Base changes and morphisms of character varieties

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Abstract

In this article, we will firstly prove a base change result of character varieties using the vanishing result of group cohomology on standard module and the good bimodule filtration structure on the coordinate ring. Then we will prove a finiteness result about morphism of character varieties after Cotner by using two ingredients: the theory of adequate morphisms and the Bruhat–Tits theory.

1 Introduction

The first aim of the article is to prove the following theorem, about the base change of character varieties:

Theorem 1.1 (Base change)**.** *Suppose that $T \to S$ is a morphism of two schemes and G is a reductive group S-scheme. Let G act on G^n by simultaneous conjugation.*

For any integer $n \geq 0$, there is an isomorphism:

$$G_T^n /\!\!/ G_T \simeq G_S^n /\!\!/ G_S \times_S T,$$

where $/\!\!/$ denotes the categorical quotient.

The proof of the theorem will use the techniques in group cohomology theory, especially the standard modules and the good filtrations.

We can view $G_S^n /\!\!/ G_S$ to be the character variety of the free group F_n. If we consider a general finitely generated group Γ, we can prove a weak version of the base change.

Theorem 1.2 (General base change)**.** *Suppose that $T \to S$ is a morphism of two noetherian schemes and G is a reductive group S-scheme. Let Γ be a finitely generated group. Then the base change morphism*

$$\mathrm{Hom}(\Gamma', G_T) /\!\!/ G_T \to \mathrm{Hom}(\Gamma, G) /\!\!/ G \times_S T$$

is finite, where $/\!\!/$ denotes the categorical quotient.

Then after Cotner ([Cot23]), We will sketch the proof of the following finiteness result of morphisms between character varieties.

Theorem 1.3 (Finiteness)**.** *Let S be a locally noetherian scheme. and $f\colon H \to G$ be a finite morphism of reductive group schemes over S. Let Γ be a finitely generated group and $\Gamma' \in \Gamma$ be a subgroup of finite index. Then the morphism*

$$\mathrm{Hom}(\Gamma, H) /\!\!/ H \to \mathrm{Hom}(\Gamma', G) /\!\!/ G$$

is finite.

As the morphism is affine, it suffices to prove that it satisfies the existence part of the valuative criterion. The proof will use the theory of adequate moduli spaces to study the categorical quotient and use the theory of Bruhat–Tits building to study the integral points of the character variety.

Those questions arise naturally in the study of Galois representations. Indeed, for a local field K, let K_I be the maximal tamely ramified extension of K. Then the Galois group $\mathrm{Gal}(K_I/K)$ has a dense subgroup $\langle \phi, \sigma \mid \phi\sigma\phi^{-1} = \sigma^p \rangle$, where ϕ is the Frobenius automorphism. Then those consequences will give some information about the moduli space of the Galois representations.

2 The base change result

The goal of the section is to prove the base change result 1.1.

The essential step of the proof is the following vanishing result of group cohomology. The idea is that the structure ring of the categorical quotient X/G is the zeroth group cohomology of the structure ring of X. It is known that the higher group cohomology of a reductive group over a characteristic zero field vanishes. Thus the main problem is over the characteristic p fields, which is the case that we will mainly consider.

Through this section, k will be a characteristic p field and $G = G_k$ will be a split reductive group scheme over k without mentioning. The Borel subgroup and the maximal torus of G will be denoted by B, T respectively. Let w_0 be the longest element of the Weyl group of G. The following theorem is essential for the proof of the base change result.

Theorem 2.1. *For an integer $i > 1$, the higher group cohomology vanishes:*

$$H^i(G, \mathcal{O}(G^n)) = 0$$

The group cohomology does not vanish for general G_k-modules if k is of characteristic p. To prove this statement, we will firstly show that the higher group cohomology vanishes for a class of objects, called the standard objects $\nabla(\lambda)$. Then we show that the module $\mathcal{O}(G^n)$ has a good filtration, i.e., has a filtration whose subquotients are standard objects.

2.1 Group cohomology

We firstly recall the notion of induced modules from B to G. The main reference here is [Cli77].

Definition 2.2 (Induced modules)**.** *Let V be a B-representation. It corresponds to a G-equivariant vector bundle over G/B, denoted by $\mathcal{V}$, whose global section is called the induced module of V, denoted by $\mathrm{Ind}_B^G V$.*

The induced modules satisfy the following properties.

Proposition 2.3.

- *Let V be a G-module and W be a B-module, then we have the Frobenius reciprocity*

$$\mathrm{Hom}_B(V, W) = \mathrm{Hom}_G(V, \mathrm{Ind}_B^G W).$$

We also have its derived version:

$$\mathrm{Ext}^i_B(V, W) = \mathrm{Ext}^i_G(V, \mathrm{Ind}_B^G W).$$

- *Let V be a G-module and W be a B-module, then we have*

$$\operatorname{Ind}_B^G(V \otimes W) \simeq V \otimes (\operatorname{Ind}_B^G W).$$

The induced module from B to G of a one dimensional module is called the standard module.

Definition 2.4 (Standard modules)**.** *Let λ be a weight of G. Then λ corresponds to a representation $k_{w_0\lambda}$ of B of dimension 1. The induced module of $k_{w_0\lambda}$, $\operatorname{Ind}_B^G k_{w_0\lambda}$ is called the standard module, denoted by $\nabla(\lambda)$.*

The standard modules satisfy the following.

Proposition 2.5.

- *$\nabla(\lambda)$ is not zero if and only if λ is dominant.*
- *For dominant λ, the highest weight of $\nabla(\lambda)$ is λ.*
- *For dominant λ, the irreducible module of highest weight λ, $L(\lambda)$ is the socle of $\nabla(\lambda)$.*
- *The dual of $\nabla(-w_0\lambda)$, $\nabla(-w_0\lambda)^*$ has highest weight λ and head $L(\lambda)$. The module is denoted by $\Delta(\lambda)$, called the costandard module.*

Thus the notion of standard module is meaningful only for dominant λ.
The proof of the vanishing result divides into some lemmas.

Lemma 2.6. *Denote the set of non-negative combinations of positive roots by Q^+. Let λ be a weight and $\lambda \notin Q^+$, then $H^n(B, k_\lambda) = 0$ for all $n \in \mathbb{N}$.*

Proof. As $B = U \rtimes T$ and T-representations are semisimple, $\mathcal{O}(U)$ is injective. There is an injective resolution of k_λ, given by the complex

$$k_\lambda \otimes \mathcal{O}(U) \xrightarrow{a^*-m^*} k_\lambda \otimes \mathcal{O}(U \times U) \to \cdots$$

where a^* is the coaction and m^* is the comultiplication. We have that $\mathcal{O}(U)$ is a polynomial ring with generator indexed by positive roots. Thus the characters of $\mathcal{O}(U)$ are in $-Q^+$. After we take B-fixed points, all modules in the complex vanish. Thus the group cohomologies are zero. □

Lemma 2.7. *The higher group cohomologies of the trivial module vanish: $H^i(B, k) = 0$ for $i \geq 1$. Moreover, $H^0(B, k) = k$*

Proof. Consider the injective resolution above. It becomes

$$k \otimes \mathcal{O}(U) \xrightarrow{a^*-m^*} k \otimes \mathcal{O}(U \times U) \to \cdots$$

When we take the T-fixed point, it becomes

$$k \xrightarrow{0} k \xrightarrow{1} k \xrightarrow{0} \cdots .$$

Thus all higher cohomologies vanish and $H^0(B, k) = k$. □

Proposition 2.8. *For a dominant λ, the higher group cohomologies of the standard objects vanish:*

$$H^i(G, \nabla(\lambda)) = 0.$$

In fact, we have a stronger result. If V is a G-module such that all weights of it are not in $-w_0\lambda + (Q^+\backslash\{0\})$,

$$H^i(G, \nabla(\lambda) \otimes V) = 0$$

for all $i \geq 1$. Moreover, for $i = 0$, $\dim H^0(G, \nabla(\lambda) \otimes V)$ is the dimension of the subspace of V of weight $-w_0\lambda$.

Proof. We have the following identities

$$H^i(G, \nabla(\lambda) \otimes V) = \mathrm{Ext}^i_G(V^*, \nabla(\lambda)) = \mathrm{Ext}^i_B(V^*, k_{w_0\lambda})) = H^i(B, V \otimes k_{w_0\lambda})$$

All B-modules have a filtration whose subquotients are one-dimensional modules. As in the condition, those weights are not in $Q^+\backslash\{0\}$. Thus $H^i(B, V \otimes k_{w_0\lambda}) = 0$ for all $i \geq 1$ by 2.6 and 2.7.

The argument about $i = 0$ follows from 2.6 and 2.7 and the long exact sequence of cohomology. □

Corollary 2.9. *Let λ, μ be two dominant weights. $H^i(G, \nabla(\lambda) \otimes \nabla(\mu)) = 0$ for all $i \geq 1$. Moreover, if $\lambda = -w_0\mu$, $\dim H^0(G, \nabla(\lambda) \otimes \nabla(\mu)) = 1$ otherwise $H^0(G, \nabla(\lambda) \otimes \nabla(\mu)) = 0$.*

Proof. For the first statement, The highest weight of $\nabla(\lambda)$ (respectively, $\nabla(\mu)$) is λ (respectively, μ). By 2.8, if $\lambda \notin -w_0\mu + Q^+\backslash\{0\}$ or $\mu \notin -w_0\lambda + Q^+\backslash\{0\}$, the theorem holds. One of the two statements must hold, otherwise $0 = w_0\lambda + \mu + (-w_0)(\lambda + w_0\mu) \in Q^+\backslash\{0\}$, a contradiction.

The case $\lambda \neq -w_0\mu$ follows similarly, For the case $\lambda = -w_0\mu$, the result follows from the fact that the highest weight of $\nabla(\mu)$ is μ and that the dimension of the highest weight subspace is of dimension 1. □

By the duality between Δ and ∇, we have the following.

