SYNTOMIC COHOMOLOGY AND p-ADIC REGULATORS

Jan Nekovář

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The aim of this work is to study the image of p-adic regulators for smooth and projective varieties over number fields and local fields. The main tool in this investigation is the syntomic cohomology of Fontaine-Messing. The present work treats only varieties with potentially good reduction at primes dividing p (as does the work of Nizioł[Ni]). An updated version of this article, which is under preparation, will treat also the case of bad reduction, generalizing the results of Langer [La 2].

Apart from the fundamental paper of Fontaine-Messing [FM], the foundational aspects of syntomic topology and cohomology are not very well documented in the literature (with the exception of [Ba], [Br]). An Appendix to the present work, still in preparation, is intended to give a summary of the key properties of syntomic cohomology. We hope that occasional references to the Appendix will not destroy the readability of the current version.

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Introduction

(0.0) Let X be a smooth projective scheme over a number field F. For an integer $i \geq 0$ one defines the L-function $L(H^i(X), s)$, associated to the cohomology of degree i of X, by the standard Euler product [1], absolutely convergent[1] (in particular, without zeroes or poles) in the region $\mathrm{Re}(s) > i/2 + 1$.

This L-function is expected to have a meromorphic continuation to $\mathbf{C}$, with the only possible pole at $i/2 + 1$ (if i is even), and a functional equation of the form

$$(L_\infty \cdot L)(H^i(X), s) = a \cdot b^s \cdot (L_\infty \cdot L)(H^i(X), i + 1 - s), \tag{0.0.0}$$

where $L_\infty(H^i(X), s)$ is a suitable product of Γ-functions.

The value $L(H^i(X), n)$ at an integer $n > (i + 1)/2$ (which is finite and non-zero[1], at least if we stay away from the possible pole at $n = i/2 + 1$) conjecturally depends only on the "motive" $M = H^i(X)(n)$ (a "pure motive" of weight $i - 2n < -1$ over F).

(0.1) According to Beilinson [Be 2], the structure morphism $X \longrightarrow \mathrm{Spec}(F)$ should underlie a morphism of "motivic sites" $X_{mot} \xrightarrow{\pi} \mathrm{Spec}(F)_{mot}$; mixed motives over F should be abelian sheaves on $\mathrm{Spec}(F)_{mot}$. The fundamental object responsible for the L-value $L(H^i(X), n)$ is believed to be not the motive $M = R^i\pi_*\mathbf{Q}(n)$ itself, but rather the motivic cohomology group $H^{i+1}(X_{mot}, \mathbf{Q}(n))$. The Leray spectral sequence for π_*

$$E_2^{a,b} = H^a(\mathrm{Spec}(F)_{mot}, R^b\pi_*\mathbf{Q}(n)) \Longrightarrow H^{a+b}(X_{mot}, \mathbf{Q}(n)) \tag{0.1.0}$$

should degenerate at E_2 to yield isomorphisms (for $n \neq (i + 1)/2$)

$$H^{i+1}(X_{mot}, \mathbf{Q}(n)) \xrightarrow{\sim} H^1(\mathrm{Spec}(F)_{mot}, R^i\pi_*\mathbf{Q}(n)) = H^1(\mathrm{Spec}(F)_{mot}, M) \tag{0.1.1}$$

(0.2) While this picture is purely conjectural and possibly too naive, it makes sense in various realizations of motives. Consider first the Hodge realization. Let A be a noetherian subring of $\mathbf{R}$ such that $A \otimes \mathbf{Q}$ is a

[1] Strictly speaking, Euler factors at primes of F in which X has bad reduction pose a problem. We ignore these difficulties.

field. Denote by $\mathrm{Spec}(F \otimes_{\mathbf{Q}} \mathbf{R})_{A-Hodge}$ the category of mixed A-Hodge structures over $F \otimes_{\mathbf{Q}} \mathbf{R}$ (*cf.* [Be 2], Sect. 7).

The Hodge realization of (0.1.0) is the spectral sequence

$$E_2^{a,b} = H^a(\mathrm{Spec}(F \otimes_{\mathbf{Q}} \mathbf{R})_{A-Hodge}, H^b((X \otimes_{\mathbf{Q}} \mathbf{R})(\mathbf{C}), A(n))) \Longrightarrow H^{i+1}_{\text{BD}}(X \otimes_{\mathbf{Q}} \mathbf{R}, A(n)), \tag{0.2.0}$$

which degenerates at E_2 and converges to the Beilinson-Deligne cohomology of $X \otimes_{\mathbf{Q}} \mathbf{R}$ (called "absolute Hodge cohomology" in [Be 2]). For $n \neq (i+1)/2$, the spectral sequence (0.2.0) boils down to an isomorphism

$$H^{i+1}_{\text{BD}}(X \otimes_{\mathbf{Q}} \mathbf{R}, A(n)) \xrightarrow{\sim} H^1(\mathrm{Spec}(F \otimes_{\mathbf{Q}} \mathbf{R})_{A-Hodge}, M_{A-Hodge}), \tag{0.2.1}$$

where

$$M_{A-Hodge} = H^i((X \otimes_{\mathbf{Q}} \mathbf{R})(\mathbf{C}), A(n)),$$

which is a pure A-Hodge structure of weight $i - 2n$ over $F \otimes_{\mathbf{Q}} \mathbf{R}$.

(0.3) Fix a prime number p and consider p-adic étale realizations. The Leray spectral sequence for $X_{et} \xrightarrow{\pi} \mathrm{Spec}(F)_{et}$ can be identified with the Hochschild-Serre spectral sequence

$$E_2^{a,b} = H^a(F, H^b(\overline{X}_{et}, \mathbf{Q}_p(n))) \Longrightarrow H^{a+b}(X_{et}, \mathbf{Q}_p(n)), \tag{0.3.0}$$

where $\overline{X} = X \otimes_F \overline{F}$ and we use continuous Galois and étale cohomology (in the sense of [Ja 1]). Denote by $F^{\cdot} H^*(X_{et}, \mathbf{Q}_p(n))$ the induced filtration on $H^*(X_{et}, \mathbf{Q}_p(n))$. The spectral sequence (0.3.0) degenerates at E_2 ([De 1]) and Weil's conjectures - proved by Deligne [De 3] - imply that $E_2^{0,b} = 0$ for $b \neq 2n$ (*cf.* [Ja 2], Lemma 3). For $n \neq (i+1)/2$ this gives edge homomorphisms

$$H^{i+1}(X_{et}, \mathbf{Q}_p(n)) = F^1 H^{i+1}(X_{et}, \mathbf{Q}_p(n)) \longrightarrow H^1(F, M_p), \tag{0.3.1}$$

where

$$M_p = H^i(\overline{X}_{et}, \mathbf{Q}_p(n)).$$

This is a continuous p-adic representation of the Galois group $G(\overline{F}/F)$, pure of weight $i - 2n$ at primes of good reduction of X. In fact, all of this is true for proper (not necessarily projective) smooth schemes over F (by [De 4] and [de J]).

(0.4) In this motivic setting, regulator maps should arise as realizations (for $n \neq (i+1)/2$).

The Hodge realization r_∞:

$$\begin{array}{ccc} H^{i+1}(X_{mot}, \mathbf{Q}(n)) & \xrightarrow{r_\infty} & H^{i+1}_{\text{BD}}(X \otimes_{\mathbf{Q}} \mathbf{R}, \mathbf{R}(n)) \\ \downarrow \wr & & \downarrow \wr \\ H^1(\mathrm{Spec}(F)_{mot}, M) & \xrightarrow{r_\infty} & H^1(\mathrm{Spec}(F \otimes_{\mathbf{Q}} \mathbf{R})_{\mathbf{R}-Hodge}, M_{\mathbf{R}-Hodge}) \end{array} \tag{0.4.0}$$

The p-adic étale realization r_p:

$$\begin{array}{ccc} H^{i+1}(X_{mot}, \mathbf{Q}(n)) & \xrightarrow{r_p} & H^{i+1}(X_{et}, \mathbf{Q}_p(n)) \\ \downarrow \wr & & \downarrow \\ H^1(\mathrm{Spec}(F)_{mot}, M) & \xrightarrow{r_p} & H^1(F, M_p) \end{array} \tag{0.4.1}$$

There is a certain assymetry between r_∞ and r_p; the Beilinson-Deligne cohomology is a local object at archimedean primes of F, while the Galois cohomology group $H^1(F, M_p)$ is a global invariant. The map r_p can be further localized at p, yielding a diagram

$$\begin{array}{ccccc} H^{i+1}(X_{mot},\mathbf{Q}(n)) & \xrightarrow{r_p} & H^{i+1}(X_{et},\mathbf{Q}_p(n)) & \longrightarrow & H^{i+1}((X\otimes_{\mathbf{Q}}\mathbf{Q}_p)_{et},\mathbf{Q}_p(n)) \\ \downarrow\wr & & \downarrow & & \downarrow(?) \\ H^1(\mathrm{Spec}(F)_{mot},M) & \xrightarrow{r_p} & H^1(F,M_p) & \xrightarrow{res_p} & H^1(F\otimes_{\mathbf{Q}}\mathbf{Q}_p,M_p) \end{array} \tag{0.4.2}$$

The third vertical arrow is defined only if a local version of $E_2^{0,i+1}$ (over $\mathbf{Q}_p$) vanishes. This is known to be true (for $n \neq (i+1)/2$) only in a special case when $X \otimes_{\mathbf{Q}} \mathbf{Q}_p$ has a smooth projective model over $\mathcal{O}_F \otimes \mathbf{Z}_p$ (see III.3.5 below).

According to a conjecture of Jannsen ([Ja 2], Conj. 1; see also Prop. II.1.7(1c) below), the restriction map res_p should be injective for $n < 0$, $n > i+1$. More generally, the composite map

$$H^1(\mathrm{Spec}(F)_{mot},M)\xrightarrow{r_p}H^1(F,M_p)\xrightarrow{res_p}H^1(F\otimes_{\mathbf{Q}}\mathbf{Q}_p,M_p) \tag{0.4.3}$$

is expected to be injective for all $n \neq (i+1)/2$. This explains why the localized map (0.4.3), not r_p itself, is usually called a p-adic regulator.

(0.5) Suppose that X admits a regular model $\mathfrak{X}$, proper and flat over the ring of integers $\mathcal{O}_F \subset F$. The motivic sites are expected to extend to

$$X_{mot} \longrightarrow \mathfrak{X}_{mot}, \qquad j : \mathrm{Spec}(F)_{mot} \longrightarrow \mathrm{Spec}(\mathcal{O}_F)_{mot},$$

giving a commutative diagram

$$\begin{array}{ccc} H^{i+1}(\mathfrak{X}_{mot},\mathbf{Q}(n)) & \longrightarrow & H^{i+1}(X_{mot},\mathbf{Q}(n)) \\ \downarrow & & \downarrow\wr \\ H^1(\mathrm{Spec}(\mathcal{O}_F)_{mot},j_*M) & \hookrightarrow & H^1(\mathrm{Spec}(F)_{mot},M) \end{array} \tag{0.5.0}$$

Recall the conjectural description of $L(H^i(X),n)$, up to a sign:

Conjecture. *Suppose that $n > (i+1)/2$ (and, if $n = i/2+1$, that $L(H^i(X),s)$ does not have a pole at $s = n$). Then*
(1) (Beilinson [Be 1, 2], Deligne [De 5]) *The regulator r_∞ induces an isomorphism*

$$r_\infty \otimes 1 : H^1(\mathrm{Spec}(\mathcal{O}_F)_{mot}, j_*M) \otimes_{\mathbf{Q}} \mathbf{R} \xrightarrow{\sim} H^1(\mathrm{Spec}(F\otimes_{\mathbf{Q}}\mathbf{R})_{\mathbf{R}-Hodge}, M_{\mathbf{R}-Hodge})$$

and $L(H^i(X),n)$ is equal to "$\det(r_\infty \otimes 1)$", up to a factor in $\mathbf{Q}^*$.
(2) (Bloch-Kato [BK]) *For each prime number p, the p-adic regulator r_p induces an isomorphism*

$$r_p \otimes 1 : H^1(\mathrm{Spec}(\mathcal{O}_F)_{mot}, j_*M) \otimes_{\mathbf{Q}} \mathbf{Q}_p \xrightarrow{\sim} H^1_f(F,M_p) \subseteq H^1(F,M_p),$$

where the subspace $H^1_f(F,M_p)$ is the generalized Selmer group defined in [BK]. *The p-adic valuation of the undetermined rational factor from* (1) *can be recovered from the isomorphism $r_p \otimes 1$.*

(0.6) Both parts of this conjecture were originally formulated in terms of a K-theoretic version of motivic cohomology. In this context, the map

$$H^{i+1}(\mathfrak{X}_{mot},\mathbf{Q}(n)) \longrightarrow H^{i+1}(X_{mot},\mathbf{Q}(n))$$

is replaced by

$$K_{2n-i-1}(\mathfrak{X})^{(n)}_{\mathbf{Q}} \longrightarrow K_{2n-i-1}(X)^{(n)}_{\mathbf{Q}},$$

where the superscript $^{(n)}$ refers to the subspace of $K_{\cdot}(-)\otimes\mathbf{Q}$ on which all Adams operations ψ^k ($k \geq 1$) act by k^n (*cf.* [So 2], [Tamme]).

The role of $H^1(\mathrm{Spec}(\mathcal{O}_F)_{mot}, j_*M)$ is played by

$$\mathrm{Im}\left[K_{2n-i-1}(\mathfrak{X})^{(n)}_{\mathbf{Q}} \longrightarrow K_{2n-i-1}(X)^{(n)}_{\mathbf{Q}}\right]$$

This group does not depend on the choice of $\mathfrak{X}$, provided at least one $\mathfrak{X}$ exists ([Be 1], 2.4.2). The maps r_∞ (resp. r_p) are given by the Chern character defined on higher K-theory, with values in the Beilinson-Deligne (resp. p-adic étale) cohomology

$$\begin{aligned} r_\infty : K_{2n-i-1}(X)^{(n)}_{\mathbf{Q}} &\longrightarrow H^{i+1}_{\mathrm{BD}}(X \otimes_{\mathbf{Q}} \mathbf{R}, \mathbf{R}(n)) \\ r_p : K_{2n-i-1}(X)^{(n)}_{\mathbf{Q}} &\longrightarrow H^{i+1}(X_{et}, \mathbf{Q}_p(n)) \end{aligned} \tag{0.6.0}$$

See [Gi], [Sc 1] for a general discussion of characteristic classes.

The conjecture of Beilinson (resp. of Bloch-Kato) then predicts that r_∞ (resp. r_p) induces an isomorphism

$$r_\infty : \left(\mathrm{Im}\left[K_{2n-i-1}(\mathfrak{X})^{(n)}_{\mathbf{Q}} \longrightarrow K_{2n-i-1}(X)^{(n)}_{\mathbf{Q}}\right]\right) \otimes_{\mathbf{Q}} \mathbf{R} \xrightarrow{\sim} H^{i+1}_{\mathrm{BD}}(X \otimes_{\mathbf{Q}} \mathbf{R}, \mathbf{R}(n)) \tag{0.6.1}$$

resp.

$$r_p : \left(\mathrm{Im}\left[K_{2n-i-1}(\mathfrak{X})^{(n)}_{\mathbf{Q}} \longrightarrow K_{2n-i-1}(X)^{(n)}_{\mathbf{Q}}\right]\right) \otimes_{\mathbf{Q}} \mathbf{Q}_p \xrightarrow{\sim} H^1_f(F, M_p) \tag{0.6.2}$$

(with a slight modification if $n = i/2 + 1$ and $L(H^i(X), s)$ has a pole at n).

(0.7) The situation at the central point $n = (i+1)/2$ of the conjectural functional equation (0.0.0) is slightly different. Let $CH^n(X)$ be the Chow group of algebraic cycles of codimension n on X, modulo rational equivalence. Let

$$\begin{aligned} CH^n(X)_0 &= \mathrm{Ker}\left[CH^n(X) \longrightarrow H^{2n}(\overline{X}_{et}, \mathbf{Q}_p(n))\right] = \\ &= \mathrm{Ker}\left[CH^n(X) \longrightarrow H^{2n}(X(\mathbf{C}), \mathbf{Q}(n))\right] \end{aligned}$$

be its subgroup represented by cycles which are homologically trivial (modulo torsion).

The p-adic regulator r_p is replaced by the p-adic Abel-Jacobi map (see [Ja 3, (9.4.1)])

$$r_p : CH^n(X)_0 \otimes \mathbf{Q} \longrightarrow F^1 H^{2n}(\overline{X}_{et}, \mathbf{Q}_p(n)) \longrightarrow H^1(F, M_p)$$

The conjecture of Bloch-Kato predicts that r_p induces an isomorphism

$$r_p \otimes 1 : CH^n(X)_0 \otimes \mathbf{Q}_p \xrightarrow{\sim} H^1_f(F, M_p) \tag{0.7.0}$$

For $n = i = 1$, this is equivalent to the finiteness of the p-primary part of the Tate-Šafarevič group of the abelian variety $A = \mathrm{Pic}^0(X/F)$ (over F). Indeed, in this case, $M_p = V_p(A)$, r_p is the usual descent map

$$A(F) \otimes \mathbf{Q} \hookrightarrow H^1(F, V_p(A))$$

and $H^1_f(F, V_p(A))$ is a $\mathbf{Q}_p$-version of the standard Selmer group.

(0.8) The conjectures (0.6.2) and (0.7.0) consist of three statements, arranged in an increasing order of difficulty:

$$\mathrm{Im}\,(r_p) \subseteq H^1_f(F, M_p) \tag{0.8.0}$$

$$\mathrm{Coker}\,(r_p \otimes 1) = 0 \tag{0.8.1}$$

$$\mathrm{Ker}(r_p \otimes 1) = 0 \tag{0.8.2}$$

Indeed, (0.8.0) is merely a consistency of the conjectures. (0.8.1) is equivalent to the finiteness of the p-primary part of a suitably generalized Tate-Šafarevič group (see [BK]); this has been verified in certain cases. By contrast, (0.8.2) seems to be beyond reach at present. The only exception is the case $n = i = 1$ in (0.7), when we are dealing with rational points on the abelian variety $A = \mathrm{Pic}^0(X/F)$.

(0.9) Let V be an arbitrary p-adic representation of $G(\overline{F}/F)$ (of finite dimension). The Selmer group $H^1_f(F, V) \subseteq H^1(F, V)$ is defined by local conditions, namely by a cartesian diagram

$$\begin{array}{ccc} H^1_f(F,V) & \hookrightarrow & H^1(F,V) \\ \downarrow & & \downarrow \\ \prod_v H^1_f(F_v,V) & \hookrightarrow & \prod_v H^1(F_v,V) \end{array} , \qquad (0.9.0)$$

in which

$$H^1_f(F_v, V) = \begin{cases} H^1_{ur}(F_v, V), & \text{if } v \not| p \\ \mathrm{Ker}\left[H^1(F_v, V) \longrightarrow H^1(F_v, V \otimes_{\mathbf{Q}_p} B_{cris})\right], & \text{if } v|p \end{cases}$$

($H^1(F_v, V)$ vanishes for an archimedean prime v).

(0.10) One of the aims of the present article is to prove a version of (0.8.0), under some mild restrictions on p. In view of (0.9.0), one is naturally led to corresponding local statements over various completions of F.

Changing perspective, we fix a (non-archimedean) prime v of F and consider a proper smooth scheme X_v over $\mathrm{Spec}(F_v)$. Let $\mathfrak{X}_v \longrightarrow \mathrm{Spec}(\mathcal{O}_v)$ be a proper model of X_v over the ring of integers of F_v.

(0.11) Assume first that $v \not| p$. Étale Chern classes then define a commutative diagram of p-adic regulators

$$\begin{array}{ccc} K_{2n-i-1}(\mathfrak{X}_v)_{\mathbf{Q}} & \longrightarrow & K_{2n-i-1}(X_v)^{(n)}_{\mathbf{Q}} \\ \downarrow & & \downarrow \\ H^{i+1}((\mathfrak{X}_v)_{et}, \mathbf{Q}_p(n)) & \longrightarrow & H^{i+1}((X_v)_{et}, \mathbf{Q}_p(n)) \end{array} \qquad (0.11.0)$$

resp. cycle classes (assuming $\mathfrak{X}_v$ regular)

$$\begin{array}{ccc} CH^n(\mathfrak{X}_v) \otimes \mathbf{Q} & \longrightarrow & CH^n(X_v) \otimes \mathbf{Q} \\ \downarrow & & \downarrow \\ H^{2n}((\mathfrak{X}_v)_{et}, \mathbf{Q}_p(n)) & \longrightarrow & H^{2n}((X_v)_{et}, \mathbf{Q}_p(n)) \end{array} \qquad (0.11.1)$$

The following statement is well known. It is a straightforward consequence of Deligne's fundamental result in Weil II ([De 4], Thm. 1) and the commutative diagrams (0.11.0-1).

Theorem A. *If $\mathfrak{X}_v$ is proper over $\mathrm{Spec}(\mathcal{O}_v)$ and $v \not| p$, then*
(1) *For $n > (i+1)/2$, $K_{2n-i-1}(\mathfrak{X}_v) \otimes \mathbf{Q}$ maps to $F^1H^{i+1}((X_v)_{et}, \mathbf{Q}_p(n))$ and*

$$\mathrm{Im}\left[r_p : K_{2n-i-1}(\mathfrak{X}_v) \otimes \mathbf{Q} \longrightarrow H^1(F_v, H^i((\overline{X}_v)_{et}, \mathbf{Q}_p(n)))\right] = 0.$$

(2) *For $n = (i+1)/2$ and $\mathfrak{X}_v$ regular,*

$$\mathrm{Im}\left[r_p : CH^n(\mathfrak{X}_v)_0 \otimes \mathbf{Q} \longrightarrow H^1(F_v, H^{2n-1}((\overline{X}_v)_{et}, \mathbf{Q}_p(n)))\right] = 0.$$

Here $CH^n(\mathfrak{X}_v)_0 = \mathrm{Ker}\left[CH^n(\mathfrak{X}_v) \longrightarrow H^{2n}((\mathfrak{X}_v \otimes_{\mathcal{O}_v} \overline{k(v)})_{et}, \mathbf{Q}_p(n))\right]$.

Remark. If $n = (i+1)/2$ and $H^{2n-1}(\overline{X}_{et}, \mathbf{Q}_p)$ satisfies the purity conjecture for the monodromy filtration at v (*e.g.* if X has a potentially good reduction at v), then $H^1(F_v, H^{2n-1}((\overline{X}_v)_{et}, \mathbf{Q}_p(n))) = 0$.

(0.12) For $v|p$, diagrams (0.11.0-1) no longer exist. Assume that $\mathfrak{X}_v$ is proper and smooth over $\mathrm{Spec}(\mathcal{O}_v)$. Instead of p-adic étale cohomology of $\mathfrak{X}_v$, consider the syntomic cohomology of Fontaine-Messing [FM]. For $p > 2$ and $n \geq 0$, Fontaine and Messing defined maps [1]

$$H^*((\mathfrak{X}_v)_{syn}, s_{\mathbf{Q}_p}(n)) \longrightarrow H^*((X_v)_{et}, \mathbf{Q}_p(n))$$

Chern classes with values in syntomic cohomology give rise to commutative diagrams of p-adic regulators

$$\begin{array}{ccc} K_{2n-i-1}(\mathfrak{X}_v)^{(n)}_{\mathbf{Q}} & \longrightarrow & K_{2n-i-1}(X_v)^{(n)}_{\mathbf{Q}} \\ \downarrow & & \downarrow \\ H^{i+1}((\mathfrak{X}_v)_{syn}, s_{\mathbf{Q}_p}(n)) & \longrightarrow & H^{i+1}((X_v)_{et}, \mathbf{Q}_p(n)) \end{array} \tag{0.12.0}$$

resp. cycle classes

$$\begin{array}{ccc} CH^n(\mathfrak{X}_v) & \longrightarrow & CH^n(X_v) \\ \downarrow & & \downarrow \\ H^{2n}((\mathfrak{X}_v)_{syn}, s_{\mathbf{Q}_p}(n)) & \longrightarrow & H^{2n}((X_v)_{et}, \mathbf{Q}_p(n)) \end{array} \tag{0.12.1}$$

Our main local result [2] (proved in a more general context, when $\mathcal{O}_v$ is replaced by an arbitrary complete discrete valuation ring of mixed characteristic $(0, p)$, with a perfect residue field) states that there is a canonical isomorphism - compatible with the Hochschild-Serre spectral sequence (0.3.0) for X_v -

$$\mathrm{Ker}\left[H^{i+1}((\mathfrak{X}_v)_{syn}, s_{\mathbf{Q}_p}(n)) \longrightarrow H^{i+1}((\overline{X}_v)_{et}, \mathbf{Q}_p(n))\right] \xrightarrow{\sim} H^1_f(F_v, H^i((\overline{X}_v)_{et}, \mathbf{Q}_p(n))) \tag{0.12.2}$$

and that

$$\mathrm{Ker}\left[H^{i+1}((\mathfrak{X}_v)_{syn}, s_{\mathbf{Q}_p}(n)) \longrightarrow H^{i+1}((X_v)_{et}, \mathbf{Q}_p(n))\right] = 0. \tag{0.12.3}$$

This implies that syntomic cohomology of a proper and smooth scheme over $\mathrm{Spec}(\mathcal{O}_v)$ displays all signs of behaviour of a "p-adic Beilinson-Deligne cohomology", which should be the abutment of a spectral sequence

$$E_2^{a,b} = H^a_f(F_v, H^b((\overline{X}_v)_{et}, \mathbf{Q}_p(n))) \Longrightarrow H^{a+b}_{\text{Б}D}(X_v, \mathbf{Q}_p(n)) \tag{0.12.4}$$

As a corollary, we obtain

Theorem B. *If $\mathfrak{X}_v$ is proper and smooth over* $\mathrm{Spec}(\mathcal{O}_v)$ *and $v|p$, then*
(1) *For $n > (i+1)/2$,*

$$\mathrm{Im}\left[r_p : (K_{2n-i-1}(\mathfrak{X}_v) \otimes \mathbf{Q})_0 \longrightarrow H^1(F_v, H^i((\overline{X}_v)_{et}, \mathbf{Q}_p(n)))\right] \subseteq H^1_f(F_v, H^i((\overline{X}_v)_{et}, \mathbf{Q}_p(n))).$$

(2) *For $n = (i+1)/2$,*

$$\mathrm{Im}\left[r_p : CH^n(X_v)_0 \otimes \mathbf{Q} \longrightarrow H^1(F_v, H^{2n-1}((\overline{X}_v)_{et}, \mathbf{Q}_p(n)))\right] \subseteq H^1_f(F_v, H^{2n-1}((\overline{X}_v)_{et}, \mathbf{Q}_p(n))).$$

Here $(K_{2n-i-1}(\mathfrak{X}_v) \otimes \mathbf{Q})_0 = \mathrm{Ker}\left[K_{2n-i-1}(\mathfrak{X}_v) \otimes \mathbf{Q} \longrightarrow H^{i+1}((\overline{X}_v)_{et}, \mathbf{Q}_p(n))\right]$.

In fact, Theorem B holds under a weaker assumption, namely that X_v has a *potentially* good reduction. Moreover, it follows from crystalline Weil conjectures for $H^{i+1}_{cris}(\mathfrak{X}_v)$ ([KM], [CLeS]) that $(K_{2n-i-1}(\mathfrak{X}_v) \otimes \mathbf{Q})_0 = K_{2n-i-1}(\mathfrak{X}_v) \otimes \mathbf{Q}$. The statement (2) of Theorem B is crucial for the construction of p-adic height pairings – see [Ne 2], [Ne 3].

(0.13) For $\mathfrak{X}_v$ smooth and projective over $\mathrm{Spec}(\mathcal{O}_v)$, P. Schneider ([Sc 2], Sect. 7) defined (for $n \geq 0$) a cohomology theory $H^*((\mathfrak{X}_v)_{et}, S_{\mathbf{Q}_p}(n))$ mapping to $H^*(X_{et}, \mathbf{Q}_p(n))$. He conjectured that, for $i \neq 2n, 2n-1$, the Hochschild-Serre spectral sequence induces an isomorphism

$$H^{i+1}((\mathfrak{X}_v)_{et}, S_{\mathbf{Q}_p}(n)) \xrightarrow{\sim} H^1_f(F_v, H^i((\overline{X}_v)_{et}, \mathbf{Q}_p(n))) \tag{0.13.0}$$

It is explained in [Sc 2] that (0.12.2) combined with a result of Kurihara [Ku] implies the following result.

[1] The case of $p = 2$ is treated the work of Tsuji [Ts].

[2] For technical reasons, we use the syntomic-étale site of the p-adic completion of $\mathfrak{X}_v$. The cohomology of $s_{\mathbf{Q}_p}(n)$ is the same for both sites.

Theorem C. *For $i \neq 2n-1$ and $0 \leq n < p-1$ Schneider's conjecture (0.13.0) holds.*

(0.14) Our methods also shed some light on the relation between the image of the cycle class map

$$cl_{X_v} : CH^n(X_v) \otimes \mathbf{Q} \longrightarrow H^{2n}((X_v)_{et}, \mathbf{Q}_p(n))$$

and the filtration $F^{\cdot}H^{2n}((X_v)_{et}, \mathbf{Q}_p(n))$ on the target (again the case of $v \not| p$ is well known).

Theorem D. *Let X_v be proper and smooth over* $\operatorname{Spec}(F_v)$.
(1) *If $v \not| p$ and either $r = 1$, or $r = \dim(X)$, or X_v has a potentially good reduction, then* $\operatorname{Im}(cl_{X_v}) \cap F^1 H^{2n}((X_v)_{et}, \mathbf{Q}_p(n)) = 0$.
(2) *Suppose that $v \not| p$ and that X_v satisfies the condition*
($*_n$) *There is a finite extension L/F_v such that $X_L := X_v \otimes_{F_v} L$ admits a regular model $\mathfrak{X}_L$, proper over* $\operatorname{Spec}(\mathcal{O}_L)$, *such that the restriction map $(CH^n(\mathfrak{X}_L) \otimes \mathbf{Q})_0 \longrightarrow (CH^n(X_L) \otimes \mathbf{Q})_0$ is surjective.*
Then $\operatorname{Im}(cl_{X_v}) \cap F^1 H^{2n}((X_v)_{et}, \mathbf{Q}_p(n)) = 0$.
(3) *If $v|p$ and if X_v has a potentially good reduction, then* $\operatorname{Im}(cl_{X_v}) \cap F^2 H^{2n}((X_v)_{et}, \mathbf{Q}_p(n)) = 0$.

The condition ($*_n$) is known to be satisfied - provided $\mathfrak{X}_L$ exists - if X_v is equidimensional and $n = \dim(X_v)$.

(0.15) Returning to the global setting, consider the cycle class map

$$cl_X : CH^n(X) \otimes \mathbf{Q} \longrightarrow H^{2n}(X_{et}, \mathbf{Q}_p(n))$$

for a smooth and projective scheme X over a number field F. The conjectural injectivity of the Abel-Jacobi map r_p from (0.7) predicts that

$$\operatorname{Im}(cl_X) \cap F^2 H^{2n}(X_{et}, \mathbf{Q}_p(n)) = 0$$

Our local results imply

Theorem E. *If X is proper and smooth over a number field F and has a potentially good reduction at all primes dividing p, then $\operatorname{Im}(cl_X) \cap F^2 H^{2n}(X_{et}, \mathbf{Q}_p(n))$ is a subquotient (as a $\mathbf{Q}_p$-vector space) of*

$$\operatorname{Ker}\left[\alpha_{S,\Sigma} : H^2(G_{F,S}, H^{2n-2}(\overline{X}_{et}, \mathbf{Q}_p(n))) \longrightarrow \bigoplus_{v \in \Sigma} H^2(F_v, H^{2n-2}(\overline{X}_{et}, \mathbf{Q}_p(n)))\right].$$

Here $G_{F,S}$ is the Galois group of the maximal extension of F, unramified outside of $S = \{v|p\} \cup \{$primes of bad reduction of $X\}$, and $\Sigma = \{v|p\} \cup \{v \in S \mid v \not| p, (*_n)$ holds for $X \otimes_F F_v\}$.

