

GEOMETRIC SATAKE EQUIVALENCES

RAJARSHI GHOSH

1. THE AFFINE GRASSMANNIAN AND PRELIMINARY REMARKS ON THE GEOMETRIC SATAKE EQUIVALENCE

To derive the geometric Satake equivalence, one must first describe the structure of the affine Grassmannian and the affine Schubert varieties. However, this is explained very well Sections 1, 2, and 3 of Zhu [24], and we will not repeat the same.

However, by way of setting up some notation we recall the following. Let G be a split reductive group over an algebraically closed field, k . Let LG denote the loop group, which is the functor sending a k -algebra R to the set $G(R((t)))$, and L^+G the positive loop group, which is the functor $R \mapsto G(R[[t]])$. LG is an ind-affine group, while L^+G is an affine group, but not of finite type. The fpqc quotient L^+G/LG is represented by an ind-projective ind-scheme, the affine Grassmannian, denote Gr_G , or simply Gr if it is clear from context, which is the group whose affine Grassmannian is being considered. These notions carry over to more general settings, where one works with a non split reductive group G over the local field $k((t))$ and replaces L^+G with other parahoric subgroups. This generalisation will be useful for the “ramified geometric Satake equivalence” which is treated in sections 8 onwards, and these generalisations in particular are recalled in Section 8.

There is a right action of the group L^+G on Gr . The Cartan-Iwasawa decomposition establishes a bijection between these L^+G -orbits and $X_*(T)_+$, after a choice of torus T and Borel $B \supset T$ are made. These orbits are finite dimensional affine bundles over classical partial flag varieties. The orbit corresponding to $\lambda \in X_*(T)_+$ under this natural bijection is denoted Gr_λ . Its closure is no longer smooth, it is the *affine Schubert variety* corresponding to λ , and usually denoted $\mathrm{Gr}_{\leq \lambda}$. On points, one has

$$\mathrm{Gr}_{\leq \lambda} = \coprod_{\lambda' \leq \lambda} \mathrm{Gr}_{\lambda'}.$$

There is a desingularisation of these affine Schubert varieties, furnished by the Demazure resolutions recalled in Section 8.1.1.

1.1. Technicalities surrounding derived categories. We will have occasion to deal with equivariant perverse sheaves on ind-schemes. Technicalities on equivariant perverse sheaves are not recalled here, the basic facts we need are summarised in Baumann-Riche [7] Appendix A.1, or Zhu [24] A.1.1, for an even shorter summary. For a more detailed exposition, we refer to Achar [1] Chapter 6, which also defines the equivariant derived category of Bernstein-Lunts. There is another approach taken in Achar-Lourenço-Richarz-Riche, where one considers (equivalently) constructible sheaves on a certain quotient stack, which they call the “Hecke stack”, for instance.

It is explained in Baumann-Riche [7] Appendix A.4 and Gaitsgory [11] Appendix, how these notions carry over to the ind-schemes dealt with in these scenarios.

In any case, let us denote by $\mathcal{S}$ the stratification of Gr given by the Schubert varieties. We work the derived category $D_{\mathcal{S}}^b(\mathrm{Gr}, \bar{\mathbb{Q}}_l)$ of bounded constructible complexes adapted to $\mathcal{S}$.

2. UNIVERSAL LOCAL ACYCLICITY AND THE FUSION PRODUCT

2.1. Universal Local Acyclicity. In this section we collect some facts concerning the notion of “Universal Local Acyclicity” (which is henceforth abbreviated “ULA”, following Zhu [24]). We will omit proofs for they are mostly easy, and sketched in the Zhu, with the exception of point 4 of Theorem 2.1 below, for whose proof one is referred to Illusie [14] Corollary 2.5.

We define ULA sheaves as follows. Let S be a scheme and $f : X \rightarrow S$ be a map. Let s be a geometric point of S , and t a geometric point of $S_{(s)}$, the strict henselization of S at s . Let j_t denote the map $X_t := X \times_S t \rightarrow X$. Moreover, we write $\Psi_{t \rightarrow s}(\mathcal{F}) := ((j_t)_*(j_t)^*\mathcal{F})|_{X_s}$, anticipating the connection with nearby cycles.

Then $\mathcal{F}$ is said to be locally acyclic with respect to $f : X \rightarrow S$ if the natural map $\mathcal{F}|_{X_s} \rightarrow \Psi_{t \rightarrow s}(\mathcal{F})$ is an isomorphism.

If $\mathcal{F}$ continues to remain locally acyclic for any base change $S' \rightarrow S$, we say that it is *universally locally acyclic*.

The basic properties of ULA’s that we require are listed below.

THEOREM 2.1 (Zhu [24]). *(1) Let S be a smooth variety over k . Then a complex $\mathcal{F}$ on S is ULA with respect to the identity map $\text{id} : S \rightarrow S$ if and only if $\mathcal{H}^i(\mathcal{F})$ on S is a local system for each i .
(2) For $f : X \rightarrow Y$ a proper morphism, one has that for any ULA $\mathcal{F}$ on X , the pushforward $f_*\mathcal{F}$ is ULA on Y .
(3) For $f : X \rightarrow Y$ a smooth surjective morphism, $f^*\mathcal{F}$ is ULA on X if and only if $\mathcal{F}$ is so on Y .
(4) Let $f_i : X_i \rightarrow S_i$ for $i = 1, 2$ be two morphisms and let $\mathcal{F}_i$ be a ULA complex with respect to f_i . Then $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ is ULA with respect to $f_1 \times f_2$.*

THEOREM 2.2 (Theorem A.2.6 of Zhu [24]). *Let $f : X \rightarrow S$, with the base S smooth and $D \subset S$ a smooth effective divisor. We write $i : Z := f^{-1}(D) \rightarrow X$ and $j : U := X - D \rightarrow X$ for the complementary open embedding. Suppose we are given $\mathcal{F}$ which is a complex on X ULA with respect to f , such that $\mathcal{F}|_U$ is perverse.*

Then $\mathcal{F} \simeq j_{!}\mathcal{F}|_U$ (in particular $\mathcal{F}$ is automatically perverse), and $i^!\mathcal{F}[1] \simeq i^*\mathcal{F}[-1]$ is perverse and ULA with respect to $f|_i$.*

2.2. The Fusion Product. We define Sat_{X^I} as the full subcategory of $\text{Perv}_{L^+G_{X^I}}(\text{Gr}_{X^I}, k)$ of sheaves ULA with respect to $\pi_I : \text{Gr}_{X^I} \rightarrow X^I$. (Here $L^+G_{X^I}$ denotes the “multi-point” version of L^+G .) For every $\zeta : J \rightarrow I$, one has a map f_ζ defined in the following manner:

$$\begin{array}{ccc} \text{Gr}_{X^I} & \xrightarrow{f_\zeta} & \text{Gr}_{X^J} \\ \downarrow & & \downarrow \\ X^I & \longrightarrow & X^J \end{array}$$

and this in turn defines a functor

$$f_\zeta^* [||I| - |J||] : \text{Sat}_{X^I} \rightarrow \text{Sat}_{X^J}.$$

Indeed, this is well defined, perversity of the pullback as above is guaranteed by repeated applications of the last statement of Theorem 2.2. That such a pullback is ULA follows immediately from the definition of the property itself.

Given a partition $(J_1, \dots, J_n)$ of J , consider $m = m_{J_1, \dots, J_n}$ defined by

$$m : \text{Gr}_{X^J}^{(J_1, \dots, J_n)} \rightarrow \text{Gr}_{X^J}.$$

Given $\mathcal{A}_i \in \text{Sat}_{X^{I_i}}$, we may form the twisted box product $\mathcal{A}_1 \tilde{\boxtimes} \dots \tilde{\boxtimes} \mathcal{A}_n$ as a perverse sheaf on $\text{Gr}_{X^J}^{(J_1, \dots, J_n)}$. Next, we define

$$\mathcal{A}_1 \boxtimes \dots \boxtimes \mathcal{A}_n := m_!(\mathcal{A}_1 \tilde{\boxtimes} \dots \tilde{\boxtimes} \mathcal{A}_n).$$

That this is a perverse sheaf, follows from the fact that it is firstly ULA by the second statement of Theorem 2.1, and that its restriction to $\text{Gr}_{X^J}|_{X^\phi}$ is $\mathcal{A}_1 \boxtimes \dots \boxtimes \mathcal{A}_n$. (Here, and what follows, we denote by X^ϕ the locus $(x_j)_{j \in J}$ of points in X^J , where no two coordinates are equal.) Now, we may appeal to Theorem 2.2 to prove perversity.

More precisely, we have proved:

PROPOSITION 2.3. (1) $\mathcal{A}_1 \boxtimes \dots \boxtimes \mathcal{A}_n \in \text{Sat}_{X^J}$. This defines the “external convolution” functor

$$\boxtimes : \text{Sat}_{X^{J_1}} \times \dots \times \text{Sat}_{X^{J_n}} \rightarrow \text{Sat}_{X^J}.$$

$$(2) \mathcal{A}_1 \boxtimes \dots \boxtimes \mathcal{A}_n \in \text{Sat}_{X^J} \simeq j_{!*}((\mathcal{A}_1 \boxtimes \dots \boxtimes \mathcal{A}_n)|_{X^\phi}).$$

In the next step we define the operation of *internal convolution*, as the composite

$$\begin{aligned} \star : \quad & \text{Sat}_{X^J}^n \rightarrow \text{Sat}_{X^J} \rightarrow \text{Sat}_{X^I}, \\ & (\mathcal{A}_1, \dots, \mathcal{A}_n) \mapsto \mathcal{A}_1 \boxtimes \dots \boxtimes \mathcal{A}_n \mapsto f_\zeta^* [|J| - n|J|](\mathcal{A}_1 \boxtimes \dots \boxtimes \mathcal{A}_n) \end{aligned}$$

(Recall that we have already discussed why the last object is indeed contained in Sat_{X^I} .)

Remark 2.4. This is one place where imposing the ULA property seems crucial. Defining internal convolution on the entire category $\text{Perv}_{L^+G_{X^I}}(\text{Gr}_{X^I}, k)$ via the above formula fails, as Zhu shows in Remark 5.4.5 of [24].

To conclude this section, observe that there is functor

$$\begin{aligned} \tau : \text{Sat}_G &\rightarrow \text{Sat}_X \\ \mathcal{A} &\mapsto \mathcal{A} \tilde{\boxtimes} \overline{\mathbb{Q}}_I[1] \end{aligned}$$

LEMMA 2.5. (1) The above functor $\mathcal{A} \rightarrow \mathcal{A}_X$ is fully faithful.

(2) One has a canonical isomorphism

$$(\mathcal{A}_1 * \dots * \mathcal{A}_n)_X \simeq (\mathcal{A}_1)_X * \dots * (\mathcal{A}_n)_X.$$

3. SEMISIMPLICITY OF THE SATAKE CATEGORY

An important step in the derivation of the geometric Satake equivalence is the fact that Sat_G is semisimple. Thus, objects in Sat_G are simply direct sums of IC_μ , where IC_μ denotes the Intersection Cohomology sheaf associated to the Schubert variety $\text{Gr}_{\leq \lambda}$.

An almost complete proof of the alleged semisimplicity can be found in Baumann-Riche [7], Chapter 4. We will recall the main steps in the argument and prove in full only the fact which is not proved therein. This is the fact that “fibres of Demazure resolutions admit pavings by affine spaces”. We give the proof of this fact although it is exactly parallel to the one in Haines [12] because the argument in *loc. cit.* a priori is set in the classical case of Schubert varieties. Further, we will require the non-split case in Section 8.

This semi-simplicity is equivalent to

PROPOSITION 3.1. For $\lambda, \mu \in X_*(T)^+$, we have

$$\text{Ext}^1(IC_\lambda, IC_\mu) = \text{Hom}_{D_{\mathcal{S}}^b(\text{Gr}_G, \overline{\mathbb{Q}}_I)}(IC_\lambda, IC_\mu[1]) = 0.$$

In the first step, one works with $\lambda = \mu$. Here, the result follows from the yoga of dealing with the “fundamental distinguished triangles” and applying cohomological functors - the argument is found in page 33 of Baumann-Riche [7] under Step 1.

Next, when neither $\lambda \leq \mu$ or $\mu \leq \lambda$ is similarly straightforward, and is Step 2 of their argument found on pages 33-34.

Finally, we assume $\lambda \geq \mu$ (the other case $\mu \geq \lambda$ would follow from Verdier duality). In this case, we replace Gr with $\text{Gr}_{\leq \lambda}$ since all sheaves we will consider will be supported on the latter. Let us denote by j_μ the inclusion of Gr_μ inside $\text{Gr}_{\leq \lambda}$. Moreover, we denote by $\mathcal{G}$ the cone

$$IC_\mu \rightarrow (j_\mu)_* \tilde{Q}_l[\dim \text{Gr}_\mu] \rightarrow \mathcal{G} \rightarrow . \quad (3.0.1)$$

The middle term is concentrated in non-negative perverse degrees, while in perverse degree 0, the first map is injective. Thus, it follows that $\mathcal{G}$ is itself concentrated in non-negative degrees. Applying the functor $\text{Hom}(IC_\lambda, -)$ to the above, one obtains an exact sequence

$$\text{Hom}(IC_\lambda, \mathcal{G}) \rightarrow \text{Hom}(IC_\lambda, IC_\mu[1]) \rightarrow \text{Hom}(IC_\lambda, (j_\mu)_* \tilde{Q}_l[\dim \text{Gr}_\mu + 1]). \quad (3.0.2)$$

The first term above is zero. We want to show that so is the last. First, we observe that $(j_\mu)^* IC_\lambda$ is concentrated in degrees $< \dim \text{Gr}_\mu$. We will in fact show that $j_\mu^* IC_\lambda$ is in fact concentrated in degrees $< \dim \text{Gr}_\mu - 1$, which would prove the claim. Observing that $\dim \text{Gr}_\lambda \equiv \dim \text{Gr}_\mu \pmod{2}$ since they lie in the same connected component, this follows from the following lemma:

LEMMA 3.2 (“Parity vanishing” - Baumann-Riche [7] Lemma 4.5). *For $\lambda \in X_*(T)^+$ the cohomology sheaf $\mathcal{H}^n(IC_\lambda) \neq 0$ implies $n \equiv \dim(\text{Gr}_\lambda) \pmod{2}$.*

The reduction to proving that the Demazure resolutions admit pavings by affine spaces is explained with full details in Section 4.2 of Baumann-Riche [7], so we proceed to show the existence of the paving.

3.1. Paving by affine spaces. In this section, we recall the argument given by Haines in [12] to prove that the fibres of the Demazure resolutions are paved by affine spaces. We assume G is quasi-split over K and let I denote an Iwahori subgroup.

Let $w = s_1 \cdots s_r$ be a word in the affine Weyl group W_{aff} . We rewrite the Demazure resolution as the subset $\widetilde{X(w)} \subset (G(K)/I)^r$ given by

$$\{(g_1 I, \dots, g_r I) \mid g_{i-1}^{-1} g_i \in \mathcal{P}_{s_i}\},$$

where $\mathcal{P}_{s_i}$ is used to denote the parahoric subgroup of $G(K)$ given by $I \cup I s_i I$. Often, we will think of the coordinates as Iwahori subgroups corresponding to the stabilisers of the points $g_i I$ in the flag variety. (These stabilisers are the Iwahori subgroups $g_i I g_i^{-1}$.) The projection onto the r -th coordinate defines the map $\pi_s : \widetilde{X(w)} \rightarrow X(w)$ whose fibres we wish to study in this section. We denote the Schubert cells by $Y(w)$ and their closures, the corresponding Schubert varieties as $X(w)$.

PROPOSITION 3.3. *For each $v \leq w$ (in the Bruhat order), the fibre over $y.I \in Y(v)$ is paved by affine spaces.*

Proof. We induct on $r = l(w)$. By translation, we may assume that $y.I = v.I$ in $Y(v) \subset G(K)/I$.

Our first step is to construct a map $p : \pi^{-1}(v.I) \rightarrow Y(v) \cup Y(v s_r)$. Indeed, for $(I_1, \dots, I_{r-1}, v.I) \in \pi^{-1}(v.I)$ it is clear that $I_{r-1} \in Y(v) \cup Y(v s_r)$, and the map p is just the projection onto the $(r-1)$ -th coordinate. The strategy is now as follows: to decompose the $\text{Im}(p) \cap Y(v) \cup Y(v s_r)$ as a union of affine spaces, and show that p is a trivial fibration over each. Next, we apply, in a fashion to be made precise below, the induction hypothesis to the fibre and conclude thence.

LEMMA 3.4. *Both $\text{Im}(p) \cap Y(v)$ and $\text{Im}(p) \cap Y(v s_r)$ are either empty, a single point, or an $\mathbb{A}^1$. Further, one of them is closed in $\text{Im}(p)$, while the other is open and dense in the same. (We call the former $\mathbb{A}_1$ and the latter $\mathbb{A}_2$.)*

Proof. We consider the image of p . These are Iwahori subgroups I' such that their relative position from $J := \nu.I$ is “at most s_r ”, in the sense that $I' \subset J \cup Js_r J$. This statement is an immediate consequence of the definition of the Demazure resolution.

Now, the union $J \cup Js_r J$ is of the form $\{\infty\} \cup \mathbb{A}^1$ such that the intersections with $Y(\nu)$ and $Y(\nu s_r)$ are one of $\emptyset, \{\infty\}$ and $\mathbb{A}^1$, but not both empty, and $\{\infty\}$ (which corresponds to the point J) is contained in exactly one of these intersections. These statements finish the proof of the claim.

Q.E.D

The following lemma is fairly immediate.

LEMMA 3.5. *Let π' denote the Demazure resolution for $s_1 \dots s_{r-1}$. Then for $i = 1, 2$, we have $p^{-1}(\mathbb{A}_i) = \pi'^{-1}(\mathbb{A}_i)$. Further, the restriction of p to each $p^{-1}(\mathbb{A}_i)$ coincides with π' to $\pi'^{-1}(\mathbb{A}_i)$.*

LEMMA 3.6. *$\pi'^{-1}(\mathbb{A}_i) \rightarrow \mathbb{A}_i$ is a trivial fibration.*

Proof. The statement is vacuously true for $\mathbb{A}_1$. To prove the statement for $\mathbb{A}_2$ we place ourselves in Case 1(a), as the other non-trivial case is entirely analogous.

Here, $\mathbb{A}_2 = \{I' | I' = \nu u s_r . I, u \in U_r\}$, where r denotes the root corresponding to s_r and U_r the corresponding root subgroup. We write an isomorphism $\pi'^{-1}(\mathbb{A}_2) \rightarrow \pi'^{-1}(\nu s_r . I) \times \mathbb{A}_2$ as follows:

$$(I_1, \dots, I_{r-2}, \nu u s_r . I) \mapsto ((\nu u^{-1} \nu^{-1} . I_1, \dots, \nu u^{-1} \nu^{-1} I_{r-2}, \nu s_r . I), \nu u s_r . I).$$

In order to check that the first factor of the right hand term above indeed defines a term of $\pi'^{-1}(\nu s_r . I)$, it suffices to check that $\nu u^{-1} \nu^{-1} \in I$. Indeed, one has that $u \in U_r$, whence $\nu u^{-1} \nu^{-1} \in U_{\nu(r)}$. Now, $\nu < \nu s_r$, which means $\nu(r)$ is also a positive root by Humphreys [16], and thus $\nu u^{-1} \nu^{-1} \in U$, and we are done.

Q.E.D

To conclude the proof of the proposition, we argue as hinted above. Indeed, by Lemma 3.5 it suffices to show that $\pi'^{-1}(\mathbb{A}_i)$ are paved by affine spaces. However, this follows from the induction hypothesis, and this concludes the proof.

Q.E.D

4. SEMI-INFINITE GEOMETRY OF THE AFFINE GRASSMANNIAN

In this section we consider the left action of LU on the Gr . The orbits under this action are indexed by $\lambda \in X_*(T)$, where the orbit corresponding to λ , denoted S_λ is the orbit of t^λ . In this section, we study, following Zhu [24] and Baumann-Riche [7] the geometry of these orbits, which turn out to neither of finite dimension or codimension.

LEMMA 4.1. (1) *The closure of S_λ in Gr is the union*

$$S_{\leq \lambda} := \cup_{\lambda' \leq \lambda} S_{\lambda'}.$$

(2) *The set*

$$S_{< \lambda} := \cup_{\lambda' < \lambda} S_{\lambda'} \subset S_{\leq \lambda}$$

is the intersection with a hyperplane section of $\mathcal{O}(1)$.

(3) *Consider the action of $\mathbb{G}_m$ on Gr via the coweight $2\rho^\vee$. The sets S_λ are precisely the attractors for this action, with corresponding fixed point t^λ .*

Proof. (1) (Following Zhu.) We begin by showing that $S_{\leq \lambda}$ is a closed subset of Gr . First, assume that G_{der} is simply connected. One observes that if $x \in S_{\leq \lambda}$ is a closed point, then the associated pair (E, β) satisfies the following property. Let V_χ be the highest weight representation of G of highest weight χ . If (E_χ, β_χ) denotes the associated family of lattices, then the pre-image under the highest weight line in E_χ is contained in $t^{\langle \chi, \lambda \rangle}$. Indeed, this is clear, once x is represented as

$ut^\mu g$ as an element in the double coset $U(K)t^\mu G(\mathcal{O})$, and it is observed that $U(K)$ acts trivially on the highest weight line. To conclude, one takes intersection across all dominant weights χ .