Corollary 2.10. *There exists a non-zero morphism $\Delta(\lambda) \to \nabla(\mu)$ for two dominant weights λ and μ if and only if $\lambda = \mu$. In this case, the image of a non-zero morphism $\Delta(\lambda) \to \nabla(\lambda)$ is the simple module $L(\lambda)$. Moreover, $\mathrm{Ext}^i_G(\Delta(\lambda), \nabla(\mu)) = 0$ for all $i \geq 1, \mu, \nu$.*

2.2 Good filtration

We now define the notion of good filtration and exploit its properties. We will prove that the structure ring $\mathcal{O}(G_k)$ has a good filtration, which will imply the vanishing result of group cohomology. The main reference here is [Kop84].

Definition 2.11. *A G-module M has a good filtration if there is a filtration (indexed by some ordinal α) of M:*

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots$$

such that for all $i \in \mathbb{N}$, the subquotient M_{i+1}/M_i is of the form $\nabla(\lambda)$ for some dominant weight i.

The following is a criterion for good filtrations.

Proposition 2.12. *Let V be a G-module such that it is a union of finitely dimensional G-modules. The following are equivalent:*

1. *V has a good filtration.*

2. *For any $i \geq 1$ and dominant weight λ of G, $H^i(G, V \otimes \nabla(\lambda)) = 0$.*

3. *For any dominant weight λ of G, $H^1(G, V \otimes \nabla(\lambda)) = 0$.*

Moreover, if these conditions hold, the number of $\nabla(\lambda)$ in the subquotients of any good filtration equals to $\dim \mathrm{Hom}_G(\Delta(\lambda), V)$.

The proof will use the induction method. We firstly make the following convention on the order structure of weights.

Definition 2.13. *The relation $\lambda \leq \mu \iff \mu - \lambda \in Q^+$ defines a partial order on the dominant weights.*

Proof.

- (1) $\implies$ (2): It follows from the long exact sequence of cohomology and the proposition 2.9.

- (2) $\implies$ (3): It is trivial.

- (3) $\implies$ (1): We may use the transfinite induction. Suppose we have constructed V_i. We then construct V_{i+1}. Note that by 2.9 and the long exact sequence of cohomology, V/V_i Let μ be the minimal weight such that $\mathrm{Hom}(L(\mu), V) \neq 0$ (where $L(\mu)$ is the simple module of highest weight μ). It suffices to prove that this induces an injection $\nabla(\mu) \to V$. There is an exact sequence

$$0 \to L(\mu) \to \nabla(\mu) \to R(\mu) \to 0.$$

For any composition factor $L(\nu)$ of $R(\mu)$, we have $\nu < \mu$. By definition of μ, $\mathrm{Hom}_G(L(\nu), V) = 0$ and then $\mathrm{Hom}_G(R(\mu), V) = 0$. For $L(\nu)$, there is an exact sequence

$$0 \to P(\nu) \to \Delta(\nu) \to L(\nu) \to 0,$$

which induces

$$\mathrm{Hom}_G(P(\nu), V) \to \mathrm{Ext}^1_G(L(\nu), V) \to \mathrm{Ext}^1_G(\Delta(\nu), V).$$

The weights of $P(\nu)$ are smaller than ν, thence than ν. Thus $\mathrm{Hom}_G(P(\nu), V) = 0$. We also have $\mathrm{Ext}^1_G(\Delta(\nu), V) = H^1(G, V \otimes \nabla(-w_0\nu)$, which is zero by the assumption in (3). Thus $\mathrm{Ext}^1_G(L(\nu), V) = 0$ and then $\mathrm{Ext}^1_G(R(\mu), V) = 0$.

Now the exact sequence for $L(\mu)$ induces a long exact sequence

$$\mathrm{Hom}_G(R(\mu), V) \to \mathrm{Hom}_G(\nabla(\mu), V) \to \mathrm{Hom}_G(L(\mu), V) \to \mathrm{Ext}^1_G(R(\mu), V).$$

Thus there is an isomorphism

$$\mathrm{Hom}_G(\nabla(\mu), V) \simeq \mathrm{Hom}_G(L(\mu), V)$$

and the inclusion $L(\mu) \to V$ induces a homomorphism $\nabla(\lambda) \to V$. The homomorphism is an inclusion, otherwise it will factor through $R(\mu)$, a contradiction. Moreover, for a limit ordinal α, if we have constructed V_β for $\beta < \alpha$, we may construct $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$. Then the transfinite induction applies and we obtain a filtration on V.

The last argument follows from 2.10. □

Proposition 2.14. *As a $G\times G$-module, $\mathcal{O}(G)$ has a good filtration. For dominant weight λ,μ, $\nabla(\lambda,\mu):=\nabla(\lambda)\boxtimes\nabla(\mu)$ appears in the subquotients if and only if $\mu=-w_0\lambda$. In the case $\mu=-w_0\lambda$, $\nabla(\lambda,\mu)$ appears once.*

Proof. For the first argument, by the criterion 2.12, it suffices to prove that for any weights λ,μ,

$$H^1(G\times G,\nabla(\lambda)\boxtimes\nabla(\mu)\otimes\mathcal{O}(G))=0.$$

Denote the two copies of G by G_1 and G_2. The Hochschild–Serre spectral sequence induces an exact sequence

$$0\to H^1(G_2,W^{G_1})\to H^1(G_1\times G_2,W)\to H^1(G_1,W)^{G_2},$$

where $W=\nabla(\lambda)\boxtimes\nabla(\mu)\otimes\mathcal{O}(G)$. There is an isomorphism

$$(\nabla(\lambda)\otimes\mathcal{O}(G))^{G_1}\simeq\nabla(\lambda)$$

as G_2-modules, induced by $\mathrm{id}\otimes\epsilon$ where ϵ is the counit map $\mathcal{O}(G)\to k$ and the coaction $\nabla(\lambda)\to\nabla(\lambda)\otimes\mathcal{O}(G)$. Thus $H^1(G_2,W^{G_1})=0$ by 2.9. Moreover, as $V\otimes\mathcal{O}(G)\simeq V_0\otimes\mathcal{O}(\mathcal{G})$ for any G-module V, where $V_0=V$ as vector spaces but the action of G on V_0 is trivial. Thus $H^1(G_1,W)=0$. Hence $H^1(G_1\times G_2,W)=0$.

For the second argument, it suffices to compute $\mathrm{Hom}_G(\Delta(\lambda)\boxtimes\Delta(\mu),\mathcal{O}(G))$. We have

$$\mathrm{Hom}_G(\Delta(\lambda)\boxtimes\Delta(\mu),\mathcal{O}(G))\simeq(\nabla(\lambda,\mu),\mathcal{O}(G))^{G\times G}\simeq(\nabla(\lambda)\otimes\nabla(\mu))^G.$$

By 2.9, $(\nabla(\lambda)\otimes\nabla(\mu))^G=0$ if $\lambda\neq-w_0\mu$ and $\dim(\nabla(\lambda)\otimes\nabla(\mu))^G=1$ if $\lambda=-w_0\mu$. $\square$

Proposition 2.15. *If both of two G-modules have good filtrations, their tensor product has a good filtration.*

The proof of the proposition can be found in [Mat90]. The idea of proof is to realize the tensor product of two standard modules $\nabla(\lambda)\otimes\nabla(\mu)$ as the global section of a line bundle on some generalized Schubert variety S (i.e., varieties of the form $\overline{Bw_1B}\times^B\times^B\cdots\times^B\overline{Bw_nB}/B\subset(G/B)^n$) and one can find some Schubert subvarieties S' such that morphisms of the form $\Gamma(S,\mathcal{L})\to\Gamma(S',\mathcal{L})$ induces the good filtration of $\nabla(\lambda)\otimes\nabla(\mu)$.

Now we turn to the reductive group schemes over $\mathbb{Z}$. Let $G=G_{\mathbb{Z}}$ be a split reductive group scheme over $\mathbb{Z}$.

Over $\mathbb{Z}$, the standard and costandard modules can still be defined.

Definition 2.16 (Standard modules)**.** *Let λ be a weight of G. Then λ corresponds to a representation $\mathbb{Z}_{w_0\lambda}$ of B of rank 1. The induced module of $\mathbb{Z}_{w_0\lambda}$, $\mathrm{Ind}_B^G\mathbb{Z}_{w_0\lambda}$ is called the standard module, denoted by $\nabla(\lambda)$. The dual module $\mathrm{Hom}_{\mathbb{Z}}(\nabla(-w_0\lambda),\mathbb{Z})$ is called the costandard module, denoted by $\Delta(\lambda)$.*

We will then use $\bar{\nabla}(\lambda)$ and $\bar{\Delta}(\lambda)$ to denote the standard and costandard modules over some $\mathbb{F}_p$.

Note that by proper base change theorem, the standard and costandard modules are free over $\mathbb{Z}$ and they satisfy the base change

$$\nabla(\lambda)\otimes_{\mathbb{Z}}\mathbb{F}_p\simeq\bar{\nabla}(\lambda)$$

and similarly for $\Delta(\lambda)$.

Lemma 2.17. *Suppose that V is a $G_{\mathbb{Z}}$-module, which is of finite type as an $\mathbb{Z}$-module. Denote $V_p := V \otimes_{\mathbb{Z}} \mathbb{F}_p$. If $H^1(G_p, V_p) = 0$ for any prime p, then $H^1(G, V) = 0$ and $V^G \otimes \mathbb{F}_p \simeq V_p^G$.*

Proof. By the universal coefficient theorem, there is an exact sequence

$$0 \to H^i(G, V) \otimes \mathbb{F}_p \to H^i(G_p, V_p) \to \mathrm{Tor}_1^{\mathbb{Z}}(H^{i+1}(G, V), \mathbb{F}_p) \to 0.$$

Taking $i = 1$, we have $H^i(G, V) \otimes \mathbb{F}_p = 0$. The group cohomology of V is given by the cohomology of the chain complex

$$0 \to V \xrightarrow{1-a^*} V \otimes \mathcal{O}(G) \xrightarrow{1-a^*+m^*} V \otimes \mathcal{O}(G) \otimes \mathcal{O}(G) \cdots$$

Thus the torsion of $H^1(G, V)$ is finite and then $H^1(G, V) = 0$. Taking $i = 0$ in the exact sequence, we have $V^G \otimes \mathbb{F}_p \simeq V_p^G$. □

Lemma 2.18. *Let λ be a dominant weight. There is an isomorphism*

$$\mathrm{Hom}_{G \times G}(\Delta(\lambda, -w_0\lambda), \mathcal{O}(G)) \simeq \mathbb{Z}.$$

Proof. By similar computations as in 2.14, we have

$$\mathrm{Hom}_{G \times G}(\Delta(\lambda, -w_0\lambda), \mathcal{O}(G)) \simeq (\nabla(\lambda) \otimes \nabla(-w_0\lambda))^G.$$

The module $(\nabla(\lambda) \otimes \nabla(-w_0\lambda))^G$ is free as a submodule of a free module and $(\nabla(\lambda) \otimes \nabla(-w_0\lambda))^G \otimes \mathbb{F}_p \simeq \mathbb{F}_p$ by 2.17 and 2.9. Thus $(\nabla(\lambda) \otimes \nabla(-w_0\lambda))^G \simeq \mathbb{Z}$. □

Proposition 2.19. *There is a filtration on $\mathcal{O}(G)$*

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V := \mathcal{O}(G)$$

such that V_i is a G-bimodule and saturated (i.e. $V_i \otimes \mathbb{Q} \cap V = V_i$) for all i and there exists λ such that the subquotients $V_i/V_{i-1} \otimes \mathbb{F}_p$ is of the form $\bar{\nabla}(\lambda, -w_0\lambda)$ for all prime p.

