Paris VI, Jussieu

M2 Memoire

From the Hodge-Tate Conjecture to p-adic Hodge Theory

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Abstract

We prove the Hodge-Tate conjecture for p-adic Abelian schemes with good reduction. We introduce two of Fontaine's period rings and state the formalism of admissible representations.

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Contents

1 Introduction

p-adic Hodge Theory made its official debut in Tate's 1967 celebrated article [11], on p-divisible groups, the first of a series of works on the subject. Although the tools for p-adic Hodge Theory such as Fontaine rings, Almost mathematics and more recently perfectoid spaces were developed as independent branches of mathematics, stretching somewhere in between commutative algebra and category theory, Tate's original results still provide the main motivation and largely drive the evolution of the subject.

Let A be an abelian scheme defined over the valuation ring K^+ of a complete discretely valued field K of mixed characteristic. Denote by C the completion of $\overline{K}$ and Γ_K, the absolute Galois group of K. Tate showed that for every $i \leq 2\dim(A)$ there exists a natural equivariant isomorphism

$$H^i_{\text{ét}}(A_{\overline{K}}, C) \cong \bigoplus_{j+k=i} H^j(A, \Omega^k_A) \otimes_K C(-k)$$

where, for every integer $j \in \mathbb{Z}$, we define $C(j) := \mathbb{Q}_p(j) \otimes_{\mathbb{Q}_p} C$, and $\mathbb{Q}_p(j)$ is the j-th tensor power of the one-dimensional p-adic representation $\mathbb{Q}_p(1)$ on which Γ_K acts as the cyclotomic character. Tate conjectured that such an equivariant decomposition should exist for any smooth projective variety defined over K. To put things in perspective, let us turn to consider the archimedean counterpart: if X is a smooth, proper complex algebraic variety, one can combine deRham's theorem with Grothendieck's theoem on algebraic deRham cohomology [6], to deduce a natural isomorphism

$$H^\bullet(X^{\text{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^\bullet_{\text{dR}}(X)$$

between the singular and deRham cohomologies. The point is that the two sides contribute complementary information on X and neither of these two structures is reducible to the other.

Tate's conjecture in degree one was finally solved by Fontaine, in 1982 [4]. The construction of Fontaine hinges on a remarkable ring, endowed with both a Galois action and a filtration. The simplest of such rings is the graded ring

$$B_{\text{HT}} := \bigoplus_{i \in \mathbb{Z}} C(i).$$

With its help, Tate's decomposition can be rewritten as an isomorphism of graded K-vector spaces

$$(H^i_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{HT}})^{G_K} \cong \bigoplus_{j+k=i} H^j(A, \Omega^k_A).$$

In 1988 Faltings extended the proof to all degrees, formally known as the C_{cris} conjecture, by the method of almost étale extensions [3]. For simplicity we let X be a smooth, connected and projective K^+-scheme which has good reduction.

The idea is to construct an intermediate cohomology $\mathcal{H}(X)$ with values in C-vector spaces, receiving maps from both étale and Hodge-Tate cohomology, and prove that the resulting natural transformations

$$H_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \to \mathcal{H}(X) \quad \text{and} \quad H_{\text{HT}}(X_K) \to \mathcal{H}(X)$$

are isomorphisms of functors.

Faltings relied on his almost purity result which was used as a black box due to the difficulty of its proof. However in 2012 Peter Scholze's discovery of perfectoid Spaces [8], gave a remarkably simple proof of the later. The original aim of the theory of perfectoid spaces was to prove Deligne's weight-monodromy conjecture over p-adic fields by reduction to the case of local fields of equal characteristic p, where the result is known. In order to do so, the theory of perfectoid spaces establishes a general framework relating geometric questions over local fields of mixed characteristic with geometric questions over local fields of equal characteristic. However, the theory of perfectoid spaces has proved to be useful in other situations and certainly many more will be found. On one hand, perfectoid spaces embody Faltings's almost purity theorem, as this is the crucial technical ingredient to Faltings's approach to p-adic Hodge theory, it is not surprising that one can prove new results in p-adic Hodge theory. Initially perfectoid spaces had more of an auxiliary role. However, it turns out that many natural constructions, which so far could not be given any geometric meaning, are perfectoid spaces in a natural way. Shimura varieties with infinite level at p, and Rapoport-Zink spaces with infinite level are two examples.