In general, recall that the components of the affine Grassmannian are indexed by certain cosets of coweights $t^\mu + \text{coroot lattice}$, and the action of $t^{-\mu}$ brings one component to that of the “identity coset”. The latter, coincides with the affine Grassmannian of the simply connected cover of the derived group, whence we are reduced to the first case.

Lastly, to show that S_λ is open dense in $S_{\leq \lambda}$ we argue as follows. Open-ness is clear from the above argument. As for density of the same, we use the customary curve for these matters, which is described below. Let α be a positive coroot, and we reduce to drawing a curve joining t^λ to $t^{\lambda-\alpha}$ using Rapoport [19] as in Zhu [24]. Set $m = (\mu, \alpha) - 1$, $t^{\lambda_m} := \begin{bmatrix} t^m & 0 \\ 0 & 1 \end{bmatrix}$, and $K_m := \text{Ad}_{t^{\lambda_m}}(L^+ \text{SL}_2) \subset L \text{SL}_2$. Finally, we set $K_m^1 = \text{Ad}_{t^{\lambda_m}}(L^> \text{SL}_2)$, so that $K_m/K_m^1 \simeq \text{SL}_2$. We write i_α for the map $\text{SL}_2 \rightarrow G$ corresponding to the coroot α . The curve we consider is then given by

$$C_{\lambda, \alpha} := ((Li_\alpha)(K_m)) \cdot t^\lambda.$$

To have a better geometric description of the curve, consider the stabiliser of this action of $(Li_\alpha)(K_m)$. It is given by

$$(Li_\alpha)(K_m) \cap t^\mu L^+ G t^{-\mu}$$

and certainly contains

$$\left\langle K_m^1, \begin{bmatrix} a & 0 \\ t^{-m}c & a^{-1} \end{bmatrix} \right\rangle,$$

where $a, c \in k$.

Thus, using the remark $K_m/K_m^1 \simeq \text{SL}_2$, we find that the curve affords a surjective map from $\text{SL}_2 / \{\text{lower triangular matrices}\} \simeq \mathbb{P}^1$, and must therefore be $\mathbb{P}^1$ itself. Let π denote the induced isomorphism

$$\text{SL}_2 / \{\text{lower triangular matrices}\} \rightarrow C_{\lambda, \alpha}.$$

The image of $[1 : 0]$ under this isomorphism is just t^λ . On the other hand, consider the image of $[0 : 1]$. In terms of the SL_2 chosen above, the point is given by

$$\begin{bmatrix} t^m & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} t^m & 0 \\ 0 & 1 \end{bmatrix}^{-1} \cdot t^\lambda.$$

However, we have the identity

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -t^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \begin{bmatrix} 1 & -t^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t^{-1} & 0 \\ 0 & t \end{bmatrix}$$

The last term $\begin{bmatrix} t^{-1} & 0 \\ 0 & t \end{bmatrix}$ corresponds to the negative coroot $-\alpha$. Upon conjugating the first three terms by t^{λ_m} , we find that they each individually land in $t^{\lambda-\alpha} L^+ G t^{-(\lambda-\alpha)}$, which is the stabiliser of t^{λ_m} under the left action of LG on the affine Grassmannian. All in all, we conclude that the image of $[0 : 1]$ under π is $t^{\lambda-\alpha}$ in the affine Grassmannian, as claimed.

- (2) (Following Zhu again.) Translating by t^λ it suffices to prove the statement for $\lambda = 0$. In this case, one has that $S_0 \subset \text{Gr}^0$ and $S_{<0} \cap \text{Gr}^0 = \emptyset$ (this follows from the first statements of the proposition below, which does not rely on the statement here). However, recall that the complement of $\text{Gr}_{<0}$ in the affine Grassmannian is a hyperplane section of $\mathcal{O}(1)$, which we call H . By the foregoing statement, one has $S_{<0} = S_{\leq 0} \cap H$, whence we are done.
- (3) This is subsumed in the first point of the following Proposition.

Q.E.D

The following dimension computations will be of interest.

PROPOSITION 4.2 (Theorem 5.2 in Baumann-Riche [7]). *Let $\lambda, \mu \in X_*(T)$ be coweights, with λ dominant. Then the following hold:*

- (1) $\text{Gr}_{\leq \lambda} \cap S_\mu$ is nonempty if and only if $t^\mu \in \text{Gr}_{\leq \lambda}$, which in turn holds if and only if $\mu \in \text{Conv}(W\lambda) \cap (\lambda + Q^\vee)$, where $\text{Conv}(W\lambda)$ denotes the convex hull of the set $W\lambda = \{w\lambda : w \in W\}$.
- (2) For μ as above, $\text{Gr}_{\leq \lambda} \cap S_\mu$ has pure dimension $\langle \rho, \lambda + \mu \rangle$.
- (3) Once again, with μ as above, $\text{Gr}_\lambda \cap S_\mu$ is open dense in $\text{Gr}_{\leq \lambda} \cap S_\mu$.
In particular, the irreducible components of $\text{Gr}_{\leq \lambda} \cap S_\mu$ are in bijection with those of $\text{Gr}_\lambda \cap S_\mu$.

Proof. We follow the argument in Baumann-Riche [7].

- (1) The second of the two equivalences in the statement is immediate, upon using the Cartan-Iwasawa decomposition.

Thus, it remains to show that $\text{Gr}_{\leq \lambda} \cap S_\mu \neq \emptyset \implies t^\mu \in \text{Gr}_{\leq \lambda}$. Now for any regular dominant weight η (such as 2ρ in the 3 from the previous Lemma) one has that

$$U(K) = \{g \in G(K) \mid \lim_{a \rightarrow 0} \eta(a) g \eta(a)^{-1} = 1\}.$$

By the Iwasawa decomposition, we have further that $G(K)$ is a union of double cosets $L(K)t^\mu G(K)$, for $\mu \in X_*(T)$. This shows that $S_\mu = \{x \in \text{Gr} \mid \lim_{a \rightarrow 0} \eta(a) \cdot x = t^\mu\}$. Moreover, this action of η leaves Gr_λ stable, proving the sought-after implication.

- (2) As a first step, we prove the following lemma.

LEMMA 4.3. $\dim \text{Gr}_{\leq \lambda} \cap S_\mu \leq \langle \rho, \lambda + \mu \rangle$.

Proof. We induct on the ordering of $\text{Conv}(W\lambda) \cap \lambda + Q^\vee$. First, for $\mu = w_0\lambda$ (where w_0 denotes the longest word of W), the claim is clear.

Next, one observes that since $S_{<\mu}$ is a hyperplane section in $S_{\leq \mu}$, the intersection $\text{Gr}_\lambda \cap S_{\leq \mu}$ has dimension at most 1 more than that of $\text{Gr}_\lambda \cap S_{<\mu}$, i.e.

$$\begin{aligned} \dim \text{Gr}_{\leq \lambda} \cap S_{\leq \mu} &\leq \dim \text{Gr}_{\leq \lambda} \cap S_{<\mu} + 1 \\ &= \max_{\mu' < \mu} \dim \text{Gr}_{\leq \lambda} \cap S_{<\mu'} + 1, \end{aligned}$$

proving the claim, since $\dim \text{Gr}_{\leq \lambda} \cap S_\mu \leq \dim \text{Gr}_{\leq \lambda} \cap S_{\leq \mu}$. Q.E.D

Now, we prove the opposite inequality. We do so via a sort of “descending” induction.

Indeed, the claim is clear for $\mu = \lambda$. Let us now choose $\mu < \lambda$. Suppose that the claim holds for all $\mu < \mu' \leq \lambda$. To that end, consider the intersection $\text{Gr}_\lambda \cap S_{\leq \mu'}$ and its hyperplane section $H := \text{Gr}_\lambda \cap S_{<\mu'}$. Here, let us choose μ' such that $\mu' - \mu$ is simple. Let C denote an irreducible component of $\text{Gr}_\lambda \cap S_\mu$. Let C' denote an irreducible component of H containing C . Suppose C' intersects $\text{Gr}_\lambda \cap S_{\mu''}$ non-trivially, containing an open dense subset of the former, for some μ'' . Clearly, $\mu'' < \mu$ by definition of H . Moreover, our hypothesis implies $\overline{\text{Gr}_{\leq \lambda} \cap S_{\mu''}} \cap \text{Gr}_\lambda \cap S_\mu$, which in turn implies that $\overline{S_{\mu''}} \cap S_\mu \neq \emptyset$, which shows that $\mu \leq \mu''$. However, our assumption was that $\mu' - \mu$ be simple, and we have already deduced that $\mu'' < \mu'$, whence $\mu'' = \mu$, and we are done.

- (3) Follows from the same argument as in the previous point. Q.E.D

5. THE WEIGHT FUNCTORS

We follow Pramod Achar [1].

PROPOSITION 5.1. *Let $\nu \in X^\vee$ and $i_\nu^\pm$ be the inclusions of the semi-infinite orbits $S_\nu^\pm \hookrightarrow \text{Gr}$. Given $\in D_{L+G}^b(\text{Gr}, k)$, one has the isomorphism*

$$R\Gamma(i_\nu^! \mathcal{F}) \xrightarrow{\sim} R\Gamma_c((i_\nu^+)^* \mathcal{F}). \quad (5.0.1)$$

Moreover, for $k \neq \langle 2\rho, \nu \rangle$, one has that the cohomology

$$H^k(S_v^-, (i_v^-)^! \mathcal{F}) \simeq H_c^k(S_v^+, (i_v^+)^* \mathcal{F})$$

vanishes.

Proof. First, the isomorphism 5.0.1 follows from Braden's hyperbolic localization theorem in the following manner. We make $\mathbb{G}_m$ act on Gr via the cocharacter $2\rho^\vee$. One observes that the fixed point set of this action is given precisely by the points t^λ .

We now prove the vanishing of cohomology. First we prove

LEMMA 5.2. $H_c^k(S_v^+, i_v^+ \mathcal{F}) = 0$ for $\mathcal{F} \in {}^p D_{L^+G}^b(\text{Gr}, k)^{\leq 0}$.

The proof is by induction on the number of L^+G orbits in the support of $\mathcal{F}$. Suppose Gr_λ is open in the support of $\mathcal{F}$. Let $i : Z := \text{Supp } \mathcal{F} - \text{Gr}_\lambda \hookrightarrow \text{Gr}$ and $j = j_\lambda$ be the complementary open embedding. One has the distinguished triangle

$$j_! j^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow i_* i^* \mathcal{F} \rightarrow . \quad (5.0.2)$$

Consider the commutative diagram:

$$\begin{array}{ccc} \text{Gr}_\lambda \cap S_v^+ & \longrightarrow & \text{Gr}_\lambda \\ \downarrow & & \downarrow \\ S_v^+ & \longrightarrow & \text{Supp } \mathcal{F} \end{array}$$

We apply the proper base change theorem to it to obtain

$$H_c^k(S_v^+, (i_v^+)^* j_! \mathcal{F}|_{\text{Gr}_\lambda}) \simeq H_c^k(S_v^+ \cap \text{Gr}_\lambda, \mathcal{F}|_{\text{Gr}_\lambda \cap S_v^+}). \quad (5.0.3)$$

Next, consider $\mathcal{F} \in {}^p D_{L^+G}^b(\text{Gr}, k)$, such that Gr_λ is open in the support of $\mathcal{F}$. Thus, by the definition of perversity, one finds that

$$\mathcal{F}|_{\text{Gr}_\lambda} \in D_{L^+G}^b(\text{Gr}_\lambda, k)^{\leq -\dim \text{Gr}_\lambda},$$

whence

$$\mathcal{F}|_{\text{Gr}_\lambda \cap S_v^+} \in D_{L^+G}^b(\text{Gr}_\lambda \cap S_v^+, k)^{\leq -\dim \text{Gr}_\lambda}$$

holds as well. However, $\dim \text{Gr}_\lambda = \langle 2\rho, \lambda \rangle$. Further, applying duality on $\mathcal{F}$, we find

$$H_c^k(S_v^+ \cap \text{Gr}_\lambda, \mathcal{F}|_{S_v^+ \cap \text{Gr}_\lambda}) = 0$$

for $k \geq -\langle 2\rho, \lambda \rangle + 2 \dim(S_v^+ \cap \text{Gr}_\lambda) = \langle 2\rho, \nu \rangle$, proving the lemma, after applying 5.0.3.

To finish, we prove

LEMMA 5.3. For $k < \langle 2\rho, \nu \rangle$ and $\mathcal{F} \in {}^p D_{L^+G}^b(\text{Gr}, k)^{\geq 0}$ one has

$$H^k(S_v^-, i_v^- \mathcal{F}) = 0.$$

Once again, we induct on the number of L^+G -orbits in the support. Given $\mathcal{F}$ as in the lemma, consider the distinguished triangle

$$i_* i^! \mathcal{F} \rightarrow \mathcal{F} \rightarrow j_* j^* \mathcal{F} \rightarrow . \quad (5.0.4)$$

From this distinguished triangle, it is clear that it suffices to show $H^k(S_v^- \cap \text{Gr}_\lambda, (i_v^-)^! j_* \mathcal{F}|_{\text{Gr}_\lambda}) = 0$, for k in the same range as above. This lets us assume $\mathcal{F} = j_* \mathcal{F}'$ for some $\mathcal{F}'$ on Gr_λ . Moreover, by truncation, and induction on cohomological amplitude, one may further reduce to the case where $\mathcal{F} = \mathcal{L}[m]$, for some local system $\mathcal{L}$ on Gr_λ .

As in the earlier lemma, we consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}_\lambda \cap S_v^- & \longrightarrow & \mathrm{Gr}_\lambda \\ \downarrow & & \downarrow \\ S_v^- & \longrightarrow & \mathrm{Supp} \mathcal{F} \end{array}$$

and apply proper base change to the same to conclude the following, where g denotes the top arrow $\mathrm{Gr}_\lambda \cap S_v^- \rightarrow \mathrm{Gr}_\lambda$,

$$H^k(\mathrm{Gr}_\lambda \cap S_v^-, g^! \mathcal{L}[m]) \simeq H^k(\mathrm{Gr}_\lambda \cap S_v^-, (i_v^-)^! j_* \mathcal{L}[m]), \quad (5.0.5)$$

and we show the former term vanishes. Now, for local systems like $\mathcal{L}$, one knows that

$$g^! \mathcal{L}[m] \in D_c^b(\mathrm{Gr}_\lambda \cap S_v^-, k)^{\geq \langle 2\rho, \lambda + v \rangle - m} \subset D_c^b(\mathrm{Gr}_\lambda \cap S_v^-, k)^{\geq \langle 2\rho, \lambda + v \rangle}.$$

Next, since $\Gamma(\mathrm{Gr}_\lambda \cap S_v^-, -)$ is left-exact, we conclude

$$H^k(\mathrm{Gr}_\lambda \cap S_v^-, g^! \mathcal{L}[m]) = 0$$

for $k < \langle 2\rho, v \rangle$, as required.

Q.E.D

In what follows, the following notation will be convenient:

Definition 5.4. Let $Q \subset X^\vee$. Then (following Achar [1]) we write S_Q^+ for the union

$$S_Q^+ := \cup_{v \in P} S_v^+.$$

LEMMA 5.5. For finite $P \subset X^\vee$, consider the locally closed $S_Q^+ \subset \mathrm{Gr}$. Then, for any $\mathcal{F} \in \mathrm{Perv}_{L^+G}(\mathrm{Gr}, k)$ one has $H_c^i(S_Q^+, \mathcal{F}|_{S_Q^+}) = 0$ unless $\langle 2\rho, \eta \rangle \leq i \leq \langle 2\rho, \eta' \rangle$ holds for $\eta, \eta' \in P$.

Proof. Once again, we induct on the cardinality of Q . Pick $v \in Q$ which is maximal. Let Q' denote the complement. Consider the distinguished triangle (where we write $\mathcal{F}$ for $\mathcal{F}|_{S_Q^+}$ for the sake of brevity)

$$j_! j^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow i_* i^* \mathcal{F} \rightarrow,$$

where we write j for the inclusion of S_v^+ into S_Q^+ , i for the complementary (closed) embedding.

We apply the functor $R\Gamma_c$ to the above triangle, and consider H_c^i for i in the range as stated in the lemma. First, $H_c^i(S_v^+, \mathcal{F}|_{S_v^+}) = 0$ by the previous proposition. Further, analogous vanishing holds for $H_c^i(S_{Q'}^+, \mathcal{F}|_{S_{Q'}^+})$ by the induction hypothesis, and we are done. Q.E.D

LEMMA 5.6. Let $v_1, \dots, v_k \in X^\vee$ and $Z = \cup_{i=1}^k \overline{S_{v_i}^+}$. For $\mathcal{F} \in \mathrm{Perv}_{L^+G}(\mathrm{Gr}, k)$, we have

$$H_c^i(Z, \mathcal{F}|_Z) = 0,$$

for $i > \max\{\langle 2\rho, v_i \rangle\}$.

Proof. (Sketch.) One easily reduces to the previous lemma, by observing that $\mathrm{Supp} \mathcal{F} \cap Z$ is contained in a finite union of $S_{v'}^+$ (for some v' 's satisfying $v' < v_i$ for some i). Q.E.D

LEMMA 5.7. Let $v_1, \dots, v_k \in X^\vee$ once again, but assume now that $\langle 2\rho, v_i \rangle = m$. Let $Z = \cup_i \overline{S_{v_i}^+}$ and $\mathcal{F} \in \mathrm{Perv}_{L^+G}(\mathrm{Gr}, k)$ be given.

Then there is a natural isomorphism

$$\bigoplus_i F_{v_i}(\mathcal{F}) \xrightarrow{\simeq} \mathbb{H}_c^m(Z, \mathcal{F}|_Z). \quad (5.0.6)$$

THEOREM 5.8. For $\mathcal{F} \in \text{Perv}_{L^+G}(\text{Gr}, k)$ we have a natural isomorphism

$$\mathbb{H}^*(\text{Gr}, \mathcal{F}) \simeq \bigoplus_{v \in X^\vee} \mathbf{F}_v(\mathcal{F}). \quad (5.0.7)$$

Proof. We may assume that $\mathcal{F}$ is supported on a single connected component of Gr . It suffices moreover to show for each m that

$$\mathbb{H}^m(\text{Gr}, \mathcal{F}) \simeq \bigoplus_{v \in X^\vee; \langle 2\rho, v \rangle = m} \mathbf{F}_v(\mathcal{F}). \quad (5.0.8)$$

To that end, let $P = \{\lambda | t^\lambda \in \text{Supp } \mathcal{F}\}$. Clearly, the above set is finite. Thus we can find μ which satisfies $\mu > v$ for each $v \in P$.

Now, let $M := \max\{\langle 2\rho, v \rangle | v \in P\}$. If $m > M$, then both sides of 5.0.8 vanish and there is nothing to prove. Thus, we assume that $m \leq M$. In particular, observe that $\langle 2\rho, \mu \rangle \geq \langle 2\rho, v \rangle + 2$, whence $\langle 2\rho, \mu \rangle \geq M + 2$ in fact.

We partition P into the sets $P_1 := \{v | \langle 2\rho, v \rangle < m\}$ and $P_2 := \{v | \langle 2\rho, v \rangle \geq m\}$. We write j for the open embedding $S_{P_2}^+ \hookrightarrow S_P^+$ and i for the complementary closed embedding. Then we have the distinguished triangle, to which we apply $R\Gamma_c$,

$$j_! j^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow i_* i^* \mathcal{F} \rightarrow .$$

We now have two separate cases:

- (1) First suppose M and m are of different parity. Then we observe, using Lemma 5.6, that $R\Gamma_c(i^* \mathcal{F})$ is supported in cohomological degrees $\leq m - 1$.

Likewise, $R\Gamma_c(j^* \mathcal{F})$ is supported in cohomological degree $\geq m$ (i.e. $\geq m + 1$), by Lemma 5.7.

Furthermore, if there is $v \in X^\vee$ for which $\langle 2\rho, v \rangle = m$, then it appears in a different connected component of Gr , whence we have that the restriction of $\mathcal{F}$ to the same is zero.

Consequently, we find that both sides of 5.0.8 vanish, and we are done.

- (2) Next, suppose m and M are of the same parity.

In this case, we observe, using Lemma 5.6 once again, that $R\Gamma_c(i^* \mathcal{F})$ is supported in cohomological degrees $\leq m - 2$.

On the other hand, we have from Lemma 5.7 and the long exact sequence in cohomology, that

$$\mathbb{H}^m(\text{Gr}, \mathcal{F}) = \bigoplus_{\{v \in P | \langle 2\rho, v \rangle = m\}} \mathbb{H}^m(S_v^+, \mathcal{F}|_{S_v^+}).$$

Moreover, if v is another weight which satisfies $\langle 2\rho, v \rangle = m$, the $\mathbf{F}_v(\mathcal{F})$ vanishes since it lies outside the support of $\mathcal{F}$.