Proof. We construct this filtration inductively. Suppose that we have constructed V_i and the manner of construction is the same as below if $i \geq 1$.

There is an commutative diagram

$$\begin{array}{ccccc} \Delta(\lambda, -w_0\lambda) & \longrightarrow & \tilde{\Delta}(\lambda, -w_0\lambda) & \longrightarrow & V/V_i \\ \downarrow & & \downarrow & & \downarrow \\ \bar{\Delta}(\lambda, -w_0\lambda) & \longrightarrow & \bar{\tilde{\Delta}}(\lambda, -w_0\lambda) & \longrightarrow & \overline{V/V_i}. \end{array}$$

where the morphism $\Delta(\lambda, -w_0\lambda)$ is induced by the inverse image of 1 in the isomorphism in 2.18, $\sim$ denotes the saturation of the image of $\Delta(\lambda, -w_0\lambda)$, in V/V_i (the saturation of M in N means $M \otimes \mathbb{Q} \cap N$) and the second arrow is the first arrow tensored by $\mathbb{F}_p$.

Choose a minimal λ such that $\bar{\nabla}(\lambda, -w_0\lambda)$ appears in the good G-bimodule filtration of $\overline{V/V_i}$ (We choose it for one prime p, but it will be minimal for any p). Take V_{i+1} be the inverse image of $\tilde{\Delta}(\lambda, -w_0\lambda)$ in V. We claim that V_{i+1} satisfies the conditions.

We firstly show that the morphism $\Delta(\lambda, -w_0\lambda) \to V/V_i$ is injective. As both sides are free modules over $\mathbb{Z}$, it suffices to show that the morphism is injective after tensoring $\mathbb{Q}$. By construction, the image of $\Delta(\lambda, -w_0\lambda) \otimes \mathbb{Q}$ in $V \otimes \mathbb{Q}$ is a simple module and $V_i \otimes \mathbb{Q}$ is the direct sum of simple modules different form $\Delta(\lambda, -w_0\lambda)$. Thus $\Delta(\lambda, -w_0\lambda) \otimes \mathbb{Q} \cap V_i = \emptyset$ and the morphism $\Delta(\lambda, -w_0\lambda) \otimes \mathbb{Q} \to \overline{V/V_i} \otimes \mathbb{Q}$ is injective.

Next, the morphism $\bar{\Delta}(\lambda, -w_0\lambda) \to \overline{V/V_i}$ is not zero and uniquely induced by the morphism $\bar{\Delta}(\lambda) \to L(\lambda) \to \bar{\nabla}(\lambda)$ by 2.10. Thus the image of $\bar{\Delta}(\lambda, -w_0\lambda)$ in $\overline{V/V_i}$ has a subquotient which is isomorphic to $L(\lambda, -w_0\lambda)$ and $\bar{\bar{\Delta}}(\lambda, -w_0\lambda) \to \overline{V/V_i}$ also has such subquotient.

Moreover, the characters of $M := \bar{\bar{\Delta}}(\lambda, -w_0\lambda)$ and of $\bar{\nabla}(\lambda, -w_0\lambda)$ are the same, as they equal to the character of $\nabla(\lambda, -w_0\lambda) \otimes \mathbb{Q}$. The rest of proof is a general argument which is valid in all highest weight categories.

The simple components of M are of the form $L(\mu, -w_0\mu)$ for $\mu \leq \lambda$ and $L(\lambda, -w_0\lambda)$ appears only once. The module M does not contain a simple submodule $L(\mu, \nu)$ such that $(\mu, \nu) < (\lambda, -w_0\lambda)$ otherwise by the proof of 2.12 there will be a submodule $\bar{\nabla}(\mu, \nu)$ of $\overline{V/V_i}$, a contradiction with the choice of λ. Thus the socle of M must be $L(\lambda, -w_0\lambda)$. There is an exact sequence

$$0 \to L(\lambda, -w_0\lambda) \to M \to M/L(\lambda, -w_0\lambda) \to 0$$

which induces a long exact sequence

$$\begin{aligned}&\mathrm{Hom}_G(M/L(\lambda, -w_0\lambda), \bar{\nabla}(\lambda, -w_0\lambda)) \to \mathrm{Hom}_G(M, \bar{\nabla}(\lambda, -w_0\lambda))\\ &\to \mathrm{Hom}_G(L(\lambda, -w_0\lambda), \bar{\nabla}(\lambda, -w_0\lambda)) \to \mathrm{Ext}^1_G(M/L(\lambda, -w_0\lambda), \bar{\nabla}(\lambda, -w_0\lambda))\end{aligned}$$

As the simple components of $M/L(\lambda, -w_0\lambda)$ are of the form $L(\mu, \nu)$ such that $(\mu, \nu) < (\lambda, -w_0\lambda)$ by 2.8 the first and the fourth term are zero and the inclusion $L(\lambda, -w_0\lambda) \subset \bar{\nabla}(\lambda, -w_0\lambda)$ induces a morphism $M \to \bar{\nabla}(\lambda, -w_0\lambda)$. The morphism must be injective as the simple components of the kernel are of the form $L(\mu, nu)$ such that $(\mu, nu) < (\lambda, -w_0\lambda)$. Then the morphism $M \to \bar{\nabla}(\lambda, -w_0\lambda)$ is an isomorphism as they have the same character. □

In fact we can prove that the subquotients of the filtration is of the form $\nabla(\lambda, -w_0\lambda)$ over $\mathbb{Z}$. It suffices to prove the following fact.

Lemma 2.20. *Let M be an G-module which is free as a Z-module. If there exists a dominant weight λ such that for all prime p, there is an isomorphism $M \otimes_{\mathbb{Z}} \mathbb{F}_p \simeq \nabla(\lambda) \otimes_{\mathbb{Z}} \mathbb{F}_p$, then there is an isomorphism $M \simeq \nabla(\lambda)$.*

Proof. We have that $\mathrm{Ext}^n_G(M, \nabla(\lambda)) = H^n(G, M^* \otimes \nabla(\lambda))$ for all $n \in \mathbb{N}$. As $M \otimes \mathbb{F}_p$ is isomorphic to $\nabla(\lambda) \otimes \mathbb{F}_p$, the highest weight of M^* is $-w_0\lambda$ and then $H^1(G, \nabla(\lambda) \otimes M^* \otimes \mathbb{F}_p) \simeq 0$ by 2.8. Then by 2.17 we have

$$\mathrm{Hom}_G(M, \nabla(\lambda)) \otimes \mathbb{F}_p \simeq \mathrm{Hom}_G(M \otimes \mathbb{F}_p, \nabla(\lambda) \otimes \mathbb{F}_p)$$

and the latter is isomorphic to $\mathbb{F}_p$ by 2.8. As $\mathrm{Hom}_G(M, \nabla(\lambda))$ is a submodule of a free module, it is a free module. Thus we have

$$\mathrm{Hom}_G(M, \nabla(\lambda)) \simeq \mathbb{Z}.$$

The inverse image of 1 induces a morphism $M \to \nabla(\lambda)$. For any prime number p, the morphism induced by tensoring with $\mathbb{F}_p$ is an isomorphism, as $\mathrm{Hom}_G(M \otimes \mathbb{F}_p, \nabla(\lambda) \otimes \mathbb{F}_p)$ is one-dimensional and the morphism tensored by p is not zero. As M and ∇ are free and of finite type, the morphism $M \to \nabla(\lambda)$ is an isomorphism. □

Thus we have proved the following theorem.

Theorem 2.21. *Let G be a split reductive group over $\mathbb{Z}$. As a $G \times G$-module, $\mathcal{O}(G)$ has a filtration whose subquotients are of the form $\nabla(\lambda, -w_0\lambda)$ and for each dominant λ, $\nabla(\lambda, -w_0\lambda)$ appears once.*

Remark 2.22. *For any ring A, a split reductive group scheme G_A over A is of the form $G_{\mathbb{Z}} \times A$ for some split reductive group scheme $G_{\mathbb{Z}}$ over $\mathbb{Z}$. One can also define the standard module over $\mathbb{A}$ by $\nabla(\lambda) := H^0(G/B, A_{w_0\lambda})$. As $\nabla(\lambda)$ is compatible with base change, we have that $\mathcal{O}(G)$ has a $G \times G$-module filtration whose subquotients are of the form $\nabla(\lambda, -w_0\lambda)$.*

Lemma 2.23. *The first group cohomology $H^1(G_{\mathbb{Z}}, \mathcal{O}(G^n_{\mathbb{Z}}))$ vanishes, where G acts on G^n by simultaneous conjugation.*

Proof. Consider the filtration introduced in 2.19. The tensor product of the $G\times G$-module filtration induces a G^{2n}-module filtration on $\mathcal{O}(G^n)$. The subquotients are of finite type and by 2.15 and induction they have good filtrations. By 2.17, the first group cohomology of all the subquotients vanish. Thus the first group cohomology of $\mathcal{O}(G^n)$ vanishes. □

Now we can prove the theorem. We firstly consider the case that $T \to S$ is flat.