1.1 Thank You

I would like to thank Jean François Dat for his guidance into the subject and Laurent Fargues for some helpful discussions on p-divisible groups. Without them I would probably be in Australia surfing and getting way too tanned.

2 p-Divisible Groups

In this section we prove a partial result of the Hodge-Tate conjecture, concerning p-divisible groups. For this reason, we recall some basic definitions and statements about group schemes. The material of this section can be found in [11]. It should be noted that Tate's initial motivation was to prove that a p-divisible group G defined over a ring of integers R is determined by its generic fiber $G \times_R K$, where K is a local field of characteristic 0 containing R.

Definition 2.1. *Let R be a commutative ring. A finite group scheme of order n over R is a group object in the category of schemes over R which is locally free of rank n.*

Example 2.2. Let $A = R[X]/(X^n - 1)$. Then $\text{Spec}(A)$ is a finite group scheme of order n over R, with multiplication given by $x \mapsto x \otimes x$ and inverse by $x \mapsto x^{n-1}$, where x is the image of X in A.

Example 2.3. Let G be a finite group. The associated constant group scheme G' as a functor takes an R-algebra T and produces the group

$$G'(T) = \{(e_g)_{g \in G} : e_g \in T,\ 1 = \sum_{g \in G} e_g,\ e_g e_h = \delta_{g,h} e_g\}.$$

The group structure comes from convolution

$$(e_g) \cdot (f_g) := \left(\sum_{g = hh'} e_h f_{h'} \right)_g$$

with unit $\mathbf{1} = (\delta_{1,g})_g$.

Throughout we will assume all group schemes are commutative, affine and R is a complete noetherian local ring. Indeed it is not always true that one can cut a non-affine group scheme into affine group scheme pieces. Our definition of affine group schemes is of mixed nature: we have an algebra A together with a group structure on the associated functor of points. We would like to work with A explicitly. Suppose now G is represented by A, then $A \otimes A$ represents $G \times G$, and we can apply Yoneda's lemma. Hence making G a group functor is the same as giving R-algebra maps:

comultiplication	$\Delta : A \to A \otimes A$
counit	$\varepsilon : A \to R$
coinverse	$S : A \to A$

such that the diagrams

$$\begin{CD} A \otimes A \otimes A @<{\mathrm{id} \otimes \Delta}<< A \otimes A \\ @A{\Delta \otimes \mathrm{id}}AA @AA{\Delta}A \\ A \otimes A @<<{\Delta}< A \end{CD} \qquad \begin{CD} R \otimes A @<{\varepsilon \otimes \Delta}<< A \otimes A \\ @AAA @AA{\Delta}A \\ A @<<{\mathrm{id}}< A \end{CD}$$

and

$$\begin{CD} A @<{(S, \mathrm{id})}<< A \otimes A \\ @AAA @AA{\Delta}A \\ R @<<{\varepsilon}< A \end{CD}$$

commute. An R-algebra A with specified maps Δ, ε and S satisfying these conditions is called a Hopf algebra. Thus affine group schemes over R correspond to Hopf algebras over R. The structure for $\mathbb{G}_m$ has $A = R[X,Y]/(XY-1)$ and it is not difficult to see the Hopf algebra structure

$$\Delta(X) = X \otimes X,\ \varepsilon(X) = 1 \text{ and } S(X) = 1/X.$$

Definition 2.4. *Let $G = \mathrm{Spec}(A)$ be a finite group scheme over R. We define its Cartier dual, as in* [1], *$G' = \mathrm{Spec}(A')$, where $A' := \mathrm{Hom}_R(A, R)$.*

