathfrak{h}^{(j)} \subset \mathfrak{g}^{(j)}$ *is a Cartan subalgebra, then* $\mathfrak{h} = \prod_{j\in J} \mathfrak{h}^{(j)}$ *is a Cartan subalgebra of* $\mathscr{D}\mathfrak{g}$ *and* $\mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{h}$ *is a Cartan subalgebra of* $\mathfrak{g}$*. All weights of* $\mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{h}$ *occurring in* M *lie in* $\mu + \lambda + Q$*, where* Q *denotes the root* *lattice of* $\mathscr{D}\mathfrak{g}$, $\mu \in \mathfrak{z}(\mathfrak{g})^*$ *is the weight by which* $\mathfrak{z}(\mathfrak{g})$ *acts on* M, $\lambda = \sum_{j\in J}\sum_{(i,t)\in I_j}\lambda_{i,t}$ *and* $\lambda_{i,t}\in\mathfrak{h}^{(j)*}$ *is the highest weight of the* $\mathfrak{g}^{(j)}$*-module* $N_{i,t}$*;*

- *if each* $M_{i,t}$ *is either one-dimensional or a minuscule* $\mathfrak{g}_{i,t}$*-module, then each* $M^{(j)}$ *is a tensor product (possibly empty) of minuscule* $\mathfrak{g}^{(j)}$*-modules.*
- *if each* M_i *is a faithful* $\mathfrak{g}_i$*-module, then the following are equivalent:*

M is a simple $\mathfrak{g}$-module $\iff \mathscr{D}\mathfrak{g} = \mathscr{D}\mathfrak{g}_1\times\cdots\times\mathscr{D}\mathfrak{g}_r \iff \forall i\neq j\quad p_{ij}(\mathscr{D}\mathfrak{g}) = \mathscr{D}\mathfrak{g}_i\times\mathscr{D}\mathfrak{g}_j$,
where p_{ij} *denotes the projection onto the* i*-th and the* j*-th factors.*

Proof. We have, for each $i\in I$, $\mathfrak{g}_i = \mathfrak{z}(\mathfrak{g}_i)\oplus\mathscr{D}\mathfrak{g}_i$, where $\mathscr{D}\mathfrak{g}_i = [\mathfrak{g}_i,\mathfrak{g}_i]$ is semisimple. Decomposing each $\mathscr{D}\mathfrak{g}_i = \prod_t \mathfrak{g}_{i,r}$ into a product of simple Lie algebras and applying Proposition 2.1 to $\mathscr{D}\mathfrak{g}\subset\mathscr{D}\mathfrak{g}_1\times\cdots\times\mathscr{D}\mathfrak{g}_m$, we deduce that $\mathscr{D}\mathfrak{g}$ is semisimple, hence $\mathfrak{g}$ is reductive and $\mathfrak{g} = \mathfrak{z}(\mathfrak{g})\oplus\mathscr{D}\mathfrak{g}$. As each element of $p_i(\mathfrak{z}(\mathfrak{g})) = \mathfrak{z}(\mathfrak{g}_i)$ acts on M_i by a scalar, the same holds for the action of each element of $\mathfrak{z}(\mathfrak{g})$ on M. The remaining statements follow from Proposition 2.1 (and the fact that $M^{(j)}$ is a simple $\mathfrak{g}^{(j)}$-module $\iff |I_j| = 1$, by [Dy2, Thm. 3.2]).

(2.3) Corollary. *For any surjective morphism of Lie algebras* $f:\mathfrak{g}\longrightarrow\overline{\mathfrak{g}}$*, the weights of any Cartan subalgebra* $\overline{\mathfrak{h}}\subset\overline{\mathfrak{g}}$ *on* $\overline{M} = M/\mathrm{Ker}(f)M$ *lie in one coset modulo the root lattice of* $(\overline{\mathfrak{g}},\overline{\mathfrak{h}})$*.*

Proof. This follows from the corresponding statement for $\mathfrak{g}$ and M, the fact that $\mathrm{Ker}(f)\subset\mathfrak{g}$ is a Lie ideal and that $\mathfrak{g}$ is isomorphic to $\mathrm{Ker}(f)\times\overline{\mathfrak{g}}$.

(2.4) Theorem. *Let* $k\supset\mathbf{Q}$ *be a complete non-discrete non-archimedean field (for example,* $k=\mathbf{Q}_\ell$*) and* V *a non-zero* $\overline{k}$*-vector space of finite dimension. If* $\mathfrak{g}\subset\mathrm{End}_{\overline{k}}(V)$ *is a* k*-Lie subalgebra of finite dimension (over* k*) such that*
(H1) $\mathfrak{g}$ *contains a dense set (in the topology induced by the non-archimedean norm on* k*) of elements that are semisimple in* $\mathrm{End}_{\overline{k}}(V)$*,*
then:
(1) $\overline{\mathfrak{g}} = \overline{k}\cdot\mathfrak{g}\subset\mathrm{End}_{\overline{k}}(V)$ *is a decomposable linear Lie algebra over* $\overline{k}$*.*
(2) *Let* $\overline{\mathfrak{m}}\subset\overline{\mathfrak{g}}$ $(\overline{\mathfrak{m}}\simeq\overline{\mathfrak{g}}/\mathfrak{n}_V(\overline{\mathfrak{g}}))$*,* $\{0\} = V_0\subsetneq V_1\subsetneq\cdots\subsetneq V_s = V$ *and* $\mathrm{gr}(V)$ *be as in Corollary 1.9 (over the base field* $\overline{k}$*). Fix a Cartan subalgebra* $\overline{\mathfrak{h}}\subset\overline{\mathfrak{m}}$*. Assume that the following condition holds:*
(H2) *all weights of* $\overline{\mathfrak{h}}$ *occurring in* $\mathrm{gr}(V)$ *lie in one coset of the root lattice of* $(\overline{\mathfrak{m}},\overline{\mathfrak{h}})$*.*
Then $\mathfrak{n}_V(\overline{\mathfrak{g}}) = 0$*,* $\overline{\mathfrak{g}} = \overline{\mathfrak{m}}$ *is a reductive Lie algebra and* V *is a semisimple* $\overline{\mathfrak{g}}$*-module.*

(2.5) Example. Let $\mathfrak{g}_0$ be a reductive Lie algebra over $k=\overline{k}$ and M a faithful simple finite-dimensional $\mathfrak{g}_0$-module. If we identify the semidirect product $\mathfrak{g} = M\rtimes\mathfrak{g}_0$ with

$$\left\{\begin{pmatrix} X & m\\ 0 & 0\end{pmatrix}\ \middle|\ X\in\mathfrak{g}_0\subset\mathrm{End}(M),\ m\in M\right\}\subset\mathrm{End}(M\oplus k),$$

then $\mathfrak{g}$ will be a decomposable linear Lie algebra, $\mathfrak{n}_V(\mathfrak{g}) = M$ and $\mathfrak{m} = \mathfrak{g}_0$. The flag $\{0\} = V_0\subsetneq V_1 = M\oplus 0\subsetneq V_2 = V = M\oplus k$ is as in Corollary 1.9(3). If $\mathfrak{g}_0 = \mathfrak{s}l_2$ and M is the standard two-dimensional representation (resp. the adjoint representation), then (H1) is satisfied and (H2) is not (resp. (H2) is satisfied and (H1) is not).

Proof of Theorem 2.4. (1) There exist a finite subextension k'/k of $\overline{k}/k$ and a k'-vector space $V'\subset V$ of finite dimension such that $V'\otimes_{k'}\overline{k} = V$ and $\mathfrak{g}\subset\mathrm{End}_{k'}(V')$. Assumption (H1) implies that the set of all elements of $\mathfrak{g}$ that are semisimple in $\mathrm{End}_{k'}(V')$ is Zariski dense in $\mathfrak{g}$, hence in

$$\mathfrak{g}\otimes_k\overline{k}\subset\mathrm{End}_{k'}(V')\otimes_k\overline{k} = \bigoplus_{\sigma:k'\longrightarrow\overline{k}}\mathrm{End}_{\overline{k}}(V'\otimes_{k',\sigma}\overline{k}).$$

Taking the projection onto the factor corresponding to the inclusion of k' into $\overline{k}$, we deduce that $\overline{\mathfrak{g}}\subset\mathrm{End}_{\overline{k}}(V)$ satisfies the assumptions of Corollary 1.9 (over $\overline{k}$), which proves (1).
(2) The arguments in the proof of (1) show that we can replace k by $\overline{k}$, $\mathfrak{g}$ by $\overline{\mathfrak{g}}$ and the non-archimedean topology in (H1) by the Zariski topology. In other words, it is enough to prove part (2) of the following theorem.

(2.6) Theorem. *Let V be a non-zero vector space of finite dimension over an algebraically closed field $k \supset \mathbf{Q}$. If $\overline{\mathfrak{g}} \subset \mathrm{End}_k(V)$ is a k-Lie subalgebra such that*
(H1-ZAR) *$\overline{\mathfrak{g}}$ contains a Zariski dense set of elements that are semisimple in $\mathrm{End}_k(V)$,*
then:
(1) *$\overline{\mathfrak{g}} \subset \mathrm{End}_k(V)$ is a decomposable linear Lie algebra.*
(2) *Let $\overline{\mathfrak{n}} = \mathfrak{n}_V(\overline{\mathfrak{g}})$ and fix $\overline{\mathfrak{m}} \subset \overline{\mathfrak{g}}$ ($\overline{\mathfrak{m}} \simeq \overline{\mathfrak{g}}/\overline{\mathfrak{n}}$), a Cartan subalgebra $\overline{\mathfrak{h}} \subset \overline{\mathfrak{m}}$ and $\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_s = V$ as in Corollary 1.9. Assume that the following condition holds:*
(H2) *all weights of $\overline{\mathfrak{h}}$ occurring in $\mathrm{gr}(V)$ lie in one coset of the root lattice of $(\overline{\mathfrak{m}}, \overline{\mathfrak{h}})$.*
Then $\overline{\mathfrak{n}} = 0$, $\overline{\mathfrak{g}} = \overline{\mathfrak{m}}$ is a reductive Lie algebra and V is a semisimple $\overline{\mathfrak{g}}$-module.

Proof. (1) See the proof of Theorem 2.4(1).
(2) There is nothing to prove if $s = 1$. Assume that $s = 2$. The Lie algebra $\overline{\mathfrak{g}} = \overline{\mathfrak{m}} \ltimes \overline{\mathfrak{n}}$ satisfies $[\overline{\mathfrak{n}}, \overline{\mathfrak{n}}] = 0$ (since $s = 2$) and the normaliser of the abelian subalgebra $\overline{\mathfrak{h}} \times \overline{\mathfrak{n}}^{\overline{\mathfrak{h}}}$ in $\overline{\mathfrak{g}}$ is equal to $\overline{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}$, where $\widetilde{\mathfrak{n}} = \{X \in \overline{\mathfrak{n}} \mid \overline{\mathfrak{h}} \cdot X \in \overline{\mathfrak{n}}^{\overline{\mathfrak{h}}}\} = \overline{\mathfrak{n}}^{\overline{\mathfrak{h}}}$ (the last equality follows from the fact that $\overline{\mathfrak{h}}$ acts semisimply on $\mathrm{gr}(V) = V_1 \oplus V/V_1$, hence also on $\mathrm{Hom}_k(V/V_1, V_1) \supset \overline{\mathfrak{n}}$). In other words, $\overline{\mathfrak{h}} \times \overline{\mathfrak{n}}^{\overline{\mathfrak{h}}}$ is an abelian Cartan subalgebra of $\overline{\mathfrak{g}}$. Proposition 1.8(4) implies, therefore, that all elements of $\overline{\mathfrak{n}}^{\overline{\mathfrak{h}}}$ act semisimply on V, hence $\overline{\mathfrak{n}}^{\overline{\mathfrak{h}}} = 0$. The equality $\overline{\mathfrak{n}} = 0$ then follows from the fact (used already in the proof of Proposition 1.3(1)) that 0 occurs as a weight of $\overline{\mathfrak{h}}$ in any non-zero simple $\overline{\mathfrak{m}}$-module whose highest weight lies in the root lattice of $(\overline{\mathfrak{m}}, \overline{\mathfrak{h}})$ (in particular, in any non-trivial simple $\overline{\mathfrak{m}}$-submodule of $\overline{\mathfrak{n}} \subset \mathrm{End}_k(\mathrm{gr}(V))$). This shows that $\overline{\mathfrak{g}}$ is equal to $\overline{\mathfrak{m}}$, which is a reductive Lie algebra acting semisimply on V. This finishes the proof if $s = 2$.

Assume now that $s > 2$. We are going to prove the statement by induction on $d = \dim(V)$. The cases $d = 1, 2$ have already been treated. Assume that $d > 2$ and that the statement has been proved for all pairs $(V, \overline{\mathfrak{g}})$ with $\dim < d$. Consider the $\overline{\mathfrak{g}}$-submodule $V' = V_{s-1} \subsetneq V$ and the Lie algebra $\overline{\mathfrak{g}}' = \mathrm{Im}(\mathrm{res} : \overline{\mathfrak{g}} \longrightarrow \mathrm{End}_k(V'))$. Note that the pair $\overline{\mathfrak{g}}' \subset \mathrm{End}_k(V')$ satisfies (H1-ZAR). The image of the radical of $\overline{\mathfrak{g}}$ under the restriction map res is contained in the radical of $\overline{\mathfrak{g}}'$, which implies that $\mathrm{res}(\overline{\mathfrak{n}}) \subset \overline{\mathfrak{n}}' = \mathfrak{n}_{V'}(\overline{\mathfrak{g}}')$. As a result, res induces a surjective morphism of reductive Lie algebras $r : \overline{\mathfrak{m}} \xrightarrow{\sim} \overline{\mathfrak{g}}/\overline{\mathfrak{n}} \twoheadrightarrow \overline{\mathfrak{g}}'/\overline{\mathfrak{n}}'$. Fix a $\overline{\mathfrak{g}}'$-stable flag $\{0\} = V'_0 \subsetneq V'_1 \subsetneq \cdots \subsetneq V'_{s'} = V'$ such that $\overline{\mathfrak{n}}'$ acts trivially on $\mathrm{gr}(V') = \oplus_{i=1}^{s'} V'_i/V'_{i-1}$. All weights of $\overline{\mathfrak{h}} \subset \overline{\mathfrak{m}}$ on the semisimple $\overline{\mathfrak{g}}'/\overline{\mathfrak{n}}'$-module $\mathrm{gr}(V')$ lie in one coset modulo the root lattice of $(\overline{\mathfrak{m}}, \overline{\mathfrak{h}})$, thanks to (H2) and the fact that $\mathrm{gr}(V) \simeq \mathrm{gr}(V') \oplus V/V'$ as $\overline{\mathfrak{m}}$-modules. After choosing a decomposition $\overline{\mathfrak{m}} \xrightarrow{\sim} \mathrm{Ker}(r) \times \overline{\mathfrak{g}}'/\overline{\mathfrak{n}}'$, we obtain the same statement for an appropriate Cartan subalgebra of $\overline{\mathfrak{g}}'/\overline{\mathfrak{n}}'$, which means that the flag $\{V'_i\}$ and the Lie algebra $\overline{\mathfrak{g}}'$ satisfy (H2). As $\dim(V') < d$, the induction hypothesis implies that $\overline{\mathfrak{n}}' = 0$, $\overline{\mathfrak{g}}'$ is a reductive Lie algebra and $V' = \mathrm{gr}(V')$ is a semisimple $\overline{\mathfrak{g}}'$-module. This means that $\{0\} \subsetneq V' \subsetneq V$ is a flag of $\overline{\mathfrak{g}}$-submodules of length $s = 2$ satisfying the assumptions of Theorem 2.6(2). However, the case $s = 2$ was already treated, which concludes the proof.

3. Semisimplicity criteria for representations of profinite groups

(3.1) Let Γ be a profinite group, V a non-zero vector space of finite dimension over $\overline{\mathbf{Q}}_\ell$ and $\rho : \Gamma \longrightarrow \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V)$ a representation (continuous, according to the convention from 0.1).

In this situation $\rho(\Gamma)$ is a compact subgroup of $\mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V)$, which implies that there exists a finite extension E of $\mathbf{Q}_\ell$ contained in $\overline{\mathbf{Q}}_\ell$ and an E-structure $V_E \subset V$ (an E-vector subspace such that $V_E \otimes_E \overline{\mathbf{Q}}_\ell = V$) for which $\rho(\Gamma) \subset \mathrm{Aut}_E(V_E)$. According to a non-archimedean version of Lie's theorem [LIE, Ch. 3, §8, no. 2, Thm. 2], $\rho(\Gamma)$ is a (compact) Lie group of finite dimension over $\mathbf{Q}_\ell$. In particular, the profinite topology on $\rho(\Gamma)$ coincides with the topology induced by the ℓ-adic valuation on $\overline{\mathbf{Q}}_\ell$. The Lie algebra $\mathrm{Lie}(\rho(\Gamma)) \subset \mathrm{End}_E(V_E) \subset \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V)$ is a $\mathbf{Q}_\ell$-Lie algebra of finite dimension.

(3.2) The following properties of ρ are equivalent:

3.2.1. ρ is semisimple.

3.2.2. There exists an open subgroup $U \subset \Gamma$ such that $\rho|_U$ is semisimple.

3.2.3. V is a semisimple $\overline{\mathbf{Q}}_\ell \otimes_{\mathbf{Q}_\ell} \mathrm{Lie}(\rho(\Gamma))$-module.

This implies that the semisimplification ρ^{ss} of ρ satisfies $\rho^{\mathrm{ss}}|_U = (\rho|_U)^{\mathrm{ss}}$, for any open subgroup $U \subset \Gamma$.

(3.3) Theorem. *Let Γ be a profinite group, $V, W_1, \ldots, W_r$ non-zero vector spaces of finite dimension over $\overline{\mathbf{Q}}_\ell$ and $\rho : \Gamma \longrightarrow \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V)$, $\rho_i : \Gamma \longrightarrow \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(W_i)$ representations. Assume that the following conditions hold:*
(A) *Each ρ_i is strongly irreducible.*
(B) *The semisimplification ρ^{ss} of ρ is isomorphic to a subrepresentation of $(\rho_1 \otimes \cdots \otimes \rho_r)^{\oplus m}$, for some $m \geq 1$.*
(C) *There exists an open subgroup $\Gamma' \subset \Gamma$ for which $\rho(\Gamma')$ contains a dense subset consisting of semisimple elements of $\mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V)$.*
Then the representation ρ is semisimple, $\rho = \rho^{\mathrm{ss}}$.

Proof. Assumption (B) (resp. (C)) for ρ implies the corresponding condition for any subquotient of ρ. It is sufficient, therefore, to consider only the case when V is an extension of two irreducible representations of Γ. In addition, we can replace Γ by any of its open subgroups, thanks to 3.2. This implies that we can assume, after shrinking Γ and passing to another subquotient if necessary, that V sits in an exact sequence of $\overline{\mathbf{Q}}_\ell[\Gamma]$-modules

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V/V_1 \longrightarrow 0,$$

in which both V_1 and V/V_1 are strongly irreducible.

We want to show that Theorem 2.4 (for $s = 2$) applies to the $\mathbf{Q}_\ell$-Lie algebra $\mathfrak{g} = \mathrm{Lie}(\rho(\Gamma)) \subset \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V)$ and the $\overline{\mathbf{Q}}_\ell$-Lie algebra $\overline{\mathfrak{g}} = \overline{\mathbf{Q}}_\ell \cdot \mathfrak{g} \subset \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V)$. Firstly, both V_1 and V/V_1 are irreducible $\overline{\mathfrak{g}}$-modules. Secondly, condition (H1) in Theorem 2.4 is a consequence of (C). It follows that $\overline{\mathfrak{g}}$ is a decomposable Lie subalgebra of $\mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V)$. We distinguish two cases.

Case (a): there exists a $\overline{\mathfrak{g}}$-submodule $0 \subsetneq W \subsetneq V$ such that $W \neq V_1$. Irreducibility of V_1 and V/V_1 implies that $W \cap V_1 = 0$ and $W + V_1 = V$, hence $W \simeq V/V_1$ is a complementary $\overline{\mathfrak{g}}$-submodule to V_1; thus $V = V_1 \oplus W \simeq V \oplus V/V_1$ is a semisimple $\overline{\mathfrak{g}}$-module, which implies that ρ is semisimple.

Case (b): V_1 is the only proper non-zero $\overline{\mathfrak{g}}$-submodule of V. This uniqueness implies that the flag $0 \subset V_1 \subset V_2 = V$ is as in Proposition 1.9(3): the nilpotent ideal $\mathfrak{n}_V(\overline{\mathfrak{g}}) \subset \overline{\mathfrak{g}}$ acts trivially on $\mathrm{gr}(V) = V_1 \oplus V/V_1$. Moreover, both V_1 and V/V_1 are simple $\overline{\mathfrak{g}}/\mathfrak{n}_V(\overline{\mathfrak{g}})$-modules.

It remains to check condition (H2). Consider the representation $\rho_0 = \rho_1 \oplus \cdots \oplus \rho_r : \Gamma \longrightarrow \prod_{i=1}^r \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(W_i)$. Assumption (B) implies that $\mathrm{Ker}(\rho_0) \subset \mathrm{Ker}(\rho^{\mathrm{ss}})$, which yields surjective morphisms $\rho_0(\Gamma) \twoheadrightarrow \rho^{\mathrm{ss}}(\Gamma)$ and $\mathrm{Lie}(\rho_0(\Gamma)) \twoheadrightarrow \mathrm{Lie}(\rho^{\mathrm{ss}}(\Gamma))$. The $\overline{\mathbf{Q}}_\ell$-Lie subalgebra

$$\overline{\mathbf{Q}}_\ell \otimes_{\mathbf{Q}_\ell} \mathrm{Lie}(\rho_0(\Gamma)) \subset \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r, \qquad \mathfrak{g}_i = \overline{\mathbf{Q}}_\ell \otimes_{\mathbf{Q}_\ell} \mathrm{Lie}(\rho_i(\Gamma)),$$

and the $\mathfrak{g}_i$-modules $M_i = W_i$ satisfy the assumptions of Proposition 2.2 (for $k = \overline{\mathbf{Q}}_\ell$), thanks to (A). Corollary 2.3 tells us that all weights of a fixed Cartan subalgebra of $\overline{\mathbf{Q}}_\ell \otimes_{\mathbf{Q}_\ell} \mathrm{Lie}(\rho^{\mathrm{ss}}(\Gamma))$ on $\mathrm{gr}(V) \subset (W_1 \otimes \cdots \otimes W_r)^{\oplus m}$ lie in one coset of the root lattice of $\overline{\mathbf{Q}}_\ell \otimes_{\mathbf{Q}_\ell} \mathrm{Lie}(\rho^{\mathrm{ss}}(\Gamma))$. The action of this Lie algebra on $\mathrm{gr}(V)$ factors through $\mathrm{Im}(\overline{\mathfrak{g}} \longrightarrow \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(\mathrm{gr}(V)))$, which is a quotient of the reductive Lie algebra $\overline{\mathfrak{g}}/\mathfrak{n}_V(\overline{\mathfrak{g}})$. This implies, as in the proof of Corollary 2.3, that condition (H2) in Theorem 2.4 is satisfied. Applying Theorem 2.4, we conclude that $V = V_1 \oplus W \simeq V \oplus V/V_1$ is a semisimple $\overline{\mathfrak{g}}$-module, as in case (a) (more precisely, the above argument shows that case (b) does not occur).

(3.4) Condition (C) in Theorem 3.3 is satisfied if there is a dense subset $\Sigma \subset \Gamma'$ with the following property: for each $g \in \Sigma$ there exist polynomials $P_1, \ldots, P_r \in \overline{\mathbf{Q}}_\ell[X]$ without multiple roots and pairwise commuting elements $u_1, \ldots, u_r \in \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V(g))$, where $V(g) \supset V$ is a finite dimensional vector space over $\overline{\mathbf{Q}}_\ell$ (depending on g) such that $P_i(u_i) = 0$ for all $i = 1, \ldots, r$, V is stable under $u_1 \cdots u_r$ and $\rho(g) = u_1 \cdots u_r|_V$.

(3.5) For representations V occurring in cohomology of Shimura varieties (see §5 and A5 below) the group Γ is the Galois group of a suitable extension of the reflex field, the set Σ consists of Frobenius elements, $P_i(X) = P_{\rho_i(g)}(X)$ is the characteristic polynomial of $\rho_i(g)$ and u_i is induced by a partial Frobenius morphism acting on the special fibre of an integral model of the Shimura variety in question. In our abstract context, a sufficient (and also necessary) condition for these characteristic polynomials to be without multiple roots is as follows.

(3.6) Proposition. *In the situation of Theorem 3.3, assume that condition (A) holds. For each* $i = 1, \ldots, r$ *denote by* $\overline{\mathfrak{g}}_i = \overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}(\rho_i(\Gamma))$ *the image of* $\mathfrak{g}_i$ *in* $\mathrm{End}_{\overline{\mathbf{Q}}_\ell}(W_i)$. *This is a reductive Lie algebra whose centre is contained in* $\overline{\mathbf{Q}}_\ell \cdot \mathrm{id}$. *If, for each* $i = 1, \ldots, r$, *a fixed Cartan subalgebra* $\overline{\mathfrak{h}}_i$ *of* $\overline{\mathfrak{g}}_i$ *acts on* W_i *without multiplicities, then there exists an open subgroup* $\Gamma_0 \subset \Gamma$ *and an open dense subset* $U_0 \subset \Gamma_0$ *such that for all* $g \in U_0$ *and all* $a \geq 1$ *the characteristic polynomials* $P_{\rho_i(g^a)}(X)$ $(i = 1, \ldots, r)$ *are without multiple roots.*

Proof. Using the notation of the proof of Theorem 3.3, set $\overline{\mathfrak{g}}_0 = \overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}(\rho_0(\Gamma)) \subset \overline{\mathfrak{g}}_1 \times \cdots \times \overline{\mathfrak{g}}_r$; then $p_i(\overline{\mathfrak{g}}_0) = \overline{\mathfrak{g}}_i$. For each $i = 1, \ldots, r$, the polynomial function $\Delta_i : \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(W_i) \longrightarrow \overline{\mathbf{Q}}_\ell$ given by the discriminant of the characteristic polynomial is not identically equal to zero on $\overline{\mathfrak{g}}_i$, which implies that $\bigcap_{i=1}^r p_i^{-1}(\Delta_i^{-1}(\overline{\mathbf{Q}}_\ell \smallsetminus \{0\}))$ is a dense Zariski open in $\overline{\mathfrak{g}}_0$, hence its intersection U with $\mathfrak{g}_0 := \mathrm{Lie}(\rho_0(\Gamma))$ is a dense Zariski open in $\mathfrak{g}_0$. Fix a sufficiently small $\mathbf{Z}_\ell$-module of finite type $T_0 \subset \mathfrak{g}_0$ stable under the Lie bracket such that the exponential map induces a homeomorphism between T_0 and $\rho_0(\Gamma_0)$ (for an open subgroup $\Gamma_0 \subset \Gamma$) and that $\rho_0(\Gamma_0)$ acts trivially on $T/2\ell T$, for some $\rho_0(\Gamma_0)$-stable $\overline{\mathbf{Z}}_\ell$-lattice $T \subset \oplus_i W_i$. The image $\exp(T_0 \cap U)$ is open and dense in $\rho_0(\Gamma_0)$ (for the profinite topology), which implies that $U_0 = \rho_0^{-1}(\exp(T_0 \cap U))$ is open and dense in Γ_0. By construction, for each $g \in U_0$ and each $i = 1, \ldots, r$, the polynomial $P_{\rho_i(g)}(X)$ has distinct roots $\lambda_1, \ldots, \lambda_{d_i}$ contained in $1 + 2\ell\overline{\mathbf{Z}}_\ell$, which implies that the powers $\lambda_1^a, \ldots, \lambda_{d_i}^a$ are also distinct, since $1 + 2\ell\overline{\mathbf{Z}}_\ell$ contains no non-trivial roots of unity.

(3.7) Theorem. *In the situation of Theorem 3.3 with* $r = 1$, *assume that conditions (A), (B) from Theorem 3.3 and (C') below hold.*
(C') *There exists an open subgroup* $\Gamma' \subset \Gamma$ *such that* $P_{\rho_1(g)}(\rho(g)) = 0 \in \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V)$ *holds for all elements* g *of a certain dense subset of* Γ'.
Then the representation ρ *is semisimple (and isomorphic to* $\rho_1^{\oplus n}$, *for some* $n \geq 1$*).*

Proof. As any submodule of $\rho_1^{\oplus m}$ is isomorphic to $\rho_1^{\oplus m'}$, we can assume that $\rho^{\mathrm{ss}} = \rho_1^{\oplus m}$. By induction it is sufficient to consider the case $m = 2$:

$$0 \longrightarrow W_1 \longrightarrow V \longrightarrow W_1 \longrightarrow 0. \tag{3.7.1}$$

We can assume that $\Gamma = \Gamma'$, by 3.2. Condition (C') then implies, by continuity, that $P_{\rho_1(g)}(\rho(g)) = 0$ for all $g \in \Gamma$.

Choose a $\overline{\mathbf{Q}}_\ell$-linear splitting of the exact sequence (3.7.1) and denote by $G \subset \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V)$ (resp. $G_1 \subset \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(W_1)$) the Zariski closure of $\rho(\Gamma)$ (resp. $\rho_1(\Gamma)$). These are (the sets of $\overline{\mathbf{Q}}_\ell$-valued points of) affine algebraic groups over $\overline{\mathbf{Q}}_\ell$ such that

$$G = \left\{ g = \begin{pmatrix} g_1 & u \\ 0 & g_1 \end{pmatrix} \,\middle|\, g_1 \in G_1,\ u \in N \right\} = N \rtimes G_1, \tag{3.7.2}$$

for a suitable $\overline{\mathbf{Q}}_\ell$-vector subspace $N \subset \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(W_1)$. Our aim is to show that $N = 0$. We have, again by continuity,

$$\forall g = \begin{pmatrix} g_1 & u \\ 0 & g_1 \end{pmatrix} \in G \quad P_{g_1}(g) = 0. \tag{3.7.3}$$

Condition (A) implies that the connected component of the identity G_1° acts irreducibly on W_1, hence G_1° is a connected reductive group over $\overline{\mathbf{Q}}_\ell$ and its centre acts on W_1 by a character. Fix a maximal torus $T \subset G_1^\circ$. If $N \neq 0$, Proposition 1.3(2) (in its version 1.5) states that there is a weight $\lambda : T \longrightarrow \mathbf{G}_m$ which occurs in W_1 with multiplicity one and $n \in N^T$ with non-zero image n_λ under the composite map

$$N^T \subset \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(W_1)^T = \bigoplus_\mu \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(W_1(\mu)) \longrightarrow \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(W_1(\lambda)).$$

Choose $t \in T$ such that $\mu(t) \neq \lambda(t)$ for all weights $\mu \neq \lambda$ of T occurring in W_1. The element

$$g = \begin{pmatrix} t & n \\ 0 & t \end{pmatrix} \in G$$

satisfies

$$P_t(X) = \prod_\mu (X - \mu(t))^{\dim W_1(\mu)}, \qquad P_t(g) = \begin{pmatrix} 0 & n' \\ 0 & 0 \end{pmatrix}, \qquad n'_\lambda = n_\lambda \prod_{\mu \neq \lambda} (\lambda(t) - \mu(t))^{\dim W_1(\mu)} \neq 0,$$

which is impossible, by (3.7.3). This contradiction shows that $N = 0$, hence $\rho = \rho_1^{\oplus 2}$, as claimed.

(3.8) Questions. (1) Does Theorem 3.3 still hold if we relax assumption (A) by merely requiring each ρ_i to be irreducible and without multiplicities after restricting to open subgroups of Γ?
(2) Is there a common generalisation of Theorem 3.7 and the special case 3.4 of Theorem 3.3?

(3.9) We are now going to show that a variant of condition (B) of Theorem 3.3 (in its version 3.4) is a consequence of the other assumptions, provided the representations ρ_i have sufficiently large image,

(3.10) Proposition. *Assume that the representations ρ and ρ_i from Theorem 3.3 satisfy the following conditions.*
(A') *Each W_i is a direct sum of simple modules for the $\overline{\mathbf{Q}}_\ell$-Lie algebra $\overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}(\rho_i(\Gamma))$ (which is then reductive); each of these simple modules is one-dimensional or minuscule.*
(C') *There exist an integer $a \geq 1$, an open subgroup $\Gamma' \subset \Gamma$ and a dense subset $\Sigma \subset \Gamma'$ such that*

$$\forall g \in \Sigma \qquad P_{(\rho_1 \otimes \cdots \otimes \rho_r)(g^a)}(\rho(g^a)) = 0 \in \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V).$$

Then: (1) *There is an open subgroup $U \subset \Gamma'$ such that $\rho^{\mathrm{ss}}|_U = (\rho|_U)^{\mathrm{ss}}$ is isomorphic to a subrepresentation of $(\rho_1 \otimes \cdots \otimes \rho_r)|_U^{\oplus m}$, for some $m \geq 1$.*
(2) *If $\rho_1 \otimes \cdots \otimes \rho_r$ is strongly irreducible and $a = 1$ or $a = 2$, then every irreducible constituent of $\rho^{\mathrm{ss}}|_{\Gamma'}$ is isomorphic to $(\rho_1 \otimes \cdots \otimes \rho_r)|_{\Gamma'} \otimes \sigma$, for some character $\sigma : \Gamma' \longrightarrow \{\pm 1\}$ satisfying $\sigma^a = 1$.*
(3) *If $a = r = 1$ and W_1 is a minuscule representation of $\overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}(\rho_1(\Gamma))$, then ρ is semisimple and $\rho|_{\Gamma'}$ is isomorphic to $\rho_1|_{\Gamma'}^{\oplus n}$, for some $n \geq 1$.*

Proof. Thanks to 3.2, we can (and will) assume that $\Gamma' = \Gamma$. By continuity, (C') implies that

$$\forall g \in \Gamma \qquad P_{(\rho_1 \otimes \cdots \otimes \rho_r)(g^a)}(\rho(g^a)) = 0 \in \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V). \tag{3.10.1}$$

(1) We proceed in several steps.

Step 1: It is enough to consider the case when ρ is irreducible. Furthermore, after shrinking Γ if necessary we can assume that ρ is strongly irreducible. This means that V is a simple $\overline{\mathfrak{g}}$-module, where $\mathfrak{g} = \mathrm{Lie}(\rho(\Gamma))$ and $\overline{\mathfrak{g}} = \overline{\mathbf{Q}}_\ell \otimes_{\mathbf{Q}_\ell} \mathfrak{g}$. As in the proof of Theorem 2.4 we deduce that $\mathfrak{g}$ (resp. $\overline{\mathfrak{g}}$) is a reductive Lie algebra over $\mathbf{Q}_\ell$ (resp. over $\overline{\mathbf{Q}}_\ell$) and each element of the centre of $\overline{\mathfrak{g}}$ acts on V by a scalar. Consider the representation $\rho_0 = \rho_1 \oplus \cdots \oplus \rho_r : \Gamma \longrightarrow \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(W_0)$, where $W_0 = W_1 \oplus \cdots \oplus W_r$ (cf. the proof of Theorem 3.3).

The subgroup $\rho(\mathrm{Ker}(\rho_0)) \subset \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V)$ is a compact Lie group of finite dimension over $\mathbf{Q}_\ell$. The formula (3.10.1) implies that $(\rho(g^a) - 1)^N = 0$ for all $g \in \mathrm{Ker}(\rho_0)$ (where $N = \dim(W_1 \otimes \cdots \otimes W_r)$), which means that the Lie ideal $\mathfrak{a} = \overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}(\rho(\mathrm{Ker}(\rho_0)))$ in $\overline{\mathbf{Q}}_\ell \cdot \mathfrak{g} = \mathrm{Im}(\overline{\mathfrak{g}} \longrightarrow \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V))$ consists of nilpotent elements, hence is a nilpotent Lie ideal [LIE, Ch. I, §4, no. 2, Cor. 3], and so $\mathfrak{a} = 0$, since $\overline{\mathbf{Q}}_\ell \cdot \mathfrak{g}$ is reductive. It follows that $\rho(\mathrm{Ker}(\rho_0))$ is a finite subgroup of $\mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V)$, hence $\mathrm{Ker}(\rho_0) \cap \mathrm{Ker}(\rho)$ is an open subgroup of $\mathrm{Ker}(\rho_0)$. The surjection $\Gamma/(\mathrm{Ker}(\rho_0) \cap \mathrm{Ker}(\rho)) \twoheadrightarrow \Gamma/\mathrm{Ker}(\rho)$ yields a canonical surjection $f : \mathfrak{g}_0 = \mathrm{Lie}(\rho_0(\Gamma)) \twoheadrightarrow \mathfrak{g}$. We consider V and all W_i as irreducible representations of the reductive $\overline{\mathbf{Q}}_\ell$-algebra $\overline{\mathfrak{g}}_0 = \overline{\mathbf{Q}}_\ell \otimes_{\mathbf{Q}_\ell} \mathfrak{g}_0$.