Proposition 2.24. *Suppose that $T \to S$ is a morphism of two affine schemes and G is a reductive group S-scheme. Let G act on G^n by simultaneous conjugation.*

Then for any integer $n \geq 0$, there is an isomorphism:

$$G^n_S /\!\!/ G_S \times_S T \to G^n_T /\!\!/ G_T,$$

where $/\!\!/$ denotes the categorical quotient.

In fact, a stronger result holds. Let M be any G_S-module. Every group cohomology of M satisfy the base change: for $i \geq 0$,

$$H^i(G_S, M) \otimes_{\mathcal{O}_S} \mathcal{O}(T) \simeq H^i(G_T, M_T)$$

where $\mathcal{O}(X)$ denote the structure ring of an affine scheme X.

Proof. As $\mathcal{O}(S) \to \mathcal{O}(T)$ is flat, we have

$$\begin{aligned} H^i(G_T, M_T) &= H^i\left(M_T \to M_T \otimes \mathcal{O}(G_T) \to \cdots\right) \\ &= \left(H^i\left(M \to M \otimes \mathcal{O}(G_S) \to \cdots\right)\right) \otimes_{\mathcal{O}(S)} \mathcal{O}(T) \\ &= H^i(G_S, M) \otimes_{\mathcal{O}(S)} \mathcal{O}(T). \end{aligned}$$

The categorical quotients are also affine and structure ring of $G^n_S /\!\!/ G_S$ is $H^0(G_S, \mathcal{O}(G^n_S))$. Thus the result for $G^n_S /\!\!/ G_S$ holds. □

Proof of the theorem 1.1. The question is local on both S and T. Thus we may assume that S and T are affine. By [DGIdhésP70], Exp. XXII, corollary 2.3, there is an étale cover $S' \to S$ such that $G_{S'} := G \times_S S'$ is split and there exists a reductive group scheme $G_{\mathbb{Z}}$ over $\mathbb{Z}$ such that $G_{S'} = G_{\mathbb{Z}} \otimes S'$. By proposition 2.24 and descent, it suffices to prove the result for $G_{S'}$ and base morphism $T' := S' \times_S T \to S$. We will then replace S and T by S' and T'.

We firstly show that

$$\mathcal{O}(G_{\mathbb{Z}}^n /\!\!/ G_{\mathbb{Z}}) \otimes_{\mathbb{Z}} A \simeq \mathcal{O}(G_A^n /\!\!/ G_A)$$

for any ring A. By universal coefficient theorem, there is an exact sequence

$$0 \to \mathcal{O}(G_{\mathbb{Z}}^n /\!\!/ G_{\mathbb{Z}}) \otimes A \to \mathcal{O}(G_A^n /\!\!/ G_A) \to \mathrm{Tor}_1^{\mathbb{Z}}(A, H^1(G_{\mathbb{Z}}, \mathcal{O}(G_{\mathbb{Z}}^n))) \to 0$$

By 2.23, $H^1(G_{\mathbb{Z}}, \mathcal{O}(G_{\mathbb{Z}}^n)) = 0$ and

$$\mathcal{O}(G_{\mathbb{Z}}^n /\!\!/ G_{\mathbb{Z}}) \otimes_{\mathbb{Z}} A \simeq \mathcal{O}(G_A^n /\!\!/ G_A).$$

Let $\mathcal{O}(X)$ denote the structure ring of an affine scheme X. Now the theorem follows:

$$\begin{aligned} \mathcal{O}(G_S^n /\!\!/ G_S) \otimes_{\mathcal{O}(S)} \mathcal{O}(T) &= \mathcal{O}(G_{\mathbb{Z}}^n /\!\!/ G_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathcal{O}(S) \otimes_{\mathcal{O}(S)} \mathcal{O}(T) \\ &= \mathcal{O}(G_{\mathbb{Z}}^n /\!\!/ G_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathcal{O}(T) \\ &= \mathcal{O}(G_T^n /\!\!/ G_T). \end{aligned}$$

□

For the character variety $\mathrm{Hom}(\Gamma, G) /\!\!/ G$, where Γ is an arbitrary finitely generated group, the base change result may not hold. However, we can still prove that the base change morphism is finite, by using the notion of adequate moduli spaces in the next section. The result can even be generalized to reductive group schemes over arbitrary noetherian rings.

Theorem 2.25 (General base change)**.** *Suppose that $T \to S$ is a morphism of two noetherian schemes and G is a reductive group S-scheme. Let $X \to S$ be an affine scheme of finite type. Then the base change morphism*

$$X_T /\!\!/ G_T \to X /\!\!/ G \times_S T$$

is finite, where $X_T := X \times_S T$ and similarly for G_T. In fact, it is an adequate homeomorphism defined in 3.1.

In particular, let Γ be a finitely generated group. Take $X = \mathrm{Hom}(\Gamma', G_T)$, the representation variety. Then the base change morphism

$$\mathrm{Hom}(\Gamma', G_T) /\!\!/ G_T \to \mathrm{Hom}(\Gamma, G) /\!\!/ G \times_S T$$

is finite.

Proof. There is a Cartesian diagram

$$\begin{array}{ccc} [X_T/G_T] & \longrightarrow & [X/G] \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X /\!\!/ G \times_S T & \longrightarrow & X /\!\!/ G \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

where $[X/G]$ denotes the quotient stack. The morphism f is an adequate moduli space by 3.13. By 3.9, the morphism f' decomposes into an adequate moduli space $[X_T/G_T] \to X_T /\!\!/ G_T$ and an adequate homeomorphism $X_T /\!\!/ G_T \to X /\!\!/ G \times_S T$. Moreover, by [Alp18], theorem 6.3.3, the morphism $X /\!\!/ G \to S$ is of finite type. Thus the base change morphism is of finite type and then it is finite. □

3 Adequate moduli spaces

In the following sections, we will prove the finiteness result after Cotner [Cot23]. The proof relies on the theory of adequate moduli spaces and the theory of Bruhat–Tits buildings. In this section we will introduce the theory of adequate moduli spaces. The main reference here is [Alp18].

Recall that when K is a field of characteristic p and G is a reductive group scheme over K, the functor $(-)^G$ of taking G-fixed point is not exact. However, we can introduce a notion of adequate ring morphisms and prove that for a reductive group scheme over an arbitrary ring, the functor $(-)^G$ will map surjective ring morphisms to adequate ring morphisms and we can establish the theory of adequate moduli spaces.

Definition 3.1 (Adequate ring morphism)**.** *A ring morphism $f\colon A \to B$ is called adequate if for any $b \in B$ there exists $a \in A$ and $n \in \mathbb{N}\backslash\{0\}$ such that $f(a) = b^n$.*

A ring morphism $f\colon A \to B$ is called universally adequate if for any ring morphism $A \to A'$ the morphism $f \otimes_A A'$ is adequate.

A ring morphism $f\colon A \to B$ is called an adequate homeomorphism if is universally adequate and satisfies $\ker(A \to B)$ *is locally nilpotent and $f \otimes \mathbb{Q}$ is an isomorphism,*

Remark 3.2. *Over $\mathbb{Q}$, a morphism is adequate if and only if it is surjective.*

One can show that the notion of universally adequate morphism satisfies the fpqc descent ([Alp18], lemma 3.1.3). Thus one can also define the notion of universally adequate morphism on sheaves of rings reasonably.

Definition 3.3 (Adequate morphism of sheaves of rings)**.** *Let $\mathcal{X}$ be an algebraic stack. A morphism of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebras $\mathcal{A} \to \mathcal{B}$ is called universally adequate if for any object $U \to \mathcal{X}$ in the lisse-étale site and any section $s \in U \to \mathcal{X}$, there exists an covering U_i of U, elements t_i and integers $N_i > 0$ such that $t_i^{N_i} = s_i$ on U_i.*

Lemma 3.4. *When $\mathcal{X}$ is an affine scheme, a morphism $\mathcal{A} \to \mathcal{B}$ between two quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebras is universally adequate if and only if the induced morphism between their global sections is universally adequate.*

We also need a notion of adequately affine morphism. We consider only those quasi-compact and quasi-separated morphisms to ensure that the quasi-coherent sheaves are stable under pushforward.

Definition 3.5. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism between two algebraic stacks. If for any surjective morphism of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebras $\mathcal{A} \to \mathcal{B}$, the morphism of their direct images $f_*\mathcal{A} \to f_*\mathcal{B}$ is universally adequate, then f is called adequately affine.*

Note that affine morphisms are adequately affine, as the pushing forward functor on quasi-coherent sheaves is exact.

We state a criterion of adequately affine morphism.

Proposition 3.6. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism between algebraic stacks, then the following are equivalent.*

1. *For any universally adequate morphism $\mathcal{A} \to \mathcal{B}$ between quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebras with kernel $\mathcal{I}$, the morphism $f_*\mathcal{A}/f_*\mathcal{I} \to f_*\mathcal{B}$ is an adequate homeomorphism.*

2. *f is adequate.*

3. *For any surjection of quasi-coherent sheaves $\mathcal{F} \to \mathcal{G}$, the morphism $f_* \operatorname{Sym} \mathcal{F} \to f_* \operatorname{Sym} \mathcal{G}$ is adequate, where* Sym *denotes the symmetric algebra over $\mathcal{O}_{\mathcal{X}}$.*

Proof. The statements (1) $\implies$ (2) $\implies$ (3) are from the definition. For (3) $\implies$ (2), suppose $\mathcal{A} \to \mathcal{B}$ to be a surjection of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebras. Note that $\operatorname{Sym} \mathcal{B} \to \mathcal{B}$ is split surjective, $f_* \operatorname{Sym} \mathcal{B} \to f_* \mathcal{B}$ is surjective. We have a commutative diagram

$$\begin{array}{ccc} f_* \operatorname{Sym} \mathcal{A} & \longrightarrow & f_* \operatorname{Sym} \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{B} \end{array}$$

Thus the morphism $f_* \operatorname{Sym} \mathcal{A} \to \mathcal{B}$ is universally adequate and then the morphism $\mathcal{A} \to \mathcal{B}$ is universally adequate.