Remarks. (i) It is believed that $\Sigma = S$.
(ii) Conjecturally, $\operatorname{Ker}(\alpha_{S,S}) = 0$. A function field analogue was proved in ([Ja 2], Thm. 4) and ([Ra], Thm. 4.1).

(0.16) There are only a handful of cases in which the vanishing of $\operatorname{Ker}(\alpha_{S,S})$ has been established. Combining Theorem E with results of [Fl], [La 1], [Wi], we obtain

Theorem F. *Let E be a modular elliptic curve over $\mathbf{Q}$, without complex multiplication (over $\overline{\mathbf{Q}}$), $p > 2$ a prime number at which E has good reduction. Then, for every $d \geq 1$, the cycle class map*

$$cl_{E^d} : CH^d(E^d) \longrightarrow H^{2d}((E^d)_{et}, \mathbf{Q}_p(d))$$

satisfies $\operatorname{Im}(cl_{E^d}) \cap F^2 H^{2d}((E^d)_{et}, \mathbf{Q}_p(d)) = 0$.

Under additional hypotheses on p, an analogue of Theorem D can be proved for cohomology with $\mathbf{Z}_p$-coefficients. Combined with vanishing results for a $\mathbf{Z}_p$-version of $\alpha_{S,S}$ ([Fl]), this can be applied to the torsion in the Chow groups:

Theorem G. *Let E be a modular elliptic curve over $\mathbf{Q}$, without complex multiplication (over $\overline{\mathbf{Q}}$). Then there is an explicit integer $c_E \geq 1$ such that, for every $d \geq 1$ and every prime p not dividing $(2d)! \cdot c_E$, we have*

$$\left(\mathrm{Im}\left[CH^d(E^d) \longrightarrow H^{2d}((E^d)_{et}, \mathbf{Z}_p(d))\right]\right)_{tors} = 0.$$

(0.17) Theorem B was independently proved by Nizioł[Ni] for $\mathfrak{X}_v$ smooth and projective over $\mathrm{Spec}(\mathcal{O}_v)$. A special case of (0.12.2) is treated by Langer-Saito ([LaSa], Thm. 6.5). Our approach is modelled on a paper of M. Gros [Gros]. However, many arguments in [Gros] are rather sketchy. The Appendix to the present paper will treat the compatibility of syntomic and étale Chern classes in (0.12.0-1) (this is one of the points where the treatment in [Gros] does not seem to be completely satisfactory).

I. Topological background

This chapter sums up technical material that will be needed later.

1. Abstract nonsense

(1.1) Notation. For a category (resp. a site) C, denote by $C^{\widehat{}}$ (resp. $C^{\sim}$) the category of presheaves (resp. sheaves) on C (we suppress the dependence on a universe). For a topos E, write $\mathrm{Ab}(E)$ for the category of abelian groups in E. If A is a ring in E, $\mathrm{Mod}(E, A)$ denotes the category of A-modules in E. Both $\mathrm{Ab}(E)$ and $\mathrm{Mod}(E, A)$ are abelian categories with enough injectives ([SGA 4], Exp. II, 6.9). If $E = (Sets)$, then we often write $(A - Mod)$ instead of $\mathrm{Mod}((Sets), A)$. Write $\Gamma_E = \Gamma(E, -) : E \longrightarrow (Sets)$ for the functor of global sections on E. Cohomology groups in E are denoted by $H^q(E, -)$ (resp. $H^q(E; X, -)$ for an object X of E). Occasionally, we shall abuse the notation and write $H^q(C, -)$ instead of $H^q(C^{\sim}, -)$ (and similarly for $H^q(C; X, -)$). See Appendix and ([SGA 4], Exp. I–VI) for further topological terminology.

(1.2) Localization. Let $\mathcal{A}$ be an abelian category. Denote by $Q(\mathcal{A}) = \mathcal{A} \otimes \mathbf{Q}$ its localization with respect to all morphisms $n \cdot 1_A : A \longrightarrow A$ (for $A \in \mathrm{Ob}(\mathcal{A})$, $n \geq 1$). It has the same class of objects as $\mathcal{A}$ and the identity induces a functor $Q : \mathcal{A} \longrightarrow Q(\mathcal{A})$. We give a list of standard properties of $Q(\mathcal{A})$.

(1) $Q(\mathcal{A})$ is an abelian category.
(2) $Q : \mathcal{A} \longrightarrow Q(\mathcal{A})$ is an additive functor, exact and essentially surjective.
(3) For every $A, B \in \mathrm{Ob}(\mathcal{A})$,

$$\mathrm{Hom}_{Q(\mathcal{A})}(Q(A), Q(B)) = \mathrm{Hom}_{\mathcal{A}}(A, B) \otimes \mathbf{Q}$$

(4) If $\mathcal{B}$ is an additive category and $u : \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor such that $u(n \cdot 1_A) = n \cdot 1_{u(A)} : u(A) \longrightarrow u(A)$ is invertible for all $A \in \mathrm{Ob}(\mathcal{A}), n \geq 1$, then u factors uniquely as

$$u : \mathcal{A} \xrightarrow{Q} Q(\mathcal{A}) \xrightarrow{v} \mathcal{B},$$

where v is an additive functor. In particular, for every commutative ring R the functor $- \otimes \mathbf{Q} : (R - Mod) \longrightarrow ((R \otimes \mathbf{Q}) - Mod)$ factors as

$$(R - Mod) \xrightarrow{Q} Q(R - Mod) \xrightarrow{\xi} ((R \otimes \mathbf{Q}) - Mod),$$

where ξ is an *exact* functor.
(5) For $A \in \mathrm{Ob}(\mathcal{A})$, $Q(A) = 0$ iff $n \cdot \mathrm{Hom}_{\mathcal{A}}(A, A) = 0$ for some $n \geq 1$.
(6) An additive functor $u : \mathcal{A} \longrightarrow \mathcal{B}$ between abelian categories $\mathcal{A}, \mathcal{B}$ induces an additive functor $Q(u) : Q(\mathcal{A}) \longrightarrow Q(\mathcal{B})$ such that the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{u} & \mathcal{B} \\ \downarrow{\scriptstyle Q_{\mathcal{A}}} & & \downarrow{\scriptstyle Q_{\mathcal{B}}} \\ Q(\mathcal{A}) & \xrightarrow{Q(u)} & Q(\mathcal{B}) \end{array}$$

is (strictly) commutative.
(7) In the situation of (6), u is exact (resp. left exact, resp. right exact) $\Longrightarrow$ $Q(u)$ is exact (resp. left exact, resp. right exact).
(8) If $\mathcal{A}$ has enough injectives (*e.g.* if $\mathcal{A} = \mathrm{Mod}(E, A)$), then $Q(\mathcal{A})$ has enough injectives and

$$\{X \in \mathrm{Ob}(Q(\mathcal{A})) \mid X \text{ injective}\} = \{Q_{\mathcal{A}}(I) \mid I \in \mathrm{Ob}(\mathcal{A}) \text{ injective}\}$$

(9) If $\mathcal{A}$ has enough injectives and if $u : \mathcal{A} \longrightarrow \mathcal{B}$ (as in (6)) preserves injectives, then $Q(u) : Q(\mathcal{A}) \longrightarrow Q(\mathcal{B})$ preserves injectives.
(10) If $\mathcal{A}$ has enough injectives and $u : \mathcal{A} \longrightarrow \mathcal{B}$ (as in (6)) is left exact, then $Q(u) : Q(\mathcal{A}) \longrightarrow Q(\mathcal{B})$ is left exact, $R^q u$, $R^q Q(u)$ are defined for all $q \geq 0$ and the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{R^q u} & \mathcal{B} \\ \downarrow{\scriptstyle Q_{\mathcal{A}}} & & \downarrow{\scriptstyle Q_{\mathcal{B}}} \\ Q(\mathcal{A}) & \xrightarrow{R^q(Q(u))} & Q(\mathcal{B}) \end{array}$$

is commutative (up to a canonical isomorphism).
(11) Denote by $C(\mathcal{A})$ (resp. $K(\mathcal{A})$, resp. $D(\mathcal{A})$) the category of complexes (resp. the category of complexes up to homotopy, resp. the derived category) of $\mathcal{A}$. The functor $Q_{\mathcal{A}} : \mathcal{A} \longrightarrow Q(\mathcal{A})$ extends to triangulated functors $Q : K(\mathcal{A}) \longrightarrow K(Q(\mathcal{A}))$ and $Q_D : D(\mathcal{A}) \longrightarrow D(Q(\mathcal{A}))$.
If X (resp. Y) is a complex bounded below (resp. bounded above), then the canonical maps

$$\mathrm{Hom}_{C(\mathcal{A})}(X, Y) \otimes \mathbf{Q} \longrightarrow \mathrm{Hom}_{C(Q(\mathcal{A}))}(Q(X), Q(Y))$$
$$\mathrm{Hom}_{K(\mathcal{A})}(X, Y) \otimes \mathbf{Q} \longrightarrow \mathrm{Hom}_{K(Q(\mathcal{A}))}(Q(X), Q(Y))$$

are both *isomorphisms*.
(12) If $\mathcal{A}$ has enough injectives, then the canonical map

$$\mathrm{Hom}_{D(\mathcal{A})}(X, Y) \otimes \mathbf{Q} \longrightarrow \mathrm{Hom}_{D(Q(\mathcal{A}))}(Q_D(X), Q_D(Y))$$

is an *isomorphism* for every $X \in \mathrm{Ob}(D^-(\mathcal{A}))$, $Y \in \mathrm{Ob}(D^+(\mathcal{A}))$.
(13) If $\mathcal{A}$ has enough injectives and $u : \mathcal{A} \longrightarrow \mathcal{B}$ is a left exact functor to an abelian category $\mathcal{B}$, then there is a canonical isomorphism

$$Q_D(\mathbf{R}u(X)) \xrightarrow{\sim} \mathbf{R}(Q(u))(Q_D(X))$$

for every $X \in \mathrm{Ob}(D^+(\mathcal{A}))$.

(1.3) Projective systems. Let C be a category. Denote by $C^{\mathbf{N}}$ the category of projective systems

$$[A_n]_{n\in\mathbf{N}} = [A_0 \longleftarrow A_1 \longleftarrow A_2 \longleftarrow \cdots]$$

(indexed by $\mathbf{N}$) in C.

If E is a topos, then $E^{\mathbf{N}}$ is a topos as well. Indeed, viewing the ordered set $\mathbf{N}$ as a category in the usual way, consider the functor $\Phi : \mathbf{N} \longrightarrow E$ sending each element of $\mathbf{N}$ to the final object e_E of E. The pull-back of the fibred topos of arrows $Fl(E) \longrightarrow E$ ([SGA 4], Exp. VI, 7.3.1) by Φ is a fibred topos $\Phi^*(Fl(E)) \longrightarrow \mathbf{N}$ over $\mathbf{N}$. The associated total topos $\mathrm{Top}(\Phi^*(Fl(E))$ (*cf.* [Il 1], VI.5.2) is canonically equivalent to $E^{\mathbf{N}}$. The functor of global sections of $E^{\mathbf{N}}$ factors as

$$\Gamma_{(E^{\mathbf{N}})} : E^{\mathbf{N}} \xrightarrow{\Gamma_E^{\mathbf{N}}} (Sets)^{\mathbf{N}} \xrightarrow{\Gamma_{\mathbf{N}}} (Sets),$$

where $\Gamma_E^{\mathbf{N}}([A_n]) = [\Gamma_E(A_n)]$ and $\Gamma_{\mathbf{N}} = \varprojlim_{\mathbf{N}}$ (*cf.* [SGA 4], Exp. VI, 7.4.15; [Il 1], VI.5.8.1(ii)).

(1.4) Continuous cohomology. For a projective system $A = [A_n] \in \mathrm{Ob}(\mathrm{Ab}(E^{\mathbf{N}}))$ we shall write, abusively, $H^q(E, [A_n])$ instead of $H^q(E^{\mathbf{N}}, [A_n])$. The spectral sequence of ([SGA 4], Exp. VI, 7.4.15; Appendix 2.3.2) for the composition of functors

$$\Gamma_{(E^{\mathbf{N}})} = \Gamma_{\mathbf{N}} \circ \Gamma_E^{\mathbf{N}} : \mathrm{Ab}(E^{\mathbf{N}}) \longrightarrow (Ab)^{\mathbf{N}} \longrightarrow (Ab)$$

becomes

$$0 \longrightarrow (R^1 \varprojlim_{\mathbf{N}})\, H^{q-1}(E, A_n) \longrightarrow H^q(E, [A_n]) \longrightarrow \varprojlim_{\mathbf{N}}\, H^q(E, A_n) \longrightarrow 0, \qquad (1.4.1)$$

as $R^q\Gamma_{\mathbf{N}} = R^q \varprojlim_{\mathbf{N}} = 0$ for $q > 1$ (in $(Ab)^{\mathbf{N}}$). We denote by $H^q_{naive}(E, [A_n])$ the third group in the short exact sequence (1.4.1).

(1.5) A morphism of topoi $u = (u^*, u_*) : E \longrightarrow E'$ induces a morphism of topoi $u^{\mathbf{N}} : E^{\mathbf{N}} \longrightarrow E'^{\mathbf{N}}$, given termwise by u. On abelian group objects, the derived functors

$$R^q u_*^{\mathbf{N}} : \mathrm{Ab}(E^{\mathbf{N}}) = \mathrm{Ab}(E)^{\mathbf{N}} \longrightarrow \mathrm{Ab}(E'^{\mathbf{N}}) = \mathrm{Ab}(E')^{\mathbf{N}}$$

can be computed termwise as well:

$$\left(R^q u_*^{\mathbf{N}}\right)([A_n]) = [R^q u_*(A_n)] \tag{1.5.1}$$

The Leray spectral sequence for $u^{\mathbf{N}}$ ([SGA 4], Exp. V, 5.3) is given by

$$E_2^{p,q} = H^p(E', [R^q u_*(A_n)]_{n\in\mathbf{N}}) \Longrightarrow H^{p+q}(E, [A_n]_{n\in\mathbf{N}}) \tag{1.5.2}$$

Denote by $F^i H^q(E, [A_n])$ the induced filtration on its abutment. The kernel of the edge homomorphism is equal to

$$\mathrm{Ker}\left[H^q(E, [A_n]) \longrightarrow E_2^{0,q}\right] = F^1 H^q(E, [A_n])$$

and (1.5.2) induces a homomorphism

$$\delta : F^1 H^q(E, [A_n]) \longrightarrow E_\infty^{1,q-1} \hookrightarrow E_2^{1,q-1} \tag{1.5.3}$$

(1.6) Mittag-Leffler conditions. Recall that a projective system $[A_n] \in \mathrm{Ob}(\mathrm{Ab}(E)^{\mathbf{N}})$ is a ML-system if, for each $n \in \mathbf{N}$, the sequence of subobjects $[\mathrm{Im}(A_{m+n} \longrightarrow A_n)]_{m\in\mathbf{N}}$ of A_n becomes stationary for $m > m(n)$. The projective system is ML-zero, if this stationary value is equal to 0 (= the final object of $\mathrm{Ab}(E)$) for all $n \in \mathbf{N}$. The projective system is an AR-system (= Artin-Rees), if the value of $m(n)$ can be taken independent of n. If $[A_n]$ is ML-zero, then ([Ja 1], Lemma 1.11)

$$H^q(E, [A_n]) = 0 \tag{1.6.1}$$

A ML-isomorphism is a homomorphism $f : [A_n] \longrightarrow [B_n]$ such that both $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are ML-zero. By (1.6.1), such an f induces an isomorphism

$$H^q(E, [A_n]) \xrightarrow{\sim} H^q(E, [B_n]) \tag{1.6.2}$$

for all $q \geq 0$. A projective system $[A_n]$ of abelian groups is called ML-p-adic if it is ML-isomorphic to the system $[A \otimes \mathbf{Z}/p^n\mathbf{Z}]$ for a suitable abelian group A.

(1.7) Group cohomology. Let G be a pro-finite group and let B_G be its classifying topos ([SGA 4], Exp. IV, 2.7). Objects of B_G are sets with a discrete left action of G (the stabilizer of each element is an open subgroup of G). Abelian groups in B_G are discrete G-modules. For a projective system of discrete G-modules $[A_n]$, we write $H^q(G, [A_n])$ for $H^q(B_G, [A_n])$.

The group G acts continuously on the projective limit $A = \varprojlim_{\mathbf{N}} A_n$ (equipped with the topology of a projective limit of discrete sets) and one can consider continuous cohomology groups $H^q_{cont}(G, A)$ in the sense of Tate [Ta], computed using continuous cochains. The basic comparison result, due to Jannsen ([Ja 1], Thm. 2.2), states that for a *Mittag-Leffler* system $[A_n]$ there is a canonical (and functorial) isomorphism

$$H^q_{cont}(G, \varprojlim_{\mathbf{N}} A_n) \xrightarrow{\sim} H^q(G, [A_n]) \tag{1.7.1}$$

(1.8) Lemma. *Let G be a pro-finite group, p a prime number such that $d := cd_p(G)$ (cf.* [Se 1], I.3.1*) is finite. If $[A_n]$ is a projective system of p-power torsion discrete G-modules, then*
(1) $H^q(G, [A_n]) = 0$, *if $q > d + 1$.*
(2) $H^{d+1}(G, [A_n]) = 0$, *if $[A_n]$ is a ML-system.*

Proof. (1) Follows from (1.4.1). For (2), we have to show that $(R^1 \varprojlim_{\mathbf{N}})\, H^d(G, [A_n]) = 0$ (again by (1.4.1)). In view of (1.6.1), we can assume that $[A_n]$ is a surjective system. In this case $\left[H^d(G, A_n)\right]_n$ is a surjective system as well, hence its $(R^1 \varprojlim_{\mathbf{N}})$ vanishes.

(1.9) Proposition. *Let G be a pro-finite group, T a topological G-module. Define a topology on $T \otimes \mathbf{Q}$ as follows: $T \otimes \mathbf{Q} = \varinjlim_m \mathrm{Im}(\alpha_m)$, where $\alpha_m : T \longrightarrow T \otimes \mathbf{Q}$ (for $m \geq 1$) is given by $\alpha_m(x) = x \otimes (1/m)$. Put the quotient topology on $\mathrm{Im}(\alpha_m)$ and the inductive limit topology on $T \otimes \mathbf{Q}$. If T_{tors} has a finite exponent, then the canonical map*

$$H^q_{cont}(G, T) \otimes \mathbf{Q} \longrightarrow H^q_{cont}(G, T \otimes \mathbf{Q})$$

is an isomorphism (for each $q \geq 0$).

Proof. As in ([Ja 1], Thm. 5.15.c).

(1.10) Remark. If p is a prime number and $[A_n]$ is a ML-p-adic projective system of discrete abelian groups, then the projective limit topology on $A = \varprojlim_{\mathbf{N}} A_n$ is the p-adic one (see [Ja 1], Lemma 4.5).

2. Continuous étale cohomology

This is a brief summary of some aspects of Jannsen's theory [Ja 1].

(2.1) For a scheme X, let X_{et} be its small étale site ([SGA 4], Exp. VII, 1.2). As in 1.4, for an object $[A_n]$ of $\mathrm{Ab}(X^{\sim}_{et})^{\mathbf{N}}$, there are continuous cohomology groups $H^q(X_{et}, [A_n])$ sitting in the exact sequence

$$0 \longrightarrow (R^1 \varprojlim_{\mathbf{N}})\, H^{q-1}(X_{et}, A_n) \longrightarrow H^q(X_{et}, [A_n]) \longrightarrow \varprojlim_{\mathbf{N}} H^q(X_{et}, A_n) \longrightarrow 0 \qquad (2.1.1)$$

As before, we denote the third group by $H^q_{naive}(X_{et}, [A_n])$.

(2.2) Let K be a (commutative) field. Fix a separable closure K_s of K and put $G_K = G(K_s/K)$. Evaluation at $\overline{\eta} : \mathrm{Spec}(K_s) \longrightarrow \mathrm{Spec}(K)$ (more precisely, at $\mathrm{Spec}(L) \longrightarrow \mathrm{Spec}(K)$ for all finite subextensions $K \subset L \subset K_s$) defines an equivalence of categories

$$\mathrm{Spec}(K)^{\sim}_{et} \xrightarrow{\approx} B_{G_K}, \qquad A \mapsto A_{\overline{\eta}},$$

inducing canonical isomorphisms

$$H^q(\mathrm{Spec}(K)_{et}, [A_n]) \xrightarrow{\sim} H^q(G_K, [(A_n)_{\overline{\eta}}]_{n \in \mathbf{N}}) \qquad (2.2.1)$$

Let $\pi : X \longrightarrow \mathrm{Spec}(K)$ be a coherent (= quasi-compact and quasi-separated) morphism (for example, a morphism of finite type). For every abelian sheaf A on X_{et} and $q \geq 0$, there is a canonical isomorphism

$$(R^q \pi_* A)_{\overline{\eta}} \xrightarrow{\sim} H^q(\overline{X}_{et}, A), \qquad (2.2.2)$$

where $\overline{X} = X \otimes_K K_s$ ([SGA 4], Exp. VIII, 5.2) (by abuse of notation, we write A instead of f^*A, for $f : \overline{X} \longrightarrow X$). If π is not coherent, then the R.H.S. of (2.2.2) has to be replaced by

$$\varinjlim_{L} H^q((X \otimes_K L)_{et}, A),$$

with L running through subextensions $K \subset L \subset K_s$, finite over K. The Leray spectral sequence (1.5.2) becomes – for coherent π – the Hochschild-Serre spectral sequence

$$E_2^{i,j} = H^i(G_K, [H^j(\overline{X}_{et}, A_n)]_{n \in \mathbf{N}}) \Longrightarrow H^{i+j}(X_{et}, [A_n]_{n \in \mathbf{N}}) \qquad (2.2.3)$$

(2.3) Under additional hypotheses, the spectral sequence (2.2.3) further simplifies ([Ja 1], Rmk. 3.5):

(2.3.1) If $[H^j(\overline{X}_{et}, A_n)]_{n \in \mathbf{N}}$ is a ML-system for all $j \geq 0$ (*e.g.* if the groups $H^j(\overline{X}_{et}, A_n)$ are finite for all $j, n \geq 0$), then

$$H^q(X_{et}, [A_n]) \xrightarrow{\sim} H^q_{naive}(X_{et}, [A_n])$$

and (1.7.1) gives

$$E_2^{i,j} = H^i_{cont}(G_K, H^j_{naive}(\overline{X}_{et}, [A_n])) \Longrightarrow H^{i+j}(X_{et}, [A_n]_{n\in\mathbf{N}}) \tag{2.3.1.1}$$

(2.3.2) If, furthermore, $H^i(G_K, M)$ is finite for any finite G_K-module M and $i \geq 0$, then the spectral sequence (2.3.1.1) and its abutment are equal to $\varprojlim_{\mathbf{N}}$ of the usual spectral sequences

$$E_2^{i,j} = H^i(G_K, H^j(\overline{X}_{et}, A_n)) \Longrightarrow H^{i+j}(X_{et}, A_n), \tag{2.3.2.1}$$

as all groups in (2.3.2.1) are finite and $\varprojlim_{\mathbf{N}}$ is exact on projective systems of finite groups. This happens, for example, if K is a finite field or a finite extension of $\mathbf{Q}_\ell$.

(2.3.3) For a prime p different from the characteristic of K we put, as usual,

$$\mathbf{Z}/p^n\mathbf{Z}(r) = \begin{cases} \mu_{p^n}^{\otimes r} & \text{if } r \geq 0 \\ \mathrm{Hom}(\mu_{p^n}, \mathbf{Z}/p^n\mathbf{Z})^{\otimes -r} & \text{if } r \leq 0 \end{cases}$$

If $\pi : X \longrightarrow \mathrm{Spec}(K)$ is of finite type, then, according to ([SGA 5], Exp. V, 5.3.1),

$$[H^i(\overline{X}_{et}, \mathbf{Z}/p^n\mathbf{Z}(r))]_{n\in\mathbf{N}} \quad \text{is a ML}-p-\text{adic system of finite groups} \tag{2.3.3.1}$$

(for every $i \geq 0, r \in \mathbf{Z}$). This implies that

$$H^j(\overline{X}_{et}, \mathbf{Z}_p(r)) \xrightarrow{\sim} H^j_{naive}(\overline{X}_{et}, \mathbf{Z}_p(r)).$$

Following Jannsen [Ja 1], we use the following notation:

$$H^q(X_{et}, \mathbf{Z}_p(r)) := H^q(X_{et}, [\mathbf{Z}/p^n\mathbf{Z}(r)])$$
$$H^q(X_{et}, \mathbf{Q}_p(r)) := H^q(X_{et}, [\mathbf{Z}/p^n\mathbf{Z}(r)]) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

(2.3.1.1) then gives a spectral sequence

$$E_2^{i,j} = H^i_{cont}(G_K, H^i(\overline{X}_{et}, \mathbf{Z}_p(r))) \Longrightarrow H^{i+j}(X_{et}, \mathbf{Z}_p(r)), \tag{2.3.3.2}$$

which, on tensoring by $\mathbf{Q}$, becomes – in view of Proposition 1.9 –

$$E_2^{i,j} = H^i_{cont}(G_K, H^i(\overline{X}_{et}, \mathbf{Q}_p(r))) \Longrightarrow H^{i+j}(X_{et}, \mathbf{Q}_p(r)) \tag{2.3.3.3}$$

Note that the topology on $H^i(\overline{X}_{et}, \mathbf{Z}_p(r))$ (resp. on $H^i(\overline{X}_{et}, \mathbf{Q}_p(r))$) is the p-adic one, according to (1.10) and (2.3.3.1),

If follows from (2.3.3.1) and Lemma 1.8 that

$$E_2^{i,j} = 0 \qquad \text{for } i > cd_p(K) \tag{2.3.3.4}$$

(for both spectral sequences (2.3.3.2–3)).

If X is smooth and projective over K, then the spectral sequence (2.3.3.3) degenerates at E_2, by ([De 1], Prop. 2.4) and the existence of a good formalism for the derived category of $\mathbf{Q}_p$-sheaves (Ekedhal [Ek], Jannsen (unpublished)). It follows from de Jong's theorem on alterations [de J] that $E_2 = E_\infty$ for X proper (not necessarily projective) and smooth over K.

(2.3.4) If, furthermore, K is as in (2.3.2), then (2.3.3.2) is the projective limit of spectral sequences

$$E_2^{i,j} = H^i(G_K, H^i(\overline{X}_{et}, \mathbf{Z}/p^n\mathbf{Z}(r))) \Longrightarrow H^{i+j}(X_{et}, \mathbf{Z}/p^n\mathbf{Z}(r)), \tag{2.3.4.1}$$

with all groups in (2.3.4.1) being finite.

(2.4) Proper base change. Let S be the spectrum of a henselian local ring, $s \hookrightarrow S$ the inclusion of the closed point. Let $\pi : X \longrightarrow S$ be a proper morphism with a special fibre $X_s = X \otimes_S s$. Denote by

$i : X_s \hookrightarrow X$ the immersion of X_s into X. If $A = [A_n]$ is a projective system of *torsion* abelian sheaves on X_{et}, then i induces an isomorphism

$$i^* : H^q(X_{et}, [A_n]) \xrightarrow{\sim} H^q((X_s)_{et}, [i^* A_n]) \qquad (\forall q \geq 0) \tag{2.4.1}$$

This follows from (2.1.1) and the usual proper base change theorem. The latter also implies that

$$(R^q \pi_*^{\mathbf{N}} A)_{\overline{s}} := [(R^q \pi_* A_n)_{\overline{s}}]_{n \in \mathbf{N}} \xrightarrow{\sim} [H^q((X_{\overline{s}})_{et}, \overline{i}^* A_n)]_{n \in \mathbf{N}} \tag{2.4.2}$$

(2.5) Let p be a prime number. Write "$\mathbf{Z}_p$" for the projective system $[\mathbf{Z}/p^n\mathbf{Z}]_{n \in \mathbf{N}}$; it is a ring in $(Sets)^{\mathbf{N}}$. For any topos E, the morphism of topoi $\Gamma_E^{\mathbf{N}} : E^{\mathbf{N}} \longrightarrow (Sets)^{\mathbf{N}}$ defines a ring $(\Gamma_E^{\mathbf{N}})^*$("$\mathbf{Z}_p$") in $E^{\mathbf{N}}$, which will be again denoted by "$\mathbf{Z}_p$". By abuse of language, we put

$$\mathrm{Mod}(E, \text{“}\mathbf{Z}_p\text{”}) := \mathrm{Mod}(E^{\mathbf{N}}, \text{“}\mathbf{Z}_p\text{”}), \qquad \mathrm{Mod}(E, \text{“}\mathbf{Q}_p\text{”}) := Q(\mathrm{Mod}(E, \text{“}\mathbf{Z}_p\text{”}))$$

These are abelian categories with enough injectives (*cf.* (1.1), (1.2.8)).

If K is a field of characteristic $char(K) \neq p$ and $G = G_K$, denote by "$\mathbf{Z}_p(r)$" (for any $r \in \mathbf{Z}$) the projective system of discrete G-modules $[\mathbf{Z}/p^n\mathbf{Z}(r)]_{n \in \mathbf{N}}$. It is a flat object of $\mathrm{Mod}(B_G, \text{“}\mathbf{Z}_p\text{”})$. Write "$\mathbf{Q}_p(r)$" for Q("$\mathbf{Z}_p(r)$"), which is a flat object of $\mathrm{Mod}(B_G, \text{“}\mathbf{Q}_p\text{”})$.

Tensor products $X \mapsto X \otimes_{\text{“}\mathbf{Z}_p\text{”}} \text{“}\mathbf{Z}_p(r)\text{”}$ (resp. $X \mapsto X \otimes_{\text{“}\mathbf{Q}_p\text{”}} \text{“}\mathbf{Q}_p(r)\text{”}$) define exact endofunctors $X \mapsto X(r)$ ("Tate twists") of $\mathrm{Mod}(B_G, \text{“}\mathbf{Z}_p\text{”})$ (resp. $\mathrm{Mod}(B_G, \text{“}\mathbf{Q}_p\text{”})$). For every X and $r, q \in \mathbf{Z}$, there are canonical isomorphisms

$$(X(r))(q) \xrightarrow{\sim} X(r+q), \qquad (X(r))(-r) \xrightarrow{\sim} X \tag{2.5.1}$$

3. Some homological algebra

This section should be skipped on the first reading. We shall need Proposition 3.5 for our main comparison results in Chapters III and IV.

(3.1) Suppose we are given the following data:

(0) An integer $i \in \mathbf{Z}$.

(1) Abelian categories with enough injectives $\mathcal{C}, \overline{\mathcal{C}}, \mathcal{D}$.

(2) Left exact functors $u : \overline{\mathcal{C}} \longrightarrow \mathcal{C}$, $\Psi : \mathcal{C} \longrightarrow \mathcal{D}$, $\Phi = \Psi \circ u : \overline{\mathcal{C}} \longrightarrow \mathcal{D}$.

(3) Distinguished triangles $\Delta : A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ (resp. $\overline{\Delta} : \overline{A} \longrightarrow \overline{B} \longrightarrow \overline{C} \longrightarrow \overline{A}[1]$) in $D^+(\mathcal{C})$ (resp. $D^+(\overline{\mathcal{C}})$).

(4) A morphism of triangles $\Delta \longrightarrow (\mathbf{R}u)(\overline{\Delta})$ in $D^+(\mathcal{C})$.