Step 2: The $\mathbf{Q}_\ell$-Lie algebra $\mathfrak{g}_0 \subset \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r$ (where $\mathfrak{g}_i = \mathrm{Lie}(\rho_i(\Gamma))$) satisfies $p_i(\mathfrak{g}_0) = \mathfrak{g}_i$ for all $i = 1, \ldots, r$. After shrinking Γ if necessary we can assume that, for each $g \in \Gamma$, all eigenvalues of $\rho(g)$ and $\rho_i(g)$ ($i = 1, \ldots, r$) are contained in $1 + 2\ell\,\overline{\mathbf{Z}}_\ell$. After taking ℓ-adic logarithms $\rho_i(\Gamma) \longrightarrow \mathrm{Lie}(\rho_i(\Gamma))$ (and similarly for $\rho(\Gamma)$) we deduce from (3.10.1) that

$$\forall X=(X_1,\dots,X_r)\in\mathfrak{g}_0\subset\mathfrak{g}_1\times\cdots\times\mathfrak{g}_r\qquad P_{(X_1,\dots,X_r)|W_1\otimes\cdots\otimes W_r}(f(X))=0\in\mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V),$$

where $(X_1,\dots,X_r)\in\mathrm{End}_{\overline{\mathbf{Q}}_\ell}(W_1)\times\cdots\times\mathrm{End}_{\overline{\mathbf{Q}}_\ell}(W_r)$ acts on $W_1\otimes\cdots\otimes W_r$ by $\sum_i 1\otimes\cdots\otimes 1\otimes X_i\otimes 1\otimes\cdots\otimes 1$. As $\mathfrak{g}_0$ is Zariski dense in $\overline{\mathfrak{g}}_0$, we obtain that

$$\forall X=(X_1,\dots,X_r)\in\overline{\mathfrak{g}}_0\subset\overline{\mathfrak{g}}_1\times\cdots\times\overline{\mathfrak{g}}_r\qquad P_{(X_1,\dots,X_r)|W_1\otimes\cdots\otimes W_r}(\overline{f}(X))=0\in\mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V),\tag{3.10.2}$$

where $\overline{f}:\overline{\mathfrak{g}}_0\longrightarrow\overline{\mathfrak{g}}\twoheadrightarrow\overline{\mathbf{Q}}_\ell\cdot\mathfrak{g}\subset\mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V)$ is induced by f.

Step 3: For each $i=1,\dots,r$ we have $W_i=\oplus_{u_i}W_{i,u_i}$, where each W_{i,u_i} is a simple module for the $\overline{\mathbf{Q}}_\ell$-Lie algebra $\overline{\mathfrak{g}}_i=\overline{\mathbf{Q}}_\ell\otimes_{\mathbf{Q}_\ell}\mathfrak{g}_i$. The $\mathbf{Q}_\ell$-Lie algebras $\mathfrak{g}_{i,u_i}=\mathrm{Lie}(\mathrm{Im}(\Gamma\longrightarrow\mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(W_{i,u_i})))$ are reductive and $\mathfrak{g}_0\subset\prod_{i=1}^r\mathfrak{g}_i\subset\prod_{i,u_i}\mathfrak{g}_{i,u_i}$ is a subalgebra projecting surjectively on each of the factors.

Applying Proposition 2.2 to $\overline{\mathfrak{g}}_0\subset\prod_{i,u_i}\overline{\mathfrak{g}}_{i,u_i}$ and the simple $\overline{\mathfrak{g}}_{i,u_i}(=\overline{\mathbf{Q}}_\ell\otimes_{\mathbf{Q}_\ell}\mathfrak{g}_{i,u_i})$-modules W_{i,u_i}, we obtain an isomorphism of $\overline{\mathbf{Q}}_\ell$-Lie algebras $h:\mathfrak{g}^{(1)}\times\cdots\times\mathfrak{g}^{(s)}\xrightarrow{\sim}\mathscr{D}\overline{\mathfrak{g}}_0$, where each $\mathfrak{g}^{(j)}$ is a simple $\overline{\mathbf{Q}}_\ell$-Lie algebra and each $\mathscr{D}\overline{\mathfrak{g}}_{i,u_i}$ is isomorphic to a product of several $\mathfrak{g}^{(j)}$'s (not necessarily distinct). Moreover, for each $u=(u_1,\dots,u_r)$ the $\mathscr{D}\overline{\mathfrak{g}}_0$-module $W_{\otimes,u}=W_{1,u_1}\otimes\cdots\otimes W_{r,u_r}$ satisfies $h^*(W_{\otimes,u})=M_u^{(1)}\boxtimes\cdots\boxtimes M_u^{(s)}$, where $M_u^{(j)}$ is a tensor product (possibly empty) of minuscule representations of the simple Lie algebra $\mathfrak{g}^{(j)}$. In addition, each element of the centre $\mathfrak{z}(\overline{\mathfrak{g}}_0)\subset\prod_{i,u_i}\mathfrak{z}(\overline{\mathfrak{g}}_{i,u_i})$ acts on both $W_{\otimes,u}$ and V by a scalar.

These properties imply that the $\overline{\mathfrak{g}}_0$-module $W_1\otimes\cdots\otimes W_r=\oplus_u W_{\otimes,u}$ is a finite direct sum of tensor products of one-dimesional or minuscule representations of $\overline{\mathfrak{g}}_0$.

Fix Cartan subalgebras $\mathfrak{h}^{(j)}\subset\mathfrak{g}^{(j)}$ and $\mathfrak{h}\subset\overline{\mathfrak{g}}_0$ such that $\mathfrak{h}\cap\mathscr{D}\overline{\mathfrak{g}}_0=h(\mathfrak{h}^{(1)}\times\cdots\times\mathfrak{h}^{(s)})$. The formula (3.10.2) applied to elements of $\mathfrak{h}$ implies that each weight of $\mathfrak{h}$ occurring in V must occur in $W_1\otimes\cdots\otimes W_r$. Applying Proposition 1.6 we deduce that V is isomorphic (as a $\overline{\mathfrak{g}}_0$-module) to a submodule of $W_1\otimes\cdots\otimes W_r$, which is equivalent to the fact that there exists an open subgroup $U\subset\Gamma$ such that $\rho|_U$ is isomorphic to a subrepresentation of $(\rho_1\otimes\cdots\otimes\rho_r)|_U$, as claimed.

(2) Denote the representation $\rho_1\otimes\cdots\otimes\rho_r$ by $\widetilde{\rho}$. According to (1) the restriction of ρ^{ss} to some open subgroup $U\subset\Gamma$ is isomorphic to a subrepresentation of $\widetilde{\rho}|_U^{\oplus m}$, hence to $\widetilde{\rho}|_U^{\oplus n}$ for some $n\le m$. It remains to show that ρ^{ss} and $\widetilde{\rho}^{\oplus n}$ are isomorphic as representations of Γ.

As ρ^{ss} is semisimple, it is enough to consider any of its simple submodules, so we can assume that $\rho^{\mathrm{ss}}=\rho$ is irreducible. The statement (1) implies, by Frobenius reciprocity, that ρ is isomorphic to a simple $\overline{\mathbf{Q}}_\ell[\Gamma]$-submodule of $\mathrm{Ind}_U^\Gamma(\widetilde{\rho}|_U)^{\oplus n}=(\widetilde{\rho}\otimes\mathbf{Z}[\Gamma/U])^{\oplus n}$, hence of $\widetilde{\rho}\otimes\sigma$, for some irreducible representation $\sigma:\Gamma/U\longrightarrow\mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V_\sigma)$ (after shrinking U we can assume that it is a normal subgroup of Γ). As

$$\mathrm{End}_{\overline{\mathbf{Q}}_\ell[\Gamma]}(\widetilde{\rho}\otimes\sigma)=\mathrm{End}_{\overline{\mathbf{Q}}_\ell[U]}(\widetilde{\rho}\otimes\sigma)^{\Gamma/U}=\Big(\mathrm{End}_{\overline{\mathbf{Q}}_\ell[U]}(\widetilde{\rho})\otimes_{\overline{\mathbf{Q}}_\ell}\mathrm{End}_{\overline{\mathbf{Q}}_\ell}(\sigma)\Big)^{\Gamma/U}=\mathrm{End}_{\overline{\mathbf{Q}}_\ell[\Gamma/U]}(\sigma)=\overline{\mathbf{Q}}_\ell,$$

the (semisimple) representation $\widetilde{\rho}\otimes\sigma$ is irreducible, hence is isomorphic to ρ. We distinguish two cases.

First case: $\sigma(s)^a=\mathrm{id}$ for all $s\in\Gamma/U$. If $a=1$, then σ is the trivial representation and we are done. If $a=2$, then $\sigma(\Gamma/U)$ is an abelian group of exponent one or two, hence $\sigma:\Gamma/U\longrightarrow\{\pm1\}$ is a trivial or a quadratic character.

Second case: there exists $s\in\Gamma$ such that $\sigma(s)^a\neq\mathrm{id}$, hence an eigenvalue $c\neq1$ of $\sigma(s)^a$. Relation (3.10.1) for $g=us$ then gives

$$\forall u\in U\qquad P_{(\widetilde{\rho}(u)S)^a}(c(\widetilde{\rho}(u)S)^a)=0\in\mathrm{End}_{\overline{\mathbf{Q}}_\ell}(\widetilde{V}),\tag{3.10.3}$$

where $\widetilde{V}=W_1\otimes\cdots\otimes W_r$ and $S=\widetilde{\rho}(s)\in\mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(\widetilde{V})$. Denote by $\widetilde{G}$ the Zariski closure of $\widetilde{\rho}(U)$ in $\mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(\widetilde{V})$. This is an affine algebraic group over $\overline{\mathbf{Q}}_\ell$ whose connected component of identity $\widetilde{G}^\circ$ acts irreducibly on $\widetilde{V}$ (thanks to the assumption on $\widetilde{\rho}$ in (2)), hence $\widetilde{G}^\circ$ is reductive. By continuity, we have

$$\forall A\in\widetilde{G}^\circ\qquad P_{(AS)^a}(c(AS)^a)=0\in\mathrm{End}_{\overline{\mathbf{Q}}_\ell}(\widetilde{V}).$$

Lemma 3.11 below shows that $S=0$, which is impossible. This contradiction implies that the second case never occurs.

(3) Combine (2) (for $a=r=1$) with Theorem 3.7.

(3.11) Lemma. *Let H be a connected reductive group over an algebraically closed field $k \supset \mathbf{Q}$ and $r : H \longrightarrow GL(V)$ an irreducible rational representation of H. If $a \geq 1$, $c \in k^\times \smallsetminus \{1\}$ and $S \in \mathrm{End}_k(V)$ satisfy*

$$\forall h \in H(k) \qquad P_{(r(h)S)^a}(c(r(h)S)^a) = 0 \in \mathrm{End}_k(V), \qquad (\star)$$

then $S = 0$.

Proof. As in 1.3, a choice of a maximal torus $T \subset H$ gives rise to weight decompositions $V = \bigoplus_\mu V(\mu)$, $\mathrm{End}_k(V) = \bigoplus_{\mu,\mu'} \mathrm{Hom}_k(V(\mu), V(\mu'))$. Denote by λ the highest weight of V (for a fixed ordering of the roots of (H, T)).

If $S \neq 0$, then there exists a non-zero irreducible rational subrepresentation $M \subset \mathrm{End}(V)$ of H containing S. Proposition 1.3(2) in the form 1.5 implies that there exists $h_0 \in H(k)$ such that the image of $h_0 S h_0^{-1}$ in $\mathrm{End}_k(V(\lambda))$ is non-zero. Condition $(\star)$ for S is invariant by conjugation by an element of $H(k)$, which means that we can replace S by $h_0 S h_0^{-1}$ and assume that the image S_λ of S in $\mathrm{End}_k(V(\lambda)) = k$ is non-zero.

Fix a cocharacter $\beta : \mathbf{G}_m \longrightarrow T$ such that $\beta(\lambda) > 0$ and $\beta(\lambda) > \beta(\mu)$ for all weights $\mu \neq \lambda$ such that $V(\mu) \neq 0$. Denote by z the standard coordinate on $\mathbf{G}_m$ and by $f(z)$ the image of $P_{(r(\beta(z))S)^a}(c(r(\beta(z))S)^a) \in \mathrm{End}_k(V)[z, z^{-1}]$ in $\mathrm{End}_k(V(\lambda))[z, z^{-1}] = k[z, z^{-1}]$. The monomial of the highest degree occurring in $f(z)$ is equal to $c^{n-1}(c-1) S_\lambda^{na} z^{\beta(\lambda)na}$, where $n = \dim(V)$. As the coefficient $c^{n-1}(c-1)S_\lambda^{na}$ is non-zero, there exists $u \in k^\times$ such that $f(u) \neq 0$, which implies that $h = \beta(u) \in H(k)$ satisfies $P_{(r(h)S)^a}(c(r(h)S)^a) \neq 0$. This contradiction with $(\star)$ shows that $S = 0$, as claimed.

(3.12) Theorem. *Assume that the representations ρ and ρ_i from Theorem 3.3 satisfy the following conditions.*

(A") *Each ρ_i is strongly irreducible and W_i is a minuscule representation of the reductive $\overline{\mathbf{Q}}_\ell$-Lie algebra $\overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}(\rho_i(\Gamma))$.*

(C") *There exist an integer $a \geq 1$, an open subgroup $\Gamma' \subset \Gamma$ and a dense subset $\Sigma \subset \Gamma'$ such that for each $g \in \Sigma$ there are pairwise commuting elements $u_1, \ldots, u_r \in \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V(g))$, where $V(g) \supset V$ is a finite dimensional vector space over $\overline{\mathbf{Q}}_\ell$ depending on g, such that V is stable under $u_1 \cdots u_r$, $\rho(g^a) = u_1 \cdots u_r|_V$ and $P_{\rho_i(g^a)}(u_i) = 0$ $(i = 1, \ldots, r)$.*

Then: (1) *Condition (C') from Proposition 3.10 holds.*

(2) *The representation ρ is semisimple.*

(3) *The restriction of ρ to a suitable open subgroup $U \subset \Gamma'$ is isomorphic to a subrepresentation of $(\rho_1 \otimes \cdots \otimes \rho_r)|_U^{\oplus m}$, for some $m \geq 1$.*

(4) *If $\rho_1 \otimes \cdots \otimes \rho_r$ is strongly irreducible and if $a = 1$ or $a = 2$, then each irreducible constituent of $\rho|_{\Gamma'}$ is isomorphic to $(\rho_1 \otimes \cdots \otimes \rho_r)|_{\Gamma'} \otimes \sigma$, for some character $\sigma : \Gamma' \longrightarrow \{\pm 1\}$ satisfying $\sigma^a = 1$.*

Proof. (1) This is a consequence of the following elementary fact: whenever $u_1 u_2 = u_2 u_1$ and $P_{A_i}(u_i) = 0$, then $P_{A_1 \otimes A_2}(u_1 u_2) = 0$.

(2) In a minuscule representation each weight occurs with multiplicity one. As a result, Proposition 3.6 applies in the situation we consider, which means that we can assume – after replacing Γ' by a open subgroup $\Gamma_0 \subset \Gamma'$ and Σ by $\Sigma \cap U_0$, for an open dense subset $U_0 \subset \Gamma_0$ – that for each $g \in \Sigma$ all polynomials $P_{\rho_i(g^a)}(X)$ are without multiple roots, hence each $u_i \in \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V(g))$ is semisimple, and so is the restriction to V of their product $\rho(g^a)$. Therefore $\rho(g)$ is semisimple and condition (C) in Theorem 3.3 is satisfied. Furthermore, condition (B) in Theorem 3.3 is satisfied if we replace Γ by a suitable open subgroup, thanks to (1) and Proposition 3.10(1). Applying Theorem 3.3 and taking into account 3.2 we deduce that ρ is semisimple.

(3),(4) Combine (2) with Proposition 3.10(1) resp. 3.10(2).

4. Semisimplicity criteria for representations of profinite groups (bis)

(4.1) Our next goal is to prove a variant of Theorems 3.3 and 3.12 (Theorems 4.4 and 4.7 below) in which assumptions (A) and (A') are not satisfied.

(4.2) Let Γ be a profinite group and $\Delta \lhd \Gamma$ an open normal subgroup. Denote by $pr : \Gamma \longrightarrow \Gamma/\Delta$ the projection map. There is a natural right action of Γ/Δ on the set of characters $\alpha : \Delta \longrightarrow \overline{\mathbf{Q}}_\ell^\times$, namely

$$\alpha^{pr(g)}(h) = \alpha(ghg^{-1}), \qquad g \in \Gamma,\, h \in \Delta. \tag{4.2.1}$$

(4.3) From now on until 4.7 we assume that Γ/Δ is a cyclic group of order $n > 1$ and fix one of its generators σ. We change notation and write Γ_n instead of Δ. For characters α as in (4.2.1) the induced representations

$$I(\alpha) = \mathrm{Ind}_{\Gamma_n}^{\Gamma}(\alpha)$$

are semisimple and have the following properties:

$$\begin{gathered}
I(\alpha)|_{\Gamma_n} \xrightarrow{\sim} \bigoplus_{i=0}^{n-1} \alpha^{\sigma^i}, \\
I(\alpha) \xrightarrow{\sim} I(\beta) \iff \exists\, i \in \mathbf{Z}/n\mathbf{Z} \quad \beta = \alpha^{\sigma^i}, \\
I(\alpha) \otimes I(\beta) = \mathrm{Ind}_{\Gamma_n}^{\Gamma}(\alpha \otimes I(\beta)|_{\Gamma_n}) \xrightarrow{\sim} \bigoplus_{i=0}^{n-1} I(\alpha\beta^{\sigma^i}), \\
I(\alpha_1) \otimes \cdots \otimes I(\alpha_r) \xrightarrow{\sim} \bigoplus_{i_2,\ldots,i_r=0}^{n-1} I(\alpha_1 \alpha_2^{\sigma^{i_2}} \cdots \alpha_r^{\sigma^{i_r}}).
\end{gathered}$$

Fix a lift $\widetilde{\sigma} \in pr^{-1}(\sigma)$ of σ and identify the representation space

$$I(\alpha) = \{f : \Gamma \longrightarrow \overline{\mathbf{Q}}_\ell \mid f(hg) = \alpha(h)f(g) \quad \forall h \in \Gamma_n, \forall g \in \Gamma\}$$

with $\overline{\mathbf{Q}}_\ell^n = \oplus_{i=1}^n \overline{\mathbf{Q}}_\ell \cdot e_i$ via

$$f \mapsto \sum_{i=1}^{n} f(\widetilde{\sigma}^{i-1})\, e_i = \begin{pmatrix} f(1) \\ f(\widetilde{\sigma}) \\ \vdots \\ f(\widetilde{\sigma}^{n-1}) \end{pmatrix}.$$

The action $(g * f)(g_1) = f(g_1 g)$ of Γ on $I(\alpha)$ then becomes

$$\forall h \in \Gamma_n \quad h(e_i) = \alpha^{\sigma^{i-1}}(h)\, e_i, \qquad \widetilde{\sigma}(e_i) = \begin{cases} e_{i-1}, & i \neq 1 \\ \alpha(\widetilde{\sigma}^n)\, e_n & i = 1. \end{cases}$$

In particular, $\widetilde{\sigma}^n \in \Gamma_n$ acts on $I(\alpha)$ by multiplication by the scalar $\alpha(\widetilde{\sigma}^n)$. This scalar depends on $\widetilde{\sigma}$, not just on σ, since

$$\forall h \in \Gamma_n \quad \alpha\left((h\widetilde{\sigma})^n\right) = \alpha^{1+\sigma+\cdots+\sigma^{n-1}}(h)\, \alpha(\widetilde{\sigma}^n).$$

The representation $I(\alpha)$ decomposes into a direct sum of n/d irreducible representations of dimension d

$$I(\alpha) = \bigoplus_{\widetilde{\alpha}} \mathrm{Ind}_{\Gamma_d}^{\Gamma}(\widetilde{\alpha}),$$

where $d = \min\{i \geq 1 \mid \alpha^{\sigma^i} = \alpha\}$ is a divisor of n, $\Gamma_d = pr^{-1}(\langle \sigma^d \rangle)$ is the inverse image of the cyclic group generated by σ^d and $\widetilde{\alpha}$ runs through all characters $\widetilde{\alpha} : \Gamma_d \longrightarrow \overline{\mathbf{Q}}_\ell^\times$ extending α.

(4.4) Theorem. *Let $\rho_i = I(\alpha_i)$ $(i = 1, \ldots, r)$, where $\alpha_1, \ldots, \alpha_r : \Gamma_n \longrightarrow \overline{\mathbf{Q}}_\ell^\times$ are characters. Assume that a representation $\rho : \Gamma \longrightarrow \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V)$ satisfies the following conditions.*
(B) *The semisimplification ρ^{ss} of ρ is isomorphic to a subrepresentation of $(\rho_1 \otimes \cdots \otimes \rho_r)^{\oplus m}$, for some $m \geq 1$.*
(S) *There is a dense subset $\Sigma \subset \Gamma$ such that, for each $g \in \Sigma$, $\rho(g)$ is a semisimple element of $\mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V)$.*
If n is a prime number, then the representation ρ is semisimple.

Proof. Note that the representations ρ_i and their tensor products are semisimple and that conditions (B) and (S) are satisfied by any subquotient of ρ. By induction, it is sufficient to consider only the case when ρ sits in an exact sequence

$$0 \longrightarrow Y \longrightarrow \rho \longrightarrow X \longrightarrow 0,$$

where X and Y are irreducible subrepresentations of $\rho_1 \otimes \cdots \otimes \rho_r$, hence $X \oplus X' = I(\alpha)$, $Y \oplus Y' = I(\beta)$ for some $\alpha = \alpha_1 \alpha_2^{\sigma^{i_2}} \cdots \alpha_r^{\sigma^{i_r}}$ and $\beta = \alpha_1 \alpha_2^{\sigma^{j_2}} \cdots \alpha_r^{\sigma^{j_r}}$. After replacing ρ by $\rho \oplus X' \oplus Y'$ we can assume that

$$0 \longrightarrow I(\beta) \longrightarrow \rho \longrightarrow I(\alpha) \longrightarrow 0,$$

where $\beta/\alpha = \varphi^{\sigma-1}$ for a suitable character $\varphi : \Gamma_n \longrightarrow \overline{\mathbf{Q}}_\ell^\times$. It is enough, therefore, to prove the following statement.

(4.5) Proposition. *If n is a prime number and if $\alpha, \beta, \varphi : \Gamma_n \longrightarrow \overline{\mathbf{Q}}_\ell^\times$ are characters such that $\beta/\alpha = \varphi^{\sigma-1}$, then any representation $\rho : \Gamma \longrightarrow \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V)$ satisfying (S) which sits in an exact sequence*

$$0 \longrightarrow I(\beta) \longrightarrow \rho \longrightarrow I(\alpha) \longrightarrow 0$$

is isomorphic to $I(\beta) \oplus I(\alpha)$.

Proof. **Step 1:** For each $h \in \Gamma_n$ the element $(h\widetilde{\sigma})^n = h_n \widetilde{\sigma}^n$ $(h_n = h(\widetilde{\sigma} h \widetilde{\sigma}^{-1}) \cdots (\widetilde{\sigma}^{n-1} h \widetilde{\sigma}^{1-n}))$ acts on both $I(\alpha)$ and $I(\beta)$ by the same scalar $\alpha(h_n \widetilde{\sigma}^n) = \beta(h_n \widetilde{\sigma}^n)$, since

$$(\beta/\alpha)(h_n \widetilde{\sigma}^n) = \varphi^{(\sigma-1)(1+\sigma+\cdots+\sigma^{n-1})}(h)\, \varphi^{\sigma-1}(\widetilde{\sigma}^n) = \varphi^{(\sigma^n-1)}(h)\, \varphi^{\sigma-1}(\widetilde{\sigma}^n) = 1.$$

It follows that

$$\{h \in \Gamma_n \mid \rho(h\widetilde{\sigma}) \text{ is semisimple}\} = \{h \in \Gamma_n \mid \rho((h\widetilde{\sigma})^n) \in \overline{\mathbf{Q}}_\ell^\times \cdot \mathrm{id}\}$$

is a closed subset of Γ_n, hence equal to Γ_n, thanks to (S). In particular, $\widetilde{\sigma}$ acts semisimply on V.

Step 2: According to Step 1 there exists a $\widetilde{\sigma}$-equivariant $\overline{\mathbf{Q}}_\ell$-linear splitting of $V \longrightarrow I(\alpha)$, which we fix. Together with the identifications $I(\alpha) \xrightarrow{\sim} \overline{\mathbf{Q}}_\ell^n \xleftarrow{\sim} I(\beta)$ from 4.3 this allows us to identify V with $\overline{\mathbf{Q}}_\ell^n \oplus \overline{\mathbf{Q}}_\ell^n$ in such a way that $g \in \Gamma$ acts on V by a matrix

$$\rho(g) = \begin{pmatrix} B & c(g)A \\ 0 & A \end{pmatrix},$$

where $A \in GL_n(\overline{\mathbf{Q}}_\ell)$ (resp. $B \in GL_n(\overline{\mathbf{Q}}_\ell)$) is given by the action of g on $I(\alpha)$ (resp. on $I(\beta)$) and

$$c \in Z^1(\Gamma, \mathrm{Hom}(I(\alpha), I(\beta))) = Z^1(\Gamma, M_n(\overline{\mathbf{Q}}_\ell))$$

is the 1-cocycle attached to the splitting. Continuity of ρ implies that the functions $g \mapsto A, B, c(g)$ are continuous, too. By construction, $c(\widetilde{\sigma}) = 0$.

Note that there is an isomorphism of Γ-modules

$$\mathrm{Hom}(I(\alpha), I(\beta)) \xrightarrow{\sim} I(\alpha)^\vee \otimes I(\beta) \xrightarrow{\sim} I(\alpha^{-1}) \otimes I(\beta) \xrightarrow{\sim} \bigoplus_{i=0}^{n-1} I(\beta/\alpha^{\sigma^i})$$

under which $I(\beta/\alpha^{\sigma^i})$ corresponds to the set of matrices $C \in M_n(\overline{\mathbf{Q}}_\ell)$ which have non-zero entries C_{ab} only on the shifted diagonal $a + i \equiv b \pmod n$, since for each $g \in \Gamma_n$ the matrix A is diagonal, with diagonal entries equal to $\alpha(g), \alpha^\sigma(g), \ldots, \alpha^{\sigma^{n-1}}(g)$ (and similarly for B, when α is replaced by β). Denote by

$$p_{I(\beta/\alpha^{\sigma^j})} : M_n(\overline{\mathbf{Q}}_\ell) \xrightarrow{\sim} \mathrm{Hom}(I(\alpha), I(\beta)) \xrightarrow{\sim} \bigoplus_{i=0}^{n-1} I(\beta/\alpha^{\sigma^i}) \longrightarrow I(\beta/\alpha^{\sigma^j})$$

the projection on the term $I(\beta/\alpha^{\sigma^j})$ (corresponding to a shifted diagonal of $M_n(\overline{\mathbf{Q}}_\ell)$).

Step 3: Assume first that $\beta = \alpha$. For each $h \in \Gamma_n \cap \Sigma$ the semisimplicity of $\rho(h) = \begin{pmatrix} A & c(h)A \\ 0 & A \end{pmatrix}$ implies that all terms on the main diagonal of $c(h)$ vanish, since A is diagonal. By continuity of c, it follows that

$$\forall h \in \Gamma_n \qquad p_{I(\beta/\alpha)}(c(h)) = 0. \tag{4.5.1}$$

Step 4: Assume now that $\beta \neq \alpha$ (hence $\varphi^\sigma \neq \varphi$). Step 1 implies that

$$\forall h \in \Gamma_n \qquad 0 = c((h\widetilde{\sigma})^n) = \sum_{j=0}^{n-1} (h\widetilde{\sigma})^j \, c(h\widetilde{\sigma}) = \sum_{j=0}^{n-1} (h\widetilde{\sigma})^j \, c(h), \tag{4.5.2}$$

since $c(h\widetilde{\sigma}) = hc(\widetilde{\sigma}) + c(h) = c(h)$.

Let $h_0, h_1 \in \Gamma_n$ be two elements such that h_0 acts trivially on $gr(V) = I(\beta) \oplus I(\alpha)$; set $h_2 = h_0 h_1$. The condition on h_0 implies that $c(h_2) = h_0 c(h_1) + c(h_0) = c(h_1) + c(h_0)$ and that the action of $h_2\widetilde{\sigma}$ on $gr(V)$ coincides with that of $h_1\widetilde{\sigma}$. Subtracting the relations (4.5.2) for h_2 and h_1, we obtain

$$0 = \sum_{j=0}^{n-1} \big((h_2\widetilde{\sigma})^j \, c(h_2) - (h_1\widetilde{\sigma})^j \, c(h_1) \big) = \sum_{j=0}^{n-1} (h_1\widetilde{\sigma})^j \, c(h_0). \tag{4.5.3}$$

Write $x := p_{I(\beta/\alpha)}(c(h_0))$ as $x = \sum_{i=1}^n x_i e_i$ ($x_i \in \overline{\mathbf{Q}}_\ell$) using the identification $I(\beta/\alpha) = I(\varphi^{\sigma-1}) \xrightarrow{\sim} \overline{\mathbf{Q}}_\ell^n$ from 4.3. The formula

$$(h_1\widetilde{\sigma})^j(e_i) = (\varphi^{\sigma-1})^{\sigma^{i-1} + \cdots + \sigma^{i-j}}(h_1) \, e_{i-j} = (\varphi^{\sigma^i - \sigma^{i-j}})(h_1) \, e_{i-j}$$

implies that (4.5.3) can be rewritten as

$$\forall h_1 \in \Gamma_n \qquad 0 = \sum_{j=0}^{n-1} (h_1\widetilde{\sigma})^j \, x = \sum_{i=1}^{n} \sum_{j=0}^{n-1} x_i (\varphi^{\sigma^i - \sigma^{i-j}})(h_1) \, e_{i-j} = \sum_{a=1}^{n} e_a / \varphi^{\sigma^a}(h_1) \left(\sum_{j=0}^{n-1} \varphi^{\sigma^{a+j}}(h_1) \, x_{a+j} \right),$$

hence

$$\forall h_1 \in \Gamma_n \qquad \sum_{i=1}^{n} \varphi^{\sigma^i}(h_1) \, x_i = 0.$$

Linear independence of the distinct characters $\varphi^\sigma, \ldots, \varphi^{\sigma^n} = \varphi$ (we are using our assumptions that $\varphi^\sigma \neq \varphi$ and n is a prime number) then implies that all $x_i = 0$. In other words,

$$p_{I(\beta/\alpha)}(c(h_0)) = x = 0.$$

Step 5: Applying Step 3 and Step 4 to the identifications $I(\alpha) \xrightarrow{\sim} I(\alpha^{\sigma^i}) \xrightarrow{\sim} \overline{\mathbf{Q}}_\ell^n$ (when $\beta/\alpha^{\sigma^i} = \varphi_i^{\sigma-1}$ for $\varphi_i = \varphi/\alpha^{1+\sigma+\cdots+\sigma^{i-1}}$) we obtain

$$\forall i = 0, \ldots, n-1 \qquad p_{I(\beta/\alpha^{\sigma^i})}(c(h_0)) = 0,$$

hence $c(h_0) = 0$ for every $h_0 \in \Gamma_n$ which acts trivially on $gr(V)$. This implies that, for every $h \in \Gamma_n$, the value $c(h)$ depends only on the image $\rho_{gr(V)}(h)$ of $\rho(h)$ in $\mathrm{Aut}(gr(V))$.

Step 6: According to Step 5, the restriction c_n of the cocycle c to Γ_n

$$c_n \in Z^1(\Gamma_n, \mathrm{Hom}(I(\alpha), I(\beta))) = \bigoplus_{i=0}^{n-1} Z^1(\Gamma_n, I(\beta/\alpha^{\sigma^i}))$$

lies in the image of the inflation map

$$\inf : Z^1(A, \mathrm{Hom}(I(\alpha), I(\beta))) = \bigoplus_{i=0}^{n-1} Z^1(A, I(\beta/\alpha^{\sigma^i})) \longrightarrow \bigoplus_{i=0}^{n-1} Z^1(\Gamma_n, I(\beta/\alpha^{\sigma^i})),$$

where $A := \rho_{gr(V)}(\Gamma_n) \subset \mathrm{Aut}(I(\alpha) \oplus I(\beta))$ is an abelian profinite group.

If $\alpha^{\sigma^i} \neq \beta$, then $H^1(A, \beta/\alpha^{\sigma^i}) = 0$, by Sah's Lemma.

If $\alpha^{\sigma^i} = \beta$, then (4.5.1) implies that $p_{I(\beta/\alpha^{\sigma^i})}\, c_n = 0$.

In either case, the cohomology class of $p_{I(\beta/\alpha^{\sigma^i})}\, c_n$ vanishes in $H^1(\Gamma_n, I(\beta/\alpha^{\sigma^i}))$, hence the cohomology class of c lies in the group

$$\mathrm{Ker}\big(\mathrm{res} : H^1(\Gamma, \mathrm{Hom}(I(\alpha), I(\beta))) \longrightarrow H^1(\Gamma_n, \mathrm{Hom}(I(\alpha), I(\beta)))\big) \xrightarrow{\sim} H^1(\Gamma/\Gamma_n, \mathrm{Hom}(I(\alpha), I(\beta))^{\Gamma_n}),$$

which is trivial, since $\mathrm{Hom}(I(\alpha), I(\beta))^{\Gamma_n}$ is a $\mathbf{Q}$-vector space. This finishes the proof of the fact that $\rho \xrightarrow{\sim} I(\beta) \oplus I(\alpha)$ (and of Theorem 4.4).

(4.6) Proposition. *Let $\rho_i = I(\alpha_i)$ $(i = 1, \dots, r)$ be as in Theorem 4.4. Let $\rho : \Gamma \longrightarrow \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V)$ be a representation satisfying the following condition.*
(S') *There exist an open subgroup $\Gamma' \subset \Gamma$ such that $pr(\Gamma') = \Gamma/\Gamma_n$ and a dense subset $\Sigma \subset \Gamma'$ such that*

$$\forall g \in \Sigma \qquad P_{(\rho_1 \otimes \cdots \otimes \rho_r)(g)}(\rho(g)) = 0 \in \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V).$$

Then there is an open subgroup $U \subset \Gamma'$ such that $pr(U) = \Gamma/\Gamma_n$ and $\rho^{\mathrm{ss}}|_U = (\rho|_U)^{\mathrm{ss}}$ is isomorphic to a subrepresentation of $(\rho_1 \otimes \cdots \otimes \rho_r)|_U^{\oplus m}$, for some $m \geq 1$.

Proof. **Step 1:** (S') holds for all subquotients of ρ, which means that we can – and will – assume that ρ is irreducible. Note that conditions (A') and (C') from Proposition 3.10 are satisfied (with $a = 1$). Step 1 of the proof of Proposition 3.10(1) shows that

$$\mathrm{Ker}(\rho) \supseteq \mathrm{Ker}(\rho_0) = \bigcap_{i=1}^{r} \mathrm{Ker}(\rho_i) = \bigcap_{i,j} \mathrm{Ker}(\alpha^{\sigma^j})$$

(where $\rho_0 = \rho_1 \oplus \cdots \oplus \rho_r$), which means that we can replace Γ by $\Gamma/\mathrm{Ker}(\rho_0)$, hence assume that Γ_n is *abelian*. In this case ρ is necessarily of the form (cf. 4.3)

$$\rho = \mathrm{Ind}_{\Gamma_d}^{\Gamma}(\widetilde{\beta}) \subset I(\beta), \qquad \widetilde{\beta} : \Gamma_d \longrightarrow \overline{\mathbf{Q}}_\ell^\times, \qquad \widetilde{\beta}|_{\Gamma_n} = \beta, \qquad d = \min\{i \geq 1 \mid \beta^{\sigma^i} = \beta\}, \qquad \rho|_{\Gamma_n} = \bigoplus_{i=0}^{d-1} \beta^{\sigma^i}.$$

Proof of Proposition 3.10(1) implies that there is an open subgroup $A \subset \Gamma_n$ such that

$$\beta|_A \subset (\rho_1 \otimes \cdots \otimes \rho_r)|_A = \bigoplus_{i_1, \dots, i_r = 0}^{n-1} \alpha_1^{\sigma^{i_1}} \cdots \alpha_r^{\sigma^{i_r}}|_A,$$

hence there exist $\{i_j\}$ for which the character $\chi = \alpha_1^{\sigma^{i_1}} \cdots \alpha_r^{\sigma^{i_r}}/\beta$ has finite order. We can replace each $\alpha_j^{\sigma^{i_j}}$ by α_j without changing $I(\alpha_j)$; then $\chi = \alpha_1 \cdots \alpha_r/\beta$.