In conclusion, we have proved 5.0.8.

Q.E.D

Finally, we will need the following observation in the identifying the maximal torus of $G^\vee$ later.

LEMMA 5.9. For dominant $\lambda \in X_*(T)$ one has that $\dim \mathbf{F}_\lambda(IC_\lambda) = 1$. Moreover, for $\mu \notin \text{Conv}(W\lambda) \cap (\lambda + Q^\vee)$ (where $Q^\vee$ denotes the coroot lattice) one has $\mathbf{F}_\mu(IC_\lambda) = 0$.

Proof. The second statement is a restatement of a part of Proposition 4.2 above, but we restate it nonetheless.

For the first statement, one has

$$\mathbf{F}_\lambda(IC_\lambda) = \mathbb{H}^{\langle 2\rho, \lambda \rangle}(S_\lambda, i_\lambda^! IC_\lambda) = \mathbb{H}_c^{-\langle 2\rho, \lambda \rangle}(S_\lambda, IC_\lambda|_{S_\lambda})(\langle 2\rho, \lambda \rangle).$$

However, $IC_\lambda|_{S_\lambda}$ is just the constant sheaf on $S_\lambda \cap \text{Gr}_\lambda$ twisted by $\langle 2\rho, \lambda \rangle$. Thus via proper base change, this hypercohomology is just the top compactly-supported cohomology of the smooth space $\text{Gr}_\lambda \cap S_\lambda$, which is of course one dimensional, being smooth and connected.

Q.E.D

6. HYPERCOHOMOLOGY AS A FIBRE FUNCTOR

The goal in this section is to derive a commutative diagram of the form

$$\begin{array}{ccc} \mathbb{H}^*(\mathrm{Gr}, \mathcal{F}) \otimes \mathbb{H}^*(\mathrm{Gr}, \mathcal{G}) & \xrightarrow{\cong} & \mathbb{H}^*(\mathrm{Gr}, \mathcal{F} \otimes \mathcal{G}) \\ \cong \downarrow & & \downarrow \cong \\ \mathbb{H}^*(\mathrm{Gr}, \mathcal{G}) \otimes \mathbb{H}^*(\mathrm{Gr}, \mathcal{F}) & \xrightarrow{\cong} & \mathbb{H}^*(\mathrm{Gr}, \mathcal{G} \otimes \mathcal{F}) \end{array}$$

and establish $\mathbb{H}^*(\mathrm{Gr}, -)$ as a fibre functor. For this, we follow Achar [1] once again.

Let σ denote the isomorphism of complexes $A \otimes^L B \rightarrow B \otimes^L A$ in the derived category of bounded complexes of k -vector spaces. We recall that this involves the “supersymmetric sign convention” in order to make the corresponding map at the level of complexes compatible with the differentials. This translates to the following issue: upon taking the cohomology, the “commutativity constraint” does not translate to the usual one in category or k -vector spaces. We will resolve this discrepancy in what follows.

LEMMA 6.1. *For $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}_{L^+G}(\mathrm{Gr}, k)$, there is a natural isomorphism*

$$\phi : R\Gamma(\mathcal{F}) \otimes^L R\Gamma(\mathcal{G}) \xrightarrow{\cong} R\Gamma(\mathcal{F} * \mathcal{G}), \quad (6.0.1)$$

making $R\Gamma : \mathrm{Perv}_{L^+G}(\mathrm{Gr}, k) \rightarrow D^b(\mathrm{Vect}_k)$ a monoidal functor.

Furthermore, if we let β' denote the isomorphism $R\Gamma(\mathcal{F} * \mathcal{G}) \rightarrow R\Gamma(\mathcal{G} * \mathcal{F})$ coming from the commutativity of the convolution product, we obtain the commutative diagram below:

$$\begin{array}{ccc} R\Gamma(\mathcal{F}) \otimes^L R\Gamma(\mathcal{G}) & \xrightarrow{\phi} & R\Gamma(\mathcal{F} * \mathcal{G}) \\ \sigma \downarrow & & \downarrow \beta' \\ R\Gamma(\mathcal{G}) \otimes^L R\Gamma(\mathcal{F}) & \xrightarrow[\phi]{} & R\Gamma(\mathcal{F} * \mathcal{G}) \end{array}$$

Proof. We work with the BD-Grassmannian and interpret the perverse sheaves as $\mathcal{F}, \mathcal{G} \in \mathrm{Sat}_{\mathbb{A}^1}$ and $\mathcal{F} * \mathcal{G} \in \mathrm{Sat}_{\mathbb{A}^2}$.

First, from the generalities concerning ULA sheaves recalled in section ??, we find that $\pi_*(\mathcal{F} * \mathcal{G})$ has constant cohomology sheaves. Now, we recall that if we work over a contractible base, as is $\mathbb{A}^2$, for any object $\mathcal{H}$ in the bounded derived category which has as local systems its cohomology sheaves, there is a natural isomorphism

$$R\Gamma(\mathcal{H}) \rightarrow \mathcal{H}_z,$$

, for any point z in the base.

In our situation, we therefore obtain isomorphisms

$$\begin{aligned} R\Gamma(\mathcal{F}) \otimes^L R\Gamma(\mathcal{G})[2] &\xrightarrow{\cong} R\Gamma(\mathcal{F} \boxtimes \mathcal{G})[2] \\ &\xrightarrow{\cong} \pi_*(\mathcal{F} * \mathcal{G})_{(x,y)} \\ &\xrightarrow{\cong} R\Gamma(\pi_*(\mathcal{F} * \mathcal{G})) \\ &\xrightarrow{\cong} \pi_*(\mathcal{F} * \mathcal{G})_{(x,x)} \\ &\xrightarrow{\cong} R\Gamma(\mathcal{F} * \mathcal{G})[2]. \end{aligned}$$

Finally, for commutativity of 6.1, one observes that the isomorphisms at each stage above commute with the maps $\mathrm{swap}_{\mathbb{A}^2} : (x, y) \mapsto (y, x)$ and its base change $\mathrm{swap}_{\mathrm{Gr}_{\mathbb{A}^2}} : \mathrm{Gr}_{\mathbb{A}^2} \rightarrow \mathrm{Gr}_{\mathbb{A}^2}$. Q.E.D

COROLLARY 6.2. For $\mathcal{F}, \mathcal{G} \in \text{Perv}_{L^+G}(\text{Gr}, k)$, we have an isomorphism

$$\phi = \phi_{\mathcal{F}, \mathcal{G}} : \mathbb{H}^*(\text{Gr}, \mathcal{F}) \otimes \mathbb{H}^*(\text{Gr}, \mathcal{G}) \xrightarrow{\sim} \mathbb{H}^*(\text{Gr}, \mathcal{F} * \mathcal{G}), \quad (6.0.2)$$

making $\mathbb{H}^*$ into a monoidal functor $\text{Perv}_{L^+G}(\text{Gr}, k) \rightarrow \text{Vect}_k$, such that there is a commutative diagram

$$\begin{array}{ccc} \mathbb{H}^*(\text{Gr}, \mathcal{F}) \otimes^L \mathbb{H}^*(\text{Gr}, \mathcal{G}) & \xrightarrow{\phi} & \mathbb{H}^*(\text{Gr}, \mathcal{F} * \mathcal{G}) \\ \sigma \downarrow & & \downarrow \beta' \\ \mathbb{H}^*(\text{Gr}, \mathcal{G}) \otimes^L \mathbb{H}^*(\text{Gr}, \mathcal{F}) & \xrightarrow{\phi} & \mathbb{H}^*(\text{Gr}, \mathcal{F} * \mathcal{G}) \end{array}$$

To address the issue observed earlier, we modify the isomorphism $\beta' : \mathcal{F} * \mathcal{G} \rightarrow \mathcal{G} \rightarrow \mathcal{F}$ as follows. We write $\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-$ (likewise for $\mathcal{G}$), where $\mathcal{F}_+$ is supported on the even dimensional L^+G -orbits, and $\mathcal{F}_-$ on the odd dimensional ones. We declare $\beta : \mathcal{F}_\pm * \mathcal{G}_\pm \rightarrow \mathcal{G}_\pm * \mathcal{F}_\pm$ equal to β' except for $\mathcal{F}_- * \mathcal{G}_-$, in which case it is $-\beta$. Now, we let τ denote the usual isomorphism in the category of vector spaces, $V \otimes W \simeq W \otimes V$. Using β , we are able to rewrite the last corollary as using τ instead of σ , as follows.

$$\begin{array}{ccc} \mathbb{H}^*(\text{Gr}, \mathcal{F}) \otimes^L \mathbb{H}^*(\text{Gr}, \mathcal{G}) & \xrightarrow{\phi} & \mathbb{H}^*(\text{Gr}, \mathcal{F} * \mathcal{G}) \\ \tau \downarrow & & \downarrow \beta \\ \mathbb{H}^*(\text{Gr}, \mathcal{G}) \otimes^L \mathbb{H}^*(\text{Gr}, \mathcal{F}) & \xrightarrow{\phi} & \mathbb{H}^*(\text{Gr}, \mathcal{F} * \mathcal{G}) \end{array}$$

This proves the following theorem

THEOREM 6.3. $\mathbb{H}^*(\text{Gr}, -) : \text{Perv}_{L^+G}(\text{Gr}, k) \rightarrow \text{Vect}_k$ is a fibre functor (i.e. an exact, faithful, symmetric monoidal functor).

7. IDENTIFICATION OF THE DUAL GROUP FOLLOWING FARGUES-SCHOLZE [10] CHAPTER VI.10

By the theorem of Tannakian reconstruction, we obtain a group scheme $\tilde{G}$ over $\bar{\mathbb{Q}}_l$ such that we have an equivalence

$$S : (\text{Sat}_G, *) \simeq (\text{Rep}_{\tilde{G}}, \otimes).$$

Our aim is to prove the following theorem.

THEOREM 7.1. $\tilde{G}$ is a connected reductive group such that there exists an isomorphism $\tilde{G} \simeq G^\vee$.

First, we observe that the $\tilde{G}$ is of finite type. Indeed, let $\lambda_1, \dots, \lambda_n$ be any finite set of cocharacters which generate $X_*(T)_+$ as a monoid. Then any representation of $\tilde{G}$ can be found as a direct summand of a large tensor power of $S(IC_{\lambda_1}) \oplus \dots \oplus S(IC_{\lambda_n})$, which follows from the properties of Sat_G . Further, there is clearly no subcategory of Sat_G , generated by a nontrivial object X (i.e. contains subquotients of direct sums of X) which is stable under $*$. This shows that $\tilde{G}$ must in fact be connected. Next, the category Sat_G being semi-simple implies that $\tilde{G}$ must in fact be reductive.

The Theorem is evidently true in the case $G = T$ a torus. To proceed further, we will make use of the constant term functors that we will define below.

7.1. The constant term functor. Let $P \subset G$ be a parabolic subgroup and M its corresponding Levi subgroup. In this section, we will work with not only with the categories Sat_G and Sat_M but also with $\text{Sat}_{G,X}$ and $\text{Sat}_{M,X}$ where for simplicity we will assume that X is a smooth affine curve - we may assume it is in fact $\mathbb{A}^1$. Since Levi subgroups exist for parabolics Zariski locally, we may assume that there is also an inclusion $M \hookrightarrow P \subset G$. This allows to view L^+G -equivariant objects on Gr_G as L^+M -equivariant ones. We also define the correction function

$$\deg_P : \text{Gr}_M \rightarrow \pi_0(\text{Gr}_M) \simeq \pi_1(M) \rightarrow \mathbb{Z}$$

as the composite of the Kottwitz map followed by the pairing with $\langle 2\rho_G - 2\rho_M, - \rangle$.

Now, for an object of $\mathcal{F} \in D_c^b(\mathrm{Gr}_M, \bar{\mathbb{Q}}_l)$, we define $\mathcal{F}[\mathrm{deg}_p]$ as follows. If $\mathcal{F}$ is supported on a single component of Gr_M , say C , then $\mathcal{F}[\mathrm{deg}_p]$ is defined to be the shift of $\mathcal{F}$ by $\mathrm{deg}_p(C)$. For general $\mathcal{F}$, we extend this definition linearly.

Consider now the diagram

$$\begin{array}{ccc} & \mathrm{Gr}_P & \\ p \swarrow & & \searrow q \\ \mathrm{Gr}_M & & \mathrm{Gr}_G \end{array}$$

(Also, note that there is a similar diagram defined over X for the Beilinson-Drinfeld Grassmannians.)

Now we are ready to define

$$CT_P : D_{c, L^+G}^b(\mathrm{Gr}_G, \bar{\mathbb{Q}}_l) \rightarrow D_{c, L^+M}^b(\mathrm{Gr}_M, \bar{\mathbb{Q}}_l)$$

as $CT_P := p_! q^* [\mathrm{deg}_p]$.

An immediate property of this functor is the following:

LEMMA 7.2. *Suppose $P' \subset P$ and $Q = P' \cap M$. Then $CT'_P = CT_Q \circ CT_P$.*

The next important property is that CT_P is t-exact for the perverse t-structure. To see this, by Lemma 7.2 above, we reduce to the case of $P = B$, a Borel subgroup. In this case, the statement follows from 5.1, because Gr_T is discrete.

Next, we want to show that CT_P is compatible with convolution. To that end, we remark that CT_P may be defined in an entirely analogous manner over X^I , from $\mathrm{Sat}_{G, X^I} \rightarrow \mathrm{Sat}_{M, X^I}$.¹ Then, the compatibility of convolution with the constant term functor follows from Theorem 2.2.

7.2. Back to the identification of $G^\vee$. There is therefore the map $T^\vee \rightarrow \hat{G}$, commuting with fibre functors. We show further that this map is a closed embedding. This follows from the fact that every representation of $T^\vee$ appears as a direct summand of some representation of $\hat{G}$.

Moreover, this makes $T_{\mathbb{Q}_l}^\vee$ into a maximal torus of $\hat{G}$. We observe that this maximal torus is the stabiliser of the grading on the fibre functor F by cohomological degree. We may similarly define a Borel subgroup of $\hat{G}$ as the stabiliser of the same filtration.

7.3. The case $G = \mathrm{PGL}_2$. We assume now that $G = \mathrm{PGL}_2$. Consider the miniscale cocharacter $\mu : \mathbb{G}_m \rightarrow G$ giving rise to the miniscale Schubert cell of Gr_G which is isomorphic to $\mathbb{P}^1$. However, the l -adic cohomology of $\mathbb{P}^1$ implies that

$$F(IC_\mu) = H^0(\mathbb{P}^1) \oplus H^2(\mathbb{P}^1) = \bar{\mathbb{Q}}_l \oplus \bar{\mathbb{Q}}_l(-1). \quad (7.3.1)$$

By construction $F(IC_\mu)$ arises from a representation of $\hat{G} \rightarrow \mathrm{GL}(\bar{\mathbb{Q}}_l \oplus \bar{\mathbb{Q}}_l(-1))$. Next, it is clear that the maximal torus acts with weights ± 1 and thus lands in $\mathrm{SL}(\bar{\mathbb{Q}}_l \oplus \bar{\mathbb{Q}}_l(-1))$. The same is true for B which preserves the grading, and likewise for B^{opp} which preserves the opposite grading - this shows that the image of $\hat{G} \rightarrow \mathrm{GL}(\bar{\mathbb{Q}}_l \oplus \bar{\mathbb{Q}}_l(-1))$ lands inside SL .

LEMMA 7.3. *The induced map*

$$\hat{G} \rightarrow \mathrm{SL}(\bar{\mathbb{Q}}_l \oplus \bar{\mathbb{Q}}_l(-1)) \quad (7.3.2)$$

is an isomorphism.

The fact that 7.3.2 is an isomorphism over $\mathbb{Q}_l$ is clear, because it is a rank 1 reductive group surjecting onto SL_2 .

¹It is not entirely clear to me how to deduce the ULA property - hopefully this can be fixed soon.

7.4. Reduction to G of adjoint type. Let us denote by G_{ad} the adjoint group, i.e. G modulo its centre. For instance, when G is reductive of rank 1, this would be PGL_2 .

In this subsection we assume that Theorem 7.1 holds for G_{ad} and we deduce the same for G .

Of course, we have a map $\mathrm{Gr}_G \rightarrow \mathrm{Gr}_{G_{ad}}$, which is an isomorphism when restricted to connected components of the former. There are also the two Kottwitz maps, $\mathrm{Gr}_G \rightarrow \pi_1(G)$ and $\mathrm{Gr}_{G_{ad}} \rightarrow \pi_1(G_{ad})$ whose compatibility gives rise to a map $\mathrm{Gr}_{G_{ad}} \times_{\pi_1(G_{ad})} \pi_1(G)$. In terms of Tannakian group therefore, we have the following conclusions. Let us denote by Z the split group of multiplicative type whose character group is given by $\pi_1(G)$. Then we have

$$\hat{G} \simeq \widehat{G_{ad}} \times^{\pi_1(G_{ad})} Z \simeq G_{ad}^\vee \times^{\pi_1(G_{ad})} Z, \quad (7.4.1)$$

which is precisely the description of $G^\vee$; moreover the pinnings of $G_{ad}^\vee$ induce pinnings of $G^\vee$.

7.5. Arbitrary G of adjoint type. We now assume that G is an arbitrary semisimple group of adjoint type. Let a denote any simple coroot, and M_a its centraliser, the corresponding Levi subgroup. The constant term factor $CT : \mathrm{Sat}_G \rightarrow \mathrm{Sat}_{M_a}$ gives rise to a map $\hat{M}_a \rightarrow \hat{G}$. The dual torus $T^\vee$ is contained inside both $\hat{M}_a$ and $\hat{G}$, and it is such that the map $\hat{M}_a \rightarrow \hat{G}$ respects these inclusions. In fact, this turns a into a root of $\hat{G}$. Indeed, the root group inside $\hat{M}_a$ is not contained in any normal subgroup, whence the map $\hat{M}_a \rightarrow \hat{G}$ cannot kill the same. Thus we find that there is a root space in the Lie algebra of $\hat{G}$ such that $T^\vee$ acts via a . Further, considering once again the inclusions $T^\vee \rightarrow \hat{M}_a \rightarrow \hat{G}$, we find that the coroot $a^\vee : \mathbb{G}_m \rightarrow T^\vee$ gives rise to a coroot of $\hat{G}$.

As a consequence, we obtain that $\Phi(G^\vee, T^\vee) \subset \Phi(\hat{G}, T^\vee)$ and that $W \subset \hat{W}$, where we denote by $\hat{W}$ the Weyl group of $\hat{G}$.

Now, to show that there is an isomorphism of $\hat{G}_{\mathbb{Q}_l}$ with $G_{\mathbb{Q}_l}^\vee$, we must show that the inclusion of root systems above is in fact an equality. However, this is immediate upon studying the differences in weights of the nonzero weight spaces in the highest weight representations.

8. THE RAMIFIED GEOMETRIC SATAKE EQUIVALENCE FOLLOWING ZHU [25]: THE SETUP

Our next goal is to derive the "ramified geometric Satake equivalence" following Zhu [25]. The setting we work in is as follows. Let k be an algebraically closed field and $F = k((t))$. Let G be a reductive group over F , which splits over $\tilde{F}$, a tamely ramified extension of F . Now, by Steinberg's theorem, G is quasi-split. Further, we know that in this case the centraliser of A , is a maximal torus T . We fix a choice of $B \supset T$.

In what follows, we denote by H a split Chevalley group scheme with the same root data, which we denote by $(X, \Phi, X^\vee, \Phi^\vee)$, as that of the G over $\tilde{F}$. We fix an isomorphism $G \otimes \tilde{F} \simeq H \otimes \tilde{F}$, where the Galois action on the left corresponds to the following action on the right. Let Γ denote the Galois group (which is some cyclic group) of the extension $\tilde{F}/F$. Then, there is a map $\psi : \Gamma \rightarrow \mathrm{Out} H$ such that $\gamma \in \Gamma$ acts on $H \otimes \tilde{F}$ by the automorphism $\psi(\gamma) \otimes \gamma$.

We will frequently use the notation $X_*(T)$. This will denote the $X_*(T_{\tilde{F}})$ rather than the actual group of co-characters of T over F . Consider the natural composition $X_*(A) \rightarrow X_*(T) \rightarrow X_*(T)_\Gamma$. This induces an isomorphism $\mathbb{R} \otimes_{\mathbb{Z}} X_*(A) \simeq \mathbb{R} \otimes_{\mathbb{Z}} X_*(T)_\Gamma$.

We make the following important definition:

Definition 8.1. We set

$$X_*(T)_\Gamma^+ = \{\lambda \in X_*(T)_\Gamma \mid \langle a^{-1}(\lambda), \alpha \rangle \geq 0, \forall \alpha \in \Phi^+\}.$$

Remark 8.2. It is not true that an element of $X_*(T)_\Gamma^+$ always admits a lift to $X_*(T)^+$, as shown in Remark 2.7 of Achar-Lourenço-Richarz-Riche [3]. As explained therein, this seems to have been overlooked by Zhu, in our primary reference [25]. The remedy is to first work with the adjoint group, before deducing it for other groups. The claim is in fact true in the case of the adjoint group, thanks to the second part of Lemma 2.6 of Achar-Lourenço-Richarz-Riche [3], which we reproduce below in the form we will utilise.