(5) Objects $E \in \mathrm{Ob}(D^+(\mathcal{C}))$, $\overline{E} \in \mathrm{Ob}(D^+(\overline{\mathcal{C}}))$ and an isomorphism $\rho : E \xrightarrow{\sim} (\mathbf{R}u)(\overline{E})$.

(6) Morphisms $\mu : A \longrightarrow E$ (resp. $\overline{\mu} : \overline{A} \longrightarrow \overline{E}$) in $D^+(\mathcal{C})$ (resp. $D^+(\overline{\mathcal{C}})$) such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\mu} & E \\ \downarrow & & \downarrow \wr\rho \\ (\mathbf{R}u)(\overline{A}) & \xrightarrow{\mathbf{R}u(\overline{\mu})} & (\mathbf{R}u)(\overline{E}) \end{array}$$

is commutative.

Denote the composition

$$(R^{i+1}\Psi)(A) \xrightarrow{\mu} (R^{i+1}\Psi)(E) \xrightarrow{\sim} (R^{i+1}\Phi)(\overline{E})$$

by ν and write λ^j for the map $(R^j\Psi)(B) \longrightarrow (R^j\Psi)(C)$.

We assume the following axioms hold:

(A1) $\mathrm{Ker}\left[H^j(\overline{A}) \longrightarrow H^j(\overline{B})\right] = 0$ for $j = i, i+1$.

(A2) $\overline{\mu}$ induces an isomorphism $\tau_{\leq i}\overline{A} \xrightarrow{\sim} \tau_{\leq i}\overline{E}$ and a monomorphism $\Phi(H^{i+1}(\overline{A})) \hookrightarrow \Phi(H^{i+1}(\overline{E}))$.

(A3) The canonical maps $(R^j\Psi)(B) \longrightarrow \Phi(H^j(\overline{B}))$, $(R^j\Psi)(C) \longrightarrow \Phi(H^j(\overline{C}))$ (for $j = i, i+1$) induce isomorphisms

$$\mathrm{Coker}(\lambda^i) \xrightarrow{\sim} \mathrm{Coker}\left[\Phi(H^i(\overline{B})) \longrightarrow \Phi(H^i(\overline{C}))\right]$$
$$\mathrm{Ker}(\lambda^{i+1}) \xrightarrow{\sim} \mathrm{Ker}\left[\Phi(H^{i+1}(\overline{B})) \longrightarrow \Phi(H^{i+1}(\overline{C}))\right]$$

Note that (A2) resp. (A3) follow from stronger axioms

(A2') $\overline{\mu}$ induces an isomorphism $\tau_{\leq i+1}\overline{A} \xrightarrow{\sim} \tau_{\leq i+1}\overline{E}$.

(A3') The canonical maps $(R^j\Psi)(B) \longrightarrow \Phi(H^j(\overline{B}))$, $(R^j\Psi)(C) \longrightarrow \Phi(H^j(\overline{C}))$ are isomorphisms for $j = i, i+1$.

(3.2) The spectral sequence $E_2^{p,q} = (R^p\Phi)(H^q(\overline{A})) \Longrightarrow (R^{p+q}\Phi)(\overline{A})$ defines a decreasing filtration $F^i(R^j\Phi)(\overline{A})$ and homomorphisms

$$edge : (R^j\Phi)(\overline{A}) \longrightarrow E_\infty^{0,j} \hookrightarrow E_2^{0,j} = \Phi(H^j(\overline{A}))$$
$$\delta : F^1(R^j\Phi)(\overline{A}) = \mathrm{Ker}(edge) \longrightarrow E_\infty^{1,j-1} \hookrightarrow E_2^{1,j-1} = (R^1\Phi)(H^{j-1}(\overline{A}))$$

(and similarly for $\overline{B}$, $\overline{C}$). The maps in (A3) are given by

$$(R^j\Psi)(B) = H^j(\mathbf{R}\Psi(B)) \xrightarrow{(3.1.4)} H^j(\mathbf{R}\Psi\mathbf{R}u(\overline{B})) \xrightarrow{\sim} H^j(\mathbf{R}\Phi(\overline{B})) = (R^j\Phi)(\overline{B}) \xrightarrow{edge} \Phi(H^j(\overline{B}))$$

Proposition. *For every $j \in \mathbf{Z}$, the following diagram is commutative:*

$$\begin{array}{ccc}
\mathrm{Ker}\left[(R^j\Phi)(\overline{C}) \xrightarrow{edge} \Phi(H^j(\overline{C})) \longrightarrow \Phi(H^{j+1}(\overline{A}))\right] & \longrightarrow & F^1(R^{j+1}\Phi)(\overline{A}) \\
\downarrow{\scriptstyle edge} & & \downarrow{\scriptstyle \delta} \\
\mathrm{Ker}\left[\Phi(H^j(\overline{C})) \longrightarrow \Phi(H^{j+1}(\overline{A}))\right] \xrightarrow{\partial} (R^1\Phi)(\mathrm{Im}[H^j(\overline{A}) \longrightarrow H^j(\overline{B})]) & \longleftarrow & (R^1\Phi)(H^j(\overline{A}))
\end{array}$$

Here ∂ is the coboundary map in the cohomology exact sequence of

$$0 \longrightarrow \mathrm{Im}[H^j(\overline{A}) \longrightarrow H^j(\overline{B})] \longrightarrow H^j(\overline{B}) \longrightarrow \mathrm{Ker}[H^j(\overline{C}) \longrightarrow H^{j+1}(\overline{A})] \longrightarrow 0$$

Proof. This is a derived category version of [Ja 3, Lemma 9.5].

(3.3) The axiom (A1) implies that

$$\mathrm{Im}\left[(R^i\Phi)(\overline{C}) \longrightarrow (R^{i+1}\Phi)(\overline{A})\right] \subseteq F^1(R^{i+1}\Phi)(\overline{A}), \tag{3.3.1}$$

the sequence

$$0 \longrightarrow H^i(\overline{A}) \longrightarrow H^i(\overline{B}) \longrightarrow H^i(\overline{C}) \longrightarrow 0$$

is exact and the maps

$$\Phi(H^j(\overline{A})) \longrightarrow \mathrm{Ker}\left[\Phi(H^j(\overline{B})) \longrightarrow \Phi(H^j(\overline{C}))\right] \tag{3.3.2}$$

are isomorphisms for $j = i, i+1$.

According to the axiom (A2), $\overline{\mu}$ induces isomorphisms

$$\begin{array}{ll} (R^j\Phi)(\overline{A}) \xrightarrow{\sim} (R^j\Phi)(\overline{E}), \quad H^j(\overline{A}) \xrightarrow{\sim} H^j(\overline{E}) & (\forall j \leq i) \\ F^1(R^j\Phi)(\overline{A}) \xrightarrow{\sim} F^1(R^j\Phi)(\overline{E}) & (\forall j \leq i+1) \end{array} \tag{3.3.3}$$

and a monomorphism (which is an isomorphism, if (A2') holds)

$$F^1(R^{i+1}\Phi)(\overline{A}) \hookrightarrow F^1(R^{i+1}\Phi)(\overline{E}). \tag{3.3.4}$$

By transport of structure, we obtain an exact sequence

$$0 \longrightarrow H^i(\overline{E}) \longrightarrow H^i(\overline{B}) \longrightarrow H^i(\overline{C}) \longrightarrow 0$$

Denote by

$$\partial : \Phi(H^i(\overline{C})) \longrightarrow (R^1\Phi)(H^i(\overline{E}))$$

the corresponding boundary map.

(3.4) Corollary. *Assuming (A1), (A2), the following diagram is commutative*

$$\begin{array}{ccccc}
\mathrm{Coker}\left[(R^i\Phi)(\overline{B})) \longrightarrow (R^i\Phi)(\overline{C}))\right] & \overset{(3.3.1)}{\longrightarrow} & F^1(R^{i+1}\Phi)(\overline{A}) & \overset{\overline{\mu}}{\longrightarrow} & F^1(R^{i+1}\Phi)(\overline{E}) \\
\big\downarrow{\scriptstyle edge} & & \big\downarrow{\scriptstyle \delta} & & \big\downarrow{\scriptstyle \delta} \\
\mathrm{Coker}\left[\Phi(H^i(\overline{B})) \longrightarrow \Phi(H^i(\overline{C}))\right] & \overset{\partial}{\hookrightarrow} & (R^1\Phi)(H^i(\overline{A})) & \overset{\overline{\mu}}{\longrightarrow} & (R^1\Phi)(H^i(\overline{E}))
\end{array}$$

Both maps $\overline{\mu}$ are isomorphisms, by (3.3.3).

(3.5) Proposition. *Suppose the axioms (A1)–(A3) hold. Then*
(1) *There is a commutative diagram with exact rows, with vertical maps induced by ν:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Coker}(\lambda^i) & \longrightarrow & (R^{i+1}\Psi)(A) & \longrightarrow & \mathrm{Ker}(\lambda^{i+1}) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \nu} & & \big\downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & F^1(R^{i+1}\Phi)(\overline{E}) & \longrightarrow & (R^{i+1}\Phi)(\overline{E}) & \overset{edge}{\longrightarrow} & \Phi(H^{i+1}(\overline{E})) & & \\
 & & \big\downarrow{\scriptstyle \delta} & & & & & & \\
 & & (R^1\Phi)(H^i(\overline{E})) & & & & & &
\end{array}$$

(2) *The composition $\delta \circ \beta$ is equal to the map ∂ (composed with the isomorphism of (A3)). In particular,* $\mathrm{Ker}(\delta \circ \beta) = 0$.
(3) $\mathrm{Ker}(\alpha) = 0$. *If (A2') holds, then α is an isomorphism and* $\mathrm{Coker}(edge) = 0$.
(4) $\mathrm{Ker}(\nu) = 0$.

Proof. (1),(2) The first row is the cohomology sequence of Δ. The commutativity of the right square is clear. The commutativity of the left square follows from Corollary 3.4 and the commutative diagram

$$\begin{array}{ccc}
\mathrm{Coker}(\lambda^i) & \longrightarrow & (R^{i+1}\Psi)(A) \\
\big\downarrow & & \big\downarrow \\
\mathrm{Coker}\left[(R^i\Phi)(\overline{B})) \longrightarrow (R^i\Phi)(\overline{C}))\right] & \hookrightarrow F^1(R^{i+1}\Phi)(\overline{A}) \hookrightarrow & (R^{i+1}\Phi)(\overline{A})
\end{array}$$

(3) The map α is the composition of the monomorphism $\Phi(H^{i+1}(\overline{A})) \hookrightarrow \Phi(H^{i+1}(\overline{E}))$ (which is an isomorphism, if (A2') is satisfied), and the isomorphisms (3.3.2) and (A3).
(4) Follows from $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta) = 0$.

(3.6) Define a filtration on $(R^j\Psi)(A)$ by

$$\begin{aligned}
F^0(R^j\Psi)(A) &= (R^j\Psi)(A) \\
F^1(R^j\Psi)(A) &= \mathrm{Ker}\left[(R^j\Psi)(A) \longrightarrow (R^j\Psi)(B)\right] \overset{\sim}{\longleftarrow} \mathrm{Coker}(\lambda^{j-1}) \\
F^m(R^j\Psi)(A) &= 0 \qquad (m > 1)
\end{aligned}$$

Proposition 3.5 can then be reformulated as follows:

Proposition. (1) *For every* $j \geq 0$, *we have* $\nu(F^j(R^{i+1}\Psi)(A)) \subseteq F^j(R^{i+1}\Phi)(\overline{E}))$ *and the induced map* $gr^j_F(\nu) : gr^j_F(R^{i+1}\Psi)(A) \longrightarrow gr^j_F(R^{i+1}\Phi)(\overline{E}) = E^{j,i+1-j}_\infty$ *is a monomorphism.*
(2) $\mathrm{Im}\left[gr^1_F(R^{i+1}\Psi)(A) \hookrightarrow E^{1,i}_\infty \hookrightarrow E^{1,i}_2\right] = \mathrm{Im}(\partial)$.
(3) *If (A2') is satisfied, then the map*

$$gr^0_F(R^{i+1}\Psi)(A) \hookrightarrow E^{0,i+1}_\infty \hookrightarrow E^{0,i+1}_2 = \Phi(H^{i+1}[\overline{E}))$$

is an isomorphism and $E^{0,i+1}_\infty = E^{0,i+1}_2$.

II. Local situation at $\ell \neq p$

There is nothing new in this chapter. We simply summarize well-known results on étale cohomology and p-adic regulators for varieties over ℓ-adic fields.

1. Étale cohomology

(1.1) Geometric situation. Let $\ell \neq p$ be prime numbers, $K/\mathbf{Q}_\ell$ a finite extension, $\mathcal{O}_K \subset K$ the ring of integers, k the residue field of $\mathcal{O}_K$, $I_K \subset G_K$ the inertia subgroup.
For a morphism $\pi : \mathfrak{X} \longrightarrow \mathrm{Spec}(\mathcal{O}_K)$, denote by $\pi_\eta : X \longrightarrow \mathrm{Spec}(K) = \eta$ (resp. $\pi_s : Y \longrightarrow \mathrm{Spec}(k) = s$) its generic (resp. special) fibre. Denote the corresponding open (resp. closed) immersion by $j : X \hookrightarrow \mathfrak{X}$ (resp. $i : Y \hookrightarrow \mathfrak{X}$).

(1.2) Cohomology. For a fixed $r \in \mathbf{Z}$, there is a projective system of abelian sheaves $[\mathbf{Z}/p^n\mathbf{Z}(r)]_{n\in\mathbf{N}}$ on $\mathfrak{X}_{et}$. The Leray spectral sequences (1.5.2) for π, π_η, π_s (tensored by $\mathbf{Q}$) are

$$\begin{aligned} E_2^{a,b} &= H^a(\mathrm{Spec}(\mathcal{O}_K)_{et}, [R^b\pi_*(\mathbf{Z}/p^n\mathbf{Z}(r))]_{n\in\mathbf{N}}) \otimes \mathbf{Q} \Longrightarrow H^{a+b}(\mathfrak{X}_{et}, \mathbf{Q}_p(r)) \\ {}^sE_2^{a,b} &= H^a(\mathrm{Spec}(k)_{et}, [R^b\pi_{s*}(\mathbf{Z}/p^n\mathbf{Z}(r))]_{n\in\mathbf{N}}) \otimes \mathbf{Q} \Longrightarrow H^{a+b}(Y_{et}, \mathbf{Q}_p(r)) \\ {}^\eta E_2^{a,b} &= H^a(\mathrm{Spec}(K)_{et}, [R^b\pi_{\eta*}(\mathbf{Z}/p^n\mathbf{Z}(r))]_{n\in\mathbf{N}}) \otimes \mathbf{Q} \Longrightarrow H^{a+b}(X_{et}, \mathbf{Q}_p(r)) \end{aligned} \tag{1.2.1}$$

(see (2.3.3) for the notation).
If π is a morphism of finite type, then

$$\begin{aligned} {}^sE_2^{a,b} &= H^a_{cont}(G_k, H^b(\overline{Y}_{et}, \mathbf{Z}_p(r))) \otimes \mathbf{Q} \xrightarrow{\sim} H^a_{cont}(G_k, H^b(\overline{Y}_{et}, \mathbf{Q}_p(r))) \\ {}^\eta E_2^{a,b} &= H^a_{cont}(G_K, H^b(\overline{X}_{et}, \mathbf{Z}_p(r))) \otimes \mathbf{Q} \xrightarrow{\sim} H^a_{cont}(G_K, H^b(\overline{X}_{et}, \mathbf{Q}_p(r))) \end{aligned} \tag{1.2.2}$$

by (I.1.9) and (I.2.3.3.2). According to (I.2.3.3–4), we have

$$H^q(Z_{et}, \mathbf{Z}_p(r)) \xrightarrow{\sim} H^q_{naive}(Z_{et}, \mathbf{Z}_p(r)) \tag{1.2.3}$$

for all $Z = X, \overline{X}, Y, \overline{Y}$.

(1.3) Base change. The immersions i, j induce homomorphisms

$$\begin{aligned} {}^sE_c^{a,b} \xleftarrow{i^*} E_c^{a,b} \xrightarrow{j^*} {}^\eta E_c^{a,b} \\ H^{a+b}(Y_{et}, \mathbf{Q}_p(r)) \xleftarrow{i^*} H^{a+b}(\mathfrak{X}_{et}, \mathbf{Q}_p(r)) \xrightarrow{j^*} H^{a+b}(X_{et}, \mathbf{Q}_p(r)) \end{aligned} \tag{1.3.1}$$

As $cd_p(k) = 1$ and $cd_p(K) = 2$, (I.2.3.3.4) implies that ${}^sE_2^{a,b} = 0$ for $a > 1$ and ${}^\eta E_2^{a,b} = 0$ for $a > 2$. Thus ${}^sE_2^{a,b} = {}^sE_\infty^{a,b}$.

The filtrations $F^\cdot$ induced by the spectral sequences (1.2.1) on their abutments are compatible with the maps i^*, j^* in (1.3.1). From now on, assume that π is *proper*. Then i^* induces an isomorphism of spectral sequences $E_c^{a,b}$ and ${}^sE_c^{a,b}$, by (I.2.4.1–2). This means that $F^2H^i(\mathfrak{X}_{et}, \mathbf{Q}_p(r)) = 0$ and that the map δ of (I.1.5.3) (for π_s) is an isomorphism, sitting in a commutative diagram

$$\begin{array}{ccccc} F^1H^{i+1}(Y_{et}, \mathbf{Q}_p(r)) & \xleftarrow{\sim} & F^1H^{i+1}(\mathfrak{X}_{et}, \mathbf{Q}_p(r)) & \longrightarrow & F^1H^{i+1}(X_{et}, \mathbf{Q}_p(r)) \\ \downarrow \wr & & & & \downarrow \delta \\ H^1_{cont}(G_k, H^i(\overline{Y}_{et}, \mathbf{Q}_p(r))) & & \longrightarrow & & H^1_{cont}(G_K, H^i(\overline{X}_{et}, \mathbf{Q}_p(r))) \end{array} \tag{1.3.2}$$

All of its entries (as well as $H^i(\overline{X}_{et}, \mathbf{Q}_p(r))$ and $H^i(\overline{Y}_{et}, \mathbf{Q}_p(r))$) are $\mathbf{Q}_p$-vector spaces of finite dimension, by (I.2.3.4).
Write V_s^i (resp. V_η^i) for $H^i(\overline{Y}_{et}, \mathbf{Q}_p)$ (resp. $H^i(\overline{X}_{et}, \mathbf{Q}_p)$). The graded pieces for the filtrations $F^\cdot$ satisfy

$$\begin{aligned} gr^0_F H^{i+1}(\mathfrak{X}_{et}, \mathbf{Q}_p(r)) &= E^{0,i+1}_\infty \xrightarrow{\sim} {}^sE^{0,i+1}_\infty = {}^sE^{0,i+1}_2 = \left(V^{i+1}_s(r)\right)^{G_k} \\ gr^1_F H^{i+1}(\mathfrak{X}_{et}, \mathbf{Q}_p(r)) &= E^{1,i}_\infty \xrightarrow{\sim} {}^sE^{1,i}_\infty = {}^sE^{1,i}_2 = H^1_{cont}(G_k, V^i_s(r)) \\ gr^j_F H^{i+1}(\mathfrak{X}_{et}, \mathbf{Q}_p(r)) &= 0 \qquad (\forall j > 1) \end{aligned} \tag{1.3.3.1}$$

and

$$\begin{aligned} gr^0_F H^{i+1}(X_{et}, \mathbf{Q}_p(r)) &= {}^\eta E^{0,i+1}_\infty \hookrightarrow {}^\eta E^{0,i+1}_2 = \left(V^{i+1}_\eta(r)\right)^{G_K} \\ gr^1_F H^{i+1}(X_{et}, \mathbf{Q}_p(r)) &= {}^\eta E^{1,i}_\infty = {}^\eta E^{1,i}_2 = H^1_{cont}(G_K, V^i_\eta(r)) \end{aligned} \tag{1.3.3.2}$$

The homomorphisms

$$\begin{aligned} gr^0_F(j^*) &: \left(V^{i+1}_s(r)\right)^{G_k} \longrightarrow \left(V^{i+1}_\eta(r)\right)^{G_K} \\ gr^1_F(j^*) &: H^1_{cont}(G_k, V^i_s(r)) \longrightarrow H^1_{cont}(G_K, V^i_\eta(r)) \end{aligned} \tag{1.3.4}$$

are induced by the specialization maps

$$sp^j : V^j_s \longrightarrow \left(V^j_\eta\right)^{I_K} \tag{1.3.5}$$

for $j = i+1$ (resp. $j = i$). In particular, the bottom arrow in (1.3.2) is equal to $gr^1_F(j^*)$, hence factors through the unramified Galois cohomology:

$$H^1_{cont}(G_k, V^i_s(r)) \longrightarrow H^1_{cont}(G_k, (V^i_\eta(r))^{I_K}) = H^1_{ur}(G_K, V^i_\eta(r)) \hookrightarrow H^1_{cont}(G_K, V^i_\eta(r)) \tag{1.3.6}$$

According to Deligne ([De 4], Thm. 1), $V^i_s(r)$ is mixed of weights $\leq i - 2r$. Consequently,

$$\begin{aligned} gr^0_F H^{i+1}(\mathfrak{X}_{et}, \mathbf{Q}_p(r)) &= 0, \qquad \text{if } r > (i+1)/2 \\ gr^1_F H^{i+1}(\mathfrak{X}_{et}, \mathbf{Q}_p(r)) &= 0, \qquad \text{if } r > i/2 \end{aligned} \tag{1.3.7}$$

If π is proper and smooth, then I_K acts trivially on V^i_η and sp^i is an isomorphism (for all $i \geq 0$), by the proper and smooth base change theorems.

(1.4) Proposition. *Let $\pi : \mathfrak{X} \longrightarrow \mathrm{Spec}(\mathcal{O}_K)$ be proper.*
(1) *If $r > (i+1)/2$, then $H^{i+1}(\mathfrak{X}_{et}, \mathbf{Q}_p(r)) = 0$.*
(2) *If $r = (i+1)/2$, then the edge homomorphism* $H^{2r}(\mathfrak{X}_{et}, \mathbf{Q}_p(r)) \longrightarrow V^{2r}_s(r)^{G_k}$ *is an isomorphism.*

Proof. Everything follows from (1.3.7) and (1.3.3.1).

(1.5) Purity conjecture for the monodromy filtration. Suppose that $\pi : \mathfrak{X} \longrightarrow \mathrm{Spec}(\mathcal{O}_K)$ is proper and its generic fibre $\pi_\eta : X \longrightarrow \mathrm{Spec}(K)$ is smooth. According to Grothendieck ([ST], Appendix), there is a finite extension K'/K such that $I_{K'}$ acts on $H := H^i(\overline{X}_{et}, \mathbf{Q}_p)$ unipotently, through the p-part of its tame quotient. This gives rise ([De 4], Prop. 1.6.1) to a monodromy operator $N : H \longrightarrow H(-1)$ and the corresponding monodromy filtration $M_i H$ (increasing, exhaustive and separating). It is characterized by

$$N(M_j H) \subseteq M_{j-2}(-1) \quad (j \in \mathbf{Z}); \qquad N^j : Gr^M_j(H) \xrightarrow{\sim} Gr^M_{-j}(H)(-j)$$

The purity conjecture for the monodromy filtration ([De 2], 8.1) predicts that

$H = M_i H \quad (\Longleftrightarrow M_{-i-1}H = 0)$ (1.5.1)

Over K', $Gr^M_j(H)$ is unramified and pure of weight $i + j \quad (\forall j \in \mathbf{Z})$ (1.5.2)

The characteristic polynomial $\det(1 - T \cdot Fr_{k'} \mid Gr^M_j(H))$ has coefficients in $\mathbf{Q}$ and is independent of p (1.5.3)

If true, then the weights of

$$H^i(\overline{X}_{et}, \mathbf{Q}_p)^{I_K} \subseteq H^i(\overline{X}_{et}, \mathbf{Q}_p)^{I'_K} = \mathrm{Ker}(N : H \longrightarrow H(-1)) \tag{1.5.4}$$

are contained in the set $\{0, 1, \ldots, i\}$ and all weights of

$$H^i(\overline{X}_{et}, \mathbf{Q}_p)_{I_K} \longleftarrow H^i(\overline{X}_{et}, \mathbf{Q}_p)_{I'_K} = \mathrm{Coker}(N(1) : H(1) \longrightarrow H) \tag{1.5.5}$$

are contained in $\{i, i+1, \ldots, 2i\}$.

(1.6) The conjecture (1.5.1–3) has been proved in the following cases:
(1.6.0) $i = 0$ (trivial).
(1.6.1) X has good reduction ([De 3]). In this case H is pure of weight i: $H = M_0 H$ and $M_{-1} H = 0$.
(1.6.2) $i = 1$ ([SGA 7], Exp. IX, Thm. 3.6).
(1.6.3) $\dim(X) = 2$ ([RZ], Thm. 2.13, assuming that X has semistable reduction; it follows from de Jong's theorem on alterations [de J] that this assumption is unnecessary).
(1.6.4) X is an abelian variety (by (1.6.2) and the fact that the cohomology ring of $\overline{X}$ is the exterior algebra over H^1).
(1.6.5) If X is equidimensional of $\dim(X) = d$, then (1.5.1–3) is valid for i iff it is valid for $2d - i$, by the Poincaré duality.
(1.6.6) If X has a potentially semistable reduction, then (1.5.1) is proved in [RZ] (*cf.* [Il 2, Cor. 3.3]). As in 1.6.3, the general case follows from de Jong's theorem.

(1.7) The following statement is well-known ([Ja 2], [So 1]). It shows that, in many cases, the cohomology group $H^1_{cont}(G_K, H^i(\overline{X}_{et}, \mathbf{Q}_p(r)))$ vanishes.

Proposition. (1) *If X is proper and smooth over K and the purity conjecture for the monodromy filtration holds for $H^i(\overline{X}_{et}, \mathbf{Q}_p)$, then*

(a) $H^0(G_K, H^i(\overline{X}_{et}, \mathbf{Q}_p(r))) = H^1_{ur}(G_K, H^i(\overline{X}_{et}, \mathbf{Q}_p(r))) = 0$ *for* $\begin{cases} r < 0 \\ r > i/2 \end{cases}$

In particular, the "local Euler factor" of $H^i(X)$ has no pole for these values of $s = r$.

(b) $H^2_{cont}(G_K, H^i(\overline{X}_{et}, \mathbf{Q}_p(r))) = 0$ *for* $\begin{cases} r < i/2 + 1 \\ r > i + 1 \end{cases}$

(c) $H^1_{cont}(G_K, H^i(\overline{X}_{et}, \mathbf{Q}_p(r))) = 0$ *for* $\begin{cases} r < 0 \\ r = (i+1)/2 \\ r > i + 1 \end{cases}$

(2) *If, furthermore, X has good reduction, then*
(a) $H^0(G_K, H^i(\overline{X}_{et}, \mathbf{Q}_p(r))) = H^1_{ur}(G_K, H^i(\overline{X}_{et}, \mathbf{Q}_p(r))) = 0$ *for* $r \neq i/2$.
(b) $H^2_{cont}(G_K, H^i(\overline{X}_{et}, \mathbf{Q}_p(r))) = 0$ *for* $r \neq i/2 + 1$.
(c) $H^1_{cont}(G_K, H^i(\overline{X}_{et}, \mathbf{Q}_p(r))) = 0$ *for* $r \neq i/2, i/2 + 1$.
(d) $H^{i+1}(X_{et}, \mathbf{Q}_p(r)) = 0$ *for* $r \neq i/2, (i+1)/2, i/2 + 1$.
(e) $H^{i+1}(\mathfrak{X}_{et}, \mathbf{Q}_p(r)) = H^{i+1}(Y_{et}, \mathbf{Q}_p(r)) = 0$ *for* $r \neq i/2, (i+1)/2$.
(here $\mathfrak{X}$ is a model of X, proper and smooth over $\mathrm{Spec}(\mathcal{O}_K)$*, and Y its special fibre).*
In fact, the statements (a)–(d) are true if X has only a potentially good reduction.

Proof. (1) This is a standard weight argument. Putting $V := H^i(\overline{X}_{et}, \mathbf{Q}_p(r))$, we see that V^{I_K} has no invariants (resp. coinvariants) by the Frobenius for $r < 0$ or $r > i/2$, by (1.5.4), proving (a). By local duality ([Se 1], II.5.2.2), $H^2_{cont}(G_K, V)$ is dual to $H^0(G_K, V^*(1))$. The statement (b) follows from (1.5.5), as $V^*(1)^{I_{K'}} = (V_{I_{K'}})^*(1)$ has no invariants under Frobenius for the indicated values of r. For (c), we use the formula for the local Euler characteristic ([Se 1], II.5.4.17): $\dim H^0(G_K, V) - \dim H^1(G_K, V) + \dim H^2(G_K, V) = 0$.
(2) In (a)–(d), all the groups involved satisfy Galois descent for finite Galois extensions L/K. This means that all the statements for X with a potentially good reduction are reduced, by a suitable base change, to

the case of a good reduction. The argument for (a)–(c) is then the same as in (1), simplified by the fact that V is unramified and pure of weight $i-2r$. (d) follows from the Hochschild-Serre spectral sequence and (a)–(c). For (e), observe that the two cohomology groups are isomorphic by the proper base change (2.4.1). As in (1.3), the Hochschild-Serre spectral sequence gives an exact sequence

$$0 \longrightarrow H^1_{cont}(G_k, H^i(\overline{Y}_{et}, \mathbf{Q}_p(r))) \longrightarrow H^{i+1}(Y_{et}, \mathbf{Q}_p(r)) \longrightarrow H^{i+1}(\overline{Y}_{et}, \mathbf{Q}_p(r))^{G_k} \longrightarrow 0$$

and we apply the vanishing statements (a) (resp. (c)) for the first (resp. the third) term.

(1.8) If π is *semistable*, then the specialization maps sp^i can be analysed using the spectral sequence of vanishing cycles and an explicit construction of the monodromy operator N [RZ], [Il 2]. For example, if the purity conjecture (1.5.2) for the monodromy filtration on $H^i(\overline{X}_{et}, \mathbf{Q}_p)$ holds, then sp^i is surjective.

