Elliptic curves for SNARK and proof systems

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Journées Numération, Arithmétique, Cryptographie
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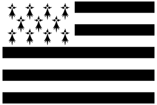


https://webusers.imj-prg.fr/~jean-claude.bajard/NAC2024/

Outline

zk-SNARK

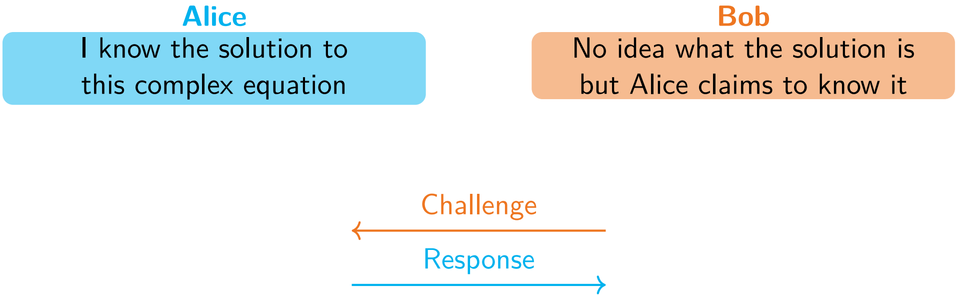
Elliptic Curves

Pairings

Pairing-friendly curves

SNARK-friendly curves

Zero-knowledge proofs (ZKP)



Alice

I know the solution to this complex equation

Bob

No idea what the solution is but Alice claims to know it

Challenge

Response

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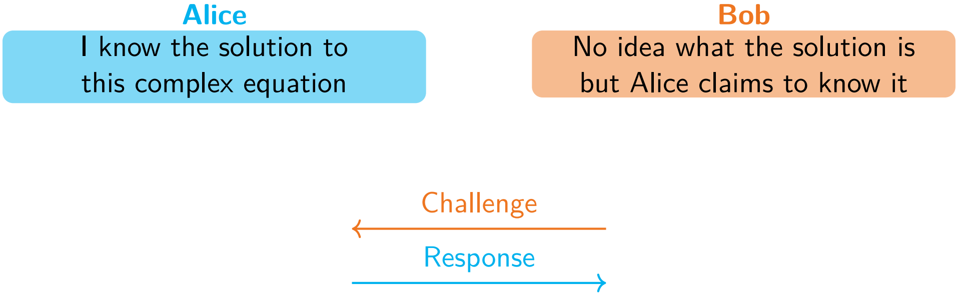
Zero-knowledge proofs (ZKP)



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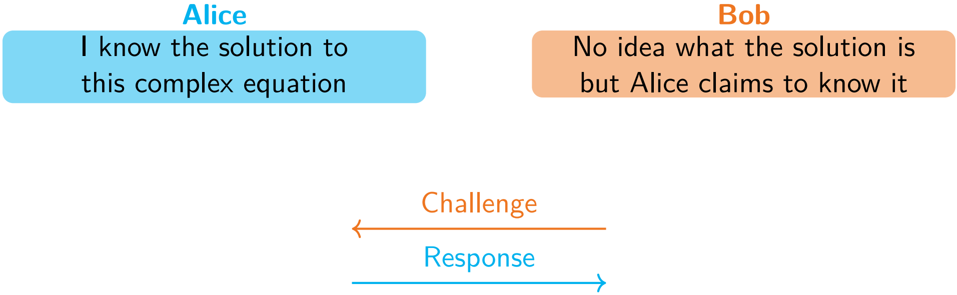

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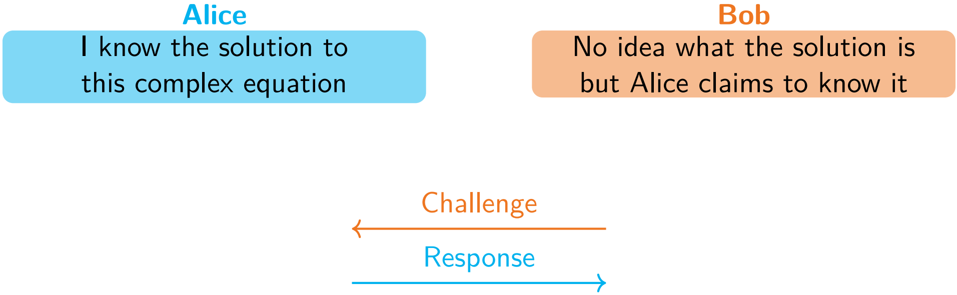

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- **Complete**: **Alice** has the solution $\implies$ **Bob** is convinced.
- **Zero-knowledge**: **Bob** does NOT learn the solution.

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Example: Sigma protocol

Alice

Bob

I know x such that $g^x = y$

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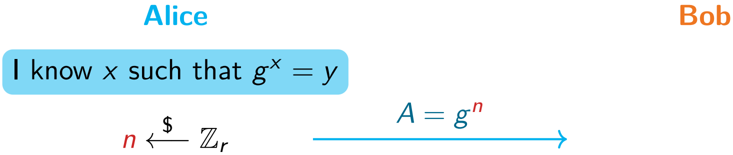
Example: Sigma protocol



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I know x such that $g^x = y$

$n \xleftarrow{\$} \mathbb{Z}_r$

$\xrightarrow{A = g^n}$

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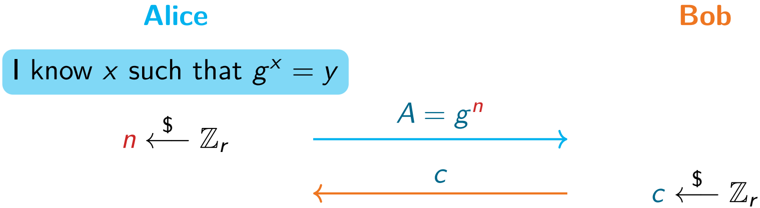