LEMMA 8.3. (1) *The composite*

$$X_*(T)_\Gamma^+ \hookrightarrow X_*(T)_\Gamma \twoheadrightarrow X_*(T)_\Gamma / W_0,$$

where $W_0 = N_G(A)(F)/T(F)$ denotes the finite Weyl group, is a bijection.

(2) *The map*

$$X_*(T)^+ \hookrightarrow X_*(T) \twoheadrightarrow X_*(T)_\Gamma$$

factors to give a map

$$X_*(T)^+ \rightarrow X_*(T)_\Gamma^+,$$

which is surjective when G^{ad} , where $G^{ad} = G/Z$ with $Z = \text{centre of } G$.

Next, we want to define when one has $\lambda \leq \mu$ for elements $\lambda, \mu \in X_*(T)_\Gamma$. In this direction, we first have

LEMMA 8.4 (Lemma 2.5 of Achar-Lourenço-Richarz-Riche [3]). *Consider the natural inclusion $\mathbb{Z}\Phi^\vee \subset X_*(T)$. Upon taking covariants, we continue to have an inclusion $(\mathbb{Z}\Phi^\vee)_\Gamma \subset X_*(T)_\Gamma$.*

Next, we observe that $(\mathbb{Z}\Phi^\vee)_\Gamma$ has a basis consisting of Γ -orbits of simple coroots. Thus, we set $\lambda \leq \mu$ if $\mu - \lambda \in (\mathbb{Z}\Phi^\vee)_\Gamma$.

8.1. Some generalities on affine Grassmannians. In this section, we recall some generalities on affine Grassmannians and “partial affine flag varieties”. Some statements will be made in greater generality of arbitrary parahoric subgroups, though our interest is ultimately in the case of special ones.

Recall that there is the Kottwitz homomorphism, constructed by Kottwitz, recalled in Pappas-Rapoport [17]

$$\kappa : T(F) \rightarrow X_*(T)_\Gamma$$

We denote by $T(K)_1$ the kernel of the Kottwitz homomorphism. This kernel is nothing but the bounded subgroup of $T(F)$ which is the connected Néron model of T over $k[[t]]$. We denote by $\tilde{W}$ the Iwahori-Weyl group of G , given by $N(K)/T(K)_1$, and by W_0 the relative Weyl group over F , defined as $N(K)/T(K)$. As remarked in Pappas-Rapoport [17], there is thus the exact sequence

$$1 \rightarrow X_*(T)_\Gamma \rightarrow \tilde{W} \rightarrow W \rightarrow 1.$$

The choice of a special vertex gives rise to a splitting of the above sequence in the form $\tilde{W} = X_*(T)_\Gamma \rtimes W_0$.

For any subgroup $A \in \{G, S, T, N\}$, we write A_{sc} for the preimage of $A \cap G_{der}$ in the simply connected cover of G_{der} . The affine Weyl group W_{aff} is the Iwahori Weyl group of the simply connected cover, G_{sc} , of the derived group. By the first paragraph of page 152 of Pappas-Rapoport [17], this is in fact a Coxeter group. It is realised as so in the following manner: let C denote a chamber in the apartment of (G, S) associated to an Iwahori subgroup. Then W_{aff} is the group generated by reflections in the walls of the same. This realisations endows W_{aff} with a length function and a Bruhat order.

Definition 8.5. *Let P be a parahoric subgroup of G , corresponding to a facet in the apartment of the maximal split torus S . Then $\tilde{W}^P$ is defined as the subgroup of $\tilde{W}$ given by $(N(K) \cap P)/T(K)_1$.*

PROPOSITION 8.6 (Proposition 8.1 of Pappas-Rapoport [17]). *There is a natural bijection*

$$P \backslash G(K) / P \xrightarrow{\sim} \tilde{W}^P \backslash W / \tilde{W}^P.$$

Writing $G(K) = P \cdot N(K) \cdot P$, the map is given by

$$PnP \mapsto \bar{n} \in W^P \backslash W / \tilde{W}^P$$

This proposition provides the necessary context for the following definitions.

Definition 8.7. *For P as above, we denote by Fl_P the flag variety associated to P , i.e. the object representing the fpqc sheaf “ LG/P ”. Let $w \in \tilde{W}^P \backslash W / \tilde{W}^P$ with representative $n_w \in N$ we define:*

- The Schubert cell $Y(w)$ associated to w to be the reduced subscheme

$$P \cdot n_w \subset \mathrm{Fl}_P.$$

- The Schubert variety $X(w)$ associated to w is the reduced subscheme of the closure of $Y(w)$.

8.1.1. *The Demazure varieties.* We fix an alcove C in the apartment corresponding to S , and let I denote the corresponding Iwahori subgroup.

PROPOSITION 8.8. *For $w \in W_{\mathrm{aff}}$, let $w = s_{i_1} \dots s_{i_r}$ denote a reduced decomposition. Then $\widehat{X(w)}$ is defined as the contracted product*

$$P_{i_1} \times^I P_{i_2} \times^I \dots \times^I P_{i_r} / I,$$

where the action of I is given by

$$(b_1, \dots, b_r) \cdot (p_1, \dots, p_r) = (p_1 b_1^{-1}, b_1 p_2 b_2^{-1}, \dots, b_{r-1} p_r b_r^{-1}).$$

This is a smooth projective variety over k of dimension $l(w) = r$.

There is a map $\widehat{X(w)} \rightarrow X(w)$ given by

$$(p_1, \dots, p_r) \mapsto p_1 \dots p_r;$$

this map is moreover a desingularisation of $X(w)$.

Proof. Most of this is Pappas-Rapoport [17], Proposition 8.8. To see that this map is well defined, we use the axiom of Tits systems:

$$BsB \cup BwB \subset BswB,$$

when $w < sw$.

To see that this map is birational, we restrict to the open locus $Is_{i_1}I \times^I \dots \times^I Is_{i_r}I$, where this map is bijective. Q.E.D

It is now a consequence of Proposition 8.8 that when ν is a special facet as above,

COROLLARY 8.9. *The orbits of Fl_ν are parametrised by $X_*(T)_\Gamma^+$. The dimension of the orbit corresponding to $\bar{\mu} \in X_*(T)_\Gamma^+$ is $(2\rho, \bar{\mu})$, and the complement of the Schubert cell $Y(\bar{\mu})$ in $X(\bar{\mu})$ is the union of Schubert varieties $X(\bar{\mu}')$ for $\bar{\mu}' < \bar{\mu}$.*

(To make sense of $(2\rho, \bar{\mu})$ we observe that the choice of Γ -action, preserving a pinning, ensures that the map $X_*(T) \rightarrow \mathbb{Z}$ descends to a map $X_*(T)_\Gamma \rightarrow \mathbb{Z}$.)

8.2. **The Satake category.** Let P_ν denote the parahoric subgroup of LG associated to the special facet ν . For the right action of P_ν on the flag variety $\mathrm{Fl}_\nu = LG/P_\nu$ as above, we may consider the category of equivariant perverse sheaves $\mathrm{Sat}_\nu := \mathrm{Perv}_{P_\nu}(\mathrm{Fl}_\nu, \mathbb{Q}_l)$. The first step towards understanding this category is the following lemma.

LEMMA 8.10. *Sat_ν is semi-simple.*

Proof. Once one obtains that the fibres of the affine Demazure resolutions are paved by affine spaces, the proof is the same as the one in the classical case. That the fibres of the affine Demazure resolutions admit pavings by affine spaces is proved in section 3.1. Q.E.D

With these preliminaries in place, our goal is to prove Theorem 12.3 below. For the sequel, we will need to consider Grassmannians over $\mathbb{A}^1$ of the Beilinson-Drinfeld kind. For that, we will construct a group scheme $\mathcal{G}$ over $\mathbb{A}^1$ satisfying certain properties. This is the task of the next section. Also, some statements hold more generally in the case of arbitrary parahoric subgroups, rather than just the special ones. Wherever, possible we will try to make these statements in this generality before interpreting them in our setting of special parahorics.

9. CONSTRUCTION OF THE GROUP SCHEME $\mathcal{G}$ AND THE $\mathbb{G}_m$ -ACTION

We write, as in Zhu [26], $\tilde{C} \rightarrow C$ for the map $[e] : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ given by $u \mapsto u^e$. The respective origins are denoted $\tilde{0}$ and 0 , and the complements $\tilde{C}^0$ and C^0 . The Galois group of this covering, which is the cyclic group of order e is denoted Γ . Let (H, T_H, B_H, X) denote a split Chevalley group scheme, as above.

Suppose we are given some facet σ_Y of the apartment corresponding to the pair (G, A) . Let us denote by $\mathcal{G}_{\sigma_Y}$ the Bruhat-Tits group scheme whose valued points gives the parahoric group corresponding to the fact σ_Y . The goal is to construct on C a group scheme $\mathcal{G}$ which satisfies the following properties as in Zhu:

- (1) $\mathcal{G}_\eta$, where η is the generic point of C , splits over a tamely ramified extension.
- (2) If F_0 is the fraction field of the completion of the local ring at 0 , we have an isomorphism $\mathcal{G}|_{F_0} \simeq G$.
- (3) For $y \neq 0$, one has that $\mathcal{G}_{\hat{\mathcal{O}}_y} \simeq H \times \hat{\mathcal{O}}_y$.
- (4) $\mathcal{G}_{\hat{\mathcal{O}}_0} = \mathcal{G}_{\sigma_Y}$.

We let Γ act on $H \times \tilde{C}$ via diagram automorphisms on the first factor, and dilation on the latter. Next, Zhu defines

$$\mathcal{G}|_{C^0} := \text{Res}_{\tilde{C}^0/C^0}(H \times C)^\Gamma.$$

Then one may glue, using Bosch-Lütkebohmert-Raynaud [6] $\mathcal{G}$ with the integral model $\mathcal{G}_{\sigma_Y}$, as we find that over the fraction field F_0 of the strict henselization of 0 , $\mathcal{G}|_{C^0}$ and $\mathcal{G}_{\sigma_Y}$ are isomorphic.

Now we construct a subgroup-scheme $\mathcal{T}$ as the identity component of

$$\tilde{\mathcal{T}} := \text{Res}_{\tilde{C}/C}(T_H \times \tilde{C})^\Gamma.$$

It is not too hard to see that $\tilde{\mathcal{T}}$ embeds into $\mathcal{G}$. Indeed, as we shall see below, the group $\mathcal{G}$ itself may be realised as some $\text{Res}_{\tilde{C}/C} \mathcal{H}$, where $\mathcal{H}$ is an appropriately chosen integral model of H on $\tilde{C}$. On the other hand, one knows that T_H is a maximal torus of such an integral model $\mathcal{H}$ by the work Bruhat-Tits [8].

LEMMA 9.1. *There exists a parahoric group scheme $\tilde{G}$ over $\tilde{C}$, equivariant with respect to the action of Γ such that*

$$\mathcal{G} \simeq (\text{Res}_{\tilde{C}/C} \tilde{G})^\Gamma.$$

Proof. Consider the embedding of the apartment $A(G, S)$ into $A(H, T_H)$. For $x \in \sigma$, the facet that we started with, a parahoric subgroup $\mathcal{G}_x$ of $H \otimes \tilde{F}$ is determined. We show that the connected component of $(\text{Res}_{\tilde{C}/C} \mathcal{G}_x)^\Gamma$ is isomorphic to $\mathcal{G}$. First, the subgroup of invariants under the Γ -action is smooth by a result of Edixhoven. Next, the subgroup of valued points is given by $\mathcal{G}_x(\tilde{\mathcal{O}})^\Gamma$, which is precisely the stabiliser of x inside $G(F)$. This finishes the proof.

Q.E.D

9.1. The global Schubert variety. First, one would want to describe a global section of the the Beilinson-Drinfeld Grassmannian $\text{Gr}_{\mathcal{G}} \rightarrow C$, whose orbit closure will be the corresponding global Schubert variety. This will be possible upon pulling back to $\tilde{C}$. However, we first need to the following proposition:

PROPOSITION 9.2 (Zhu [25] Proposition 3.4). *For each $\mu \in X_*(\mathcal{T}_\eta)$ we have a section*

$$s_\mu : \tilde{C} \rightarrow L\mathcal{T} \times_C \tilde{C}.$$

satisfying the following property. For each $\tilde{y} \in \tilde{C}(k)$ with $y = [e](\tilde{y})$, we have that the image of $s_\mu(\tilde{y}) \in L\mathcal{T}_y(k) = \mathcal{T}_{\mathcal{F}_y}(F_y)$ under the Kottwitz map $\mathcal{T}_{\mathcal{F}_y}(F_y) \rightarrow X_(\mathcal{T}_\eta)_{\text{Gal}(F_y^s/F_y)}$ coincides with the projection of μ to $X_*(\mathcal{T}_\eta)_{\text{Gal}(F_y^s/F_y)}$.*

For the construction of the Kottwitz map, we refer to Pappas-Rapoport [17]. For a shortcut using a result of Richarz, we refer to Remark 9.3

Proof. We are required to exhibit an element of $\text{Hom}(\tilde{C}, L\mathcal{T})$ satisfying certain properties. We start by constructing an element of $\text{Hom}(\tilde{C}, L\tilde{\mathcal{T}})$ and then show that the element in fact factors through $L\mathcal{T}$. We unravel what it means to be an element of $\text{Hom}(\tilde{C}, L\tilde{\mathcal{T}})$. Indeed, writing Γ for the graph of $[e] : \tilde{C} \rightarrow C$, $\hat{\Gamma}$ for the completion along the same, and $\hat{\Gamma}^0 = \hat{\Gamma} - \Gamma$, one has

$$\text{Hom}_C(\tilde{C}, L\tilde{\mathcal{T}}) = \text{Hom}_C(\hat{\Gamma}^0, \tilde{\mathcal{T}}) = \text{Hom}_C(\hat{\Gamma}^0 \times_C \tilde{C}, T_H)^\Gamma.$$

As in Zhu, we write u for the local coordinate on u on $\tilde{C}$ and v for the same on C . In these coordinates, the map on coordinate rings is given by $v \mapsto u^e$. Then $C \times \tilde{C}$ is simply the affine space $\text{Spec } k[u, v]$, and the graph is cut out by the equation $v = u^e$. As a result, we find that

$$\hat{\Gamma}^0 = \text{Spec } k[u]((v - u^e)).$$

These considerations allow us to express the coordinate ring of $\tilde{C} \times_C \hat{\Gamma}^0$ as

$$k[u_1]((v - u_1^e)) \otimes_{k[v]} k[u_2],$$

where $k[u_2]$ is viewed as a $k[v]$ -algebra via the map $v \mapsto u_2^e$. Moreover, the Galois action is such that it multiplies the second factor by a primitive e -th root of unity.

First, we consider the situation where $T = \mathbb{G}_m$. Giving an element of $\text{Hom}_C(\hat{\Gamma}^0 \times_C \tilde{C}, T_H)^\Gamma$ is equivalent to giving an invertible element of $k[u_1]((v - u_1^e)) \otimes_{k[v]} k[u_2]$, invariant under the Galois action. As motivation, $v - u_1^e$ would provide such an element. In general, we observe that $v - u_1^e$ is the norm of the element

$$x = 1 \otimes u_2 - u_1 \otimes 1,$$

from the factorisation

$$(v - u_1^e) \otimes 1 = 1 \otimes v - u_1^e \otimes 1 = (1 \otimes u_2)^e - (u_1 \otimes 1)^e = \prod_{i=0}^{e-1} (\xi^i \otimes u_2 - u_1 \otimes 1).$$

In the general case therefore, we are motivated to define the $s_\mu : \hat{\Gamma}^0 \times_C \tilde{C} \rightarrow T_H$ as follows. Let $\omega_1, \dots, \omega_n$ denote a basis of $X^*(T_H)$, and γ a generator of Γ . Then s_μ is the section satisfying

$$\omega_j(s_\mu) = \prod_{i=0}^{e-1} (\gamma^i \cdot x)^{(\mu, \gamma^i \omega_j)}.$$

We want to check next that the map $\hat{\Gamma}^0 \rightarrow \tilde{\mathcal{T}}$ lands in the neutral component of the same. Outside the origin 0, this is clear. At the origin, one has that $u_2 = 0$. Consider the composition of the section with the adjunction map to T_H . At the origin, this gives rise to a map $\hat{\Gamma}^0|_0 \rightarrow T_H^\Gamma$. It suffices to show that this lands in the identity component of the latter. For this, upon evaluating the expression for s_μ at the origin, we find that it is simply the composite:

$$\hat{\Gamma}^0|_0 \xrightarrow{u_1} \mathbb{G}_m \xrightarrow{\sum \gamma^i \cdot \mu} T_H^\Gamma.$$

The latter map being a co-character lands in the identity component of T_H^Γ , whence we are done.

It remains to remark on the relation of s_μ and the Kottwitz map. First, let us consider the expression of $\omega_j(s_\mu)$ for $y \neq 0$. The total power of a uniformiser appearing in it is clearly (μ, ω_j) , whence the claim is clear.

On the other hand, at the origin, we review the construction of an element $t_\mu \in T(k((t)))$ whose image under the Kottwitz map is μ upon projecting to $X_*(T)_\Gamma$. We first go to an extension $k((s))$ where the torus T splits. Let $t_\mu \in T(k((t)))$ be an element which is the norm of the element $\tilde{\mu}(s) \in T_{k((s))}$. Now, it is clear by construction that s_μ at $y = 0$ is also the norm of an element which is of the form $\tilde{\mu}(c.s)$, for some $c \in k$. This finishes the proof.

Q.E.D

Viewing this as a section to $\widetilde{\text{Gr}}_{\mathcal{G}}$, we may consider the closure of the $\widetilde{L^+G}$ -orbit therein. We *define* this to be the *global Schubert variety*. Let us denote for now the global Schubert variety corresponding to $\mu \in X_*(T)^+$ by M_μ . First and foremost, we observe that it is flat, being the schematic closure over a DVR. In particular, it has equi-dimensional fibres.

Remark 9.3. In the above construction, we have used a local coordinate to define the section s_μ . Zhu in [26] shows that it is not too hard to see that the image of the section in $\widetilde{\text{Gr}}_{\mathcal{G}}$ does not change. Alternatively, Richarz in [20] has shown that $\widetilde{\text{Gr}}_{\mathcal{G}}$ is ind-proper. This shows that any generic section - this would be the one giving corresponding to " t_μ " - extends to all of $\tilde{C}$, and does so uniquely.

The case where $\sigma_Y = \nu$, the special facet is of course, of special interest. In this case, the last statement of Proposition 9.2 implies that the image of the section s_μ lands in $\text{Fl}_{v, \bar{\mu}}$, where $\bar{\mu}$ denotes the image of μ in $X_*(T)_\Gamma^+$.

In this situation we in fact have the following description of the closed fibre over the global Schubert variety, which in general has a more complicated description due to Richarz [20] and [21].

LEMMA 9.4. *For $\sigma_Y = \nu$, the special vertex, we have that $M_\mu|_0$ (i.e. the fibre of the global Schubert variety M_μ over the origin) is isomorphic to the Schubert variety $\text{Fl}_{v, \leq \bar{\mu}}$ in Fl_ν .*

Proof. In this case, we have seen that the Satake category Sat_ν is semisimple. Applying the compatibility of nearby cycles with proper pushforward to the map $\widetilde{\text{Gr}}_{\mathcal{G}} \rightarrow \tilde{C}$, we find that

$$\mathbb{H}^d(\text{Gr}_{H, \leq \mu}, IC_\mu) = \mathbb{H}^d(\text{Fl}_\nu, Z(IC_\mu)),$$

where $d = \dim \text{Gr}_{H, \leq \mu} = \dim \text{Fl}_{v, \leq \bar{\mu}} = \langle 2\rho, \mu \rangle$. Since the former is one-dimensional, so is the latter. Now writing $Z(IC_\mu)$ as a direct sum of IC-sheaves as we may, by the semi-simplicity of Sat_ν , we find that $Z(IC_\mu)$ has only a single summand of the form $IC_{\mu'}$ with $d = \langle 2\rho, \mu' \rangle = \dim \text{Fl}_{v, \leq \mu'}$. Now, Lemma 9.5 below shows that $\text{Fl}_{v, \leq \bar{\mu}}$ already lies in the fibre $M_\mu|_0$, so it must be the only maximal dimensional irreducible component. Now flatness shows that M_μ is equi-dimensional. This ensures that there are no lower dimensional irreducible components, and we are done. Q.E.D

LEMMA 9.5. *For $\mu \in X_*(T)^+$ as above, if j denotes the inclusion of $\text{Fl}_{v, \leq \bar{\mu}}$ in Fl_ν , then one has $j^* Z(IC_\mu) = \bar{Q}_I[\langle 2\rho, \mu \rangle]$.*

Proof. By compatibility of nearby cycles with smooth pullback, we may restrict to the smooth open part of the global Schubert variety, which is the $\widetilde{L^+G}^{BD}$ -orbit of the section s_μ . Here, the statement is clear. Q.E.D

9.2. Construction of the $\mathbb{G}_m$ -action on $\overline{\text{Gr}}_{\mathcal{G}, \mu}$.

LEMMA 9.6. *There is a $\mathbb{G}_m$ -action on $\mathcal{G} \rightarrow C$ compatible with the dilation on C .*

Proof. We recall that we have constructed a group scheme $\tilde{G}$ on $\tilde{C}$ satisfying the following properties:

- $\tilde{G}|_{\tilde{C}^0} = H \times \tilde{C}^0$.
- $\tilde{\mathcal{G}}|_{\tilde{C}_0}$ is a parahoric group scheme for $H \otimes \tilde{F}_0$, corresponding to $x \in A(H, T_H)^\Gamma$.
- $\mathcal{G}$ is the identity component of $(\text{Res}_{\tilde{C}/C} \tilde{G})^\Gamma$.