(1.9) The discussion in (1.3.1-6) remains valid if one replaces everywhere $\mathbf{Q}_p$ by $\mathbf{Z}_p$ (and V^i_s (resp. V^i_η) by $T^i_s = H^i(\overline{Y}_{et}, \mathbf{Z}_p)$ (resp. $T^i_\eta = H^i(\overline{X}_{et}, \mathbf{Z}_p)$)). In particular,

$$\mathrm{Im}\left[\delta \circ j^* : F^1 H^{i+1}(\mathfrak{X}_{et}, \mathbf{Z}_p(r)) \longrightarrow H^1_{cont}(G_K, T^i_\eta(r))\right] \subseteq H^1_{ur}(G_K, T^i_\eta(r))$$

If π is proper and smooth, then T^i_η is unramified and $sp^i : T^i_s \xrightarrow{\sim} T^i_\eta$ is an isomorphism (for all $i \geq 0$). Consequently,

$$gr^0_F(j^*) : \left(T^{i+1}_s(r)\right)^{G_k} \longrightarrow \left(T^{i+1}_\eta(r)\right)^{G_K}$$

is an isomorphism and

$$gr^1_F(j^*) : H^1_{cont}(G_k, T^i_s(r)) \longrightarrow H^1_{cont}(G_K, T^i_\eta(r))$$

is injective, with image equal to $H^1_{ur}(G_K, T^i_\eta(r))$. This implies that the restriction map

$$j^* : H^{i+1}(\mathfrak{X}_{et}, \mathbf{Z}_p(r)) \longrightarrow H^{i+1}(X_{et}, \mathbf{Z}_p(r)) \tag{1.9.1}$$

is injective and that

$$F^k H^{i+1}(\mathfrak{X}_{et}, \mathbf{Z}_p(r)) = (j^*)^{-1}\left(F^k H^{i+1}(X_{et}, \mathbf{Z}_p(r))\right),$$

for every $k \geq 0$. In particular,

$$\mathrm{Im}(j^*) \cap F^2 H^{i+1}(X_{et}, \mathbf{Z}_p(r)) = 0. \tag{1.9.2}$$

2. Regulators

(2.1) ***K*-theory.** In the situation of (1.1), suppose that $\pi : \mathfrak{X} \longrightarrow \mathrm{Spec}(\mathcal{O}_K)$ is of finite type. Then, for $r > (i+1)/2$, there are étale Chern classes ([Gi])

$$\begin{array}{ccc} K_{2r-i-1}(\mathfrak{X}) & \xrightarrow{j^*} & K_{2r-i-1}(X) \\ \downarrow{\scriptstyle c_{i+1,r}} & & \downarrow{\scriptstyle c_{i+1,r}} \\ H^{i+1}(\mathfrak{X}_{et}, \mathbf{Z}/p^n\mathbf{Z}(r)) & \xrightarrow{j^*} & H^{i+1}(X_{et}, \mathbf{Z}/p^n\mathbf{Z}(r)) \end{array}$$

which factor through $K_{2r-i-1}(\mathfrak{X}, \mathbf{Z}/p^n\mathbf{Z})$ (resp. $K_{2r-i-1}(X, \mathbf{Z}/p^n\mathbf{Z})$). Taking projective limit and using (1.2.3), we obtain a commutative diagram

$$\begin{array}{ccc} K_{2r-i-1}(\mathfrak{X}) \otimes \mathbf{Q} & \xrightarrow{j^*} & K_{2r-i-1}(X) \otimes \mathbf{Q} \\ \downarrow{\scriptstyle ch_{i+1,r}} & & \downarrow{\scriptstyle ch_{i+1,r}} \\ H^{i+1}(\mathfrak{X}_{et}, \mathbf{Q}_p(r)) & \xrightarrow{j^*} & H^{i+1}(X_{et}, \mathbf{Q}_p(r)) \end{array} \tag{2.1.1}$$

with $ch_{i+1,r} = (-1)^{r-1}/(r-1)! \cdot c_{i+1,r}$ (the scalar factor ensures that the Chern character ch is compatible with products). This map was denoted by r_p in the Introduction.

(2.2) Theorem. *If $\pi : \mathfrak{X} \longrightarrow \mathrm{Spec}(\mathcal{O}_K)$ is proper and $r > (i+1)/2$, then*

$$\mathrm{Im}\left[K_{2r-i-1}(\mathfrak{X}) \otimes \mathbf{Q} \xrightarrow{ch \circ j^*} H^{i+1}(X_{et}, \mathbf{Q}_p(r))\right] = 0$$

Proof. In the commutative diagram (2.1.1), the group $H^{i+1}(\mathfrak{X}_{et}, \mathbf{Q}_p(r))$ vanishes, by Proposition 1.4.1.

(2.3) Algebraic cycles. Let us now consider the "central point" $r = (i+1)/2$. Assume that $\pi : \mathfrak{X} \longrightarrow \mathrm{Spec}(\mathcal{O}_K)$ is a separated morphism of finite type with $\mathfrak{X}$ regular. Cycle classes define a commutative diagram

$$\begin{array}{ccc} CH^r(\mathfrak{X}) \otimes \mathbf{Q} & \xrightarrow{j^*} & CH^r(X) \otimes \mathbf{Q} \\ \Big\downarrow{\scriptstyle cl_{\mathfrak{X}}} & & \Big\downarrow{\scriptstyle cl_X} \\ H^{2r}(\mathfrak{X}_{et}, \mathbf{Q}_p(r)) & \xrightarrow{j^*} & H^{2r}(X_{et}, \mathbf{Q}_p(r)) \end{array} \tag{2.3.1}$$

The vertical maps can be defined in several ways (see [Grot], [Sa] in the smooth case); one possibility is to use K-theory:

$$cl_{\mathfrak{X}} : CH^r(\mathfrak{X}) \otimes \mathbf{Q} \xrightarrow{\sim} K_0(\mathfrak{X})^{(r)}_{\mathbf{Q}} \longrightarrow H^{2r}(\mathfrak{X}_{et}, \mathbf{Q}_p(r))$$
$$cl_X : CH^r(X) \otimes \mathbf{Q} \xrightarrow{\sim} K_0(X)^{(r)}_{\mathbf{Q}} \longrightarrow H^{2r}(X_{et}, \mathbf{Q}_p(r))$$

These isomorphisms are due to Grothendieck ([SGA 6], Exp. XIV, 4.1) and Soulé ([So 2], Thm. 4(iv); [GiSo], Thm. 8.2); the maps from K-theory to étale cohomology are given by $(-1)^{r-1}/(r-1)! \cdot c_r$, where c_r is the r-th Chern class of a vector bundle ([SGA 5], Exp. VII, Prop. 3.4; [Ja 1], Thm. 6.12).

Denote the subgroups of homologically trivial cycles by

$$(CH^r(X) \otimes \mathbf{Q})_0 := \mathrm{Ker}\left[CH^r(X) \otimes \mathbf{Q} \longrightarrow H^{2r}(\overline{X}_{et}, \mathbf{Q}_p(r))\right]$$
$$(CH^r(\mathfrak{X}) \otimes \mathbf{Q})_0 := \mathrm{Ker}\left[CH^r(\mathfrak{X}) \otimes \mathbf{Q} \longrightarrow H^{2r}((\mathfrak{X} \otimes_{\mathcal{O}_K} \mathcal{O}_K^{ur})_{et}, \mathbf{Q}_p(r))\right]$$

Suppose, furthermore, that π is proper. Then

$$H^{2r}((\mathfrak{X} \otimes_{\mathcal{O}_K} \mathcal{O}_K^{ur})_{et}, \mathbf{Q}_p(r)) \xrightarrow{\sim} H^{2r}(\overline{Y}_{et}, \mathbf{Q}_p(r)) = V_s^{2r}(r) \tag{2.3.2}$$

by the proper base change (I.2.4.1).

(2.4) Theorem. *Assume $\pi : \mathfrak{X} \longrightarrow \mathrm{Spec}(\mathcal{O}_K)$ is proper and $\mathfrak{X}$ is regular. Then*

$$\mathrm{Im}\left[(CH^r(\mathfrak{X}) \otimes \mathbf{Q})_0 \xrightarrow{j^*} (CH^r(X) \otimes \mathbf{Q})_0 \xrightarrow{cl_X} H^{2r}(X_{et}, \mathbf{Q}_p(r))\right] = 0.$$

Proof. By (2.3.2) and Proposition 1.4.2, we have $(CH^r(\mathfrak{X}) \otimes \mathbf{Q})_0 = \mathrm{Ker}(cl_{\mathfrak{X}})$. We conclude by (2.3.1).

(2.5) As in 1.5, assume that π is proper with smooth generic fibre π_η. A folklore conjecture states that

$$cl_X((CH^r(X) \otimes \mathbf{Q})_0) = 0. \tag{2.5.1}$$

According to (1.3.3.2),

$$(CH^r(X) \otimes \mathbf{Q})_0 = cl_X^{-1}(F^1 H^{2r}(X_{et}, \mathbf{Q}_p(r))),$$

which means that (2.5.1) is equivalent to

$$\mathrm{Im}(cl_X) \cap F^1 H^{2r}(X_{et}, \mathbf{Q}_p(r)) = 0. \tag{2.5.2}$$

Note that the purity conjecture for the monodromy filtration on $H^{2r-1}(\overline{X}_{et}, \mathbf{Q}_p)$ would imply that

$$F^1 H^{2r}(X_{et}, \mathbf{Q}_p(r)) = F^2 H^{2r}(X_{et}, \mathbf{Q}_p(r)), \tag{2.5.3}$$

by (1.3.3.2) and Proposition 1.7.1(c).

(2.6) Proposition. *Let π be proper and π_η smooth. The conjecture (2.5.1) holds in the following cases.*
(i) *X is equidimensional of dimension $\dim(X) = r$ (zero cycles).*
(ii) *$r = 1$ (divisors).*
(iii) *$\mathfrak{X}$ is regular and the map $j^* : (CH^r(\mathfrak{X}) \otimes \mathbf{Q})_0 \longrightarrow (CH^r(X) \otimes \mathbf{Q})_0$ is surjective.*
(iv) *π is proper and smooth (good reduction).*

Proof. (i) [Be 3, 2.2], [Ra, Prop. 3.2]. (ii) There is a canonical isomorphism $\alpha : A(K) \otimes \mathbf{Q} \xrightarrow{\sim} (CH^1(X) \otimes \mathbf{Q})_0$, where $A = \mathrm{Pic}^0(X/F)$. The composition $cl_X \circ \alpha$ is the descent map

$$A(K) \otimes \mathbf{Q} \longrightarrow A(K)\widehat{\otimes}\mathbf{Q}_p \hookrightarrow H^1_{cont}(G_K, V_p(A)).$$

The group $A(K)$ has a subgroup of finite index, isomorphic to the direct sum of several copies of $\mathbf{Z}_\ell$. As $\ell \neq p$, this implies that the completed tensor product $A(K)\widehat{\otimes}\mathbf{Q}_p$ vanishes.
(iii) This follows from Theorem 2.4.
(iv) In this case the map j^* is surjective, since sp^{2r} is an isomorphism. We conclude by (iii).

(2.7) Consider the base change map $u : X \otimes_K L \longrightarrow X$ for a finite field extension L/K of degree d. There are contravariant (resp. covariant) maps u^* (resp. u_*) between Chow groups, étale cohomology and the terms of the Hochschild-Serre spectral sequences of X resp. $X \otimes_K L$. They satisfy the usual identitites:

$$u_* \circ u^* = d \cdot 1; \qquad u^* \circ u_* = \sum_{g \in G(L/K)} g \quad (if\ L/K\ is\ Galois)$$

This implies that the restriction maps

$$u^* : CH^r(X) \otimes \mathbf{Z}[1/d] \longrightarrow CH^r(X \otimes_K L) \otimes \mathbf{Z}[1/d] \tag{2.7.1}$$

and

$$u^* : F^j H^m(X_{et}, \mathbf{Z}_p(r)) \otimes \mathbf{Z}[1/d] \longrightarrow F^j H^m((X \otimes_K L)_{et}, \mathbf{Z}_p(r)) \otimes \mathbf{Z}[1/d] \tag{2.7.2}$$

are both injective. Moreover, for L/K Galois, the image of u^* coincides with $G(L/K)$-invariants of the R.H.S. It follows from the injectivity of (2.7.2) that the conditions (iii) and (iv) of Proposition 2.6 can be replaced by
(iii') There is a finite extension L/K and a proper regular model $\mathfrak{X}_L$ of $X \otimes_K L$ over $\mathrm{Spec}(\mathcal{O}_L)$ such that the map

$$j^* : (CH^r(\mathfrak{X}_L) \otimes \mathbf{Q})_0 \longrightarrow (CH^r(X \otimes_K L) \otimes \mathbf{Q})_0$$

is surjective.
(iv') X has potentially good reduction.

(2.8) Proposition. *If L/K is a finite extension of degree prime to p such that $X \otimes_K L$ has good reduction, then*

$$\mathrm{Im}(cl_X) \cap F^2 H^{2n}(X_{et}, \mathbf{Z}_p(n)) = 0.$$

Proof. For $X \otimes_K L$ this follows from (1.9.2); we can descend to X thank to the injectivity of (2.7.2).

(2.9) It follows from a recent work of K. Künnemann [Kn] that (2.5.1) holds for abelian varieties with potentially totally toric reduction.

Another (partial) result in this direction is proved in ([GrSc], Prop. 7.2), where the authors show that the class of a modified diagonal on the triple product of a semistable curve lies in the image of j^*.

(2.10) Proof of Theorems A and D(1-2). Theorem A is a combination of Theorems 2.2 and 2.4. Theorem D(1-2) follows from Proposition 2.6 and remarks in 2.7.

III. Local situation at p

In this chapter we prove a comparison result (Theorem 3.2) that relates syntomic cohomology (in the case of good reduction) to the Bloch-Kato exponential map. Similar result (Theorem 5.2) is proved for cohomology with $\mathbf{Z}_p$-coefficients, under more restrictive hypotheses. As a consequence we obtain upper bounds on the images of p-adic regulators.

1. p-Adic Galois representations

In this section we recall Fontaine's machinery relating crystalline and étale cohomology and also the exponential map of Bloch-Kato.

(1.1) Notation. Let k be a perfect field of characteristic $char(k) = p > 0$; $W = W(k)$ the ring of Witt vectors of k; $K_0 = W[p^{-1}]$ the fraction field of W; σ the lifting of the absolute Frobenius $a \mapsto a^p$ to W and K_0; K a finite totally ramified extension of K_0. We fix an algebraic closure $\overline{K}$ of K and write G for the Galois group $G(\overline{K}/K)$. For any scheme S over Spec(W) put $S_n = S \otimes_W W_n$.

(1.2) Fontaine [Fo 2, 4] constructed topological K_0-algebras B_{cris}, B_{dR} equipped with the following structure:

(i) A continuous K_0-linear action of G on B_{cris} and B_{dR}.

(ii) A commutative diagram of (G-equivariant and injective) continuous K_0-algebra homomorphisms

$$\begin{array}{ccccc} K_0^{ur} & \hookrightarrow & B_{cris}^+ & \hookrightarrow & B_{cris} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{K} & \hookrightarrow & B_{dR}^+ & \hookrightarrow & B_{dR} \end{array}$$

(iii) A σ-linear continuous bijective $\mathbf{Q}_p$-algebra homomorphism $f : B_{cris} \longrightarrow B_{cris}$, commuting with G and preserving B_{cris}^+.

(iv) An exhaustive and separated decreasing filtration $F^i B_{dR}$ (by $\overline{K}[G]$-submodules of B_{dR}) with $B_{dR}^+ = F^0 B_{dR}$. Warning: $B_{cris}^+ \neq B_{dR}^+ \cap B_{cris}$.

(v) A $\mathbf{Q}_p[G]$-equivariant injective map

$$\mathbf{Q}_p(1) = \Big(\varprojlim_n \mu_{p^n}(\overline{K})\Big) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \hookrightarrow B_{cris}^+$$

such that, for any $0 \neq t \in \mathbf{Q}_p(1) \subset B_{cris}^+ \subset B_{dR}^+$, we have $f(t) = pt$, $F^n B_{dR} = t^n B_{dR}^+$ $(n \in \mathbf{Z})$, $B_{cris} = B_{cris}^+[t^{-1}]$ and $B_{dR} = B_{dR}^+[t^{-1}]$.

(vi) $(B_{cris})^G = \big(B_{cris}^+\big)^G = K_0, \qquad (B_{dR})^G = \big(B_{dR}^+\big)^G = K.$

(1.3) These rings have a crystalline interpretation (see Appendix for the notation). For a subextension L/K of $\overline{K}/K$ and $n, r \geq 0$, put

$$\begin{aligned} B_{n,L} :=& H^0((\mathrm{Spec}(\mathcal{O}_L)/W_n)_{\mathrm{cris}}, \mathcal{O}_{\mathrm{Spec}(\mathcal{O}_L)/W_n}) = H^0(\mathrm{Spf}(\mathcal{O}_L)_{syn-et}, \mathcal{O}_n^{\mathrm{cris}}) \\ J_{n,L}^{[r]} :=& H^0((\mathrm{Spec}(\mathcal{O}_L)/W_n)_{\mathrm{cris}}, J_{\mathrm{Spec}(\mathcal{O}_L)/W_n}^{[r]}) = H^0(\mathrm{Spf}(\mathcal{O}_L)_{syn-et}, J_n^{[r]}) \end{aligned}$$

(here $\mathcal{O}_L$ is the ring of integers of L and Spf$(\mathcal{O}_L)$ denotes the formal p-adic completion of Spec$(\mathcal{O}_L)$). $B_{n,L}$ is a W_n-algebra and $J_{n,L}^{[r]}$ an ideal of $B_{n,L}$ with a divided power structure. Put

$$\begin{aligned} B_n &:= \varinjlim_L B_{n,L}, & A_{cris} &:= \varprojlim_n B_n \\ J_{B_n}^{[r]} &:= \varinjlim_L J_{n,L}^{[r]}, & J_{A_{cris}}^{[r]} &:= \varprojlim_n J_{B_n}^{[r]}, \end{aligned}$$

where L runs through all *finite* extesions of K inside $\overline{K}$. By [FM, II.2.2] we have

$$B_n = B_{n,\overline{K}}, \qquad J^{[r]}_{B_n} = J^{[r]}_{n,\overline{K}}$$

It is proved in ([Fo 3], Thm. 1(ii)) that

$$B_n = A_{cris} \otimes \mathbf{Z}/p^n\mathbf{Z} \qquad (n \geq 0)$$

and the same argument shows that

$$J^{[r]}_{B_n} = J^{[r]}_{A_{cris}} \otimes \mathbf{Z}/p^n\mathbf{Z} \qquad (n \geq 0)$$

The rings B^+_{cris}, B^+_{dR} are defined as

$$B^+_{cris} := A_{cris} \otimes \mathbf{Q} = A_{cris} \otimes_W K_0$$
$$B^+_{dR} := \varprojlim_r \left(\mathbf{Q} \otimes \varprojlim_n \left(B_n/J^{[r]}_{B_n}\right)\right) = \varprojlim_r \left(\mathbf{Q} \otimes \left(A_{cris}/J^{[r]}_{A_{cris}}\right)\right)$$

The embedding $B^+_{cris} \hookrightarrow B^+_{dR}$ is induced by the canonical maps

$$A_{cris} = \varprojlim_n B_n \longrightarrow \varprojlim_n \left(B_n/J^{[r]}_n\right)$$

The ring B_{dR} has a canonical topology, defined in [Fo 4]. Its subrings $A_{cris}, B_{cris}, B^+_{dR}$ are equipped with the induced topology. The topology induced on $\overline{K}, A_{cris}$ is coarser than the p-adic topology.
Recall from [FM, III.3.1] that there are homomorphisms

$$\mu_{p^n} \longrightarrow s_n(1) \longrightarrow J^{[1]}_n$$

of sheaves on $\mathrm{Spec}(W_n)_{syn}$. They define maps $\mu_{p^n}(L) \longrightarrow J^{[1]}_{n,L}$ and, passing to the limit, homomorphisms

$$\mathbf{Z}_p(1) \longrightarrow J^{[1]}_{A_{cris}} \hookrightarrow A_{cris}, \qquad \mathbf{Q}_p(1) \longrightarrow B^+_{cris} \tag{1.3.1}$$

If $t \neq 0$ is any element in the image of $\mathbf{Q}_p(1)$, then $B_{cris} = B^+_{cris}[t^{-1}]$ and $B_{dR} = B^+_{dR}[t^{-1}]$.
The action of the crystalline Frobenius f on $B_{n,L}$ defines, in the limit, the operator $f : B^+_{cris} \longrightarrow B^+_{cris}$. By definition of $s_n(1)$, $f = p$ on $\mathbf{Q}_p(1) \subset B^+_{cris}$; it extends to B_{cris} by putting $f(t^{-1}) = p^{-1}t^{-1}$.

(1.4) Denote by *Rep*(G) the abelian category of p-adic representations of G (= vector spaces of finite dimension over $\mathbf{Q}_p$ with a continuous linear action of G) and by MF_K the additive category of filtered Dieudonné modules over K [1]. Recall that an object of MF_K is a vector space D of finite dimension over K_0, equipped with a σ-linear bijective endomorphism $f : D \longrightarrow D$ and with a separated exhaustive decreasing filtration F^iD_K of $D_K = D \otimes_{K_0} K$ by K-subspaces. Morphisms in MF_K are K_0-linear maps compatible with f and the filtrations F^i.
Fontaine [Fo 1, 4] defines a functor $V : MF_K \longrightarrow$ *Rep*(G) by

$$V(D) = (D \otimes_{K_0} B_{cris})^{f=1} \cap F^0\left(D_K \otimes_K B_{dR}\right) \tag{1.4.1}$$

Here $f = f \otimes f$ on $D \otimes_{K_0} B_{cris}$ and

$$F^i\left(D_K \otimes_K B_{dR}\right) = \sum_{a+b=i} \mathrm{Im}\left(F^aD_K \otimes_K F^bB_{dR} \hookrightarrow D_K \otimes_K B_{dR}\right). \tag{1.4.2}$$

For $V \in \mathrm{Ob}(Rep(G))$ he defines

$$D(V) = \left(V \otimes_{\mathbf{Q}_p} B_{cris}\right)^G, \qquad D_{dR}(V) = \left(V \otimes_{\mathbf{Q}_p} B_{dR}\right)^G \tag{1.4.3}$$

[1] A terminology suggested in [Og] is "Fontaine modules".

$D(V)$ is a vector space over $K_0 = (B_{cris})^G$ with a σ-linear action of $f = 1 \otimes f$. $D_{dR}(V)$ is a vector space over $K = (B_{dR})^G$ with a decreasing filtration by K-subspaces

$$F^i D_{dR}(V) = \big(V \otimes_{\mathbf{Q}_p} F^i B_{dR}\big)^G \tag{1.4.4}$$

It is shown in [Fo 1] that the canonical map $D(V) \otimes_{K_0} K \longrightarrow D_{dR}(V)$ is injective and that

$$\dim_{K_0}(D(V)) \leq \dim_K(D_{dR}(V)) \leq \dim_{\mathbf{Q}_p}(V) \tag{1.4.5}$$

The full subcategory of *crystalline representations* $Rep_{cris}(G) \subset Rep(G)$ consists of those V which satisfy

$$\dim_{K_0}(D(V)) = \dim_{\mathbf{Q}_p}(V) \qquad (\overset{(1.4.5)}{\Longrightarrow} = \dim_K(D_{dR}(V)))$$

Then D becomes a functor $D : Rep_{cris}(G) \longrightarrow MF_K$ with

$$D(V)_K = D_{dR}(V), \qquad F^i D(V)_K = F^i D_{dR}(V).$$

(1.5) The full subcategory of *admissible* filtered Dieudonné modules $MF_K^{ad} \subset MF_K$ is, by definition, the essential image of D. According to a fundamental result of Fontaine ([Fo 1], Thm. 3.6.5), the functors V and D are quasi-inverse to each other and define an equivalence of categories $Rep_{cris}(G) \approx MF_K^{ad}$.

More precisely, for $V \in \mathrm{Ob}(Rep_{cris}(G))$ and $D = D(V)$ (resp. $D \in \mathrm{Ob}(MF_K^{ad})$ and $V = V(D)$), there is a canonical isomorphism

$$D \otimes_{K_0} B_{cris} \xrightarrow{\sim} V \otimes_{\mathbf{Q}_p} B_{cris}, \tag{1.5.1}$$

compatible with the action of G and f. Here $g \in G$ acts by $1 \otimes g$ (resp. $g \otimes g$) on the L.H.S. (resp. R.H.S.) and f acts as $f \otimes f$ (resp. $1 \otimes f$) on the L.H.S. (resp. R.H.S.). The induced isomorphism

$$D_K \otimes_K B_{dR} = D \otimes_{K_0} B_{dR} \xrightarrow{\sim} V \otimes_{\mathbf{Q}_p} B_{dR} \tag{1.5.2}$$

is compatible with the filtrations: $F^i(D_K \otimes_K B_{dR})$, defined in (1.4.2), corresponds to $V \otimes_{\mathbf{Q}_p} F^i B_{dR}$.

The functors $D \mapsto F^i D_K$, $D \mapsto D_K / F^i D_K$ are exact on MF_K^{ad} and (1.5.2) induces isomorphisms

$$F^i D_K \xrightarrow{\sim} \big(V \otimes_{\mathbf{Q}_p} F^i B_{dR}\big)^G, \qquad D_K / F^i D_K \xrightarrow{\sim} \big(V \otimes_{\mathbf{Q}_p} (B_{dR}/F^i B_{dR})\big)^G \tag{1.5.3}$$

(1.6) Both $Rep_{cris}(G)$ and MF_K^{ad} are $\mathbf{Q}_p$-linear rigid tensor categories and the functors D, V preserve this structure. For example, we have

$$\begin{aligned} D(V_1 \otimes_{\mathbf{Q}_p} V_2) &= D(V_1) \otimes_{K_0} D(V_2) \qquad (\text{with } f = f \otimes f) \\ D_{dR}(V_1 \otimes_{\mathbf{Q}_p} V_2) &= D_{dR}(V_1) \otimes_K D_{dR}(V_2) \end{aligned} \tag{1.6.1}$$

with the convolution filtration (defined as in (1.4.2)) and

$$\begin{aligned} D(V^*) &= \mathrm{Hom}_{K_0}(D(V), K_0) \qquad (\text{with } f(d^*) = \sigma \circ d^* \circ f^{-1}) \\ D_{dR}(V^*) &= \mathrm{Hom}_K(D_{dR}(V), K) \end{aligned} \tag{1.6.2}$$

with $F^i(\mathrm{Hom}_K(D_{dR}(V), K)) = (F^{1-i} D_{dR}(V))^\perp$.

The unit object of $Rep(G)$ (resp. MF_K^{ad}) is $\mathbf{Q}_p$ with trivial action of G (resp. $D(\mathbf{Q}_p) = K_0$ with $f = \sigma$, $D_{dR}(\mathbf{Q}_p) = K = F^0 \supset F^1 = 0$). The representation on p-power roots of unity

$$\mathbf{Q}_p(1) = \mathbf{Z}_p(1) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p = \Big(\varprojlim_n \mu_{p^n}(\overline{K})\Big) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

is crystalline, hence all its tensor powers $\mathbf{Q}_p(n) = \mathbf{Q}_p(1)^{\otimes n}$ ($n \in \mathbf{Z}$) are crystalline as well. The corresponding filtered modules are

$$\begin{aligned} D(\mathbf{Q}_p(n)) &= K_0\, e_{-n} \quad \text{with } f = p^{-n}\sigma \\ D_{dR}(\mathbf{Q}_p(n)) &= K\, e_{-n} = F^{-n} \supset F^{-n+1} = 0 \end{aligned} \tag{1.6.3}$$

If V is a crystalline representation, then its Tate twists $V(n) = V \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(n)$ ($n \in \mathbf{Z}$) are crystalline as well and (1.6.1-3) give

$$\begin{aligned} D(V(n)) &= D(V) \otimes_{K_0} K_0\, e_{-n} \xrightarrow{\sim} D(V) \qquad (\text{with } f \text{ replaced by } p^{-n} f) \\ D_{dR}(V(n)) &= D_{dR}(V) \otimes_K K\, e_{-n} \xrightarrow{\sim} D_{dR}(V) \end{aligned} \tag{1.6.4}$$

with $F^i D_{dR}(V(n)) = F^{i+n} D_{dR}(V) \otimes_K K\, e_{-n} \xrightarrow{\sim} F^{i+n} D_{dR}(V)$. In fact, the formulas (1.6.4) can be used to define Tate twists $D\{n\}$ for objects of MF_K.

(1.7) Bloch-Kato theory. Let V be a crystalline representation of G. For $q \geq 0$, define

$$H^q_f(G, V) := \mathrm{Ext}^q_{Rep_{cris}(G)}(\mathbf{Q}_p, V) \quad \left(\xrightarrow{\sim} \mathrm{Ext}^q_{MF_K^{ad}}(D(\mathbf{Q}_p), D(V))\right) \tag{1.7.1}$$

These Yoneda Ext-groups can be interpreted in terms of derived functors as follows ([Hu], Thm. 2.6): the Ind-category $\mathcal{A} = \mathrm{Ind}(Rep_{cris}(G))$ (see [SGA 4], Exp. I, 8.2.1) is an abelian category with enough injectives, the functor $F : \mathcal{A} \longrightarrow (Ab)$ given by

$$F((A_i)_{i \in I}) = \varinjlim_I \mathrm{Hom}_{Rep_{cris}(G)}(\mathbf{Q}_p, V)$$

(for any small filtered category I) is left exact and, for every $q \geq 0$,

$$H^q_f(K, -) = R^q F|_{Rep_{cris}(G)} \tag{1.7.2}$$

It is shown in [BK] (see also [FoPR], Prop. 3.3.7) that

$$\begin{aligned} &H^0_f(K, V) = H^0(G, V) = V^G \\ &H^1_f(K, V) = \mathrm{Ker}\left[H^1_{cont}(G, V) \longrightarrow H^1_{cont}(G, V \otimes_{\mathbf{Q}_p} B_{cris})\right] \\ &H^q_f(K, V) = 0 \qquad (q \geq 2) \end{aligned} \tag{1.7.3}$$

More precisely, $H^q_f(K, V)$ are cohomology groups of the complex

$$C^{\cdot}_f(V) = \left[D(V) \xrightarrow{(1-f, \mathrm{can})} D(V) \oplus D_{dR}(V)/F^0\right] \tag{1.7.4}$$

The isomorphism

$$V^G \xrightarrow{\sim} H^0(C^{\cdot}_f(V)) = D(V)^{f=1} \cap F^0 D_{dR}(V)$$

follows from (1.2.(vi)), (1.4.1); the isomorphism $H^1_f(K, V) \xrightarrow{\sim} H^1(C^{\cdot}_f(V))$ can be described as follows [Ne 2]: given an extension in $Rep_{cris}(G)$

$$0 \longrightarrow V \longrightarrow E \longrightarrow \mathbf{Q}_p \longrightarrow 0,$$

chose a K_0-linear section s of the surjection $D(E) \longrightarrow D(\mathbf{Q}_p) = K_0$ and associate to (the extension class of) E the class of $[(f-1)s(1), -s(1) \bmod F^0 D_{dR}(E)] \in C^1_f(V)$ in $H^1(C^{\cdot}_f(V))$, which does not depend on the choice of s; observe that $D_{dR}(V)/F^0 \xrightarrow{\sim} D_{dR}(E)/F^0$ (*cf.* [Ne 2], Prop. 1.21).

The original approach of Bloch and Kato (which makes sense for representation V that are de Rham, i.e. those satisfying $\dim_{\mathbf{Q}_p}(V) = \dim_K(D_{dR}(V))$) was based on the exact sequence ([BK], 1.17.2)

$$0 \longrightarrow \mathbf{Q}_p \longrightarrow B_{cris} \xrightarrow{(1-f, \mathrm{can})} B_{cris} \oplus B_{dR}/B^+_{dR} \longrightarrow 0, \tag{1.7.5}$$

which, on tensoring with V, gives a commutative diagram with exact rows (*cf.* 1.5)

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & V & \longrightarrow & V\otimes_{\mathbf{Q}_p} B_{cris} & \xrightarrow{(1-1\otimes f,\mathrm{can})} & (V\otimes_{\mathbf{Q}_p} B_{cris}) & \bigoplus & (V\otimes_{\mathbf{Q}_p}(B_{dR}/B_{dR}^+)) & \longrightarrow & 0\\
 & & \Vert & & \downarrow\wr & & & & \downarrow\wr & & \\
0 & \longrightarrow & V & \longrightarrow & D(V)\otimes_{K_0} B_{cris} & \xrightarrow{(1-f\otimes f,\mathrm{can})} & (D(V)\otimes_{K_0} B_{cris}) & \bigoplus & \frac{D_{dR}(V)\otimes_K B_{dR}}{F^0(D_{dR}(V)\otimes_K B_{dR})} & \longrightarrow & 0
\end{array}
\tag{1.7.6}
$$

The map $(1-f,\mathrm{can})$ in (1.7.5) admits a continuous section ([BK], 1.18), thus (1.7.6) gives rise to an exact cohomology sequence ([BK], 3.8.4)

$$
\begin{aligned}
0 \longrightarrow & H^0(G,V) \longrightarrow D(V) \xrightarrow{(1-f,\mathrm{can})} D(V)\oplus D_{dR}(V)/F^0 \xrightarrow{\partial}\\
&\longrightarrow \mathrm{Ker}\left[H^1_{cont}(G,V)\longrightarrow H^1_{cont}(G,V\otimes_{\mathbf{Q}_p} B_{cris})\right] \longrightarrow 0
\end{aligned}
\tag{1.7.7}
$$

The coboundary map ∂ is the *exponential map* of Bloch-Kato

$$
\exp_V : D(V)\oplus D_{dR}(V)/F^0 \longrightarrow H^1_f(K,V) \tag{1.7.8}
$$

Replacing V by $V(r)$ (for $r\in\mathbf{Z}$), we get exact sequences (*cf.* (1.6.4))

$$
0\longrightarrow V(r)\longrightarrow D(V)\otimes_{K_0} B_{cris} \xrightarrow{(1-p^{-r}f\otimes f,\mathrm{can})} (D(V)\otimes_{K_0} B_{cris}) \bigoplus \frac{D_{dR}(V)\otimes_K B_{dR}}{F^r(D_{dR}(V)\otimes_K B_{dR})}\longrightarrow 0 \tag{1.7.9}
$$

$$
0\longrightarrow H^0(G,V(r))\longrightarrow D(V)\xrightarrow{(1-p^{-r}f,\mathrm{can})} D(V)\oplus D_{dR}(V)/F^r \longrightarrow H^1_f(K,V(r))\longrightarrow 0 \tag{1.7.10}
$$

If $T\subset V$ is a $\mathbf{Z}_p$-lattice stable by G_K, then $H^1_f(K,T)$ is defined as the preimage of $H^1_f(K,V)$ in $H^1(K,T)$.