Step 2: By continuity, the equality in (S') holds for all $g \in \Gamma'$, in particular for all $g \in \Gamma' \cap pr^{-1}(\sigma)$. Fix $\widetilde{\sigma} \in \Gamma' \cap pr^{-1}(\sigma)$ and write $g = \widetilde{\sigma} h$, where $h \in \Gamma'_n = \Gamma' \cap \Gamma_n$. If we denote by $\mathbf{N}\alpha$ the product $\alpha\alpha^{\sigma} \cdots \alpha^{\sigma^{n-1}}$ (for any character α of Γ_n), then

$$P_{(\rho_1 \otimes \cdots \otimes \rho_r)(\widetilde{\sigma} h)}(X) = (X^n - (\mathbf{N}(\alpha_1 \cdots \alpha_r))(h)\,(\alpha_1 \cdots \alpha_r)(\widetilde{\sigma}^n))^{n^{r-1}}$$

and

$$P_{\rho(\widetilde{\sigma} h)}(X) = X^d - \left(\widetilde{\beta}\widetilde{\beta}^{\sigma} \cdots \widetilde{\beta}^{\sigma^{d-1}}\right)(h)\,\widetilde{\beta}(\widetilde{\sigma}^d), \qquad \rho(\widetilde{\sigma} h)^n = (\mathbf{N}\beta)(h)\,\beta(\widetilde{\sigma}^n) \cdot \mathrm{id}_V.$$

Condition (S') then yields

$$\forall h \in \Gamma'_n \qquad (\mathbf{N}\chi)(h) = (\mathbf{N}(\alpha_1 \cdots \alpha_r/\beta))\,(h) = (\alpha_1 \cdots \alpha_r/\beta)(\widetilde{\sigma}^n)^{-1} = \chi(\widetilde{\sigma}^n)^{-1},$$

which is equivalent to $\chi(\widetilde{\sigma}^n) = 1$ and $(\mathbf{N}\chi)|_{\Gamma'_n} = 1$, hence to

$$\forall h \in \Gamma'_n \qquad \chi((\widetilde{\sigma} h)^n) = 1.$$

Step 3: The following open subgroup of Γ'_n

$$U_n = \{h \in \Gamma'_n \mid \forall i \in \mathbf{Z} \quad \chi^{\sigma^i}(h) = 1\} = \Gamma' \cap \bigcap_{i=0}^{n-1} \mathrm{Ker}(\chi^{\sigma^i})$$

is stable under the conjugation action by the cyclic group $\langle\sigma\rangle$. The extension class $[\Gamma'] \in H^2(\langle\sigma\rangle, \Gamma'_n)$ of

$$1 \longrightarrow \Gamma'_n \longrightarrow \Gamma' \longrightarrow \langle\sigma\rangle \longrightarrow 1$$

is equal to the image of $\widetilde{\sigma}^n \in (\Gamma'_n)^{\sigma=1}$ under the periodicity isomorphism (depending on σ)

$$(\Gamma'_n)^{\sigma=1}/(1 + \sigma + \cdots + \sigma^{n-1})\Gamma'_n = \widehat{H}^0(\langle\sigma\rangle, \Gamma'_n) \xrightarrow{\sim} H^2(\langle\sigma\rangle, \Gamma'_n).$$

The conclusion of Step 2 implies that $\widetilde{\sigma}^n \in U_n$, hence

$$[\Gamma'] \in \mathrm{Im}\left(H^2(\langle\sigma\rangle, U_n) \longrightarrow H^2(\langle\sigma\rangle, \Gamma'_n)\right),$$

which means that there is a commutative diagram of group extensions

$$\begin{array}{ccccccc}
1 \longrightarrow & U_n & \longrightarrow & U & \longrightarrow & \langle\sigma\rangle & \longrightarrow 1 \\
 & \big\downarrow & & \big\downarrow & & \| & \\
1 \longrightarrow & \Gamma'_n & \longrightarrow & \Gamma' & \longrightarrow & \langle\sigma\rangle & \longrightarrow 1 \\
 & \big\downarrow & & \big\downarrow & & \| & \\
1 \longrightarrow & \Gamma_n & \longrightarrow & \Gamma & \longrightarrow & \langle\sigma\rangle & \longrightarrow 1
\end{array}$$

such that $\beta|_{U_n} = (\alpha_1 \cdots \alpha_r)|_{U_n}$. We have $\rho \subset I(\beta)$, hence

$$\rho|_U \subset I(\beta)|_U = \mathrm{Ind}_{U_n}^U(\beta|_{U_n}) = \mathrm{Ind}_{U_n}^U(\alpha_1 \cdots \alpha_r|_{U_n}) \subset \bigotimes_{i=1}^r \mathrm{Ind}_{U_n}^U(\alpha_i|_{U_n}) = (\rho_1 \otimes \cdots \otimes \rho_r)|_U,$$

as claimed.

(4.7) Theorem. *Let $\rho_i = I(\alpha_i)$ $(i = 1, \ldots, r)$ be as in Theorem 4.4. Assume that n is a prime number and that $\alpha_i/\alpha_i^{\sigma^j}$ is a character of infinite order, for each $i = 1, \ldots, r$ and $j = 1, \ldots, n-1$. Let $\rho : \Gamma \longrightarrow \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V)$ be a representation satisfying the following condition.*
(S") *There exist an open subgroup $\Gamma' \subset \Gamma$ such that $pr(\Gamma') = \Gamma/\Gamma_n$ and a dense subset $\Sigma \subset \Gamma'$ such that for each $g \in \Sigma$ there are pairwise commuting elements $u_1, \ldots, u_r \in \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V(g))$, where $V(g) \supset V$ is a finite-dimensional vector space over $\overline{\mathbf{Q}}_\ell$ depending on g, such that V is stable by $u_1 \cdots u_r$, $\rho(g) = u_1 \cdots u_r|_V$ and $P_{\rho_i(g)}(u_i) = 0$ $(i = 1, \ldots, r)$.*
Then: (1) *Condition (S') from Proposition 4.6 holds.*
(2) *The representation ρ is semisimple.*
(3) *The restriction of ρ to a suitable open subgroup $U \subset \Gamma'$ satisfying $pr(U) = \Gamma/\Gamma_n$ is isomorphic to a subrepresentation of $(\rho_1 \otimes \cdots \otimes \rho_r)|_U^{\oplus m}$, for some $m \geq 1$.*

Proof. (1) See the proof of Theorem 3.12(1).
(2) Thanks to (1), Proposition 4.6 applies to ρ, which yields an open subgroup $U \subset \Gamma'$ such that $pr(U) = \Gamma/\Gamma_n$ for which $\rho|_U$ satisfies condition (B) in Theorem 4.4 (where we replace $(\Gamma, \Gamma_n, \alpha_i)$ by $(U, U_n = U \cap \Gamma_n, \alpha_i|_{U_n})$). Condition (S") implies that, for each element g of $\Sigma \cap U \cap pr^{-1}(\sigma)$ (which is a dense subset of $U \cap pr^{-1}(\sigma)$), we have $\rho(g) = u_1 \cdots u_r|_V$ with pairwise commuting $u_i \in \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V(g))$ satisfying $0 = P_{\rho_i(g)}(u_i) = u_i^n - \alpha_i(g^n)\mathrm{id}$. It follows that each u_i is semisimple, and so is the restriction of their product $\rho(g)$. After replacing σ by other generators of $\Gamma/\Gamma_n = U/U_n$ we obtain the same statement for all $g \in \Sigma \cap (U \smallsetminus U_n)$. The assumption on $\alpha_i/\alpha_i^{\sigma^j}$ implies that for all elements g of a suitable open dense subset $U_n' \subset U_n$ the characteristic polynomials $P_{\rho_i(g)}(X)$ $(i = 1, \ldots, r)$ have distinct roots. As above, this implies that $\rho(g)$ is semisimple for each $g \in \Sigma \cap U_n'$. This means that condition (S) in Theorem 4.4 is also satisfied by $\rho|_U$ (with Σ replaced by $(\Sigma \cap (U \smallsetminus U_n)) \cup (\Sigma \cap U_n'))$. We deduce from Theorem 4.4 that $\rho|_U$, hence ρ as well, is semisimple.
(3) Combine (2) with Proposition 4.6 (which applies, by (1)).

5. Cohomology of quaternionic Shimura varieties

(5.1) In this section we apply the abstract results of §3 and §4 to Galois representations occurring in étale cohomology of quaternionic Shimura varieties. Historically, this was the first class of Shimura varieties of dimension dim > 1 to which the Langlands(-Kottwitz) method was applied [La], [VSL].

(5.2) Let $F \subset \overline{\mathbf{Q}} \subset \mathbf{C}$ be a totally real number field of degree $r = [F : \mathbf{Q}]$. Fix a quaternion algebra D over F which is not totally definite. Let H (resp. $G = R_{F/\mathbf{Q}}(H)$) be the group $D^\times$, viewed as an algebraic group over F (resp. over $\mathbf{Q}$). Define G^* to be the fibre product

$$\begin{array}{ccc} G^* & \longrightarrow & G \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Nrd}} \\ \mathbf{G}_{m,\mathbf{Q}} & \longrightarrow & R_{F/\mathbf{Q}}(\mathbf{G}_{m,F}), \end{array}$$

where Nrd is the reduced norm.

(5.3) The set of infinite primes of F naturally decomposes as

$$\{v : F \hookrightarrow \mathbf{R}\} = \{v \mid D_v \simeq M_2(\mathbf{R})\} \cup \{v \mid D_v \simeq \mathbf{H}\} = \Omega \cup \Omega^c, \qquad |\Omega| = t \geq 1, \qquad |\Omega^c| = r - t.$$

We fix the corresponding isomorphisms

$$D \otimes \mathbf{R} \simeq M_2(\mathbf{R})^\Omega \times \mathbf{H}^{\Omega^c}, \qquad G(\mathbf{R}) \simeq GL_2(\mathbf{R})^\Omega \times (\mathbf{H}^\times)^{\Omega^c}.$$

Let

$$h : \mathbf{S} = R_{\mathbf{C}/\mathbf{R}}\mathbf{G}_{m,\mathbf{C}} \to G_{\mathbf{R}} \simeq GL(2)_{\mathbf{R}}^\Omega \times (\mathbf{H}^\times)^{\Omega^c} \tag{5.3.1}$$

be the standard morphism $x+iy \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{\Omega} \times 1^{\Omega^c}$. Its $G(\mathbf{R})$-conjugacy class $\mathscr{X}$ is naturally identified with $(\mathbf{C} \smallsetminus \mathbf{R})^{\Omega}$ and h with the point $\{i\}^{\Omega}$.

(5.4) The Shimura variety $Sh_K(G, \mathscr{X})$ has dimension t and its reflex field $E = E(G, \mathscr{X})$ is equal to

$$E = \mathbf{Q}\big(\sum_{v\in\Omega} v(a) \mid a \in F\big) \subset F^{gal} \subset \overline{\mathbf{Q}} \subset \mathbf{C}.$$

Equivalently, $\Gamma_E = \mathrm{Gal}(\overline{\mathbf{Q}}/E) = \{\gamma \in \Gamma_{\mathbf{Q}} \mid \gamma(\Omega) = \Omega\}$, if we consider $\Omega \subset \mathrm{Hom}(F, \mathbf{C}) = \mathrm{Hom}(F, \overline{\mathbf{Q}})$. Define an intermediate field $E \subset E' \subset F^{gal}$ to be the fixed field of $\Gamma_{E'} = \mathrm{Gal}(\overline{\mathbf{Q}}/E') = \{\gamma \in \Gamma_{\mathbf{Q}} \mid \forall v \in \Omega\ \gamma(v) = v\}$.

If $t = 1$ (the case of Shimura curves), then $E = E' = F$. If $t = r$ (the essentially PEL case), then $E = \mathbf{Q}$ and $E' = F^{gal}$.

(5.5) Our main objects of interest will be the cohomology groups $H^*_{et}(Sh(G, \mathscr{X}) \otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell})$ (in the notation of Introduction), where

$$\xi : G_{\mathbf{C}} = GL(2)_{\mathbf{C}}^{\mathrm{Hom}(F,\mathbf{R})} \longrightarrow GL(N)_{\mathbf{C}}$$

is an irreducible rational representation such that $\xi|_{Z_{\mathbf{C}}}$ factors through $N_{F/\mathbf{Q}}$. Explicitly,

$$\xi = \otimes_{v|\infty}\xi_v, \qquad \xi_v = \mathrm{Sym}^{k_v-2}(\mathrm{Std}^\vee) \otimes (\det \circ \mathrm{Std}^\vee)^{(w-k_v)/2} : GL(2)_{\mathbf{C}} \longrightarrow GL(k_v - 1)_{\mathbf{C}}, \tag{5.5.1}$$

where $\mathrm{Std}^\vee$ is the dual of the standard two-dimensional representation of $GL(2)$, $k : \mathrm{Hom}(F, \mathbf{R}) \longrightarrow \mathbf{Z}_{\geq 2}$, $w \in \mathbf{Z}$ and $\forall v|\infty\ \ k_v \equiv w \pmod 2$. The reason why we consider $\mathrm{Std}^\vee$ rather then Std is explained by the discussion in A5.6 below.

The corresponding ℓ-adic sheaf $\mathscr{L}_{\xi,\ell}$ is pure of weight $t(w-2)$.

(5.6) If $t = r$ (i.e., if the quaternion algebra D is totally indefinite), then the morphism (5.3.1) factors through $G^*_{\mathbf{R}}$. Its $G^*(\mathbf{R})$-conjugacy class $\mathscr{X}^*$ is naturally identified with $(\mathcal{H}^+)^{\mathrm{Hom}(F,\mathbf{R})} \cup (\mathcal{H}^-)^{\mathrm{Hom}(F,\mathbf{R})} \subset (\mathbf{C} \smallsetminus \mathbf{R})^{\mathrm{Hom}(F,\mathbf{R})}$, where $\mathcal{H}^+$ and $\mathcal{H}^-$ denote the upper and lower half planes in $\mathbf{C}$, respectively.

The Shimura variety $Sh(G^*, \mathscr{X}^*)$ is of PEL type. It is defined over the common reflex field $E = \mathbf{Q}$ of $(G^*, \mathscr{X}^*)$ and $(G, \mathscr{X})$.

(5.7) Proposition. *If $t = r$, then the morphism $i : Sh(G^*, \mathscr{X}^*) \longrightarrow Sh(G, \mathscr{X})$ defined by the inclusion $(G^*, \mathscr{X}^*) \subset (G, \mathscr{X})$ induces an isomorphism of the connected components $Sh(G^*, \mathscr{X}^*)^{an,+} \simeq Sh(G, \mathscr{X})^{an,+}$ containing $\mathscr{X}^+ \times \{1\}$, where $(\mathscr{X}^*)^+ = \mathscr{X}^+ = (\mathcal{H}^+)^{\mathrm{Hom}(F,\mathbf{R})} \subset \mathscr{X}^* \subset \mathscr{X}$. Moreover, the map i is an open immersion.*

Proof. This is a well-known consequence of Chevalley's theorem on units (cf. [De2, Cor. 2.0.12]). According to [De1, Prop. 1.15], the map i is an open immersion. It is enough, therefore, to show that for any pair of open compact subgroups $K_1^* \subset G^*(\widehat{\mathbf{Q}})$, $K_1 \subset G(\widehat{\mathbf{Q}})$ such that $K_1^* \subset K_1$ there is a smaller pair $(K_2^*, K_2) \subset (K_1^*, K_1)$ for which the following diagram can be completed by a diagonal morphism making the two triangles commutative.

$$\begin{array}{ccc}
(K_2^* \cap G^*(\mathbf{Q})_+)\backslash(\mathscr{X}^*)^+ & \longrightarrow & (K_2 \cap G(\mathbf{Q})_+)\backslash\mathscr{X}^+ \\
\downarrow & \swarrow & \downarrow \\
(K_1^* \cap G^*(\mathbf{Q})_+)\backslash(\mathscr{X}^*)^+ & \longrightarrow & (K_1 \cap G(\mathbf{Q})_+)\backslash\mathscr{X}^+
\end{array}$$

Above, $G(\mathbf{Q})_+ = G(\mathbf{Q}) \cap G(\mathbf{R})_+$ (resp. $G^*(\mathbf{Q})_+ = G^*(\mathbf{Q}) \cap G^*(\mathbf{R})_+$), where $G(\mathbf{R})_+$ (resp. $G^*(\mathbf{R})_+$) is the subgroup of elements of $G(\mathbf{R})$ (resp. of $G^*(\mathbf{R})$) whose reduced norm lies in $(\mathbf{R}_+^\times)^{\mathrm{Hom}(F,\mathbf{R})}$ (resp. in the diagonally embedded $\mathbf{R}_+^\times$). In concrete terms, it is enough to find (K_2^*, K_2) such that

$$K_2 \cap G(\mathbf{Q})_+ \subset (K_1^* \cap G^*(\mathbf{Q})_+)Z(\mathbf{R}),$$

where $Z \subset G$ is the centre. We can assume that

$$K_1 \cap G(\mathbf{Q}) \subset (1 + MO_B) \cap O_B^\times, \qquad K_1^* \cap G^*(\mathbf{Q}) \supset (1 + M'O_B)^{\mathrm{Nrd}=1}$$

for some O_F-order $O_B \subset B$ and integers $3 \mid M \mid M'$. According to [C, Thm. 1] there exists an integer $N \geq 1$ such that $(1 + M'NO_F) \cap O_F^\times \subset (O_F^\times \cap (1 + M'O_F))^2$. Taking $K_2 = (1 + M'N\widehat{O}_B) \cap K_1$ and $K_2^* = K_2 \cap K_1^*$, the reduced norm of any $a \in K_2 \cap G(\mathbf{Q}) \subset 1 + M'NO_B$ is of the form $\mathrm{Nrd}(a) = u^{-2}$ for some $u \in O_F^\times \cap (1 + M'O_F)$. It follows that

$$a = (au)u^{-1} \in (1 + M'NO_B)^{\mathrm{Nrd}=1} Z(\mathbf{R}) \subset (K_1^* \cap G^*(\mathbf{Q})_+) Z(\mathbf{R}),$$

as required.

In fact, a more sophisticated version of the above argument [TX, Lemma 2.5] shows that one can choose K_2 and K_2^* so that $Sh_{K_2^*}(G^*, \mathscr{X}^*)^{an,+} = Sh_{K_2}(G, \mathscr{X})^{an,+}$. However, we are not going to use this refined statement.

(5.8) Corollary. (1) *There is a $G(\widehat{\mathbf{Q}})$ -equivariant isomorphism $Sh(G^*, \mathscr{X}^*) \times^U G(\widehat{\mathbf{Q}}) \simeq Sh(G, \mathscr{X})$, where $U \subset G(\widehat{\mathbf{Q}}) = \widehat{D}^\times$ is the stabiliser of $Sh(G^*, \mathscr{X}^*)$.*
(2) *For any ξ as in 5.5 there is a $\Gamma_{\mathbf{Q}} \times G(\widehat{\mathbf{Q}})$-equivariant isomorphism*

$$H^j_{et}(Sh(G, \mathscr{X}) \otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell}) \simeq \mathrm{Ind}_U^{G(\widehat{\mathbf{Q}})} H^j_{et}(Sh(G^*, \mathscr{X}^*) \otimes_E \overline{\mathbf{Q}}, i^*\mathscr{L}_{\xi,\ell}),$$

with smooth induction on the right hand side.

(5.9) If $D \not\simeq M_2(F)$, then $Sh_K(G, \mathscr{X})^{an}$ is compact (for each open compact subgroup $K \subset G(\widehat{\mathbf{Q}})$) and the formula (0.4.1) applies (with $m(\pi) = 1$ in (0.3.2), by the multiplicity one theorem for automorphic representations of $D_{\mathbf{A}}^\times$):

$$H^i = H^i_{et}(Sh(G, \mathscr{X}) \otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell}) = \bigoplus_{\pi^\infty} V^i(\pi^\infty) \otimes \pi^\infty, \tag{5.9.1}$$

where $\pi_\infty \otimes \pi^\infty$ is an automorphic representation of $G(\mathbf{A}) = D_{\mathbf{A}}^\times$ such that

$$\forall v \in \Omega \quad H^*(\mathfrak{g}l_2, O(2)\mathbf{R}^\times; \pi_v \otimes \xi_v) \neq 0 \tag{5.9.2}$$

and

$$\forall v \in \Omega^c \quad \pi_v \simeq \xi_v^\vee. \tag{5.9.3}$$

If $D \simeq M_2(F)$, then $Sh_{K^*}(G^*, \mathscr{X}^*)$ and $Sh_K(G, \mathscr{X})$ are Hilbert modular varieties and the formula (5.9.1) applies to the intersection cohomology of the Baily-Borel compactification $j : Sh_K(G, \mathscr{X}) \hookrightarrow Sh_K(G, \mathscr{X})_{BB} = Sh_K(G, \mathscr{X}) \cup \{\text{cusps}\}$:

$$H^i = \varinjlim_K (H^i)^K = \varinjlim_K H^i_{et}(Sh_K(G, \mathscr{X})_{BB} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_{!*}\mathscr{L}_{\xi,\ell}) = \varinjlim_K H^i(Sh_{K,BB}),$$

with $\pi = \pi_\infty \otimes \pi^\infty$ being a discrete ($\Longleftrightarrow$ cuspidal or one-dimensional) automorphic representation of $GL(\mathbf{A}_F)$ [BC]. As in the case $D \not\simeq M_2(F)$, the Galois representation $(H^i)^K$ is pure of weight $i + t(w-2)$ at all unramified primes not dividing ℓ.

The canonical map $(H^i)^K = H^i(Sh_{K,BB}) \longrightarrow H^i(Sh_K) = H^i_{et}(Sh_K(G, \mathscr{X}) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell})$ is almost always injective (see Proposition A6.17 below), the only exception being the case $i = 2r$, $k_v = 2$ for all $v \mid \infty$, when $H^{2r}(Sh_{K,BB})$ is dual to a Tate twist of $H^0(Sh_{K,BB}) = H^0(Sh_K)$.

(5.10) For fixed $v \in \Omega$, the condition (5.9.2) can be made explicit as follows.

(5.10.1) If $\dim(\pi_v) < \infty$, then necessarily $\dim(\pi) = 1$ and $\pi = \chi \circ \mathrm{Nrd}$ for some $\chi : \mathbf{A}_F^\times / F^\times \longrightarrow \mathbf{C}^\times$. The central and infinitesimal characters of $\xi_v \otimes (\chi_v \circ \mathrm{Nrd})$ must be trivial [BW, Thm. I.5.3], hence $k_v = 2$ and $\chi_v^2(a)a^{2-w} = 1$ for all $a \in F_v^\times = \mathbf{R}^\times$, which implies that $\chi \| \cdot \|_F^{1-w/2}$ is a character of finite order. In this case

$$\dim H^i(\mathfrak{g}l_2, O(2)\mathbf{R}^\times; \pi_v \otimes \xi_v) = \begin{cases} 1, & i = 0, 2 \\ 0, & \text{otherwise.} \end{cases}$$

(5.10.2) If $\dim(\pi_v) = \infty$, then the matching of central and infinitesimal characters of π_v and the dual of ξ_v [BW, Thm. I.5.3] implies that π_v is a discrete series representation of weight k_v (and appropriate central character). In this case

$$\dim H^i(\mathfrak{g}l_2, O(2)\mathbf{R}^\times; \pi_v \otimes \xi_v) = \begin{cases} 2, & i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

(5.11) Combining 5.10 with (5.9.3) and the strong multiplicity one theorem for automorphic representations of $D^\times_{\mathbf{A}}$ (specifically, the fact that $m(\pi) = 1$ and π is determined by π^∞), we deduce that $V^i(\pi^\infty) \neq 0$ precisely in the following two mutually exclusive cases.

(A) $\pi = \chi \circ \mathrm{Nrd}$, where $\chi : \mathbf{A}^\times_F/F^\times \longrightarrow \mathbf{C}^\times$ is a character such that $\chi\|\cdot\|_F^{1-w/2}$ is of finite order, $\dim(\xi) = 1$ ($\iff \forall v|\infty\ \ k_v = 2$) and

$$\dim V^i(\pi^\infty) = \begin{cases} \binom{t}{j}, & i = 2j \ (0 \leq j \leq t) \\ 0, & \text{otherwise.} \end{cases}$$

This corresponds to the universal cohomology classes given by the cohomology of the dual compact symmetric space $\mathbf{P}^1(\mathbf{C})^t$.

(B) π corresponds by the Jacquet-Langlands correspondence to a representation $\Pi = JL(\pi)$ of $GL_2(\mathbf{A}_F)$ attached (up to a twist) to a holomorphic cuspidal Hilbert modular newform ϕ (this still holds in the case $D \simeq M_2(F)$ when $\Pi = \pi$, since π was necessarily cuspidal). In this case

$$V(\pi^\infty) = V^t(\pi^\infty), \qquad \dim V(\pi^\infty) = 2^t.$$

In the case (A) we consider $\chi\|\cdot\|_F^{1-w/2}$ as a Galois character of finite order $\Gamma_F = \mathrm{Gal}(\overline{\mathbf{Q}}/F) \longrightarrow \mathbf{C}^\times \simeq \overline{\mathbf{Q}}_\ell^\times$, via the reciprocity map $\mathrm{rec}_F : \mathbf{A}^\times_F/F^\times \longrightarrow \Gamma_F^{ab}$. By abuse of language, we denote by $\chi : \Gamma_F \longrightarrow \overline{\mathbf{Q}}_\ell^\times$ the tensor product of this Galois character of finite order by $\overline{\mathbf{Q}}_\ell(w/2-1)$ (this is the ℓ-adic Galois representation attached to the algebraic Hecke character χ, [Sc, ch. 0, §5], [HT, p. 20]).

In the case (B) denote by $\rho_\pi = \rho_\Pi : \Gamma_F \longrightarrow GL_2(\overline{\mathbf{Q}}_\ell)$ the Galois representation attached to Π. It is irreducible, unramified outside $\ell\,\mathrm{cond}(\Pi)$ and satisfies

$$P_{\rho_\pi(\mathrm{Fr}(P))}(X) = X^2 - a_P X + \omega(P)(NP) \tag{5.11.1}$$

for all finite primes $P \nmid \ell\,\mathrm{cond}(\Pi)$ of F. Above, ω denotes the central character of Π (hence also of π) and a_P (resp. $\omega(P)$) is the eigenvalue of the Hecke operator T_P (resp. S_P) acting on the spherical vector of $\pi_P = \Pi_P$ (see A1.6 below). The character ω satisfies $\omega_v(a)a^{2-w} = 1$ for all $v \mid \infty$, which implies that $\omega\|\cdot\|_F^{2-w}$ has finite order. The Galois representation ρ_π is pure of weight $3 - w$ at all $P \nmid \ell\,\mathrm{cond}(\Pi)$.

The Langlands-Kottwitz method yields the following information ([La], [VSL], [R], [BL]).

(5.11.2) For $\pi = \chi \circ \mathrm{Nrd}$ as in (A) the dual representation is $\pi^\vee = \chi^{-1} \circ \mathrm{Nrd}$ and

$$V^{2j}(\pi^\infty)^{\mathrm{ss}} \simeq \Big(\bigwedge\nolimits^j \mathrm{Ind}_\Omega\, \overline{\mathbf{Q}}_\ell(-1)\Big) \otimes \det\big(\mathrm{Ind}_\Omega(\chi^{-1})\big) = \Big(\bigwedge\nolimits^j \mathrm{Ind}_\Omega\, \overline{\mathbf{Q}}_\ell(-1)\Big) \otimes \mathrm{Ind}^\otimes_\Omega(\chi^{-1}).$$

(5.11.3) If $JL(\pi) = \Pi$ is attached to π as in (B), then

$$V^t(\pi^\infty)^{\mathrm{ss}} \simeq \mathrm{Ind}^\otimes_\Omega(\rho_{\pi^\vee}),$$

where the dual representation $\pi^\vee$ satisfies $JL(\pi^\vee) = \Pi^\vee \simeq \Pi \otimes \omega^{-1}$ and $\rho_{\pi^\vee} \simeq \rho_\pi^\vee(-1)$ (cf. (A5.6.3); this Galois representation is pure of weight $w - 1$).

Had we used h^{-1} instead of h, then $\rho_{\pi^\vee}$ would have to be replaced by ρ_π.

(5.12) Partial (tensor) induction. In (5.11.2-3) we have denoted by Ind_Ω and $\mathrm{Ind}^\otimes_\Omega$, respectively, the partial induction and partial tensor induction functors which associate to a representation of Γ_F of dimension m a representation of Γ_E of dimension tm (resp. m^t). They are defined as follows.

The set $X = \mathrm{Hom}(F, \overline{\mathbf{Q}}) = \mathrm{Hom}(F, \mathbf{R}) = \Omega \cup \Omega^c$ is naturally identified with $\Gamma_{\mathbf{Q}}/\Gamma_F$. A choice of a section $s : X \longrightarrow \Gamma_{\mathbf{Q}}$ of the canonical projection defines an injective group morphism

$$i_s : \Gamma_{\mathbf{Q}} \hookrightarrow S_X \ltimes \Gamma_F^X, \qquad i_s(\gamma) = (\sigma, \delta), \qquad \gamma(s(x)) = s(\sigma(x))\delta(x).$$

By definition,

$$\Gamma_E = i_s^{-1}\big((S_\Omega \ltimes \Gamma_F^\Omega) \times (S_{\Omega^c} \ltimes \Gamma_F^{\Omega^c})\big).$$

Let M be any $A[\Gamma_F]$-module (where A is an arbitrary commutative ring). The wreath product $S_\Omega \ltimes \Gamma_F^\Omega$ acts naturally on $M^{\oplus\Omega}$ and $M^{\otimes\Omega}$. We let $S_{\Omega^c} \ltimes \Gamma_F^{\Omega^c}$ act trivially and define

$$\mathrm{Ind}_\Omega(M) = i_s^*\big(M^{\oplus\Omega}\big), \qquad \mathrm{Ind}_\Omega^\otimes(M) = i_s^*\big(M^{\otimes\Omega}\big).$$

The isomorphism classes of these two Γ_E-modules do not depend on s.

For $\gamma \in \Gamma_{E'}$ the image of $i_s(\gamma)$ in $S_\Omega \ltimes \Gamma_F^\Omega$ lies in Γ_F^Ω; it is equal to $(s(x)^{-1}\gamma s(x))_{x\in\Omega}$. It follows that

$$\mathrm{Ind}_\Omega(M)|_{\Gamma_{E'}} \simeq \bigoplus_{x\in\Omega} {}^{s(x)}M, \qquad \mathrm{Ind}_\Omega^\otimes(M)|_{\Gamma_{E'}} \simeq \bigotimes_{x\in\Omega} {}^{s(x)}M, \tag{5.12.1}$$

where we have denoted by ${}^{s(x)}M$ the pull-back of M via the map

$$\mathrm{int}(s(x))^{-1} : \Gamma_{E'} \longrightarrow s(x)^{-1}\Gamma_{E'}s(x) \subset \Gamma_F, \qquad \gamma \mapsto s(x)^{-1}\gamma s(x).$$

(5.13) It follows from (5.12.1) and (5.11.2-3) that

$$V^{2j}(\pi^\infty)^{\mathrm{ss}}|_{\Gamma_{E'}} \simeq \Big(\overline{\mathbf{Q}}_\ell(-j) \otimes \bigotimes_{x\in\Omega} {}^{s(x)}\chi^{-1}\Big)^{\oplus\binom{t}{j}} \tag{5.13.1}$$

in the case (A) and

$$V^t(\pi^\infty)^{\mathrm{ss}}|_{\Gamma_{E'}} \simeq \bigotimes_{x\in\Omega} {}^{s(x)}\rho_{\pi^\vee} \tag{5.13.2}$$

in the case (B).

The finiteness of the class number implies that $H^1_g(k, \mathbf{Q}_\ell) = 0$ for any number field k. As a result, any representation $\sigma : \Gamma_k \longrightarrow GL_n(\overline{\mathbf{Q}}_\ell)$ which is de Rham at all primes above ℓ and for which $\sigma^{\mathrm{ss}} \simeq \alpha^{\oplus n}$ for some character $\alpha : \Gamma_k \longrightarrow \overline{\mathbf{Q}}_\ell^\times$ is necessarily semisimple, $\sigma = \sigma^{\mathrm{ss}}$. In particular, (5.13.1) together with 3.2 imply that, in the case (A), each $V^{2j}(\pi^\infty)$ is semisimple. Consequently, we can replace $V^{2j}(\pi^\infty)^{\mathrm{ss}}$ by $V^{2j}(\pi^\infty)$ in (5.11.2).

(5.14) In 5.11-13 we have summed up the information about the Galois representations $V^i(\pi^\infty)$ which can be obtained by the Langlands-Kottwitz method. In fact, this method yields (5.11.3) in a form which does not assume the existence of the Galois representation $\rho_{\pi^\vee}$.

We are now going to revisit the representations $V^i(\pi^\infty)|_{\Gamma_{E'}}$, by applying Eichler-Shimura relations together with the abstract results of §3 and §4 (but, unlike in the Langlands-Kottwitz method, assuming the existence of $\rho_{\pi^\vee}$).

(5.15) Fix π as in 5.11, i.e., assume that $V^i(\pi^\infty) \neq 0$. Fix a neat open compact subgroup $K \subset G(\widehat{\mathbf{Q}})$ such that $(\pi^\infty)^K \neq 0$. This implies that

$$0 \neq V^i(\pi^\infty) \otimes (\pi^\infty)^K \subset H^i_{et}(Sh_K(G, \mathscr{X}) \otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell}). \tag{5.15.1}$$

There exists a finite set S of primes of F such that $K = K_S K^S$, where $K^S = \prod_{v\notin S} K_v$ with $(K_v, G(F_v)) \simeq (GL_2(O_{F,v}), GL_2(F_v))$ for all $v \notin S$. We can, and will, assume that S contains all infinite primes, all primes dividing 2ℓ and all primes at which $F/\mathbf{Q}$ and D are ramified. Denote by $\mathbf{Q}_S/\mathbf{Q}$ the maximal subextension of $\overline{\mathbf{Q}}/\mathbf{Q}$ which is unramified at all rational primes not lying below S. Note that $\mathbf{Q}_S \supset F^{gal}$.

Fix an intermediate field $E' \subset \widetilde{E} \subset F^{gal}$. We are going to consider primes P_S of $\mathbf{Q}_S$ which are unramified in $\mathbf{Q}_S/\mathbf{Q}$ and which satisfy

$$\mathrm{Fr}_{\mathbf{Q}_S/\mathbf{Q}}(P_S) \in \mathrm{Gal}(\mathbf{Q}_S/\widetilde{E}). \tag{5.15.2}$$

Denote by

$$p = P_S \cap \mathbf{Z}, \qquad P = P_S \cap O_E, \qquad \widetilde{P} = P_S \cap O_{\widetilde{E}}$$

the respective primes of $\mathbf{Q}$, E and $\widetilde{E}$ below P_S. It follows from the condition (5.15.2) that

$$\mathbf{Q}_p = E_P = \widetilde{E}_{\widetilde{P}}.$$

Moreover, for each $x \in \Omega$

$$s(x)^{-1}\mathrm{Fr}_{\mathbf{Q}_S/\mathbf{Q}}(P_S)s(x) \in s(x)^{-1}\mathrm{Gal}(\mathbf{Q}_S/\widetilde{E})s(x) \subset s(x)^{-1}\mathrm{Gal}(\mathbf{Q}_S/E')s(x) \subset \mathrm{Gal}(\mathbf{Q}_S/F),$$

which implies that the following primes of F

$$P_x = s(x)^{-1}P_S \cap O_F \qquad (x \in \Omega) \tag{5.15.3}$$

are distinct and satisfy $F_{P_x} = \mathbf{Q}_p$.

(5.16) Eichler-Shimura relations. The following statements are discussed in A6 below.

In the situation of 5.15, the Shimura variety $Sh_K(G, \mathscr{X})$ has (for sufficiently small K^p) a canonical model S_K over $O_{E,P} = \mathbf{Z}_p$. Denote by $S_K^\circ = S_K \otimes_{\mathbf{Z}_p} \mathbf{F}_p$ its special fibre.

The Frobenius morphism $\varphi : S_K^\circ \longrightarrow S_K^\circ$ has degree $\deg(\varphi) = p^t$ and the action of $\mathrm{Fr}(P_S)$ on $H^i(Sh_K) = H^i_{et}(Sh_K(G, \mathscr{X}) \otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell}) \simeq H^i_{et}(S_K^\circ \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p, \mathscr{L}^\circ_{\xi,\ell})$ is given by the action $(\varphi \otimes \mathrm{id})^*$ of $\varphi \otimes \mathrm{id} : S_K^\circ \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p \longrightarrow S_K^\circ \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p$.

A cohomological form of Eichler-Shimura relations in our situation asserts that $(\varphi \otimes \mathrm{id})^* \in \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(H^i(Sh_K))$ naturally decomposes as a product of pairwise commuting cohomological partial Frobenius automorphisms

$$(\varphi \otimes \mathrm{id})^* = \prod_{x \in \Omega} \varphi_x^*, \qquad \varphi_x^* \in \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(H^i(Sh_K)), \qquad \varphi_x^* \varphi_y^* = \varphi_y^* \varphi_x^*$$

satisfying the following quadratic equations, which generalise (0.8.1):

$$\forall x \in \Omega \quad Q_x(\varphi_x^*) = 0, \qquad Q_x(Y) = Y^2 - (T_{P_x}/S_{P_x})Y + p/S_{P_x}. \tag{5.16.1}$$

The relations (5.16.1) are proved in A6.4 and A6.19 (the case $t = r$) and A6.14 (the case $t < r$) below for $\widetilde{E} = F^{gal}$.

In the totally indefinite case $t = r$ (see A6.4) the automorphisms φ_x^* are given by the action of $\varphi_x \otimes \mathrm{id}$ on $H^i(Sh_K)$, where $\varphi_x : S_K^\circ \longrightarrow S_K^\circ$ are mutually commuting ($\varphi_x\varphi_y = \varphi_y\varphi_x$) geometric partial Frobenius morphisms of degree $\deg(\varphi_x) = NP_x = p$. In the case $t < r$ we construct geometric partial Frobenius morphisms on the special fibre of a closely related unitary Shimura variety, and then transfer the corresponding cohomological partial Frobenius automorphisms to the quaternionic Shimura variety (see A6.13).