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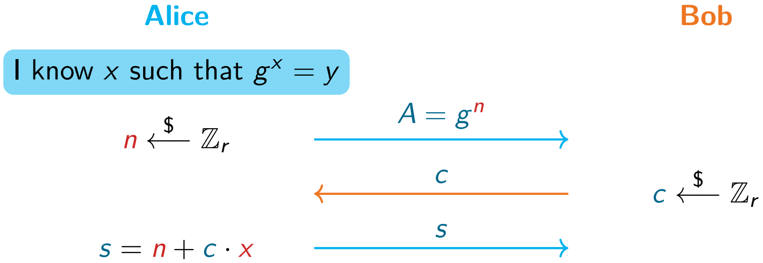

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$s = n + c \cdot x$

s

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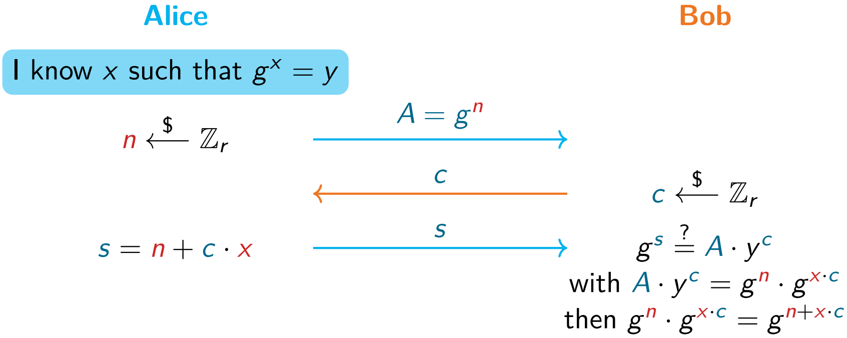

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$\xleftarrow{c}$

$c \xleftarrow{\$} \mathbb{Z}_r$

$s = n + c \cdot x$

$\xrightarrow{s}$

$g^s \overset{?}{=} A \cdot y^c$

with $A \cdot y^c = g^n \cdot g^{x \cdot c}$

then $g^n \cdot g^{x \cdot c} = g^{n + x \cdot c}$

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Non-Interactive Zero-Knowledge (NIZK) Sigma protocol

Alice **Bob**

I know x such that $g^{x} = y$

$$n \xleftarrow{\$} \mathbb{Z}_r$$

$$g \quad ; A = g^{n}$$

$$c = H(A, y)$$

$$s = n + c \cdot x$$

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$$c = H(A, y)$$

$$\underbrace{s = n + c \cdot x}_{\text{Prove}}$$

$$\xrightarrow{\qquad\qquad} $$

$$\underbrace{\pi = (A, c, s)}_{\text{proof}}$$

$$g^s \overset{?}{=} A \cdot y^c$$

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ZKP literature landmarks

- First ZKP work [GMR85]
- Non-Interactive ZKP [BFM88]
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- SNARK with universal and updatable setup [GKM$^+$18], [MBKM19] (Sonic), [GWC19] (PlonK), [CHM$^+$20] (Marlin), ...

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What is a zero-knowledge proof?

“I have a *sound*, *complete* and *zero-knowledge* proof that a statement is true”. [GMR85]

Sound

False statement $\implies$ cheating prover cannot convince honest verifier.

Complete

True statement $\implies$ honest prover convinces honest verifier.

Zero-knowledge

True statement $\implies$ verifier learns nothing other than statement is true.

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zk-SNARK: Zero-Knowledge Succinct Non-interactive ARgument of Knowledge

"I have a *computationally sound*, *complete*, *zero-knowledge*, **succinct**, **non-interactive** proof that a statement is true and that I know a related secret".

Succinct

A proof is very short and easy to verify.

Non-interactive

No interaction between the prover and verifier for proof generation and verification (except the proof message).

ARgument of Knowledge

Honest verifier is convinced that a computationally bounded prover knows a secret information.

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Preprocessing zk-SNARK for NP language

F: public `NP` program, x, z: public inputs, w: private input (witness)

$z := F(x, w)$

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A zk-SNARK consists of algorithms S, P, V s.t. for a security parameter λ:

$$\begin{array}{llll} \mathit{Setup}: & (pk, vk) & \leftarrow & S(F, 1^{\lambda}) \\ \mathit{Prove}: & \pi & \leftarrow & P(x, z, w, pk) \\ \mathit{Verify}: & \mathtt{false/true} & \leftarrow & V(x, z, \pi, vk) \end{array}$$

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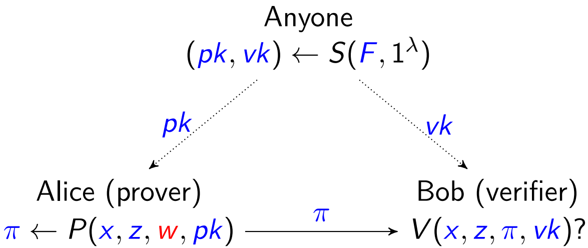

Anyone

$(pk, vk) \leftarrow S(F, 1^{\lambda})$

Alice (prover)

$\pi \leftarrow P(x, z, w, pk)$

Bob (verifier)

$V(x, z, \pi, vk)?$

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zk-SNARKs in a nutshell

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3. Use homomorphic hiding cryptography to blindly verify the polynomial equation.

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2. Use Schwartz–Zippel lemma to succinctly verify the polynomial equation with high probability.
3. Use homomorphic hiding cryptography to blindly verify the polynomial equation.
4. Make the protocol non-interactive.

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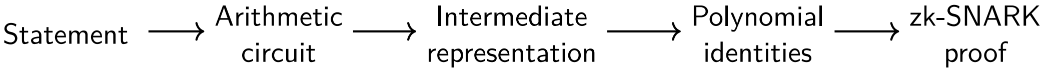
Data flow



Statement → Arithmetic circuit → Intermediate representation → Polynomial identities → zk-SNARK proof

(knowledge of a
hash function preimage,
a discrete log...)