2. Geometric situation

In this section we recall Fontaine's crystalline conjecture and its "syntomic" proof. We then make preparations for the proof of comparison results of Section 3.

(2.1) In the notation of (1.1), let $\mathfrak{X}\longrightarrow \mathrm{Spec}(\mathcal{O}_K)$ be a proper smooth morphism with a generic (resp. special) fibre $X=\mathfrak{X}\otimes_{\mathcal{O}_K} K$ (resp. $Y=\mathfrak{X}\otimes_{\mathcal{O}_K} k$). Write $\widehat{\mathfrak{X}}=\varprojlim_n(\mathfrak{X}\otimes\mathbf{Z}/p^n\mathbf{Z})$ (resp. $\widehat{\overline{\mathfrak{X}}}$) for the p-adic completion of $\mathfrak{X}$ (resp. of $\overline{\mathfrak{X}}=\mathfrak{X}\otimes_{\mathcal{O}_K}\mathcal{O}_{\overline{K}}$) and put $\overline{X}=X\otimes_K\overline{K}$.
We recall from Appendix various syntomic-étale ([FM], III.4) and étale topoi and morphisms between them:

$$
\widehat{\mathfrak{X}}^{\sim}_{syn-et}\longrightarrow \mathfrak{X}^{\sim}_{syn-et}\longleftarrow X^{\sim}_{et}, \tag{2.1.1}
$$

projective limits of (2.1.1) for $\mathfrak{X}\otimes_{\mathcal{O}_K}\mathcal{O}_L$ (for finite extensions $K\subset L\subset\overline{K}$)

$$
\widehat{\overline{\mathfrak{X}}}^{\sim}_{syn-et}\longrightarrow \overline{\mathfrak{X}}^{\sim}_{syn-et}\longleftarrow \overline{X}^{\sim}_{et} \tag{2.1.2}
$$

and a G-equivariant version of (2.1.2)

$$
(\widehat{\overline{\mathfrak{X}}}^{\sim}_{syn-et},G)\longrightarrow (\overline{\mathfrak{X}}^{\sim}_{syn-et},G)\longleftarrow (\overline{X}^{\sim}_{et},G) \tag{2.1.3}
$$

There is a commutative diagram of morphisms of topoi

$$
\begin{array}{ccccc}
\widehat{\overline{\mathfrak{X}}}^{\sim}_{syn-et} & \xrightarrow{v} & (\widehat{\overline{\mathfrak{X}}}^{\sim}_{syn-et},G) & \xrightarrow{\pi} & \widehat{\mathfrak{X}}^{\sim}_{syn-et}\\
\downarrow{\scriptstyle\Gamma_{\overline{\mathfrak{X}}}} & & \downarrow{\scriptstyle\Gamma_{\overline{\mathfrak{X}}}} & & \downarrow{\scriptstyle\Gamma_{\mathfrak{X}}}\\
(Sets) & \xrightarrow{v} & B_G & \xrightarrow{\Gamma_G} & (Sets)
\end{array}
\tag{2.1.4}
$$

and similar diagrams for $\mathfrak{X}^{\sim}_{syn-et}$ and $X^{\sim}_{et}$. In (2.1.4), B_G is the category of discrete G-sets, $v^*=$ "forget the action of G", $(\Gamma_G)_*=(-)^G$, $(\Gamma_G)^*=$ "let G act trivially". The étale version of π defines an equivalence of categories $\pi^* : X^{\sim}_{et} \approx (\overline{X}^{\sim}_{et}, G)$, which underlies the Hochschild-Serre spectral sequence. See Appendix for more details.

(2.2) The crystalline conjecture. For $i \geq 0$, denote by

$$D^i := H^i((Y/K_0)_{\rm cris}, \mathcal{O}_{Y/K_0}) = H^i(\mathbf{R}\Gamma_{\mathbf{N}}\mathbf{R}\Gamma((Y/W_n)_{\rm cris}, \mathcal{O}_{Y/W_n})) \otimes_W K_0$$
$$\widetilde{D}^i := H^i((\mathfrak{X}_1/K_0)_{\rm cris}, \mathcal{O}_{\mathfrak{X}_1/K_0}) = H^i(\mathbf{R}\Gamma_{\mathbf{N}}\mathbf{R}\Gamma((\mathfrak{X}_1/W_n)_{\rm cris}, \mathcal{O}_{\mathfrak{X}_1/W_n})) \otimes_W K_0$$

the crystalline cohomology of Y (resp. $\mathfrak{X}_1$) with coefficients in K_0. The canonical isomorphism

$$D^i_K = D^i \otimes_{K_0} K \xrightarrow{\sim} H^i_{dR}(X/K) = H^i(X_{Zar}, \Omega^{\cdot}_{X/K})$$

proved in ([BO 2], Cor. 2.5) makes D^i an object of the category MF_K; the filtration $F^r D^i_K$ is given by the Hodge filtration

$$F^r H^i_{dR}(X/K) = H^i(X_{Zar}, \sigma_{\geq r}\Omega^{\cdot}_{X/K})$$

Put

$$V^i := H^i(\overline{X}_{et}, \mathbf{Q}_p) \xrightarrow{\sim} H^i_{naive}(\overline{X}_{et}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

(*cf.* I.2.3.3). The *crystalline conjecture* of Fontaine [Fo 2] asserts that
(2.2.1) D^i is admissible.
(2.2.2) V^i is a crystalline representation of G.
(2.2.3) There are canonical isomorphisms $D(V^i) \xrightarrow{\sim} D^i$, $V(D^i) \xrightarrow{\sim} V^i$.

This conjecture was proved under some restrictions ($K = K_0$ and $p > \dim(X)$) in [FM] and in full generality in [Fa]. Combining the methods of [FM] with results on p-adic vanishing cycles, the conjecture was proved in [KaM] (resp. [Ts]) for $p > 2\dim(X) + 1$ (resp. all p).

(2.3) Various syntomic sheaves of crystalline origin form "$\mathbf{Z}_p$"-modules (*cf.* I.2.5, Appendix):

$$\mathcal{O}^{cris}_{“\mathbf{Z}_p”} = \left[\mathcal{O}^{cris}_n\right]_{n\in\mathbf{N}}, \quad J^{[r]}_{“\mathbf{Z}_p”} = \left[J^{[r]}_n\right]_{n\in\mathbf{N}}, \quad J'^{[r]}_{“\mathbf{Z}_p”} = \left[J'^{[r]}_n\right]_{n\in\mathbf{N}}, \quad s_{“\mathbf{Z}_p”}(r) = [s_n(r)]_{n\in\mathbf{N}}$$

are objects of $\mathrm{Mod}(\widehat{\mathfrak{X}}^{\sim}_{syn-et}, “\mathbf{Z}_p”)$. We denote by $\mathcal{O}^{cris}_{“\mathbf{Q}_p”} = Q(\mathcal{O}^{cris}_{“\mathbf{Z}_p”})$, $J^{[r]}_{“\mathbf{Q}_p”} = Q(J^{[r]}_{“\mathbf{Z}_p”})$ *etc.* their images in $\mathrm{Mod}(\widehat{\mathfrak{X}}^{\sim}_{syn-et}, “\mathbf{Q}_p”)$. Recall from [FM, III.4.1] and Appendix that all of these sheaves have the same cohomology on the syntomic-etale site of $\widehat{\mathfrak{X}}$ as on the syntomic site of $\mathfrak{X}$ (and similarly for $\overline{\mathfrak{X}}$).

Proposition. (1) *For $0 \leq r < p$ there is an exact sequence in* $\mathrm{Mod}(\widehat{\mathfrak{X}}^{\sim}_{syn-et}, “\mathbf{Z}_p”)$

$$0 \longrightarrow s_{“\mathbf{Z}_p”}(r) \longrightarrow J^{[r]}_{“\mathbf{Z}_p”} \xrightarrow{1-f_r} \mathcal{O}^{cris}_{“\mathbf{Z}_p”} \longrightarrow 0$$

(2) *For every $r \geq 0$ there is a commutative diagram with exact rows in* $\mathrm{Mod}(\widehat{\mathfrak{X}}^{\sim}_{syn-et}, “\mathbf{Q}_p”)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & s_{“\mathbf{Q}_p”}(r) & \longrightarrow & J^{[r]}_{“\mathbf{Q}_p”} & \xrightarrow{1-p^{-r}f} & \mathcal{O}^{cris}_{“\mathbf{Q}_p”} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle (id,0)} & & \\ 0 & \longrightarrow & s_{“\mathbf{Q}_p”}(r) & \longrightarrow & \mathcal{O}^{cris}_{“\mathbf{Q}_p”} & \xrightarrow{(1-p^{-r}f,\mathrm{can})} & \mathcal{O}^{cris}_{“\mathbf{Q}_p”} \oplus \mathcal{O}^{cris}_{“\mathbf{Q}_p”}/J^{[r]}_{“\mathbf{Q}_p”} & \longrightarrow & 0 \end{array}$$

Proof. (1) This follows from [FM, III.1.1] and the fact that $(i_{n+r})_* : \left(\mathfrak{X}^{\sim}_{n+r}\right)_{syn} \longrightarrow \widehat{\mathfrak{X}}^{\sim}_{syn-et}$ is an exact functor ([FM, III.4.1], Appendix). (2) The exactness of the first row is obtained as in (1), observing that $J^{[r]}_{“\mathbf{Z}_p”}/J'^{[r]}_{“\mathbf{Z}_p”}$ is killed by p^r; an easy diagram chase proves the exactness of the second row.

(2.4) Fix $r \geq 0$. According to Proposition 2.3, there is a distinguished triangle

$$s_{“\mathbf{Q}_p”}(r) \longrightarrow \mathcal{O}^{cris}_{“\mathbf{Q}_p”} \xrightarrow{(1-p^{-r}f,\mathrm{can})} \mathcal{O}^{cris}_{“\mathbf{Q}_p”} \oplus \mathcal{O}^{cris}_{“\mathbf{Q}_p”}/J^{[r]}_{“\mathbf{Q}_p”} \longrightarrow s_{“\mathbf{Q}_p”}(r)[1] \tag{2.4.1}$$

in $D^+(\mathrm{Mod}(\widehat{\mathfrak{X}}^{\sim}_{syn-et}, “\mathbf{Q}_p”))$. Applying $\mathbf{R}(Q(\Gamma^{\mathbf{N}}(\widehat{\mathfrak{X}}_{syn-et})))$, we obtain a distinguished triangle

$$\mathbf{R}(Q(\Gamma^{\mathbf{N}}(\widehat{\mathfrak{X}}_{syn-et})))(s_{“\mathbf{Q}_p”}(r)) \longrightarrow \mathbf{R}(Q(\Gamma^{\mathbf{N}}(\widehat{\mathfrak{X}}_{syn-et})))(\mathcal{O}^{cris}_{“\mathbf{Q}_p”}) \xrightarrow{(1-p^{-r}f,\mathrm{can})}$$
$$\longrightarrow \mathbf{R}(Q(\Gamma^{\mathbf{N}}(\widehat{\mathfrak{X}}_{syn-et})))(\mathcal{O}^{cris}_{“\mathbf{Q}_p”}) \oplus \mathbf{R}(Q(\Gamma^{\mathbf{N}}(\widehat{\mathfrak{X}}_{syn-et})))(\mathcal{O}^{cris}_{“\mathbf{Q}_p”}/J^{[r]}_{“\mathbf{Q}_p”}) \longrightarrow \mathbf{R}(Q(\Gamma^{\mathbf{N}}(\widehat{\mathfrak{X}}_{syn-et})))(s_{“\mathbf{Q}_p”}(r))[1] \tag{2.4.2}$$

in $D^+(\mathrm{Mod}(Sets, “\mathbf{Q}_p”))$, which will be denoted by

$$\Delta_r : A_r \longrightarrow B_r \longrightarrow C_r \longrightarrow A_r[1] \tag{2.4.3}$$

In fact, all A_r, B_r, C_r are objects of $D^b(\mathrm{Mod}(Sets, “\mathbf{Q}_p”))$, (see Appendix).

Apply the functor π^* from (2.1.4) to the triangle (2.4.1) and take $\mathbf{R}(Q(\Gamma^{\mathbf{N}}(\widehat{\overline{\mathfrak{X}}}_{syn-et})))$. We obtain a distinguished triangle

$$\mathbf{R}(Q(\Gamma^{\mathbf{N}}(\widehat{\overline{\mathfrak{X}}}_{syn-et})))(s_{“\mathbf{Q}_p”}(r)) \longrightarrow \mathbf{R}(Q(\Gamma^{\mathbf{N}}(\widehat{\overline{\mathfrak{X}}}_{syn-et})))(\mathcal{O}^{cris}_{“\mathbf{Q}_p”}) \xrightarrow{(1-p^{-r}f,\mathrm{can})}$$
$$\longrightarrow \mathbf{R}(Q(\Gamma^{\mathbf{N}}(\widehat{\overline{\mathfrak{X}}}_{syn-et})))(\mathcal{O}^{cris}_{“\mathbf{Q}_p”}) \oplus \mathbf{R}(Q(\Gamma^{\mathbf{N}}(\widehat{\overline{\mathfrak{X}}}_{syn-et})))(\mathcal{O}^{cris}_{“\mathbf{Q}_p”}/J^{[r]}_{“\mathbf{Q}_p”}) \longrightarrow \mathbf{R}(Q(\Gamma^{\mathbf{N}}(\widehat{\overline{\mathfrak{X}}}_{syn-et})))(s_{“\mathbf{Q}_p”}(r))[1] \tag{2.4.4}$$

in $D^+(\mathrm{Mod}(B_G, “\mathbf{Q}_p”))$, which will be denoted by

$$\overline{\Delta}_r : \overline{A}_r \longrightarrow \overline{B}_r \longrightarrow \overline{C}_r \longrightarrow \overline{A}_r[1] \tag{2.4.5}$$

Note that $C_r = B_r \oplus C'_r$, $\overline{C}_r = \overline{B}_r \oplus \overline{C}'_r$ for

$$C'_r = \mathbf{R}(Q(\Gamma^{\mathbf{N}}(\widehat{\mathfrak{X}}_{syn-et})))(\mathcal{O}^{cris}_{“\mathbf{Q}_p”}/J^{[r]}_{“\mathbf{Q}_p”}), \quad \overline{C}'_r = \mathbf{R}(Q(\Gamma^{\mathbf{N}}(\widehat{\overline{\mathfrak{X}}}_{syn-et})))(\mathcal{O}^{cris}_{“\mathbf{Q}_p”}/J^{[r]}_{“\mathbf{Q}_p”})$$

(2.5) Proposition. *For every $q, r \geq 0$, there are canonical maps between distinguished triangles*

$$\left(Q(\Gamma^{\mathbf{N}}_G)^*\Delta_r\right)(q) \longrightarrow \overline{\Delta}_{r+q}, \qquad Q(\Gamma^{\mathbf{N}}_G)^*\Delta_r \longrightarrow \overline{\Delta}_{r+q}(-q)$$

in $\mathrm{Mod}(B_G, “\mathbf{Q}_p”)$.

Proof. For $q = 0$ there is nothing to prove: for every abelian sheaf A on $\widehat{\mathfrak{X}}_{syn-et}$ there is a canonical map

$$\Gamma^*_G \mathbf{R}\Gamma(\widehat{\mathfrak{X}}_{syn-et}, A) \xrightarrow{\mathrm{can}} \mathbf{R}\Gamma(\widehat{\overline{\mathfrak{X}}}_{syn-et}, \pi^* A)$$

induced by π.
Suppose that $q \geq 1$. The canonical map

$$“\mathbf{Z}_p(1)” = [\mu_{p^n}(\overline{K})]_{n\in\mathbf{N}} \longrightarrow [H^0(\widehat{\overline{\mathfrak{X}}}_{syn-et}, s_n(1))]_{n\in\mathbf{N}}$$

in $\mathrm{Mod}(B_G, “\mathbf{Z}_p”)$ is compatible with products, defining a map

$$“\mathbf{Z}_p(q)” \longrightarrow [H^0(\widehat{\overline{\mathfrak{X}}}_{syn-et}, s_n(q))]_{n\in\mathbf{N}} \longrightarrow \mathbf{R}\Gamma(\widehat{\overline{\mathfrak{X}}}_{syn-et}, s_{“\mathbf{Z}_p”}(q))$$

in $D^+(\mathrm{Mod}(B_G, “\mathbf{Z}_p”))$. Together with the cup product

$$\overline{A}_r \overset{\mathbf{L}}{\otimes}_{“\mathbf{Q}_p”} \overline{A}_q = \mathbf{R}(Q(\Gamma^{\mathbf{N}}(\widehat{\overline{\mathfrak{X}}}_{syn-et})))(s_{“\mathbf{Q}_p”}(r)) \overset{\mathbf{L}}{\otimes} \mathbf{R}(Q(\Gamma^{\mathbf{N}}(\widehat{\overline{\mathfrak{X}}}_{syn-et})))(s_{“\mathbf{Q}_p”}(q)) \xrightarrow{\cup}$$
$$\xrightarrow{\cup} \mathbf{R}(Q(\Gamma^{\mathbf{N}}(\widehat{\overline{\mathfrak{X}}}_{syn-et})))(s_{“\mathbf{Q}_p”}(r+q)) = \overline{A}_{r+q},$$

this induces a map

$$\alpha_{r,q} : \left(Q(\Gamma_G^{\mathbf{N}})^* A_r\right)(q) \xrightarrow{\text{can}} \overline{A}_r(q) = \overline{A}_r \overset{\mathbf{L}}{\otimes}_{\text{“}\mathbf{Q}_p\text{”}} \text{“}\mathbf{Q}_p(q)\text{”} \longrightarrow \overline{A}_r \overset{\mathbf{L}}{\otimes}_{\text{“}\mathbf{Q}_p\text{”}} \overline{A}_q \xrightarrow{\cup} \overline{A}_{r+q}$$

(see Appendix for the discussion of cup products and comments on the simultaneous appearance of $\mathbf{R}\Gamma$ and $\overset{\mathbf{L}}{\otimes}$).

In the same vein, the morphisms $s_{\text{“}\mathbf{Z}_p\text{”}}(q) \longrightarrow J^{[q]}_{\text{“}\mathbf{Z}_p\text{”}} \longrightarrow \mathcal{O}^{cris}_{\text{“}\mathbf{Z}_p\text{”}}$ and the cup product associated to $\mathcal{O}^{cris}_{\text{“}\mathbf{Z}_p\text{”}} \times \mathcal{O}^{cris}_{\text{“}\mathbf{Z}_p\text{”}} \longrightarrow \mathcal{O}^{cris}_{\text{“}\mathbf{Z}_p\text{”}}$ give rise to a map

$$\beta_{r,q} : \left(Q(\Gamma_G^{\mathbf{N}})^* B_r\right)(q) \xrightarrow{\text{can}} \overline{B}_r(q) = \overline{B}_r \overset{\mathbf{L}}{\otimes}_{\text{“}\mathbf{Q}_p\text{”}} \text{“}\mathbf{Q}_p(q)\text{”} \longrightarrow \overline{B}_r \overset{\mathbf{L}}{\otimes}_{\text{“}\mathbf{Q}_p\text{”}} \overline{A}_q \longrightarrow \overline{B}_r \overset{\mathbf{L}}{\otimes}_{\text{“}\mathbf{Q}_p\text{”}} \overline{B}_q \xrightarrow{\cup} \overline{B}_{r+q},$$

compatible with $\alpha_{r,q}$. As f acts as p^q on the image of “$\mathbf{Q}_p(q)$” in $\mathcal{O}^{cris}_{\text{“}\mathbf{Q}_p\text{”}}$, we have

$$1 - p^{r+q} f \circ \beta_{r,q} = 1 - p^{-r} \beta_{r,q} \circ f \tag{2.5.1}$$

Finally, the cup product associated to

$$\mathcal{O}^{cris}_{\text{“}\mathbf{Q}_p\text{”}} / J^{[r]}_{\text{“}\mathbf{Q}_p\text{”}} \times J^{[q]}_{\text{“}\mathbf{Q}_p\text{”}} \longrightarrow J^{[q]}_{\text{“}\mathbf{Q}_p\text{”}} / J^{[r+q]}_{\text{“}\mathbf{Q}_p\text{”}} \longrightarrow \mathcal{O}^{cris}_{\text{“}\mathbf{Q}_p\text{”}} / J^{[r+q]}_{\text{“}\mathbf{Q}_p\text{”}}$$

and $s_{\text{“}\mathbf{Q}_p\text{”}}(q) \longrightarrow J^{[q]}_{\text{“}\mathbf{Q}_p\text{”}}$ define a map

$$\gamma'_{r,q} : \left(Q(\Gamma_G^{\mathbf{N}})^* C'_r\right)(q) \xrightarrow{\text{can}} \overline{C}'_r(q) = \overline{C}'_r \overset{\mathbf{L}}{\otimes}_{\text{“}\mathbf{Q}_p\text{”}} \text{“}\mathbf{Q}_p(q)\text{”} \longrightarrow \overline{C}'_r \overset{\mathbf{L}}{\otimes}_{\text{“}\mathbf{Q}_p\text{”}} \mathbf{R}\Gamma(\widehat{\overline{\mathfrak{X}}}_{syn-et}, J^{[q]}_{\text{“}\mathbf{Q}_p\text{”}}) \xrightarrow{\cup} \overline{C}'_{r+q},$$

Thank to (2.5.1), the map

$$\gamma_{r,q} = (\beta_{r,q}, \gamma'_{r,q}) : \left(Q(\Gamma_G^{\mathbf{N}})^* C_r\right)(q) \longrightarrow \overline{C}_{r+q}$$

together with $\alpha_{r,q}$ and $\beta_{r,q}$ define a map of triangles

$$\left(Q(\Gamma_G^{\mathbf{N}})^* \Delta_r\right)(q) \longrightarrow \overline{\Delta}_{r+q}$$

Using (I.2.5.1), we get the second map.

(2.6) Let $u : (E', A') \longrightarrow (E, A)$ be a morphism of ringed topoi such that $u^* : \mathrm{Mod}(E, A) \longrightarrow \mathrm{Mod}(E', A')$ is *exact* ($\iff$ A' is a flat $u^{-1}(A)$-module). The “trivial duality” ([Ber], V.3.3.1; [BO1], Prop. 7.7) gives a canonical (and functorial) isomorphism

$$\mathrm{Hom}_{D(\mathrm{Mod}(E',A'))}(u^*(X), Y') \xrightarrow{\sim} \mathrm{Hom}_{D(\mathrm{Mod}(E,A))}(X, \mathbf{R}u_*(Y')) \tag{2.6.1}$$

for all $X \in \mathrm{Ob}(D^-(\mathrm{Mod}(E, A)))$, $Y' \in \mathrm{Ob}(D^+(\mathrm{Mod}(E', A')))$. From (I.1.2.12–13) we get a canonical isomorphism

$$\mathrm{Hom}_{D(Q(\mathrm{Mod}(E',A')))}(Q(u)^*(Q_D(X)), Q_D(Y')) \xrightarrow{\sim} \mathrm{Hom}_{D(Q(\mathrm{Mod}(E,A)))}(Q_D(X), \mathbf{R}Q(u)_*(Q_D(Y'))) \tag{2.6.2}$$

For $(E', A') = ((B_G)^{\mathbf{N}}, \text{“}\mathbf{Z}_p\text{”})$, $(E, A) = ((Sets)^{\mathbf{N}}, \text{“}\mathbf{Z}_p\text{”})$ and $u = \Gamma_G^{\mathbf{N}}$, we know that all constituents A_r, B_r, C_r of Δ_r (resp. $\overline{A}_{r+q}$, $\overline{B}_{r+q}$, $\overline{C}_{r+q}$ of $\overline{\Delta}_{r+q}$) are of the form $Q_D(X)$ for some $X \in \mathrm{Ob}(D^b(\mathrm{Mod}(E, A)))$ (resp. $Q_D(Y')$ for some $Y' \in \mathrm{Ob}(D^+(\mathrm{Mod}(E', A'))))$ (2.4).

It follows from Proposition 2.5 and (2.6.2) that, for all $r, q \geq 0$, there is a canonical map of distinguished triangles

$$\Delta_r \longrightarrow \mathbf{R}Q(\Gamma_G^{\mathbf{N}})_* \left(\overline{\Delta}_{r+q}(-q)\right) \tag{2.6.3}$$

in $D^+(Q(\mathrm{Mod}((Sets)^{\mathbf{N}}, \text{“}\mathbf{Z}_p\text{”}))) = D^+(\mathrm{Mod}(Sets, \text{“}\mathbf{Q}_p\text{”}))$.

(2.7) There is a diagram of functors

$$\begin{array}{ccccc} \mathrm{Mod}(B_G, \text{“}\mathbf{Q}_p\text{”}) & \xrightarrow{Q(\Gamma_{\mathbf{N}})} & Q(\mathbf{Z}_p[G]-Mod) & \xrightarrow{\xi_G} & (\mathbf{Q}_p[G]-Mod) \\ \downarrow{\scriptstyle Q((\Gamma_G^{\mathbf{N}})_*)} & & \downarrow{\scriptstyle Q((\Gamma_G)_*)} & & \downarrow{\scriptstyle (\Gamma_G)_*} \\ \mathrm{Mod}(Sets, \text{“}\mathbf{Q}_p\text{”}) & \xrightarrow{Q(\Gamma_{\mathbf{N}})} & Q(\mathbf{Z}_p-Mod) & \xrightarrow{\xi} & (\mathbf{Q}_p-Mod), \end{array} \tag{2.7.1}$$

in which $\Gamma_G = (-)^G$, $\Gamma_{\mathbf{N}} = \varprojlim_{\mathbf{N}}$ and ξ, ξ_G are the (exact) functors from (I.1.2.4). The first square of (2.7.1) is commutative; in the second square, there is a canonical morphism of functors

$$\xi \circ Q((\Gamma_G)_*) \longrightarrow (\Gamma_G)_* \circ \xi_G \tag{2.7.2}$$

such that

$$\xi \circ Q((\Gamma_G)_*)(X) \longrightarrow (\Gamma_G)_* \circ \xi_G(X)$$

is a monomorphism for every $X \in \mathrm{Ob}(Q(\mathbf{Z}_p[G]-Mod))$ (this is just the canonical map $Y^G \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \longrightarrow (Y \otimes_{\mathbf{Z}_p} \mathbf{Q}_p)^G$).
Put

$$\begin{aligned} \Psi &= \xi \circ Q(\Gamma_{\mathbf{N}}) : \mathrm{Mod}(Sets, \text{“}\mathbf{Q}_p\text{”}) \longrightarrow (\mathbf{Q}_p - Mod) \\ \Phi &= \Psi \circ Q((\Gamma_G^{\mathbf{N}})_*) : \mathrm{Mod}(B_G, \text{“}\mathbf{Q}_p\text{”}) \longrightarrow (\mathbf{Q}_p - Mod) \\ \Theta &= \xi_G \circ Q(\Gamma_{\mathbf{N}}) : \mathrm{Mod}(B_G, \text{“}\mathbf{Q}_p\text{”}) \longrightarrow (\mathbf{Q}_p[G] - Mod) \end{aligned} \tag{2.7.3}$$

(all of these functors are left exact) and write $v^* : (\mathbf{Q}_p[G] - Mod) \longrightarrow (\mathbf{Q}_p - Mod)$ for the functor “forget the action of G”. There are canonical morphisms of functors (induced by (2.7.2))

$$\Phi \longrightarrow (\Gamma_G)_* \circ \Theta \longrightarrow v^* \circ \Theta \tag{2.7.4}$$

such that both arrows in

$$\Phi(X) \longrightarrow (\Gamma_G)_* \circ \Theta(X) \longrightarrow v^* \circ \Theta(X) \tag{2.7.5}$$

are monomorphisms (for every $X \in \mathrm{Ob}(\mathrm{Mod}(B_G, \text{“}\mathbf{Q}_p\text{”}))$).

(2.8) We show that syntomic(-étale) cohomology fits into the abstract framework of I.3.1.
(0) Fix an integer $i \in \mathbf{Z}$.
(1),(2) We have categories $\mathcal{C} = \mathrm{Mod}(Sets, \text{“}\mathbf{Q}_p\text{”})$, $\overline{\mathcal{C}} = \mathrm{Mod}(B_G, \text{“}\mathbf{Q}_p\text{”})$, $\mathcal{D} = (\mathbf{Q}_p - Mod)$ and functors $u = Q((\Gamma_G^{\mathbf{N}})_*)$, Ψ, Φ from (2.7.3).
(3),(4) Fix $r \geq 0$ and choose $q \geq 0$ such that $r + q > i$ (as $cd_p(\overline{X}_{et}) = 2\dim(X)$, we can treat all relevant i's simultaneously by taking q such that $r + q > 2\dim(X)$). By (2.4) and (2.6.3), there are distinguished triangles $\Delta = \Delta_r$, $\overline{\Delta} = (\overline{\Delta})_{r+q}(-q)$ and a morphism of triangles $\Delta \longrightarrow (\mathbf{R}u)(\overline{\Delta})$. As in (2.4), we have $C = B \oplus C'$, $\overline{C} = \overline{B} \oplus \overline{C}'$.
(5) Étale cohomology defines objects $E = \mathbf{R}(Q(\Gamma^{\mathbf{N}}(X_{et})))(\text{“}\mathbf{Z}_p(r)\text{”}) \in \mathrm{Ob}(D^+(\mathcal{C}))$, $\overline{E} = \mathbf{R}(Q(\Gamma^{\mathbf{N}}(\overline{X}_{et})))(\text{“}\mathbf{Z}_p(r)\text{”}) \in \mathrm{Ob}(D^+(\overline{\mathcal{C}}))$. The isomorphism $\rho : E \xrightarrow{\sim} (\mathbf{R}u)(\overline{E})$ comes from the equivalence of categories $\pi^* : X_{et}^{\sim} \approx (\overline{X}_{et}^{\sim}, G)$.
(6) The morphisms $\mu : A \longrightarrow E$ (resp. $\overline{\mu} : \overline{A} \longrightarrow \overline{E}$) are furnished by the Fontaine-Messing map ([FM], III.5.1; Appendix). The diagram in I.3.1.6 is commutative, as the Fontaine-Messing map is compatible with cup products, the Galois action and with the map $\mu_{p^n} \longrightarrow s_n(1)$.