The relations (5.16.1) imply that the action of $\mathrm{Fr}(P_S) = (\varphi \otimes \mathrm{id})^* = \prod_{x \in \Omega} \varphi_x^*$ on $H^i(Sh_K)$ satisfies the following equation of degree 2^t:

$$Q(\varphi^*) = 0, \qquad Q = \bigotimes_{x \in \Omega} Q_x, \tag{5.16.2}$$

where $\bigotimes$ is the "Rankin-Selberg product": writing formally $Q_x(Y) = (Y - \alpha_{x,1})(Y - \alpha_{x,2})$, then

$$Q(Y) = \prod_{i : \Omega \longrightarrow \{1,2\}} \big(Y - \prod_{x \in \Omega} \alpha_{x,i(x)}\big). \tag{5.16.3}$$

In the essentially PEL case $t = r$ (when $E = \mathbf{Q}$ and $E' = \widetilde{E} = F^{gal}$) the formula (5.16.2) follows from [Mo, Cor. 4.2.15] and [W, Theorem, p. 44] (taking into account Corollary 5.8(2)).

In the isotropic case $D \simeq M_2(F)$ the relations (5.16.1-2) also hold for the action of the φ_x^* on $H^i = H^i(Sh_{K,BB})$, in the notation of 5.9 (see A6.19).

(5.17) For $\pi = \chi \circ \mathrm{Nrd}$ in the case (A), the formulas (5.16.1) and (5.16.2) imply, respectively (thanks to (5.15.1)) that

$$\forall x \in \Omega \qquad \big(\varphi_x^* - p\,\chi^{-1}(P_x)\big)\big(\varphi_x^* - \chi^{-1}(P_x)\big)\big|_{V^i(\pi^\infty)\otimes(\pi^\infty)^K} = 0 \tag{5.17.1}$$

resp.

$$\prod_{I\subset\Omega}\Big(\mathrm{Fr}(P_S) - p^{|I|}\chi^{-1}\big(\prod_{x\in\Omega} P_x\big)\Big)\Big|_{V^i(\pi^\infty)\otimes(\pi^\infty)^K} = 0. \tag{5.17.2}$$

(5.18) Theorem. *For $\pi = \chi\circ\mathrm{Nrd}$ in the case (A), the relation (5.16.2) (which holds for $\widetilde{E} = F^{gal}$, thanks to [Mo] and [W] in the case $t = r$ and (A6.14.3) in the case $t < r$) implies (5.13.1). In other words, the Eichler-Shimura relation for the usual Frobenius morphism (for P_S as in (5.15.2)) determines the isomorphism class of each $V^i(\pi^\infty)|_{\Gamma_{\widetilde{E}}}$ (thanks to the remarks at the end of 5.13).*

Proof. As H^{2j} is pure of weight $2j + t(w-2)$ and $\chi\|\cdot\|_F^{1-w/2}$ is of finite order, it follows from (5.17.2) that, for each P_S satisfying (5.15.2), $\mathrm{Fr}(P_S)$ acts on $V^{2j}(\pi^\infty)^{\mathrm{ss}}$ by the scalar $p^j\chi^{-1}(\prod_{x\in\Omega} P_x)$. The set of elements $\mathrm{Fr}(P_S)$ is dense in $\mathrm{Gal}(\mathbf{Q}_S/\widetilde{E})$, which implies that

$$V^{2j}(\pi^\infty)^{\mathrm{ss}}|_{\Gamma_{\widetilde{E}}} \simeq \big(\overline{\mathbf{Q}}_\ell(-j)\otimes\bigotimes_{x\in\Omega}{}^{s(x)}\chi^{-1}\big)^{\oplus\binom{t}{j}},$$

as claimed. Note that one can rewrite the above formula in a more succint form as

$$V(\pi^\infty)^{\mathrm{ss}}|_{\Gamma_{\widetilde{E}}} \simeq \big(\chi^{-1}\otimes(\overline{\mathbf{Q}}_\ell\oplus\overline{\mathbf{Q}}_\ell(-1))\big)^{\otimes t}\big|_{\Gamma_{\widetilde{E}}}.$$

As remarked in 5.13, the finiteness of the class number implies that each $V^{2j}(\pi^\infty)^{\mathrm{ss}} = V^{2j}(\pi^\infty)$ is semisimple.

(5.19) If we are in the case (B) and $JL(\pi) = \Pi$, write $\Omega = \{x_1,\ldots,x_t\}$ and $\rho_i = {}^{s(x_i)}\rho_{\pi^\vee} : \Gamma_{E'} \longrightarrow GL_2(\overline{\mathbf{Q}}_\ell)$ $(1 \le i \le t)$. Denote by $\rho : \Gamma_{E'} \longrightarrow GL_{2^t}(\overline{\mathbf{Q}}_\ell)$ the action of $\Gamma_{E'}$ on $V^t(\pi^\infty)$.

The relation (5.16.1) implies (again thanks to (5.15.1)) that, for each P_S satisfying (5.15.2),

$$\rho(\mathrm{Fr}(P_S)) = u_1\cdots u_t, \qquad u_i = \varphi_{x_i}^*, \qquad u_iu_j = u_ju_i, \qquad P_{\rho_i(\mathrm{Fr}(P_S))}(u_i) = 0. \tag{5.19.1}$$

Similarly, the relation (5.16.2) implies that

$$P_{(\rho_1\otimes\cdots\otimes\rho_t)(\mathrm{Fr}(P_S))}\big(\rho(\mathrm{Fr}(P_S))\big) = 0, \tag{5.19.2}$$

hence

$$\forall g \in \Gamma_{\widetilde{E}} \qquad P_{(\rho_1\otimes\cdots\otimes\rho_t)(g)}\big(\rho(g)\big) = 0, \tag{5.19.3}$$

by the Čebotarev density theorem.

(5.20) Theorem. *Assume that we are in the case (B) and $JL(\pi) = \Pi$.*

(1) *The relation (5.16.2) implies that there exists a finite extension $E''/\widetilde{E}$ and an integer $m \ge 1$ such that*

$$V^t(\pi^\infty)^{\mathrm{ss}}|_{\Gamma_{E''}} \subset \Big(\bigotimes_{x\in\Omega}{}^{s(x)}\rho_{\pi^\vee}\Big)\Big|_{\Gamma_{E''}}^{\oplus m}.$$

If, in addition, ϕ has complex multiplication by a totally imaginary quadratic extension M of F, then E'' satisfies $M^{gal} \not\subset (M^{gal})^+E''$, where M^{gal} denotes the Galois closure of M in $\overline{\mathbf{Q}}$ and $(M^{gal})^+$ the maximal

totally real subfield of the CM field M^{gal}.
(2) *If ϕ does not have complex multiplication and if the weights $(k_x)_{x\in\Omega}$ are distinct, then the representation $\bigotimes_{x\in\Omega}{}^{s(x)}\rho_{\pi^\vee}$ of $\Gamma_{E'}$ is strongly irreducible and the relation (5.16.2) implies that (5.13.2) holds after restricting to $\Gamma_{\widetilde{E}}$.*
(3) *The relation (5.16.1) implies that the representation $V^t(\pi^\infty)$ is semisimple.*

Proof. We must distinguish two cases.

If ϕ has complex multiplication, then $\rho_{\pi^\vee} = \mathrm{Ind}_{\Gamma_M}^{\Gamma_F}(\alpha) = I(\alpha)$, where $M \subset \overline{\mathbf{Q}}$ is a totally imaginary quadratic extension of F and $\alpha : \Gamma_M \longrightarrow \overline{\mathbf{Q}}_\ell^\times$ a character. Let M' be the Galois closure of M in $\overline{\mathbf{Q}}$. It is also a CM field and its maximal totally real subfield F' is a Galois extension of $\mathbf{Q}$ containing F^{gal} (hence $\widetilde{E}$). Using the notation from 5.19, we have, for each $i = 1, \ldots, t$,

$$\rho_i\big|_{\Gamma_{F'}} = \mathrm{Ind}_{\Gamma_{M'}}^{\Gamma_{F'}}(\alpha_i), \qquad \alpha_i = {}^{s(x_i)}\big(\alpha\big|_{\Gamma_{M'}}\big).$$

Thanks to (5.19.3) (resp. (5.19.1)), Proposition 4.6 (resp. Theorem 4.7(2)) applies to the restrictions of ρ and ρ_i to $\Gamma_{F'}$ (note that $n = 2$, $\widetilde{\sigma} = c$ is the complex conjugation and the character α_i/α_i^c has infinite order, since the Hodge-Tate weights of α and α^c are distinct). The statement (1) (resp. (3)) of the theorem follows.

If ϕ has no complex multiplication, then $\overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}(\rho_{\pi^\vee}(\Gamma_F)) = \mathfrak{gl}_2(\overline{\mathbf{Q}}_\ell)$ and $\rho_{\pi^\vee}$ is strongly irreducible. Thanks to (5.19.3) (resp. (5.19.1)), Proposition 3.10(1) (resp. Theorem 3.12(2)) applies to the representations $\rho = V^t(\pi^\infty)$ and ρ_i of $\Gamma = \Gamma' = \mathrm{Gal}(\mathbf{Q}_S/\widetilde{E})$ and to $\Sigma = \{\mathrm{Fr}(P_S)\}$. The statement (1) (resp. (3)) of the theorem follows. The statement (2) will follow from Proposition 3.10(2) once we show that $\bigotimes_{x\in\Omega}{}^{s(x)}\rho_{\pi^\vee} = \rho_1 \otimes \cdots \otimes \rho_t$ is strongly irreducible.

Fix $1 \leq i \neq j \leq t$. The $\overline{\mathbf{Q}}_\ell$-Lie algebra

$$\mathfrak{g} = \overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}\big((\rho_i \oplus \rho_j)(\Gamma)\big) \subset \overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}(\rho_i(\Gamma)) \oplus \overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}(\rho_j(\Gamma)) = \mathfrak{gl}_2(\overline{\mathbf{Q}}_\ell) \times \mathfrak{gl}_2(\overline{\mathbf{Q}}_\ell)$$

satisfies $p_1(\mathfrak{g}) = p_2(\mathfrak{g}) = \mathfrak{gl}_2(\overline{\mathbf{Q}}_\ell)$. If the desired strong irreducibility statement does not hold, the last part of Proposition 2.2 implies (together with Proposition 2.1) that $\mathscr{D}(\mathfrak{g}) \subset \mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell) \times \mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$ is the graph of a Lie algebra isomorphism $\mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell) \xrightarrow{\sim} \mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$. Every automorphism of $\mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$ is inner, which means that, after conjugating ρ_j by a suitable matrix $A \in GL_2(\overline{\mathbf{Q}}_\ell)$, $\mathscr{D}(\mathfrak{g})$ will coincide with the diagonally embedded $\mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$. Moreover, the determinants of ρ_i and ρ_j differ by a character of finite order, which implies that $\mathfrak{g}$ itself coincides with the diagonally embedded $\mathfrak{gl}_2(\overline{\mathbf{Q}}_\ell)$. As a result, the restrictions of ρ_i and ρ_j to an appropriate open subgroup $U \subset \Gamma$ are isomorphic (this is equivalent, by Schur's Lemma, to the existence of a character of finite order $\alpha : \Gamma \longrightarrow \overline{\mathbf{Q}}_\ell^\times$ such that $\rho_i \simeq \rho_j \otimes \alpha$, but we do not need this fact). This is impossible, since the Hodge-Tate weights of ρ_i (with respect to $E' \subset \overline{\mathbf{Q}} \subset \mathbf{C} \simeq \overline{\mathbf{Q}}_\ell$) are equal to $(1-(w+k_{x_i})/2, (k_{x_i}-w)/2)$, hence are different from those of ρ_j. This contradiction concludes the proof of (2).

(5.21) Remarks. (1) The proof of Theorem 5.20(2) shows that the conclusion holds more generally, namely, if we assume that ϕ has no complex multiplication and ${}^{s(x)}\rho_{\pi^\vee} \not\simeq {}^{s(y)}\rho_{\pi^\vee} \otimes \alpha$ for any $x \neq y \in \Omega$ and any character of finite order α of $\Gamma_{E'}$.
(2) Is it possible to deduce from Theorem 5.20(2) the full statement of the restriction of (5.13.2) to $\Gamma_{\widetilde{E}}$ (for ϕ without complex multiplication) by letting ϕ vary in a ℓ-adic family?

(5.22) Corollary. *For every ξ as in 5.5, the action of Γ_E on $H^i_{et}(Sh_K(G, \mathscr{X}) \otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell})$ in the case $D \not\simeq M_2(F)$ (resp. on $H^i_{et}(Sh_K(G, \mathscr{X})_{BB} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_{!*}\mathscr{L}_{\xi,\ell})$ in the case $D \simeq M_2(F)$) is semisimple, and the same result holds for $Sh_{K^*}(G^*, \mathscr{X}^*)$ (and $i^*\mathscr{L}_{\xi,\ell}$)) if $D \otimes \mathbf{R} \simeq M_2(\mathbf{R})^r$.*

6. Cohomology of quaternionic Shimura varieties (bis)

(6.1) In this section we investigate the cohomology of $Sh(G^*, \mathscr{X}^*)$ with coefficients in local systems that do not come from $Sh(G, \mathscr{X})$. The notation is as in §5. We assume throughout §6 that $D \otimes \mathbf{R} \simeq M_2(\mathbf{R})^r$ (i.e., that $\Omega = X$, $t = r$).

(6.2) An irreducible algebraic representation ξ^* of $G^*_{\mathbf{C}}$ is a restriction to $G^*_{\mathbf{C}}$ of a representation $\bigotimes_{v\in X}\xi_v$ of $G_{\mathbf{C}}$, where $\xi_v = \mathrm{Sym}^{k_v-2}(\mathrm{Std}^\vee)\otimes(\det\circ\mathrm{Std}^\vee)^{m_v} : GL(2)_{\mathbf{C}} \longrightarrow GL(k_v-1)_{\mathbf{C}}$ ($k_v\geq 2$, $m_v\in\mathbf{Z}$). The corresponding ℓ-adic local system $\mathscr{L}_{\xi^*,\ell} = \mathscr{L}_{\underline{k},\underline{m}}$ on $Sh_{K^*}(G^*,\mathscr{X}^*)$ (for small enough $K^*\subset G^*(\widehat{\mathbf{Q}})$) is pure of weight $w^* = \sum_{v\in X}(k_v-2+2m_v)$ and satisfies $\mathscr{L}_{\underline{k},\underline{m}} = \mathscr{L}_{\underline{k},0}(-\sum_{v\in X}m_v)$. This implies that $\mathscr{L}_{\underline{k},\underline{m}}$ is a Tate twist of $i^*\mathscr{L}_{\xi,\ell}$ for some ξ as in (5.5.1) iff

$$\forall v,v'\in X \qquad k_v\equiv k_{v'} \pmod 2 \tag{6.2.1}$$

(in other words, iff $\underline{k}=(k_v)_{v\in X}$ is a "motivic weight" in the language of [BR]).

(6.3) We fix $\underline{k}$ and $\underline{m}=(m_v)_{v\in X}$ and write $H^i = H^i_{et}(Sh(G^*,\mathscr{X}^*)\otimes_{\mathbf{Q}}\overline{\mathbf{Q}},\mathscr{L}_{\underline{k},\underline{m}})$ in the case $D\not\simeq M_2(F)$ (resp. $H^i = H^i_{et}(Sh(G^*,\mathscr{X}^*)_{BB}\otimes_{\mathbf{Q}}\overline{\mathbf{Q}}, j_{!*}\mathscr{L}_{\underline{k},\underline{m}})$ in the case $D\simeq M_2(F)$).

If the weight $\underline{k}$ is motivic, Corollary 5.22 implies that H^i is a semisimple $\Gamma_{\mathbf{Q}}$-module. If $\underline{k}$ is not motivic, we are going to prove in Corollary 6.20 below an analogous result for the non CM part of H^i. The techniques of §5 do not apply in this case directly, only after a passage to an auxiliary totally imaginary quadratic extension F_c of F. For a good prime p that splits completely in F_c/F one cannot define the partial Frobenius morphism φ_x on the special fibre of $Sh_{K^*}(G^*,\mathscr{X}^*)$ at p, only a certain twist of its square φ_x^2. Similarly, there is no Galois representation of Γ_F attached to a Hilbert modular form of non-motivic weight involved in the decomposition of H^r, but there is a Galois representation of Γ_{F_c} attached to a suitable twist of the base change of the Hilbert modular form to F_c [BR]. Working over F_c with such twisted objects leads to a proof of a variant of Theorem 5.20, but only in the non CM case (see Theorem 6.19 below). Instead of the Eichler-Shimura relation for the action of φ_x on the cohomology of $Sh_K(G,\mathscr{X})$ given in A6.4 below we use the results of [Mo] and [W], which apply to the action of the twisted version of φ_x^2. This method also works for motivic weights; it reproves the non CM case of Theorem 5.20 for $t=r$.

(6.4) As in §5, there is a $\Gamma_{\mathbf{Q}}\times G^*(\widehat{\mathbf{Q}})$-equivariant decomposition

$$H^i = \bigoplus_{(\pi^*)^\infty} V^i((\pi^*)^\infty)\otimes(\pi^*)^\infty,$$

where $\pi^* = \pi^*_\infty\otimes(\pi^*)^\infty$ is an automorphic representation of $G^*(\mathbf{A})$ such that π^*_∞ is cohomological for ξ^* in degree i and π^* is one-dimensional or cuspidal (which is automatic if $D\not\simeq M_2(F)$). We are going to investigate the ℓ-adic representation $V^i((\pi^*)^\infty)$ of $\Gamma_{\mathbf{Q}}$, which is pure of weight $i+w^*$ at all unramified primes. As in 5.11, $V^i((\pi^*)^\infty)\neq 0$ only in the following two cases.

(6.5) Case (A): $\dim(\pi^*)=1$. In this case $\pi^* = \chi^*\circ\mathrm{Nrd}$, $\chi^* : \mathbf{A}^\times/\mathbf{Q}^\times \longrightarrow \mathbf{C}^\times$, $k_v=2$ for all $v\in X$, $\mathscr{L}_{\xi^*,\ell} = \overline{\mathbf{Q}}_\ell(-m) = (i^*\mathscr{L}_{\xi,\ell})(-m)$, $m=\sum_v m_v$, ξ is the trivial representation of $G_{\mathbf{C}}$ and $\chi^*_\infty(a)=a^{2m}$.

We can assume that $m_v=0$ for all $v\in X$; then χ^* can be identified (via the reciprocity map) with a Galois character of finite order $\chi^* : \Gamma_{\mathbf{Q}}\longrightarrow\mathbf{C}^\times\simeq\overline{\mathbf{Q}}_\ell^\times$. The arguments from 5.18 show that the Eichler-Shimura relation (proved in [Mo] and [W]) for the usual Frobenius acting on H^i (at good primes that split completely in $F/\mathbf{Q}$) implies that $V^i((\pi^*)^\infty)=0$ if $i\not\in\{0,2,\ldots,2r\}$ and

$$V^{2j}((\pi^*)^\infty)\big|_{\Gamma_{F^{gal}}} \simeq \overline{\mathbf{Q}}_\ell(-j)\otimes(\chi^*)^{-1}\big|^{\oplus m_j}_{\Gamma_{F^{gal}}} \qquad (0\leq j\leq r). \tag{6.5.1}$$

(6.6) Case (B): $\dim(\pi^*)=\infty$, π^* cuspidal.

As the restriction of π^*_∞ to $SL_2(F\otimes\mathbf{R})$ is of infinite dimension and cohomological in degree i for ξ^*, we have necessarily that $i=r$ and $\pi^*_\infty|_{SL_2(F\otimes\mathbf{R})}$ is a direct sum of tensor products $\bigotimes_{v\in X}$ of (holomorphic or antiholomorphic) discrete series representations of weight k_v of $SL_2(F_v)=SL_2(\mathbf{R})$.

The central character $\omega_{\pi^*} : \mathbf{A}^\times/\mathbf{Q}^\times \longrightarrow \mathbf{C}^\times$ of π^* satisfies $(\omega_{\pi^*})_\infty = \omega_{\xi^*}^{-1} : a\mapsto a^{w^*}$, which implies that $\pi^*(-w^*/2) = \pi^*\otimes(\|\cdot\|_{\mathbf{Q}}\circ\mathrm{Nrd})^{-w^*/2}$ has central character of finite order. As in [LSc, Prop. 3.5], every cuspidal automorphic form on $G^*(\mathbf{A})$ extends to a cuspidal automorphic form on $G(\mathbf{A})=D^\times_{\mathbf{A}}$. As a result, there exists a cuspidal automorphic representation π of $D^\times_{\mathbf{A}}$ with central character ω_π of finite order such that $\pi^*(-w^*/2)$ is isomorphic to a quotient of the restriction of π to $G^*(\mathbf{A})$. Fix such a π.

For each $v\in X$, π_v is a discrete series representation of weight k_v and central character $\omega_{\pi_v}=(\mathrm{sgn})^{k_v}$ of $GL_2(\mathbf{R})$. The Jacquet-Langlands transfer $JL(\pi)$ of π to $GL_2(\mathbf{A}_F)$ is cuspidal, since π is cuspidal in the case $D\simeq M_2(F)$.

(6.7) The representation $JL(\pi)$ corresponds, up to a twist, to a cuspidal holomorphic Hilbert modular newform ϕ of weight $\underline{k} = (k_v)_{v\in X}$. Blasius and Rogawski [BR] attached compatible systems of Galois representations to suitable twists of $JL(\pi)$ by Hecke characters. Their setup is the following.

Fix an auxiliary imaginary quadratic field E_0 and let $F_c = E_0F$ (if ϕ has complex multiplication by a totally imaginary quadratic extension M of F, assume that $E_0 \not\subset M^{gal}$). Fix an embedding of E_0 into $\mathbf{C}$ (as $F \subset \overline{\mathbf{Q}} \subset \mathbf{C}$ by assumption, this defines a distinguished embedding $F_c \hookrightarrow \overline{\mathbf{Q}} \subset \mathbf{C}$) and denote by $\{\sigma_x : F_c \hookrightarrow \mathbf{C}\}_{x\in X}$ the induced CM type of F_c.

According to Proposition A6.15 there exists a character $\psi : \mathbf{A}_{F_c}^\times/F_c^\times \longrightarrow \mathbf{C}^\times$ satisfying

$$\psi\big|_{\mathbf{A}_F^\times} = \omega_\pi^{-1}, \qquad \forall x \in X \quad \psi_x(a) = (\sigma_x(a)/|\sigma_x(a)|)^{k_x};$$

fix such a ψ. The twisted base change

$$\Pi = BC_{F_c/F}(JL(\pi)) \otimes \psi \tag{6.7.1}$$

is a cuspidal automorphic representation of $GL_2(\mathbf{A}_{F_c})$ (since $JL(\pi)$ does not have CM by F_c) such that

$$\Pi^\vee \simeq \Pi^c, \qquad \omega_\Pi = \psi/\psi^c, \qquad \forall x \in X \quad (\omega_\Pi)_x(a) = (\sigma_x(a)/\overline{\sigma_x(a)})^{k_x}$$

(above, c denotes the non-trivial element of $\mathrm{Gal}(F_c/F)$ and $(\psi^c)(a) = \psi(c^{-1}(a))$).

According to [BR, Thm. 2.6.1] there exists a semisimple Galois representation

$$\rho_\Pi = \rho_{\Pi,\ell} : \Gamma_{F_c} \longrightarrow GL_2(\overline{\mathbf{Q}}_\ell)$$

such that (note that our normalisations – including the values of $(k_x)_{x\in X}$ – differ from those of [BR])

$$L_v(\rho_\Pi, s) = L_v(\Pi, \mathrm{Std}, s - 1/2) \qquad\qquad (\forall v \nmid \ell\,\mathrm{cond}(\Pi)\,\mathrm{cond}(\psi) D_{F_c}). \tag{6.7.2}$$

(6.8) From now on, until 6.19, assume that $V^r((\pi^*)^\infty) \neq 0$. There exists an open compact subgroup $K \subset G(\widehat{\mathbf{Q}}) = \widehat{D}^\times$ such that $(\pi^\infty)^K \neq 0 \neq ((\pi^*)^\infty)^{K^*}$, where $K^* = K \cap G^*(\widehat{\mathbf{Q}})$; fix such a K.

Let S be a finite set of primes of F satisfying the properties listed in 5.15; we require, in addition, all primes ramified in F_c/F to be contained in S. Let p be a rational prime not lying below S. After shrinking K_S if necessary, there exists a smooth quasi-projective model S_{K^*} (projective if $D \not\simeq M_2(F)$) of $Sh_{K^*}(G^*, \mathscr{X}^*)$ over $\mathbf{Z}_{(p)}$ constructed in [Ko, §5] (cf. A6.3 below). As in 5.16, $H^i(Sh_{K^*})$ is isomorphic to $H^i_{et}(S^\circ_{K^*} \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p, \mathscr{L}^\circ_{\xi^*,\ell})$, where $S^\circ_{K^*}$ denotes the special fibre of S_{K^*} (this is also true in the case $D \simeq M_2(F)$, as explained in A5.11.2).

(6.9) Let P_S be a prime of $\mathbf{Q}_S$ unramified in $\mathbf{Q}_S/\mathbf{Q}$ such that

$$\mathrm{Fr}_{\mathbf{Q}_S/\mathbf{Q}}(P_S) \in \mathrm{Gal}(\mathbf{Q}_S/F_c^{gal}), \tag{6.9.1}$$

where F_c^{gal} denotes the Galois closure of F_c in $\overline{\mathbf{Q}}$. The rational prime $p = P_S \cap \mathbf{Z}$ is then as in 6.8; moreover, it splits completely in $F_c/\mathbf{Q}$. Extend each element $\sigma_x : F_c \hookrightarrow \mathbf{C}$ ($x \in X$) of the CM type of F_c to an element $s(x) \in \Gamma_{\mathbf{Q}}$. As in 5.15 and A5.9, we obtain primes above p in F and F_c, respectively, given by

$$P_x = s(x)^{-1}P_S \cap O_F, \qquad P'_x = s(x)^{-1}P_S \cap O_{F_c}$$

that depend only on σ_x and such that

$$pO_F = \prod_{x\in X} P_x, \qquad P_xO_{F_c} = P'_xP''_x, \qquad F_{P_x} = (F_c)_{P'_x} = \mathbf{Q}_p.$$

(6.10) The pair $(G(\mathbf{Q}_p), K_p)$ is isomorphic to $\prod_{x\in X}(GL_2(F_{P_x}), GL_2(O_{F,P_x})) = (GL_2(\mathbf{Q}_p), GL_2(\mathbf{Z}_p))^X$; its subgroup $(G^*(\mathbf{Q}_p), K_p^*)$ corresponds to the elements whose determinant lies in the diagonally embedded sugroup $(\mathbf{Q}_p^\times, \mathbf{Z}_p^\times) \subset (\mathbf{Q}_p^\times, \mathbf{Z}_p^\times)^X$.

As in A4.3, the cocharacter $\mu : \mathbf{G}_{m,\mathbf{Q}_p} \longrightarrow G_{\mathbf{Q}_p} \simeq GL(2)^X_{\mathbf{Q}_p}$ from A3.2 (attached to the Shimura datum h) decomposes as $\mu = (\mu_{P_x})_{x\in X}$, where $\mu_{P_x}(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. The corresponding parabolic $P_\mu \subset G_{\mathbf{Q}_p}$ (see A1 below) is the product of upper Borel subgroups; its Levi subgroup M is the product of the diagonal maximal tori in $GL(2)_{\mathbf{Q}_p}$. We identify M with its set of $\mathbf{Q}_p$-points and we write $M^* = M \cap G^*(\mathbf{Q}_p)$, $L = M \cap K_p$ and $L^* = M^* \cap K_p^*$.

The maps $K_p^* g K_p^* \mapsto K_p g K_p$ (resp. $mL^* \mapsto mL$) define embeddings of Hecke algebras

$$\mathcal{H}(G^*(\mathbf{Q}_p)//K_p^*, \mathbf{Q}) \hookrightarrow \mathcal{H}(G(\mathbf{Q}_p)//K_p, \mathbf{Q}) = \bigotimes_{P|p} \mathcal{H}(H(F_P)//K_P, \mathbf{Q}) \simeq \mathcal{H}(GL_2(\mathbf{Q}_p)//GL_2(\mathbf{Z}_p), \mathbf{Q})^X$$

and

$$\mathcal{H}(M^*//L^*, \mathbf{Q}) \hookrightarrow \mathcal{H}(M//L, \mathbf{Q}) = \bigotimes_{P|p} \mathcal{H}(M_P//L_P, \mathbf{Q}) \simeq \mathcal{H}((\mathbf{Q}_p^\times \times \mathbf{Q}_p^\times)//(\mathbf{Z}_p^\times \times \mathbf{Z}_p^\times), \mathbf{Q})^X$$

that are compatible with the twisted Satake transforms

$$\overline{S}_\mu : \mathcal{H}(G^*(\mathbf{Q}_p)//K_p^*, \mathbf{Q}) \longrightarrow \mathcal{H}(M^*//L^*, \mathbf{Q}), \qquad \overline{S}_\mu : \mathcal{H}(G(\mathbf{Q}_p)//K_p, \mathbf{Q}) \longrightarrow \mathcal{H}(M//L, \mathbf{Q})$$

from A1.4. These Hecke algebras contain the following important elements (with $P = P_x$, $x \in X$, $NP = p$):

- $S_P, T_P, T_{P^2} \in \mathcal{H}(H(F_P)//K_P, \mathbf{Q})$, where $S_P = K_P \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} K_P$, $T_P = K_P \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_P$ and $T_{P^2} = K_P \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} K_P = T_P^2 - (NP+1)S_P$.
- $T_P^2/S_P, S_p = \prod_{P|p} S_P \in \mathcal{H}(G^*(\mathbf{Q}_p)//K_p^*, \mathbf{Q})$.
- $\varphi_P = \mu_P(p)^{-1} L_P \in \mathcal{H}(M_P//L_P, \mathbf{Q})$ (the partial Frobenius at P)
- $\varphi = \mu(p)^{-1} L^* = \prod_{P|p} \varphi_P \in \mathcal{H}(M^*//L^*, \mathbf{Q})$ (the total Frobenius)
- $\widetilde{\varphi}_P = \varphi_P^2 \prod_{P' \neq P} S_{P'}^{-1} = \varphi_P^2 S_P S_p^{-1} \in \mathcal{H}(M^*//L^*, \mathbf{Q})$ (twisted square of φ_P), $\prod_{P|p} \widetilde{\varphi}_P = \varphi^2 S_p^{1-r}$.

The important point is that the partial Frobenius φ_P does not lie in $\mathcal{H}(M^*//L^*, \mathbf{Q})$, but its twisted square $\widetilde{\varphi}_P$ does.

(6.11) For a non-zero polynomial $Q \in A[Y]$ with coefficients in a commutative ring A and $a \in A^\times$ let

$$(R_a Q)(Y) = a^{\deg(Q)} Q(a^{-1}Y); \text{ then } (R_{a_1} Q_1) \otimes (R_{a_2} Q_2) = R_{a_1 a_2}(Q_1 \otimes Q_2), \tag{6.11.1}$$

where the Rankin-Selberg product $Q_1 \otimes Q_2$ is defined analogously as in (5.16.3). If Q is monic, factor it formally as $Q(Y) = (Y - \alpha_1) \cdots (Y - \alpha_n)$ and let

$$Q^{(2)}(Y) = (Y - \alpha_1^2) \cdots (Y - \alpha_n^2) \in A[Y].$$

The Eichler-Shimura relations in our situation are consequences of the following abstract polynomial identity (see A1.6–A.1.8):

$$\overline{S}_{\mu_P}(Q_P)\big|_{Y=\varphi_P} = 0 \in \mathcal{H}(M_P//L_P, \mathbf{Q}), \tag{6.11.2}$$

where $P = P_x$, $NP = p$ and

$$Q_P(Y) = Y^2 - (T_P/S_P)Y + (NP)/S_P \in \mathcal{H}(H(F_P)//K_P, \mathbf{Q}).$$

It implies that

$$\overline{S}_\mu\big(\bigotimes_{P|p} Q_P\big)\big|_{Y=\varphi} = 0 \in \mathcal{H}(M^*//L^*, \mathbf{Q}) \tag{6.11.3}$$

and

$$\overline{S}_{\mu_P}\big(Q_P^{(2)}\big)\big|_{Y=\varphi_P^2} = 0 \in \mathcal{H}(M_P//L_P, \mathbf{Q}),$$

hence

$$\overline{S}_{\mu_P}\big(R_{S_P/S_p}Q_P^{(2)}\big)\big|_{Y=\widetilde{\varphi}_P} = 0 \in \mathcal{H}(M^*//L^*, \mathbf{Q}), \qquad R_{S_P/S_p}Q_P^{(2)}(Y) = Y^2 + (2NP - T_P^2/S_P)Y/S_p + (NP/S_p)^2. \tag{6.11.4}$$

(6.12) These polynomials are related to the Euler factors of the Galois representation ρ_Π as follows. If we write $P = P_x$, $P' = P'_x$ and $P'' = P''_x$, then

$$L_{P'}(BC_{F_c/F}(JL(\pi)), \mathrm{Std}, s)^{-1} = L_P(\pi, \mathrm{Std}, s)^{-1} = 1 - T_P p^{-1/2-s} + S_P p^{-2s}\big|_{\pi_P^{K_P}}$$

and the relations (6.7.1-2) imply that

$$P_{\rho_\Pi(\mathrm{Fr}(P'))}(Y) = Y^2 - T_P\psi(P')Y + pS_P\psi(P')^2\big|_{\pi_P^{K_P}}.$$

For $x \in X$ let ρ_x be the representation $\rho_x = {}^{s(x)}\rho_{\Pi^\vee} = \rho_{\Pi^\vee} \circ \mathrm{int}(s(x))^{-1} : \Gamma_{F_c^{gal}} \longrightarrow GL_2(\overline{\mathbf{Q}}_\ell)$, as in 5.12. Its Euler factor at P_S is given by

$$P_{\rho_x(\mathrm{Fr}(P_S))}(Y) = P_{\rho_\Pi^\vee(P'_x)}(Y) = P_{\rho_\Pi(P''_x)}(Y) = Y^2 - T_{P_x}\psi(P''_x)Y + pS_{P_x}\psi(P''_x)^2\big|_{\pi_{P_x}^{K_{P_x}}} = \big(R_{S_{P_x}\psi(P''_x)}Q_{P_x}\big)\big|_{\pi_{P_x}^{K_{P_x}}}, \tag{6.12.1}$$

since $\Pi^\vee \simeq \Pi^c \simeq \Pi \otimes \omega_\Pi^{-1} \simeq \Pi \otimes (\psi^c/\psi)$.

(6.13) Proposition. *There exists a character $\chi : \Gamma_{F_c^{gal}} \longrightarrow \overline{\mathbf{Q}}_\ell^\times$ with the following property: for every P_S satisfying (6.9.1) we have*

$$\chi(\mathrm{Fr}(P_S)) = p^{w^*/2}\prod_{x\in X}\psi(P'_x), \qquad \Big(\bigotimes_{x\in X} Q_{P_x}\Big)(Y)\big|_{(\pi_p^*)^{K_p^*}} = P_{(\chi\otimes\bigotimes_{x\in X}\rho_x)(\mathrm{Fr}(P_S))}(Y).$$

Proof. The character

$$\widetilde{\chi} : \prod_{x\in X}\big(\psi\|\cdot\|_{F_c}^{-w^*/2r}\big) \circ N_{F_c^{gal}/F_c} \circ s(x)^{-1} : \mathbf{A}_{F_c^{gal}}^\times/(F_c^{gal})^\times \longrightarrow \mathbf{C}^\times$$

sends a uniformiser at $P_S \cap O_{F_c^{gal}}$ to $p^{w^*/2}\prod_{x\in X}\psi(P'_x)$. The infinity type of $\widetilde{\chi}$ is algebraic: if $\tau : F_c^{gal} \hookrightarrow \mathbf{C}$ extends $E_0 \hookrightarrow \mathbf{C}$, then

$$\widetilde{\chi}_\tau(a) = |\tau(a)|^{-w^*}(\tau(a)/|\tau(a)|)^k = \tau(a)^{r-m}\overline{\tau(a)}^{r-k-m},$$

where $k = \sum_{x\in X} k_x$ and $w^* = \sum_{x\in X}(k_x - 2 + 2m_x) = k - 2r + 2m$. The ℓ-adic character $\chi : \Gamma_{F_c^{gal}} \longrightarrow \overline{\mathbf{Q}}_\ell^\times$ attached to $\widetilde{\chi}$ ([Sc, ch. 0, §5], [HT, p. 20]) then satisfies $\chi(\mathrm{Fr}(P_S)) = p^{w^*/2}\prod_{x\in X}\psi(P'_x)$, as claimed. The remaining statement follows from the fact that

$$R_{p^{-w^*/2}}\Big(\bigotimes_{x\in X} Q_{P_x}\Big)\big|_{(\pi_p^*)^{K_p^*}} = \Big(\bigotimes_{x\in X} Q_{P_x}\Big)\big|_{\pi_p^{K_p}} = \bigotimes_{x\in X}\big(Q_{P_x}\big|_{\pi_{P_x}^{K_{P_x}}}\big) = \bigotimes_{x\in X}\big(R_{a_x}P_{\rho_x(\mathrm{Fr}(P_S))}\big) = R_a P_{(\bigotimes\rho_x)(\mathrm{Fr}(P_S))},$$

where

$$a_x = \psi(P''_x)^{-1}S_{P_x}\big|_{\pi_{P_x}^{K_{P_x}}} = \psi(P''_x)^{-1}(\omega_\pi)_{P_x}(P_x) = \psi(P'_x), \qquad a = \prod_{x\in X} a_x = p^{-w^*/2}\chi(\mathrm{Fr}(P_S)).$$

(6.14) Proposition. *Denote by* ρ *the* ℓ*-adic representation of* $\Gamma_{\mathbf{Q}}$ *given by its action on* $V^r((\pi^*)^\infty)$*. The dense subset* $\Sigma = \{\mathrm{Fr}(P_S) \mid P_S$ *as in (6.9.1)*$\} \subset \Gamma_{F_c^{gal}}$ *has the following properties.*

(1) $\forall g \in \Sigma \qquad P_{(\chi\otimes\bigotimes_{x\in X}\rho_x)}(\rho(g)) = 0.$

(2) *For every* $g \in \Sigma$ *there exist mutually commuting endomorphisms* $u_x \in \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V^r((\pi^*)^\infty)$ $(x \in X)$ *and non-zero scalars* $c, c_x \in \overline{\mathbf{Q}}_\ell^\times$ *such that*

$$\rho(g^2) = c\prod_{x\in X} u_x, \qquad \forall x \in X \quad P_{\rho_x(g^2)}(c_x u_x) = 0.$$

Proof. If $P_S \mid p$ is as in (6.9.1), then the group G^* splits over $\mathbf{Q}_p$ and the main result of [W] (see also [Mo, Cor. 4.2.5]) applies. Explicitly, there is a stack (in fact, a scheme, after imposing a level structure outside p) of p-isogenies $p-\mathrm{Isog}_{K^{*p}} \longrightarrow S_{K^*}\times S_{K^*}$ (where S_{K^*} is the Kottwitz model of Sh_{K^*} over $\mathbf{Z}_{(p)}$) and a commutative diagram (see A4 below for the notation and the sign conventions, which differ from those in [W])

$$\begin{array}{ccc}\mathcal{H}(G^*(\mathbf{Q}_p)_-//K_p^*,\mathbf{Q}) & \xrightarrow{h} & \mathbf{Q}[p-\mathrm{Isog}_{K^{*p}}\otimes\mathbf{Q}_p] \\ \downarrow{\scriptstyle \overline{S}_\mu} & & \downarrow{\scriptstyle\sigma} \\ \mathcal{H}(M_-^*//L^*,\mathbf{Q}) & \xrightarrow{\overline{h}} & \mathbf{Q}[p-\mathrm{Isog}_{K^{*p}}\otimes\mathbf{F}_p]\end{array}$$

equipped with compatible $G^*(\widehat{\mathbf{Q}})$-equivariant actions on $H^r(Sh_{K^*}) = H^r(Sh_{K^*}(G^*,\mathscr{X}^*)\otimes_{\mathbf{Q}}\overline{\mathbf{Q}},\mathscr{L}_{\xi^*,\ell})$ (hence also on $V^r((\pi^*)^\infty)\otimes(\pi^*)^\infty \subset (H^r)^{K^*}$, since $(H^r)^{K^*} = H^r(Sh_{K^*})$ if $D \not\simeq M_2(F)$ and $(H^r)^{K^*}$ injects into $H^r(Sh_{K^*})$ if $D \simeq M_2(F)$, by Proposition A6.17).