For (2) $\implies$ (1), suppose $\phi \colon \mathcal{A} \to \mathcal{B}$ to be a universally adequate morphism of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebras. We may assume that $\mathcal{A} \to \mathcal{B}$ is injective by replacing $\mathcal{A}$ by its image in $\mathcal{B}$. For any affine cover $V \to \mathcal{Y}$, we may choose a presentation $R \rightrightarrows U \to \mathcal{X} \times_{\mathcal{Y}} V$ such that U is affine. We have a commutative diagrams of sections

$$\begin{array}{ccccc} f_*\mathcal{A}(V) & \longrightarrow & \mathcal{A}(U) & \rightrightarrows & \mathcal{A}(R) \\ \downarrow & & \downarrow & & \downarrow \\ f_*\mathcal{B}(V) & \longrightarrow & \mathcal{B}(U) & \rightrightarrows & \mathcal{B}(R). \end{array}$$

For any $s \in f_*\mathcal{B}(V)$, there exists $\tilde{t} \in \mathcal{A}(U)$ such that $s^n = \tilde{t}$ in $\mathcal{B}(U)$ as ϕ is universally adequate and V is affine. Then by the injectivity and the sheaf property, $\tilde{t}$ comes from an element $t \in f_*\mathcal{A}(V)$ and $s^n = t$ in $f_*\mathcal{B}(V)$. Thus $f_*\mathcal{A} \to f_*\mathcal{B}$ is universally adequate. □

Some properties of adequately affine morphism are enumerated here.

Proposition 3.7. 1. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ and $g \colon \mathcal{X} \to \mathcal{Y}$ are morphisms between algebraic stacks, if f is affine, then $g \circ f$ is adequately affine if g is adequately affine.*

2. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ and $g \colon \mathcal{X} \to \mathcal{Y}$ are morphisms between algebraic stacks, if g is affine, then $g \circ f$ is adequately affine if and only if f is adequately affine.*

3. *The class of adequately affine morphisms is closed under composition, fpqc descent and base change.*

Proof. The first statement follows from the fact that the pushforward functor along affine morphism is exact.

For the second statement, as the statement is local on $\mathcal{Z}$ and $\mathcal{Y}$, we may assume that $\mathcal{Y}$ and $\mathcal{Z}$ are affine. Notice that a morphism of quasi-coherent algebras on a affine scheme is adequately affine if and only if the morphism between their global section is adequately affine. The statement follows from the fact that the pushforward does not change the global section.

For the third statement, the stability under composition follows from 3.6. The stability under fpqc descent is from the flat base change and the fact that the fpqc descent of a universally adequate morphism is again universally adequate. For the stability under the base change, suppose that there is a 2-Cartesian diagram of algebraic stacks

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g'} & \mathcal{X} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

and f is adequately affine. The statement is local on $\mathcal{Y}$ and $\mathcal{Y}'$ and we may assume that $\mathcal{Y}$ and $\mathcal{Y}'$ are affine schemes. Then g and g' are affine morphisms. Then the statement follows from the first two statement. □

Definition 3.8 (Adequate moduli space)**.** *Let $f\colon \mathcal{X} \to Y$ be a quasi-compact and quasi-separated morphism between an algebraic stack $\mathcal{X}$ and an algebraic space Y. f is called an adequate moduli space if the following holds:*

- $f_*\mathcal{O}_{\mathcal{X}} \simeq \mathcal{O}_Y$.
- *f is adequately affine.*

We enumerate some properties of adequate moduli spaces.

Proposition 3.9.
1. *The class of adequate moduli spaces are closed under composition.*
2. *For an adequate moduli space $f\colon \mathcal{X} \to Y$ and a quasi-coherent $\mathcal{O}_Y$-algebra $\mathcal{B}$, the morphism $\mathcal{B} \to f_*f^*\mathcal{B}$ is an adequate homeomorphism.*
3. *Suppose there is a 2-Cartesian diagram of algebraic stacks*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g'} & \mathcal{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

If f is an adequate moduli space, then the morphism g' decompose into an adequate moduli space $\mathcal{X}' \to \operatorname{Spec}_{Y'}(f_*\mathcal{O}_{\mathcal{X}'})$ and an adequate homeomorphism $\operatorname{Spec}_{Y'}(f_*\mathcal{O}_{\mathcal{X}'}) \to Y'$.

Proof. Both of the two constituents of an adequate moduli spaces are closed under composition by proposition 3.7. Thus the first statement holds.

For the second, note that the morphism is an isomorphism for $\mathcal{O}_Y$ and thus an isomorphism for any free $\mathcal{O}_Y$-algebra. For a quasi-coherent $\mathcal{O}_Y$-algebra $\mathcal{B}$, choose a surjection $\mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$ is a polynomial algebra over $\mathcal{O}_Y$. Then the statement follows from proposition 3.6.

For the third, as adequately affine morphisms are stable under base changes and left compositions with affine morphisms by proposition 3.7. It suffices to prove the statement about being adequate homeomorphism. The morphism is identified with $g_*\mathcal{O}_{Y'} \to f_*f^*g_*\mathcal{O}_{Y'}$. Thus it is finite by (2). □

Now we can state the two main theorem of this section.

The first states some properties of the adequate moduli spaces.

Theorem 3.10. *Let $f\colon \mathcal{X} \to Y$ be an adequate moduli space, then*

1. *f is surjective.*
2. *f is universally closed.*
3. *For two closed substacks $\mathcal{Z}_1$, $\mathcal{Z}_2$ of $\mathcal{X}$, $\mathrm{im}(Z_1) \cap \mathrm{im}(Z_2) = \mathrm{im}(\mathcal{Z}_1 \cap \mathcal{Z}_2)$, where* im *denotes the schematic image.*
4. *For an algebraically closed field k, define an equivalence relation on $\mathcal{X}(k)$ by $x_1 \sim x_2$ if and only if $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \emptyset$, then the map $\mathcal{X}(k)/\sim \to Y(k)$ is bijective.*

Proof.

1. Suppose there is a point $\mathrm{Spec}(k) \to Y$, by 3.9, the morphism $\mathcal{X} \times_Y k \to k$ decomposes into an adequate moduli space and an adequate homeomorphism. Thus $\mathcal{X} \times_Y k$ cannot be empty.
2. For a closed substack $\mathcal{Z}$ of $\mathcal{X}$, by 3.6, the morphism $\mathcal{Z} \to \mathrm{im}(\mathcal{Z})$ decomposes into an adequate moduli space $\mathcal{Z} \to \mathrm{Spec}_Y(\mathcal{O}_{\mathcal{Z}})$ and an adequate homeomorphism $\mathrm{Spec}_Y(\mathcal{O}_{\mathcal{Z}}) \to \mathrm{im}(\mathcal{Z})$. Thus the morphism $\mathcal{Z} \to \mathrm{im}(\mathcal{Z})$ is surjective and then $\mathrm{im}(\mathcal{Z})$ is closed. Moreover, the morphism is universally closed as the notion of adequate moduli space is preserved under base change up to an adequate homeomorphism.
3. By definition, it suffices to show that for any two quasi-coherent ideals $\mathcal{I}_1$, $\mathcal{I}_2$ of $\mathcal{O}_{\mathcal{X}}$, the morphism
$$\mathcal{O}_Y/f_*(\mathcal{I}_1 + \mathcal{I}_2) \to \mathcal{O}_Y/f_*\mathcal{I}_1 + f_*\mathcal{I}_2$$
is an adequate homeomorphism.It is equivalent to show that for any affine open $U \subset Y$ and any $s \in \Gamma(U, f_*(I_1 + I_2))$, there exists an integer $n > 0$ and $t \in \Gamma(U, f_*\mathcal{I}_1 + f_*\mathcal{I}_2)$ such that t^n is mapped to s. This statement follows formally from the adequate affineness of f.
4. If two k-points $x, y \in \mathcal{X}(k)$ satisfy $\overline{\{x\}} \cap \overline{\{y\}} \neq \emptyset$, then by (2)(3), $\overline{\{f(x)\}} \cap \overline{\{f(y)\}}$ is not empty. However, as Y is an algebraic space, its k-points are closed. Thus $f(x) = f(y)$. Then we conclude with (1).

□

The second is about adequateness of the classifying stack of G.

Theorem 3.11. *Let S be an algebraic space, and G be a reductive group scheme over S, then then morphism $BG \to S$ is an adequate moduli space, where BG is the classifying stack of G.*

Remark 3.12. *If a group scheme satisfies the property that $BG \to S$ is an adequate moduli space, it is called geometrically reductive. In fact on can show that any geometrically reductive group is an extension of a finite group scheme and a reductive group scheme.*

Proof. By [DGIdhésP70] Exp. XXII, corollary 2.3, there exists an étale cover $S' \to S$ such that $G \times_{S'} S$ is split and there exists a split reductive group scheme G' $\mathbb{Z}$ such that $G' \times S' \simeq G \times_S S'$. Thus it suffices to show the case $S = \mathrm{Spec}(\mathbb{Z})$. By descent, it suffices to show that the statement holds for $S = \mathrm{Spec}(A)$, where A is a discrete valuation ring with algebraically closed field k. Consider the radical $R(G)$ of G. As $R(G)$ is a torus, it is geometrically reductive. One can show that extensions and quotients of geometrically reductive group schemes are still geometrically reductive ([Alp18], proposition 9.5.1).

Thus G can be replaced by the universal cover of $G/R(G)$. Thus we may assume that G is simply connected.

G has the resolution property: for any G-module M there is a G-module surjection $V \to M$ such that V is free as an A-module. By [Alp18], lemma 9.2.5 (another version of the criterion 3.6) to prove G is geometrically reductive, it suffices to show that for any finite type G-module V which is free as an A-module and a G-equivariant morphism $x\colon R \to V$, there is an integer n and $f \in \mathrm{Sym}^n(V^*)$ such that $f \circ x = 1$.

By [Alp18], lemma 9.7.2 and lemma 9.7.4, there exists an integer m and a G-equivariant morphism

$$\phi\colon V \to \nabla(m\rho) \otimes \nabla(m\rho)^\vee \to A,$$

where ρ is the half sum of all positive roots and ∇ is the standard module, such that $\phi \circ x$ is the identity. Thus one can choose f to be the composition of ϕ and the determinant $\det\colon \nabla(m\rho) \otimes \nabla(m\rho)^\vee \to A$. □

The theorem implies the following.