(QAP: Quadratic
Arithmetic Program)

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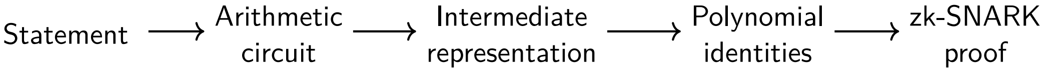

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Group $\langle g \rangle$ of order r,
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$$g^s \overset{?}{=} A \cdot y^c$$

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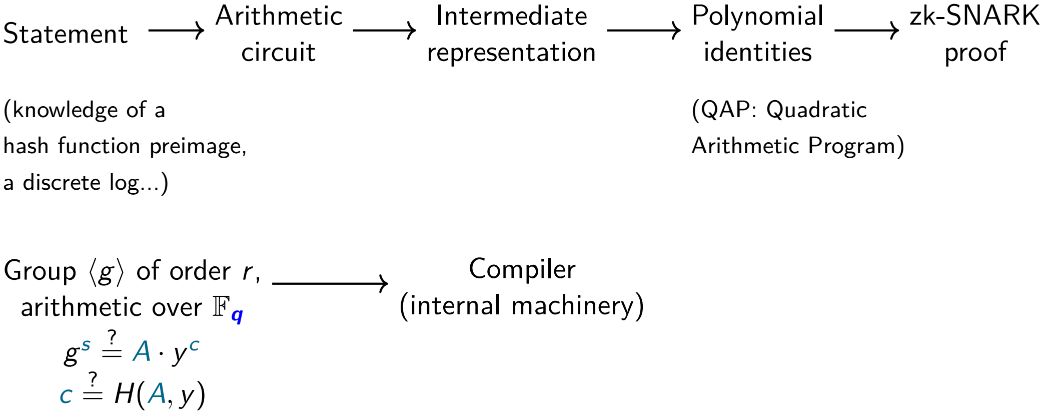

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Group $\langle g \rangle$ of order r, arithmetic over $\mathbb{F}_{q}$ ⟶ Compiler (internal machinery)

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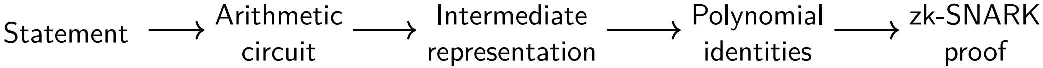

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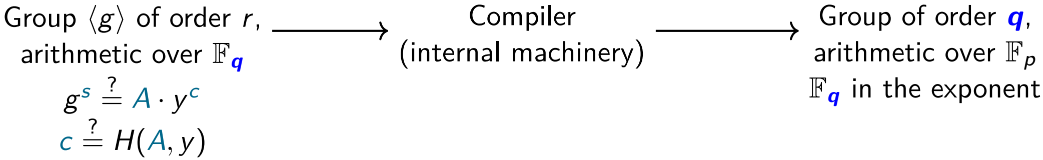

Group $\langle g \rangle$ of order r, arithmetic over $\mathbb{F}_{\boldsymbol{q}}$

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→ Compiler (internal machinery) →

Group of order $\boldsymbol{q}$, arithmetic over $\mathbb{F}_p$, $\mathbb{F}_{\boldsymbol{q}}$ in the exponent

Outline

zk-SNARK

Elliptic Curves

Pairings

Pairing-friendly curves

SNARK-friendly curves

Elliptic curves in cryptography

- 1985 (published in 1987) Hendrik Lenstra Jr., Elliptic Curve Method (ECM) for integer factoring
- 1985, Koblitz, Miller: Elliptic Curves over a finite field form a group suitable for Diffie–Hellman key exchange
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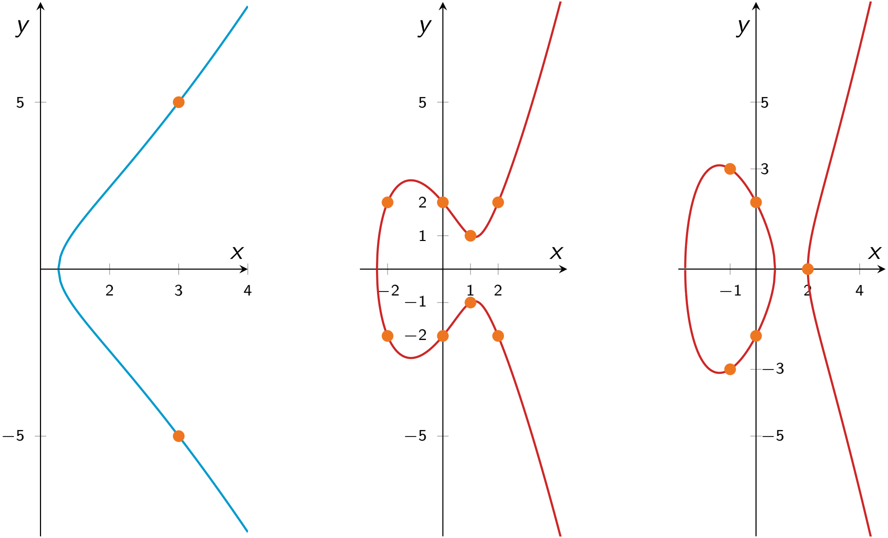
- 2016: NIST Post-Quantum competition
  Isogenies on elliptic curves