(1) The action of $\mathrm{Fr}(P_S)$ on $H^r(Sh_{K^*})$ is given by the action of $\varphi \in \mathcal{H}(M_-^*//L^*,\mathbf{Q})$. Letting the relation (6.11.3) act on $V^r((\pi^*)^\infty)\otimes(\pi^*)^\infty$ we obtain the Eichler-Shimura relation for the usual Frobenius

$$\Big(\bigotimes_{x\in X} Q_{P_x}\Big)\big|_{(\pi_p^*)^{K_p^*}}\left(\mathrm{Fr}(P_S)|_{V^r((\pi^*)^\infty)\otimes(\pi^*)^\infty}\right) = 0 \in \mathrm{End}(V^r((\pi^*)^\infty)\otimes(\pi^*)^\infty),$$

which is equivalent to

$$P_{(\chi\otimes\bigotimes_{x\in X}\rho_x)(\mathrm{Fr}(P_S))}(\rho(\mathrm{Fr}(P_S))) = 0,$$

thanks to Proposition 6.13.

(2) As explained in A5.5, the $G^*(\widehat{\mathbf{Q}})$-equivariance implies that the action of each $\widetilde{\varphi}_{P_x} \in \mathcal{H}(M_-^*//L^*,\mathbf{Q})$ on $V^r((\pi^*)^\infty)\otimes(\pi^*)^\infty$ is of the form $u_x\otimes\mathrm{id}$, where $u_x \in \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V^r((\pi^*)^\infty))$ (but we do not really need this fact) and $u_xu_y = u_yu_x$. The relation (6.11.4) combined with (6.12.1) implies that u_x is a root of the polynomial

$$\left(R_{S_{P_x}/S_p}Q_{P_x}^{(2)}\right)\big|_{(\pi_p^*)^{K_p^*}} = R_{p^{w^*}}\left(R_{S_{P_x}/S_p}Q_{P_x}^{(2)}\right)\big|_{(\pi_{P_x})^{K_{P_x}}} = R_{p^{w^*}}R_{S_{P_x}/S_p}R_{S_{P_x}^{-2}\psi(P_x'')^{-2}}P_{\rho_x(\mathrm{Fr}(P_S))^2}\big|_{(\pi_{P_x})^{K_{P_x}}} =$$
$$= R_{c_x^{-1}}P_{\rho_x(\mathrm{Fr}(P_S)^2)}, \qquad c_x = p^{-w^*}\psi(P_x'')^2S_pS_{P_x}\big|_{(\pi_{P_x})^{K_{P_x}}} \in \overline{\mathbf{Q}}_\ell^\times.$$

Therefore $P_{\rho_x(\mathrm{Fr}(P_S)^2)}(c_xu_x) = 0$. Finally

$$\rho(\mathrm{Fr}(P_S)^2) = \varphi^2|_{V^r((\pi^*)^\infty)} = S_p^{r-1}\prod_{x\in X}\widetilde{\varphi}_{P_x}|_{V^r((\pi^*)^\infty)} = c\prod_{x\in X}u_x, \qquad c = S_p^{r-1}|_{\pi_p^{K_p}}.$$

(6.15) Corollary. *If* P_S *is as in (6.9.1) and if, for each* $x \in X$*, the polynomial* $P_{\rho_x(\mathrm{Fr}(P_S)^2)}$ *has two distinct roots, then* $\rho(\mathrm{Fr}(P_S))$ *acts semisimply on* $V^r((\pi^*)^\infty)$*.*

(6.16) Note that, if ϕ has complex multiplication by a (totally imaginary) quadratic extension M of F and if $\mathrm{Fr}(P_S)|_M \neq \mathrm{id}$, then $\rho_x(\mathrm{Fr}(P_S)) = \begin{pmatrix}0 & *\\ * & 0\end{pmatrix}$ and $P_{\rho_x(\mathrm{Fr}(P_S)^2)}(Y) = (Y-a_x)^2$ has a double root, for all $x \in X$.

In general, the representations $\rho_\Pi : \Gamma_{F_c} \longrightarrow GL_2(\overline{\mathbf{Q}}_\ell)$ and $\rho_x = {}^{s(x)}\rho_{\Pi^\vee} = {}^{s(x)}\rho_{\Pi^c} : \Gamma_{F_c^{gal}} \longrightarrow GL_2(\overline{\mathbf{Q}}_\ell)$ have the following properties.

(6.17) Proposition. (1) *Let $\lambda \mid \ell$ be the prime of F_c above ℓ induced by the fixed embeddings $F_c \subset \overline{\mathbf{Q}} \subset \mathbf{C} \simeq \overline{\mathbf{Q}}_\ell$. For every $x \in X$ the restriction of the representation ρ_x to $\Gamma_{(F_c)_\lambda}$ is Hodge-Tate. Its two Hodge-Tate weights are distinct; their difference is equal to $k_x - 1$.*
(2) *The representation ρ_Π is irreducible.*
(3) *If ϕ does not have complex multiplication, then the representation ρ_Π is strongly irreducible and, for each $x \in X$, $\overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}(\mathrm{Im}(\rho_x)) \supset \mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$.*

Proof. (1) This is a special case of a general compatibility between the Hodge-Tate weights and automorphic weights for n-dimensional ℓ-adic Galois representations ρ_Π of Γ_{F_c} attached to self-dual ($\Pi^\vee \simeq \Pi^c$) regular algebraic automorphic representations Π of $GL_n(\mathbf{A}_{F_c})$, proved in [CH, Thm. 3.2.5].
(2) The argument from [T, Prop. 3.1] in the case of motivic weight applies.
(3) If ρ_Π is not strongly irreducible, then it is induced from a character $\Gamma_L \longrightarrow \overline{\mathbf{Q}}_\ell^\times$, where $[L : F_c] = 2$. It follows that $\Pi \otimes \alpha \simeq \Pi$, where $\alpha : \mathbf{A}_{F_c}^\times / F_c^\times \longrightarrow \{\pm 1\}$ is the quadratic character attached to L/F_c. Therefore $\alpha^c = \alpha$, which implies that $\alpha = \beta \circ N_{F_c/F}$ for some $\beta : \mathbf{A}_F^\times / F^\times \longrightarrow \{\pm 1\}$ and $BC_{F_c/F}(JL(\pi)) = BC_{F_c/F}(JL(\pi) \otimes \beta)$, hence $JL(\pi) \simeq JL(\pi) \otimes \beta$ or $JL(\pi) \simeq JL(\pi) \otimes \beta\eta$, where η is the quadratic character of $\mathbf{A}_F^\times / F^\times$ attached to F_c/F; thus $JL(\pi)$ has complex multiplication by $\beta \neq 1$ or by $\beta\eta \neq 1$.

If ρ_Π is strongly irreducible, then the $\overline{\mathbf{Q}}_\ell$-Lie algebra $\overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}(\mathrm{Im}(\rho_{\Pi^\vee})) \subset \mathfrak{gl}_2(\overline{\mathbf{Q}}_\ell)$ acts irreducibly on $\overline{\mathbf{Q}}_\ell^2$, which means that it must contain $\mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$.

(6.18) Proposition. *If ϕ does not have complex multiplication, then there exists an open subgroup $U \subset \Gamma_{F_c^{gal}}$ and a dense subset $\Sigma_U \subset U$ such that each element of Σ_U acts semisimply on $V^r((\pi^*)^\infty)$.*

Proof. Combine Proposition 6.17(3) with Proposition 3.6 (for $\rho_i = \rho_x$ and $a = 2$) and Corollary 6.15.

(6.19) Theorem. *Assume that we are in the case (B) and $BC_{F_c/F}(JL(\pi)) \otimes \psi = \Pi$.*
(1) *There exists a finite extension E''/F_c^{gal} and an integer $m \geq 1$ such that*

$$V^r((\pi^*)^\infty)^{\mathrm{ss}}|_{\Gamma_{E''}} \subset \left(\bigotimes_{x \in X} {}^{s(x)}\rho_{\Pi^\vee} \right) \otimes \chi \Big|_{\Gamma_{E''}}^{\oplus m}.$$

If, in addition, ϕ has complex multiplication by a totally imaginary quadratic extension M of F, then E'' satisfies $M^{gal} \not\subset (M^{gal})^+ E''$.
(2) *If ϕ does not have complex multiplication and if the weights $(k_x)_{x \in X}$ are distinct, then the representation $\bigotimes_{x \in X} {}^{s(x)}\rho_{\pi^\vee}$ of $\Gamma_{F_c^{gal}}$ is strongly irreducible and*

$$V^r((\pi^*)^\infty)^{\mathrm{ss}}|_{\Gamma_{F_c^{gal}}} = \left(\bigotimes_{x \in X} {}^{s(x)}\rho_{\Pi^\vee} \right) \otimes \chi \Big|_{\Gamma_{F_c^{gal}}}^{\oplus m}.$$

(3) *If ϕ does not have complex multiplication, then the representation $V^r((\pi^*)^\infty)$ of $\Gamma_{\mathbf{Q}}$ is semisimple.*

Proof. The arguments used in the proof of Theorem 5.20 apply, with references to 5.19 to be replaced by those to Proposition 6.14 and Proposition 6.17. In concrete terms, (1) is a consequence of Proposition 6.14(1) and Proposition 3.10(1) (resp. and Proposition 4.6) if ϕ does not (resp. does) have complex multiplication. The statement (2) follows from Proposition 3.10(2) applied to $\rho_i = \rho_x$ and $a = 1$; the assumptions (A') and (C') are consequences of Proposition 6.17(3) and Proposition 6.14(1), respectively, and the strong irreducibility is a consequence of the argument from 5.20 that uses the Hodge-Tate weights. The statement (3) follows from Theorem 3.12(2) applied to $\rho_i = \rho_x$ and $a = 2$; the assumptions (A") and (C") are satisfied, respectively, thanks to Proposition 6.17(3) and Proposition 6.14(2).

(6.20) Corollary. *For every ξ^* as in 6.2, the action of $\Gamma_{\mathbf{Q}}$ on the non CM part of $H^i_{et}(Sh_{K^*}(G^*, \mathscr{X}^*) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathscr{L}_{\xi^*,\ell})$ in the case $D \not\simeq M_2(F)$ (resp. on $H^i_{et}(Sh_{K^*}(G^*, \mathscr{X}^*)_{BB} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_{!*}\mathscr{L}_{\xi^*,\ell})$ in the case $D \simeq M_2(F)$) is semisimple.*

Appendix: Eichler-Shimura relations

In this Appendix we indicate how the methods of [FC] and [W] for proving Eichler-Shimura relations in the split case also work for partial Frobenius morphisms. We try to keep consistent sign conventions, but it is conceivable that some of the formulas hold only up to a sign. This is not important, however, for the applications to semisimplicity in the main body of this article.

A1. The Satake transform (the split case)

(A1.1) In this section we recall various versions of the Satake transform in the simplest possible setting. Let $\mathcal{K}$ be a finite extension of $\mathbf{Q}_p$ with ring of integers O. Let $|\cdot|$ be the normalised valuation on $\mathcal{K}$ (so that the valuation of any uniformiser ϖ is equal to $|\varpi| = q^{-1}$, where q is the cardinality of the residue field $O/\varpi O$).

(A1.2) Let $\underline{G}$ be a split connected reductive group over $\mathcal{K}$; it is the general fibre of a group scheme over O (which will also be denoted by $\underline{G}$) with reductive special fibre.

Assume that we are given a cocharacter $\mu : \mathbf{G}_{m,O} \longrightarrow \underline{G}$ (defined over O). Fix a maximal torus $\underline{T} \subset \underline{G}$ such that μ factors through $\underline{T}$. This defines subgroup schemes $\underline{T} \subset \underline{M} \subset \underline{P}_\mu \subset \underline{G}$, where $\underline{M}$ is the centraliser of μ in $\underline{G}$ and $\underline{P}_\mu$ is a parabolic subgroup of $\underline{G}$ with Levi subgroup $\underline{M}$ and unipotent radical $\underline{N}_\mu$ characterised by the fact that $\mathrm{Lie}(\underline{N}_\mu)$ is the direct sum of those root spaces $\mathrm{Lie}(\underline{G})^\alpha$ with respect to $\underline{T}$ for which $\alpha \circ \mu > 0$.

We obtain the corresponding groups of points $T = \underline{T}(\mathcal{K}) \subset M = \underline{M}(\mathcal{K}) \subset P_\mu = \underline{P}_\mu(\mathcal{K}) = M \ltimes N_\mu \subset G = \underline{G}(\mathcal{K}) \supset K = \underline{G}(O)$, where $N_\mu = \underline{N}_\mu(\mathcal{K})$. The modulus morphism

$$\delta_\mu : M \longrightarrow q^{\mathbf{Z}}, \qquad m \mapsto |\det(\mathrm{Ad}(m) \mid \mathrm{Lie}(N_\mu))|$$

satisfies $\delta_{\mu^{-1}} = \delta_\mu^{-1}$.

(A1.3) Let $\mathcal{H}(G//K, \mathbf{Q})$ be the Hecke algebra of locally constant functions with compact support $f : G \longrightarrow \mathbf{Q}$ satisfying $f(kgk') = f(g)$ for all $g \in G$ and $k, k' \in K$. The product is given by the convolution

$$(f_1 * f_2)(h) = \int_G f_1(g) f_2(g^{-1}h)\, dg,$$

where dg is the Haar measure on G giving K volume 1. The algebra $\mathcal{H}$ is commutative and the characteristic function $\mathrm{char}(K)$ of K is its unit. We define similarly $\mathcal{H}(M//M \cap K, \mathbf{Q})$.

(A1.4) Let du be the Haar measure on N_μ normalised by giving $N_\mu \cap K$ volume 1. We define two twisted Satake transforms

$$\overline{S}_\mu, \widetilde{S}_\mu : \mathcal{H}(G//K, \mathbf{Q}) \longrightarrow \mathcal{H}(M//(M \cap K), \mathbf{Q}),$$

related by $\widetilde{S}_\mu = \delta_\mu \cdot \overline{S}_\mu$, by the formulas

$$(\overline{S}_\mu f)(m) = \int_{N_\mu} f(mu)\, du, \qquad (\widetilde{S}_\mu f)(m) = \int_{N_\mu} f(um)\, du,$$

and the usual (normalised) Satake transform

$$S_\mu = \delta_\mu^{1/2} \cdot \overline{S}_\mu = \delta_\mu^{-1/2} \cdot \widetilde{S}_\mu : \mathcal{H}(G//K, \mathbf{Q}) \longrightarrow \mathcal{H}(M//(M \cap K), \mathbf{Q}) \otimes \mathbf{Z}[q^{\pm 1/2}].$$

(A1.5) The Hecke polynomial. In the special case when $\underline{M} = \underline{T}$ the parabolic subgroup $\underline{P}_\mu = \underline{B} = \underline{T} \ltimes \underline{U}$ is a Borel subgroup and the normalised Satake transform induces an isomorphism

$$S : \mathcal{H}(G//K, \mathbf{Z}) \otimes \mathbf{Z}[q^{\pm 1/2}] \xrightarrow{\sim} \Big(\mathcal{H}(T//(T \cap K), \mathbf{Z}) \otimes \mathbf{Z}[q^{\pm 1/2}]\Big)^W.$$

The target group is canonically identified with $R(\widehat{G}) \otimes \mathbf{Z}[q^{\pm 1/2}]$, where $R(\widehat{G})$ is the Grothendieck ring of algebraic representations of the complex dual group $\widehat{G}$, via the bijections

$$\mathbf{Z}[X_*(\underline{T})] \xrightarrow{\sim} \mathcal{H}(T//(T\cap K),\mathbf{Z}), \qquad [\lambda]\mapsto \mathrm{char}(\lambda(\varpi)(T\cap K)),$$
$$R(\widehat{G}) \xrightarrow{\sim} \mathbf{Z}[X^*(\widehat{T})]^W = \mathbf{Z}[X_*(\underline{T})]^W, \qquad [V]\mapsto \mathrm{Trace}(V|_{\widehat{\underline{T}}}).$$

Using this isomorphism, we define, for any algebraic representation V of $\widehat{G}$, the Hecke polynomial ("the characteristic polynomial")

$$H_V(X) := \sum_{k=0}^{\dim V} (-1)^k [\Lambda^k V]\, X^k \in (\mathcal{H}(G//K,\mathbf{Z})\otimes \mathbf{Z}[q^{\pm 1/2}])[X], \qquad \widetilde{H}_V(X) = X^{\dim V} H_V(1/X).$$

For any cocharacter $\lambda \in X_*(\underline{T})$ (considered as a character of the dual torus $\widehat{T}\subset\widehat{G}$) there exists $w\in W$ such that $w\lambda$ will lie in the positive Weyl chamber with respect to $\underline{B}$. We denote by V_λ the irreducible representation of $\widehat{G}$ with highest weight $w\lambda$ and we let

$$H_\lambda(X) = H_{V_\lambda}(X), \qquad \widetilde{H}_\lambda(X) = \widetilde{H}_{V_\lambda}(X).$$

For example, if $\underline{G} = \underline{T}$ is a torus, then

$$H_\lambda(X) = 1 - \mathrm{char}(\lambda(\varpi)K)X, \qquad \widetilde{H}_\lambda(X) = X - \mathrm{char}(\lambda(\varpi)K). \tag{A1.5.1}$$

(A1.6) Consider the following toy model: $\underline{G} = GL(2)$ and $\lambda(t) = \begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}$. In this case $\underline{T}$ is the diagonal torus, V_λ is the standard two-dimensional representation of $\widehat{G} = GL_2(\mathbf{C})$ and

$$H_\lambda(X) = \left(1 - X\,\mathrm{char}\big(\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}(T\cap K)\big)\right)\left(1 - X\,\mathrm{char}\big(\begin{pmatrix}1 & 0\\ 0 & \varpi\end{pmatrix}(T\cap K)\big)\right)$$
$$= 1 - q^{-1/2}T_\varpi X + S_\varpi X^2,$$

where

$$T_\varpi = \mathrm{char}\big(K\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}K\big), \qquad S_\varpi = \mathrm{char}\big(K\begin{pmatrix}\varpi & 0\\ 0 & \varpi\end{pmatrix}K\big) \in \mathcal{H}(G//K,\mathbf{Z}).$$

Similarly, $V_{\lambda^{-1}}$ is the dual of V_λ, hence $V_{\lambda^{-1}} \xrightarrow{\sim} V_\lambda \otimes (\Lambda^2 V_\lambda)^{-1}$ and

$$H_{\lambda^{-1}}(X) = 1 - q^{-1/2}(T_\varpi/S_\varpi)X + (1/S_\varpi)X^2, \qquad \widetilde{H}_{\lambda^{-1}}(X) = X^2 - q^{-1/2}(T_\varpi/S_\varpi)X + (1/S_\varpi),$$

$$q\widetilde{H}_{\lambda^{-1}}(q^{-1/2}X) = X^2 - (T_\varpi/S_\varpi)X + (q/S_\varpi). \tag{A1.6.1}$$

(A1.7) Proposition ([Bu, Prop. 3.4], [W, Prop. 2.9]). *If the cocharacter μ in A1.2 is minuscule, then $(\overline{S}_\mu(\widetilde{H}_{\mu^{-1}}))(q^{-\langle\rho,\mu\rangle}\mathrm{char}(M\mu(\varpi)^{-1}M)) = 0 \in \mathcal{H}(M//(M\cap K),\mathbf{C})$, where $2\rho\in\mathbf{Z}[X^*(\underline{T})]$ denotes the sum of all positive roots of $(\underline{G},\underline{B},\underline{T})$.*

(A1.8) In the situation of A1.6 the cocharacter $\mu=\lambda$ is minuscule, $\underline{P}_\mu = \underline{B}$ is the upper triangular Borel subgroup, $\langle\rho,\mu\rangle = 1/2$ and

$$(\overline{S}_\mu(\widetilde{H}_{\mu^{-1}}))(X) = \left(X - q^{-1/2}\,\mathrm{char}\big(\begin{pmatrix}\varpi^{-1} & 0\\ 0 & 1\end{pmatrix}(T\cap K)\big)\right)\left(X - q^{1/2}\,\mathrm{char}\big(\begin{pmatrix}1 & 0\\ 0 & \varpi^{-1}\end{pmatrix}(T\cap K)\big)\right).$$

A2. Hecke correspondences and their action on cohomology

(A2.1) Let $(G, \mathscr{X})$ be a pure Shimura datum, let $K \subset G(\widehat{\mathbf{Q}})$ be an open compact subgroup. Throughout this Appendix (with the exception of A5.6, A5.11-12 and A6.16-20) we assume that G^{der} is anisotropic, hence the Shimura variety $Sh_K = Sh_K(G, \mathscr{X})$ is projective (and smooth if K is small enough) over the reflex field $E = E(G, \mathscr{X})$.

Any diagram $(Sh_K \xleftarrow{q_1} Z \xrightarrow{q_2} Sh_K)$ with finite morphisms q_i defines a correspondence $cl(Z) = (q_1, q_2)_*(Z) \in Corr(Sh_K)_{\mathbf{Q}} = CH^{\dim Sh_K}(Sh_K \times Sh_K)_{\mathbf{Q}}$. The product of correspondences is given by $A \circ B = (p_{14})_*((A \times B) \cdot \Delta_{23})$. For example, any finite morphism $\alpha : Sh_K \longrightarrow Sh_K$ has a graph $\Gamma_\alpha = cl(Sh_K \xleftarrow{\mathrm{id}} Sh_K \xrightarrow{\alpha} Sh_K)$ and its transpose ${}^t\Gamma_\alpha = cl(Sh_K \xleftarrow{\alpha} Sh_K \xrightarrow{\mathrm{id}} Sh_K)$ satisfying $\Gamma_\alpha \circ \Gamma_\beta = \Gamma_{\beta\circ\alpha}$ and ${}^t\Gamma_\alpha \circ {}^t\Gamma_\beta = {}^t\Gamma_{\alpha\circ\beta}$.

For every reasonable cohomology theory H^* with coefficients in a field of characteristic zero (such as $H^* = H^*_{et}(- \otimes_E \overline{\mathbf{Q}}, \overline{\mathbf{Q}}_\ell)$) the ring of correspondences $Corr(Sh_K)_{\mathbf{Q}}$ naturally acts on the left on $H^*(Sh_K)$ via the formula

$$L(A) : x \mapsto (p_1)_*([A] \cup p_2^*(x)),$$

where $[A] \in H^*(Sh_K \times Sh_K)$ is the cohomology class of a correspondence $A \in Corr(Sh_K)_{\mathbf{Q}}$. In particular, $L(Sh_K \xleftarrow{q_1} Z \xrightarrow{q_2} Sh_K) = (q_1)_* \circ q_2^* : H^*(Sh_K) \longrightarrow H^*(Sh_K)$, $L(\Gamma_\alpha) = \alpha^*$, $L({}^t\Gamma_\alpha) = \alpha_*$, $L(A \circ B) = L(A) \circ L(B)$.

(A2.2) For any $g \in G(\widehat{\mathbf{Q}})$ the diagram

$$\begin{array}{ccc} Sh_{gKg^{-1}\cap K} & \xrightarrow{[\cdot g]} & Sh_{K\cap g^{-1}Kg} \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle pr_2} \\ Sh_K & & Sh_K \end{array}$$

(where $[\cdot g]$ denotes the standard right action of g on the tower $\{Sh_K\}$) defines a Hecke correspondence

$$[KgK] = (Sh_K \xleftarrow{pr_1\circ[\cdot g^{-1}]} Sh_{K\cap g^{-1}Kg} \xrightarrow{pr_2} Sh_K) \in Corr(Sh_K)_{\mathbf{Q}}$$

depending only on the double coset $KgK \in K\backslash G(\widehat{\mathbf{Q}})/K$. Define the global Hecke algebra $\mathcal{H}(G(\widehat{\mathbf{Q}})/\!/K, \mathbf{Q})$ as in A1.3, with K of volume 1. The $\mathbf{Q}$-linear extension of the map $\mathrm{char}(KgK) \mapsto [KgK]$ defines a ring homomorphism $\mathcal{H}(G(\widehat{\mathbf{Q}})/\!/K, \mathbf{Q}) \longrightarrow Corr(Sh_K)_{\mathbf{Q}}$. The corresponding left action on cohomology

$$L([KgK]) = (pr_1 \circ [\cdot g^{-1}])_* \circ pr_2^* = (pr_1)_* \circ [\cdot g]^* \circ pr_2^*$$

corresponds to the natural left action of $G(\widehat{\mathbf{Q}})$ on $H^*(Sh_K)$ given by $L(g) = [\cdot g]^*$.

(A2.3) As a multivalued map $pr_2 \circ [\cdot g] \circ (pr_1)^{-1}$, the Hecke correspondence is given, using the standard notation $[x, \gamma]_K$ for the class of $(x, \gamma) \in \mathscr{X} \times G(\widehat{\mathbf{Q}})$ in Sh_K, by

$$[x, \gamma]_K \mapsto \sum [x, \gamma g_i g]_K, \qquad K = \coprod_i g_i(K \cap gKg^{-1}), \qquad KgK = \coprod_i g_i g K.$$

See A5.5 below for the action of Hecke correspondences on cohomology with coefficients in a local system.

A3. The PEL data

(A3.1) Assume that we are given the following data: $(B, *, V, \langle\ ,\ \rangle_F)$, where B is a finite-dimensional simple $\mathbf{Q}$-algebra, $*$ is a $\mathbf{Q}$-linear positive involution on B, $F = Z(B)^{*=\mathrm{id}}$ (a totally real number field), V is a non-zero left B-module of finite type and $\langle\ ,\ \rangle_F : V \times V \longrightarrow F$ is a non-degenerate alternating F-bilinear form such that $\langle bv, v' \rangle_F = \langle v, b^* v' \rangle_F$ for all $b \in B$ and $v, v' \in V$.

The centre $Z(B) = F_c$ of B is equal either to F, or to a totally imaginary quadratic extension of F. Set $\langle\ ,\ \rangle = \mathrm{Tr}_{F/\mathbf{Q}} \circ \langle\ ,\ \rangle_F : V \times V \longrightarrow \mathbf{Q}$; this is a non-degenerate alternating $\mathbf{Q}$-bilinear form satisfying the same hermitian property as $\langle\ ,\ \rangle_F$. The centraliser $C = \mathrm{End}_B(V)$ is a simple $\mathbf{Q}$-algebra with centre F_c and an F-linear involution $\#$ given by the adjoint with respect to $\langle\ ,\ \rangle_F$.

Let $H = GSp_B(V, \langle\ ,\ \rangle_F)$ be the algebraic group over F whose points with values in any F-algebra S are given by

$$\begin{aligned} H(S) &= \{h \in GL_B(V \otimes_F S) \mid \exists \nu(h) \in S^\times\ \forall v, v' \in V\ \langle hv, hv' \rangle_F = \nu(h) \langle v, v' \rangle_F\} \\ &= \{h \in (C \otimes_F S)^\times \mid hh^\# = \nu(h) \in S^\times\} \end{aligned}$$

and let G^* be the algebraic group over $\mathbf{Q}$ such that

$$G^*(R) = \{g \in (C \otimes R)^\times \mid gg^\# = \nu(h) \in R^\times\}$$

for all $\mathbf{Q}$-algebras R. As in 5.2, there is a cartesian diagram

$$\begin{array}{ccc} G^* & \longrightarrow & G \\ \Big\downarrow{\scriptstyle \nu} & & \Big\downarrow{\scriptstyle \nu} \\ \mathbf{G}_{m,\mathbf{Q}} & \longrightarrow & R_{F/\mathbf{Q}}(\mathbf{G}_{m,F}), \end{array}$$

where $G = R_{F/\mathbf{Q}}(H)$. We assume, from now on, that the group G^{der} is anisotropic.

(A3.2) Recall [Ko, §1] that $(B \otimes_F \overline{F}, * \otimes \mathrm{id})$ is isomorphic either to $\mathrm{End}(W) \times \mathrm{End}(W)^{op}$ with $(a, b)^* = (b, a)$ (type (A)), or to $\mathrm{End}(W)$ with $*$ being the adjoint map with respect to a symmetric (resp. alternating) bilinear form on W (type (C)) (resp. type (BD)).

From now on, assume that our datum is of type (A) (when $F_c \neq F$) or (C) (when $F_c = F$). This implies that H, G and G^* are connected reductive groups and that the derived group of H (hence of G) is simply connected ([Mi, Prop. 8.7]). Furthermore, there exists a morphism of $\mathbf{R}$-algebras $h : \mathbf{C} \longrightarrow C \otimes \mathbf{R}$ such that $h(\overline{z}) = h(z)^\#$ for which the symmetric $\mathbf{R}$-bilinear form $\langle v, h(i)v' \rangle : V_\mathbf{R} \times V_\mathbf{R} \longrightarrow \mathbf{R}$ is positive definite. The morphism h is unique up to conjugation by an element $c \in (C \otimes \mathbf{R})^\times$ such that $cc^\# = 1$ ([Ko, Lemma 4.3], [Mi, Prop. 8.12]).

It follows that h defines a Shimura datum $(G^*, \mathscr{X}^*)$ (resp. $(G, \mathscr{X})$), where $\mathscr{X}^*$ (resp. $\mathscr{X}$) is the $G^*(\mathbf{R})$-conjugacy class (resp. the $G(\mathbf{R})$-conjugacy class) of h. The real group $G_\mathbf{R} = \prod_{v|\infty} H \otimes_{F,v} \mathbf{R}$ is isomorphic to $\prod_v GSp(2n)_\mathbf{R}$ (resp. to $\prod_v GU(a_v, b_v)$, $a_v + b_v = n$) if $(B, *)$ is of type (C) (resp. of type (A)).

The action of $h(i)$ defines a complex structure on $V_\mathbf{R}$, hence a Hodge decomposition $V_\mathbf{C} = V^{-1,0} \oplus V^{0,-1}$ of weight -1, with $h(z) \otimes \mathrm{id}$ acting as z (resp. $\overline{z}$) on $V^{-1,0}$ (resp. on $V^{0,-1}$). The cocharacter $\mu = \mu_h : \mathbf{G}_{m,\mathbf{C}} \longrightarrow G_\mathbf{C}$ attached to h acts on $V_\mathbf{C}$ as follows: $\mu(z)$ acts as $z \cdot \mathrm{id}$ (resp. as id) on $V^{-1,0}$ (resp. on $V^{0,-1}$).

The common reflex field $E = E(G^*, \mathscr{X}^*) = E(G, \mathscr{X}) \subset \overline{\mathbf{Q}} \subset \mathbf{C}$ is the field generated over $\mathbf{Q}$ by the coefficients of the characteristic polynomial

$$\det(X_1 \alpha_1 + \cdots + X_t \alpha_t \mid V^{-1,0}), \tag{A3.2.1}$$

for any $\mathbf{Q}$-basis $\{\alpha_j\}$ of B.

(A3.3) The arguments in [Ko, §7] show that the group G satisfies the Hasse principle. The key point is a description of the torus $T = G/G^{der}$ in terms of tori ${}_kT = R_{k/\mathbf{Q}}(\mathbf{G}_{m,k})$. If $(B, *)$ is of type (C), then ν induces an isomorphism $\nu : T \xrightarrow{\sim} {}_FT$. If it is of type (A) and $n \geq 1$ is as in A3.2, then the map "determinant" together with ν induce an isomorphism $T \xrightarrow{\sim} \{(a, b) \in {}_{F_c}T \times {}_FT \mid N_{F_c/F}(a) = b^n\}$. For $n = 2k + 1$ (resp. $n = 2k$), the map $(a, b) \mapsto ab^{-k}$ (resp. $(a, b) \mapsto (ab^{-k}, b)$) defines an isomorphism $\beta : T \xrightarrow{\sim} {}_{F_c}T$ satisfying $N_{F_c/F} \circ \beta = \nu$ (resp. $\beta = (\beta_1, \nu) : T \xrightarrow{\sim} \mathrm{Ker}(N_{F_c/F} : {}_{F_c}T \longrightarrow {}_FT) \times {}_FT$). All tori ${}_kT$ have trivial H^1 and the Hasse principle holds for the norm $N_{F_c/F}$. It follows that T satisfies the Hasse principle, hence so does G (cf. [Mi, Lemma 8.20, 8.21]).

(A3.4) Unramified local data at p. Let p be a prime number; fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Write $\widehat{\mathbf{Q}}^{(p)} = (\prod_{\ell \neq p} \mathbf{Z}_\ell) \otimes \mathbf{Q}$ for the ring of finite adeles outside p. Assume that each term in the decomposition $B_{\mathbf{Q}_p} = \prod_{P|p} B \otimes_F F_P$ is a matrix algebra over an unramified extension of $\mathbf{Q}_p$ (in particular, p is unramified in $F_c/\mathbf{Q}$). This assumtion will be further strengthened in A4.3 below.

Assume, furthermore, that we are given the following data: an open compact subgroup $K^p \subset G(\widehat{\mathbf{Q}}^{(p)})$, a $*$-stable $O_F \otimes \mathbf{Z}_{(p)}$-order $O_B \subset B$ such that $O_B \otimes \mathbf{Z}_p$ is a maximal order in $B_{\mathbf{Q}_p}$ and an $O_F \otimes \mathbf{Z}_p$-lattice $\Lambda \subset V_{\mathbf{Q}_p}$ which is self-dual (up to a scalar in $(F \otimes \mathbf{Q}_p)^\times$) with respect to $\langle\ ,\ \rangle_F$. Fix such a Λ and let $K_p = \{g \in G(\mathbf{Q}_p) \mid g(\Lambda) = \Lambda\}$, $K = K_p K^p \subset G(\widehat{\mathbf{Q}})$. As in [Ko, p.390], the characteristic polynomial $\det(b \mid V^{-1,0})$ in (A3.2.1) will have coefficients in $O_E \otimes \mathbf{Z}_{(p)}$, if we choose $\{\alpha_j\}$ to be a $\mathbf{Z}_{(p)}$-basis of O_B.

(A3.5) A moduli problem. A p-integral model S_K of $Sh_K(G, \mathscr{X})$ (for sufficiently small K^p) can be constructed as follows (see [R, 2.14] and [TX, 2.3-2.8, 4.6] for special cases).