Corollary 3.13. *Let S be an algebraic space, $X \to S$ be an affine morphism and G be a reductive group scheme acting on X. Then the morphism $[X/G] \to X /\!\!/ G$ is an adequate moduli spaces, where $[X/G]$ is the quotient stack and $X /\!\!/ G$ is the categorical quotient* $\mathrm{Spec}_S(\mathcal{O}_X^G)$.

Proof. We may assume that S is affine as the statement is local on S. The morphism $[X/G] \to BG$ is affine and thus the composition $[X/G] \to BG \to S$ is adequately affine. By 3.4, the morphism $[X/G] \to X /\!\!/ G$ is also adequately affine as $X /\!\!/ G$ is affine. Moreover, the morphism $[X/G] \to X /\!\!/ G$ induces an isomorphism of global sections by definition. Thus $[X/G] \to X /\!\!/ G$ is an adequate moduli space. □

The following result is about A-points on categorical quotient.

Proposition 3.14. *Let S be an locally noetherian scheme, G be a reductive group scheme over S and $X \to S$ be an affine morphism of finite type. For any A-point x of $X /\!\!/ G$, there exists a local extension $A \subset A'$ and an A'-point of X such that the diagram*

$$\begin{array}{ccc} \mathrm{Spec}(A') & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & X /\!\!/ G \end{array}$$

commutes.

Proof. Denote $[X/G]$ the quotient stack. The morphism $[X/G] \to X /\!\!/ G$ is an adequate moduli space by corollary 3.13. Thus it is universally closed by theorem 3.10 (2). As $G \to S$ is quasi-compact, the stack $[X/G]$ is quasi-separated and of finite type. Thus the morphism $[X/G] \to X /\!\!/ G$ satisfies the existence part of the valuative criterion by [Sta23, Tag 0CLX].

Suppose there is an A-point of $X//G$, which induces a K-point of $X /\!\!/ G$. By theorem 3.10 (1), after a finite extension of L/K, the K-point lifts to $[X/G]$. Then we may replace

K by L and A by a discrete valuation ring $A' \subset L$ such that the fraction field of A' is L. Now we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & [X/G] \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & X /\!\!/ G. \end{array}$$

The diagram is just the diagram appeared in the valuative criterion. Thus there exists a morphism $\mathrm{Spec}(A) \to [X/G]$ such that the diagram commutes. Moreover, as $X \to [X/G]$ is an étale G-torsor, after extending A again we may lift the A-point to X. □

4 Bruhat–Tits buildings

In this section, we briefly sketch the constructions and the consequences about the Bruhat–Tits building. As the theory is standard and the proof is complicated, we will omit some proofs and only illustrated the ideas.

The theory of Bruhat–Tits building is devoted to classify the open bounded subgroups of a reductive group G over a Henselian discrete valuation field K and study their properties. Roughly speaking, the Bruhat–Tits building is a complex with G-action. Each facet of it corresponds to an open bounded subgroup of G, namely the stablizer of it. Moreover, for those open bounded subgroup, on can find a reductive group scheme $\mathcal{G}$ over A, the integer ring of K, such that $\mathcal{G}$ is an integral model of G.

We firstly recall the notion of simplicial complexes.

Definition 4.1 (Simplicial complex).

- *A simplicial complex is a pair $(V, \mathcal{B})$ where V is a set and $\mathcal{B}$ is a set of non-empty subsets of V, such that for all $x \in V$ we have $\{x\} \in \mathcal{B}$ and if $\emptyset \neq A \subset B \in \mathcal{B}$ implies that $A \in \mathcal{B}$. The elements of $\mathcal{B}$ are called facets of the simplicial complex.*

- *A polysimplicial complex is of the form $(V_1, \mathcal{B}_1) \times \cdots \times (V_n, \mathcal{B}_n)$, where each of $(V_i, \mathcal{B}_\rangle)$ is a simplicial complex. The elements of the set $\mathcal{B}_1 \times \mathcal{B}_n$ are called facets. If there is no risk of confusion, we will denote $\mathcal{B}$ for a simplicial complex or a polysimplicial complex.*

- *For $A, B \subset \mathcal{B}$ and $A \subset B$, A is called a face of B. The case for polysimplicial complexes is similar.*

- *For $A \subset B \in \mathcal{B}$, the codimension of A in B is length of a maximal chain $A = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = B$, denoted* $\mathrm{codim}(A, B)$.

- *The maximal facets of a polysimplicial complex are called chambers. A chamber complex is a polysimplicial complex such that every element is contained in a chamber and for any two chambers C, C', there exists a chain of chambers $C_0 = C, C_1, \cdots, C_n = C'$ such that $\mathrm{codim}(C_i \cap C_{i+1}, C_i) = \mathrm{codim}(C_i, C_{i+1}) = 1$ for each i. For a chamber complex, the codimension of a facet A is the codimension of it in a chamber containing it. The condition for the chamber complex ensures that the codimension is well-defined.*

- *The dimension of a facet B is the maximum of* $\mathrm{codim}(A, B)$, *where $A \subset B$.*

One can naturally define the notion of subcomplexes and the morphism and isomorphisms between polysimplicial complexes.

For each simplicial complex, one can form a metric space corresponding to it, called its geometric realization. Each facet of it corresponds to a standard simplex and the simplices are glued with each other via their common faces. For polysimplicial complexes the case is similar. Each facet of it corresponds to the product of standard simplices.

Then we turn to the definition of buildings.

Definition 4.2 (Building). *A building is a chamber complex $\mathcal{B}$ equipped with a collection of its subcomplexes, called the apartments of it, satisfying the following conditions.*

1. *The chamber complex is thick, i.e., each facets of codimension* 1 *is contained in at least three chambers.*
2. *Every apartment is a thin chamber complex, i.e., each facets of codimension* 1 *in it is contained in exactly two chambers.*
3. *Any two chambers belongs to an apartment.*
4. *For any two apartments $\mathcal{A}_1$, $\mathcal{A}_2$ and facets $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{A}_1 \cap \mathcal{A}_2$, there is an isomorphism of $\mathcal{A}_1 \to \mathcal{A}_\in$ such that it fixes $\mathcal{F}_1$ and $\mathcal{F}_2$ pointwisely.*

The following is a simple example of a building.

Example 4.3. *Consider the reductive group $G = \mathrm{SL}_2$ over any field k. Let $(V, \mathcal{B})$ be as the following:*

- *V is the set of all Borel subgroups of G. Then $V \simeq \mathbb{P}^1(k)$.*
- *$\mathcal{B}$ is the set of all singletons in V.*

Moreover, the apartments are those subsets containing two elements in V. Then the conditions are trivially true.

In fact, one can associate a building to each Tits system. The simple example above is just the special case for $G = \mathrm{SL}_2$.

Definition 4.4 (The building associated to a Tits system). *Let (G, B, N, S) be an irreducible Tits system. The building is as the following:*

- *V is the set of maximal proper parabolic subgroups of G.*
- *The elements of $\mathcal{B}$ are of the form $\{P_1, \cdots, P_n\}$ such that the intersection of those parabolic subgroups is again parabolic.*
- *Let $\mathcal{C} \subset \mathcal{B}$ be the subcomplex consisting of the subsets of those standard parabolic subgroups. Then $\mathcal{C}$ is a chamber, called the standard chamber.*
- *Let $\mathcal{A}$ be the union of all the N-conjugates of $\mathcal{C}$. The apartments are those G-conjugates of $\mathcal{A}$ and $\mathcal{A}$ is called the standard apartment.*

For general Tits systems, we define the building of it to be the product of the buildings of the irreducible components of it.

One can show that the construction is exactly a building by using the theory of Tits systems. One kind of examples of the Tits system is $(G(k), B(k), N(k).S)$, where G is a split reductive group over a field k, B is its Borel subgroup, N is the normalizer of the maximal torus and $S \in N(k)$ is liftings of the simple reflections of the Weyl group $W = N(k)/T(k)$. This kind of example is "of finite type".

There is also a kind of examples, which are "affine". For a split reductive group G over k, one can take $G = G(k((t)))$ be the loop group and B is its Iwahori subgroup. N is the normalizer of the torus $T(k)$ and S is liftings of the simple reflections of the affine Weyl group $N/T(k)$. The case that we will introduce in the Bruhat–Tits theory is a generalization of the case for loop groups. That is, the case for $G = G(K)$ where K is a discrete valuation field.

We will introduce two examples to illustrate the idea. The first example is the simplest case $G = \mathrm{SL}_2$. Then G is split.

Example 4.5 (Example for SL_2)**.** *Let K be a discrete valuation field, A be its integer ring, $\mathfrak{m}$ be the maximal ideal of A and π a uniformizer of A. Consider the group $G(K) = \mathrm{SL}_2(K)$. The cocharacter group of the maximal torus T is $X_*(T) \simeq \mathbb{Z}$ For a root b of G, we define the morphism $u_b \colon \mathbb{G}_a \to G$ to be*

$$x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

whose image is denoted U_b.

We firstly construct the standard apartment $\mathcal{A}$. The total space of $\mathcal{A}$ is $X_*(T)\otimes\mathbb{R} \simeq \mathbb{R}$, in which the zero dimensional cells are the elements of $\frac{1}{2}X_*(T)$ and the one dimensional cells are the segments joining the consecutive points.

For each $x \in X_*(T) \otimes \mathbb{R}$, we assign it to the subgroup $\mathcal{P}_x$ of $G(K)$ generated by $T(A)$ and $u_b(\mathfrak{m}^{-[\langle b,x\rangle]})$. Explicitly, we have that

$$\mathcal{P}_x = \begin{pmatrix} A & \mathfrak{m}^{-[\langle a,x\rangle]} \\ \mathfrak{m}^{-[\langle -a,x\rangle]} & A \end{pmatrix}$$

where a is the positive root and $[-]$ is the floor function. For $x \in \frac{1}{2}X_*(T)$, the group $\mathcal{P}_x$ can be shown as a maximal compact subgroup. For $x < y < x + \frac{1}{2}$ where $x \in \frac{1}{2}\mathbb{Z}$, the group $\mathcal{P}_y$ is the intersection of $\mathcal{P}_x$ and $\mathcal{P}_{x+\frac{1}{2}}$, which is called the Iwahori subgroup. The group $\mathcal{P}_x$ depends only on the facet where x is. Thus the group $\mathcal{P}_{\mathcal{F}}$ for a facet $\mathcal{F}$ is well-defined. The idea is that the zero dimensional facets correspond to the points x such that $[\langle a, x\rangle]$ jumps.