Examples of elliptic curves

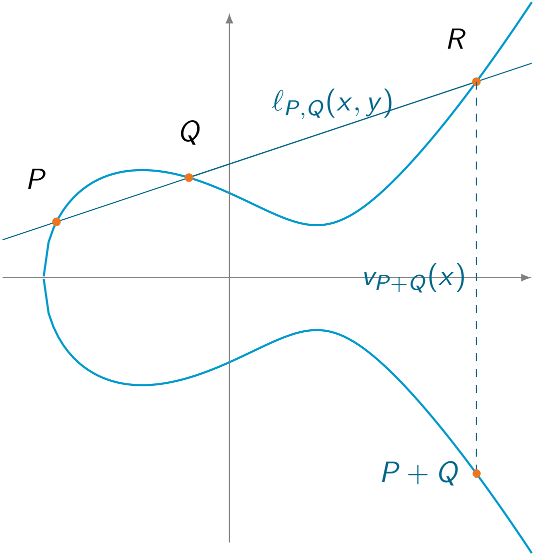


$y^2 = x^3 - 2$

$y^2 = x^3 - 4x + 4$

$y^2 = x^3 - 6x + 4$

Chord and tangent rule



$P(x_1, y_1),\ Q(x_2, y_2),\ x_1 \neq x_2$

slope $\lambda = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$

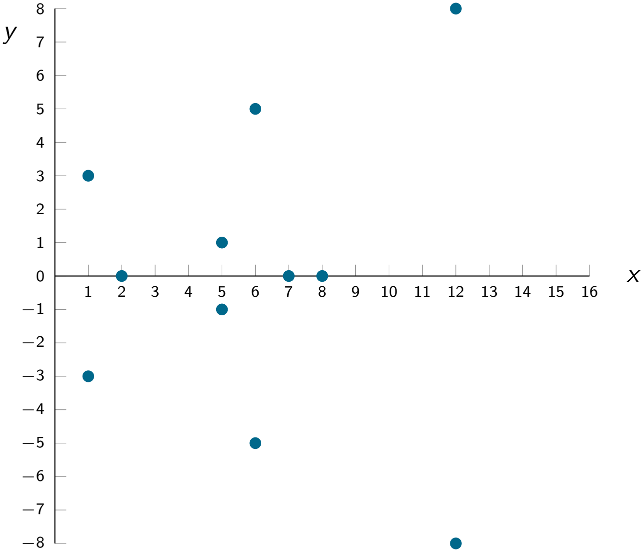
line L through P and Q has equation

$L\colon y = \lambda(x - x_1) + y_1$

$P, Q, R \in L \cap E$

Elliptic curves over finite fields

$$E/\mathbb{F}_{17} : y^2 = x^3 + x + 7$$



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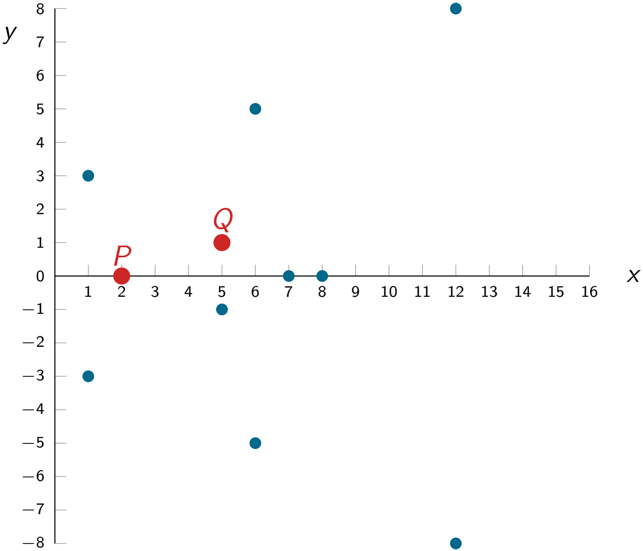

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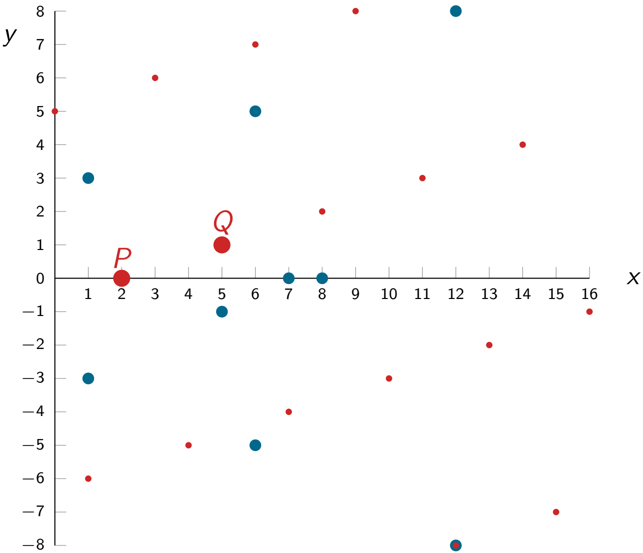

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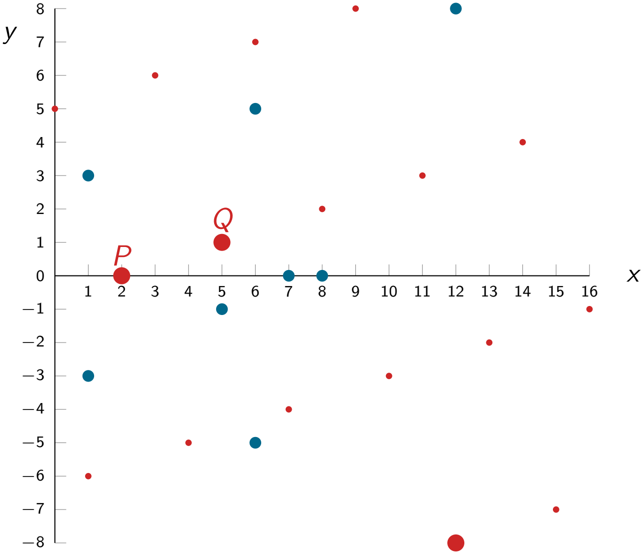