Thus, it suffices to show that there is a $\mathbb{G}_m$ -action on $\tilde{G}$, compatible with dilation on $\tilde{C}$ as well as the action of Γ . To begin with, there is a natural action of $\mathbb{G}_m$ on $H \times \tilde{C}^0$ acting by scaling on the second factor. We show that this action extends uniquely extends to all of $\tilde{C}$. Uniqueness takes care of the Γ action.

Let $m : \mathbb{G}_m \times \tilde{C} \rightarrow \tilde{C}$ be the action map, and p the projection. We wish to show that there is an isomorphism of group schemes $p^* \tilde{G} \simeq m^* \tilde{G}$. The partial action described above gives us

$$c : m^* \tilde{G}|_{\mathbb{G}_m \times \tilde{C}^0} \simeq p^* \tilde{G}|_{\mathbb{G}_m \times \tilde{C}^0}.$$

Uniqueness of the extension of c follows from flatness of $\tilde{G}$ and the fact that it is constrained to coincide with the above over $\tilde{C}^0$.

As remarked by Zhu, we need to prove that the induced map on coordinate rings

$$c^* : \mathcal{O}_{m^*\tilde{G}}[u^{-1}] \rightarrow \mathcal{O}_{p^*\tilde{G}}[u^{-1}]$$

takes $\mathcal{O}_{m^*\tilde{G}}$ to $\mathcal{O}_{p^*\tilde{G}}$. We check this isomorphism at every closed point of $\mathbb{G}_m \times \{0\}$. To that end let $r \in \mathbb{G}_m(k)$. We show that there is an extension to r of the isomorphism on C^0 , i.e. we construct an isomorphism $r^*\tilde{G} \simeq G$. To do this, we use the property of Bruhat-Tits 1.7.6 to show that the map induced by $\tilde{G}(\tilde{F}_0) \rightarrow \tilde{G}(\tilde{F}_0)$ takes $\tilde{G}(\tilde{\mathcal{O}}_x)$ to $\tilde{G}(\tilde{\mathcal{O}}_x)$. Consider the root subgroups of $L\tilde{G}$ constructed in Pappas-Rapoport [17] and Faltings [9]. By their very definition, it is clear that they are left invariant under the scaling action $t \mapsto r.t$ (where t is the local variable for the disc around 0). Therefore, the $\mathbb{G}_m$ -action does not change the parahoric subgroup of H we are dealing with and this finishes the proof.

Q.E.D

Now, we work with the pullback $\widetilde{\text{Gr}}_{\mathcal{G}}$ of $\text{Gr}_{\mathcal{G}}$ to $\tilde{C} \rightarrow C$. Equipped with the $\mathbb{G}_m$ -action on $\mathcal{G}$, we are able to describe the sought $\mathbb{G}_m$ -action on $\widetilde{\text{Gr}}_{\mathcal{G}}$

10. THE SUBGROUP OF FIXED POINTS $(H^\vee)^I$

The theorem we prove in this section is a consequence of Haines [13], but our proof follows that of Richarz [20].

We place ourselves in the following more abstract setting: let I be a finite group whose order is coprime to that of the characteristic of k , our base field. Let H be a connected reductive group over k endowed with a pinning (H, B, T, X) . The group I is allowed to act on H via automorphisms which fix this pinning. Thus, in our case, $I = \Gamma$ and H is really $H^\vee$.

THEOREM 10.1 (Haines[13], Richarz [20], Prasad-Yu [18], Steinberg [22]). *In the above setting, the group of fixed points under the action, H^I , is a (possibly disconnected) reductive group. Moreover, we have:*

- (1) $(H^{I,0}, B^{I,0}, T^{I,0}, X)$ is a pinning;
- (2) There is a bijection between the connected components of T^I and H^I .

Proof. (1) First, it follows from Prasad-Yu [18] that $H^{I,0}$ is a reductive group. Next, we show that $B^{I,0}$ is a Borel subgroup of $H^{I,0}$. Indeed, consider first the quotient H^I/B^I . It is easy to see that there is an embedding

$$H^I/B^I \hookrightarrow (H/B)^I,$$

whence H^I/B^I is projective. On the other hand, $H^{I,0}/B^{I,0} \rightarrow H^I/B^I$ is at most finite onto its image, whence $H^{I,0}/B^{I,0}$ is proper, and it follows that $B^{I,0}$ is a Borel subgroup (since it is clearly connected and solvable).

Now let U denote the unipotent radical of B . We show below that U^I is connected - let us assume this statement for the moment. Writing $B = T \rtimes U$, one finds that $B^I = T^I \rtimes U^I$. The aforementioned connectedness of U^I ensures that $B^{I,0} = T^{I,0} \rtimes U^I$. We conclude from this that $T^{I,0}$ is a maximal torus of $H^{I,0}$.

We return to the matter of the connectedness of U^I . Factoring the root system of H as a product of irreducible root systems, we reduce to the case where this root system of H is itself irreducible. Since we have assumed that the action of I preserves a pinning, it also preserves the corresponding choice of base of the associated root datum. First, let us analyse the orbits of the action of the group I . We assume that it is a subgroup of the group of diagram automorphisms of H . The latter group is one of $\mathbb{Z}/2, \mathbb{Z}/3$ or S_3 . In the last case, which is precisely the case of D_4 , the orbits of the action are the same as that of the action of the $\mathbb{Z}/3$ subgroup of S_3 . Thus, we assume I is cyclic and generated by σ . Following Steinberg [22], denote by S_α for a positive

root α , the set of positive roots whose projection into $\mathfrak{g}^I$ are proportional to that of α . Studying each case separately, we find that the only possibilities are the following:²

- $S_\alpha = \text{orbit of } \alpha$. Moreover, no two elements of S_α sum to another root. This implies that U_β, U_γ commute for $\beta, \gamma \in S_\alpha$.
- $S_\alpha = \{\alpha, \sigma\alpha, \beta = \alpha + \sigma\alpha\}$. This case map hold only in type A_{2n} .

Once again, analysing each case shows that the set $\{\alpha, \sigma\alpha\}$ is part of an irreducible root subsystem fixed by σ .

Recall that there is an isomorphism of schemes

$$\prod U_\alpha \rightarrow U,$$

where the positive roots α may be taken in any order, a priori. However, we choose the following order. Consider the collection of sets (taken without multiplicity) $\{S_\alpha\}_{\alpha \text{ is a positive root}}$. Let us order the sets $S_1, \dots, S_k$ arbitrarily. Within each S_i , we order the roots arbitrarily, except that when $S_i = \{\alpha, \sigma\alpha, \beta = \alpha + \sigma\alpha\}$, we ensure that β appears last. This decomposition shows the following. Let U_i^I be the subscheme of U^I consisting of elements of the form $\prod x_\beta$, where $\beta \in S_i$ and $x_\beta \in U_\beta$, and ordered as above. Then $U^I = \prod U_i^I$.

The final claim is that each U_i^I is connected. This argument is once again as in Steinberg [22]. Suppose S_i is the orbit of a single root α . Writing $x \in U_i$ as

$$x = x_\alpha(k_\alpha) x_{\sigma\alpha}(k_{\sigma\alpha}) \cdots,$$

and using the fact that each $U_{\sigma^i\alpha}$ and $U_{\sigma^j\alpha}$ commute, we find that $x \in U_i^I$ if and only if each $k_{\sigma^i\alpha}$ are equal. Thus, $U_i^I \simeq \mathbb{G}_a$ via $k \mapsto \prod_{\beta \in S_i} x_\beta(k)$. In the latter case, following Steinberg, we adjust the homomorphism so that we have the commutator relation

$$[x_\alpha(k), x_{\sigma\alpha}(l)] = x_\beta(kl).$$

Once again in this case we find that $U_i^I \simeq \mathbb{G}_a$ given by $k \mapsto x_\alpha(0) x_{\sigma\alpha}(0) x_\beta(k)$. This finishes the proof.

- (2) For this proof, we follow Richarz [20]. Let U_- denote the unipotent radical of the Borel subgroup B_- opposite to B , and $B_- = T \rtimes U_-$. It suffices to show

$$U_-^I \times T^I \times U^I \rightarrow H^I \tag{10.0.1}$$

is an open dense immersion. Open-ness is clear. For the rest, we have that we have the natural map $N^I \rightarrow W^I$ is a surjection, where W is the Weyl group of H and N is the normaliser of T in H .

Now, for each element $w \in W$, we pick a lift $n_w \in N$. In case $w \in W^I$, by the foregoing, we pick $n_w \in N^I$. Let $U_w = U \cap n_w^{-1} U_- n_w$. The Bruhat decomposition allows us to write each element of $x \in H$ uniquely as $un_w b$, for some $w \in W$, $u \in U_w$ and $b \in B$. Moreover, if $x \in H^I$, then in fact each of n_w, u, b are I -invariant by the above. In other words,

$$H^I = \coprod_{w \in W^I} U_w^I n_w B^I.$$

Let w_0 denote the longest element in W . It is clearly I -invariant. Thus, firstly $U_{w_0} = U$ and must therefore be the largest among all the double cosets, in terms of dimension, and likewise for the I -invariants $U_{w_0}^I$. In particular this means $U_{w_0}^I n_{w_0} B^I$ is dense in H^I . However, $U_{w_0}^I n_{w_0} B^I$

²As far as I can tell, making the following distinction on the basis of "proportional projections to $\mathfrak{g}^I$ " is perhaps not the right perspective, at least for this part of the proof. Rather, it seems to me to be the following:

- Orbits of roots are such that no two members of the same orbit add to another root.
- The other case, which happens only in the A_{2n} as described above.

is isomorphic to $U_-^I \times T^I \times U^I$ whence they have the same number of connected components. The former being dense and the latter open, we conclude the density of the latter as well.

Q.E.D

Let Q_+ denote the positive elements of the root lattice inside $X^*(T)$. We will describe the representations of H^I in terms of $X^*(T)_I$, so we denote by $(Q_I)_+$ the projection to Q_+ to the same. We write for weights $\mu, \lambda \in X^*(T)_I, \mu \leq \lambda$, when $\lambda - \mu \in (Q_I)_+$.

LEMMA 10.2. *Suppose H and I are as above. Assume further that our base field k is algebraically closed. Then, the irreducible representations of H^I are given precisely by the highest weight representations $\lambda \in X^*(T^I)_+$, where the multiplicity of the highest weight is 1.*

Proof. We denote by H_0 the connected component of H^I . By the foregoing, this is a connected reductive group. Thus by Jantzen [15], its irreducible representations are indexed by the highest weights $\bar{\mu} \in X^*(T_I^0)_+$. Let us call these representations $W_{\bar{\mu}}$. Our first claim is that if there exists any irreducible representation V_{μ} of H^I of highest weight μ , it must appear as a direct summand of $\text{Ind}_{H_0}^{H^I} W_{\bar{\mu}}$. Indeed, consider the restriction of V_{μ} to H_0 . Clearly it has a direct summand of the form $W_{\bar{\mu}}$. Now, by Frobenius reciprocity, we obtain

$$\text{Hom}_{H^I}(V_{\mu}, \text{Ind}_{H_0}^{H^I} W_{\bar{\mu}}) = \text{Hom}_{H_0}(\text{Res } V_{\mu}, W_{\bar{\mu}}) \neq 0,$$

whence the claim follows.

Our next claim is that there is a representation of H^I whose restriction to H_0 is $W_{\bar{\mu}}$. Consider once again

$$\text{Ind}_{H_0}^{H^I} W_{\bar{\mu}} = \bigoplus_{\sigma \in H^I/H_0} \sigma \cdot W_{\bar{\mu}}.$$

The diagonal subset $\{(\sigma \cdot v)_{\sigma \in H^I/H_0} | v \in W_{\bar{\mu}}\}$ clearly fits the bill. We call this H^I -representation W .

Now, any T^I/T_0 representation may be viewed as an H^I representation via the sequence of maps $H^I \rightarrow H^I/H_0 \simeq T^I/T_0$. The representations of T^I/T_0 , on the other hand are precisely its characters, being an abelian group, and these characters are $|T^I/T_0|$ in number. Let χ be one such character, viewed as an H^I -representation. We claim that

$$\text{Ind}_{H_0}^{H^I} W_{\bar{\mu}} = \bigoplus_{\chi \in X^*(T^I/T_0)} \chi \otimes W.$$

Indeed, Frobenius reciprocity, as we have already applied once above, shows that each term on the right is indeed a direct summand at least. Dimension considerations show that these are in fact all. Further, among the representations $\chi \otimes W$, it is also clear that precisely one of them realises the T^I -weight μ . Thus, combining all the observations above, the lemma is proved.

Q.E.D

11. THE CENTRAL FUNCTOR Z

11.1. Useful notation. In most of this section, we follow Achar-Riche [2]. Moreover, their notation is rather convenient and we adopt the same. In the present subsection, we provide the definitions.

We define the set of symbols as the set $\mathcal{S} := \{\emptyset, \underline{0}, \underline{y}, \underline{y} \cup \underline{0}\}$. We will use y to denote an R -point of C , for k -algebras R . Γ_y . Likewise, Γ_0 is used to denote the graph $\{0\} \times \text{Spec } R$. Completions along these closed subsets are denoted using a “hat”, for instance, the completion along Γ_y is written $\hat{\Gamma}_y$. We denote by Γ_y^0 the complement $\hat{\Gamma}_y - \Gamma_y$, and so for 0 . For a sequence $\underline{S}$ of symbols $S_1, \dots, S_n \in \mathcal{S}$, we make the following definition

Definition 11.1. $\text{Gr}_{\mathcal{G}}(S_1, \dots, S_n)$ is defined as the functor which assign to a k -algebra R , the set (up to isomorphism classes) of data $(y, \mathcal{E}_1, \dots, \mathcal{E}_n, \beta_1, \dots, \beta_n)$, where

- $y \in C(R)$;

- $\mathcal{E}_i$ are $\mathcal{G}_R$ bundles on $\widehat{\Gamma_y \cup \Gamma_0}$;
- β_i are isomorphisms $\mathcal{E}_{i-1} \xrightarrow{\cong} \mathcal{E}_i$ on $\hat{\Gamma}_{S_i}^0$, where

$$\hat{\Gamma}_{S_i}^0 = \begin{cases} \widehat{\Gamma_y \cup \Gamma_0} - \Gamma_0, & \text{if } S_i = \underline{0} \\ \widehat{\Gamma_y \cup \Gamma_0} - \Gamma_y, & \text{if } S_i = \underline{y} \\ \widehat{\Gamma_y \cup \Gamma_0} - \Gamma_0 \cup \Gamma_y, & \text{if } S_i = \underline{y} \cup \underline{0} \\ \widehat{\Gamma_y \cup \Gamma_0}, & \text{if } S_i = \emptyset \end{cases}.$$

Simply via composition, we obtain maps

$$\mathrm{Gr}_{\mathcal{G}}(S_1, \dots, S_n) \rightarrow \mathrm{Gr}_{\mathcal{G}}(S_1, \dots, S_i \cup S_{i+1}, \dots, S_n),$$

where $S_i \cup S_j$ is defined in the obvious way.

11.2. The functor Z . We start by considering the group scheme $\mathcal{G}$ that has been constructed over the curve C . At the origin 0 , we have that the restriction of $\mathcal{G}$ to $\mathrm{Spec} \hat{\mathcal{O}}_0$ is the Bruhat-Tits group scheme associated to the facet σ_Y say. We denote by $P \subset G(K)$ the associated parahoric subgroup.

In what follows, we consider, for a sequence $S = (S_1, \dots, S_n)$ of sets of symbols $\underline{0}, \underline{y}$, the Grassmannians $\mathrm{Gr}_{\mathcal{G}}(S_1, \dots, S_n)$. Its fibre over the origin 0 is given by a certain $LG \times^P \dots \times^P LG \times^P \mathrm{Fl}_P$, where the number factors is governed by the sequence S and Fl_P denotes the flag variety “ $G(K)/P$ ”. As observed by Zhu [26], it is more convenient to work with the pullback of $\mathrm{Gr}_{\mathcal{G}}(S)$ to $\tilde{C}$, which we denote by $\widetilde{\mathrm{Gr}}_{\mathcal{G}}(S)$. Outside the origin, since the covering $\tilde{C} \rightarrow C$ is étale, we have that $\widetilde{\mathrm{Gr}}_{\mathcal{G}}(S)$ is simply $\mathrm{Gr}_H \times C^0$ - the Galois action is such that Γ acts on the first factor via the map $\Gamma \rightarrow \Xi$ into the diagram automorphisms of H , while on the latter it is the obvious one. At $\tilde{0}$, the fibre is the same as over 0 , namely the twisted product $LG \times^P \dots \times^P LG \times^P \mathrm{Fl}_P$.

As in Achar-Riche, we may consider the nearby cycles functor

$$\Psi_S : D_{L^+ \mathcal{G}^{BD}}^b(\mathrm{Gr}_{\mathcal{G}}(S)|_{C^0}, \tilde{Q}_I) \rightarrow D_P^b(LG \times^P \dots \times^P LG \times^P \mathrm{Fl}_P, \tilde{Q}_I).$$

Consideration on the interaction of nearby cycles with monodromy are made in the following section.

We recall that we have defined, for $i \leq j$ the proper maps $\mu_{i,j} : \mathrm{Gr}_{\mathcal{G}}(S_1, \dots, S_n) \rightarrow \mathrm{Gr}_{\mathcal{G}}(S_1, \dots, S_i \cup S_{i+1} \cup \dots \cup S_j, S_{j+1}, \dots, S_n)$. For us, the key point is the following incarnation of the compatibility of Ψ with proper pushforward:

$$(\mu_{i,j})_{0,*} \circ \Psi_{(S_1, \dots, S_n)} \simeq \Psi_{(S_1, \dots, S_i \cup \dots \cup S_j, \dots, S_n)} \circ \mu_{i,j}|_{C^0,*}.$$

Similarly, the compatibility of Ψ with smooth pullback is utilised in the following lemma, to which we will appeal several times.

LEMMA 11.2 (Achar-Riche [2] Lemma 2.4.2). *Let S' and S'' be sequences as above and $S = S' \cup S''$. Suppose we have $\mathcal{F} \in D_{L^+ \mathcal{G}^{BD}|_{C^0}}^b(\widetilde{\mathrm{Gr}}_{\mathcal{G}}(S'), \tilde{Q}_I)$ and $\mathcal{G} \in D_{L^+ \mathcal{G}^{BD}|_{C^0}}^b(\widetilde{\mathrm{Gr}}_{\mathcal{G}}(S''), \tilde{Q}_I)$. Then, setting $S = S' \cup S''$ we have an isomorphism of functors*

$$\Psi_S(\mathcal{F} \boxtimes \mathcal{G}) \simeq \Psi_{S'}(\mathcal{F}) \boxtimes \Psi_{S''}(\mathcal{G}).$$

Proof. The argument that follows is not entirely rigorous, but after presenting the same we comment on how to tweak the notation to recover a proof.

One knows that construction of nearby cycles commutes with formation of external tensor product. This gives us an isomorphism

$$\Psi_{\widetilde{\mathrm{Gr}}_{\mathcal{G}}(S') \times_C \widetilde{\mathrm{Gr}}_{\mathcal{G}}(S'')}(\mathcal{F} \boxtimes^L \mathcal{G}) \simeq \Psi_{\widetilde{\mathrm{Gr}}_{\mathcal{G}}(S')}(\mathcal{F}) \boxtimes^L \Psi_{\widetilde{\mathrm{Gr}}_{\mathcal{G}}(S'')}(\mathcal{G}).$$

However, one has a $L\mathcal{G}^{BD}$ -principal bundle

$$\mathrm{Gr}_{\mathcal{G}}(S') \times_C \mathrm{Gr}_{\mathcal{G}}(S'') \rightarrow \mathrm{Gr}_{\mathcal{G}}(S') \times_C^{L\mathcal{G}^{BD}} \mathrm{Gr}_{\mathcal{G}}(S'') \simeq \mathrm{Gr}_{\mathcal{G}}(S),$$

and an analogous one upon pulling back to $\tilde{C}$. Thus the result follows from the compatibility of nearby cycles with smooth pullback.