Next we consider the affine Weyl group and the affine roots. The affine Weyl group is defined to be the group $N(K)/T(A)$ where N is the normalizer of the maximal torus. The group $N(K)$ will conjugate $\mathcal{P}_x$ to another $\mathcal{P}_y$ for some $\mathcal{P}$ and there is a natural action of $N(k)$ on $\mathcal{A}$. The affine roots are defined to be those affine functions on $X_*(T) \otimes \mathbb{R}$ of the form $b + \mathbb{Z}$. The affine Weyl group is isomorphic to the subgroup of the automorphism group of $\mathcal{A}$ generated by the reflections corresponding to the affine roots.

The Bruhat–Tits building of $\mathrm{GL}_2(K)$ is defined to be $G(K)\times\mathcal{A}/\sim$, where $\sim$ is defined by

$$(g, x) \sim (h, y) \iff \exists n \in N(K) : y = nx,\ g^{-1}hn \in \mathcal{P}_x.$$

The next example is $G = \mathrm{SU}_3$. In this case G is quasi-split but not split. We should consider the relative root system, which is not reduced.

Example 4.6 (Example for SU_3)**.** *Let L/K be a quadratic extension of Henselian discrete valuation fields. Let π be a uniformizer of K, A be the integer ring of A and ϖ be a uniformizer of L. Let ω be the valuation on K (and so on L) such that $\omega(\pi) = 1$. Define $\mathrm{SU}_3(K)$ to be the group*

$$\{g \in \mathrm{SL}_3(L) \mid \bar{g} = Jg^{-t}J^{-1}\}, \quad J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

As the map $g \mapsto JgJ^{-1}$ preserves the Borel subgroup of $\mathrm{SL}_3(L)$, the Borel subgroup descends to the Borel subgroup of SU_3.

The maximal torus T of G is $\{\mathrm{diag}(x, \bar{x}/x, \bar{x}^{-1}) \mid x \in L\}$ and the maximal split torus of G is $\{\mathrm{diag}(x, 0, x^{-1}) \mid x \in K\}$. The character groups of T and S are $X^*(T) \simeq \mathbb{Z}^2$ and $X^*(S) \simeq \mathbb{Z}$ and the map $X^*(T) \to X^*(S)$ can be identified with $(x, y) \mapsto x + y$. The relative roots is $\{-2a, a, a, 2a\}$ where a is the generator of $X_*(S)$. The unipotent subgroup U_{2a} corresponding to $2a$ is

$$\left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid x + \bar{x} = 0 \right\}$$

and similarly for $-2a$. The unipotent subgroup U_a corresponding to a is

$$\left\{ u_a(u, v) := \begin{pmatrix} 1 & u & v \\ 0 & 1 & \bar{u} \\ 0 & 0 & 1 \end{pmatrix} \mid v + \bar{v} = u\bar{u} \right\},$$

and similarly for $-a$. We have that the quotient $U_{a/2a} := U_a/U_{2a}$ is isomorphic to L.

We then define a filtration on the subgroups U_a and U_{2a}. The filtration on U_{2a} is defined to be

$$U_{2a,r} := \{u_a(0, v) \mid \omega(v) \geq r + \mu$$

where $\mu := 0$ if L/K is unramified and $\mu := -\omega(\bar{(\varpi)}/\pi - 1)$ if L/K is ramified. The filtration of U_a is defined to be

$$U_{a,r} := \{u_a(u, v) \mid \omega(v)/2 \geq r + \mu/2\}.$$

The induced filtration on $U_{a,2a}$ coincides with the filtration on L by valuations. Similarly one can define the filtrations on U_{-a} and U_{-2a}.

As in the case for SL_2, we define the total space of the standard apartment $\mathcal{A}$ to be the affine space $X_*(S) \otimes \mathbb{R} \simeq \mathbb{R}$ and the zero dimensional facets of $\mathcal{A}$ are those x such that the filtration $U_{b,\langle x,b \rangle}$ jumps for some root b. The results are

- *When L/K is unramified, the jump set is $\mathbb{Z}$ for both $U_{a/2a}$ and U_{2a}. Thus the set of the zero dimensional facets is $\mathbb{Z}$. The affine roots are $\{\pm a + \mathbb{Z}\} \cup \{\pm 2a + \mathbb{Z}\}$.*

- *When L/K is ramified, the jump set is $\mathbb{Z} + \frac{1}{2}$ for U_{2a} and $\frac{1}{2}\mathbb{Z}$ for U_a. Thus the set of the zero dimensional facets is $\frac{1}{2}\mathbb{Z}$. The affine roots are $\{\pm a + \frac{1}{2}\mathbb{Z}\} \cup \{\pm 2a + \frac{1}{2} + \mathbb{Z}\}$.*

For $x \in \mathcal{A}$, its corresponding group $\mathcal{P}_x$ is defined to be the group generated by $T(A)$ and $U_{b,\langle -b,x\rangle}$ for $b = \pm a, \pm 2a$.

The affine Weyl group can still be defined as the subgroup of the automorphism group of $\mathcal{A}$, generated by the reflections corresponding to the affine roots. The Bruhat–Tits building is then $G(K) \times \mathcal{A}/\sim$, where $\sim$ is defined by

$$(g, x) \sim (h, y) \iff \exists n \in N(K) : y = nx,\ g^{-1}hn \in \mathcal{P}_x.$$

Now we turn to the general case. We will only discuss the case for quasi-split groups for simplicity. In fact, any reductive group over a strictly Henselian discrete valuation field is quasi-split ([KP23], corollary 2.3.8). Then for an arbitrary reductive group over a Henselian discrete valuation field one can pass to the maximal unramified extension of it and then use the technique "unramified descent" (introduced in [KP23], chapter 9).

We firstly discuss the split reductive groups. We will imitate the case for SL_2, The issue here is that the isomorphism $\mathbb{G}_\mathrm{a} \simeq U_b$ for a root b is not canonical. Thus there is not a canonical filtration on $U_b(K)$. Instead of this, we will associate a weak Chevalley system to a filtration.

Let G be a split reductive group over a discrete valuation field K and $S = T$ be a maximal torus of G. Denote $V(S) = X_*(S) \otimes \mathbb{R}$. Let A be the integer ring of K.

Definition 4.7. *Given a weak Chevalley system $o = (G, (X_a)_{a\in\Phi})$, each X_a defines an isomorphism $\mathbb{G}_a \simeq U_a$. Then there is a valuation $\phi_{o,a} \colon U_a(K) \to \mathbb{G}_a \to \mathbb{Z} \cup \{\infty\}$ on $U_a(K)$ and we define the filtration of $U_a(K)$ by*

$$U_{a,o,r}(K) = \phi_{o,a}^{-1}([r, \infty]).$$

Moreover, for $v \in V(S)$, we can define the translation of a valuation by v. Define $\phi_{o+v,a} \colon U_a(K) \to \mathbb{Z} \cup \{\infty\}$ to be $\phi_{o,a} + a(v)$. Moreover, the valuation is defined by

$$U_{a,o+v,r}(K) = \phi_{o,a}^{-1}([r - a(v), \infty]).$$

The proof of the following lemma can be found in [KP23], lemma 6.1.13.

Lemma 4.8. *The valuations induced by different choices of weak Chevalley systems differ only by a translation. That is, for two weak Chevalley systems o and o', there exists $v \in V(S)$ such that $\phi_{o+v,a} = \phi_{o'}$ for any root a.*

Definition 4.9 (Standard apartment)**.** *The total space of the standard apartment $\mathcal{A}$ is the affine space $o + V(S)$, viewed as a subspace of the space of valuations, where o is a weak Chevalley system.*

The definition is independent of the choice of the weak Chevalley system by 4.8.

The case for a quasi-split group is more complicated. Let G be a quasi-split reductive group over a Henselian discrete valuation field K, T be a maximal torus and $S \subset T$ be a maximal split torus. Let Φ, $\tilde{\Phi}$ be the relative and absolute root system respectively. Fix a weak Chevalley–Steinberg system (defined in [KP23], definition 2.9.12). For $a \in \Phi$ such that $a/2, 2a \notin \Phi$, we have $U_a(K) \simeq K$. For $a \in \Phi$ such that $2a \in \Phi$, the case is similar to the case in SU_3. We have $U_{a/2a}(K) \simeq L$ for a quadratic extension L/K and $U_{2a}(K) \simeq \{x \in L \mid x + \bar{x} = 0\}$. Then one can again define the filtrations as in the case for SU_3 ([KP23], construction 6.1.21).

Let $N = N_G(T)$ be the normalizer of the maximal torus. Then there is a natural action of $N(K)$ on the weak Chevalley systems and the action induces an action of $N(K)$ on the space $\mathcal{A}$.

The simplicial complex structure is also defined to be the "jumps" and we will then construct it. For this purpose, we firstly define the affine roots, which will also be some affine functions as in the examples.

Definition 4.10 (Affine roots)**.** *For an affine function $\psi\colon \mathcal{A} \to \mathbb{R}$ with slope a, where a is a root, define U_ψ to be $U_{a,x,\psi(x)}$ for some $x \in \mathcal{A}$ (the group U_ψ is independent of the choice of x).*

Define $U_{\psi^+} = \cup_{\psi'>\psi} U_{\psi'}$ and the affine roots to be

$$\Psi = \{\psi \in \mathcal{A}^* \mid \mathrm{d}\psi \in \Phi, U_\psi \nsubseteq U_{2\psi}U_{\psi^+}\},$$

where Φ is the relative root system and $U_{2\psi} = 1$ if $2\mathrm{d}\psi \notin \Phi$.

The affine roots induce automorphisms of the space $\mathcal{A}$ and the group generating by them is called the affine Weyl group.

The we can define the simplicial complex structure of $\mathcal{A}$.

Definition 4.11 (Simplicial complex structure)**.** *The polysimplicial complex structure of $\mathcal{A}$ is defined as follows. Denote $\mathcal{H}_\psi$ the zero space of an affine function ψ.*

- *The chambers of the complex are the connected components of the space*

$$\mathcal{A} \backslash \bigcup_{\psi \in \Psi} \mathcal{H}_\psi$$

.