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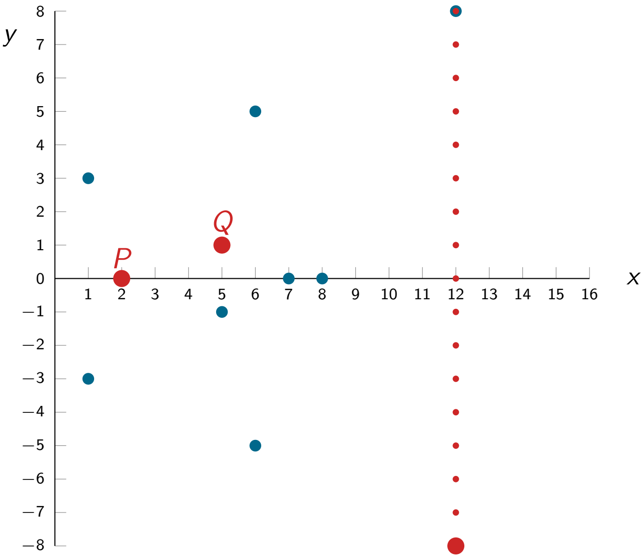

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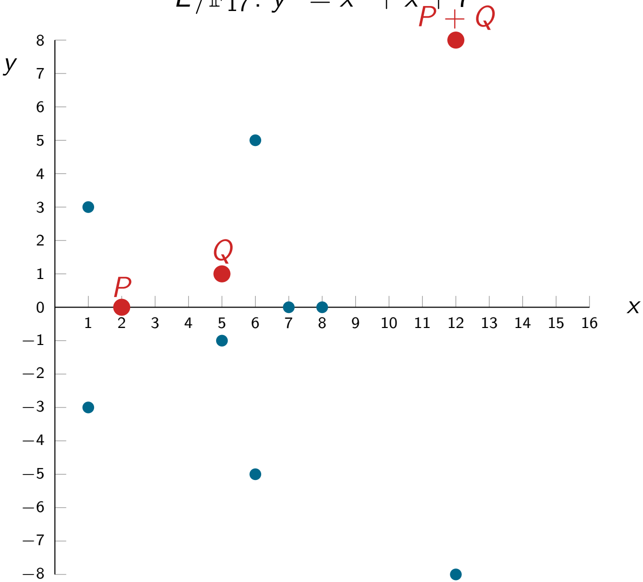

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SNARK-friendly curves

What is a pairing?

$(\mathbf{G}_1,+),(\mathbf{G}_2,+),(\mathbf{G}_T,\cdot)$ three cyclic groups of large prime order n
Pairing: map $e : \mathbf{G}_1 \times \mathbf{G}_2 \to \mathbf{G}_T$

1. bilinear: $e(P_1+P_2,Q)=e(P_1,Q)\cdot e(P_2,Q)$, $e(P,Q_1+Q_2)=e(P,Q_1)\cdot e(P,Q_2)$
2. non-degenerate: $e(G_1,G_2)\neq 1$ for $\langle G_1\rangle=\mathbf{G}_1$, $\langle G_2\rangle=\mathbf{G}_2$
3. efficiently computable.

Most often used in practice:

$$e([a]P,[b]Q)=e([b]P,[a]Q)=e(P,Q)^{ab}\ .$$

$\rightsquigarrow$ Many applications in asymmetric cryptography.

Pairing setting: elliptic curves

$$E/\mathbb{F}_p : \ y^2 = x^3 + ax + b, \ a, b \in \mathbb{F}_p, \ p \geq 5$$

- proposed in 1985 by Koblitz, Miller
- $E(\mathbb{F}_p)$ has an efficient group law (chord an tangent rule) → $\mathbf{G}_1$
- $\#E(\mathbb{F}_p) = p + 1 - t$, trace t: $|t| \leq 2\sqrt{p}$
- efficient group order computation (*point counting*)

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- $\#E(\mathbb{F}_p) = p + 1 - t$, trace t: $|t| \leq 2\sqrt{p}$
- efficient group order computation (*point counting*)
- large subgroup of prime order n s.t. $n \mid p + 1 - t$ and n coprime to p
- $E(\mathbb{F}_p)[n] = \{P \in E(\mathbb{F}_p) \colon [n]P = \mathcal{O}\}$ has order n
- $E[n] \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ (for crypto)
- only generic attacks against DLP on well-chosen genus 1 and genus 2 curves
- optimal parameter sizes

Tate pairing

From its definition to its efficient implementation

- John Tate, 1958
- Stephen Lichtenbaum, 1969
- Victor Miller, 1986, Miller algorithm for f_P
- Frey–Rück, 1994: the MOV attack with the Tate pairing instead of the Weil pairing
- Harasawa, Shikata, Suzuki, Imai, 1999, 161467 s (112 days)
  163-bit supersingular curve, $\mathbf{G}_T \subset \mathbb{F}_{p^2}$ of 326 bits.
- Antoine Joux, 2000: how to compute Miller algorithm more efficiently
  1 s on a supersingular 528-bit curve, $\mathbf{G}_T \subset \mathbb{F}_{p^2}$ of 1055 bits

Cryptographic pairing

Modified Weil or Tate pairing on an elliptic curve

Discrete logarithm problem with one more dimension.

$$e \; : \; E(\mathbb{F}_p)[n] \times E(\mathbb{F}_{p^k})[n] \longrightarrow \mathbb{F}_{p^k}^*, \quad e([a]P, [b]Q) = e(P, Q)^{ab}$$

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Attacks

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Attacks

- inversion of e : hard problem (exponential)

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Modified Weil or Tate pairing on an elliptic curve

Discrete logarithm problem with one more dimension.

$$e \;:\; E(\mathbb{F}_p)[n] \times E(\mathbb{F}_{p^k})[n] \longrightarrow \mathbb{F}^*_{p^k}, \quad e([a]P, [b]Q) = e(P, Q)^{ab}$$

Attacks

- inversion of e : hard problem (exponential)
- discrete logarithm computation in $E(\mathbb{F}_p)$: hard problem (exponential, in $O(\sqrt{n})$)

Cryptographic pairing

Modified Weil or Tate pairing on an elliptic curve

Discrete logarithm problem with one more dimension.