(2.9) In Sections 2.9–14 we show that objects defined in 2.8 verify the axioms (A1)–(A3) of I.3.1. Almost by definition, we have

$$R^j\Psi(A) = H^j(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Q}_p}(r)) \qquad (\forall j \geq 0) \tag{2.9.1}$$

According to ([KaM], Lemma 4.5(1)),

$$R^j\Psi(C') = D^j_K/F^r \qquad (\forall j \geq 0) \tag{2.9.2}$$

It is shown in ([KaM], Prop. 1.2, Prop. 1.3(1)) that (for $j \geq 0$)

$$\begin{aligned} \Theta(H^j(\overline{B}_{r+q})) &= D^j \otimes_{K_0} B^+_{cris} \\ \Theta(H^j(\overline{C}'_{r+q})) &= (D^j_K \otimes_K B^+_{dR})/F^{r+q}(D^j_K \otimes_K B^+_{dR}), \end{aligned} \tag{2.9.3}$$

which implies that, for $j \leq r+q$,

$$\begin{aligned} \Theta(H^j(\overline{B})) &= D^j \otimes_{K_0} B^+_{cris}(-q) \xrightarrow{\sim} D^j \otimes_{K_0} t^{-q}B^+_{cris} \hookrightarrow D^j \otimes_{K_0} B_{cris} \\ \Theta(H^j(\overline{C}')) &= \frac{D^j_K \otimes_K B^+_{dR}}{F^{r+q}(D^j_K \otimes_K B^+_{dR})}(-q) \xrightarrow{\sim} \frac{D^j_K \otimes_K t^{-q}B^+_{dR}}{F^r(D^j_K \otimes_K t^{-q}B^+_{dR})} \hookrightarrow \frac{D^j_K \otimes_K B_{dR}}{F^r(D^j_K \otimes_K B_{dR})} \end{aligned} \tag{2.9.4}$$

Here t is a generator of $\mathbf{Q}_p(1) \subset B^+_{cris}$. Note that the last map in (2.9.4) is indeed injective: for $i \leq r+q$ we have

$$F^r(D^j_K \otimes_K F^{-q}B_{dR}) = \sum_{k\leq j} F^k D^j_K \otimes_K F^{r-k}(F^{-q}B_{dR}) = \sum_{k \leq j} F^k D^j_K \otimes_K F^{r-k}B_{dR} = F^r(D^j_K \otimes_K B_{dR})$$

(2.10) Vanishing cycles and the crystalline conjecture. Consider the map induced by μ on cohomology:

$$H^j(\overline{A}) = Q([H^j(\widehat{\overline{\mathfrak{X}}}_{syn-et}, s_n(r+q))(-q)]_{n\in\mathbf{N}}) \longrightarrow Q([H^i(\overline{X}_{et}, \mathbf{Z}/p^n\mathbf{Z}(r))]_{n\in\mathbf{N}}) = H^j(\overline{E}) \tag{2.10.1}$$

A fundamental theorem on vanishing cycles, proved in an increasing degree of generality by Kato [Ka], Kurihara [Ku] and Tsuji [Ts], asserts that (2.10.1) is an *isomorphism* for all $j \leq r+q$, proving thus (A2). According to (I.2.3.3.1), the projective system $[H^j(\overline{X}_{et}, \mathbf{Z}/p^n\mathbf{Z}(r))]_{n\in\mathbf{N}}$ is ML-p-adic, hence the last group in (2.10.1) is isomorphic to $Q([T^j(r) \otimes \mathbf{Z}/p^n\mathbf{Z}]_{n\in\mathbf{N}})$, where

$$T^j = H^j(\overline{X}_{et}, \mathbf{Z}_p) \xrightarrow{\sim} \varprojlim_n H^j(\overline{X}_{et}, \mathbf{Z}/p^n\mathbf{Z})$$

This implies that (2.10.1) induces an isomorphism

$$\Theta(H^j(\overline{A})) \xrightarrow{\sim} V^j(r), \qquad (\forall j \leq i+1) \tag{2.10.2}$$

which, together with (2.9.4), shows that the complex

$$\Theta(H^j(\overline{A})) \longrightarrow \Theta(H^j(\overline{B})) \longrightarrow \Theta(H^j(\overline{C}))$$

is isomorphic to the first row of the following commutative diagram

$$\begin{array}{ccccccccc} & & V^j(r) & \xrightarrow{\alpha} & D^j \otimes_{K_0} t^{-q}B^+_{cris} & \xrightarrow{(1-p^{-r},\text{can})} & (D^j \otimes_{K_0} t^{-q}B^+_{cris}) & \oplus & \frac{D^j_K \otimes_K t^{-q}B^+_{dR}}{F^r(D^j_K \otimes_K t^{-q}B^+_{dR})} \\ & & & & \big\downarrow & & & & \big\downarrow \\ 0 & \longrightarrow & V(D^j(r)) & \longrightarrow & D^j \otimes_{K_0} B_{cris} & \xrightarrow{(1-p^{-r},\text{can})} & (D^j \otimes_{K_0} B_{cris}) & \oplus & \frac{D^j_K \otimes_K B_{dR}}{F^r(D^j_K \otimes_K B_{dR})} \end{array} \tag{2.10.3}$$

(here V is Fontaine's functor defined in (1.4.1)). Let us rapidly recall how one deduces from (2.10.3) the crystalline conjecture (2.2) (see [KaM, 3.2], [Ts] for more details): an argument based on Poincaré duality ([FM], 6.3) shows that the map α in (2.10.3) is *injective*; a dimension count then implies that α induces isomorphisms

$$V^j(r) \xrightarrow{\sim} V(D^j(r)), \qquad V^j \otimes_{\mathbf{Q}_p} B_{cris} \xrightarrow{\sim} D^j \otimes_{K_0} B_{cris}, \qquad (\forall j \leq i+1)$$

which proves the crystalline conjecture (taking, *e.g.*, $i = 2\dim(X)$).

(2.11) Put $K^j := \mathrm{Ker}[\Phi(H^j(\overline{A})) \longrightarrow \Phi(H^j(\overline{B}))]$. As (2.8.4) and α in (2.10.3) are injective, it follows that

$$\Phi(K^j) = \mathrm{Ker}\left[\Phi(H^j(\overline{A})) \longrightarrow \Phi(H^j(\overline{B}))\right] = 0 \qquad (\forall j \leq i+1) \tag{2.11.1}$$

However, $K^j \subseteq H^j(\overline{A}) \xrightarrow{\sim} Q([T^j(r) \otimes \mathbf{Z}/p^n\mathbf{Z}]_{n\in\mathbf{N}})$, with T^j a $\mathbf{Z}_p$-module of finite type. For such K^j, (2.11.1) implies that $K^j = 0$ for all $j \leq i+1$. This proves the axiom (A1) of I.3.1.

(2.12) As crystalline cohomology depends only on reduction $\pmod p$, we have

$$R^j\Psi(B) = H^j((\mathfrak{X}/W)_{\mathrm{cris}}, \mathcal{O}_{\mathfrak{X}/W}) \otimes_W K_0 = H^j((\mathfrak{X}_1/W)_{\mathrm{cris}}, \mathcal{O}_{\mathfrak{X}_1/W}) \otimes_W K_0 = \widetilde{D}^j$$

Our aim is to compare $\widetilde{D}^j$ and D^j, using the "Frobenius trick" of [BO 2].
For a scheme T over $\mathrm{Spec}(\mathbf{F}_p)$ denote by $F_T : T \longrightarrow T$ its absolute Frobenius morphism. Consider the commutative diagram

$$\begin{array}{ccc} Y & \overset{g}{\hookrightarrow} & \mathfrak{X}_1 \\ \downarrow{\scriptstyle F_Y} & & \downarrow{\scriptstyle F_{\mathfrak{X}_1}} \\ Y & \overset{g}{\hookrightarrow} & \mathfrak{X}_1 \end{array}$$

and the induced maps on crystalline cohomology

$$\begin{array}{ccc} D^j & \overset{g^*}{\longleftarrow} & \widetilde{D}^j \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle f'} \\ D^j & \overset{g^*}{\longleftarrow} & \widetilde{D}^j \end{array}$$

For sufficiently large integer $n >> 0$ there is a morphism $\rho : \mathfrak{X}_1 \longrightarrow Y$ (depending on n, which will be fixed) such that

$$\rho \circ g = F_Y^n, \qquad g \circ \rho = F_{\mathfrak{X}_1}^n.$$

The induced map on cohomology $\rho^* : D^j \longrightarrow \widetilde{D}^j$ satisfies $f' \circ \rho^* = \rho^* \circ f$, $g^* \circ \rho = f^n$, $\rho^* \circ g^* = f'^n$. The crystalline Frobenius $f : D^j \longrightarrow D^j$ is bijective, which implies that $g^* : \widetilde{D}^j \longrightarrow D^j$ is surjective.
According to [BO 2], the canonical map

$$\widetilde{D}^j = R^j\Psi(B) \longrightarrow R^j\Psi(C') \overset{(2.9.2)}{=} D^j_K/F^r$$

factors as $\widetilde{D}^j \xrightarrow{g^*} D^j \xrightarrow{\mathrm{can}} D^j_K/F^r$.

(2.13) Lemma. *For every $0 \neq \alpha \in K_0$, the map g^* induces isomorphisms*

$$(\widetilde{D}^j)^{\alpha f'=1} \xrightarrow{\sim} (D^j)^{\alpha f=1}, \qquad \widetilde{D}^j/(\alpha f'-1)\widetilde{D}^j \xrightarrow{\sim} D^j/(\alpha f-1)D^j$$

and a quasi-isomorphism

$$\left[\widetilde{D}^j \xrightarrow{(\alpha f'-1, \mathrm{can}\circ g^*)} \widetilde{D}^j \oplus D^j_K/F^r\right] \xrightarrow{Qis} \left[D^j \xrightarrow{(\alpha f-1, \mathrm{can})} D^j \oplus D^j_K/F^r\right].$$

Proof. (Following a remark of P. Berthelot) We apply the Snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(g^*) & \longrightarrow & \widetilde{D}^j & \xrightarrow{g^*} & D^j & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \alpha f'-1} & & \Big\downarrow{\scriptstyle \alpha f'-1} & & \Big\downarrow{\scriptstyle \alpha f-1} & & \\ 0 & \longrightarrow & \mathrm{Ker}(g^*) & \longrightarrow & \widetilde{D}^j & \xrightarrow{g^*} & D^j & \longrightarrow & 0 \end{array}$$

We first observe that g^* indeed induces an isomorphism between the kernels (resp. cokernels) of the second and third vertical maps (it is invertible, its inverse being $\alpha^n\rho^*$). This implies that the first vertical arrow must be an isomorphism. Another application of the Snake lemma, this time to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(g^*) & \longrightarrow & \widetilde{D}^j & \xrightarrow{g^*} & D^j & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \alpha f'-1} & & \Big\downarrow{\scriptstyle (\alpha f'-1, \mathrm{can}\circ g^*)} & & \Big\downarrow{\scriptstyle (\alpha f-1, \mathrm{can})} & & \\ 0 & \longrightarrow & \mathrm{Ker}(g^*) & \longrightarrow & \widetilde{D}^j \oplus D^j_K/F^r & \xrightarrow{(g^*, id)} & D^j \oplus D^j_K/F^r & \longrightarrow & 0, \end{array}$$

concludes the proof.

(2.14) Observe that, for $j \leq i+1$, the composite map

$$\begin{aligned} \widetilde{D}^j = & R^j\Psi(B) \longrightarrow \Phi(H^j(\overline{B})) \overset{incl}{\hookrightarrow} (\Gamma_G)_* \circ \Theta(H^j(\overline{B})) \xrightarrow{(2.9.4)\sim} \\ & \xrightarrow{\sim} (D^j \otimes_{K_0} t^{-q} B^+_{cris})^G \overset{incl}{\hookrightarrow} (D^j \otimes_{K_0} B_{cris})^G = D^j \end{aligned}$$

resp.

$$\begin{aligned} D^j_K/F^r = & R^j\Psi(C') \longrightarrow \Phi(H^j(\overline{C}')) \overset{incl}{\hookrightarrow} (\Gamma_G)_* \circ \Theta(H^j(\overline{C}')) \xrightarrow{(2.9.4)\sim} \\ & \xrightarrow{\sim} \left(\frac{D^j_K \otimes_K F^{-q} B_{dR}}{F^r(D^j_K \otimes_K F^{-q} B_{dR})}\right)^G \overset{incl}{\hookrightarrow} \left(\frac{D^j_K \otimes_K B_{dR}}{F^r(D^j_K \otimes_K B_{dR})}\right)^G \overset{(1.5.3)}{=} D^j_K/F^r \end{aligned}$$

is equal to g^* resp. to the identity map. This implies that all four inclusion maps $\overset{incl}{\hookrightarrow}$ above are equalities. The axiom (A3) then follows from Lemma 2.13 (for $\alpha = p^{-r}$).

3. The Main Comparison Theorem

In this section we prove Theorems B, C and D(3).

(3.1) The assumptions of (2.1) (including $p > 2$) are in force. For $i, r \geq 0$, denote the map

$$D^i \xrightarrow{(1-p^{-r}, \mathrm{can})} D^i \oplus D^i_K/F^r$$

by $\lambda_{i,r}$. From (2.9.1–2) and the cohomology sequence of Δ_r we obtain an exact sequence

$$0 \longrightarrow \mathrm{Coker}(\lambda_{i,r}) \longrightarrow H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Q}_p}(r)) \longrightarrow \mathrm{Ker}(\lambda_{i+1,r}) \longrightarrow 0 \qquad (3.1.1)$$

The crystalline conjecture (2.2) and (1.7.3–4) yield isomorphisms

$$\mathrm{Ker}(\lambda_{i+1,r}) \xrightarrow{\sim} H^0(G_K, V^{i+1}(r)) \qquad (3.1.2.1)$$

$$\exp_{V^i(r)} : \mathrm{Coker}(\lambda_{i,r}) \xrightarrow{\sim} H^1_f(K, V^i(r)) \qquad (3.1.2.2)$$

The Hochschild-Serre spectral sequence gives rise to an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & F^1H^{i+1}(X_{et}, \mathbf{Q}_p(r)) & \longrightarrow & H^{i+1}(X_{et}, \mathbf{Q}_p(r)) & \xrightarrow{edge} & H^0(G_K, V^{i+1}(r)) \\ & & \Big\downarrow{\scriptstyle \delta} & & & & \\ & & H^1_{cont}(G_K, V^i(r)) & & & & \end{array} \qquad (3.1.3)$$

We are now ready to state our main p-adic comparison result, relating (3.1.1) and (3.1.3).

(3.2) Theorem. *Let $i, r \geq 0$. The Fontaine-Messing map $\nu : H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Q}_p}(r)) \longrightarrow H^{i+1}(X_{et}, \mathbf{Q}_p(r))$ is injective and gives rise to a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Coker}(\lambda_{i,r}) & \longrightarrow & H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Q}_p}(r)) & \longrightarrow & \mathrm{Ker}(\lambda_{i+1,r}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\nu} & & \downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & F^1 H^{i+1}(X_{et}, \mathbf{Q}_p(r)) & \longrightarrow & H^{i+1}(X_{et}, \mathbf{Q}_p(r)) & \longrightarrow & H^0(G_K, V^{i+1}(r)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\delta} & & & & & & \\
 & & H^1_{cont}(G_K, V^i(r)) & & & & & &
\end{array}$$

in which α is an isomorphism, $\delta \circ \beta = \exp_{V^i(r)}$ is injective and $\mathrm{Im}(\delta \circ \beta) = H^1_f(K, V^i(r))$.

Proof. Everything follows from Proposition I.3.5, applied to the data (2.8). The axioms of I.3.1 were verified in (2.9)–(2.12).

(3.3) Denote by F^j_{et} the filtration induced on $H^{i+1}(X_{et}, \mathbf{Q}_p(r))$ by the Hochschild-Serre spectral sequence. As $cd_p(k) = 1$ ([Se 1], II.2.2.3), we have $cd_p(K) \leq 2$ ([Se 1], II.4.3.12), hence $F^j_{et} = 0$ for $j > 2$ (I.2.3.3.4) and $E^{1,b}_2 = E^{1,b}_\infty$ for all $b \geq 0$ (recall that, for X smooth and *projective* over K, the Hochschild-Serre spectral sequence degenerates at E_2 (I.2.3.3)).

As in I.3.6, we define a filtration F^j_{syn} on $H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Q}_p}(r))$ by

$$\begin{aligned}
F^0_{syn} &= H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Q}_p}(r)) \\
F^1_{syn} &= \mathrm{Ker}\left[H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Q}_p}(r)) \xrightarrow{h} H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, \mathcal{O}^{cris}_{\mathbf{Q}_p})\right] \\
F^2_{syn} &= 0
\end{aligned}$$

(the map h is induced by $s_n(r) \hookrightarrow J^{[r]}_n \hookrightarrow \mathcal{O}^{cris}_n$). From I.3.6 we get

Proposition. (1) *For all $j \geq 0$, $\nu(F^j_{syn}) \subseteq F^j_{et}$.*
(2) *$gr^0_F(\nu) : F^0_{syn}/F^1_{syn} \xrightarrow{\sim} F^0_{et}/F^1_{et} = E^{0,i+1}_\infty = E^{0,i+1}_2 = H^0(G_K, V^{i+1}(r))$ is an isomorphism.*
(3) *$gr^1_F(\nu) : F^1_{syn} \hookrightarrow F^1_{et}/F^2_{et} = E^{1,i}_\infty = E^{1,i}_2 = H^1_{cont}(G_K, V^i(r))$ is injective, with image equal to $H^1_f(K, V^i(r))$.*

(3.4) p-adic Beilinson-Deligne cohomology. In an ideal world, one would hope that there is a canonical object $\mathbf{R}\Gamma_{\mathrm{cris}}(\overline{X}_{et}, \mathbf{Q}_p(r))$ in $D^b(Rep_{cris}(G))$ with cohomology groups $H^i(\overline{X}_{et}, \mathbf{Q}_p(r))$; the p-adic Beilinson-Deligne cohomology of X would then be defined as

$$H^i_{\mathrm{BD}}(X, \mathbf{Q}_p(r)) = \mathrm{Hom}_{D^b(Rep_{cris}(G))}(\mathbf{Q}_p, \mathbf{R}\Gamma_{\mathrm{cris}}(\overline{X}_{et}, \mathbf{Q}_p(r))[i])$$

The associated spectral sequence

$$E^{a,b}_2 = H^a_f(K, V^b(r)) \Longrightarrow H^{a+b}_{\mathrm{BD}}(X, \mathbf{Q}_p(r))$$

would degenerate into exact sequences

$$0 \longrightarrow H^1_f(K, V^i(r)) \longrightarrow H^{i+1}_{\mathrm{BD}}(X, \mathbf{Q}_p(r)) \longrightarrow H^0(G_K, V^{i+1}(r)) \longrightarrow 0.$$

Theorem 3.2 shows that $H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Q}_p}(r))$ behaves exactly like this conjectural "p-adic Beilinson-Deligne cohomology", as it sits in an exact sequence

$$0 \longrightarrow H^1_f(K, V^i(r)) \longrightarrow H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Q}_p}(r)) \longrightarrow H^0(G_K, V^{i+1}(r)) \longrightarrow 0.$$

It would be interesting to carry out Beilinson's construction [Be 2] of the Beilinson-Deligne cohomology (called "absolute Hodge cohomology" in [Be 2]) in the p-adic context.

(3.5) If K is a finite extension of $\mathbf{Q}_p$ ($\iff$ k is a finite field), then the crystalline Weil conjectures predict that the action of f on D^i should be pure of weight i, *i.e.* all eigenvalues of the semi-simplification of the K_0-linear map $f^{[K_0:\mathbf{Q}_p]} : D^i \longrightarrow D^i$ should be algebraic integers with all archimedean absolute values equal to $p^{[K_0:\mathbf{Q}_p]i/2}$. This was proved by Katz and Messing [KM] for $\mathfrak{X}$ smooth and *projective* over $\mathrm{Spec}(\mathcal{O}_K)$, and by Chiarellotto and Le Stum [CLeS] in general. As a consequence, the crystalline conjecture, (1.7.4) and local duality imply that

$$\begin{aligned} H^0(G_K, V^i(r)) &= 0 \qquad \text{for} \quad r \neq i/2 \\ H^2_{cont}(G_K, V^i(r)) &= 0 \qquad \text{for} \quad r \neq i/2+1 \end{aligned} \tag{3.5.1}$$

It follows from (3.5.1) that the Hochschild-Serre spectral sequence induces isomorphisms

$$\delta : H^{i+1}(X_{et}, \mathbf{Q}_p(r)) \xrightarrow{\sim} H^1_{cont}(G_K, V^i(r)) \qquad (r \neq (i+1)/2) \tag{3.5.2}$$

and Theorem 3.2 reduces to

(3.6) Corollary. *Suppose that K is a finite extension of $\mathbf{Q}_p$ and that $\mathfrak{X}$ is proper and smooth over* $\mathrm{Spec}(\mathcal{O}_K)$*. Then, for every $r \neq (i+1)/2$, the isomorphism (3.5.2) and the Fontaine-Messing map ν induce an isomorphism*

$$\delta \circ \nu : H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Q}_p}(r)) \xrightarrow{\sim} H^1_f(K, V^i(r))$$

(3.7) It is explained in Appendix how to construct Chern classes

$$K_j(\mathfrak{X}) \longrightarrow H^{2i-j}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Q}_p}(r)),$$

which are homomorphisms for $j > 0$ and are compatible with p-adic étale Chern classes on X. The corresponding Chern character maps ([Sc 1], p. 28) define syntomic regulators (resp. cycle classes), which are multiplicative and compatible with p-adic étale regulators (resp. cycle classes) on X.

For $r > (i+1)/2$, put

$$K_{2r-i-1}(\mathfrak{X})_0 := \mathrm{Ker}\left[K_{2r-i-1}(\mathfrak{X}) \longrightarrow H^0(G_K, H^{i+1}(\overline{X}_{et}, \mathbf{Q}_p(r)))\right]$$

and write

$$\delta \circ r_p \circ j^* : K_{2r-i-1}(\mathfrak{X})_0 \longrightarrow H^1_{cont}(G_K, H^i(\overline{X}_{et}, \mathbf{Q}_p(r)))$$

for the map induced by the p-adic étale regulator and the Hochschild-Serre spectral sequence.

(3.8) Theorem. *Under the assumptions of (2.1),*
(1) $\mathrm{Im}\left[CH^r(X)_0 \xrightarrow{r_p} H^1_{cont}(G_K, H^{2r-1}(\overline{X}_{et}, \mathbf{Q}_p(r)))\right] \subseteq H^1_f(K, H^{2r-1}(\overline{X}_{et}, \mathbf{Q}_p(r)))$.
(2) *The cycle class map $cl_X : CH^r(X) \longrightarrow H^{2r}(X_{et}, \mathbf{Q}_p(r))$ satisfies $\mathrm{Im}(cl_X) \cap F^2 H^{2r}(X_{et}, \mathbf{Q}_p(r)) = 0$.*
(3) *For $r > (i+1)/2$,*
$\mathrm{Im}\left[\delta \circ r_p \circ j^* : K_{2r-i-1}(\mathfrak{X})_0 \longrightarrow H^1_{cont}(G_K, H^i(\overline{X}_{et}, \mathbf{Q}_p(r)))\right] \subseteq H^1_f(K, H^i(\overline{X}_{et}, \mathbf{Q}_p(r)))$.
(4) *If K is a finite extension of $\mathbf{Q}_p$, then $K_{2r-i-1}(\mathfrak{X})_0 = K_{2r-i-1}(\mathfrak{X})$ for all $r > (i+1)/2$.*

Proof. (1),(2) This follows from Theorem (3.2), the compatibility of étale and syntomic cycle classes (Appendix) and the fact that $j^* : CH^r(\mathfrak{X}) \longrightarrow CH^r(X)$ is surjective.
(3) Follows from Theorem 3.2 and the compatibility of étale and syntomic regulators (Appendix).
(4) In this case, $H^0(G_K, H^{i+1}(\overline{X}_{et}, \mathbf{Q}_p(r))) = 0$ vanishes, by crystalline Weil conjectures (3.5.1).

(3.9) If L/K is a finite Galois extension, then the canonical maps

$$H^0(K,V) \longrightarrow H^0(L,V)^{G(L/K)}$$
$$H^1_f(K,V) \longrightarrow H^1_f(L,V)^{G(L/K)}$$

are isomorphisms (for every $\mathbf{Q}_p$-representation V of G_K). This implies that the statements (1), (2) of Theorem 3.8 hold true under a weaker assumption that X has a potentially good reduction.

(3.10) Proof of Theorems B, C, D(3). The statements of Theorem B and Theorem D(3) follow from Theorem 3.8 and (3.9). Theorem C follows from Corollary 3.6 and [Ku], as explained in ([Sc 2], Sect. 7).

4. Integral theory

What follows is an integral version of results of Sect. 2. Throughout Sect. 4, $K = K_0$. From 4.13 on, $p > 2$.

(4.1) We recall the main results of the theory of Fontaine-Laffaille [FL]. Denote by $MF_{W,tf}$ the category of triples (M, F^iM, φ_i), where

(0) M is a W-module of finite type.

(1) $(F^iM)_{i\in\mathbf{Z}}$ is a decreasing filtration of M by W-submodules, which are direct summands and satisfy $F^iM = M$ (resp. $F^iM = 0$) for $i << 0$ (resp. $i >> 0$).

(2) $\varphi_i : F^iM \longrightarrow M$ are σ-linear maps with $\varphi_i|_{F^{i+1}M} = p\varphi_{i+1}$.

(3) $\sum_{i\in\mathbf{Z}} \varphi_i(F^iM) = M$.

A morphism $(M, F^iM, \varphi_{i,M}) \longrightarrow (N, F^iN, \varphi_{i,N})$ is a W-linear map $\alpha : M \longrightarrow N$ satisfying $\alpha(F^iM) \subseteq F^iN$, $\alpha \circ \varphi_{i,M} = \varphi_{i,N} \circ \alpha$ (for all $i \in \mathbf{Z}$). Denote by $MF_{W,lf}$ the full subcategory of $MF_{W,tf}$ consisting of objects with $length_W(M) < \infty$ ($\Longleftrightarrow$ M is W-torsion). Both of these categories are abelian and $\mathbf{Z}_p$-linear. All morphisms are strictly compatible with filtrations and all functors $M \mapsto F^iM$ are exact.

(4.2) For a pair of integers $a \leq b$, denote by $MF_W^{[a,b]}$ (resp. $MF_{W,tors}^{[a,b]}$) the full subcategory of $MF_{W,tf}$ (resp. of $MF_{W,lf}$) consisting of objects with $F^aM = M$, $F^{b+1} = 0$. Denote by $MF_W^{[a,b[}$ (resp. $MF_W^{]a,b]}$) the full subcategory of $MF_W^{[a,b]}$ consisting of M with no non-zero quotients N satisfying $N = F^bN$ (resp. no non-zero subobjects satisfying $F^{a+1}N = 0$). (And similarly with $MF_{W,tors}^{[a,b[}$, resp. $MF_{W,tors}^{]a,b]}$).
For every $n \in \mathbf{Z}$ there is a functor $M \mapsto M\{n\}$ ("Tate twist"), given by

$$F^i(M\{n\}) = F^{i+n}M, \qquad \varphi_{i,M\{n\}} = \varphi_{i+n,M}$$

This defines an equivalence of categories between $MF_W^{[a,b]}$ and $MF_W^{[a-n,b-n]}$.
The category $MF_{W,tf}$ admits internal Hom and $\otimes$. The unit object for $\otimes$ is

$$\mathbb{1} = M, \quad F^iM = \begin{cases} W, & \text{if } i \leq 0 \\ 0, & \text{if } i > 0 \end{cases}, \quad \varphi_i = \begin{cases} p^{-i}\sigma, & \text{if } i \leq 0 \\ 0, & \text{if } i > 0 \end{cases}$$

(4.3) The formulas $D := M \otimes_W K$, $F^iD := F^iM \otimes_W K$, $f|_{F^iD} := \varphi_i \otimes p^i\sigma$ define a functor

$$- \otimes_W K : MF_{W,tf} \longrightarrow MF_K, \tag{4.3.1}$$

compatible with internal Hom, $\otimes$ and Tate twists. According to Laffaille ([L], Thm. 3.2), the essential image of (4.3.1) is the category of *weakly admissible* filtered Dieudonné modules MF_K^f. This category was defined by Fontaine, who also proved ([Fo 1], Prop. 4.4.5) that MF_K^f contains MF_K^{ad} and conjectured that these two categories coincide. If we define $MF_K^{[a,b]}$ and its (weakly) admissible versions as in 4.2, then [FL, Thm. 8.4] says that

$$MF_K^{f,[a,b]} = MF_K^{ad,[a,b]}, \qquad \text{if } 0 \leq b - a < p \tag{4.3.2}$$

(4.4) We recall the definition and basic properties of functors ψ_r and Λ_r studied by Kato [Ka].

(4.4.1) Fix $n \geq 0$. Recall (1.3) that $B_n = A_{cris} \otimes \mathbf{Z}/p^n\mathbf{Z}$ has a decreasing filtration by $J^{[r]}_{B_n}$ $(r \geq 0)$ and that, for $0 \leq r < p$, the crystalline Frobenius f defines σ-linear maps $f_r : J^{[r]}_{B_n} \longrightarrow B_n$ such that $p^r f_r = f$. Let M be an object of $MF^{[0,p-1]}_{W,tors}$, killed by p^n. Put on $M \otimes_W A_{cris} = M \otimes_{W_n} B_n$ the convolution filtration

$$F^r(M \otimes_{W_n} B_n) = \sum_{0 \leq i \leq r} \Big((F^i M) \otimes_{W_n} J^{[r-i]}_{B_n} \Big)$$

and define, for $0 \leq r < p$, σ-linear maps $f_r : F^r(M \otimes_W A_{cris}) \longrightarrow M \otimes_W A_{cris}$ by

$$f_r \Big| (F^i M) \otimes_{W_n} J^{[r-i]}_{B_n} = \varphi_{i,M} \otimes f_{r-i} \qquad 0 \leq i \leq r)$$

This is well-defined and independent on n, as all quotients $J^{[i]}_{B_n}/J^{[i+1]}_{B_n}$ are free W_n-modules.

(4.4.2) Definition. For $M \in \mathrm{Ob}(MF^{[0,p-1]}_{W,tors})$ and $0 \leq r < p$, define a $\mathbf{Z}_p[G_K]$-module $\psi_r(M)$ (resp. $\Lambda_r(M)$) to be the kernel (resp. the cokernel) of the map

$$1 - f_r : F^r(M \otimes_W A_{cris}) \longrightarrow M \otimes_W A_{cris}$$

For $M \in \mathrm{Ob}(MF^{[0,p-1]}_W)$ and $0 \leq r < p$, put $\psi_r(M) = \varprojlim_n(\psi_r(M/p^n M))$.

(4.4.3) By ([Ka], II.3.2), a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in $MF^{[0,p-1]}_{W,tors}$ induces an exact sequence of $\mathbf{Z}_p[G_K]$-modules

$$0 \longrightarrow \psi_r(M') \longrightarrow \psi_r(M) \longrightarrow \psi_r(M'') \longrightarrow \Lambda_r(M') \longrightarrow \Lambda_r(M) \longrightarrow \Lambda_r(M'') \longrightarrow 0$$

(for every $0 \leq r < p$).