- *The codimension 1 facets are the connected components of the space*

$$\bigcup_{\psi \in \Psi} \mathcal{H}_\psi - \bigcup_{\psi \neq \psi' \in \Psi} \mathcal{H}_\psi \cap \mathcal{H}_{\psi'}.$$

- *Et cetera.*

Now we can define the Bruhat–Tits building. We introduce the following notations. Z is the central torus of G. For a torus T over K, $T(K)_b$ is the maximal bounded subgroup of $T(K)$. For a torus T over K, choose a field extension $L^{\mathrm{nr}}/K^{\mathrm{nr}}/K$ such that T splits over L^{nr}, $T(K)_0 := T(K) \cap \mathrm{Nm}_{L^{\mathrm{nr}}/K^{\mathrm{nr}}} T(L^{\mathrm{nr}})_b$.

As in the examples, we define the Bruhat–Tits building to some quotient.

Definition 4.12 (Bruhat–Tits building)**.** *Define for $x \in \mathcal{A}$ $\mathcal{P}_x$ to be the subgroup of $G(K)$ generated by $U_{a,x,0}$, $a \in \Phi$ and $Z(K)^0$. The Bruhat–Tits building $\mathcal{B}(G)$ is the quotient $G(K) \times \mathcal{A}/\sim$, where $\sim$ is defined by*

$$(g, x) \sim (h, y) \iff \exists n \in N(K) : y = nx,\ g^{-1}hn \in \mathcal{P}_x.$$

$G(K)$ acts on $\mathcal{B}(G)$ by acting on the first component.

Remark 4.13. *In [KP23], the Bruhat–Tits building is defined by the building associated to some Tits system ([KP23], definition 7.6.1). The definition here is equivalent as explained in [KP23], remark 7.6.5.*

For a field extension L/K, there is a natural map $\mathcal{B}(G) \to \mathcal{B}(G_L)$, induced by the map between their standard apartments $X_*(S(G)) \otimes \mathbb{R} \to X_*(S(G_L)) \otimes \mathbb{R}$ (note that the scalar extension of a weak Chevalley system is again a weak Chevalley system). The details can be found in [KP23], section 7.9.2.

We then study the points of the Bruhat–Tits buildings.

The special points are defined for an affine root system.

Definition 4.14. *For an affine root system Ψ defined over an affine affine space A, $x \in A$ is called special if for any $\psi \in \Psi$ there exists $\psi' \in \Psi$ such that $\mathrm{d}\psi = \mathrm{d}\psi'$ and $\psi'(x) = 0$.*

For a Bruhat–Tits building, one can define the following conditions on its points.

Definition 4.15. *For $\mathcal{B}(G)$, a point $x \in \mathcal{B}(G)$ is called*

- *special, if its conjugate in the standard apartment is special;*
- *superspecial, if the image of x in $\mathcal{B}(G_L)$ is special for any finite unramified extension L/K.*
- *absolutely special, if the image of x in $\mathcal{B}(G_L)$ is special for any finite separable extension L/K.*
- *hyperspecial, if x is superspecial and G splits over some unramified extension of K.*

Remark 4.16. *As mentioned in [KP23], remark 7.11.2, if G splits over some unramified extension of K, the notions superspecial, absolutely special and hyperspecial are equivalent.*

Example 4.17 (Example for SL_2)**.** *In 4.5, the special points in $\mathcal{A}$ are elements in $\frac{1}{2}\mathbb{Z}$. They are also superspecial, absolutely special and hyperspecial.*

Example 4.18 (Example for SU_3)**.** *In 4.6, for points in $\mathcal{A}$,*

- *If L/K is unramified, then the special points are $\frac{1}{2}\mathbb{Z}$. $\mathrm{SU}_3 \simeq \mathrm{SL}_3$ over L. Then the hyperspecial points are $\mathbb{Z}$ and the same for superspecial and absolutely special.*
- *If L/K is ramified, the special points are $\frac{1}{4}\mathbb{Z}$. In this case, SU_3 is never split over an unramified extension. The superspecial points are $\frac{1}{4}\mathbb{Z}$ and the absolutely special points are $\frac{1}{2}\mathbb{Z}$.*

The following proposition ([KP23],corollary 7.11.5) will be used in the proof of the finiteness theorem 1.3.

Proposition 4.19. *Any two absolutely special points are conjugate by the adjoint group $G_{\mathrm{ad}}(K)$.*

Finally we discuss the relation between the bounded subgroups and the points on the Bruhat–Tits building.

Proposition 4.20. *Any bounded subgroup $\mathcal{K}$ of $G(K)$ is contained in a maximal open bounded subgroup*

Proof. Choose a faithful representation $G(K) \to \mathrm{GL}(V)$, where V is a vector space over K. Choose an $\mathcal{A}$-lattice Λ in V. The set $\mathcal{K}\Lambda$ is bounded and an A-module. Thus $\mathcal{K}\Lambda$ is a lattice Λ' in V. Then $\mathcal{K}$ is contained in an open bounded subgroup $\mathrm{GL}(\Lambda') \cap G(K)$ of $G(K)$.

By Zorn's lemma, it suffices to prove that an increasing union $\bigcup_{i \in I} B_i$ of bounded subgroups is again bounded. If not, the union will contain an element whose some eigenvalue ζ satisfies $\omega(\zeta) < 0$ by [KP23], lemma 2.2.11 as the union is a subgroup. However, none of B_i contains such element, a contradiction. □

Definition 4.21. *For a torus T over K, the group $T(K)^1$ is defined to be the maximal bounded subgroup of T.*

For a reductive group G, denote G_{sc} to be the simply connected cover of $G/Z(G)$. Denote $G(K)^\natural$ the image $G_{sc}(K) \to G$. The group $G(K)^1$ is defined to be $G(K)^\natural Z(G)^1$.

Then we have that any bounded subgroup is contained in $G(K)^1$.

Theorem 4.22. *For $x \in \mathcal{B}(G)$, define $G(K)^1_x$ the stablizer of x in $G(K)^1$.*

1. *Any maximal open bounded subgroup $\mathcal{K}$ of $G(K)$ is of the form $G(K)^1_x$ for some point x, where x is the barycenter of some facet.*
2. *For any vertex x, $G(K)^1_x$ is a maximal open bounded subgroup of $G(K)$.*

Sketch of proof. (1) One can show that the Bruhat–Tits building $\mathcal{B}(G)$ is complete ([KP23], theorem 4.2.10) and non-positively curved ([KP23], proposition 4.2.7). Thus by the Bruhat–Tits fixed point theorem ([KP23], theorem 1.1.15) $\mathcal{K}$ is contained in such $G(K)^1_x$ and one concludes with the maximality of $\mathcal{K}$.

(2) The group $G(K)^1_x$ is bounded by [KP23], theorem 7.7.1. For a bounded subgroup $\mathcal{K}$ containing $G(K)^1_x$, it is contained in some $G(K)^1_y$. We then conclude that $x = y$ as x is the only point fixed by $G(K)^1_x$. □

5 The finiteness result

In this section, we will proof the finiteness theorem 1.3 after Cotner.

The proof relies on the following lemma.

Lemma 5.1. *Let G be a reductive group scheme over A and $B \subset G(K)$ a bounded subgroup, We may extend A such that there is an element $g \in G(K)$ satisfying $Ad(g)(B) \subset G(A)$.*

Proof. By 4.20 and 4.22, B is contained in a maximal bounded open subgroup U of G and U is the stablizer of some $x \in \mathcal{B}(G^0_K)$, where $\mathcal{B}(G^0_K)$ is the Bruhat–Tits building of G^0_K and x is the barycenter of some facet of $\mathcal{B}(G^0_K)$. By [Lar95], lemma 2.4, there is a local extension $A \subset A'$ such that the inverse image x' of x along $\mathcal{B}(G^0_{K'}) \to \mathcal{B}(G^0_K)$ is a hyperspecial point. We may also extend A such that G_A is hyperspecial. Thus by 4.19, $G(A)$ and the subgroup corresponding to x' are conjugate by $G(K)$ after extending A. □

We turn to the proof of the finiteness theorem 1.3.

Proof. We may assume that S is affine as the result is local on S. Then $G^n /\!\!/ G$ and $H^n /\!\!/ H$ are affine. Moreover, the morphism is of finite type. Thus by the valuative criterion, it suffices to prove that for any discrete valuation ring A with fraction field K and any commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathrm{Hom}(\Gamma, H) /\!\!/ H \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Hom}(\Gamma', G) /\!\!/ G \end{array}$$

There exists a local extension $A \subset A'$ with fraction field K' and a morphism $\mathrm{Spec}(A) \to H^n /\!\!/ H$ such that the diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & \mathrm{Hom}(\Gamma, H) /\!\!/ H \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & \mathrm{Hom}(\Gamma', G) /\!\!/ G \end{array}$$
$$\mathrm{Spec}(A') \dashrightarrow \mathrm{Hom}(\Gamma, H) /\!\!/ H$$

commutes.

By lemma 3.14, we may and do extend A such that the morphism $\mathrm{Spec}(A) \to \mathrm{Hom}(\Gamma', G) /\!\!/ G$ comes from a group homomorphism $\psi\colon \Gamma' \to G(A)$ and $\mathrm{Spec}(K) \to \mathrm{Hom}(\Gamma, H) /\!\!/ H$ comes from a group homomorphism $\phi\colon \Gamma \to H(K)$ and we may choose ϕ, ψ such that $\psi_{\bar{K}}$ and $\phi_{\bar{K}}$ live in closed orbits of $\mathrm{Hom}(\Gamma', G(\bar{K}))$ and $\mathrm{Hom}(\Gamma, H(\bar{K}))$ respectively. Thus we may extend A such that ψ_K and $f \circ \phi|_{\Gamma'}$ are conjugate by $G(K)$.

As $\psi(\Gamma') \subset G(A)$ is bounded, $f \circ \phi(\Gamma)$ is also bounded and $\phi(\Gamma)$ is bounded. Note that the boundedness is preserved by local extension of discrete valuation rings. Then by lemma 5.1 after extending A there is a group homomorphism $\phi'\colon \Gamma \to H(A)$ such that ϕ' and ϕ are conjugate by $G(K)$ and ϕ' induces the desired dashed morphism. □

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