$$e \ : \ E(\mathbb{F}_p)[n] \times E(\mathbb{F}_{p^k})[n] \longrightarrow \mathbb{F}_{p^k}^*, \quad e([a]P, [b]Q) = e(P, Q)^{ab}$$

Attacks

- inversion of e : hard problem (exponential)
- discrete logarithm computation in $E(\mathbb{F}_p)$: hard problem (exponential, in $O(\sqrt{n})$)
- discrete logarithm computation in $\mathbb{F}_{p^k}^*$: **easier, subexponential** → take a large enough field

Jens Groth's proof composition [Gro16]

Given an instance $\Phi = (a_0, \ldots, a_\ell) \in \mathbb{F}_r^\ell$ of a public $\texttt{NP}$ program F

- $(pk, vk) \leftarrow S(F, \tau, 1^\lambda)$ where

$$vk = (vk_{\alpha,\beta}, \{vk_{\pi_i}\}_{i=0}^{\ell}, vk_\gamma, vk_\delta) \in \mathbf{G}_T \times \mathbf{G}_1^{\ell+1} \times \mathbf{G}_2 \times \mathbf{G}_2$$

- $\pi \leftarrow P(\Phi, w, pk)$ where

$$\pi = (A, B, C) \in \mathbf{G}_1 \times \mathbf{G}_2 \times \mathbf{G}_1 \qquad (O_\lambda(1))$$

- $0/1 \leftarrow V(\Phi, \pi, vk)$ where V is

$$e(A, B) = vk_{\alpha,\beta} \cdot e(vk_x, vk_\gamma) \cdot e(C, vk_\delta) \qquad (O_\lambda(|\Phi|)) \tag{1}$$

and $vk_x = \sum_{i=0}^{\ell} [a_i] vk_{\pi_i}$ depends only on the instance Φ and $vk_{\alpha,\beta} = e(vk_\alpha, vk_\beta)$ can be computed in the trusted setup for $(vk_\alpha, vk_\beta) \in \mathbf{G}_1 \times \mathbf{G}_2$.

slide Y. El Housni

Applications not in cryptocurrencies

ZK Microphone: Trusted audio in the age of deepfakes

https://ethglobal.com/showcase/zk-microphone-8161v

Proving sound authenticity

Using ZK Proofs to Fight Disinformation

https://iacr.org/submit/files/slides/2023/rwc/rwc2023/13/slides.pdf

Proving image authenticity

A Tool for Proving Software Vulnerabilities in Zero Knowledge

https://galois.com/blog/2024/02/introducing-cheesecloth-a-tool-for-proving-software-vulnerabilities-in-zero-knowledge/

Outline

zk-SNARK

Elliptic Curves

Pairings

Pairing-friendly curves

SNARK-friendly curves

First ordinary pairing-friendly curves: MNT

Miyaji, Nakabayashi, Takano, $\#E(\mathbb{F}_p) = p(u) + 1 - t(u) = q(u)$

$$k = 3 \begin{cases} t(u) = -1 \pm 6u \\ q(u) = 12u^2 \mp 6u + 1 \\ p(u) = 12u^2 - 1 \\ Dy^2 = 12u^2 \pm 12u - 5 \end{cases}$$

$$k = 4 \begin{cases} t(u) = -u,\ \ u + 1 \\ q(u) = u^2 + 2u + 2,\ \ u^2 + 1 \\ p(u) = u^2 + u + 1 \\ Dy^2 = 3u^2 + 4u + 4 \end{cases} \qquad k = 6 \begin{cases} t(u) = 1 \pm 2u \\ q(u) = 4u^2 \mp 2u + 1 \\ p(u) = 4u^2 + 1 \\ Dy^2 = 12u^2 - 4u + 3 \end{cases}$$

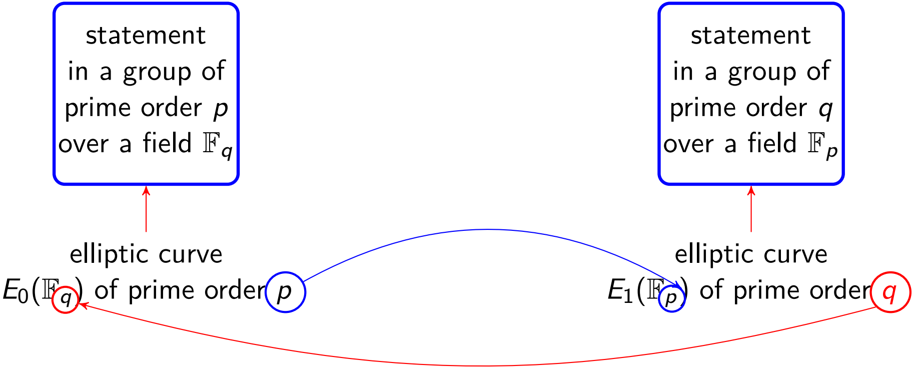
CODA [MS18]:

$k = 6$, 753 bits, $E_6 \approx 137$ bits of security, $D = -241873351932854907$, seed $u =$

`0xaa3a58eb20d1fec36e5e772ee6d3ff28c296465f137300399db8a5521e18d33581a262716214583d3b89820dd0c000`

$k = 4$, 753 bits, $E_4 \approx 113$ bits of security

Cycle of curves: unlimited chains of SNARKs [BCTV14]



statement in a group of prime order p over a field $\mathbb{F}_q$

statement in a group of prime order q over a field $\mathbb{F}_p$

elliptic curve $E_0(\mathbb{F}_q)$ of prime order p

elliptic curve $E_1(\mathbb{F}_p)$ of prime order q

MNT-4 and MNT-6 curves form a cycle

$k = 4$, MNT-4 parameters $\quad t_4 = -v, \quad q_4 = v^2 + 1, \quad p_4 = v^2 + v + 1$
$k = 6$, MNT-6 parameters $\quad t_6 = 1 - 2u, \quad q_6 = 4u^2 + 2u + 1, \quad p_6 = 4u^2 + 1$

$$\begin{array}{c} q_4 = p_6 \\ \text{and} \\ p_4 = q_6 \end{array} \iff \begin{array}{c} v = 2u \\ \text{and} \\ q_4,\ q_6 \text{ are primes} \end{array}$$

Unique known cycle of pairing-friendly curves.
Impossibility results:

Alessandro Chiesa, Lynn Chua, and Matthew Weidner.
On cycles of pairing-friendly elliptic curves.
SIAM Journal on Applied Algebra and Geometry, 3(2):175–192, 2019.