The aforementioned compatibilities are indeed satisfied when we honest finite type schemes. To that end, let m be such that the support of $\mathcal{G}$ is contained in $\widetilde{\text{Gr}_{\mathcal{G}}|_{C^0}}$ is contained in $(\widetilde{\text{Gr}_{\mathcal{G}}})_m|_{C^0}$. Let j_m be an integer such that the action of $L^+\mathcal{G}^{BD}$ factors through $L_{j_m}^+\mathcal{G}^{BD}$. Recall we have also constructed an $L_{j_m}^+\mathcal{G}^{BD}$ principal bundle $\text{Gr}_{\mathcal{G}}^{j_m}(S') \rightarrow \text{Gr}_{\mathcal{G}}(S')$. We denote by $\text{Gr}_{\mathcal{G}}^{j_m}(S')_m$ the pre-image of $\text{Gr}_{\mathcal{G}}(S')_m$. We replace $\mathcal{F}$ by its pullback to $\text{Gr}_{\mathcal{G}}^{j_m}(S')_m$, $\text{Gr}_{\mathcal{G}}(S')$ with $\text{Gr}_{\mathcal{G}}^{j_m}(S')_m$, $\text{Gr}_{\mathcal{G}}(S'')$ with $\text{Gr}_{\mathcal{G}}(S'')_m$, and $L^+\mathcal{G}^{BD}$ with $L_{j_m}^+\mathcal{G}^{BD}$. Q.E.D

Combining the above, one obtains

LEMMA 11.3. *Let S_1 and S_2 be single sets of symbols. Then, for $\mathcal{F} \in D_{L^+\mathcal{G}^{BD}}^b(\widetilde{\text{Gr}_{\mathcal{G}}}(S_1))$ and $\mathcal{G} \in D_{L^+\mathcal{G}^{BD}}^b(\text{Gr}_{\mathcal{G}}(S_2))$, then there is a natural isomorphism*

$$\Psi_{S_1 \cup S_2}(\mathcal{F} *_{C^0} \mathcal{G}) \simeq \Psi_{S_1}(\mathcal{F}) *^P \Psi_{S_2}(\mathcal{G})$$

in $D_P^b(\text{Gr}_{\mathcal{G}}(S_1 \cup S_2)_0)$.

Remark 11.4. Similar statements hold when $\mathcal{F} \in D_c^b(\text{Gr}_{\mathcal{G}}(S_1))$ removing equivariance in the final derived category appearing above.

We are now in a position to define the important functor Z :

Definition 11.5. *We define*

$$Z : D_{L^+H}^b(\text{Gr}_H, \bar{\mathbb{Q}}_l) \rightarrow D_P^b(\text{Fl}_P, \bar{\mathbb{Q}}_l)$$

given by the assignment

$$\mathcal{A} \mapsto \Psi_{\underline{y}}(\mathcal{A} \boxtimes^L \mathbb{Q}_{C^0}[1]).$$

Since the nearby cycles functor Ψ is t-exact (as is smooth pullback upon twisting) for the perverse t-structure, we may interpret the functor as one of $\text{Perv}_{L^+H}(\text{Gr}_H, \bar{\mathbb{Q}}_l) \rightarrow \text{Perv}_P(\text{Fl}_P, \bar{\mathbb{Q}}_l)$.

12. PROOF OF THE RAMIFIED SATAKE

First, we will show that Z is monoidal with respect to convolution. In other words, we will show that the following holds

$$Z(\mathcal{A} *^{L^+H} \mathcal{B}) \simeq Z(\mathcal{A}) *^P Z(\mathcal{B}). \quad (12.0.1)$$

Next, we will show that Z is a central functor in the sense of Bezrukavnikov. We recall that a monoidal functor $F : (\mathcal{C}, \otimes) \rightarrow (\mathcal{D}, \otimes)$ is said to be central if the following properties are satisfied

- (1) There is a functorial isomorphism $F(X) \otimes Y \simeq Y \otimes F(X)$, for $X \in \mathcal{C}, Y \in \mathcal{D}$;
- (2) **Property 1.** For $X \in \mathcal{C}$ and $Y, Y' \in \mathcal{D}$, the following diagram commutes:

$$\begin{array}{ccc} F(X) \otimes Y \otimes Y' & \longrightarrow & Y \otimes F(X) \otimes Y' \\ & \searrow & \downarrow \\ & & Y \otimes Y' \otimes F(X) \end{array}$$

where each arrow is the one arising in the previous point.

(3) **Property 2.** For $X, X' \in \mathcal{C}$ and $Y \in \mathcal{D}$, we have a commutative diagram of the form:

$$\begin{array}{ccccc}
 F(X) \otimes F(X') \otimes Y & \xrightarrow{\quad\quad\quad} & F(X) \otimes Y \otimes F(X') & \xrightarrow{\quad\quad\quad} & Y \otimes F(X) \otimes F(X') \\
 & \searrow & & & \swarrow \\
 & F(X \otimes X') \otimes Y & \xrightarrow{\quad\quad\quad} & Y \otimes F(X \otimes X') &
 \end{array}$$

(4) **Property 3.** For $X, X' \in \mathcal{C}$, there is the following commutative diagram:

$$\begin{array}{ccc}
 F(X) \otimes F(X') & \longrightarrow & F(X') \otimes F(X) \\
 \downarrow & & \downarrow \\
 F(X \otimes X') & \longrightarrow & F(X' \otimes X)
 \end{array}$$

These axioms are checked for Z in the section 14.

Next, once one has that $Z : \text{Sat}_H \rightarrow \text{Sat}_v$ is a monoidal functor, recalling from classical geometric Satake that the first category Sat_H is equivalent to $\text{Rep}_{H^\vee}$, an application of Proposition 1 of Bezrukavnikov [5] (the following statement is almost a restatement in our situation) gives us:

PROPOSITION 12.1. *First, there is an algebraic group $\tilde{G} \subset H^\vee$ such that there is equivalence of categories*

$$S : \text{Sat}_v \rightarrow \text{Rep}_{\tilde{G}}$$

for which $S \circ Z$ is isomorphic to the restriction functor $\text{Res}_{\tilde{G}}^{H^\vee}$.

We are in fact able to make this more precise. Consider the action of the Galois group Γ on $\widetilde{\text{Gr}}_G|_{\mathbb{G}_m}$ (here and elsewhere whenever it is clear from context, $\mathbb{G}_m$ will be used to denote the complement of the origin in $\mathbb{A}^1$).

Before proceeding further, let us recall that there is a Galois action on $H^\vee$, via pinned automorphisms, as explained in Section 7. For each $\gamma \in \Gamma$ the action of γ on Gr_H induces an isomorphism $\omega^\gamma : \text{Sat}_H \rightarrow \text{Sat}_H$. Taken together, this induces an action of Γ on Sat_H by tensor automorphisms. This in turn induces an action of Γ on $H^\vee$, preserving the pinning from 7, as explained therein.

On the other hand, we have the following result, Lemma 4.5 from Zhu [25]

LEMMA 12.2. *For $f : H_2 \rightarrow H_1$ a homomorphism of algebraic groups and ω^f the induced tensor functor from $\text{Rep}_{H_1} \rightarrow \text{Rep}_{H_2}$. Suppose $I \subset \text{Aut}(H_1)$ acts on $\text{Rep}(H_1)$ via tensor automorphisms - for $\gamma \in I$, we write ω^γ for the corresponding tensor automorphism. Moreover, suppose $\omega^f \circ \omega^\gamma \simeq \omega^f$, then f factors through $H_1^I \subset H_1$.*

We apply this in our setting to the inclusion $\tilde{G} \subset H^\vee$, and the action of Γ , to obtain that $\tilde{G} \subset (H^\vee)^\Gamma$. Next, semisimplicity of the Satake category shows that $\tilde{G}$ must have identity component reductive. Thus to show that equality $\tilde{G} = (H^\vee)^\Gamma$ holds, it suffices to show that restriction induces an isomorphism on representation rings. Indeed, for any proper inclusion of algebraic groups, we may find a representation of the larger one where the smaller is realised as the stabiliser of a line, by a theorem of Chevalley. Taking the subspace generated by this line, we may assume, when the larger group is reductive, that the representation is irreducible, whence we see that restriction cannot induce an isomorphism on representation rings.

We will actually take this approach to proving the equality $\tilde{G} = (H^\vee)^\Gamma$ only when $G = G^{ad}$, where $G^{ad} = G/Z(G)$, as usual. The general case will be deduced from this by general considerations.

First, we observe that irreducible representations of both groups are indexed by $X_*(T)_\Gamma^+$. Let $[V_\mu]$ and $[W_\mu]$ denote the corresponding objects in the Grothendieck groups of $\tilde{G}$ and $(H^\vee)^\Gamma$ respectively.

We will show inductively that

$$\text{Res}_{\tilde{G}}^{(H^\vee)^\Gamma} [W_\mu] = [V_\mu] + \sum_{\mu' < \mu} c_{\mu\mu'} [V_{\mu'}]. \quad (12.0.2)$$

For this, let us denote by $\tilde{\mu}$ a lift of μ to $X_*(T)_\Gamma^+$, which is possible because we are working in the case $G = G^{ad}$. Also, we denote by U_μ (rather than $U_{\tilde{\mu}}$, to lighten notation) the irreducible representation of $H^\vee$ of highest weight $\tilde{\mu}$. We will study the restriction of U_μ to each of $\tilde{G}$ and $(H^\vee)^\Gamma$. Studying the $T^\vee$ -weight spaces of U_μ , it is clear that the latter restriction is of the form $[W_\mu] + \sum_{\mu' < \mu} d_{\mu\mu'} [W_{\mu'}]$. For the moment, let us grant that the former restriction, i.e. to $\tilde{G}$, is of the form $[V_\mu] + \sum_{\mu' < \mu} e_{\mu\mu'} [V_{\mu'}]$. As restriction to $\tilde{G}$ from $H^\vee$ is the composition of first restricting to $(H^\vee)^\Gamma$, followed by further restriction to $\tilde{G}$, we are able to derive an equation for $\text{Res}_{\tilde{G}}^{(H^\vee)^\Gamma} [W_\mu]$. By the induction hypothesis for $\mu' < \mu$ (and that the corresponding analogues of 12.0.2 hold), we are able to deduce that 12.0.2 holds for μ too.

It remains to remark how 12.0.2 proves that Res induces an isomorphism on Grothendieck rings of $(H^\vee)^\Gamma$ and $\tilde{G}$. Indeed, the equation 12.0.2 shows that the matrix giving the map on the respective Grothendieck rings is “strictly upper triangular” with respect to the bases both indexed by $X_*(T)_\Gamma^+$. It is easy to see that these matrices are invertible whence we are done.

We now prove the fact taken for granted above, namely $\text{Res}_{\tilde{G}}^{H^\vee} [U_\mu] = [V_\mu] + \sum_{\mu' < \mu} [V_{\mu'}]$. For this we resort to the geometric setting. Under the geometric Satake equivalence for the split group H , the irreducible representation U_μ for $H^\vee$ corresponds to $IC_{\tilde{\mu}}$. By Lemma 9.5 we find that in $Z(IC_{\tilde{\mu}})$ the irreducible object IC_μ of Sat_v appears with multiplicity 1. Moreover, since the closed fibre, by 9.4, is irreducible and equal to the Schubert variety corresponding to the weight $\mu \in X_*(T)_\Gamma^+$, we find that the other summands appearing are of the form $IC_{\mu'}$ with $\mu' < \mu$. Here we also use the compatibility of nearby cycles with proper pushforward, to first compute the nearby cycle on global Schubert variety corresponding to $\tilde{\mu}$ before pushing forward to the entirety of Fl_v .

Finally, we prove the statement for general G . Our approach is modelled on that of Achar-Lourenço-Richarz-Riche [3]. Let us denote by Sat'_v the Satake category for G^{ad} with respect to the image of v in the corresponding Bruhat-Tits building. The Kottwitz homomorphism $\text{Gr}_{G^{ad}} \rightarrow \pi_1(G^{ad})_\Gamma$ endows Sat'_v with a $\pi_1(G^{ad})_\Gamma$ -grading. Thus, we have a map from $D(\pi_1(G^{ad})_\Gamma)$ (the diagonal group with group of characters $\pi_1(G^{ad})_\Gamma$) to $\tilde{G}^{ad} = (H^{ad})^\vee_\Gamma$. On the other hand, one has an isomorphism $\text{Gr}_G \simeq \text{Gr}_{G^{ad}} \times_{\pi_1(G^{ad})_\Gamma} \pi_1(G^{ad})_\Gamma$. Thus, objects of Sat_v are objects of Sat'_v with the $D(\pi_1(G^{ad})_\Gamma)$ -grading extended to a $D(\pi_1(G)_\Gamma)$ -grading. This realises the Tannakian group $\tilde{G}$ as the twisted product

$$D(\pi_1(G)_\Gamma) \times^{D(\pi_1(G^{ad})_\Gamma)} (H^{ad})^\vee_\Gamma.$$

Identifying $Z((H^\vee)^\Gamma) = \pi_1(G)_\Gamma$ (and likewise for the adjoint group) we find that the above group is isomorphic to $(H^\vee)^\Gamma$ due to Proposition 6.8 of Achar-Lourenço-Richarz-Riche [4].³

THEOREM 12.3. *The category Sat_v has the natural structure of a tensor category. Further, there is an equivalence of category*

$$S : (\text{Sat}_v, *^{P_v}) \simeq (\text{Rep}_{(H^\vee)^\Gamma}, \otimes),$$

such that the following diagram commutes.

$$\begin{array}{ccc} (\text{Sat}_H, *^{L^+H}) & \xrightarrow{Z} & (\text{Sat}_v, *^{P_v}) \\ \downarrow & & \downarrow S \\ (\text{Rep}_H, \otimes) & \xrightarrow{\text{Res}} & (\text{Rep}_{(H^\vee)^\Gamma}, \otimes) \end{array}$$

(The arrow on the left is the one arising from the geometric Satake equivalence for split groups already dealt with.)

³The proof of this is perhaps not too difficult, but I have not worked out the details yet.

13. MONODROMIC SHEAVES AND TRIVIALITY OF THE MONODROMY OPERATOR

For the setup of monodromic sheaves, we follow Achar-Riche [2] and Verdier [23]. Let $\mathbb{G}_m$ act on a scheme Y . For each $y \in Y$, let us denote by t_y the orbit map $\mathbb{G}_m \rightarrow Y$ given by $s \mapsto s.y$. Now, we say that an honest constructible sheaf $\mathcal{F}$ on Y is *monodromic* if its pullback via t_y , for each $y \in Y$ is a tame local system on $\mathbb{G}_m$. Further, $\mathcal{F} \in D_c^b(X, k)$ is said to be monodromic, if each cohomology sheaf $\mathcal{H}^i(\mathcal{F})$ is monodromic, for every i .

Verdier in [23] constructs a canonical map

$$\mu_{\mathcal{F}} : \pi_1^{\text{tame}}(\mathbb{G}_m) \rightarrow \text{Aut } \mathcal{F}.$$

The situation in which we will use this is the following. Suppose $f : X \rightarrow \mathbb{A}^1$ is a scheme over $\mathbb{A}^1$ endowed with an action of $\mathbb{G}_m$ which makes f equivariant for this action (where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by scaling). Let $X^\times$ denote $f^{-1}(\mathbb{A}^1 - \{0\})$.

PROPOSITION 13.1. *Suppose we have $\mathcal{F} \in \text{Perv}_{\mathbb{G}_m}(X^\times, k)$. Then the perverse sheaf $\Psi_f(\mathcal{F})$ on X_0 is monodromic.*

Further, the action of the Galois group $\Gamma = \text{Gal}(\bar{K}/K)$ factors through one of $\pi_1^{\text{tame}}(\mathbb{G}_m)$. This latter action is given by $\alpha \mapsto \mu_{\Psi_f(\mathcal{F})}(\alpha^{-1})$.

PROPOSITION 13.2.

These statements carry over to the setting where we deal with ind-schemes, but sheaves with finite dimensional support.

Indeed, let us recall our setting. We have constructed a group scheme $\mathcal{G} \rightarrow \mathbb{A}^1$, and we wish to consider the affine Grassmannian associated to it $\text{Gr}_{\mathcal{G}} \rightarrow \mathbb{A}^1$. Let $\widetilde{\text{Gr}}_{\mathcal{G}}$ denote the pullback of $\text{Gr}_{\mathcal{G}}$ to $\mathbb{A}^1$ via the map $e : \mathbb{A}^1 \rightarrow \mathbb{A}^1$. There is a natural action of $\mathbb{G}_m$ on $\widetilde{\text{Gr}}_{\mathcal{G}}$ such that the structure map $\widetilde{\text{Gr}}_{\mathcal{G}} \rightarrow \mathbb{A}^1$ is equivariant with respect to the scaling action on $\mathbb{A}^1$. Pulling back via e generically splits $\mathcal{G}$, so that we have an isomorphism $\widetilde{\text{Gr}}_{\mathcal{G}} \simeq \text{Gr}_H \times (\mathbb{A}^1 - \{0\})$. On the other hand, restricting to the origin, we have that $\widetilde{\text{Gr}}_{\mathcal{G}}|_0 \simeq \text{Fl}_v$. Moreover, the restriction of the $\mathbb{G}_m$ action to Fl_v coincides with the action of the rotation torus.

PROPOSITION 13.3. *The monodromy of $Z(\mathcal{F})$ is zero.*

Proof. We have that $\mathcal{F} \boxtimes \bar{\mathbb{Q}}_l[1]$ is $\mathbb{G}_m$ equivariant, whence $Z(\mathcal{F}) = \Psi(\mathcal{F} \boxtimes \bar{\mathbb{Q}}_l[1])$ is monodromic. However, it is even $\mathbb{G}_m$ -equivariant. Indeed, by semisimplicity of Sat_v , one has that $Z(\mathcal{F})$ is the direct sum of certain IC sheaves on Fl_v . These IC sheaves are equivariant with respect to the action of the rotational torus. This action, as remarked above is the restriction of the global $\mathbb{G}_m$ action to the fibre over the origin. As a result, we conclude that the monodromic perverse sheaf $Z(\mathcal{F})$ is indeed equivariant. However, this precisely means that the monodromy of is trivial by 13.2. Q.E.D

PROPOSITION 13.4. *Suppose i denotes the inclusion of the closed fibre $\widetilde{\text{Gr}}_{\mathcal{G}}|_0 \rightarrow \widetilde{\text{Gr}}_{\mathcal{G}}$ and j the complementary open embedding. We have then a canonical isomorphism*

$$Z(\mathcal{F}) \simeq i^* j_{!*}(\mathcal{F} \boxtimes \bar{\mathbb{Q}}_l).$$

Proof. We have the exact sequence

$$i^* j_{*}(\mathcal{F} \boxtimes \bar{\mathbb{Q}}_l[1]) \rightarrow \Psi(\mathcal{F} \boxtimes \bar{\mathbb{Q}}_l[1]) \xrightarrow{1-\text{monodromy}} \Psi(\mathcal{F} \boxtimes \bar{\mathbb{Q}}_l[1]) \rightarrow .$$

The second map above vanishes by the foregoing. Taking perverse cohomology of the above, we find that the corresponding terms for $\mathcal{F} \boxtimes \bar{\mathbb{Q}}_l$ vanish outside degrees 0 and 1, and in these two places, they are both isomorphic to $Z(\mathcal{F})$. However, ${}^p\mathcal{H}^0(i^* j_{*}(\mathcal{F} \boxtimes \bar{\mathbb{Q}}_l[1])) \simeq i^* j_{!*}(\mathcal{F} \boxtimes \bar{\mathbb{Q}}_l)$, whence we are done. Q.E.D

14. BRAIDING COMPATIBILITIES

14.1. The associativity constraint. As before, we consider the diagram

$$\begin{array}{ccc}
 & \mathrm{Fl}_P \times \mathrm{Fl}_P \times \mathrm{Fl}_P & \\
 \swarrow & & \searrow \\
 \mathrm{Fl}_P \times \mathrm{Fl}_P \times {}^P\mathrm{Fl}_P & & \mathrm{Fl}_P \times {}^P\mathrm{Fl}_P \times \mathrm{Fl}_P \\
 \searrow & & \swarrow \\
 & \mathrm{Fl}_P \times {}^P\mathrm{Fl}_P \times {}^P\mathrm{Fl}_P &
 \end{array}$$

where the maps in question are surjective smooth maps (LP -bundles in fact). Thus, the fact that pulling back $(\mathcal{A} * \mathcal{B}) * \mathcal{C}$ and $\mathcal{A} * (\mathcal{B} * \mathcal{C})$ to $\mathrm{Fl}_P \times \mathrm{Fl}_P \times \mathrm{Fl}_P$ gives the identity $\mathcal{A} \boxtimes \mathcal{B} \boxtimes \mathcal{C}$ translates to the fact that the two sheaves were canonically isomorphic to begin with.