(4.5) Lemma. *Let $M \in \mathrm{Ob}(MF^{[0,p-1]}_W)$, $0 \leq r < p$. Then*
(1) If $r > 0$, then the canonical map $\psi_{r-1}(M)(1) \longrightarrow \psi_r(M)$ (induced by $\mu_{p^n}(\mathcal{O}_{\overline{K}}) \hookrightarrow J^{[1]}_{B_n}$) is injective. If $F^r M = 0$, then it is an isomorphism.
(2) Assume that $r < p-1$, $length_W(M) < \infty$ and either: (a) $F^{r+1}M = 0$, or (b) $F^r M = M$. Then $\Lambda_r(M) = 0$.

Proof. First of all, we can assume that $k = \overline{k}$, as replacing M by $M \otimes_W W(\overline{k}) \in \mathrm{Ob}(MF^{[0,p-1]}_{W(\overline{k})})$ does not change $\psi_r(M)$ nor $\Lambda_r(M)$. In this case, all statements are proved in ([Ka], II.3.4.1, 3.7.1, 3.8.1) for simple objects of $MF^{[0,p-1]}_{W(\overline{k})}$ (note that the statements of ([Ka], II.3.4, 3.7) are true even for $r = p-1$, by ([FL], Thm. 5.3)). The general case follows by dévissage and (4.4.3).

(4.6) We are now ready to formulate a covariant version of the main result of Fontaine and Laffaille ([FL], Thm. 5.3, Thm. 6.1) (*cf.* [Fo 3], [Ka]). Denote by $Rep_{\mathbf{Z}_p}(G_K)$ the category of $\mathbf{Z}_p$-modules of finite type with a continuous linear action of G_K (recall that $K = K_0$).

Proposition. (1) *The formula $T(M) := \psi_{p-1}(M)(1-p)$ defines an exact and faithful functor $T : MF^{[0,p-1]}_W \longrightarrow Rep_{\mathbf{Z}_p}(G_K)$.*
(2) If $M \in \mathrm{Ob}(MF^{[0,p-1]}_{W,tors})$, then $T(M)$ is finite and $length_W(M) = length_{\mathbf{Z}_p}(T(M))$.
(3) If $M \in \mathrm{Ob}(MF^{[0,p-1]}_W)$ is free over W, then $T(M)$ is free over $\mathbf{Z}_p$ and $\mathrm{rk}_W(M) = \mathrm{rk}_{\mathbf{Z}_p}(T(M))$.
(4) The restriction of T to $MF^{[0,p-1[}_W$ (resp. $MF^{]0,p-1]}_W$) is fully faithful, inducing an equivalence of categories between $MF^{[0,p-1[}_W$ (resp. $MF^{]0,p-1]}_W$) and its essential image.

(5) *For every* $M \in \mathrm{Ob}(MF_W^{[0,p-1]})$, *the filtered Dieudonné module* $D := W \otimes_K M$ *is admissible and there is a canonical isomorphism* $V(D) \xrightarrow{\sim} T(M) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

For example, for $0 \le r < p$, $\mathbb{1}\{-r\}$ is an object of $MF_W^{[0,p-1]}$ and $T(\mathbb{1}\{-r\}) = \mathbf{Z}_p(-r)$.
The isomorphism in (5) is given by

$$\psi_{p-1}(M)(1-p) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \xrightarrow{\sim} \left(F^{p-1}(D \otimes_{K_0} B^+_{cris})(1-p)\right)^{f_{p-1}=1} \xrightarrow{\sim} F^0(D \otimes_{K_0} t^{1-p} B^+_{cris})^{f=1} \xrightarrow{\sim}$$
$$\xrightarrow{\sim} F^0(D \otimes_{K_0} B_{cris})^{f=1}$$

(4.7) For $M \in \mathrm{Ob}(MF_{W,tf})$, the complex

$$C^\cdot(M) = \left[F^0 M \xrightarrow{1-\varphi_0} M\right]$$

(in degrees 0, 1) computes the Ext-groups $\mathrm{Ext}(\mathbb{1}, M)$ in the category $MF_{W,tf}$. The isomorphisms between $H^i(C^\cdot(M))$ and $\mathrm{Ext}^i_{MF_{W,tf}}(\mathbb{1}, M)$ can be described as follows.
For $i = 0$, a morphism $\alpha : \mathbb{1} \longrightarrow M$ is uniquely determined by its value $\alpha(1) \in (F^0 M)^{\varphi_0 = 1} = H^0(C^\cdot(M))$.
For $i = 1$, an extension $0 \longrightarrow M \longrightarrow E \longrightarrow \mathbb{1} \longrightarrow 0$ gives an exact sequence of W-modules

$$0 \longrightarrow F^0 M \longrightarrow F^0 E \longrightarrow W \longrightarrow 0$$

Choosing a W-linear splitting $s : W \longrightarrow F^0 E$, we obtain an element $m = (\varphi_{0,E} - 1)(s(1)) \in M$, the class of which in $M/(1-\varphi_{0,M})F^0 M = H^1(C^\cdot(M))$ depends only on the isomorphism class of E. As $H^1(C^\cdot(M))$ is right exact, it follows that $\mathrm{Ext}^i(\mathbb{1}, M)$ vanish for $i > 1$.
Replacing M by $M\{r\}$, we see that the complex

$$[F^r M \xrightarrow{1-\varphi_r} M]$$

computes $\mathrm{Ext}(\mathbb{1}, M\{r\}) = \mathrm{Ext}^i(\mathbb{1}\{-r\}, M)$ (for every $r \in \mathbf{Z}$).
Put $D := M \otimes_W K$, which is an object of MF_K^f. The maps $incl : F^0 D \hookrightarrow D$, $(id, 0) : D \longrightarrow D \oplus D/F^0 D$ define a quasi-isomorphism between $C^\cdot(M) \otimes_W K = [F^0 D \xrightarrow{1-f} D]$ and the complex

$$\left[D \xrightarrow{(1-f,\mathrm{can})} D \oplus D/F^0 D\right]$$

of 1.7.4.

(4.8) For $M \in \mathrm{Ob}(MF_W^{[0,p-1]})$, put $T := T(M)$. If $0 \le r < p$, then the functor T induces homomorphisms (by the discussion in 4.7)

$$\alpha_{r,M} : (F^r M)^{\varphi_r = 1} \xrightarrow{\sim} \mathrm{Hom}_{MF_{W,tf}}(\mathbb{1}\{-r\}, M) = \mathrm{Hom}_{MF_W^{[0,p-1]}}(\mathbb{1}\{-r\}, M) \hookrightarrow$$
$$\hookrightarrow \mathrm{Hom}_{Rep_{\mathbf{Z}_p}(G_K)}(\mathbf{Z}_p(-r), T) = T(r)^{G_K}$$

(injective) and

$$\beta_{r,M} : \mathrm{Coker}\,[F^r M \xrightarrow{1-\varphi_r} M] \xrightarrow{\sim} \mathrm{Ext}^1_{MF_{W,tf}}(\mathbb{1}\{-r\}, M) = \mathrm{Ext}^1_{MF_W^{[0,p-1]}}(\mathbb{1}\{-r\}, M) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^1_{Rep_{\mathbf{Z}_p}(G_K)}(\mathbf{Z}_p(-r), T) = H^1_{cont}(G_K, T(r))$$

The filtered Dieudonné module $D := M \otimes_W K$ is admissible (4.3.2) and the canonical map $V := T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \longrightarrow V(D)$ is an isomorphism (4.6.5). Comparing (4.7) with (1.7.4), we see that the following diagrams are commutative (the vertical maps are given by $- \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$):

$$\begin{array}{ccc} (F^rM)^{\varphi_r=1} & \stackrel{\alpha_{r,M}}{\longrightarrow} & T(r)^{G_K} \\ \downarrow & & \downarrow \\ (F^rD)^{p^{-r}f=1} & \stackrel{1.7}{\longrightarrow} & V(r)^{G_K} \end{array}$$

$$\begin{array}{ccccc} \mathrm{Coker}\,[F^rM \xrightarrow{1-\varphi_r} M] & & \xrightarrow{\beta_{r,M}} & & H^1_{cont}(G_K, T(r)) \\ \downarrow & & & & \downarrow \\ \mathrm{Coker}\,[F^rD \xrightarrow{1-p^{-r}f} D] & \xrightarrow{\sim} & \mathrm{Coker}\,[D \xrightarrow{(1-p^{-r}f,\mathrm{can})} D \oplus D/F^rD] & \xrightarrow{exp_{V(r)}} & H^1_{cont}(G_K, V(r)) \end{array}$$

As $\mathrm{Im}(exp_{V(r)}) = H^1_f(K, V(r))$, this shows that $\mathrm{Im}(\beta_{r,M}) \subseteq H^1_f(K, T(r))$.

If $M \in \mathrm{Ob}(MF_W^{[0,p-1[})$ and $0 \le r < p-1$ (i.e. $\mathbb{1}\{-r\} \in \mathrm{Ob}(MF_W^{[0,p-1[})$), then $\alpha_{r,M}$ is an isomorphism and $\beta_{r,M}$ is injective, by Proposition 4.6.4.

(4.9) Lemma. ([BK], Lemma 4.5) *Let $M \in \mathrm{Ob}(MF_W^{[0,p-1[})$, $0 \le r < p-1$. Put $T := T(M)$. If M is torsion-free ($\Longleftrightarrow$ T is torsion-free), then $\beta_{r,M}$ induces an isomorphism*

$$\mathrm{Coker}\,[F^rM \xrightarrow{1-\varphi_r} M] \xrightarrow{\sim} H^1_f(K, T(r)) \subseteq H^1_{cont}(G_K, T(r))$$

Proof. [BK] [1] One must show that the cokernel of the map

$$\mathrm{Coker}\,[F^rM \xrightarrow{1-\varphi_r} M] \longrightarrow H^1_{cont}(K, T(r))$$

is torsion-free. This follows from the commutative diagram with exact rows

$$\begin{array}{ccccccc} M/(1-\varphi_r)F^rM & \xrightarrow{p} & M/(1-\varphi_r)F^rM & \longrightarrow & (M/pM)/(1-\varphi_r)(F^r(M/pM)) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \beta_{r,M}} & & \downarrow{\scriptstyle \beta_{r,M}} & & \downarrow{\scriptstyle \beta_{r,M/pM}} & & \\ H^1_{cont}(G_K, T(r)) & \xrightarrow{p} & H^1_{cont}(G_K, T(r)) & \longrightarrow & H^1(G_K, T/pT(r)) & & \end{array}$$

and the injectivity of all vertical arrows $\beta_{r,-}$ (by (4.8)).

(4.10) Geometric case. Let $\mathfrak{X}$ be a proper and smooth scheme over $\mathrm{Spec}(W)$, $X = \mathfrak{X} \otimes_W K$, $\overline{X} = X \otimes_K \overline{K}$, $\mathfrak{X}_n = \mathfrak{X} \otimes \mathbf{Z}/p^n\mathbf{Z}$, $Y = \mathfrak{X}_1$, $\overline{\mathfrak{X}} = \mathfrak{X} \otimes_W \mathcal{O}_{\overline{K}}$, $\overline{Y} = \overline{\mathfrak{X}} \otimes_{\mathcal{O}_{\overline{K}}} \overline{k}$. Put $d = \dim(Y)$.

For $q, r, n \ge 0$, denote

$$\begin{aligned} M^q_n &:= H^q((\mathfrak{X}_n)_{syn}, \mathcal{O}^{cris}_n) = H^q((Y/W_n)_{\mathrm{cris}}, \mathcal{O}_{Y/W_n}) = H^q_{dR}(\mathfrak{X}_n/W_n) \\ F^rM^q_n &:= H^q((\mathfrak{X}_n)_{syn}, J^{[r]}_n) = H^q((\mathfrak{X}_n)_{Zar}, \sigma_{\ge r}\Omega^{\cdot}_{\mathfrak{X}_n/W_n}) \\ T^q_n &:= H^q(\overline{X}_{et}, \mathbf{Z}/p^n\mathbf{Z}); \qquad T^q := H^q(\overline{X}_{et}, \mathbf{Z}_p) \\ M^q &:= H^q(\mathfrak{X}_{syn}, \mathcal{O}^{cris}_{\mathbf{Z}_p}) = H^q((Y/W)_{\mathrm{cris}}, \mathcal{O}_{Y/W}) \\ F^rM^q &:= H^q(\mathfrak{X}_{syn}, J^{[r]}_{\mathbf{Z}_p}) \end{aligned}$$

By ([BO 1], 7.24.3), we have

$$M^q \xrightarrow{\sim} \varprojlim_n (M^q_n), \quad F^rM^q \xrightarrow{\sim} \varprojlim_n (F^rM^q_n), \quad T^q \xrightarrow{\sim} \varprojlim_n (T^q_n)$$

and the projective system $[M^q_n]_{n\in\mathbf{N}}$ (resp. $[T^q_n]_{n\in\mathbf{N}}$) is AR-isomorphic to $[M^q \otimes \mathbf{Z}/p^n\mathbf{Z}]$ (resp. $[T^q \otimes \mathbf{Z}/p^n\mathbf{Z}]$). All F^rM^q are W-modules of finite type, T^q are objects of $Rep_{\mathbf{Z}_p}(G_K)$.

[1] Note that there is a misprint in ([BK], Lemma 4.5); it should read H^1_f, not H^1_e.

(4.11) Proposition. *If* $\min(q,d) < p$, *then*
(1) *The canonical maps* $F^{r+1}M_n^q \longrightarrow F^r M_n^q$, $F^{r+1}M^q \longrightarrow F^r M^q$ *are injective* $\forall n, r \geq 0$).
(2) $(M_n^q, F^r M_n^q, \varphi_r = f_r)$ *(resp.* $(M^q, F^r M^q, \varphi_r = f_r)$*) is an object of* $MF_{W,tors}^{[0,\min(q,d)]}$ *(resp.* $MF_W^{[0,\min(q,d)]}$*).*

Proof. ([FM], II.2.7), ([Ka], II.2.5, I.1.8), ([Fo 3], Prop. 1.3).

(4.12) Proposition. *Let* $q, n \geq 0$, $0 \leq r < p$. *Then*
(1) $H^q(\overline{\mathfrak{X}}_{syn}, \mathcal{O}_n^{cris}) \xrightarrow{\sim} H^q_{dR}(\mathfrak{X}_n/W_n) \otimes_{W_n} B_n = M_n^q \otimes_{W_n} B_n$.
(2) *If* $\min(q+1,d) < p$, *then*

(i) $H^q(\overline{\mathfrak{X}}_{syn}, J_n^{[r]}) \xrightarrow{\sim} F^r(H^q_{dR}(\mathfrak{X}_n/W_n) \otimes_{W_n} B_n) = F^r(M_n^q \otimes_{W_n} B_n)$.

(ii) *There is an exact sequence*

$$0 \longrightarrow \Lambda_r(M_n^{q-1}) \longrightarrow H^q(\overline{\mathfrak{X}}_{syn}, s_n(r)) \longrightarrow \psi_r(M_n^q) \longrightarrow 0$$

(iii) *If* $p-1 > r \geq \min(q-1,d)$, *then* $H^q(\overline{\mathfrak{X}}_{syn}, s_n(r)) \xrightarrow{\sim} \psi_r(M_n^q)$.

Proof. (1) ([FM], III.1.3), ([Ka], I.4.6, I.1.8).
(2)(i) ([FM], III.1.5.i), ([Ka], II.4.1, I.1.8). We recall the argument: there are spectral sequences

$$E_1^{a,b} = H^b(\mathfrak{X}_n, \Omega^a_{\mathfrak{X}_n/W_n}) \Longrightarrow M_n^{a+b}$$
$${}'E_1^{a,b} = E_1^{a,b} \otimes_{W_n} J_{B_n}^{[r-a]} \Longrightarrow H^{a+b}((\overline{\mathfrak{X}}_n)_{syn}, J_n^{[r]})$$

According to Proposition 4.11.1, we have $E_1^{a,b} = E_\infty^{a,b}$ for $a+b < p-1$ (or even for all $a, b \geq 0$, provided $d < p$). As $J_{B_n}^{[r-a]}$ is flat over W_n, ${}'E_c^{a,b} = E_c^{a,b} \otimes_{W_n} J_{B_n}^{[r-a]}$ for all $a, b \geq 0$, $c \geq 1$. This implies the claim.
(ii) This follows form (1), (i) and the exact cohomology sequence of $0 \longrightarrow s_n(r) \longrightarrow J_n^{[r]} \xrightarrow{1-f_r} \mathcal{O}_n^{cris} \longrightarrow 0$.
(iii) By Lemma 4.5.2, $\Lambda_r(M_n^{q-1}) = 0$.

(4.13) Proposition. *(*[Ka]*) For* $0 \leq q \leq r < p-1$ *and* $n \geq 0$, *the Fontaine-Messing map induces isomorphisms*

$$H^q(\overline{\mathfrak{X}}_{syn}, s_n(r)) \xrightarrow{\sim} H^q(\overline{X}_{et}, \mathbf{Z}/p^n\mathbf{Z}(r)) = T_n^q(r)$$
$$H^q(\overline{\mathfrak{X}}_{syn}, s_{\mathbf{Z}_p}(r)) \xrightarrow{\sim} H^q(\overline{X}_{et}, \mathbf{Z}_p(r)) = T^q(r)$$

Proof. The first statement is proved by Kato ([Ka], I.4.3). The second statement follows by passing to the projective limit, as $R^1 \varprojlim_n T_n^q(r) = 0$ by I.2.3.3.1.

(4.14) Corollary. *For* $0 \leq q \leq r < p-1$ *and* $n \geq 0$, *the Fontaine-Messing map and (4.12) induce isomorphisms*

$$T_n^q \xrightarrow{\sim} \psi_r(M_n^q)(-r) \xrightarrow{\sim} T(M_n^q)$$
$$T^q \xrightarrow{\sim} \psi_r(M^q)(-r) \xrightarrow{\sim} T(M^q)$$

(The second isomorphism comes from Lemma 4.5.1.)

Proof. Combine Proposition 4.13, Proposition 4.12.2.(iii) and Lemma 4.5.1.

(4.15) As in (2.8), we invoke the axiomatic setting of I.3.1.
(0) Fix an integer $i \in \mathbf{Z}$ (to be specified later).
(1),(2) We have categories $\mathcal{C} = \mathrm{Mod}(Sets, \text{“}\mathbf{Z}_p\text{”})$, $\overline{\mathcal{C}} = \mathrm{Mod}(B_G, \text{“}\mathbf{Z}_p\text{”})$, $\mathcal{D} = (\mathbf{Z}_p - Mod)$ and functors $u = (\Gamma_G^{\mathbf{N}})_*$, $\Psi = \Gamma_{\mathbf{N}} = \varprojlim_{\mathbf{N}}$, $\Phi = \Psi \circ u = \varprojlim_{\mathbf{N}}(-)^G$.
(3),(4) For an integer $0 \leq r < p$, there are distinguished triangles

$$\Delta_r : \mathbf{R}\Gamma^{\mathbf{N}}(\widehat{\mathfrak{X}}_{syn-et}, s_{\text{“}\mathbf{Z}_p\text{”}}(r)) \longrightarrow \mathbf{R}\Gamma^{\mathbf{N}}(\widehat{\mathfrak{X}}_{syn-et}, J^{[r]}_{\text{“}\mathbf{Z}_p\text{”}}) \xrightarrow{1-f_r} \mathbf{R}\Gamma^{\mathbf{N}}(\widehat{\mathfrak{X}}_{syn-et}, \mathcal{O}^{cris}_{\text{“}\mathbf{Z}_p\text{”}}) \longrightarrow$$
$$\longrightarrow \mathbf{R}\Gamma^{\mathbf{N}}(\widehat{\mathfrak{X}}_{syn-et}, s_{\text{“}\mathbf{Z}_p\text{”}}(r))[1]$$

in $D^+(\mathcal{C})$ and

$$\overline{\Delta}_r : \mathbf{R}\Gamma^{\mathbf{N}}(\widehat{\overline{\mathfrak{X}}}_{syn-et}, s_{\text{“}\mathbf{Z}_p\text{”}}(r)) \longrightarrow \mathbf{R}\Gamma^{\mathbf{N}}(\widehat{\overline{\mathfrak{X}}}_{syn-et}, J^{[r]}_{\text{“}\mathbf{Z}_p\text{”}}) \xrightarrow{1-f_r} \mathbf{R}\Gamma^{\mathbf{N}}(\widehat{\overline{\mathfrak{X}}}_{syn-et}, \mathcal{O}^{cris}_{\text{“}\mathbf{Z}_p\text{”}}) \longrightarrow$$
$$\longrightarrow \mathbf{R}\Gamma^{\mathbf{N}}(\widehat{\overline{\mathfrak{X}}}_{syn-et}, s_{\text{“}\mathbf{Z}_p\text{”}}(r))[1]$$

in in $D^+(\overline{\mathcal{C}})$. By Appendix, Δ_r lies in $D^b(\mathcal{C})$. If $q \geq 0$ is an integer satisfying $r+q<p$, then the same argument as in (2.5) gives morphisms of triangles

$$\left(\Gamma^{\mathbf{N}}_G\right)^* \Delta_r \longrightarrow \overline{\Delta}_{r+q}(-q)$$

in in $D^+(\overline{\mathcal{C}})$ and

$$\Delta_r \longrightarrow \mathbf{R}\left(\Gamma^{\mathbf{N}}_G\right)_* \left(\overline{\Delta}_{r+q}(-q)\right)$$

in $D^+(\mathcal{C})$. This defines a morphism $\Delta \longrightarrow (\mathbf{R}u)(\overline{\Delta})$ for $\Delta := \Delta_r$, $\overline{\Delta} := \overline{\Delta}_{r+q}(-q)$.
(5) As before, étale cohomology defines objects $E = \mathbf{R}\Gamma^{\mathbf{N}}(X_{et}, \text{“}\mathbf{Z}_p(r)\text{”})$ (resp. $\overline{E} = \mathbf{R}\Gamma^{\mathbf{N}}(\overline{X}_{et}, \text{“}\mathbf{Z}_p(r)\text{”})$) of $D^+(\mathcal{C})$ (resp. $D^+(\overline{\mathcal{C}})$) and an isomorphism $\rho : E \xrightarrow{\sim} (\mathbf{R}u)(\overline{E})$.
(6) The morphisms $\mu : A \longrightarrow E$ (resp. $\overline{\mu} : \overline{A} \longrightarrow \overline{E}$) are given by the Fontaine-Messing map ([FM], III.5.1; Appendix).

(4.16) We shall verify the axioms (A1), (A2'), (A3') of I.3.1, under the assumptions

$$0 \leq i+1, r < p-1$$

We choose $q \geq 0$ such that $i+1 \leq r+q < p-1$.
By Proposition 4.11 and 4.12,

$$\begin{aligned} H^j(\overline{B}) &= \left[F^{r+q}(M^j_n \otimes_{W_n} B_n)(-q)\right]_{n\in\mathbf{N}} \qquad (\forall j \leq p-2) \\ H^j(\overline{C}) &= \left[M^j_n \otimes_{W_n} B_n(-q)\right]_{n\in\mathbf{N}} \qquad (\forall j \geq 0) \end{aligned} \tag{4.16.1}$$

Lemma 4.5.2 and Proposition 4.11 then imply

$$\operatorname{Ker}\left[H^j(\overline{A}) \longrightarrow H^j(\overline{B})\right] = \operatorname{Coker}\left[H^{j-1}(\overline{B}) \longrightarrow H^{j-1}(\overline{C})\right] = 0$$

for all $j \leq r+q+1$, proving (A1). According to Proposition 4.13, μ induces isomorphisms

$$H^j(\overline{A}) \xrightarrow{\sim} H^j(\overline{E}) \qquad (\forall j \leq r+q),$$

which gives (A2'). Now to (A3'): by definition,

$$(R^j\Psi)(B) = F^r M^j, \qquad (R^j\Psi)(C) = M^j \qquad (\forall j \geq 0) \tag{4.16.2}$$

It follows from (4.16.1) that

$$\begin{aligned} \Phi(H^j(\overline{B})) &= \varprojlim_n \left(F^{r+q}(M^j_n \otimes_W A_{cris})(-q)\right)^G \qquad (\forall j \leq p-2) \\ \Phi(H^j(\overline{C})) &= \varprojlim_n \left(M^j_n \otimes_W A_{cris}(-q)\right)^G \qquad (\forall j \geq 0) \end{aligned} \tag{4.16.3}$$

Recall that, for $n \geq 0$ and $0 \leq q < p-1$,

$$(A_{cris} \otimes \mathbf{Z}/p^n\mathbf{Z}(-q))^G = \mathbf{Z}/p^n\mathbf{Z} \cdot t^q, \tag{4.16.4}$$

where t is a $\mathbf{Z}_p$-basis of $\mathbf{Z}_p(1) \hookrightarrow A_{cris}$. Combining (4.16.2–4), we get (A3').

(4.17) In fact, the assumptions of (4.16) can be slightly altered, to cover the value $i+1 = p-1$ as well. Assume that

$$0 \le r < p-1, \qquad \dim(\mathfrak{X}) < p \tag{4.17.1}$$

Then all M^j are objects of $MF_W^{[0,p-2]}$, by Proposition 4. 11(2). Consider the objects defined in (4.15), for $q = p-2-r = i-r$. The formulas (4.16.1) hold for all $j \ge 0$ and the same arguments as in the rest of (4.16) give (A1), (A3') and show that $\overline{\mu}$ induces an isomorphism $\tau_{\le i}\overline{A} \xrightarrow{\sim} \tau_{\le i}\overline{E}$.
Our aim is to show that $\Phi(H^{i+1}(\overline{A})) \longrightarrow \Phi(H^{i+1}(\overline{E}))$ is injective. Observing that both maps

$$\alpha_{r,M^{p-1}} : \mathrm{Ker}(\lambda'_{p-1,r}) = \left(F^r M^{p-1}\right)^{\varphi_r=1} \longrightarrow \Phi(H^{i+1}(\overline{A})) = (T(M^{p-1})(r))^{G_K}$$
$$\mathrm{Ker}(\lambda_{p-1,r}) = \mathrm{Ker}(\lambda'_{p-1,r}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \longrightarrow \Phi(H^{i+1}(\overline{E})) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p = H^0(G_K, V^{i+1}(r))$$

are isomorphisms, by (4.8) and (3.1.2.1) respectively, we see that (A2) is satisfied, provided

$$F^r M^{p-1} \text{ is torsion} - \text{free} \tag{4.17.2}$$

5. The Integral Comparison Theorem

(5.1) The assumptions of (4.1) (including $p > 2$) are in force. For $0 \le r < p$ and $j \ge 0$, denote the map $1-\varphi_r : F^r M^j \longrightarrow M^j$ by $\lambda'_{j,r}$. As in (3.1), the cohomology sequence of $\Delta = \Delta_r$ becomes

$$0 \longrightarrow \mathrm{Coker}(\lambda'_{i,r}) \longrightarrow H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Z}_p}(r)) \longrightarrow \mathrm{Ker}(\lambda'_{i+1,r}) \longrightarrow 0$$

According to (4.8) and Proposition 4.11, for j satisfying $\min(j,d) < p$ there are injective homomorphisms

$$\alpha_{r,M^j} : \mathrm{Ker}(\lambda'_{j,r}) \hookrightarrow H^0(K, T(M^j)(r))$$
$$\beta_{r,M^j} : \mathrm{Coker}(\lambda'_{j,r}) \hookrightarrow H^1_f(K, T(M^j)(r))$$

Moreover, Corollary 4.14 says that $T(M^j)$ is canonically isomorphic to T^j, provided $0 \le j < p-1$.

(5.2) Theorem. *Assume that $0 \le i+1, r < p-1$. Then:*
(1) *The Fontaine-Messing map $\nu : H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Z}_p}(r)) \longrightarrow H^{i+1}(X_{et}, \mathbf{Z}_p(r))$ is injective and gives rise to a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Coker}(\lambda'_{i,r}) & \longrightarrow & H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Z}_p}(r)) & \longrightarrow & \mathrm{Ker}(\lambda'_{i+1,r}) & \longrightarrow & 0 \\
 & & \downarrow \beta & & \downarrow \nu & & \downarrow \alpha & & \\
0 & \longrightarrow & F^1 H^{i+1}(X_{et}, \mathbf{Z}_p(r)) & \longrightarrow & H^{i+1}(X_{et}, \mathbf{Z}_p(r)) & \longrightarrow & H^0(G, T^{i+1}(r)) & \longrightarrow & 0 \\
 & & \downarrow \delta & & & & & & \\
 & & H^1_{cont}(G, T^i(r)) & & & & & &
\end{array}$$

(2) $\alpha = \alpha_{r,M^{i+1}}$ is an isomorphism, $\delta \circ \beta = \beta_{r,M^i}$ is injective and $\mathrm{Im}(\delta \circ \beta) \subseteq H^1_f(K, T^i(r))$. If M^i is torsion-free ($\Longleftrightarrow$ T^i is torsion-free), then $\mathrm{Im}(\delta \circ \beta) = H^1_f(K, T^i(r))$.

Proof. This follows from Proposition I.3.5 (applied to the data (4.15)), (4.8) and Lemma 4.9. The axioms of I.3.1 were verified in (4.16).

(5.3) As in (3.3), Theorem 5.2 can be reformulated in terms of the filtrations $F^j_{et} = F^j H^{i+1}(X_{et}, \mathbf{Q}_p(r))$ (with $F^j_{et} = 0$ for $j > 2$) and F^j_{syn} defined by

$$
\begin{aligned}
F^0_{syn} &= H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Z}_p}(r))\\
F^1_{syn} &= \operatorname{Ker}\left[H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Z}_p}(r)) \longrightarrow H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, J^{[r]}_{\mathbf{Z}_p})\right] \overset{(4.11.1)}{=}\\
&= \operatorname{Ker}\left[H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Z}_p}(r)) \longrightarrow H^{i+1}(\widehat{\mathfrak{X}}_{syn-et}, \mathcal{O}^{cris}_{\mathbf{Z}_p})\right]\\
F^2_{syn} &= 0
\end{aligned}
$$

Proposition. *Let $0 \le i+1, r < p-1$. Then the Fontaine-Messing map ν satisfies*
(1) $\nu(F^j_{syn}) \subseteq F^j_{et} \quad (\forall j \ge 0)$.
(2) $gr^0_F(\nu) : F^0_{syn}/F^1_{syn} \xrightarrow{\sim} F^0_{et}/F^1_{et} = E^{0,i+1}_\infty = E^{0,i+1}_2 = H^0(G, T^{i+1}(r))$ *is an isomorphism.*
(3) $gr^1_F(\nu) : F^1_{syn} \hookrightarrow F^1_{et}/F^2_{et} = E^{1,i}_\infty = E^{1,i}_2 = H^1_{cont}(G_K, T^i(r))$ *is injective, with image contained in* $H^1_f(K, T^i(r))$.
(4) *If T^i is torsion-free ($\Longleftrightarrow$ M^i is torsion-free), then the image of $gr^1_F(\nu)$ is equal to $H^1_f(K, T^i(r))$.*

(5.4) Suppose that $p > \dim(\mathfrak{X}) - 1$, $0 \le r < p$. The map

$$
(-1)^{r-1} c^{syn}_r/(r-1)! : \left(K_0(\mathfrak{X}) \otimes \mathbf{Z}\left[\frac{1}{(\dim(\mathfrak{X})-1)!}\right]\right)^{(r)} \longrightarrow H^{2r}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Z}_p}(r)),
$$

composed with the isomorphism ([So 2], Thm. 4(iv); [GiSo], Thm. 8.2)

$$
CH^r(\mathfrak{X}) \otimes \mathbf{Z}\left[\frac{1}{(\dim(\mathfrak{X})-1)!}\right] \xrightarrow{\sim} \left(K_0(\mathfrak{X}) \otimes \mathbf{Z}\left[\frac{1}{(\dim(\mathfrak{X})-1)!}\right]\right)^{(r)}
$$

defines a cycle class map $cl_{\mathfrak{X}}$ which makes the following diagram commutative:

$$
\begin{array}{ccc}
CH^r(\mathfrak{X}) \otimes \mathbf{Z}\left[\frac{1}{(\dim(\mathfrak{X})-1)!}\right] & \xrightarrow{j^*} & CH^r(X) \otimes \mathbf{Z}\left[\frac{1}{(\dim(\mathfrak{X})-1)!}\right] \\
\downarrow{\scriptstyle cl_{\mathfrak{X}}} & & \downarrow{\scriptstyle cl_X} \\
H^{2r}(\widehat{\mathfrak{X}}_{syn-et}, s_{\mathbf{Z}_p}(r)) & \xrightarrow{\nu} & H^{2r}(X_{et}, \mathbf{Z}_p(r))
\end{array}
\tag{5.4.1}
$$

(5.5) Proposition. *Suppose that $p > \max(2r+1, \dim(\mathfrak{X})-1)$. Then*

$$
\operatorname{Im}(cl_X) \cap F^2 H^{2r}(X_{et}, \mathbf{Z}_p(r)) = 0.
$$

Proof. This follows from Proposition 5.3, the diagram (5.4.1) and the surjectivity of the map j^* in (5.4.1).