Marta Bellés-Muñoz, Jorge Jiménez Urroz, and Javier Silva.
Revisiting cycles of pairing-friendly elliptic curves.
In Helena Handschuh and Anna Lysyanskaya, editors, *CRYPTO 2023, Part II*, volume 14082 of *LNCS*, pages 3–37. Springer, Heidelberg, August 2023.

Very popular pairing-friendly curves: Barreto-Naehrig (BN)

$$E_{BN} : \ y^2 = x^3 + b, \ p \equiv 1 \bmod 3, \ D = -3 \text{ (ordinary)}$$

$$\begin{aligned} p &= 36x^4 + 36x^3 + 24x^2 + 6x + 1 \\ t &= 6x^2 + 1 \\ q &= p + 1 - t = 36x^4 + 36x^3 + 18x^2 + 6x + 1 \\ t^2 - 4p &= -3(6x^2 + 4x + 1)^2 \ \rightarrow \text{ no CM method needed} \end{aligned}$$

Comes from the Aurifeuillean factorization of Φ_{12} :

$\Phi_{12}(6x^2) = q(x)q(-x)$

Security level	$\log_2 q$	finite field	k	$\log_2 p$	$\deg P, \ p = P(u)$	ρ
102	256	3072	12	256	4	1
123	384	4608	12	384	4	1
132	448	5376	12	448	4	1

Formerly BN-254 in Euthereum with seed `0x44e992b44a6909f1`

BLS12

Barreto, Lynn, Scott method.
Becomes more and more popular, replacing BN curves

$$E_{\mathrm{BLS}}: \; y^2 = x^3 + b, \; p \equiv 1 \bmod 3, \; D = -3 \text{ (ordinary)}$$

$$\begin{aligned} p &= (u-1)^2/3(u^4 - u^2 + 1) + u \\ t &= u + 1 \\ q &= (u^4 - u^2 + 1) = \Phi_{12}(u) \\ p + 1 - t &= \underbrace{(u-1)^2/3}_{\text{cofactor}}(u^4 - u^2 + 1) \end{aligned}$$

$$t^2 - 4p = -3y(u)^2 \rightarrow \text{no CM method needed}$$

BLS12-381 (Zcash [Bow17]) with seed `-0xd201000000010000`
BLS12-377 (Zexe [BCG+18]) with seed `0x8508c00000000001`

Outline

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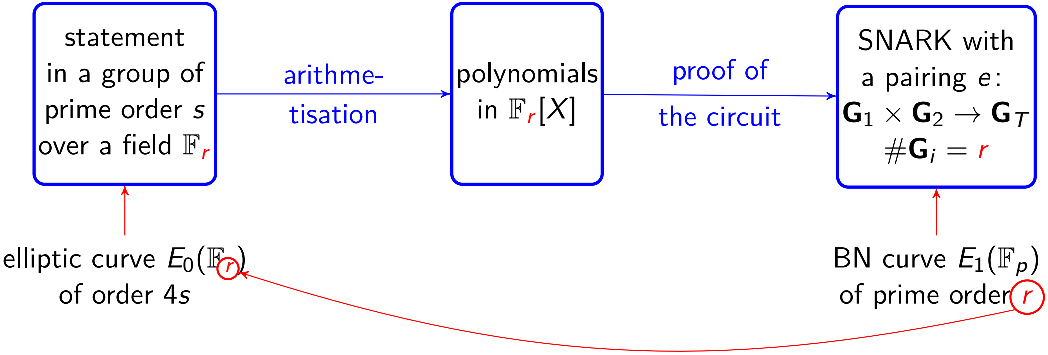
SNARK-friendly curves

CØCØ embedded curve: Kosba et al. construction [KZM+15]



statement in a group of prime order s over a field $\mathbb{F}_r$

arithme-tisation

polynomials in $\mathbb{F}_r[X]$

proof of the circuit

SNARK with a pairing $e \colon \mathbf{G}_1 \times \mathbf{G}_2 \to \mathbf{G}_T$ $\#\mathbf{G}_i = r$

elliptic curve $E_0(\mathbb{F}_r)$ of order $4s$

BN curve $E_1(\mathbb{F}_p)$ of prime order r

CØCØ: given r, search for a curve E_0 over $\mathbb{F}_r$ of order 4 times a prime

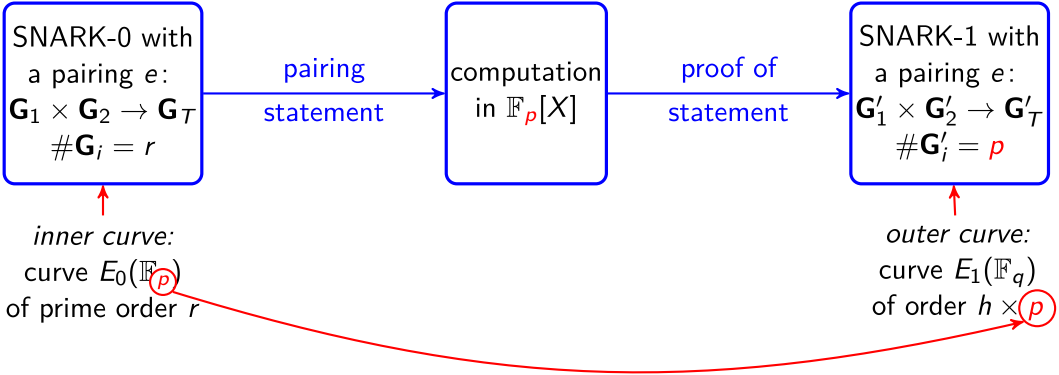
Embedded SNARK-friendly curves

Usually a twist-secure elliptic curve in Montgomery or (twisted) Edwards form

Input: field $\mathbb{F}_p$
Output: an embedded curve of order $4s$ or $8s$ with prime s
Procedure: Increment the curve coefficient(s) until a suitable curve is found

CØCØ [KZM+15] with BN-254a,
JubJub [ZCa21] or Bandersnatch [MSZ21] with BLS12-381, ...