In what follows, as in Achar-Riche [2], we use α to denote the commutativity constraint:

$$\alpha : \mathcal{A} * (\mathcal{B} * \mathcal{C}) \xrightarrow{\sim} (\mathcal{A} * \mathcal{B}) * \mathcal{C}.$$

Next, recall that we have an isomorphism $\widetilde{\mathrm{Gr}}_{\mathcal{G}}|_{C^0} \simeq \mathrm{Gr}_H \times C^0$. This lets us define

$$\tau_H : D_{L^+H}^b(\mathrm{Gr}_H) \rightarrow D_{L^+\mathcal{G}^{BD}}^b(\widetilde{\mathrm{Gr}}_{\mathcal{G}}|_{C^0}),$$

given by

$$\mathcal{A} \mapsto \mathcal{A} \boxtimes {}^L\tilde{Q}_I[1].$$

Thus, $Z(\mathcal{A}) = \Psi(\tau_H(\mathcal{A}))$.

14.2. Commutativity and a compatibility. We define

$$C : D_{L^+H}^b(\mathrm{Gr}_H) \rightarrow D_P^b(\mathrm{Fl}_P)$$

as the functor

$$(\mathcal{A}, \mathcal{B}) \mapsto \Psi_{\underline{y} \cup \underline{0}}(\mathcal{A} \boxtimes {}^L\mathcal{B} \boxtimes {}^L\tilde{Q}_I).$$

The functor has obvious partially equivariant counterparts, denoted by the same letter C .

PROPOSITION 14.1. *Suppose $\mathcal{A} \in D_{L^+H}^b(\mathrm{Gr}_H)$ and $\mathcal{B} \in D_P^b(\mathrm{Fl}_P)$. Then there are canonical isomorphisms:*

$$Z(\mathcal{A}) *^P \mathcal{B} \simeq C(\mathcal{A}, \mathcal{B}) \simeq \mathcal{B} *^P Z(\mathcal{A}).$$

Proof. Over C^0 , we have the diagram of canonical isomorphisms:

$$\begin{array}{ccc}
 & \mathrm{Gr}_H \times \mathrm{Fl}_P \times C^0 & \\
 \swarrow \text{dashed} & \downarrow \text{dashed} & \searrow \text{dashed} \\
 \widetilde{\mathrm{Gr}}_{\mathcal{G}}(\underline{y}, \underline{0})|_{C^0} & & \mathrm{Gr}_{\mathcal{G}}(\underline{0}, \underline{y})|_{C^0} \\
 \searrow \text{solid} & \downarrow \text{solid} & \swarrow \text{solid} \\
 & \widetilde{\mathrm{Gr}}_{\mathcal{G}}(\underline{y} \cup \underline{0})|_{C^0} &
 \end{array}$$

where the solid arrows represent maps which are restrictions of analogous ones defined over the entirety of C .

Now, compatibility of convolution with proper pushforward applied to the left-hand side of the diagram gives rise to the isomorphism

$$C(\mathcal{A}, \mathcal{B}) \simeq \Psi(\mathcal{A} \boxtimes {}^L\tilde{Q}_I) *^P \Psi(\mathcal{B}) = Z(\mathcal{A}) *^P \mathcal{B},$$

and likewise for the right-hand side of the same. This finishes the proof.

Q.E.D

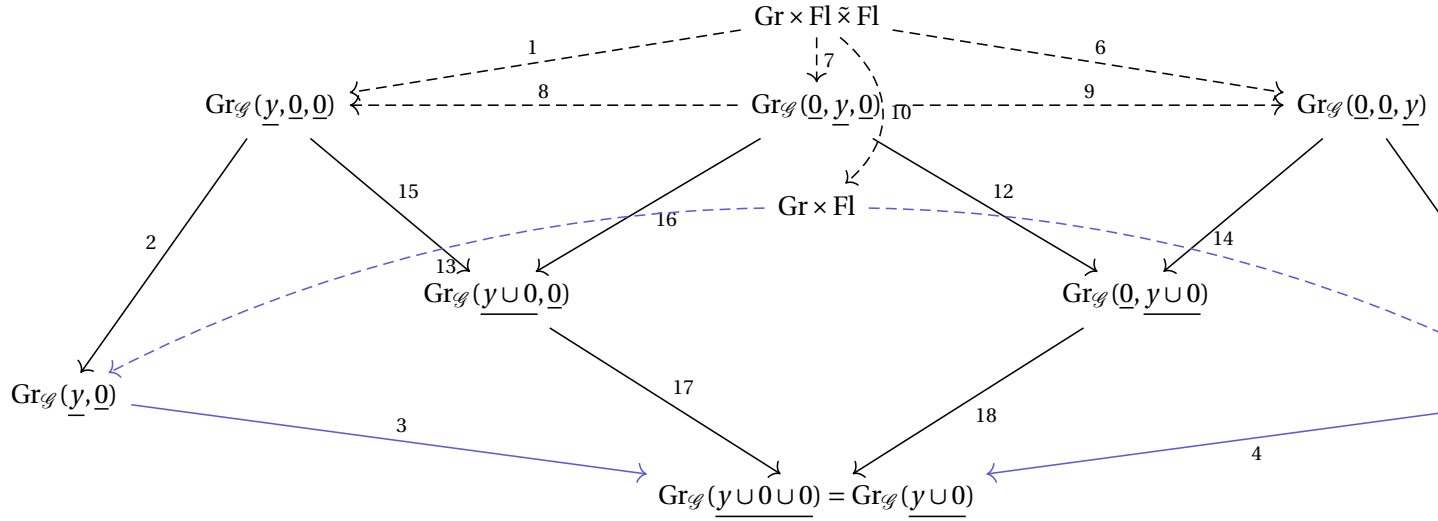
THEOREM 14.2 (Achar-Riche [2] Theorem 3.2.3). For $\mathcal{A} \in D_{L^+H}^b(\mathrm{Gr}_H)$ and $\mathcal{B} \in D_P^b(\mathrm{Fl}_P)$ as before, there is a natural isomorphism

$$\sigma_{\mathcal{A},\mathcal{B}}: Z(\mathcal{A}) *^P \mathcal{B} \xrightarrow{\sim} \mathcal{B} *^P Z(\mathcal{A})$$

such that for $\mathcal{A} \in D_{L^+H}^b(\mathrm{Gr}_H)$ and $\mathcal{B}_1, \mathcal{B}_2 \in D_P^b(\mathrm{Fl}_P)$ the following diagram commutes

$$\begin{array}{ccc} Z(\mathcal{S}) * \mathcal{T}_1 * \mathcal{T}_2 & \xrightarrow{\sigma_{\mathcal{S},\mathcal{T}_1} * \mathrm{Id}} & \mathcal{T}_1 * Z(\mathcal{S}) * \mathcal{T}_2 \\ & \searrow \sigma_{\mathcal{S},\mathcal{T}_1 * \mathcal{T}_2} & \downarrow \mathrm{Id} * \sigma_{\mathcal{S},\mathcal{T}_2} \\ & & \mathcal{T}_1 * \mathcal{T}_2 * Z(\mathcal{S}) \end{array}$$

Proof. We have the following commutative diagram:



In this diagram, the dashed arrows represent maps which exist only generically. Moreover, the blue arrows are precisely the ones which come up in the Proposition 14.1. We start with the sheaf $\mathcal{S} \boxtimes \mathcal{T}_1 \boxtimes \mathcal{T}_2$. We pushforward along each of these arrows and then apply the nearby cycles functor on them at various. We know that there are natural isomorphisms arising from the compatibility of the nearby cycle functor with proper pushforward.

For instance, applying the nearby cycles functor to $\mathcal{S} \boxtimes \mathcal{T}_1 \boxtimes \mathcal{T}_2$ on $\widetilde{\mathrm{Gr}}_g(\underline{y}, \underline{0}, \underline{0})$ gives $Z(\mathcal{S}) \boxtimes \mathcal{T}_1 \boxtimes \mathcal{T}_2$, while the same on $\widetilde{\mathrm{Gr}}_g(\underline{0}, \underline{0}, \underline{y})$ yields $\mathcal{T}_1 \boxtimes \mathcal{T}_2 \boxtimes Z(\mathcal{S})$. For both of these statements, we use Lemma 11.2. Pushing forward the former object along arrows 15 and 17 above, and the latter along 14 and 18, we obtain an isomorphism between $Z(\mathcal{S}) *^P \mathcal{T}_1 *^P \mathcal{T}_2$ and $\mathcal{T}_1 *^P \mathcal{T}_2 *^P Z(\mathcal{S})$. However, this isomorphism arises out of the blue arrows, which is precisely the diagram used in Proposition 14.1 to construct the isomorphism $\sigma_{\mathcal{S},\mathcal{T}_1 * \mathcal{T}_2}$.

On the other hand, there is another way to factor this isomorphism, which is by using $\mathrm{Gr}_g(\underline{0}, \underline{y}, \underline{0})$. Arrows 15 and 16 provide the isomorphism, after pushforward along 17,

$$\sigma_{\mathcal{S},\mathcal{T}_1} * \mathrm{Id}.$$

Similarly, arrows 12 and 14, followed by pushforward along 18 gives

$$\mathrm{Id} * \sigma_{\mathcal{S},\mathcal{T}_2}.$$

In summary, we have obtained the commutativity of the diagram 14.2.

Q.E.D

COROLLARY 14.3 (t-exactness of convolution). *For $\mathcal{A} \in \text{Perv}_{L^+H}(\text{Gr}_H), \mathcal{B} \in \text{Perv}_P(\text{Fl}_P)$, we have that $Z(\mathcal{A}) * \mathcal{B}$ is also perverse.*

Proof. As $Z(\mathcal{A}) * \mathcal{B}$ is described by taking nearby cycles of the sheaf $\mathcal{A} \boxtimes \mathcal{B} \boxtimes \tilde{Q}_I[1]$, which is itself perverse, the corollary follows from the t-exactness of the nearby cycle functor. Q.E.D

14.3. Property 2.

THEOREM 14.4. *For $\mathcal{A} \in D_{L^+H}^b(\text{Gr}_H), \mathcal{B} \in D_{L^+H}^b(\text{Gr}_H)$, there is a natural isomorphism in $D_P^b(\text{Fl}_P)$ as follows*

$$\phi_{\mathcal{A}, \mathcal{B}} : Z(\mathcal{A}) *^P Z(\mathcal{B}) \xrightarrow{\cong} Z(\mathcal{A} *^{L^+H} \mathcal{B}).$$

Moreover, one has a commutative diagram, for $\mathcal{A}_1, \mathcal{A}_2 \in D_{L^+H}^b(\text{Gr}_H)$ and $\mathcal{B} \in D_P^b(\text{Fl}_P)$:

$$\begin{array}{ccc} Z(\mathcal{A}_1) *^P Z(\mathcal{A}_2) *^P \mathcal{F} & \longrightarrow & Z(\mathcal{A}_1 *^{L^+H} \mathcal{A}_2) *^P \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{F} *^P Z(\mathcal{A}_1 *^{L^+H} \mathcal{A}_2) & & \\ \downarrow & & \\ Z(\mathcal{A}_1) *^P \mathcal{F} *^P Z(\mathcal{A}_2) & \longrightarrow & \mathcal{F} *^P Z(\mathcal{A}_1) *^P Z(\mathcal{A}_2) \end{array}$$

where the arrows are the natural ones we have discussed above.

Proof. This is similar to the previous theorem.

We have a commutative diagram as in Achar-Riche [2]

$$\begin{array}{ccccc} & & \text{Gr}_H \times \text{Gr}_H \times \tilde{C}^0 & & \\ & \swarrow v & \downarrow \eta & \searrow \mu & \\ \text{Gr}_{\mathcal{G}}(\underline{y}, \underline{y})|_{\tilde{C}^0} & & & & \text{Gr}_H \times \tilde{C}^0 \\ & \searrow \mu & \downarrow & \swarrow v & \\ & & \text{Gr}_{\mathcal{G}}(\underline{y})|_{\tilde{C}^0} & & \end{array}$$

Let us start with the complex $\mathcal{A} \boxtimes \mathcal{B} \boxtimes \tilde{Q}_I[1]$ on $\text{Gr}_H \times \text{Gr}_H$. Pushing forward along the left side of the diagram gives us first $\tau(\mathcal{A}) \boxtimes_{\tilde{C}^0} \tau(\mathcal{B})$. Pushing forward further along the map labelled μ gives us $\tau(\mathcal{A}) *_{\tilde{C}^0} \tau(\mathcal{B})$.

A similar procedure along the right half of the diagram yields an isomorphism of the form

$$\tau(\mathcal{A}) *_{\tilde{C}^0} \tau(\mathcal{B}) \simeq \eta_*(\mathcal{A} \boxtimes \mathcal{B} \boxtimes \tilde{Q}_I) \simeq \tau(\mathcal{A} *^{L^+H} \mathcal{B}).$$

Applying the nearby cycles functor and Lemma 11.3, we obtain therefore, an isomorphism

$$\phi_{\mathcal{A}, \mathcal{B}} : Z(\mathcal{A}) *^P Z(\mathcal{B}) \xrightarrow{\cong} Z(\mathcal{A} *^{L^+H} \mathcal{B}).$$

The commutativity of the diagram in the statement of the theorem is very similar to the one in the previous theorem. So for now we omit it. Q.E.D

14.4. Nearby cycles over 2-dimensional bases, as in Achar-Riche [2]. In this section, we recall the definitions, and properties of nearby cycles over 2-dimensional bases in an ad-hoc fashion, as used in Achar-Riche [2]. We refer to loc. cit. for most proofs.

We start with $f : X \rightarrow A^2$ a separated map. We denote by $X^{\times \times}$ and X^{00} the preimages of $\mathbb{G}_m \times \mathbb{G}_m$ and $(0, 0)$ respectively. Let $\mathbf{i}$ and $\mathbf{j}$ denote the inclusions $X^{00} \rightarrow X$ and $X^{\times \times} \rightarrow X$ respectively.

Definition 14.5 ([2] Definition 9.4.1). Let $\mathcal{F}$ be a perverse sheaf on $X^{\times \times}$. We say $\Upsilon_f(\mathcal{F})$ is well-defined if the following hold:

- (1) Let $F_{a,b,n} := {}^p \mathcal{H}^n \mathbf{i}_* (\mathcal{F} \otimes^L (f^{\times \times})^* (\mathcal{L}_a \boxtimes \mathcal{L}_b))$. Then whenever we have the map $F_{a,b,-2} \rightarrow F_{a',b',-2}$, for $a \leq a'$ and $b \leq b'$, it is an isomorphism.
- (2) Given a, b , the maps $F_{a,b,n} \rightarrow F_{a,b,n}$ for $n \neq -2$ vanish for all large enough a' and b' .

In that case, we define $\Upsilon_f(\mathcal{F}) = F_{a,b,-2}$, unambiguously.

Remark 14.6. The second condition above need only be checked for $n = -1, 0$ for it is not too hard to see that $F_{a,b,n}$ vanishes for $n \neq -2, -1, 0$, [2] Lemma 9.4.2.

First, we wish to study the behaviour of Υ with respect to proper pushforward and smooth pullback. The statement for the former is postponed for the moment, see below.

LEMMA 14.7. Suppose we have $g : Y \rightarrow X$ smooth of relative dimension d and $f : X \rightarrow \mathbb{A}^2$. Suppose $\mathcal{F}$ is a perverse sheaf on $X^{\times \times}$ for which $\Upsilon(\mathcal{F})$ is well defined. Then the natural morphism $g^* \Upsilon_f(\mathcal{F})[d] \rightarrow \Upsilon_{f \circ g}(g^* \mathcal{F}[d])$ is an isomorphism.

Our aim is to compare this with "iterated" nearby cycles and "diagonal" ones. To that end, we introduce the following notation, with the help of the following diagrams. First, consider the natural inclusions:

$$\begin{array}{ccccc}
 \mathbb{G}_m \times \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \times \mathbb{G}_m & \longleftarrow & \{0\} \times \mathbb{G}_m \\
 \downarrow & & \downarrow & \searrow & \downarrow \\
 \mathbb{G}_m \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^2 - \{(0,0)\} & & \{0\} \times \mathbb{A}^1 \\
 \uparrow & \searrow & \downarrow & \swarrow & \uparrow \\
 \mathbb{G}_m \times \{0\} & \longrightarrow & \mathbb{A}^1 \times \{0\} & \longrightarrow & \{0,0\} \\
 & & \searrow & \swarrow & \\
 & & \mathbb{A}^2 & &
 \end{array}$$

Restricting X over each of these terms we obtain the following.

$$\begin{array}{ccccc}
 X^{\times \times} & \xrightarrow{j_1} & X^{* \times} & \xleftarrow{i_1} & X^{0 \times} \\
 j_2 \downarrow & & j_2' \downarrow & \searrow j_2'' & \downarrow j_2^0 \\
 X^{\times *} & \xrightarrow{j_1'} & X^0 & & X^{0*} \\
 \uparrow i_2 & \searrow j_1'' & \downarrow j_1' & \swarrow h & \uparrow i_2' \\
 X^{\times 0} & \xrightarrow{j_1^0} & X^{*0} & \xrightarrow{i_2'} & X^{00} \\
 & & \searrow i_2'' & \swarrow i_1' & \\
 & & X & &
 \end{array}$$

Let $f_1 := pr_1 \circ f|_{X^{* \times}} : X^{* \times} \rightarrow \mathbb{A}^1$ and $f_2 := pr_2 \circ f|_{X^{0*}} : X^{0*} \rightarrow \mathbb{A}^1$. We define the iterated nearby cycle functor as

$$\Psi_f^2(\mathcal{F}) := \Psi_{f_2}^{un} \circ \Psi_{f_1}^{un}(\mathcal{F}).$$

In the cases of interest, it will turn out that $\Psi_{f_i}^{un}(\mathcal{F}) = \Psi_{f_i}(\mathcal{F})$ are in fact unipotent, so in what follows, we will drop the superscript "un" and assume we are working only with Ψ^{un} . Indeed, the monodromy will in fact be trivial, as we see from Proposition 13.3.

LEMMA 14.8 ([2] Lemma 9.4.4). If $\Upsilon_f(\mathcal{F})$ is well defined, then there is a natural transformation

$$\Upsilon_f(\mathcal{F}) \rightarrow \Psi_f^2(\mathcal{F}).$$

We want a condition to check if this map is an isomorphism. To that end, we introduce the following notion.

Definition 14.9 ([2] Definition 9.4.6). Let $f_1^\times := f_1 \circ j_1$ and $f_2^\times := f_2 \circ j_2^0$. We say that $\mathcal{F} \in D_c^b(X^{\times\times}, \bar{\mathbb{Q}}_l)$ is "iterated clean" if

$$\mathbf{i}^* j_{2,*} j_{1,!} \mathcal{F} = 0.$$

We have the following lemma which is a direct consequence of the proper and smooth base change theorems.

LEMMA 14.10. Suppose X, Y and separated schemes over k . Let f, g be maps $X \rightarrow \mathbb{A}^2$ and $Y \rightarrow X$ respectively.

- (1) If g is proper and $\mathcal{F} \in D_c^b(Y^{\times\times}, \bar{\mathbb{Q}}_l)$ is iterated clean, then so is $g_* \mathcal{F}$.
- (2) If g is smooth and $\mathcal{F} \in D_c^b(X, \bar{\mathbb{Q}}_l)$ is iterated clean, then so is $g^* \mathcal{F}$. If g is also surjective, the converse holds as well.

PROPOSITION 14.11 ([2] Proposition 9.4.8). Let $\mathcal{F}$ be a perverse sheaf on $X^{\times\times}$. Then the natural map $\Upsilon_f(\mathcal{F}) \rightarrow \Psi_f^2(\mathcal{F})$ is an isomorphism.

LEMMA 14.12 (Compatibility of Υ with proper pushforward, [2] Corollary 9.4.9). Let $\mathcal{F}$ be a perverse sheaf on $Y^{\times\times}$, and assume that f, g, X , and Y are as above. There is a natural map (if the $\Upsilon(\mathcal{F})$ is well defined)

$$\Upsilon_{f \circ g}(\mathcal{F}) \rightarrow \Upsilon_f(g_* \mathcal{F}).$$

Moreover, if $\mathcal{F}$ is iterated clean, then $\Upsilon(\mathcal{F})$ is well defined and this map is in fact an isomorphism.

Finally, we consider the possibility of restricting a perverse sheaf on $X^{\times\times}$ where $f : X \rightarrow \mathbb{A}^2$, to the diagonal $\mathbb{A}^1$ and take nearby cycles with respect to the induced map $f_\Delta : X_\Delta \rightarrow \mathbb{A}^1$. We use k denote the inclusion $X_\Delta \hookrightarrow X$. When we work with two schemes X and Y , we use notation such as k_X and k_Y where their meaning is clear.

LEMMA 14.13. Suppose $\mathcal{F}$ is a perverse sheaf on $X^{\times\times}$ such that $k^* \mathcal{F}[-1]$ is perverse and that $\Upsilon_f(\mathcal{F})$ is well defined. Then there is a natural morphism

$$\Upsilon_f(\mathcal{F}) \rightarrow \Psi_{f_\Delta}(k^* \mathcal{F}[-1]).$$

In the case of products discussed below, we find that this morphism is an isomorphism.