For any $\alpha \in (\widehat{F}^{(p)})^\times = (F \otimes \widehat{\mathbf{Q}}^{(p)})^\times$ consider the following moduli stack $\mathscr{M}_{\alpha,K^p}$ over the category of locally noetherian schemes S over $O_E \otimes \mathbf{Z}_{(p)}$. Its objects over S are quadruples $(A, \iota, \lambda, \overline{(\eta, u)})$, where

- A is an abelian scheme over S up to prime-to-p isogeny (notation: A is an object of $(AV/S) \otimes \mathbf{Z}_{(p)}$);
- $\lambda : A \longrightarrow \widehat{A}$ is a $\mathbf{Z}_{(p)}$-polarisation of degree prime to p;
- $\iota : O_B \longrightarrow \mathrm{End}(A)$ is a $*$-morphism (with respect to $*$ on O_B and the Rosati involution coming from λ on $\mathrm{End}(A)$);
- for a fixed geometric point s of (every connected component of) S, $\overline{(\eta, u)}$ is a $\pi_1(S, s)$-invariant K^p-level structure. By the latter we mean a K^p-orbit of pairs (η, u), where $u \in (\widehat{O}_F^{(p)})^\times$ and $\eta : V \otimes \widehat{\mathbf{Q}}^{(p)} \xrightarrow{\sim} V^{(p)}(A_s)$ is a $B \otimes \widehat{\mathbf{Q}}^{(p)}$-linear isomorphism such that the Weil pairing $\langle\ ,\ \rangle_\lambda : V^{(p)}(A_s) \times V^{(p)}(A_s) \longrightarrow \widehat{\mathbf{Q}}^{(p)}$ on $V^{(p)}(A_s) = \mathbf{Q} \otimes \prod_{\ell \neq p} T_\ell(A_s)$ attached to λ satisfies

$$\langle \eta(x), \eta(y) \rangle_\lambda = \mathrm{Tr}_{F/\mathbf{Q}}(\alpha u \langle x, y \rangle_F).$$

An element $g \in K^p$ acts on (η, u) by $(\eta, u)g = (\eta \circ g, u\,\nu(g))$.

- the Kottwitz determinant condition should be satisfied: $\det(b \mid \mathrm{Lie}(A)) = \det(b \mid V^{-1,0})$ as polynomial functions on O_B.

Morphisms between $(A, \iota, \lambda, \overline{(\eta, u)})$ and $(A', \iota', \lambda', \overline{(\eta', u')})$ are given by O_B-linear isomorphisms $f : A \xrightarrow{\sim} A'$ in $(AV/S) \otimes \mathbf{Z}_{(p)}$ such that $\lambda = f^*(\lambda')$ $(= \widehat{f} \circ \lambda' \circ f)$ and $\overline{(\eta', u')} = \overline{(f \circ \eta, u)}$.

Note that the degree of λ is determined by the above conditions. Moreover, $(A, \iota, \lambda, \overline{(\eta, u)})$ has no non-trivial automorphism if K^p is small enough (cf. [R, 2.13]), which we assume, from now on.

This implies, as in [Ko, §5] and [R], that $\mathscr{M}_{\alpha,K^p}$ is represented by a smooth quasi-projective scheme over $O_E \otimes \mathbf{Z}_{(p)}$, which will be denoted by M_{α,K^p}.

There is an action of the group of totally positive units $O_{F,+}^\times$ on $\mathscr{M}_{\alpha,K^p}$ given by the formula

$$\varepsilon \cdot (A, \iota, \lambda, \overline{(\eta, u)}) = (A, \iota, \iota(\varepsilon)\lambda, \overline{(\eta, \varepsilon u)}).$$

If $\varepsilon = N_{F_c/F}(\varepsilon')$ for some $\varepsilon' \in O_{F_c}^\times \cap K^p \subset F_c^\times = Z_G(\mathbf{Q}) \subset G(\widehat{\mathbf{Q}}^{(p)})$, then multiplication by $\iota(\varepsilon')$ on A defines an isomorphism

$$[\iota(\varepsilon')] : (A, \iota, \lambda, \overline{(\eta, u)}) \xrightarrow{\sim} \varepsilon^{-1} \cdot (A, \iota, \lambda, \overline{(\eta, u)}).$$

It follows that the finite abelian group

$$\Delta = O_{F,+}^\times / N_{F_c/F}(O_{F_c}^\times \cap K^p)$$

acts on M_{α,K^p}. It turns out that, after replacing K^p by a suitable open subgroup, the group Δ will act on the scheme M_{α,K^p} freely by permuting its connected components. This is proved in [Ki] in general and in A3.9-10 below in the cases (C) and (A even).

In particular, the quotient scheme $M_{\alpha,K^p}/\Delta$ exists and is quasi-projective and smooth over $O_E \otimes \mathbf{Z}_{(p)}$.

(A3.6) The moduli problem over C. Following [Ko, §8], we define a map

$$M_{\alpha,K^p}(\mathbf{C}) \longrightarrow Sh_K(G,\mathscr{X})(\mathbf{C}) = G(\mathbf{Q})\backslash(\mathscr{X} \times G(\mathbf{Q}_p)/K_p \times G(\widehat{\mathbf{Q}}^{(p)})/K^p)$$

(see also [TX, 2.4], [Mi, 6.3, 6.9]).

If $(A, \lambda, \iota, \overline{(\eta, u)})$ is a quadruple over $\mathbf{C}$, then $H = H_1(A, \mathbf{Q})$ is a B-module via ι, equipped with a skew-Hermitian pairing $\langle\ ,\ \rangle_{H,\lambda} : H \times H \longrightarrow F$ such that $\mathrm{Tr}_{F/\mathbf{Q}} \circ \langle\ ,\ \rangle_{H,\lambda}$ is induced by λ.

As in [Ko, p. 338-339], one checks that $H_{\mathbf{Q}_v}$ and $V_{\mathbf{Q}_v}$ are isomorphic skew-Hermitian $B_{\mathbf{Q}_v}$-modules, for all places v of $\mathbf{Q}$. For $v \neq \infty, p$ this follows from the existence of η; for $v = \infty$ one uses the determinant condition and [Ko, Lemma 4.2]. For $v = p$, $T_p(A) = H_1(A, \mathbf{Z}_p) \subset H_{\mathbf{Q}_p}$ is a self-dual $O_B \otimes \mathbf{Z}_p$-lattice and a variant of [Ko, Lemma 7.2] applies.

The validity of the Hasse principle for G implies that there is a B-linear isomorphism $a : H \xrightarrow{\sim} V$, unique up to left multiplication by $G(\mathbf{Q})$, which sends $\langle\ ,\ \rangle_{H,\lambda}$ to an $F^\times$-multiple of $\langle\ ,\ \rangle_F$. We fix such an isomorphism.

The natural complex structure h_A on $H_{\mathbf{R}} = \mathrm{Lie}(A)$ defines a complex structure $ah_A = (z \mapsto a \circ h_A(z) \circ a^{-1})$ on V which lies in $\mathscr{X}$ (when interpreted as a pure real Hodge structure of weight -1 on $V_{\mathbf{R}}$), thanks to [Ko, Lemma 4.2].

At p, $a(T_p(A)) \subset V_{\mathbf{Q}_p}$ is an $O_B \otimes \mathbf{Z}_p$-lattice, self-dual up to a scalar in $(F \otimes \mathbf{Q}_p)^\times$. A variant of [Ko, Lemma 7.3] shows that there exists $g_p \in G(\mathbf{Q}_p)$ (with $g_p K_p$ depending only on $a(T_p(A))$) such that $a(T_p(A)) = g_p \Lambda$. Equivalently, $\eta_p = a^{-1} \circ g_p : V_{\mathbf{Q}_p} \xrightarrow{\sim} H_{\mathbf{Q}_p}$ satisfies $T_p(A) = \eta_p(\Lambda)$.

Finally, $a \circ \eta : V \otimes \widehat{\mathbf{Q}}^{(p)} \xrightarrow{\sim} V \otimes \widehat{\mathbf{Q}}^{(p)}$ is an element of $G(\widehat{\mathbf{Q}}^{(p)})$.

The map $M_{\alpha,K^p}(\mathbf{C}) \longrightarrow Sh_K(G,\mathscr{X})(\mathbf{C})$ given by sending $(A, \lambda, \iota, \overline{(\eta, u)})$ to

$$[ah, g_p K_p, a \circ \eta]_K \in G(\mathbf{Q})\backslash(\mathscr{X} \times G(\mathbf{Q}_p)/K_p \times G(\widehat{\mathbf{Q}}^{(p)})/K^p) = Sh_K(G,\mathscr{X})(\mathbf{C}) \qquad (A3.6.1)$$

is well-defined and factors through the quotient $M_{\alpha,K^p}(\mathbf{C})/\Delta$.

(A3.7) A p-integral model S_K of $Sh_K(G,\mathscr{X})$. Choose a (finite) set $\Sigma = \{\alpha\} \subset (\widehat{F}^{(p)})^\times$ of representatives of the double cosets

$$(\widehat{F}^{(p)})^\times = \coprod_{\alpha \in \Sigma} (O_F \otimes \mathbf{Z}_{(p)})_+^\times \, \alpha \, (\widehat{O}_F^{(p)})^\times.$$

The maps (A3.6.1) induce a bijection ([Fa, Prop. 3.6.3])

$$\coprod_{\alpha \in \Sigma} (M_{\alpha,K^p}(\mathbf{C})/\Delta) \xrightarrow{\sim} Sh_K(G,X)(\mathbf{C}), \qquad (A3.7.1)$$

which implies that the smooth quasi-projective $O_E \otimes \mathbf{Z}_{(p)}$-scheme

$$S_K := \coprod_{\alpha \in \Sigma} (M_{\alpha,K^p}/\Delta) = M_{K^p}/\Delta, \qquad M_{K^p} = \coprod_{\alpha \in \Sigma} M_{\alpha,K^p}$$

is a model of $Sh_K(G,X)$. Moreover, (A3.7.1) identifies $S_K \otimes E$ with a canonical model of $Sh_K(G,h)$, not of $Sh_K(G,h^{-1})$ – see the discussion in [Mi, p. 347].

(A3.8) Recall that G^{der} is anisotropic, by assumption, hence $Sh_K(G,\mathscr{X})$ is compact. In this case each M_{α,K^p} (hence S_K, too) is projective over $O_E \otimes \mathbf{Z}_{(p)}$, by [Ko, p. 392] if C is a division algebra, and by [L, Thm. 4.6] in general.

(A3.9) Deligne's description of the set of geometric connected components $\pi_0 Sh_K(G,\mathscr{X})(\mathbf{C})$ ([De1, Thm. 2.4], [Mi, Thm. 5.17]) in terms of the map $G \longrightarrow G/G^{der} = T$ yields the following bijections (depending on a choice of a connected component of $\mathscr{X}$):

$$\pi_0 Sh_K(G,\mathscr{X})(\mathbf{C}) \xrightarrow{\sim} (O_F \otimes \mathbf{Z}_{(p)})^\times \backslash \widehat{F}^{(p),\times}/\nu(K^p)$$

in the case (C),

$$\pi_0 Sh_K(G,\mathscr{X})(\mathbf{C}) \xrightarrow{\sim} (O_{F_c} \otimes \mathbf{Z}_{(p)})^\times \backslash \widehat{F}_c^{(p),\times}/\beta(K^p)$$

in the case (A odd) (when $n = 2k+1$) and

$$\pi_0 Sh_K(G, \mathscr{X})(\mathbf{C}) \xrightarrow{\sim} U_1(K^p) \times U_2(K^p),$$
$$U_1(K^p) = \mathrm{Ker}(N : (O_{F_c} \otimes \mathbf{Z}_{(p)})^\times \longrightarrow (O_F \otimes \mathbf{Z}_{(p)})^\times_+) \backslash \mathrm{Ker}(N : \widehat{F}_c^{(p),\times} \longrightarrow \widehat{F}^{(p),\times})/\beta_1(K^p)$$
$$U_2(K^p) = (O_F \otimes \mathbf{Z}_{(p)})^\times_+ \backslash \widehat{F}^{(p),\times}/\nu(K^p)$$

in the case (A even) (when $n = 2k$).

In the case (C),

$$\pi_0 M_{\alpha,K^p}(\mathbf{C}) = (\widehat{O}_F^{(p)})^\times/\nu(K^p) = \widehat{O}_F^\times/\nu(K),$$

$$\pi_0(M_{\alpha,K^p}/\Delta)(\mathbf{C}) = O_{F,+}^\times \backslash \widehat{O}_F^\times/\nu(K)$$

and the stabiliser in Δ of any connected component of $M_{\alpha,K^p}(\mathbf{C})$ is equal to

$$\Delta_0(K) = \left(O_{F,+}^\times \cap \nu(K)\right)/(O_F^\times \cap K)^2$$

(see [TX, 2.3, 2.4] in the case of Hilbert modular varieties, when $H = GL(2)_F$).

Similarly, in the case (A even),

$$\pi_0 M_{\alpha,K^p}(\mathbf{C}) = U_1(K^p) \times (\widehat{O}_F^{(p)})^\times/\nu(K^p) = U_1(K^p) \times \widehat{O}_F^\times/\nu(K),$$

$$\pi_0(M_{\alpha,K^p}/\Delta)(\mathbf{C}) = U_1(K^p) \times O_{F,+}^\times \backslash \widehat{O}_F^\times/\nu(K)$$

and the stabiliser in Δ of any connected component of $M_{\alpha,K^p}(\mathbf{C})$ is equal to

$$\Delta_0(K) = \left(O_{F,+}^\times \cap \nu(K)\right)/N_{F_c/F}(O_{F_c}^\times \cap K).$$

(A3.10) Proposition ([TX, Lemma 2.5]). *Assume that $(B,*)$ is of type (C) or (A even). After replacing K^p by a suitable open subgroup if necessary one can achieve $\Delta_0(K) = 0$, hence Δ will act on each $M_{\alpha,K^p}(\mathbf{C})$ freely by permuting certain connected components.*

(A3.11) The Frobenius morphism. The absolute Frobenius morphism

$$\varphi : S_K \otimes O_E/pO_E \longrightarrow S_K \otimes O_E/pO_E \qquad (A3.11.1)$$

is induced by the relative Frobenius morphism $F_A : A \longrightarrow A^{(p)}$ on abelian schemes in characteristic p. More precisely, let S be a (locally noetherian) scheme over O_E/pO_E and $(A, \iota, \lambda, \eta, u)$ a 5-tuple representing an element of $M_{\alpha,K^p}(S)$. There is a canonical action $\iota^{(p)} : O_B \longrightarrow \mathrm{End}(A^{(p)})$ compatible with ι via F_A and a $\mathbf{Z}_{(p)}$-polarisation $\lambda^{(p)} : A^{(p)} \longrightarrow \widehat{A^{(p)}}$ satisfying $F_A^*(\lambda^{(p)}) = p\lambda$. The formula

$$\varphi(A, \iota, \lambda, \eta, u) = (A^{(p)}, \iota^{(p)}, \lambda^{(p)}, F_A \circ \eta, pu)$$

gives an explicit description of

$$\varphi : M_{\alpha,K^p} \otimes O_E/pO_E \longrightarrow M_{\alpha,K^p} \otimes O_E/pO_E$$

and of the restriction of the map (A3.11.1) to $(M_{\alpha,K^p}/\Delta) \otimes O_E/pO_E$.

(A3.12) Partial Frobenius morphisms. One can write the map φ in (A3.11.1) as a product $\varphi = \prod_{P|p} \varphi_P$ of mutually commuting partial Frobenius morphisms

$$\varphi_P : S_K \otimes O_E/pO_E \longrightarrow S_K \otimes O_E/pO_E,$$

for primes $P \mid p$ of F above p (see [TX, 4.6] in the case of Hilbert modular varieties).

Fix a totally positive element $c \in F_+^\times$ such that $v_P(c) = 1$ and $v_{P'}(c) = 0$ for all $P' \mid p$, $P' \neq P$. Let S and $(A, \iota, \lambda, \eta, u)$ be as in A3.11. Consider $A' = A/\mathrm{Ker}(F_A)[P]$ and denote by $f_P : A \longrightarrow A'$ the quotient map. Again, there is a canonical morphism $\iota' : O_B \longrightarrow \mathrm{End}(A')$ induced by ι and f_P. We define

$$\varphi_P(A, \iota, \lambda, \eta, u) = (A', \iota', \lambda', \eta', u'), \tag{A3.12.1}$$

where $c\lambda = f_P^*(\lambda')$, $\eta' = f_P \circ \eta$ and $c\alpha u = \alpha' u'$. The recipe (A3.12.1) is compatible with the right K^p-action on the pairs (η, u), with isomorphisms and with the action of Δ. However, it depends on the choice of c.

If we replace c by $\widetilde{c}$, then $\widetilde{c} = \varepsilon c$ with $\varepsilon \in O_{F,+}^\times$ and the 5-tuple $(A', \iota', \lambda', \eta', u')$ is replaced by $(A', \iota', \varepsilon\lambda', \eta', \varepsilon u')$. This implies that the above formula gives rise to a well-defined partial Frobenius morphism

$$\varphi_P : (M_{\alpha, K^p}/\Delta) \otimes O_E/pO_E \longrightarrow (M_{\alpha', K^p}/\Delta) \otimes O_E/pO_E.$$

hence to $\varphi_P : S_K \otimes O_E/pO_E \longrightarrow S_K \otimes O_E/pO_E$.

A4. The p-isogenies ([FC, VII.3-4], [W, §3-5])

We continue to assume that K^p is sufficiently small.

(A4.1) Let $v \mid p$ be the prime of $E \subset \overline{\mathbf{Q}}$ defined by the fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. As in [W, §3], a *$p$-isogeny* $f : (A_1, \iota_1, \lambda_1, \overline{(\eta_1, u_1)}) \longrightarrow (A_2, \iota_2, \lambda_2, \overline{(\eta_2, u_2)})$ between objects $(A_j, \iota_j, \lambda_j, \overline{(\eta_j, u_j)})$ of $\mathscr{M}_{\alpha_j, K^p}(S)$ (for a scheme S over O_{E_v}) is an O_B-linear isogeny in $(AV/S) \otimes \mathbf{Z}_{(p)}$ of p-power degree such that $f \circ \overline{\eta}_1 = \overline{\eta}_2$ and $f^*(\lambda_2) = c\lambda_1$ for some $c \in F_+^\times$ such that $v_P(c) \geq 0$ for all $P \mid p$ in F.

A morphism $f \longrightarrow f'$ between two p-isogenies is given by a pair of O_B-linear isomorphisms $g_j : A_j \longrightarrow A'_j$ $(j = 1, 2)$ in $(AV/S) \otimes \mathbf{Z}_{(p)}$ satisfying

$$g_2 \circ f = f' \circ g_1, \qquad g_j^*(\lambda'_j) = \lambda_j, \qquad g_j \circ \overline{\eta}_j = \overline{\eta}'_j \qquad (j = 1, 2).$$

The p-isogenies (for all possible combinations of α_1 and α_2) form a stack of groupoids $p - \mathrm{Isog}_{K^p}$ over (Sch/O_{E_v}), equipped with canonical projections $p_j : p - \mathrm{Isog}_{K^p} \longrightarrow \mathscr{M}_{K^p} = \coprod_\alpha \mathscr{M}_{\alpha, K^p}$ $(j = 1, 2)$ sending f to $(A_j, \iota_j, \lambda_j, \overline{(\eta_j, u_j)})$. As in [W], the restriction of each p_j to the substack $p - \mathrm{Isog}_{K^p}^m$ of p-isogenies with fixed value of $m = (m_P = v_P(c))_{P \mid p}$ is represented by a proper surjective map (which is finite étale when restricted to (Sch/E_v)). As a result, $p - \mathrm{Isog}_{K^p}$ is, in fact, a scheme equipped with a morphism $(p_1, p_2) : p - \mathrm{Isog}_{K^p} \longrightarrow M_{K^p} \times M_{K^p}$, where $M_{K^p} = \coprod_\alpha M_{\alpha, K^p}$.

The composition of isogenies

$$(p - \mathrm{Isog}_{K^p}) \times_{p_2, M_{K^p}, p_1} (p - \mathrm{Isog}_{K^p}) \longrightarrow p - \mathrm{Isog}_{K^p}, \qquad f_1, f_2 \mapsto f_2 \circ f_1$$

defines a ring structure on $\mathbf{Q}[p - \mathrm{Isog}_{K^p}/S]$, the $\mathbf{Q}$-vector space on the set of connected components of $p - \mathrm{Isog}_{K^p}(S)$ (see [FC, p. 252]).

(A4.2) p-isogenies in characteristic zero. If f is a p-isogeny over $S = \mathrm{Spec}(\mathbf{C})$, then f identifies $H_1(A_1, \mathbf{Q}) = H_1(A_2, \mathbf{Q}) = H$ and induces an injection $T_p(f) : H_1(A_1, \mathbf{Z}_p) = T_p(A_1) \hookrightarrow H_1(A_2, \mathbf{Z}_p) = T_p(A_2) \subset H_{\mathbf{Q}_p}$. The corresponding elements $g_{p,j} \in G(\mathbf{Q}_p)$ from A3.6 satisfy $g_{p,1}(\Lambda) = a(T_p(A_1)) \subset a(T_p(A_2)) = g_{p,2}(\Lambda)$, hence $g_{p,2} = g_{p,1} g$ with $g \in G(\mathbf{Q}_p)_- = \{u \in G(\mathbf{Q}_p) \mid g^{-1}(\Lambda) \subset \Lambda\}$. We define the *type* of f to be the double coset $K_p g K_p \in K_p \backslash G(\mathbf{Q}_p)_- / K_p$; it depends only on f.

If $\mathbf{C}$ is replaced by an arbitrary algebraically closed field of characteristic zero, then there are isomorphisms $\varphi_j : T_p(A_j) \xrightarrow{\sim} \Lambda$. They satisfy $\varphi_2 \circ T_p(f) \circ \varphi_1^{-1}(\Lambda) = g^{-1}\Lambda$ for some $g \in G(\mathbf{Q}_p)_-$; we define the type of f to be again $K_p g K_p$. The type of the geometric fibres of any p-isogeny over a base S over $\mathrm{Spec}(E)$ is locally constant on S.

Note a sign change compared to [W, 4.1]; this is forced on us by the formula (A3.6.1), which relates the moduli problem to the canonical model of the Shimura variety.

As in [FC, p. 253] and [W, 4.2], define a map

$$h : \mathcal{H}(G(\mathbf{Q}_p)_- // K_p, \mathbf{Q}) \longrightarrow \mathbf{Q}[p - \mathrm{Isog}_{K^p}/E]$$

by sending the characteristic function of any double coset K_pgK_p to the union of the connected components on which the p-isogeny has type K_pgK_p. This is a ring morphism (if we let K_p have volume 1) and its composition with

$$(p_1, p_2) : \mathbf{Q}[p - \mathrm{Isog}_{K^p}/E] \longrightarrow Corr(M_{K^p} \otimes E)_{\mathbf{Q}}$$

is given by $\mathrm{char}(K_pgK_p) \mapsto (pr \times pr)^* \circ [KgK]$, in the notation of A2.2 (where $pr : M_{K^p} \otimes E \longrightarrow S_K \otimes E$ is the map (A3.6.1)).

In particular, the action of $h(g)$ on étale cohomology of $M_{K^p} \otimes \overline{E}$ leaves stable the image under pr^* of étale cohomology of $S_K \otimes \overline{E} = Sh_K \otimes_E \overline{E}$ and acts on the latter as the Hecke operator $L([KgK])$.

(A4.3) From now on, we impose the following additional assumption:

$$p \text{ splits completely in } F_c/\mathbf{Q}. \qquad (A4.3.1)$$

This implies that $E_v = \mathbf{Q}_p$, $k(v) = \mathbf{F}_p$, $F \otimes \mathbf{Q}_p = \prod_{P|p} F_P$, $F_P = \mathbf{Q}_p$, $B_{\mathbf{Q}_p} \simeq \prod_i M_{n_i}(\mathbf{Q}_p)$, $O_B \otimes \mathbf{Z}_p \simeq \prod_i M_{n_i}(\mathbf{Z}_p)$, each group $H \otimes_F F_P$ is split over $F_P = \mathbf{Q}_p$ (cf. the discussion in [Mi, 8.5-8.6]), G splits over $\mathbf{Q}_p$. Of course, $G(\mathbf{Q}_p) = \prod_{P|p} = H(F_P)$ and $K_p = \prod_{P|p} K_P$, where K_P is a maximal compact subgroup of $H(F_P)$.

As in [W, 5.1], the conjugacy class $[\mu]$ of the cocharacter μ_h from A3.2 (considered over $\overline{\mathbf{Q}}_p$, via the given embeddings $\overline{\mathbf{Q}}_p \hookleftarrow \overline{\mathbf{Q}} \subset \mathbf{C}$) contains a cocharacter defined over $\mathbf{Q}_p$, which extends to $\mu : \mathbf{G}_{m,\mathbf{Z}_p} \longrightarrow \underline{G}$, where $\underline{G}$ is a reductive model of G over $\mathbf{Z}_p$ defined by Λ. The decomposition $\Lambda = \prod_{P|p} \Lambda_P$ defines, for each $P \mid p$ in F, a cocharacter $\mu_P : \mathbf{G}_{m,O_{F_P}} = \mathbf{G}_{m,\mathbf{Z}_p} \longrightarrow \underline{H}_P$, where $\underline{H}_P$ is a reductive model of $H \otimes_F F_P$ over $O_{F_P} = \mathbf{Z}_p$.

Fix μ as above; then $\Lambda = \Lambda^{-1,0} \oplus \Lambda^{0,-1}$, where $\mu(z) = z \cdot \mathrm{id}$ (resp. $\mu(z) = \mathrm{id}$) on $\Lambda^{-1,0}$ (resp. on $\Lambda^{0,-1}$). In [W], $\Lambda^{-1,0}$ is denoted by Λ_0 and $\Lambda^{0,-1}$ by Λ_1. The centraliser $\underline{M} = \{g \in \underline{G} \mid g(\underline{\Lambda}^{i,j}) = \underline{\Lambda}^{i,j}\}$ is a Levi factor of the parabolic $\underline{P}_\mu = \{g \in \underline{G} \mid (\underline{\Lambda}^{-1,0}) = \underline{\Lambda}^{-1,0}\}$ attached to μ as in A1.2. Let $L = \underline{M}(\mathbf{Z}_p) = K_p \cap M$, where $M = \underline{M}(\mathbf{Q}_p)$.

(A4.4) Ordinary p-isogenies ([FC, VII.4], [W, §5]). A p-isogeny over a field of characteristic p is *ordinary* if A_1 (hence A_2, too) is an ordinary abelian variety. A general p-isogeny is ordinary if its fibres over points in characteristic p are ordinary. They form a subscheme $p - \mathrm{Isog}^{ord}_{K^p}$ of $p - \mathrm{Isog}_{K^p}$.

Let $(A, \iota, \lambda, \overline{(\eta, u)}) \in \mathscr{M}_{\alpha, K^p}(k)$, where k is an algebraically closed field of characteristic p. If A is ordinary, it is shown in [W, 5.2-3] that there are O_B-linear isomorphisms

$$T_p(A) \xrightarrow{\sim} T_p(\widehat{A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}_p}(\Lambda^{-1,0}, \mathbf{Z}_p) \xrightarrow{\sim} \Lambda^{0,-1},$$

with the first isomorphism induced by λ, the third by $\langle\ ,\ \rangle_F$ and with O_B acting on the third term by $(b \cdot u)(x) = u(b^* x)$. Above, $T_p(A) = \varprojlim_n A(k)[p^n]$ denotes the physical Tate module of A.

This implies that a p-ordinary isogeny f over k gives rise to $m_1 \in \mathrm{End}_{O_B \otimes \mathbf{Z}_p}(\Lambda^{0,-1})$ and its dual $\widehat{f}$ to $m_0^* \in \mathrm{End}_{O_B \otimes \mathbf{Z}_p}(\mathrm{Hom}_{\mathbf{Z}_p}(\Lambda^{-1,0}, \mathbf{Z}_p))$, hence to $m_0 \in \mathrm{End}_{O_B \otimes \mathbf{Z}_p}(\Lambda^{-1,0})$. We define *the type of* f to be the double coset $L(m_0, m_1)^{-1} L \in L \backslash M_- / L$ (note the change of sign with respect to [W, 5.3]), where $M_- = \{m \in M \mid m^{-1}(\Lambda^{i,j}) \subset \Lambda^{i,j}\}$.

For example, the Frobenius isogeny $f = F_A$ has purely multiplicative kernel, which means that $m_1 = p$ and $m_0 = 1$ ([W, 5.9]), so $(m_0, m_1) = \mu(p)$ and the type of f in our sense is equal to $L\mu(p)^{-1}L$.

More generally, if $P \mid p$ is a prime of F above p, then the invariant (m_0, m_1) attached to the isogeny $f_P : A \longrightarrow A/\mathrm{Ker}(F_A)[P]$ used in the definition of the partial Frobenius morphism $\varphi_P : S_K \otimes k(v) = S_K \otimes \mathbf{F}_p \longrightarrow S_K \otimes k(v)$ is equal to $(m_0, m_1) = \mu_P(p)$; thus the type of f_P is $L\mu_P(p)^{-1}L = L_P \mu_P(p)^{-1} L_P$.

The relation between the type of a p-isogeny in characteristic zero and the type of its reduction modulo p (assumed to be ordinary) is explained in [FC, p. 263]: let O be a complete DVR of mixed characteristic with residue field $k = \overline{k} \supset \mathbf{F}_p$. Let f be a p-isogeny over O with ordinary special fibre $\overline{f}$. The Barsotti-Tate objects $\Lambda'_i = T_p(A_{i/O})$ sit in exact sequences

$$0 \longrightarrow (\Lambda'_i)_{mult} \longrightarrow \Lambda'_i \longrightarrow (\Lambda'_i)_{et} \longrightarrow 0$$

and f induces injections $(\Lambda'_1)_? \hookrightarrow (\Lambda'_2)_?$ $(? = \emptyset, mult, et)$. If we consider $\Lambda'_i \subset H_{\mathbf{Q}_p} = H_1(A_i, \mathbf{Q}_p)$ and if we fix an isomorphism $a : H_{\mathbf{Q}_p} \xrightarrow{\sim} V_{\mathbf{Q}_p}$, then we obtain lattices $a(\Lambda'_1) = g_1(\Lambda) \subset a(\Lambda'_2) = g_2(\Lambda) \subset V_{\mathbf{Q}_p}$, where $g = g_1^{-1} g_2 \in \underline{P}_\mu(\mathbf{Q}_p)_- = (P_\mu)_-$. By definition, the type of f is equal to $K_p g K_p$. If we denote by $m(g) \in M_-$ the projection of g onto the Levi part of P_μ, then the type of $\overline{f}$ will be equal to $L(m_0, m_1)^{-1} L = Lm(g)L$.

The arguments in [W, 5.5-7] show that the restrictions of the projections $p_1, p_2 : p - \mathrm{Isog}^{ord}_{K^p} \otimes \mathbf{F}_p \longrightarrow M^{ord}_{K^p} \otimes \mathbf{F}_p$ to the subscheme of ordinary p-isogenies of a fixed type LmL are finite and flat and their geometric fibres have pure multiplicity given by explicit constants $mult_i(LmL)$.

As in [W, 5.8], one defines a map

$$\overline{h} : \mathcal{H}(M_-//L, \mathbf{Q}) \longrightarrow \mathbf{Q}[p - \mathrm{Isog}^{ord}_{K^p} \otimes k(v)] = \mathbf{Q}[p - \mathrm{Isog}^{ord}_{K^p} \otimes \mathbf{F}_p]$$

by sending the characteristic function of LmL to $(1/mult_1(LmL))$ times the union of all connected components corresponding to ordinary p-isogenies of type LmL (again, L is of volume 1).

In particular, the correspondence $\overline{h}(\mathrm{char}(L\mu(p)^{-1}L))$ (resp. $\overline{h}(\mathrm{char}(L\mu_P(p)^{-1}L))$) on $M_{K^p} \otimes k(v) = M_{K^p} \otimes \mathbf{F}_p$ is equal to the pullback by $pr \times pr$ of the graph of the Frobenius morphism $\varphi : S_K \otimes \mathbf{F}_p \longrightarrow S_K \otimes \mathbf{F}_p$ (resp. of the graph of the partial Frobenius $\varphi_P : S_K \otimes \mathbf{F}_p \longrightarrow S_K \otimes \mathbf{F}_p$). As a result, its action on étale cohomology of $M_{K^p} \otimes \overline{\mathbf{F}}_p$ leaves stable the image under pr^* of étale cohomology of $S_K \otimes \overline{\mathbf{F}}_p$ and its action on the latter coincides with the action of $\varphi \otimes \mathrm{id}$ (resp. of $\varphi_P \otimes \mathrm{id}$).

(A4.5) Proposition ([FC, p. 263], [W, Prop. 5.10]). *The following diagram commutes (the map σ is given by specialisation of cycles).*

$$\begin{array}{ccc} \mathcal{H}(G(\mathbf{Q}_p)_-//K_p, \mathbf{Q}) & \xrightarrow{h} & \mathbf{Q}[p - \mathrm{Isog}^{ord}_{K^p} \otimes E_v] \\ \downarrow{\scriptstyle \overline{S}_\mu} & & \downarrow{\scriptstyle \sigma} \\ \mathcal{H}(M_-//L, \mathbf{Q}) & \xrightarrow{\overline{h}} & \mathbf{Q}[p - \mathrm{Isog}^{ord}_{K^p} \otimes \mathbf{F}_p] \end{array}$$

Proof. As in [FC, p. 263], this follows from the discussion in A4.4 relating the types of f and $\overline{f}$. Note that [FC] work with classical objects, such as $\Gamma\backslash\mathbb{H}_g$, defined as quotients by a left action, whereas Shimura varieties $Sh_K = Sh/K$ are quotients by a right action. This accounts for a sign change in the formulas involving the action of the Hecke algebra. More precisely, one uses the Iwasawa decomposition $G(\mathbf{Q}_p) = \underline{P}_\mu(\mathbf{Q}_p)K_p = P_\mu K_p$ to determine the number of cosets $K_p g K_p/K_p$ lying in the fibre of the map

$$G(\mathbf{Q}_p)/K_p = P_\mu/(K_p \cap P_\mu) \longrightarrow M/L, \qquad g \mapsto m(g)$$

above a fixed class mL. As $P_\mu = MU$ (with $U \cap K_p$ of volume 1), the above number is equal to

$$\int_U (\mathrm{char}(K_p g K_p))(mu)\, du = (\overline{S}_\mu \mathrm{char}(K_p g K_p))(m),$$

as claimed.

A5. Eichler-Shimura relations for partial Frobenii

(A5.1) The principal geometric result of [W, §6] is the relative density theorem for the moduli problem giving rise to an integral model of $Sh_{K^*}(G^*, \mathscr{X}^*)$. The theorem states that the ordinary p-isogenies in characteristic p are dense in all p-isogenies, provided the group G^* is split over $\mathbf{Q}_p$. Using Serre-Tate theory, this is deduced from a deformation statement about p-isogenies of principally quasi-polarised $O_B \otimes \mathbf{Z}_p$-modules [W, Prop. 6.10], which is then sufficient to prove only for $O_B \otimes \mathbf{Z}_p = \mathbf{Z}_p$ or $\mathbf{Z}_p \times \mathbf{Z}_p$.

This reduction argument to a special case of [W, 6.10] is also valid in our situation (i.e., for the union of the Kottwitz models $M_{K^p} = \coprod_\alpha M_{\alpha, K^p}$), which yields an equality $\mathbf{Q}[p - \mathrm{Isog}_{K^p} \otimes \mathbf{F}_p] = \mathbf{Q}[p - \mathrm{Isog}^{ord}_{K^p} \otimes \mathbf{F}_p]$. The commutative diagram in Proposition A4.5 then becomes

$$\begin{array}{ccc} \mathcal{H}(G(\mathbf{Q}_p)_-//K_p,\mathbf{Q}) & \xrightarrow{h} & \mathbf{Q}[p-\mathrm{Isog}_{K^p}\otimes E_v] \\ \big\downarrow{\scriptstyle \overline{S}_\mu} & & \big\downarrow{\scriptstyle \sigma} \\ \mathcal{H}(M_-//L,\mathbf{Q}) & \xrightarrow{\overline{h}} & \mathbf{Q}[p-\mathrm{Isog}_{K^p}\otimes \mathbf{F}_p] \end{array} \tag{A5.1.1}$$

(A5.2) Both Hecke algebras decompose into tensor products:

$$\mathcal{H}(G(\mathbf{Q}_p)_-//K_p,\mathbf{Q}) = \bigotimes_{P|p} \mathcal{H}(H(F_P)_-//K_P,\mathbf{Q})$$
$$\mathcal{H}(M_-//L,\mathbf{Q}) = \bigotimes_{P|p} \mathcal{H}(M_{P-}//L_P,\mathbf{Q}).$$

The discussion in A4.4 justifies the following definition: the *partial Frobenius at* P in $\mathbf{Q}[p-\mathrm{Isog}_{K^p}\otimes \mathbf{F}_p]$ is defined as

$$\varphi_P = \overline{h}(L_P\mu_P(p)^{-1}L_P).$$

Their product is equal to

$$\varphi = \prod_{P|p} \varphi_P = \overline{h}(L\mu(p)^{-1}L).$$

(A5.3) Theorem (the Eichler-Shimura relation for partial Frobenius morphisms). *If p splits completely in $F_c/\mathbf{Q}$ and satisfies the assumptions from A3.4, then the following relation holds, for every prime $P \mid p$ in F:*

$$\widetilde{H}_{\mu_P^{-1}}(p^{-\langle \rho_P,\mu_P\rangle}\varphi_P) = 0 \in \mathbf{Q}[p-\mathrm{Isog}_{K^p}\otimes \mathbf{F}_p],$$

where ρ_P denotes the half sum of all positive roots of $H\otimes_F F_P$.

Proof. Combine (A5.1.1) with Proposition A1.7.

(A5.4) Similarly, the full Frobenius φ satisfies the “Rankin-Selberg product” of the relations A5.3, in the sense of (5.16.2). This relation differs by a sign from the one stated in [W], but it is compatible with the reciprocity law giving the Galois action on H^0, which is dual to the action on π_0, hence is given by the reciprocity morphism attached to μ^{-1}, rather than to μ.