(5.6) Assume that $0 \le r < p-1$, $i+1 = p-1$, $\dim(\mathfrak{X}) < p$ and that $F^r H^{p-1}_{dR}(\mathfrak{X}/W)$ is torsion-free. Then the conclusions of (5.2)–(5.5) are still valid with the following modifications: in Theorem 5.2, the map α is injective and there is no zero at the end of the second row in (1). In Proposition 5.3(2), the map $gr^0_F(\nu)$ is injective. In Proposition 5.5, we take $r = (i+1)/2$. All this follows again from Proposition I.3.5 and 4.17.

IV. Cycle classes and filtrations on Chow groups

In this chapter we treat varieties over number fields, by putting together local results of Chapters II and III,

1. Beilinson's philosophy

(1.1) let $F/\mathbf{Q}$ be a finitely generated extension of transcendence degree $d \geq 0$. If $\pi : X \longrightarrow \mathrm{Spec}(F)$ is a smooth and projective scheme, then the conjectural spectral sequence (0.1.0)

$$E_2^{a,b} = H^a(\mathrm{Spec}(F)_{mot}, R^b\pi_*\mathbf{Q}(n)) \Longrightarrow H^{a+b}(X_{mot}, \mathbf{Q}(n))$$

is expected to degenerate at E_2. It is also expected that $E_2^{a,b} = 0$ for $a > d+1$ ([Ja 4], 4.12(c)). Consequently, the induced filtration $F^{\cdot}H^*(X_{mot}, \mathbf{Q}(n))$ should satisfy

$$\begin{aligned} gr_F^j H^m(X_{mot}, \mathbf{Q}(n)) &\xrightarrow{\sim} H^j(\mathrm{Spec}(F)_{mot}, R^{m-j}\pi_*\mathbf{Q}(n)) \\ F^{d+2}H^m(X_{mot}, \mathbf{Q}(n)) &= 0 \end{aligned} \tag{1.1.1}$$

The corresponding p-adic étale realization, given by the Hochschild-Serre spectral sequence (0.3.0), degenerates at E_2 and defines a filtration on $H^*(X_{et}, \mathbf{Q}_p(n))$ satisfying

$$gr_F^j H^m(X_{et}, \mathbf{Q}_p(n)) \xrightarrow{\sim} H^j_{cont}(G_F, H^{m-j}(\overline{X}_{et}, \mathbf{Q}_p(n)))$$

The p-adic realization map

$$H^m(X_{mot}, \mathbf{Q}(n)) \longrightarrow H^m(X_{et}, \mathbf{Q}_p(n)) \tag{1.1.2}$$

is expected to be *strictly* compatible with the filtrations, thus inducing *injective* maps

$$gr_F^j H^m(X_{mot}, \mathbf{Q}(n)) \hookrightarrow gr_F^j H^m(X_{et}, \mathbf{Q}_p(n)) \tag{1.1.3}$$

(1.2) In order to make (1.1.3) a testable statement, one replaces motivic cohomology by its K-theoretic version

$$H^m(X_{mot}, \mathbf{Q}(n)) \stackrel{?}{=} K_{2n-m}(X)^{(n)}_{\mathbf{Q}}$$

We shall be particularly interested in the case $m = 2n$, when

$$K_0(X)^{(n)}_{\mathbf{Q}} \xrightarrow{\sim} CH^n(X) \otimes \mathbf{Q}$$

and the map (1.1.2) is replaced by the cycle class map

$$cl_X : CH^n(X) \otimes \mathbf{Q} \longrightarrow H^{2n}(X_{et}, \mathbf{Q}_p(n))$$

The filtration $F^iH^{2n}(X_{mot}, \mathbf{Q}(n))$ should then correspond to the mysterious filtration on $CH^n(X) \otimes \mathbf{Q}$, studied extensively by Beilinson, Bloch, Jannsen, Murre, Raskind, S. Saito and many other people (see [Ja 4] and [Ra] for references). The conjecture (1.1.1) predicts that

$$F^{d+2}(CH^n(X) \otimes \mathbf{Q}) = 0 \tag{1.2.1}$$

and 1.1.3 can be reformulated as

$$F^j(CH^n(X) \otimes \mathbf{Q}) = cl_X^{-1}\left(F^jH^{2n}(X_{et}, \mathbf{Q}_p(n))\right) \tag{1.2.2}$$

This implies, among other things, that

$$\mathrm{Ker}(cl_X) = 0 \tag{1.2.3}$$
$$\mathrm{Im}(cl_X) \cap F^{d+2}H^{2n}(X_{et}, \mathbf{Q}_p(n)) = 0 \tag{1.2.4}$$

The injectivity of cl_X is a very difficult problem. We shall investigate (1.2.4) in the special case $d = 0$ (i.e. when F is a number field).

2. Cycle class maps and étale cohomology

(2.1) Fix a prime number p. Let $F/\mathbf{Q}$ be a finite extension, $\pi_F : X \longrightarrow \mathrm{Spec}(F)$ a proper and smooth map. Let S be a finite set of primes of F such that $\{v|p\} \subseteq S$ and that there is a proper and smooth model $\pi : \mathfrak{X} \longrightarrow \mathrm{Spec}(\mathcal{O}_{F,S}) = \mathrm{Spec}(\mathcal{O}_F) - S$ of X. Let $G_{F,S}$ be the Galois group of the maximal extension of F unramified outside of $S \cup \{v|\infty\}$. Denote the map $\mathrm{Spec}(F) \longrightarrow \mathrm{Spec}(\mathcal{O}_{F,S})$ by j.

For every $i, n \geq 0$, $r \in \mathbf{Z}$, the sheaf $A = R^i\pi_*(\mathbf{Z}/p^n\mathbf{Z}(r))$ on $\mathrm{Spec}(\mathcal{O}_{F,S})_{et}$ is locally constant and constructible, thus $A = j_*j^*A$, $j^*A = R^i\pi_{F*}(\mathbf{Z}/p^n\mathbf{Z}(r)) = H^i(\overline{X}_{et}, \mathbf{Z}/p^n\mathbf{Z}(r))$ (which is a finite $G_{F,S}$-module) and

$$H^*(\mathrm{Spec}(\mathcal{O}_{F,S})_{et}, A) = H^*(G_{F,S}, H^i(\overline{X}_{et}, \mathbf{Z}/p^n\mathbf{Z}(r)))$$

([Mi, II.2.9]). Using I.2.3, we get a similar statement for continuous cohomology

$$H^*(\mathrm{Spec}(\mathcal{O}_{F,S})_{et}, [R^i\pi_*(\mathbf{Z}/p^n\mathbf{Z}(r))]_{n\in\mathbf{N}}) = H^*_{cont}(G_{F,S}, T^i(r)) \tag{2.1.1}$$

Here $T^i(r) = H^i(\overline{X}_{et}, \mathbf{Z}_p)$. Of course, as $G_{F,S}$ satisfies the condition I.2.3.2, the continuous cohomology group $H^j_{cont}(G_{F,S}, T^i(r))$ coincides with the naive one and is a $\mathbf{Z}_p$-module of finite type.

(2.2) It follows from (2.1.1) that the Leray spectral sequence for π_* is just

$$E_2^{a,b} = H^a_{cont}(G_{F,S}, T^b(r)) \Longrightarrow H^{a+b}(\mathfrak{X}_{et}, \mathbf{Z}_p(r))$$

(*cf.* [Ja 2, Lemma 6]). It defines a filtration $F^{\cdot}\mathcal{H}^{n,r}$ on $\mathcal{H}^{n,r} = H^n(\mathfrak{X}_{et}, \mathbf{Z}_p(r))$. Similarly, the Leray spectral sequence for π_{F*}

$${}'E_2^{a,b} = H^a_{cont}(G_F, T^b(r)) \Longrightarrow H^{a+b}(X_{et}, \mathbf{Z}_p(r))$$

defines a filtration $F^{\cdot}H^{n,r}$ on $H^{n,r} = H^n(X_{et}, \mathbf{Z}_p(r))$. The restriction map $j^* : \mathcal{H}^{n,r} \longrightarrow H^{n,r}$ is compatible with the filtrations.

As $cd_p(G_F) = cd_p(G_{F,S}) = 2$, we have

$$\begin{aligned} gr^0_F(\mathcal{H}^{n,r}) &= E_\infty^{0,n} = E_3^{0,n} \hookrightarrow E_2^{0,n} \\ gr^1_F(\mathcal{H}^{n,r}) &= E_\infty^{1,n-1} = E_2^{1,n-1} \\ gr^2_F(\mathcal{H}^{n,r}) &= F^2\mathcal{H}^{n,r} = E_\infty^{2,n-2} = E_3^{2,n-2} \\ F^3\mathcal{H}^{n,r} &= 0 \end{aligned} \tag{2.2.1}$$

(and similarly for $H^{n,r}$). It follows from Weil's conjectures [De 3] that $E_2^{0,n} = {}'E_2^{0,n}$ are finite groups for $n \neq 2r$, thus

$$E_\infty^{2,n-2} \text{ differs from } E_2^{2,n-2} \text{ by a finite group} \qquad (n \neq 2r - 1) \tag{2.2.2}$$

(and similarly for ${}'E_2$). It follows from remarks at the end of (2.1) that all groups $E_2^{a,b}$, $\mathcal{H}^{n,r}$ are $\mathbf{Z}_p$-modules of finite type.

As

$$E_2^{0,n} = T^n(r)^{G_{F,S}} = T^n(r)^{G_F} = {}'E_2^{0,n}$$
$$E_2^{1,n-1} = H^1_{cont}(G_{F,S}, T^{n-1}(r)) \overset{inf}{\hookrightarrow} H^1_{cont}(G_F, T^{n-1}(r)) = {}'E_2^{1,n-1},$$

we see that both maps

$$gr^0_F(j^*) : gr^0_F(\mathcal{H}^{n,r}) \longrightarrow gr^0_F(H^{n,r})$$
$$gr^1_F(j^*) : gr^1_F(\mathcal{H}^{n,r}) \longrightarrow gr^1_F(H^{n,r})$$

are *injective*. As a result, we have

$$F^k\mathcal{H}^{n,r} = (j^*)^{-1}(F^kH^{n,r}) \qquad (k = 0, 1, 2) \tag{2.2.3}$$

(2.3) For every $r \geq 0$, there is a commutative diagram of cycle class maps (*cf.* [Sa], Sect. 5)

$$\begin{array}{ccccc} & CH^r(\mathfrak{X}) & \xrightarrow{j^*} & CH^r(X) & \\ & \downarrow cl_{\mathfrak{X}} & & \downarrow cl_X & \\ \mathcal{H}^{2r,r} = & H^{2r}(\mathfrak{X}_{et}, \mathbf{Z}_p(r)) & \xrightarrow{j^*} & H^{2r}(X_{et}, \mathbf{Z}_p(r)) & = H^{2r,r}, \end{array}$$

in which the top horizontal arrow is surjective. Combined with (2.2.3), this implies that

$$\mathrm{Im}(cl_X) \cap F^kH^{2r,r} = j^*(\mathrm{Im}(cl_{\mathfrak{X}}) \cap F^k\mathcal{H}^{2r,r}) \qquad (k = 0, 1, 2) \tag{2.3.1}$$

(2.4) We are also interested in torsion phenomena. Recall a well-known

Lemma. *Let F be a field finitely generated over $\mathbf{Q}$, X a separated scheme of finite type over F.*
(1) For every $a \geq 0$ there is an integer $C' \geq 1$ such that $H^a(\overline{X}_{et}, \mathbf{Z}_p)_{tors}$ is finite for every prime number p and vanishes for $p \not| C'(X, a)$.
(2) If X is proper and smooth over F, then, for every $a \geq 0$, $b \in \mathbf{Z}$, $a \neq 2b$, there is an integer $C(X, a, b) \geq 1$ such that $\left(H^a(\overline{X}_{et}, \mathbf{Z}_p(b)) \otimes \mathbf{Q}_p/\mathbf{Z}_p\right)^{G_F}$ is finite for every prime number p and vanishes for $p \not| C(X, a, b)$.

Proof. (1) Choose an embedding $\sigma : \overline{F} \hookrightarrow \mathbf{C}$ and put $Y = \overline{X} \otimes_{\overline{F},\sigma} \mathbf{C}$. The smooth base change and comparison theorem with classical cohomology tell us that

$$H^a(\overline{X}_{et}, \mathbf{Z}_p) = H^a(Y(\mathbf{C}), \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_p$$

However, $H^a(Y(\mathbf{C}), \mathbf{Z})$ is a finitely generated abelian group, so its torsion is finite.
(2) There exists a subring $R \hookrightarrow F$, finitely generated as a $\mathbf{Z}$-algebra, and a proper and smooth model $\mathfrak{X} \longrightarrow \mathrm{Spec}(R)$ of X. Let v be a closed point of $\mathrm{Spec}(R)$, with residue field $k(v)$. Let p be a prime, different from the characteristic ℓ_v of $k(v)$. Then $H^a(\overline{X}_{et}, \mathbf{Z}_p(b))$ is isomorphic to $H^a((\overline{X} \otimes_R \overline{k}(v))_{et}, \mathbf{Z}_p(b))$, by the smooth and proper base change theorems. By Weil's conjectures [De 4], the polynomial $P_v(T) = \det(1 - Fr(v)T \,|\, H^a(\overline{X}_{et}, \mathbf{Q}_p(b))$ has coefficients in $\mathbf{Z}[1/\ell_v]$, is independent of p and $P_v(1) \neq 1$ (as $H^a(\overline{X}_{et}, \mathbf{Q}_p(b))$ is pure of weight $a - 2b \neq 0$ at v). The group $\left(H^a(\overline{X}_{et}, \mathbf{Z}_p(b)) \otimes \mathbf{Q}_p/\mathbf{Z}_p\right)^{G_F}$ is annihilated by $P_v(1)$ by the Cayley-Hamilton theorem, so we can take $C(X, a, b) = \ell_v \cdot (\text{numerator of } P_v(1))$.

(2.5) Lemma. *In the situation of 2.1, we have*

$$\left(gr^0_F H^n(X_{et}, \mathbf{Z}_p(r))\right)_{tors} = \left(gr^0_F H^n(\mathfrak{X}_{et}, \mathbf{Z}_p(r))\right)_{tors} = 0 \qquad (p \not| C'(X, n))$$
$$\left(gr^1_F H^n(X_{et}, \mathbf{Z}_p(r))\right)_{tors} = \left(gr^1_F H^n(\mathfrak{X}_{et}, \mathbf{Z}_p(r))\right)_{tors} = 0 \qquad (p \not| C(X, n-1, r)C'(X, n-1), \quad n \neq 2r + 1)$$

Proof. As $(E_2^{0,n})_{tors} = ({}'E_2^{0,n}))_{tors} = H^n(X_{et}, \mathbf{Z}_p(r))^{G_F}_{tors}$, the first statement follows from Lemma 2.4.(1) and (2.2.1). For the second statement, observe that

$$(E_2^{1,n-1})_{tors} \hookrightarrow ('E_2^{1,n-1})_{tors} = H^1(G_F, H^{n-1}(\overline{X}_{et}, \mathbf{Z}_p(r))_{tors}) \oplus H^1(G_F, H^{n-1}(\overline{X}_{et}, \mathbf{Z}_p(r))/tors)_{tors}$$

The first term vanishes for $p \not| C'(X, n-1)$, by Lemma 2.4.(1). The second term is isomorphic to ([Ta], Prop. 2.3)

$$\left(H^{n-1}(\overline{X}_{et}, \mathbf{Z}_p(r)) \otimes \mathbf{Q}_p/\mathbf{Z}_p\right)^{G_F} /\mathrm{Div},$$

which vanishes for $p \not| C(X, n-1, r)$, $n \neq 2r+1$, by Lemma 2.4.(2). We conclude again by (2.2.1).

(2.6) Corollary. *If X is equidimensional of dimension d and $A = \mathrm{Alb}(X)$ is the Albanese variety of X, then*

$$\begin{aligned} \left(gr^0_F H^{2d}(X_{et}, \mathbf{Z}_p(d))\right)_{tors} &= 0 \qquad \text{(for all } p) \\ \left(gr^1_F H^{2d}(X_{et}, \mathbf{Z}_p(d))\right)_{tors} &= 0 \qquad (\text{for } p \not| \sharp(A(F)_{tors})) \end{aligned}$$

Proof. The trace map induces an isomorphism $H^{2d}(\overline{X}_{et}, \mathbf{Z}_p(d)) \xrightarrow{\sim} \mathbf{Z}_p^c$, thus $C'(X, 2d) = 1$. As

$$H^{2d-1}(\overline{X}_{et}, \mathbf{Z}_p(d)) \xrightarrow{\sim} T_p(A),$$

we have

$$\left(H^{2d-1}(\overline{X}_{et}, \mathbf{Z}_p(d)) \otimes \mathbf{Q}_p/\mathbf{Z}_p\right)^{G_F} = A(F)_{p^\infty}.$$

Both claims now follow from Lemma 2.5.

3. Proof of Theorems E, F, G

(3.1) We begin with Theorem E, following its notation and assumptions. Fix a proper and smooth model $\mathfrak{X} \longrightarrow \mathrm{Spec}(\mathcal{O}_{F,S})$ of X as in (2.1). Writing V^i for $H^i(\overline{X}_{et}, \mathbf{Q}_p)$, then we have (by I.2.3.3 and (2.2.2))

$$\begin{aligned} F^2 H^{2n}(X_{et}, \mathbf{Q}_p(n)) &= H^2_{cont}(G_F, V^{2n-2}(n)) \\ F^2 H^{2n}(\mathfrak{X}_{et}, \mathbf{Q}_p(n)) &= H^2_{cont}(G_{F,S}, V^{2n-2}(n)) \end{aligned}$$

For every prime v of F, the restriction map

$$res_v : H^{2n}(X_{et}, \mathbf{Q}_p(n)) \longrightarrow H^{2n}((X_v)_{et}, \mathbf{Q}_p(n))$$

(where $X_v = X \otimes_F F_v$) is compatible with the filtrations F^j on both sides. Again by I.2.3.3, we have

$$F^2 H^{2n}((X_v)_{et}, \mathbf{Q}_p(n)) = H^2_{cont}(G_{F_v}, V^{2n-2}(n))$$

Applying II.2.5.3 and III.3.8-9, we see that

$$\mathrm{Im}(cl_{\mathfrak{X}}) \cap F^2 H^{2n}(\mathfrak{X}_{et}, \mathbf{Q}_p(n)) \subseteq \mathrm{Ker}(\alpha_{S,\Sigma})$$

Theorem E now follows from a $\mathbf{Q}_p$-version of 2.3.1.

(3.2) For Theorem F, we apply Theorem E, with $F = \mathbf{Q}$ and $X = E^d$. According to Proposition II.2.6(i), we have $\Sigma = S$. Poitou-Tate duality and Poincaré duality for $\overline{X}$ show that $\mathrm{Ker}(\alpha_{S,S})$ is the $\mathbf{Q}_p$-dual of

$$\mathrm{Ker}\left[\beta_V : H^1_{cont}(G_{\mathbf{Q},S}, V) \longrightarrow \bigoplus_{\ell \in S} H^1_{cont}(G_{\mathbf{Q}_\ell}, V)\right],$$

where $V = H^2(\overline{X}_{et}, \mathbf{Q}_p(1))$. Künneth formula gives

$$V = \mathbf{Q}_p{}^{\binom{d+1}{2}} \oplus W^{\binom{d}{2}},$$

where

$$W = \mathrm{Sym}^2\left(H^1(\overline{E}_{et}, \mathbf{Q}_p)\right)(1).$$

Class field theory shows that $\mathrm{Ker}(\beta_{\mathbf{Q}_p}) = 0$. The vanishing of $\mathrm{Ker}(\beta_W)$ follows from the results of [Fl], [La 1, Lemma 2.5], [Wi].

(3.3) We now turn to the proof of Theorem G, with the following value of c_E:

$$c_E = 2N_E(\deg(\Phi))(\sharp(E(\mathbf{Q})_{tors}))\prod_{\ell\in T_1}(\ell-1)(-\mathrm{ord}_\ell(j(E)))\prod_{\ell\in T_2}\ell \qquad (3.3.1)$$

Here N_E is the conductor of E, $\Phi : X_0(N_E) \longrightarrow E$ some modular parametrization of E and

$$T_1 = \{\ell \,|\, E \text{ has a potentially multiplicative reduction at } \ell\}$$
$$T_2 = \{\ell \,|\, G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}(E_\ell) \text{ is not surjective}\}$$

Note that, according to ([Se 2], Thm. 2), the set T_2 is finite. If E is semistable (i.e. N_E is square-free), then $T_2 \subseteq \{2,3,5,7\}$ ([Se 2], Prop. 21).

If $d = 1$, then $CH^1(E) = \mathrm{Pic}(E)$, $CH^1(E)_0 = \mathrm{Pic}^0(E) = E(\mathbf{Q})$ and the kernel of $cl_E : CH^1(E) \longrightarrow H^2(E_{et}, \mathbf{Z}_p(1))$ is equal to the prime-to-p torsion in $E(\mathbf{Q})$; thus $\mathrm{Im}(cl_E)_{tors} = E(\mathbf{Q})_{p^\infty}$.

Suppose now that $d > 1$ and $p \not| (2d)! \cdot c_E$ (in particular, $p > 3$). Put $X = E^d$ $(= \mathrm{Alb}(X))$. As $p \not| \sharp((E^d)_{tors})$, Cor. 2.6 implies that

$$H^{2d}(\mathfrak{X}_{et}, \mathbf{Z}_p(d))_{tors} \subseteq \left(F^2H^{2d}(\mathfrak{X}_{et}, \mathbf{Z}_p(d))\right)_{tors}$$

Here we take $\mathfrak{X}$ as in (2.1), with $S = \{\ell | pN_E\}$. Write, as before, $T^i = H^i(\overline{X}_{et}, \mathbf{Z}_p)$. Each T^i is torsion-free by Künneth formula, thus the spectral sequence in (2.2) satisfies $E_2^{0,i} = 0$ for $i \neq 2d$. This shows that

$$F^2H^{2d}(\mathfrak{X}_{et}, \mathbf{Z}_p(d)) = E_2^{2,2d-2} = H^2_{cont}(G_{\mathbf{Q},S}, T^{2d-2}(d))$$

In fact the same argument applies locally: for any $n \geq 1$ and any prime ℓ, we have

$$\left(V^{2n-1}(n)\right)^{G_{\mathbf{Q}_\ell}} = 0,$$

by II.1.6.4 and II.1.7(1a) (resp. III.3.5.1) for $\ell \neq p$ (resp. $\ell = p$). As T^{2n-1} is torsion-free, we obtain

$$F^2H^{2n}((X_\ell)_{et}, \mathbf{Z}_p(n)) = H^2_{cont}(G_{\mathbf{Q}_\ell}, T^{2n-2}(n)) \qquad (3.3.2)$$

(here $X_\ell = X \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$).

Fix now $\ell \in S$ and consider the restriction map

$$res_\ell : H^2_{cont}(G_{\mathbf{Q},S}, T^{2d-2}(d)) \longrightarrow H^2_{cont}(G_{\mathbf{Q}_\ell}, T^{2d-2}(d))$$

By Künneth formula,

$$T^{2d-2}(d) = \mathbf{Z}_p(1)^d \oplus \left(T_p(E))^{\otimes 2}\right)^{\binom{d}{2}}.$$

It follows that

$$H^2_{cont}(G_{\mathbf{Q}_\ell}, T^{2d-2}(d))_{tors} = H^2_{cont}(G_{\mathbf{Q}_\ell}, (T_p(E))^{\otimes 2})^{\binom{d}{2}}_{tors}. \qquad (3.3.3)$$

Write H for $\mathrm{Im}(cl_{\mathfrak{X}})_{tors}$. We claim that

$$res_\ell(H) = 0 \qquad (\forall \ell \in S) \tag{3.3.4}$$

Case $\ell = p$: In this case (3.3.4) follows from (3.3.2) and Proposition III.5.5.

Case $\ell \neq p$: By ([Si], IV.10.3 and V.5.3), the greatest common divisor of degrees $[K : \mathbf{Q}_\ell]$ of extensions over which E acquires semistable reduction divides 24. As $p > 3$, we can choose such an extension $K/\mathbf{Q}_\ell$ of degree d prime to p.

Subcase $\ell \in T_1$: By Lemma 3.4 below, the group (3.3.3) vanishes.

Subcase $\ell \notin T_1 \cup \{p\}$: E has a potentially good reduction at ℓ. As above, we can choose an extension $K/\mathbf{Q}_\ell$ of degree prime to p over which E acquires good reduction. In this case (3.3.4) follows from Proposition II.2.8 and (3.3.2).

So we see that H is contained in (the torsion subgroup of)

$$\mathrm{Ker}\left[\alpha : H^2_{cont}(G_{\mathbf{Q},S}, T) \longrightarrow \bigoplus_{\ell \in S} H^2_{cont}(G_{\mathbf{Q}_\ell}, T)\right]$$

for $T = T^{2d-2}(d)$. By Poitou-Tate duality, $\mathrm{Ker}(\alpha)$ is the Pontryagin dual of

$$\mathrm{Ker}\left[\beta : H^1_{cont}(G_{\mathbf{Q},S}, T^*(1) \otimes \mathbf{Q}_p/\mathbf{Z}_p) \longrightarrow \bigoplus_{\ell \in S} H^1_{cont}(G_{\mathbf{Q}_\ell}, T^*(1) \otimes \mathbf{Q}_p/\mathbf{Z}_p)\right]$$

We have

$$T^*(1) = \mathbf{Z}_p{}^{\binom{d+1}{2}} \oplus \left(\mathrm{Sym}^2(T_p(E))(-1)\right)^{\binom{d}{2}}$$

The first factor does not contribute to $\mathrm{Ker}(\beta)$ by class field theory; as for the second factor,

$$\mathrm{Ker}\left[H^1_{cont}(G_{\mathbf{Q},S}, \mathrm{Sym}^2(T_p(E))(-1) \otimes \mathbf{Q}_p/\mathbf{Z}_p) \longrightarrow \bigoplus_{\ell \in S} H^1_{cont}(G_{\mathbf{Q}_\ell}, \mathrm{Sym}^2(T_p(E))(-1) \otimes \mathbf{Q}_p/\mathbf{Z}_p)\right]$$

vanishes by ([Fl], Thm. 1). Theorem G is proved.

(3.4) Lemma. *Let K be a finite extension of $\mathbf{Q}_\ell$, k the residue field of K, E an elliptic curve over K with a potentially multiplicative reduction. For a prime $p \neq \ell$, put $T = T_p(E)$, $V = T \otimes \mathbf{Q}$. If $p \nmid (\sharp(k^*))(-\mathrm{ord}_K(j(E)))$ (here $\mathrm{ord}_K : K^* \longrightarrow \mathbf{Z}$ is a surjective valuation and $j(E)$ the j-invariant of E), then*
(1) $H^0(G_K, V^{\otimes 2}/T^{\otimes 2}(-1)) \xrightarrow{\sim} \mathbf{Q}_p/\mathbf{Z}_p$.
(2) $H^2_{cont}(G_K, T^{\otimes 2}) \xrightarrow{\sim} \mathbf{Z}_p$.

Proof. It is sufficient to prove (1); the statement (2) then follows by local duality, as $T^*(1) \xrightarrow{\sim} T$. Our assumptions imply that E is a quadratic twist of a Tate curve. Such a twist does not change the Galois representation $T^{\otimes 2}$, so we can assume that E itself is a Tate curve over K, with Tate's parameter $q_E \in K^*$. The boundary map

$$\delta : \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow H^1(G_K, \mathbf{Q}_p/\mathbf{Z}_p(1)) = K^* \otimes \mathbf{Q}_p/\mathbf{Z}_p$$

associated to the standard exact sequence of G_K-modules

$$0 \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p(1) \xrightarrow{\alpha} V/T \xrightarrow{\beta} \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow 0$$

is given by $\delta(a) = q_E \otimes a$. As $p \nmid (\sharp(k^*))$, the valuation ord_K defines an isomorphism $\mathrm{ord}_K \otimes id : K^* \otimes \mathbf{Q}_p/\mathbf{Z}_p \xrightarrow{\sim} \mathbf{Q}_p/\mathbf{Z}_p$. This implies that δ is an isomorphism as well, since $p \nmid \mathrm{ord}_K(q_E) = -\mathrm{ord}_K(j(E))$. Observing that

$$H^0(G_K, \mathbf{Q}_p/\mathbf{Z}_p(1)) = H^0(G_K, \mathbf{Q}_p/\mathbf{Z}_p(-1)) = 0$$

(again by $p \not| (\sharp(k^*))$), we see that various cohomology sequences associated to the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{Q}_p/\mathbf{Z}_p(1) & \xrightarrow{id\otimes\alpha(-1)} & V/T & \xrightarrow{id\otimes\beta(-1)} & \mathbf{Q}_p/\mathbf{Z}_p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha(-1)\otimes id} & & \downarrow{\scriptstyle \alpha(-1)\otimes id} & & \downarrow{\scriptstyle \alpha(-1)\otimes id} & & \\
0 & \longrightarrow & V/T & \xrightarrow{id\otimes\alpha(-1)} & V^{\otimes 2}/T^{\otimes 2}(-1) & \xrightarrow{id\otimes\beta(-1)} & V/T(-1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta(-1)\otimes id} & & \downarrow{\scriptstyle \beta(-1)\otimes id} & & \downarrow{\scriptstyle \beta(-1)\otimes id} & & \\
0 & \longrightarrow & \mathbf{Q}_p/\mathbf{Z}_p & \xrightarrow{id\otimes\alpha(-1)} & V/T(-1) & \xrightarrow{id\otimes\beta(-1)} & \mathbf{Q}_p/\mathbf{Z}_p(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

give $H^0(G_K, V/T) = \mathrm{Ker}(\delta) = 0$ and isomorphisms

$$\begin{array}{ccc}
 & & \mathbf{Q}_p/\mathbf{Z}_p \\
 & & \downarrow \wr \\
H^0(G_K, V^{\otimes 2}/T^{\otimes 2}(-1)) & \xrightarrow{\sim} & H^0(G_K, V/T(-1)),
\end{array} \tag{3.3.5}$$

which proves the statement (1) of the Lemma (to see that the horizontal map in (3.3.5) is an isomorphism, note that the boundary map

$$H^0(G_K, V/T(-1)) \longrightarrow H^1(G_K, V/T)$$

composed with the isomorphism $H^0(G_K, \mathbf{Q}_p/\mathbf{Z}_p) \xrightarrow{\sim} H^0(G_K, V/T(-1))$ vanishes for trivial reasons).

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Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
16 Mill Lane
Cambridge CB2 1SB
United Kingdom

e-mail: nekovar@pmms.cam.ac.uk