PROPOSITION 14.14 ([2] Proposition 9.4.13). Let X, Y be separated k -schemes of finite type equipped with maps $f : X \rightarrow \mathbb{A}^2$ and $g : Y \rightarrow X$.

- (1) If g is proper, $\mathcal{F} \in \text{Perv}(Y^{\times\times}, \bar{\mathbb{Q}}_l)$ is iterated clean, and that the complexes

$$k_Y^* \mathcal{F}[-1], \quad g_* \mathcal{F}, \quad k_X^* g_* \mathcal{F}[-1],$$

are also perverse. Suppose further that the natural map $\Upsilon_{f \circ g}(\mathcal{F}) \rightarrow \Psi_{(f \circ g)_\Delta}(k_Y^* \mathcal{F}[-1])$ is an isomorphism. Then so is

$$\Upsilon_f(\mathcal{F}) \rightarrow \Psi_{f_\Delta}(k_X^* g_* \mathcal{F}[-1]).$$

- (2) Now suppose that g is smooth of relative dimension d . $\mathcal{F} \in \text{Perv}(X^{\times\times}, k)$ is such that $k_X^* \mathcal{F}[-1]$ is perverse and suppose that $\Upsilon_f(\mathcal{F})$ is well defined. Then $k_Y^* g^* \mathcal{F}[d-1]$ is perverse and if the natural map $\Upsilon_f(g_* \mathcal{F}) \rightarrow \Psi_{f_\Delta}(k_X^* \mathcal{F}[-1])$ is an isomorphism, then

$$\Upsilon_{f \circ g}(g^* \mathcal{F}[r]) \rightarrow \Psi_{(f \circ g)_\Delta}(k_Y^* g^* \mathcal{F}[d-1])$$

is also an isomorphism. For surjective g , the converse also holds.

Next, we deal with the situation where $X \rightarrow \mathbb{A}^2$ arises as a product. As always, we adopt notation from Achar-Riche [2], Section 9.4.6. Indeed, suppose Y_1 and Y_2 are finite type k -schemes, equipped with maps $f_1 : Y_1 \rightarrow \mathbb{A}^1$ and $f_2 : Y_2 \rightarrow \mathbb{A}^1$. We take for X the product $Y_1 \times Y_2$ and the map $f = f_1 \times f_2 : X \rightarrow \mathbb{A}^2$. We define $u_i : Y_i^\times \hookrightarrow Y_i$ and $s_i : Y_{i,0} \rightarrow Y_i$ for the inclusion maps.

LEMMA 14.15 ([2] Lemma 9.4.15). Suppose $\mathcal{F}_i$ are perverse sheaves on Y_i for $i = 1, 2$. If $\mathcal{F}_1 \boxtimes^L \mathcal{F}_2$ is perverse, it is iterated clean.

Let $p_i : Y_1 \times Y_2 \rightarrow Y_i$ and $p_i^0 : Y_{3-i,0} \times Y_i \rightarrow Y_i$ be the projections. We may compute iterated nearby cycles as follows:

LEMMA 14.16 ([2] Lemma 9.4.16). Let $\mathcal{F}_i \in D_c^b(Y_i^\times, \bar{\mathbb{Q}}_l)$. Then there exists a canonical isomorphism

$$\Psi_{f_1}(\mathcal{F}_1) \boxtimes^L \Psi_{f_2}(\mathcal{F}_2) \xrightarrow{\sim} \Psi_{f_2 \circ p_2^0} \circ \Psi_{f_1 \circ p_1}(\mathcal{F}_1 \boxtimes^L \mathcal{F}_2).$$

Further, if $\mathcal{F}_i$ are perverse such that $\mathcal{F}_1 \boxtimes^L \mathcal{F}_2$ is also perverse, then we have

$$\Psi_f^2(\mathcal{F}_1 \boxtimes^L \mathcal{F}_2) \simeq \Psi_{f_1}(\mathcal{F}_1) \boxtimes^L \Psi_{f_2}(\mathcal{F}_2).$$

Moreover, $\Upsilon_f(\mathcal{F}_1 \boxtimes^L \mathcal{F}_2)$ is well defined and the natural morphism

$$\Upsilon_f(\mathcal{F}_1 \boxtimes^L \mathcal{F}_2) \rightarrow \Psi_{f_1}(\mathcal{F}_1) \boxtimes^L \Psi_{f_2}(\mathcal{F}_2)$$

is an isomorphism.

The final ingredient we need, is the relation to diagonal nearby cycles in this case.

PROPOSITION 14.17 (Proposition 9.4.18 from [2]). Suppose for perverse sheaves $\mathcal{F}_i$ on Y_i , the complexes $\mathcal{F}_1 \boxtimes^L \mathcal{F}_2$ and $k^*(\mathcal{F}_1 \boxtimes^L \mathcal{F}_2)[-1]$ are perverse. Then $\Upsilon_f(\mathcal{F}_1 \boxtimes^L \mathcal{F}_2)$ is well defined and the natural map $\Upsilon_f(\mathcal{F}_1 \boxtimes^L \mathcal{F}_2) \rightarrow \Psi_{f_\Delta}(k^*(\mathcal{F}_1 \boxtimes^L \mathcal{F}_2)[-1])$ is an isomorphism.

14.5. Property 3. First, we need to adjust the sign of the commutativity constraint for reasons similar to those that have been mentioned previously. This modification is as follows. Suppose, for simplicity, that the two complexes $\mathcal{A}$ and $\mathcal{B}$ are supported on single components of $\text{Gr}_{\mathcal{G}}$ and Fl_P respectively. If even one of these components is even, then no change is made. On the other hand, if both components are odd, we insert a minus sign into the definition of $\sigma_{\mathcal{A}, \mathcal{B}}$ as in ??.

THEOREM 14.18. For $\mathcal{A}, \mathcal{B} \in \text{Perv}_{L^+H}(\text{Gr}_H)$ the following diagram commutes:

$$\begin{array}{ccc} Z(\mathcal{A}) *^P Z(\mathcal{B}) & \longrightarrow & Z(\mathcal{A} *^{L^+H} \mathcal{B}) \\ \downarrow & & \downarrow \\ Z(\mathcal{B}) *^P Z(\mathcal{A}) & \longrightarrow & Z(\mathcal{B} *^{L^+H} \mathcal{A}) \end{array}$$

(The maps in question are the obvious ones.)

14.5.1. Certain affine Grassmannians over 2-dimensional bases.

14.5.2. Proof of Theorem 14.18. We start by making the following definitions:

- $F(\mathcal{A}, \mathcal{B}) := \Upsilon_{\text{Gr}_{\mathcal{G}}(\underline{x}, \underline{y})}(\tau(\mathcal{A}) \tilde{\boxtimes} \tau(\mathcal{B}));$
- $F_2(\mathcal{A}, \mathcal{B}) = \Psi_{\text{Gr}_{\mathcal{G}}(\underline{x}, \underline{y})}^2(\tau(\mathcal{A}) \tilde{\boxtimes} \tau(\mathcal{B}));$
- $F_d(\mathcal{A}, \mathcal{B}) := \Psi_{\text{Gr}_{\mathcal{G}}(\underline{x}, \underline{y})|_{\Delta C}}((i^{\Delta \times})^*(\tau(\mathcal{A}) \tilde{\boxtimes} \tau(\mathcal{B})[-1])).$

Likewise, we have objects on $\text{Gr}_{\mathcal{G}}(\underline{x} \cup \underline{y})$ defined as

- $F'(\mathcal{A}, \mathcal{B}) := \Upsilon_{\text{Gr}_{\mathcal{G}}(\underline{x} \cup \underline{y})}(\tau(\mathcal{A}) * \tau(\mathcal{B}));$
- $F'_2(\mathcal{A}, \mathcal{B}) = \Psi_{\text{Gr}_{\mathcal{G}}(\underline{x} \cup \underline{y})}^2(\tau(\mathcal{A}) * \tau(\mathcal{B}));$
- $F'_d(\mathcal{A}, \mathcal{B}) := \Psi_{\text{Gr}_{\mathcal{G}}(\underline{x} \cup \underline{y})|_{\Delta C}}((i^{\Delta \times})^*(\tau(\mathcal{A}) * \tau(\mathcal{B})[-1])).$

LEMMA 14.19. For $\mathcal{A}, \mathcal{B} \in \text{Perv}_{L^+H}(\text{Gr}_H)$ there are natural isomorphisms

$$F_2(\mathcal{A}, \mathcal{B}) \xleftarrow{\cong} F(\mathcal{A}, \mathcal{B}) \xrightarrow{\cong} F_d(\mathcal{A}, \mathcal{B}) \quad (14.5.1)$$

and

$$F'_2(\mathcal{A}, \mathcal{B}) \xleftarrow{\cong} F'(\mathcal{A}, \mathcal{B}) \xrightarrow{\cong} F'_d(\mathcal{A}, \mathcal{B}). \quad (14.5.2)$$

These isomorphisms give rise to the following commutative diagram

$$\begin{array}{ccccc} \mu_* F_2(\mathcal{A}, \mathcal{B}) & \longleftarrow & \mu_* F(\mathcal{A}, \mathcal{B}) & \longrightarrow & \mu_* F_d(\mathcal{A}, \mathcal{B}) \\ \downarrow & & \downarrow & & \downarrow \\ F'_2(\mathcal{A}, \mathcal{B}) & \longleftarrow & F'(\mathcal{A}, \mathcal{B}) & \longrightarrow & F'_d(\mathcal{A}, \mathcal{B}) \end{array}$$

Proof. A direct consequence of iterated cleanliness.

Q.E.D

Our first step now will be to recast the commutative diagram of Lemma 14.19 in terms of more familiar objects. Indeed, we will show that the same identifies with the following:

$$\begin{array}{ccccc} & & \text{id} & & \\ & \xleftarrow{\quad} & & \xrightarrow{\quad} & \\ Z(\mathcal{A}) *^P Z(\mathcal{B}) & \longleftarrow & \mu_* F(\mathcal{A}, \mathcal{B}) & \longrightarrow & Z(\mathcal{A}) *^P Z(\mathcal{B}) \\ \downarrow & & \downarrow & & \downarrow \\ C(\mathcal{B}, Z(\mathcal{A})) & \longleftarrow & F'(\mathcal{A}, \mathcal{B}) & \longrightarrow & Z(\mathcal{A}) *^{L^+H} \mathcal{B} \end{array}$$

In greater detail, the following are clear from definition:

- $F_2(\mathcal{A}, \mathcal{B}) = Z(\mathcal{A}) \tilde{\boxtimes} Z(\mathcal{B})$;
- Likewise for $F_d(\mathcal{A}, \mathcal{B}) = Z(\mathcal{A}) \tilde{\boxtimes} Z(\mathcal{B})$;
- $F'_2(\mathcal{A}, \mathcal{B}) = C(\mathcal{B}, Z(\mathcal{A}))$;
- $F'_d(\mathcal{A}, \mathcal{B}) = Z(\mathcal{A}) *^{L^+H} \mathcal{B}$.
- the left vertical arrow is the one arising in Proposition 14.1;
- the right vertical arrow is the one arising in Theorem 14.4.

That the topmost arrow is the identity follows from its construction.

Thus far, we have constructed two dimensional nearby cycles by doing so first in the direction of $\{0\} \times \tilde{C}$ and then in that of $\tilde{C} \times \{0\}$. Reversing their order, we are led to consider objects $G_2(\mathcal{A}, \mathcal{B})$ and $G'_2(\mathcal{A}, \mathcal{B})$, which provide the respective replacements of $F_2(\mathcal{A}, \mathcal{B})$ and $F'_2(\mathcal{A}, \mathcal{B})$. There are isomorphisms, respectively as follows:

$$G_2(\mathcal{A}, \mathcal{B}) \simeq Z(\mathcal{A}) \tilde{\boxtimes} Z(\mathcal{B}); \quad G'_2(\mathcal{A}, \mathcal{B}) \simeq C(\mathcal{A}, Z(\mathcal{B})); \quad \mu_* G_2(\mathcal{A}, \mathcal{B}) \simeq G'_2(\mathcal{A}, \mathcal{B}).$$

Now, consider the automorphism “swap” of $\text{Gr}_{\mathcal{G}}(x, y)$ which swaps the two points x and y . This is the identity on the pre-image of the diagonal. Thus, over the origin $(0, 0)$ a canonical isomorphism is induced

$$\Upsilon_{\text{Gr}_{\mathcal{G}}(\underline{x}, \underline{y})}(\tau(\mathcal{A}) * \tau(\mathcal{B})) \xrightarrow{\cong} \Upsilon_{\text{Gr}_{\mathcal{G}}(\underline{x}, \underline{y})}(\text{swap}^*(\tau(\mathcal{A}) * \tau(\mathcal{B}))),$$

and hence a canonical isomorphism also of the form

$$F'(\mathcal{A}, \mathcal{B}) \xrightarrow{\cong} F'(\mathcal{B}, \mathcal{A}),$$

as $F'(\mathcal{A}, \mathcal{B})$ is canonical identified with $\Upsilon_{\text{Gr}_{\mathcal{G}}(\underline{x}, \underline{y})}(\text{swap}^*(\tau(\mathcal{A}) * \tau(\mathcal{B})))$.

On the diagonal outside the origin, the following isomorphism is induced.

$$F'_d(\mathcal{A}, \mathcal{B}) \xrightarrow{\cong} F'_d(\mathcal{B}, \mathcal{A}).$$

We make a closer examination of the above isomorphism. F'_d is constructed by first restricting $\tau(\mathcal{A}) \boxtimes \tau(\mathcal{B})$ to the diagonal, and then taking nearby cycles towards the origin. The aforementioned restriction is, by construction, $\mathcal{A} *^{L^+H} \mathcal{B}$. The automorphism swap therefore induces the commutativity constraint on $\sigma_{\mathcal{A}, \mathcal{B}} : \mathcal{A} *^{L^+H} \mathcal{B} \xrightarrow{\sim} \mathcal{B} *^{L^+H} \mathcal{A}$. Applying nearby cycles thus gives the isomorphism $Z(\sigma)$ between the two objects above:

$$F'_d(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} F'_d(\mathcal{B}, \mathcal{A}).$$

Along the two axes, swap induces an isomorphism

$$F'_2(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} G'_2(\mathcal{B}, \mathcal{A}).$$

There are natural isomorphisms of both objects with $C(\mathcal{B}, Z(\mathcal{A}))$ such that the obvious composite is the identity - this is a direct consequence of the construction. We consider the composite

$$\mu_* F_2(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} F'_2(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} G'_2(\mathcal{B}, \mathcal{A}) \xrightarrow{\sim} \mu_* G_2(\mathcal{B}, \mathcal{A}).$$

By construction, once again, we find that this composite coincides with the commutativity constraint

$$\sigma_{\mathcal{A}, Z(\mathcal{B})} : Z(\mathcal{A}) *^P Z(\mathcal{B}) \simeq Z(\mathcal{B}) *^P Z(\mathcal{A}).$$

We assemble all the isomorphisms described above to obtain the following commutative diagram.

$$\begin{array}{ccccccc}
Z(\mathcal{A}) *^P Z(\mathcal{B}) & & & & & & Z(\mathcal{B}) *^P Z(\mathcal{A}) \\
& \searrow & & \xrightarrow{\text{id}} & & \swarrow & \\
& & C(\mathcal{B}, Z(\mathcal{A})) & \longrightarrow & F'_2(\mathcal{A}, \mathcal{B}) & \longrightarrow & G'_2(\mathcal{B}, \mathcal{A}) & \longrightarrow & C(\mathcal{B}, Z(\mathcal{A})) \\
& & & & \downarrow & & \downarrow & & \\
& & & & F'(\mathcal{A}, \mathcal{B}) & \longrightarrow & G'(\mathcal{B}, \mathcal{A}) & & \\
& & & & \downarrow & & \downarrow & & \\
& & Z(\mathcal{A} *^{L^+H} \mathcal{B}) & \longrightarrow & F'_d(\mathcal{A}, \mathcal{B}) & \longrightarrow & F'_d(\mathcal{B}, \mathcal{A}) & \longrightarrow & Z(\mathcal{B} *^{L^+H} \mathcal{A}) \\
& & & & \searrow & & \swarrow & & \\
& & & & & & & & Z(\sigma)
\end{array}$$

That the left and rightmost oblique arrows are those appearing in Theorem 14.4 follows from the commutative diagram 14.5.2. This finishes the proof.

REFERENCES

- [1] P. Achar, Perverse sheaves and applications in representation theory, Mathematical Surveys and Monographs, no. 258, Amer. Math. Soc., Providence, RI, 2021.
- [2] Pramod Achar and Simon Riche. Central sheaves on affine flag varieties. Book in preparation, preliminary version available at <https://lmbp.uca.fr/riche/central.pdf>.
- [3] Pramod N. Achar and João Lourenço and Timo Richarz and Simon Riche, A modular ramified geometric Satake equivalence, 2024. <https://arxiv.org/abs/2403.10651>
- [4] Pramod Achar, João Lourenço, Timo Richarz, and Simon Riche. A modular ramified Satake equivalence, 2023.
- [5] R. Bezrukavnikov, On tensor categories attached to cells in affine Weyl groups, Representation theory of algebraic groups and quantum groups, 69–90, Adv. Stud. Pure Math., 40, Math. Soc. Japan, Tokyo, 2004.
- [6] S. Bosch, W. Lütkebohmert, and M. Raynaud, Néron models, Ergebnisse der Mathematik und ihrer Grenzgebiete 21, Springer-Verlag, 1990.
- [7] Pierre Baumann and Simon Riche, Notes on the geometric Satake equivalence, Relative aspects in representation theory, Langlands functoriality and automorphic forms, Lecture Notes in Math., vol. 2221, Springer, Cham, 2018, pp. 1–134. MR3839695
- [8] F. Bruhat and J. Tits, Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d’une donnée radicielle valuée, Inst. Hautes Études Sci. Publ. Math. No. 60 (1984), 197–376.

- [9] Gerd Faltings, Algebraic loop groups and moduli spaces of bundles, *J. Eur. Math. Soc. (JEMS)* 5 (2003), no. 1, 41–68, DOI 10.1007/s10097-002-0045-x. MR1961134.
- [10] Laurent Fargues and Peter Scholze. Geometrization of the local Langlands correspondence. Preprint arXiv:2102.13459, 2021.
- [11] Gaitsgory, D. Construction of central elements in the affine Hecke algebra via nearby cycles. *Invent. math.* 144, 253–280 (2001). <https://doi.org/10.1007/s002220100122>
- [12] T. Haines, A proof of the Kazhdan–Lusztig purity theorem via the decomposition theorem of BBD, expository note available on the author’s web page
- [13] Haines, T. “On Satake parameters for representations with parahoric fixed vectors.” *International Mathematical Research Notices* (2014): preprint arXiv:1402.3812. .
- [14] Luc Illusie, Around the Thom-Sebastiani theorem, available at <http://www.math.upsud.fr/illusie/Illusie-nearby-cycles27d.pdf>
- [15] J. C. Jantzen, *Representations of Algebraic Groups*, second edition, *Mathematical Surveys and Monographs* 107, Amer. Math. Soc., 2003.
- [16] J. Humphreys, *Introduction to Lie Algebras and Representation Theory*, *Graduate Texts in Mathematics*, Springer-Verlag, 1972.
- [17] George Pappas and Michael Rapoport. Twisted loop groups and their affine flag varieties. *Adv. Math.*, 219(1):118–198, 2008. With an appendix by Thomas J. Haines and Michael Rapoport.
- [18] G.Prasad, J.K.Yu, On quasi-reductive group schemes, *Journal of Algebraic Geometry*, 15 (2006), 507–549.
- [19] Michael Rapoport, A positivity property of the Satake isomorphism, *Manuscripta Math.* 101 (2000), no. 2, 153–166, DOI 10.1007/s002290050010. MR1742251
- [20] Timo Richarz. Affine Grassmannians and geometric Satake equivalences. *Int. Math. Res. Not. IMRN*, 12:3717–3767, 2016.
- [21] Timo Richarz. Schubert varieties in twisted affine flag varieties and local models. *J. Algebra*, 375:121–147, 2013.
- [22] Steinberg, R. Endomorphisms of Linear Algebraic Groups. *Memoirs of the AMS* 80, 1968. <http://www.ams.org/books/memo/0080/>.
- [23] J.-L. Verdier, Spécialisation de faisceaux et monodromie modérée, *Analysis and topology on singular spaces*, II, III (Luminy, 1981), *Astérisque*, vol. 101, Soc. Math. France, Paris, 1983, pp. 332–364.
- [24] Zhu, Xinwen. “An introduction to affine Grassmannians and the geometric Satake equivalence”. [arXiv:1603.05593](https://arxiv.org/abs/1603.05593)
- [25] Xinwen Zhu. The geometric Satake correspondence for ramified groups. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(2):409–451, 2015.
- [26] Xinwen Zhu, On the coherence conjecture of Pappas and Rapoport, *Ann. of Math. (2)* 180 (2014), no. 1, 1–85, DOI 10.4007/annals.2014.180.1.1. MR3194811