(A5.5) The Eichler-Shimura relation for the action on cohomology. The centre Z_G of G contains the torus ${}_FT$ and the quotient torus $Z_G/{}_FT$ is anisotropic over $\mathbf{R}$. Fix an irreducible algebraic representation $\xi : G_{\mathbf{C}} \longrightarrow GL(V_\xi)$ such that $\xi|_{{}_FT} = N^m_{F/\mathbf{Q}}$ for some $m\in\mathbf{Z}$. This condition implies that $\xi(Z_G(\mathbf{Q})\cap K) = \{1\}$, for all sufficiently small open compact subgroups $K\subset G(\widehat{\mathbf{Q}})$, hence the local sections of $G(\mathbf{Q})\backslash(\mathscr{X}\times V_\xi\times G(\widehat{\mathbf{Q}})/K)$ over $Sh_K(\mathbf{C})$ define a localy constant sheaf of complex vector spaces $\mathscr{L}_\xi$ on $Sh_K(\mathbf{C})$.

As in 0.1, fix an isomorphism $\mathbf{C}\xrightarrow{\sim}\overline{\mathbf{Q}}_\ell$. It is explained in [HT, III.2] how to attach to ξ_ℓ a smooth ℓ-adic étale sheaf $\mathscr{L}_{\xi,\ell}$ on Sh_K. This sheaf is $G(\widehat{\mathbf{Q}})$-equivariant ([HT, III.2], [Ko, §6]), which means that there is a natural left action of $G(\widehat{\mathbf{Q}})$ and $\mathcal{H}(G(\widehat{\mathbf{Q}})//K,\mathbf{Q})$ on $H^i(Sh_K) = H^i_{et}(Sh_K(G,\mathscr{X})\otimes_E\overline{\mathbf{Q}},\mathscr{L}_{\xi,\ell})$. Moreover, ξ_ℓ can be obtained by a suitable tensor construction from the representation $V_{\mathbf{Q}_\ell}$ (which coincides with $V_\ell(A)$ for the abelian variety A appearing in the moduli problems $\mathscr{M}_{\alpha,K^p}$). This implies that the ring $\mathbf{Q}[p-\mathrm{Isog}_{K^p}\otimes E]$ also acts on $H^i(M_{K^p}) = H^i(M_{K^p}\otimes\overline{\mathbf{Q}}, pr^*\mathscr{L}_{\xi,\ell})$ (cf. [FC, p. 253]). Moreover, $\mathscr{L}_{\xi,\ell}$ extends to the (proper) integral model S_{K^p} and $H^i(M_{K^p})$ is isomorphic to the étale cohomology of the special fibre of M_{K^p}, which is equipped with an action of $\mathbf{Q}[p-\mathrm{Isog}_{K^p}\otimes\mathbf{F}_p]$. The maps h and σ in (A5.1.1) are compatible with the actions on $pr^*H^i(Sh_K)\subset H^i(M_{K^p})$ of the various rings appearing in the diagram, thanks to the discussion at the end of A4.2.

These compatibilities yield, together with Theorem A5.3, the Eichler-Shimura relation

$$\forall P \mid p \qquad \widetilde{H}_{\mu_P^{-1}}(p^{-\langle \rho_P, \mu_P\rangle}\varphi_P|_{H^i(Sh_K)}) = 0 \in \mathrm{End}(H^i(Sh_K)). \qquad (A5.5.1)$$

The action of φ_P on étale cohomology of the special fibre can be defined directly (thanks to the compatibility alluded to at the end of A4.4), by observing that $\mathscr{L}_{\xi,\ell}$ is obtained by a limit procedure from finite Galois étale covers $S_{K'} \longrightarrow S_K$ ($K' = K_p K'^p$, where K'^p is a suitable open normal subgroup of K^p) and that φ_P from A3.12 acts compatibly on both $S_{K'} \otimes \mathbf{F}_p$ and $S_K \otimes \mathbf{F}_p$.

The decomposition (0.4.1) yields $H^i(Sh_K) = \bigoplus V^i(\pi^\infty) \otimes (\pi^\infty)^K$, where π^∞ is the non-archimedean component of an automorphic representation π of $G(\mathbf{A})$ and $V^i(\pi^\infty)$ is a finite-dimensional ℓ-adic representation of Γ_E.

The action of each φ_P on $H^i(Sh_K)$ commutes with the action of both $G(\widehat{\mathbf{Q}}^{(p)})$ (by the functorial definition of φ_P in (A3.12.1)) and $\mathcal{H}(G(\mathbf{Q}_p)_-//K_p, \mathbf{Q})$ (since the Hecke algebra $\mathcal{H}(M_-//L, \mathbf{Q})$ is commutative). As a result, each term $V^i(\pi^\infty) \otimes (\pi^\infty)^K \subset H^i(Sh_K)$ is φ_P-stable. Furthermore, $\dim \mathrm{End}_{G(\widehat{\mathbf{Q}}^{(p)})}(\pi^\infty)^{K_p} = 1$ by Schur's Lemma, which implies that $\varphi_P \in \mathrm{End}\, V^i(\pi^\infty)$.

Assume that $(\pi^\infty)^K \neq 0$. If we write $\pi^\infty = \otimes' \pi_v$, where π_v is a representation of $H(F_v)$, then $\dim(\pi_P^{K_P}) = 1$ and we obtain from (A5.5.1) the following relation (the notation means that we replace each element of the Hecke algebra $\mathcal{H}(G(\mathbf{Q}_p)_-//K_p, \mathbf{Q})$ by its eigenvalue on $\pi_P^{K_P}$):

$$\left(\widetilde{H}_{\mu_P^{-1}}\Big|_{\pi_P^{K_P}}\right)\left(p^{-\langle \rho_P, \mu_P\rangle}\varphi_P|_{V^i(\pi^\infty)\otimes(\pi^\infty)^K}\right) = 0 \in \mathrm{End}(V^i(\pi^\infty) \otimes (\pi^\infty)^K). \qquad (A5.5.2)$$

(A5.6) A toy model: $GL(2)$**.** Let us discuss the relation (A5.5.2) in the simplest case $F = B = \mathbf{Q}$ and $V = \mathbf{Q}^2$, when $G = H = G^* = GL(2)_{\mathbf{Q}}$ and the Shimura varieties Sh_K are modular curves. They are not compact, but the relation still holds for $V^1(\pi^\infty)$ contributing to the cuspidal cohomology $H^1_!$ discussed in 0.8.

The standard two-dimensional representation Std of $GL(2)$ corresponds to $H_1(A)$ of the universal abelian variety (= elliptic curve), hence is of weight -1. Its dual $\mathrm{Std}^\vee$ corresponds to $H^1(A)$ and is of weight 1.

A general irreducible algebraic representation of G is of the form $\xi = \mathrm{Sym}^{k-2}(\mathrm{Std}^\vee) \otimes (\det \circ \mathrm{Std}^\vee)^{(w-k)/2}$, where $k \geq 2$ and $w \in \mathbf{Z}$, $w \equiv k \pmod 2$. Its central character is $\omega_\xi(x) = x^{2-w}$. The sheaf $\mathscr{L}_{\xi,\ell}$ is pure of weight $w-2$, hence $V^1(\pi^\infty) \subset H^1_!$ is pure of weight $w-1$. If π is a cuspidal automorphic representation of $GL(2, \mathbf{A})$ such that π_∞ is cohomological for ξ, then $\omega_{\pi_\infty}(x) = x^{w-2}$, which implies that the central character $\omega_\pi : \mathbf{A}^\times/\mathbf{Q}^\times \longrightarrow \mathbf{C}^\times$ of π is of the form $\omega_\pi = \chi \| \cdot \|_{\mathbf{Q}}^{w-2}$, where χ is a character of finite order, which will be identified with a Dirichlet character such that $\chi(-1) = (-1)^w$.

If $p \neq \ell$ is a prime such that π_p is unramified, then the Hecke operators T_p and S_p defined in A1.6 have the following eigenvalues on $\pi_p^{K_p}$ ($K_p = GL(2, \mathbf{Z}_p)$):

$$S_p\big|_{\pi_p^{K_p}} = \omega_{\pi_p}(p) = |p|_p^{2-w}\chi(p) = p^{w-2}\chi(p), \qquad T_p\big|_{\pi_p^{K_p}} = a_p.$$

The respective local L-factors at p of π and of the Galois representation $\rho_\pi : \Gamma_{\mathbf{Q}} \longrightarrow GL_2(\overline{\mathbf{Q}}_\ell)$ attached to π are given by

$$L_p(\pi, s)^{-1} = 1 - p^{-1/2}a_p p^{-s} + p^{w-2}\chi(p)p^{-2s} = (1 - p^{-1/2}T_p p^{-s} + S_p p^{-2s})\big|_{\pi_p^{K_p}}$$

and

$$L_p(\rho_\pi, s)^{-1} = \det(1 - p^{-s}\rho_\pi(\mathrm{Fr}(p))) = 1 - a_p p^{-s} + p^{w-1}\chi(p)p^{-2s} = L_p(\pi, s-1/2)^{-1} = (1-\alpha p^{-s})(1-\beta p^{-s})$$

(where $\mathrm{Fr}(p)$ is the geometric Frobenius). The formula (A1.6.1) together with the relation A5.3 imply that the action $\varphi_p^* \in \mathrm{Aut}(H^1_!)$ of φ_p on $H^1_!$ satisfies

$$Q_p(\varphi_p^*) = 0, \qquad Q_p(X) = X^2 - (T_p/S_p)X + p/S_p, \qquad (A5.6.1)$$

which means that $\mathrm{Fr}(p)|(V^1(\pi^\infty) \otimes (\pi^\infty)^K) = \varphi_p^*|(V^1(\pi^\infty) \otimes (\pi^\infty)^K)$ is a root of the polynomial

$$Q_p(X)\big|_{\pi_p^{K_p}} = (X - p\alpha^{-1})(X - p\beta^{-1}) = \det(X - \rho_\pi^\vee(-1)(\mathrm{Fr}(p)).$$

This being true for almost all p, the Čebotarev density theorem implies that

$$P_{\rho_\pi^\vee(-1)}(V^1(\pi^\infty)) = 0. \tag{A5.6.2}$$

In fact, $V^1(\pi^\infty)$ is two-dimensional, isomorphic to $\rho_\pi^\vee(-1)$, which is, in turn, isomorphic to $\rho_{\pi^\vee}$. Indeed, $\pi^\vee \simeq \pi \otimes \omega_\pi^{-1}$ and

$$S_p\big|_{(\pi_p^\vee)^{K_p}} = 1/(p^{w-2}\chi(p)) = p\alpha^{-1}\beta^{-1}, \qquad T_p\big|_{(\pi_p^\vee)^{K_p}} = a_p/(p^{w-2}\chi(p)) = p(\alpha^{-1} + \beta^{-1}),$$
$$L_p(\pi^\vee, s)^{-1} = (1 - p^{1/2}\alpha^{-1}p^{-s})(1 - p^{1/2}\beta^{-1}p^{-s}),$$
$$L_p(\rho_{\pi^\vee}, s)^{-1} = (1 - p\alpha^{-1}p^{-s})(1 - p\beta^{-1}p^{-s}) = L_p(\rho_\pi^\vee(-1), s)^{-1}.$$

The relation (A5.6.2) thus reads as follows:

$$P_{\rho_{\pi^\vee}}(V^1(\pi^\infty)) = 0. \tag{A5.6.3}$$

Note that $\pi_\infty^\vee$ is cohomological for $\xi^\vee = \mathrm{Sym}^{k-2}(\mathrm{Std}) \otimes (\det \circ \mathrm{Std})^{(w-k)/2} \simeq \xi \otimes (\det \circ \mathrm{Std})^{w-2}$, which means that $\mathscr{L}_{\xi^\vee,\ell}$ is pure of weight $2 - w$ and $V^1((\pi^\vee)^\infty)$ is pure of weight $3 - w$. We deduce from (A5.6.3) that

$$P_{\rho_\pi}(V^1((\pi^\vee)^\infty)) = 0. \tag{A5.6.4}$$

In fact, $V^1((\pi^\vee)^\infty) = \rho_\pi$.

(A5.7) Shimura varieties of type (A). Assume that $(B, *)$ is of type (A). In this case F_c is a CM field and $[F_c : F] = 2$. For each prime $v \mid \infty$ of F fix an embedding $\sigma_v : F_c \hookrightarrow \overline{\mathbf{Q}} \subset \mathbf{C}$ extending $v : F \hookrightarrow \mathbf{R}$; then $\Phi = \{\sigma_v\}$ is a CM type of F_c. This induces an isomorphism

$$B \otimes \mathbf{R} = \prod_{v\mid\infty} B \otimes_{F,v} \mathbf{R} = \prod_{v\mid\infty} B \otimes_{F_c,\sigma_v} \mathbf{C} \xrightarrow{\sim} \prod_{v\mid\infty} M_N(\mathbf{C})$$

under which $V^{-1,0} \xrightarrow{\sim} \bigoplus_{v\mid\infty} \left((\mathbf{C}^N)^{a_v} \oplus (\overline{\mathbf{C}}^N)^{b_v}\right)$, where $a_v + b_v = n$ and $C \otimes \mathbf{R} \xrightarrow{\sim} \prod_{v\mid\infty} M_n(\mathbf{C})$.

As explained in [HT, I.6], there is a canonical isomorphism $H \otimes_F F_c \xrightarrow{\sim} C^\times \times \mathbf{G}_{m,F_c}$, where we consider $C^\times = GL_B(V)$ as an algebraic group over F_c. This induces an isomorphism

$$H \otimes_F k \xrightarrow{\sim} (C^\times)_k \times \mathbf{G}_{m,k}, \tag{A5.7.1}$$

for any field embedding $F_c \hookrightarrow k$. In particular, the choice of Φ yields an isomorphism

$$G_\mathbf{C} = \prod_{v\mid\infty} (H \otimes_F F_c) \otimes_{F_c,\sigma_v} \mathbf{C} \xrightarrow{\sim} \prod_{v\mid\infty} (GL(n)_\mathbf{C} \times \mathbf{G}_{m,\mathbf{C}})$$

under which $\mu = \mu_h$ is given by $\mu(z) = ((zI_{a_v}, I_{b_v})_{v\mid\infty}, z)$.

(A5.8) Let $\xi, \mathscr{L}_\xi$ and $\mathscr{L}_{\xi,\ell}$ be as in A5.5. Assume that $\pi = \pi_\infty \otimes \pi^\infty$ is an automorphic representation of $G(\mathbf{A}) = H(\mathbf{A}_F)$ such that π_∞ is cohomological for ξ. We further assume that π admits a transfer to a **cuspidal** automorphic representation (Π, ψ) of $GL(n, \mathbf{A}_{F_c}) \times \mathbf{A}_{F_c}^\times$ (cf. [HT, Thm. VI.2.1]). All we need to know is that: (a) $\Pi^\vee \simeq \Pi^c$; (b) Π is cohomological for a suitable algebraic representation ξ' of $R_{F_c/\mathbf{Q}}(GL(n))_\mathbf{C}$; (c) $\psi = \omega_\pi^c$ is an algebraic Hecke character of F_c; (d) let $u \nmid \infty$ be a prime of F that splits in F_c as $u = ww^c$. The inclusion $F_c \hookrightarrow (F_c)_w = F_u$ defines, by (A5.7.1), an isomorphism $H(F_u) \xrightarrow{\sim} GL(n, F_u) \times F_u^\times$. If the representation π_u of the left hand side is unramified, then it is isomorphic to the representation (Π_u, ψ_u) of the right hand side.

The cuspidality of Π together with (a) and (b) imply [CH] that there is a Galois representation

$$\rho_\Pi : \Gamma_{F_c} \longrightarrow GL_n(\overline{\mathbf{Q}}_\ell)$$

such that

$$L_w(\Pi, \mathrm{Std}_n, s) = L_w(\rho_\Pi, s + (n-1)/2),$$

for all primes $w \nmid \ell\infty$ at which Π is unramified. Similarly, one can attach to ψ a one-dimensional Galois representation $\rho_\psi : \Gamma_{F_c} \longrightarrow \overline{\mathbf{Q}}_\ell^\times$ such that $L(\psi, s) = L(\rho_\psi, s)$.

(A5.9) We are going to make the relation (A5.5.2) explicit in terms of ρ_Π and ρ_ψ. Fix a finite set of primes $S \supset \{\ell, \infty\}$ of $\mathbf{Q}$ such that $F_c/\mathbf{Q}$, H and π are ramified only at places above S.

Let $\mathbf{Q}_S \supset F_c^{gal}$ be as in 5.15 and let P_S be a prime of $\mathbf{Q}_S$ not above S such that

$$\mathrm{Fr}_{\mathbf{Q}_S/\mathbf{Q}}(P_S) \in \mathrm{Gal}(\mathbf{Q}_S/F_c^{gal}). \tag{A5.9.1}$$

Then $P_S \cap \mathbf{Z} = (p)$, where $p \notin S$ is a prime that splits completely in $F_c/\mathbf{Q}$. Fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ inducing P_S; then all the assumptions on p imposed in A3 and A4 are satisfied.

Fix $F_c \subset \overline{\mathbf{Q}}$. For each prime $v \mid \infty$ in F extend $\sigma_v : F_c \hookrightarrow \overline{\mathbf{Q}}$ to an automorphism $\widetilde{\sigma}_v : \overline{\mathbf{Q}} \xrightarrow{\sim} \overline{\mathbf{Q}}$. The prime $P'_v = \widetilde{\sigma}_v^{-1} P_S \cap O_{F_c}$ of F_c above p depends only on σ_v, and $P_v = P'_v \cap O_F$ splits in F_c as $P_v = P'_v (P'_v)^c$.

For a representation ρ of $\mathrm{Gal}(\mathbf{Q}_S/F_c^{gal})$ denote by ${}^{\widetilde{\sigma}_v}\rho$ the representation $({}^{\widetilde{\sigma}_v}\rho)(g) = \rho(\widetilde{\sigma}_v^{-1} g \widetilde{\sigma}_v)$ of the same group; then

$$({}^{\widetilde{\sigma}_v}\rho)(\mathrm{Fr}(P_S)) = \rho(\widetilde{\sigma}_v^{-1}\mathrm{Fr}(P_S)\widetilde{\sigma}_v) = \rho(\mathrm{Fr}(P'_v)).$$

(A5.10) Fix $v \mid \infty$ in F and consider the relation (A5.5.2) for $P = P_v$. Firstly, the embedding $F_c \hookrightarrow (F_c)_{P'_v} = F_P = \mathbf{Q}_p$ gives a canonical isomorphism $H(F_P) \xrightarrow{\sim} GL(n, F_P) \times F_P^\times$ under which μ_P can be chosen as $\mu_P(z) = ((zI_a, I_b), z)$, where $a = a_v$ and $b = b_v = n - a$. Secondly, the number $\langle 2\rho_P, \mu_P\rangle$ is equal to the dimension of the symmetric space for $SU(a,b)$, namely, to $ab = a(n-a)$. Thirdly, $V_{\mu_P^{-1}}$ is the representation $(\bigwedge^a \mathrm{Std}_n^\vee) \otimes \mathrm{Std}_1^\vee$ of the dual group $\widehat{GL(n)} \times \widehat{\mathbf{G}_m} = GL(n, \mathbf{C}) \times \mathbf{C}^\times$.

Write

$$L_{P'_v}(\rho_\Pi, s) = \prod_{i=1}^n (1 - \alpha_i p^{-s})^{-1};$$

then

$$L_{P'_v}(\Pi, \mathrm{Std}_n, s) = \prod_{i=1}^n (1 - t_i p^{-s})^{-1}, \qquad t_i = p^{(1-n)/2}\alpha_i.$$

The Satake parameters of $\Pi_{P'_v}$ are given, therefore, by the semisimple element $t = \mathrm{diag}(t_1, \ldots, t_n)$ of the dual $GL(n)$. Similarly, the Satake parameter of $\psi_{P'_v}$ is equal to $u = \rho_\psi(\mathrm{Fr}(P'_v))$. By definition (and by the fact that $\pi_P \simeq (\Pi, \psi)_{P'_v}$),

$$\widetilde{H}_{\mu_P^{-1}}(X)\big|_{\pi_P^{K_P}} = \det\Big(X - (t,u)\big|(\bigwedge^a \mathrm{Std}_n^\vee) \otimes \mathrm{Std}_1^\vee)\Big) = \prod_{|I|=a} (X - t_I^{-1}u^{-1}) = \prod_{|I|=a} (X - p^{(n-1)a/2}\alpha_I^{-1}u^{-1}),$$

where $I \subset \{1, \ldots, n\}$, $t_I = \prod_{i\in I} t_i$ and $\alpha_I = \prod_{i \in I}\alpha_i$. This implies that

$$p^C \widetilde{H}_{\mu_P^{-1}}(p^{-a(n-a)/2}X)\big|_{\pi_P^{K_P}} = \det\Big(X - \mathrm{Fr}(P'_v)\big|(\bigwedge^a \rho_\Pi^\vee) \otimes \rho_\psi^\vee(a(a+1)/2 - an)\Big), \tag{A5.10.1}$$

for some $C \in \mathbf{Z}$. Consider the representation

$$\rho_v = {}^{\widetilde{\sigma}_v}\Big((\bigwedge^{a_v} \rho_\Pi^\vee) \otimes \rho_\psi^\vee(a_v(a_v+1)/2 - a_v n)\Big) : \mathrm{Gal}(\mathbf{Q}_S/F_c^{gal}) \longrightarrow GL_{n_v}(\overline{\mathbf{Q}}_\ell).$$

The right hand side of (A5.10.1) is then equal to $\det(X - \rho_v(\mathrm{Fr}(P_S))) = P_{\rho_v(\mathrm{Fr}(P_S))}(X)$ and the relation (A5.5.2) reads as follows:

$$P_{\rho_v(\mathrm{Fr}(P_S))}(\varphi_{P_v}\big|_{V^i(\pi^\infty)\otimes(\pi^\infty)^K}) = 0. \qquad (A5.10.2)$$

Of course,

$$\prod_{v|\infty}(\varphi_{P_v}\big|_{V^i(\pi^\infty)\otimes(\pi^\infty)^K}) = \mathrm{Fr}(P_S)\big|_{V^i(\pi^\infty)\otimes(\pi^\infty)^K}.$$

(A5.11) The isotropic case. What happens in the general PEL situation of A3 if we drop the assumption made in A3.1 that the group G^{der} is anisotropic (but if we keep the assumptions from A3.2 and A3.4)? If G^{der} is isotropic, then $Sh_K(G,\mathscr{X})$ is no longer proper over E and the discussion in A5.5 needs to be modified as follows (as explained to the author by B. Stroh).

The (pull-back of) the sheaf $\mathscr{L}_{\xi,\ell}$ extends to the union of the Kottwitz models $M_{K^p} = \coprod_{\alpha\in\Sigma} M_{\alpha,K^p}$ from A3.5 and there is a canonical $\Gamma_E \times G(\widehat{\mathbf{Q}})$-equivariant isomorphism

$$H^i_{et}(M_{K^p}\otimes\overline{\mathbf{Q}}, pr^*(\mathscr{L}_{\xi,\ell})) \simeq H^i_{et}(M_{K^p}\otimes\overline{k(v)}, pr^*(\mathscr{L}_{\xi,\ell})) \qquad (A5.11.1)$$

([FC, Thm. VI.6.1] in the case of Siegel modular varieties, [LS, Thm. 6.1] in general). The point is that the cohomology of $pr^*(\mathscr{L}_{\xi,\ell})$ is, up to a Tate twist, a direct summand of the cohomology of the constant sheaf $\overline{\mathbf{Q}}_\ell$ on a suitable Kuga-Sato variety. The integral model of the Kuga-Sato variety over M_{K^p} admits a smooth toroidal compactification whose boundary is a relative normal crossing divisor, which means that the general result of [SGA $4\frac{1}{2}$, Th. de finitude, App. 1.3.3(i)] applies.

Passing to Δ-invariants, one obtains from (A5.11.1) an isomorphism

$$H^i_{et}(Sh_K\otimes_E\overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell}) \simeq H^i_{et}(S_K\otimes\overline{k(v)}, \mathscr{L}_{\xi,\ell}). \qquad (A5.11.2)$$

Under the assumption (A4.3.1) the arguments of [W] establish the existence of the diagram (A5.1.1) even in the isotropic case. Hecke correspondences on Sh_K (resp. the p-isogenies on M_{K^p}) are proper correspondences in the sense that their projections on each of the factors in $Sh_K\times Sh_K$ (resp. $M_{K^p}\times M_{K^p}$ are proper (and generically finite). As explained in [FC, VII.2], one can generalise the definitions recalled in A2 to this situation and define

- an action of $\mathcal{H}(G(\mathbf{Q}_p)_-//K_p,\mathbf{Q})$ on $H^i_{et}(Sh_K\otimes_E\overline{\mathbf{Q}},\mathscr{L}_{\xi,\ell})$;
- an action of $\mathbf{Q}[p-\mathrm{Isog}_{K^p}\otimes E_v]$ on $H^i_{et}(M_{K^p}\otimes\overline{\mathbf{Q}}, pr^*(\mathscr{L}_{\xi,\ell}))$;
- an action of $\mathbf{Q}[p-\mathrm{Isog}_{K^p}\otimes \mathbf{F}_p]$ on $H^i_{et}(M_{K^p}\otimes\overline{k(v)}, pr^*(\mathscr{L}_{\xi,\ell}))$;

as in A5.5, these actions are compatible with the isomorphisms (A5.11.1-2). As a result, the Eichler-Shimura relation (A5.5.1) holds (if p splits completely in $F_c/\mathbf{Q}$) for the action of φ_P on $H^i(Sh_K) = H^i_{et}(Sh_K\otimes_E \overline{\mathbf{Q}},\mathscr{L}_{\xi,\ell})$.

(A5.12) The decompositions (0.3.2) and (0.4.1) no longer hold in the isotropic case. For any irreducible smooth representation π^∞ of $G(\widehat{\mathbf{Q}})$ one can consider the π^∞-eigenspace in $H^i(Sh) = \varinjlim_K H^i(Sh_K)$, namely,

$$H^i(Sh)[\pi^\infty] = \mathrm{Im}(V^i(\pi^\infty)\otimes\pi^\infty \hookrightarrow H^i(Sh)), \qquad V^i(\pi^\infty) = \mathrm{Hom}_{G(\widehat{\mathbf{Q}})}(\pi^\infty, H^i(Sh)). \qquad (A5.12.1)$$

In general, the action of $G(\widehat{\mathbf{Q}})$ on $H^i(Sh)$ is not semisimple (as pointed out by the referee, this happens already for $G = GL(2)_{\mathbf{Q}}$, $i=1$ and $\xi=1$, when some π^∞ can occur as a subquotient but not as a submodule). This means that, a priori, $H^i(Sh)[\pi^\infty]$ could be smaller than the corresponding generalised eigenspace (in other words, the space $\mathrm{Hom}_{G(\widehat{\mathbf{Q}})}(\pi^\infty, H^i(Sh)/H^i(Sh)[\pi^\infty])$ could be non-zero).

As in the isotropic case, $V^i(\pi^\infty)$ is of finite dimension over $\overline{\mathbf{Q}}_\ell$ and the action of Γ_E on $H^i(Sh)$ gives rise to a representation

$$\Gamma_E \longrightarrow \mathrm{Aut}_{G(\widehat{\mathbf{Q}})}(H^i(Sh)[\pi^\infty]) = \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V^i(\pi^\infty)).$$

If $K = K_p K^p$ is as in A5.11 (with p split in $F_c/\mathbf{Q}$) and $(\pi^\infty)^K \neq 0$, then the $G(\widehat{\mathbf{Q}})$-equivariance of the φ_P's implies again that the subspace $V^i(\pi^\infty) \otimes (\pi^\infty)^K = H^i(Sh)[\pi^\infty]^K \subset H^i(Sh)^K = H^i(Sh_K)$ is φ_P-stable and that each φ_P acts on it through an action $\varphi_P \in \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V^i(\pi^\infty))$ on $V^i(\pi^\infty)$.

As in A5.5, restricting (A5.5.1) to $H^i(Sh)[\pi^\infty]^K$ yields the formula (A5.5.2) for the space $V^i(\pi^\infty)$ defined in (A5.12.1).

An important subspace of $V^i(\pi^\infty)$ arises as follows. The analytic intersection cohomology of the Baily-Borel compactification $j : Sh_K(G, \mathscr{X}) \hookrightarrow Sh_K(G, \mathscr{X})_{BB}$ is isomorphic to the L^2-cohomology of $Sh_K(G, \mathscr{X})^{an}$ ([Lo], [SS]) and the latter space admits a decomposition analogous to (0.3.2), by [BC, Th. A] combined with [BG, Prop. 5.6]. As a result, there is a $\Gamma_E \times G(\widehat{\mathbf{Q}})$-equivariant decomposition of the intersection étale cohomology

$$H^i(Sh_{BB}) = \varinjlim_{K} H^i_{et}(Sh_K(G, \mathscr{X})_{BB} \otimes_E \overline{\mathbf{Q}}, j_{!*}\mathscr{L}_{\xi,\ell})$$

of the form

$$H^i(Sh_{BB}) = \bigoplus_{\pi = \pi_\infty \otimes \pi^\infty} m_{disc}(\pi)\, H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes \xi) \otimes \pi^\infty = \bigoplus_{\pi^\infty} V^i_{disc}(\pi^\infty) \otimes \pi^\infty, \qquad (A5.12.2)$$

where π runs through discrete automorphic representations of $G(\mathbf{A})$ and $m_{disc}(\pi)$ denotes the multiplicity of $\pi' = (\omega'_\pi)^{-1}\pi$ in the discrete part $L^2_{disc}(G, \omega_\pi) \subset L^2(G, \omega_\pi)$.

In general, the canonical Γ_E-equivariant map $V^i_{disc}(\pi^\infty) \longrightarrow V^i(\pi^\infty)$ induced by $H^i(Sh_{BB}) \longrightarrow H^i(Sh)$ is not injective, nor surjective (cf. Proposition A6.17). It would be of interest to define compatible actions of various rings in the diagram (A5.1.1) on the intersection étale cohomology of $Sh_K(G, \mathscr{X})_{BB}$ (for which, again, isomorphisms analogous to (A5.11.1-2) hold, thanks to [LS, Thm. 6.1]) and deduce the formula (A5.5.2) on $V^i_{disc}(\pi^\infty)$. All that we can say at the moment is that $\overline{V}^i_{disc}(\pi^\infty) = \mathrm{Im}(V^i_{disc}(\pi^\infty) \longrightarrow V^i(\pi^\infty)) \subset V^i(\pi^\infty)$ is Γ_E-stable and that, for P_S as in (A5.9.1), the action of $\mathrm{Fr}(P_S)$ on $\overline{V}^i_{disc}(\pi^\infty)$ is given by the restriction of the action of $\prod_{P|p} \varphi_P \in \mathrm{Aut}(V^i(\pi^\infty))$, with each φ_P satisfying (A5.5.2).

This issue does not arise for the contribution of cuspidal representations π to (A5.12.2), since cuspidal cohomology injects into $H^i(Sh)$.

A6. Quaternionic Shimura varieties

(A6.1) Throughout A6 we assume that $F \subset \overline{\mathbf{Q}} \subset \mathbf{C}$, $r = [F : \mathbf{Q}]$, D, Ω, $t = |\Omega|$, E and ξ are as in 5.1-5.5. In A6.1–A6.14 we assume that $D \neq M_2(F)$. The Shimura varieties involved and their integral models will then all be proper.

Denote by $d \mapsto \overline{d}$ the main involution on D. Let $v \mid p$ be the prime of E induced by a fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Let $K = K_S K^S \subset (D \otimes \widehat{\mathbf{Q}})^\times$ be an open compact subgroup as in 5.15 (with S containing all primes of F dividing $2\ell\infty$ and all primes at which $F/\mathbf{Q}$ and D ramify).

The action of Γ_E on $H^i_{\xi,K} = H^i_{et}(Sh_K(D^\times) \otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell})$ then factors through $\mathrm{Gal}(\mathbf{Q}_S/E)$. The goal of A6 is to verify the Eichler-Shimura relation (5.16.1) for $\widetilde{E} = F^{gal}$, namely, that for every prime P_S of $\mathbf{Q}_S$ satisfying $\mathrm{Fr}_{\mathbf{Q}_S/\mathbf{Q}}(P_S) \in \mathrm{Gal}(\mathbf{Q}_S/F^{gal})$ ($\iff$ $p = P_S \cap \mathbf{Z}$ does not lie below S and p splits completely in $F/\mathbf{Q}$) the action of $\mathrm{Fr}(P_S)$ on $H^i_{\xi,K}$ can be written as

$$\mathrm{Fr}(P_S)|_{H^i_{\xi,K}} = \prod_{x \in \Omega} \varphi^*_x, \qquad \varphi^*_x \varphi^*_y = \varphi^*_y \varphi^*_x, \qquad (\varphi^*_x)^2 - (T_{P_x}/S_{P_x})\varphi^*_x + p/S_{P_x} = 0. \qquad (A6.1.1)$$

These relations follow directly from Theorem A5.3 if $t = r$ (see A6.4), but require an auxiliary unitary Shimura variety if $t < r$ (see A6.14).

(A6.2) The PEL data in the case $t = r$. In the totally indefinite case $D \otimes \mathbf{R} \simeq M_2(\mathbf{R})^r$ one only needs to use the fact that the main involution $A \mapsto \overline{A} = \mathrm{Tr}(A) \cdot I - A$ on $M_2(\mathbf{R})$ is not positive, but is conjugate to the positive involution $A \mapsto A^t$ by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Consider the following PEL data of type (C), in the sense of A3: $B = D$, $* =$ a positive involution on D ($\iff$ $d^* = u\overline{d}u^{-1}$ for some $u \in D^\times$, unique up to $F^\times$, such that $\overline{u} = -u$ and $\mathrm{Nrd}(u) = -u^2 \in F_+^\times$), $V = D$ with a left action of D given by $d \cdot x = xd^*$. The F-bilinear form $\langle x, y\rangle_F = \mathrm{Trd}(xu\overline{y})$ on V is skew-symmetric and satisfies $\langle d\cdot x, y\rangle_F = \langle x, d^* \cdot y\rangle_F$.

The centraliser $C = \mathrm{End}_B(V)$ is isomorphic to D, with $d \in D$ acting by left multiplication $x \mapsto dx$. As $\langle dx, y\rangle_F = \langle x, \overline{d}y\rangle_F$, the involution $\#$ on C coincides with the main involution on D and

$$GSp_B(V, \langle\ ,\ \rangle_F) = D^\times, \qquad \nu = \mathrm{Nrd}.$$

A morphism $h : \mathbf{C} \longrightarrow C \otimes \mathbf{R} = D \otimes \mathbf{R}$ as in A3.2 is given, for example, by $h(i) = u/\sqrt{-u^2}$ (with a totally positive square root), hence is conjugate to the one in (5.3.1). This identifies $Sh(D^\times)$ with the Shimura variety attached to the above PEL data $(B, *, V, \langle\ ,\ \rangle_F)$ of type (C).

(A6.3) Given a rational prime p which does not lie below S, we can choose $u \in D^\times$ in A6.2 in such a way that $u^2 \in (O_F \otimes \mathbf{Z}_p)^\times$, which implies that unramified local data O_B and Λ as in A3.4 exist. We can assume that K_S is sufficiently small; then the construction from A3.7 yields a smooth projective model S_K of $Sh_K(D^\times)$ over O_{E_v}, to which the local system $\mathscr{L}_{\xi,\ell}$ naturally extends. Let $S_K^\circ = S_K \otimes k(v)$ be the special fibre of S_K.

(A6.4) The Eichler-Shimura relation in the case $t = r$. In the situation of A6.3, assume, in addition, that $p = P_S \cap \mathbf{Z}$, where P_S is a prime of $\mathbf{Q}_S$ such that $\mathrm{Fr}_{\mathbf{Q}_S/\mathbf{Q}}(P_S) \in \mathrm{Gal}(\mathbf{Q}_S/F^{gal})$. This is equivalent to requiring p to split completely in $F/\mathbf{Q}$; in fact, $pO_F = \prod_{x\in X} P_x$, where P_x are as in (5.15.3) (and $F_{P_x} = \mathbf{Q}_p$).

The absolute Frobenius morphism $\varphi : S_K^\circ = S_K \otimes \mathbf{F}_p \longrightarrow S_K^\circ$ decomposes as a product of mutually commuting partial Frobenius morphisms $\varphi = \prod_{x\in X} \varphi_{P_x}$ defined in A3.12. The action of $\mathrm{Fr}(P_S)$ on $H^i = H^i_{et}(S_K^\circ \otimes \overline{\mathbf{F}}_p, \mathscr{L}_{\xi,\ell})$ is given by the action $(\varphi \otimes \mathrm{id})^*$ of $\varphi \otimes \mathrm{id} : S_K^\circ \otimes \overline{\mathbf{F}}_p \longrightarrow S_K^\circ \otimes \overline{\mathbf{F}}_p$. It follows that

$$\mathrm{Fr}(P_S)|_{H^i} = \prod_{x\in X} \varphi_x^*, \qquad \varphi_x^* = (\varphi_{P_x} \otimes \mathrm{id})|_{H^i}, \qquad \varphi_x^*\varphi_y^* = \varphi_y^*\varphi_x^*.$$

Theorem A5.3 applies to each φ_{P_x} and yields, thanks to (A1.6.1), the sought for Eichler-Shimura relation $(\varphi_x^*)^2 - (T_{P_x}/S_{P_x})\varphi_x^* + p/S_{P_x} = 0 \in \mathrm{End}(H^i)$.