2-chains of elliptic curves



SNARK-0 with a pairing e: $\mathbf{G}_1 \times \mathbf{G}_2 \to \mathbf{G}_T$ $\#\mathbf{G}_i = r$

pairing statement

computation in $\mathbb{F}_p[X]$

proof of statement

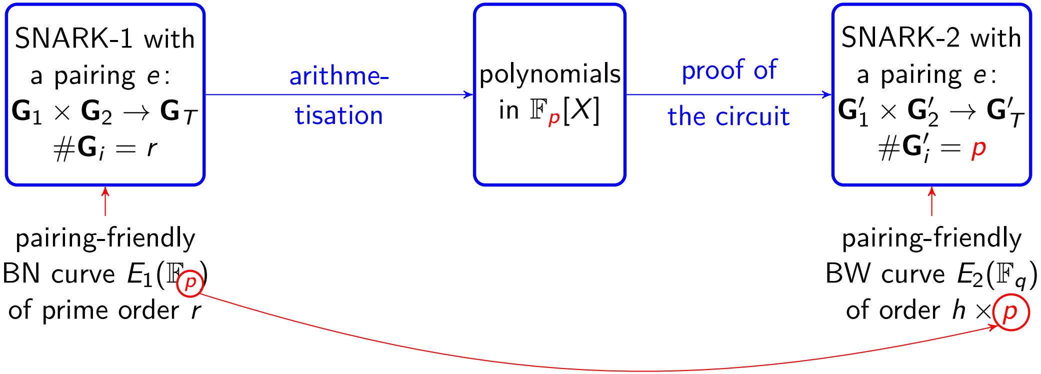
SNARK-1 with a pairing e: $\mathbf{G}'_1 \times \mathbf{G}'_2 \to \mathbf{G}'_T$ $\#\mathbf{G}'_i = p$

inner curve: curve $E_0(\mathbb{F}_p)$ of prime order r

outer curve: curve $E_1(\mathbb{F}_q)$ of order $h \times p$

Given p, search for a pairing-friendly curve E_1 of order $h \cdot p$ over a field $\mathbb{F}_q$

Geppetto construction [CFH$^+$15]



SNARK-1 with a pairing e: $\mathbf{G}_1 \times \mathbf{G}_2 \to \mathbf{G}_T$ $\#\mathbf{G}_i = r$

arithmetisation

polynomials in $\mathbb{F}_p[X]$

proof of the circuit

SNARK-2 with a pairing e: $\mathbf{G}'_1 \times \mathbf{G}'_2 \to \mathbf{G}'_T$ $\#\mathbf{G}'_i = p$

pairing-friendly BN curve $E_1(\mathbb{F}_p)$ of prime order r

pairing-friendly BW curve $E_2(\mathbb{F}_q)$ of order $h \times p$

Geppetto: given p, search for a pairing-friendly curve BW6 (Brezing–Weng) of order $h \cdot p$ over a field $\mathbb{F}_q$

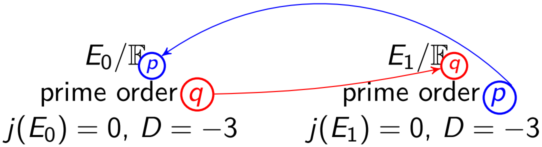
2-chains of pairing-friendly curves

- Geppetto [CFH+15]: BN254b + BW6-509
- Zexe [BCG+18]: BLS12-377 + CP6-782
- BLS12-377 + BW6-761 [EHG20] for Gorth16
- BLS24-315 + BW6-633 [EHG22] For KZG / universal SNARK

More plain/hybrid cycles of curves

Plain cycles: 2 plain prime-order elliptic curves (no pairing)

secp256k1/secq256k1 https://moderncrypto.org/mail-archive/curves/2018/000992.html
HALO: Tweedledum/tweedledee curves https://github.com/daira/tweedle
HALO2: Pallas-Vesta – Pasta curves https://github.com/zcash/pasta_curves



$E_0/\mathbb{F}_p$
prime order q
$j(E_0) = 0,\ D = -3$

$E_1/\mathbb{F}_q$
prime order p
$j(E_1) = 0,\ D = -3$

Hybrid cycles: a plain curve and a BN pairing-friendly curve, both prime order

BN254-Grumpkin https://hackmd.io/@aztec-network/ByzgNxBfd
BN382-plain https://github.com/o1-labs/zexe/tree/master/algebra/src/bn_382
Pluto (BN446) - Eris https://github.com/daira/pluto-eris/

Conclusion

Statement	SNARK 1	SNARK 2
embedded curve	inner curve	outer curve
CØCØ [KZM+15]	BN254a Ethereum	
E_0	BN254b	BW6-509 Geppetto [CFH+15]
Jubjub [ZCa21] Bandersnatch [MSZ21]	BLS12-381 [Bow17]	
E_0'	BLS12-377 [BCG+18]	CP6-782 [BCG+18] BW6-761 [EHG20]
E_0''	BLS24-315	BW6-633 [EHG22]

Survey paper [AEHG23]

Diego F. Aranha, Youssef El Housni, and Aurore Guillevic.
A survey of elliptic curves for proof systems.
Des. Codes Cryptogr., Special Issue: Mathematics of Zero-Knowledge:1–46, December 2022. ePrint 2022/586

Félicitations Jean-Claude et bonne retraite bientôt en Bretagne !

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