(A6.5) The auxiliary PEL data in the case $1 \le t < r$. In this case the quaternionic Shimura variety $Sh(D^\times)$ is not of the form considered in A3-A5, but it can be related to other Shimura varieties defined in terms of suitable PEL data of type (A) (and signature $(1,1)^t \times (2,0)^{r-t}$). The following construction, which differs from the standard one ([De1, §6], [R, p. 11]), was communicated to the author by C. Cornut.

Fix injections $F \hookrightarrow F_c \hookrightarrow D$, where F_c is a CM field and $[F_c : F] = 2$. Fix elements $\eta \in F_c^\times$ and $j \in D^\times$ such that $\eta^* = -\eta$, $j^* = -j$, $j\eta = \eta^* j = -\eta j$, where $*$ denotes the main involution of D (whose restriction to F_c is positive). For every infinite prime $v \mid \infty$ of F we have $j_v^2 \in F_v^\times = \mathbf{R}^\times$ and $\mathrm{sgn}(j_v^2) = +1$ (resp. $= -1$) if $v \in \Omega$ (resp. if $v \in \Omega^c$).

Consider the following PEL data of type (A) in the sense of A3: $(B, *) = (F_c, *)$, $V = D$ as a left F_c-module, $\langle x, y\rangle_F = \mathrm{Trd}(\eta x y^*)$. Explicitly,

$$\forall x_1, x_2, y_1, y_2 \in F_c \qquad \langle x_1 + x_2 j, y_1 + y_2 j\rangle_F = \mathrm{Tr}_{F_c/F}(\eta(x_1y_1^* - j^2x_2y_2^*)).$$

The morphism

$$\tau : F_c^\times \times D^\times \longrightarrow GL_{F_c}(V), \qquad (\lambda, d) \mapsto (x \mapsto \lambda x d^*)$$

gives rise to an exact sequence of algebraic groups over F

$$1 \longrightarrow \mathbf{G}_{m,F} \xrightarrow{\Delta} R_{F_c/F}(\mathbf{G}_{m,F_c}) \times D^\times \xrightarrow{\tau} H = GSp_B(V, \langle\ ,\ \rangle_F) \longrightarrow 1, \qquad (A6.5.1)$$

where $\Delta(a) = (a, a^{-1})$. Note that $\mathbf{G}_{m,F}$ has trivial H^1, which implies that the map τ in (A6.5.1) is surjective on adelic and F-rational points.

(A6.6) The previous data define a CM type $\Phi = \{\sigma_v : F_c \hookrightarrow \mathbf{Q}\}_{v|\infty}$ of F_c characterised by $\sigma_v(\eta) = -\mathrm{sgn}(j_v^2)|\eta_v|i$. Denote by $\sigma_v^\vee : \mathbf{C} \simeq F_c \otimes_F F_v (\hookrightarrow D \otimes_F F_v)$ the maps induced by the inverse of σ_v. Let

$$h = (h_{F_c}, h_D) : \mathbf{C} \longrightarrow (F_c \otimes \mathbf{R}) \times (D \otimes \mathbf{R}) \tag{A6.6.1}$$

be the morphism whose components $h_v : \mathbf{C} \longrightarrow (F_c \otimes_F F_v) \times (D \otimes_F F_v)$ are given as follows: $h_v = (1, \sigma_v^\vee)$ (resp. $h_v = (\sigma_v^\vee, 1)$) if $v \in \Omega$ (resp. if $v \in \Omega^c$).

Explicitly,

$$h_v(i)(x_v + y_v j_v) = \frac{\eta_v}{|\eta_v|} (x_v - y_v \operatorname{sgn}(j_v^2) j_v),$$

$$\forall x_1, x_2, y_1, y_2 \in F_c \otimes \mathbf{R} \qquad \langle x_1 + x_2 j, h(i)(y_1 + y_2 j) \rangle_F = \sum_{v|\infty} |\eta_v| \mathrm{Tr}_{F_c \otimes_F F_v / F_v}(x_{1,v} y_{1,v}^* + |j_v^2| x_{2,v} y_{2,v}^*),$$

which means that h satisfies the positivity property from A3.2. Moreover, h_D is conjugate to the morphism (5.3.1) and

$$V_v^{-1,0} = \begin{cases} \overline{F_c \otimes_F F_v} + (F_c \otimes_F F_v) j_v & v \in \Omega, \\ (F_c \otimes_F F_v) + (F_c \otimes_F F_v) j_v & v \in \Omega^c, \end{cases}$$

which implies that the signatures of H in the sense of A5.7 are equal to $(a_v, b_v) = (1,1)$ (resp. $(2,0)$) if $v \in \Omega$ (resp. if $v \in \Omega^c$).

(A6.7) The base change of the exact sequence (A6.5.1) to F_c becomes isomorphic, via (A5.7.1), to

$$1 \longrightarrow \mathbf{G}_{m,F_c} \xrightarrow{\Delta_c} \mathbf{G}_{m,F_c} \times \mathbf{G}_{m,F_c} \times GL_{F_c}(V) \xrightarrow{\tau_c} GL_{F_c}(V) \times \mathbf{G}_{m,F_c} \longrightarrow 1, \tag{A6.7.1}$$

where $\Delta_c(a) = (a, a, a^{-1})$ and $\tau_c(a_1, a_2, g) = (a_1 g, a_1 a_2 \det(g))$. Above, we have identified $\mathbf{G}_{m,F_c} \otimes_F F_c$ with $\mathbf{G}_{m,F_c} \times \mathbf{G}_{m,F_c}$ in the usual way: for every F_c-algebra R, the corresponding isomorphism $(F_c \otimes_F R)^\times \xrightarrow{\sim} R^\times \times R^\times$ sends $a \otimes r$ to $(ar, a^* r)$.

The cocharacters

$$\mu = (\mu_x : \mathbf{G}_{m,\mathbf{C}} \longrightarrow H \otimes_{F,\sigma_x} \mathbf{C})_{x \in X} : \mathbf{G}_{m,\mathbf{C}} \longrightarrow G_{\mathbf{C}}$$

attached to the morphism h from (A6.6.1) are given, up to conjugation, by

$$\forall x \in X \qquad \mu_x(z) = \begin{cases} \tau_c(1, 1, \begin{pmatrix} z & \\ & 1 \end{pmatrix}) & x \in \Omega, \\ \tau_c(z, 1, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}) & x \in \Omega^c. \end{cases} \tag{A6.7.2}$$

(A6.8) Assume that v is a finite prime of F at which D splits and which splits in F_c/F as $vO_{F_c} = v'v''$. Choose one of the factors (say, v') and identify F_v as an F_c-algebra via $F_c \hookrightarrow (F_c)_{v'} = F_v$. The sequence (A6.7.1) then gives

$$1 \longrightarrow \mathbf{G}_{m,F_v} \longrightarrow \mathbf{G}_{m,(F_c)_{v'}} \times \mathbf{G}_{m,(F_c)_{v''}} \times GL(2)_{F_v} \xrightarrow{\tau_c} H \otimes_F F_v \longrightarrow 1, \tag{A6.8.1}$$

with $H \otimes_F F_v$ split over F_v.

Consider a cocharacter $\mu : \mathbf{G}_{m,F_v} \longrightarrow H \otimes_F F_v$ given by one of the formulas in (A6.7.2). The corresponding Hecke polynomial of μ^{-1} is as follows.

(Case Ω): If $\mu(z) = \tau_c(1, 1, \begin{pmatrix} z & \\ & 1 \end{pmatrix})$, then (A1.6.1) implies that

$$(Nv)\, \widetilde{H}_{\mu^{-1}}((Nv)^{-1/2} Y) = Y^2 - (T_v/S_v) Y + (Nv)/S_v. \tag{A6.8.2}$$

(Case Ω^c): If $\mu(z) = \tau_c(z, 1, I_2)$, then (A1.5.1) implies that

$$\widetilde{H}_{\mu^{-1}}(Y) = Y - 1/S_{v'}, \qquad S_{v'} = \mathrm{char}(\varpi_{v'} O^\times_{(F_c)_{v'}}) \in \mathcal{H}((F_c)^\times_{v'} // O^\times_{(F_c)_{v'}}, \mathbf{Z}). \tag{A6.8.3}$$

(A6.9) Quaternionic and unitary Shimura data [R, §1]. Consider the following algebraic groups over F: $F_c^\times = R_{F_c/F}\mathbf{G}_{m,F_c}$, $D^\times$ and $\tau : (F_c^\times \times D^\times)/\Delta(F^\times) \xrightarrow{\sim} H = GSp_{F_c}(V)$. Their respective restrictions of scalars to $\mathbf{Q}$ (notably $G = R_{F/\mathbf{Q}}(H)$) are equipped with the Shimura data h_{F_c}, h_D (conjugate to the one from (5.3.1)) and $h_G = \tau_{\mathbf{R}} \circ h$, where $h = h_{F_c} \times h_D$ was defined in (A6.6.1). Their reflex fields are equal to

$$E(G, h_G) = E(F_c^\times, h_{F_c}) = E(\Phi|\Omega^c) = \{\gamma \in \Gamma_{\mathbf{Q}} \mid \forall v \in \Omega^c \ \ \gamma\sigma_v = \sigma_v\} \supset E(D^\times, h_D) = E.$$

The morphism τ induces a map

$$Sh(F_c^\times) \times (Sh(D^\times) \otimes_E E(\Phi|\Omega^c)) \longrightarrow Sh(G),$$

which is $\widehat{F}_c^\times \times \widehat{D}$-equivariant (in particular, $\Delta(\widehat{F}^\times)$ acts along its fibres).

If $K \subset \widehat{D}^\times$ is an open compact subgroup which is small enough in the sense that $K = K_q K^q$ for a prime q such that $K_q \cap (F \otimes \mathbf{Q}_q)^\times \subset 1 + q^e(O_F \otimes \mathbf{Z}_q)$, where $e = 1$ if $q > 2$ (resp. $e = 2$ if $q = 2$), then there exist open compact subgroups $K(F_c) \subset \widehat{F}_c^\times$ (defined in [R, (1.4)]) and $K(G) = \tau(K(F_c) \times K) \subset G(\widehat{\mathbf{Q}})$ such that

$$Sh_{K(F_c)}(F_c^\times) \times (Sh_K(D^\times) \otimes_E E(\Phi|\Omega^c)) \longrightarrow Sh_{K(G)}(G)$$

is a Galois covering with Galois group $\Delta(F^\times \backslash \widehat{F}^\times / (K \cap \widehat{F}^\times))$. In particular, its fibres are the $\Delta(\widehat{F}^\times)$-orbits.

(A6.10) The Künneth formula. Fix an algebraic representation ξ' of $({}_{F_c}T)_{\mathbf{C}}$ whose restriction to $({}_F T)_{\mathbf{C}}$ coincides with ω_ξ $(= N_{F/\mathbf{Q}}^{2-w})$; then $\xi' \otimes \xi = \xi_G \circ \tau$ for an algebraic representation ξ_G of $G_{\mathbf{C}}$. The Künneth formula combined with the discussion in the previous paragraph implies that the cohomology groups

$$H^*_\xi = H^*_{et}(Sh(D^\times) \otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell}), \qquad H^*_{\xi'} = H^*_{et}(Sh(F_c^\times) \otimes_{E(\Phi|\Omega^c)} \overline{\mathbf{Q}}, \mathscr{L}_{\xi',\ell}),$$
$$H^*_{\xi_G} = H^*_{et}(Sh(G) \otimes_{E(\Phi|\Omega^c)} \overline{\mathbf{Q}}, \mathscr{L}_{\xi_G,\ell})$$

are related as follows (note that $Sh(F_c^\times)$ has dimension zero):

$$H^i_{\xi_G} \xrightarrow{\sim} \left(H^0_{\xi'} \otimes H^i_\xi\right)^{\Delta(\widehat{F}^\times)}. \tag{A6.10.1}$$

Assume that π is as in 5.11 and 5.15: it is an automorphic representation of $D^\times_{\mathbf{A}}$ such that π_∞ is cohomological (in degree i) with respect to ξ. Fix an open compact subgroup $K \subset \widehat{D}^\times$ such that $(\pi^\infty)^K \neq 0$. As in 5.15,

$$0 \neq V^i(\pi^\infty) \otimes (\pi^\infty)^K \subset H^i_{\xi,K} = (H^i_\xi)^K,$$

with $\widehat{F}^\times$ acting on this subspace by ω_{π^∞}.

Assume, furthermore, that $\chi : \mathbf{A}^\times_{F_c}/F_c^\times \longrightarrow \mathbf{C}^\times$ is a character such that

$$\chi_\infty = (\xi')^{-1}, \qquad \chi|_{\mathbf{A}_F^\times} = \omega_\pi \tag{A6.10.2}$$

(these two conditions are compatible, since $\omega_{\pi_\infty} = \omega_\xi^{-1}$). Denote by $\chi^\infty : \widehat{F}_c^\times \longrightarrow \mathbf{C}^\times \simeq \overline{\mathbf{Q}}_\ell^\times$ its finite part.

If $K(F_c) \subset \widehat{F}_c^\times$ is sufficiently small in the sense that

$$F_c^\times \cap K(F_c) \subset O^\times_{F,+}, \qquad K(F_c) \subset \mathrm{Ker}(\chi^\infty), \tag{A6.10.3}$$

then the χ^∞-eigenspace

$$V(\chi^\infty) = V^0(\chi^\infty) = \{f \in H^0_{\xi'} \mid \forall a \in \widehat{F}_c^\times \ \ a \cdot f = \chi^\infty(a) f\}$$

satisfies $\dim(V(\chi^\infty) \otimes (\chi^\infty)^{K(F_c)}) = 1$.

The representation $\chi \otimes \pi$ of $\mathbf{A}_{F_c}^\times \times D_{\mathbf{A}}^\times$ is of the form $\chi \otimes \pi = \pi_G \circ \tau$, where π_G is an automorphic representation of $G(\mathbf{A}) = H(\mathbf{A}_F)$, with $(\pi_G)_\infty$ cohomological (in degree i) with respect to ξ_G. The previous discussion combined with (A6.10.1) implies that, for K sufficiently small,

$$V^i(\pi_G^\infty) \otimes (\pi_G^\infty)^{K(G)} = V(\chi^\infty) \otimes V^i(\pi^\infty) \otimes (\chi^\infty \otimes (\pi^\infty)^K)^{\Delta(\widehat{F}^\times)} = V(\chi^\infty) \otimes V^i(\pi^\infty) \otimes (\chi^\infty \otimes (\pi^\infty)^K) \neq 0, \tag{A6.10.4}$$

hence

$$V^i(\pi_G^\infty) = V(\chi^\infty) \otimes V^i(\pi^\infty) \neq 0.$$

In particular, there are canonical identifications

$$\mathrm{End}(V^i(\pi_G^\infty)) = \mathrm{End}(V^i(\pi^\infty)), \qquad \mathrm{End}(V^i(\pi_G^\infty) \otimes (\pi_G^\infty)^{K(G)}) = \mathrm{End}(V^i(\pi^\infty) \otimes (\pi^\infty)^K). \tag{A6.10.5}$$

(A6.11) The partial Frobenii. Assume that $K = K_S K^S \subset \widehat{D}^\times$ is as in 5.15 and that K_S is sufficiently small. Let p be a rational prime which does not lie below S and which satisfies the following two conditions: $\eta^2, j^2 \in (O_F \otimes \mathbf{Z}_p)^\times$ and every prime $P \mid p$ above p in F splits in F_c/F: $PO_{F_c} = P'P''$.

These conditions imply that there exist unramified data O_B and Λ at p (as in A3.4) for the PEL data from A6.5. As before, this gives a smooth projective model $S_{K(G)}$ of $Sh_{K(G)}(G)$ over the ring of integers of $E(\Phi|\Omega^c)_v$, where $v \mid p$ (note that $K(G)_p \simeq \tau((O_{F_c} \otimes \mathbf{Z}_p)^\times, GL_2(O_F \otimes \mathbf{Z}_p))$ in this case; cf. [R, (1.4)]).

Moreover, the construction in A3.12 defines, for primes P of F above p, partial Frobenius morphisms $\varphi_P : S^\circ_{K(G)} = S_{K(G)} \otimes k(v) \longrightarrow S^\circ_{K(G)}$ satisfying $\varphi = \prod_{P|p} \varphi_P$.

(A6.12) The Eichler-Shimura relation for the unitary Shimura variety. Denote by S_c the following finite set of primes of F: $S_c = S \cup \{v \mid v$ ramified in F_c/F or $\mathrm{ord}_v(\eta^2) \neq 0$ or $\mathrm{ord}_v(j^2) \neq 0\}$ and assume that P_{S_c} is a finite prime of $\mathbf{Q}_{S_c} \subset \overline{\mathbf{Q}}$ such that

$$\mathrm{Fr}_{\mathbf{Q}_{S_c}/\mathbf{Q}}(P_{S_c}) \in \mathrm{Gal}(\mathbf{Q}_{S_c}/F_c^{gal}). \tag{A6.12.1}$$

Extend each $\sigma_x : F_c \hookrightarrow \overline{\mathbf{Q}}$ ($x \in X$) to an element $\widetilde{\sigma}_x \in \Gamma_{\mathbf{Q}}$. As in A5.9 (and 5.15) we obtain primes $P'_x = \widetilde{\sigma}_x^{-1} P_{S_c} \cap O_{F_c}$ and $P_x = \widetilde{\sigma}_x^{-1} P_{S_c} \cap O_F$ of F_c and F, respectively, which depend only on σ_x and lie above a rational prime p satisfying the conditions from A6.11. Moreover, (A6.12.1) implies that p splits completely in $F_c/\mathbf{Q}$:

$$pO_F = \prod_{x \in X} P_x, \qquad P_x O_{F_c} = P'_x P''_x, \qquad \mathbf{Q}_p = F_{P_x} = (F_c)_{P'_x} = E(\Phi|\Omega^c)_v.$$

The discussion from A6.8 applies to each $v = P_x$ and $v' = P'_x$. After identifying $H \otimes_F F_{P_x}$ with $GL(2)_{\mathbf{Q}_p} \times GL(1)_{\mathbf{Q}_p}$ as in (A6.8.1), the cocharacters $\mu_{P_x} : \mathbf{G}_{m,F_{P_x}} \longrightarrow H \otimes_F F_{P_x}$ from A4.3 can be chosen as in A6.8, with the case Ω (resp. the case Ω^c) occurring if $x \in \Omega$ (resp. if $x \in \Omega^c$).

Theorem A5.3 applies to the action of each partial Frobenius $\varphi_{P_x} : S^\circ_{K(G)} = S_{K(G)} \otimes \mathbf{F}_p \longrightarrow S^\circ_{K(G)}$ on $H^i_{\xi_G,K(G)} = (H^i_{\xi_G})^{K(G)} = H^i_{et}(S^\circ_{K(G)} \otimes \overline{\mathbf{F}}_p, \mathscr{L}_{\xi_G,\ell})$ and yields, thanks to (A6.8.2) and (A6.8.3),

$$\forall x \in \Omega \qquad Q_x(\varphi_{P_x} \otimes \mathrm{id})\big|_{H^i_{\xi_G,K(G)}} = 0, \qquad Q_x(Y) = Y^2 - (T_{P_x}/S_{P_x})Y + p/S_{P_x}. \tag{A6.12.2}$$

$$\forall x \in \Omega^c \qquad (\varphi_{P_x} \otimes \mathrm{id} - 1/S_{P'_x})\big|_{H^i_{\xi_G,K(G)}} = 0. \tag{A6.12.3}$$

The action of $\mathrm{Fr}_{\mathbf{Q}_{S_c}/\mathbf{Q}}(P_{S_c})$ on $H^i_{\xi_G,K(G)}$ is given by the action of $\varphi \otimes \mathrm{id} = \prod_{x \in X} (\varphi_{P_x} \otimes \mathrm{id})$.

We have $H^i_{\xi_G,K(G)} = \bigoplus V^i(\pi_G^\infty) \otimes (\pi_G^\infty)^{K(G)}$. If

$$V^i(\pi_G^\infty) \otimes (\pi_G^\infty)^{K(G)} \neq 0, \tag{A6.12.4}$$

then $\pi_G \circ \tau = \chi \otimes \pi$, where χ and π are as in (A6.10.4). For all $x \in X$, the action of $\varphi_{P_x} \otimes \mathrm{id}$ on $V^i(\pi_G^\infty) \otimes (\pi_G^\infty)^{K(G)} \subset H^i_{\xi_G,K(G)}$ defines, via the identification (A6.10.5), elements

$$\varphi_x^* \in \mathrm{End}(V^i(\pi^\infty) \otimes (\pi^\infty)^K), \qquad \varphi_x^*\varphi_y^* = \varphi_y^*\varphi_x^*$$

satisfying

$$\forall x \in \Omega \quad Q_x(\varphi_x^*) = 0, \qquad \forall x \in \Omega^c \quad \varphi_x^* = \chi^\infty(P_x')^{-1}. \tag{A6.12.5}$$

On the other hand, the action of $g = \mathrm{Fr}(P_{S_c})$ satisfies

$$\prod_{x \in X} (\varphi_{P_x} \otimes \mathrm{id})\big|_{V^i(\pi_G^\infty)\otimes(\pi_G^\infty)^{K(G)}} = g\big|_{V^i(\pi_G^\infty)\otimes(\pi_G^\infty)^{K(G)}} = g\big|_{V(\chi^\infty)} \otimes g\big|_{V^i(\pi^\infty)\otimes(\pi^\infty)^K},$$
$$g\big|_{V(\chi^\infty)} = \prod_{x\in\Omega^c} \chi^\infty(P_x')^{-1}$$

(the last equality holds thanks to the reciprocity map for the Shimura variety $Sh(F_c^\times)$ [R, p. 10]), which implies, thanks to (A6.12.5), that

$$\prod_{x\in\Omega} \varphi_x^* = \mathrm{Fr}(P_{S_c})\big|_{V^i(\pi^\infty)\otimes(\pi^\infty)^K}. \tag{A6.12.6}$$

One can show that, under the above assumptions, each φ_x^* ($x \in \Omega$) is induced by a geometric morphism $\varphi_x : S_K^\circ \otimes \overline{\mathbf{F}}_p \longrightarrow S_K^\circ \otimes \overline{\mathbf{F}}_p$ (using the discussion in [R, p. 48-49]), but we are not going to use this fact.

(A6.13) It is convenient to carry out the constructions in A6.12 for $K = K_S = K^S = K_pK^p$ (where $p \notin S_c$ is fixed) and then pass to the limit $K^p \longrightarrow \{1\}$ (in other words, replace K^p by an arbitrarily small open subgroup $K'^p \subset K^p$ and work with all groups of the form $K' = K_pK'^p \subset K$).

This yields, for all $x \in \Omega$, $G(\widehat{\mathbf{Q}}^{(p)})$-equivariant automorphisms φ_x^* of the space

$$(H^i_{\xi_G})^{K_p(F_c)\times K_p} = (H^0_{\xi'} \otimes H^i_\xi)^{\Delta(\widehat{F}^\times)\cdot(K_p(F_c)\times K_p)} = \bigoplus_{\chi,\pi} (V(\chi^\infty)\otimes(\chi^\infty)^{K_p(F_c)})\otimes V^i(\pi^\infty)\otimes(\pi^\infty)^{K_p}, \tag{A6.13.1}$$

where $\chi : \mathbf{A}_{F_c}^\times / F_c^\times \longrightarrow \mathbf{C}^\times$ is unramified at all primes above p, $\chi_\infty = (\xi')^{-1}$, π is an automorphic representation of $D_{\mathbf{A}}^\times$ such that π_∞ is cohomological in degree i with respect to ξ ($\Longrightarrow \omega_{\pi_\infty} = \omega_\xi^{-1}$), $\pi_p = \otimes_{x\in X}\pi_{P_x}$ is unramified and $\chi_{\mathbf{A}_F^\times} = \omega_\pi$.

As

$$\dim V(\chi^\infty) \otimes (\chi^\infty)^{K_p(F_c)} = \dim V(\chi^\infty) \otimes \chi^\infty = 1,$$

$G(\widehat{\mathbf{Q}}^{(p)})$-equivariance of φ_x^* together with irreducibility of π^∞ and Schur's Lemma imply (as in A5.5) that

$$\varphi_x^* \in \mathrm{End}_{G(\widehat{\mathbf{Q}}^{(p)})}((V(\chi^\infty)\otimes(\chi^\infty)^{K_p(F_c)}) \otimes V^i(\pi^\infty)\otimes(\pi^\infty)^{K_p}) =$$
$$= \mathrm{End}_{(D\otimes\widehat{\mathbf{Q}}^{(p)})^\times}(V^i(\pi^\infty)\otimes(\pi^\infty)^{K_p}) = \mathrm{End}\, V^i(\pi^\infty).$$

We have not shown that $\varphi_x^* \in \mathrm{End}\, V^i(\pi^\infty)$ is independent of χ. Such an independence follows from the geometric description of φ_x^* that was alluded to at the end of A6.12, but we do not need it for the applications in 5.18–5.22.

We deduce from (A6.12.5) and (A6.12.6) that

$$\forall x \in \Omega \quad Q_x(\varphi_x^*) = 0 \in \mathrm{End}\, V^i(\pi^\infty), \qquad \prod_{x\in\Omega}\varphi_x^* = \mathrm{Fr}(P_{S_c}) \in \mathrm{End}\, V^i(\pi^\infty). \tag{A6.13.2}$$

(A6.14) The Eichler-Shimura relation for the quaternionic Shimura variety. The relations (A6.13.2) give what we need for (5.16.1) in the case $t < r$, provided we can reverse the above arguments and find $F_c \hookrightarrow D$ and χ for which (A6.12.4) holds.

In order to do that, assume that we are given $K = K_S K^S$ as in 5.15 (with K_S sufficiently small) and a prime P_S of $\mathbf{Q}_S$ satisfying (5.15.2) with $\widetilde{E} = F^{gal}$ (which implies that the rational prime p below P_S splits completely in $F/\mathbf{Q}$). There are infinitely many totally imaginary quadratic extensions F_c/F such that all primes of F above p (resp. all primes at which D is ramified) are split (resp. inert or ramified) in F_c/F. Fix such an extension F_c and embeddings $F \hookrightarrow F_c \hookrightarrow D$ (they exist, by construction). There also exist elements $\eta, j \in D$ as in A6.5 satisfying $\eta^2, j^2 \in (O_F \otimes \mathbf{Z}_p)^\times$. By construction, p splits completely in $F_c/\mathbf{Q}$, which implies that the prime P_S extends to a prime P_{S_c} satisfying (A6.12.1). Fix ξ' as in A6.10.

Finally, for every automorphic representation $\pi = \pi_\infty \otimes \pi^\infty$ of $D^\times_{\mathbf{A}}$ such that

$$0 \neq V^i(\pi^\infty) \otimes (\pi^\infty)^K \subset H^i_{\xi,K} = (H^i_\xi)^K, \qquad (A6.14.1)$$

Proposition A6.15 below implies that there exists a character $\chi : \mathbf{A}^\times_{F_c}/F_c^\times \longrightarrow \mathbf{C}^\times$ which is unramified at all primes above p and which satisfies $\chi|_{\mathbf{A}^\times_F} = \omega_\pi$ and $\chi_\infty = (\xi')^{-1}$. The pair (χ, π) then contributes to the sum (A6.13.1) and the arguments in A6.13 give mutually commuting elements (for $x \in \Omega$)

$$\varphi^*_x \in \operatorname{End} V^i(\pi^\infty), \qquad \forall x \in \Omega \quad Q_x(\varphi^*_x) = 0, \qquad \prod_{x\in\Omega} \varphi^*_x = \operatorname{Fr}(P_S)\big|_{V^i(\pi^\infty)}. \qquad (A6.14.2)$$

As a result, (5.6.1) holds (for $\widetilde{E} = F^{gal}$). The relations (A6.14.2) imply the Eichler-Shimura relation (5.16.2) for the full Frobenius, namely, that

$$Q(\operatorname{Fr}(P_S))\big|_{H^i_{\xi,K}} = 0, \qquad (A6.14.3)$$

where $Q = \bigotimes_{x\in\Omega} Q_x$ is the Rankin-Selberg polynomial defined in (5.16.3).

(A6.15) Proposition. (1) *For every pair of characters $\alpha : \mathbf{A}^\times_F/F^\times \longrightarrow \mathbf{C}^\times$ and $\beta : (F_c \otimes \mathbf{R})^\times \longrightarrow \mathbf{C}^\times$ satisfying $\beta|_{(F\otimes\mathbf{R})^\times} = \alpha_\infty$ there exists a character $\chi : \mathbf{A}^\times_{F_c}/F_c^\times \longrightarrow \mathbf{C}^\times$ such that $\chi|_{\mathbf{A}^\times_F} = \alpha$ and $\chi_\infty = \beta$.*
(2) *For every finite set T of finite primes of F there exists χ as in (1) satisfying $\operatorname{ord}_w(\operatorname{cond}(\chi)) = \operatorname{ord}_w(\operatorname{cond}(\alpha))$ for all primes w of F_c above T. In particular, if α is unramified at $v \in T$, then χ is unramified at all $w \mid v$.*

Proof. (1) Denote by $C_k = \mathbf{A}^\times_k/k^\times$ the idele class group of a number field k and by $C^1_k = \operatorname{Ker}(\|\cdot\|_k : C_k \longrightarrow \mathbf{R}^\times_+)$ its compact subgroup of unit norm ideles. We have

$$C_{F_c} \supset C_F \cdot (F_c \otimes \mathbf{R})^\times = C_F \cdot U(1)^r, \qquad C^1_{F_c} \supset C^1_F \cdot U(1)^r, \qquad C_F \cap U(1)^r = C^1_F \cap U(1)^r = \{\pm 1\}^r.$$

The compactness of the groups involved implies that the product morphism $m : C^1_F \times U(1)^r \longrightarrow C^1_{F_c}$ ($m(x,y) = xy$) is strict: it induces a topological isomorphism between the compact groups $\operatorname{Coim}(m) = (C^1_F \times U(1)^r)/\operatorname{Ker}(m)$ with quotient topology and $\operatorname{Im}(m) = C^1_F \cdot U(1)^r$ with topology induced from $C^1_{F_c}$.

The compatibility between α and β implies that the restriction of $\alpha \otimes \beta : C^1_F \times U(1)^r \longrightarrow U(1)$ to $\operatorname{Ker}(m) = \{(a, -a) \mid a \in \{\pm 1\}^r\}$ is trivial. It follows that there exists a unique character $C^1_F \cdot U(1)^r \simeq \operatorname{Coim}(m) \longrightarrow U(1)$ whose restriction to the first (resp. to the second) factor is given by α (resp. by β). Such a character extends to a character $\psi : C^1_{F_c} \longrightarrow U(1)$.

Fix an infinite prime v_∞ of F and define continuous sections

$$s : \mathbf{R}^\times_+ \longrightarrow (F \otimes \mathbf{R})^\times_+ \longrightarrow \mathbf{A}^\times_F, \qquad s_c : \mathbf{R}^\times_+ \longrightarrow (F \otimes \mathbf{R})^\times_+ \longrightarrow \mathbf{A}^\times_{F_c}$$

of $\|\cdot\|_F$ and $\|\cdot\|_{F_c}$, respectively, by $s(t)_v = 1$ for all $v \neq v_\infty$ and $s(t)_{v_\infty} = t$, and $s_c(t) = s(t^{1/2})$. The formula

$$\chi(x) = \beta(s_c(\|x\|_{F_c}))\, \psi(x/s_c(\|x\|_{F_c})), \quad x \in \mathbf{A}^\times_{F_c}$$

then defines a character $\chi : C_{F_c} \longrightarrow \mathbf{C}^\times$ with the required properties.
(2) Let χ be as in (1). For each $v \in T$, the restriction χ_v of χ to $(F_c \otimes_F F_v)^\times$ satisfies $\chi_v|_{F_v^\times} = \alpha_v$. Recall the following elementary fact: given finite abelian groups $H \subset G \supset G_1$ and a character $\lambda : G \longrightarrow U(1)$ such

that $\lambda_{H\cap G_1} = 1$, then there exists a character $\lambda' : G \longrightarrow U(1)$ such that $\lambda'|_H = \lambda|_H$ and $\lambda'|_{G_1} = 1$. Applying this statement to $G = O^\times_{F_c\otimes_F F_v} \supset H = O^\times_{F_v}$ (rather, to their quotients by a suitable piece of the canonical filtration on G) and $\lambda = \chi_v$ we deduce that there exists a character of finite order $\gamma^{(v)}$ of $(F_c \otimes_F F_v)^\times/F_v^\times$ such that $\mathrm{ord}_w(\mathrm{cond}(\gamma^{(v)}\chi_v)) = \mathrm{ord}_w(\mathrm{cond}(\alpha))$, for all $w \mid v$ in F_c. The value $\gamma^{(v)}(x)$ depends only on $x^{1-c} = x/x^c \in {}^{1-c}(F_c \otimes_F F_v)^\times \subset (F_c \otimes_F F_v)^\times$, where c is the non-trivial element of $\mathrm{Gal}(F_c/F)$. The character $x^{1-c} \mapsto \gamma^{(v)}(x)$ of ${}^{1-c}(F_c \otimes_F F_v)^\times$ extends to a character of finite order $\delta^{(v)}$ of $(F_c \otimes_F F_v)^\times$, which means that $\gamma^{(v)} = {}^{1-c}\delta^{(v)}$. There exists a global character of finite order $\delta : C_{F_c} \longrightarrow \mathbf{C}^\times$ such that $\delta_v = \delta^{(v)}$ for all $v \in T$; the character $\chi' = ({}^{1-c}\delta)\chi$ then has the required properties.

(A6.16) The Eichler-Shimura relation in the case $D \simeq M_2(F)$. In this case $H = GL(2)_F$ and Sh_K is a Hilbert modular variety of dimension $t = r$ and reflex field $E = \mathbf{Q}$. It is defined by the simplest possible PEL data of type (C): $B = F$, $* = \mathrm{id}$, $V = F^2$, $\langle\ ,\ \rangle_F =$ the standard symplectic form.

The Baily-Borel compactification $j : Sh_K \hookrightarrow Sh_{K,BB} = Sh_K \cup \{\text{cusps}\}$ has as a boundary a reduced zero-dimensional scheme of cusps. For any ξ as in (5.5.1), the canonical $G(\widehat{\mathbf{Q}})$-equivariant map

$$\mathrm{can}_i : H^i(Sh^{an}_{K,BB}) = H^i(Sh^{an}_{K,BB}, j_{!*}\mathscr{L}_\xi) = H^i(Sh^{an}_{K,BB}, \tau_{\leq r-1}\mathbf{R}j_*\mathscr{L}_\xi) \longrightarrow H^i(Sh^{an}_K, \mathscr{L}_\xi) = H^i(Sh^{an}_K)$$

has the following property.

(A6.17) Proposition. *The map* can_i *is injective in all cases except when* $i = 2r$ *and* $k_v = 2$ *for all* $v \mid \infty$.

Proof. The exact sequence

$$\cdots \longrightarrow H^i(Sh^{an}_{K,BB}) \longrightarrow H^i(Sh^{an}_K) \longrightarrow H^0(\{\text{cusps}\}^{an}, (\tau_{\geq r}\mathbf{R}j_*\mathscr{L}_\xi)[i]) \longrightarrow H^{i+1}(Sh^{an}_{K,BB}) \longrightarrow \cdots$$

implies that can_i is an isomorphism for $i < r$ and injective for $i = r$. In the decomposition (A5.12.2) of $\varinjlim_{K} H^i(Sh^{an}_{K,BB})$, only cuspidal or one-dimensional automorphic representations $\pi = \pi_\infty \otimes \pi^\infty$ of $GL_2(\mathbf{A}_F)$ appear, with π_∞ cohomological in degree i for ξ. For cuspidal π we have $i = r$, when the injectivity of can_i has been proved. It remains to investigate the (non-)injectivity of can_i (for $i > r$) when restricted to the cohomology classes corresponding to one-dimensional π, i.e., to the universal cohomology classes in the case when $k_v = 2$ for all $v \mid \infty$. This is done, for example, in [Fr, Lemma III.5.6] (in the classical language) or in [Ha, Prop. 3.2.4].

(A6.18) In the exceptional case $i = 2r$ and $k_v = 2$ for all $v \mid \infty$ the corresponding étale sheaf $\mathscr{L}_{\xi,\ell}$ is a Tate twist of the constant sheaf $\overline{\mathbf{Q}}_\ell$ and the étale intersection cohomology group $H^{2r}_{et}(Sh_{K,BB} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_{!*}\mathscr{L}_{\xi,\ell})$ is dual to a Tate twist of $H^0_{et}(Sh_{K,BB} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_{!*}\overline{\mathbf{Q}}_\ell) = H^0_{et}(Sh_K \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \overline{\mathbf{Q}}_\ell)$.

(A6.19) If p is a rational prime that splits completely in $F/\mathbf{Q}$ and such that $K = K_pK^p$ with $K_p \simeq GL_2(O_F \otimes \mathbf{Z}_p)$, then the discussion in A5.11 implies that the Eichler-Shimura relation A6.4 holds for the action on étale cohomology of the open Hilbert modular variety $H^i_{et}(Sh_K \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell})$. It follows from Proposition A6.17 that these relations also hold for the action on $H^i_{et}(Sh_{K,BB} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_{!*}\mathscr{L}_{\xi,\ell})$, unless we are in the exceptional case. However, A6.18 implies that they also hold in the exceptional case.

(A6.20) In the case $D \simeq M_2(F)$, both Proposition A6.17 and A6.18 remain valid for the Hilbert modular variety attached to $(G^*, \mathscr{X}^*)$ and for any ξ^* from 6.2. As a result, the Eichler-Shimura relation used in the proof of Proposition 6.14 holds, thanks to the discussion in A6.19.

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