Suppose now we take some finite commutative G, represented by A. The Hopf algebra structure has the following maps:

$$\begin{aligned}
&\Delta : A \to A \otimes A \\
&\varepsilon : A \to R \\
&S : A \to A \\
&m : A \otimes A \to A \\
&u : R \to A
\end{aligned}$$

When we dualize, we get on A' a very similar collection of maps:

$$\begin{aligned}
&m' : A' \to A' \otimes A' \\
&u' : A' \to R \\
&S' : A' \to A' \\
&\Delta' : A' \otimes A' \to A' \\
&\varepsilon' : R \to A'
\end{aligned}$$

It is not difficult to verify that A' is indeed a Hopf algebra.

Remark 2.5. By identifying R with R' and $(A \otimes_R A)'$ with $A' \otimes_R A'$, dualizing the multiplication and inverse maps, we see indeed that the above definition provides us with a group scheme. Furthermore there is a canonical isomorphism $G \cong (G')'$.

We know what G' is Hopf-algebraically but we do not yet have any intrinsic description of the functor G'. We attempt to compute $G'(R)$, the R-algebra of maps $A' \to R$. By duality any linear map $\phi : A' \to R$ has the form $\phi_b(f) = f(b)$ for some b in A. On a product, $\phi_b(fg) = \phi_b\Delta'(f \otimes g) = \Delta'(f \otimes g)(b) = (f \otimes g)(\Delta b)$, while $\phi_b(f)\phi_b(g) = f(b)g(b) = (f \otimes g)(b \otimes b)$. Since elements $f \otimes g$ span $A' \otimes A'$, the duality theory shows that ϕ_b preserves products iff $\Delta b = b \otimes b$. Similarly, since ε is the unit of A', we have ϕ_b preserving unit iff $1 = \phi_b(\varepsilon) = \varepsilon(b)$. Thus $G'(R) = \mathrm{Hom}(G, \mathbb{G}_m)$. But now we can evaluate $G'(T)$ for any R-algebra T simply by base change. The functor G_T is represented by $A \otimes_R T$, so $(G_T)'$ by $(A \otimes_R T)' = A' \otimes_R T$ which also represents $(G')_T$. Hence $G'(T) = (G')_T(T) = (G_T)'(T) = \mathrm{Hom}_T(G \otimes_R T, \mathbb{G}_m)$.

Definition 2.6. *Let p be a prime number, and h a non-negative integer. A p-divisible group scheme G over R of height h is given by*

$$G = \varinjlim_{i_\nu} G_\nu$$

where

1. *G_ν is a finite group scheme over R of order $p^{\nu h}$*
2. *for each $\nu \geq 0$,*
$$0 \to G_\nu \overset{i_\nu}{\to} G_{\nu+1} \overset{[p^\nu]}{\to} G_{\nu+1}$$
is exact where $[p^\nu]$ is multiplication.

Remark 2.7. The composite of the i_ν gives closed immersions $i_{\nu,t} : G_\nu \to G_{\nu+t}$.

We note that $G = \varinjlim_{i_\nu} G_\nu$ is not a scheme but a formal R-scheme, which we will discuss shortly. We treat it here as a functor of points on R-algebras

$$G : T \mapsto G(T) = \varinjlim_{i_\nu} G_\nu(T).$$

In literature Grothendieck calls p-divisible groups, Barsotti-Tate groups.

Lemma 2.8. *The kernel of $[p^\nu] : G_{\nu+t} \to G_{\nu+t}$ is the subscheme G_ν.*

Proof. We proceed by induction on t. For $t = 1$ this is just part of the definition. Assuming the lemma is true up to t we have

$$G_{\nu+t+1}[p^\nu] = G_{\nu+t+1}[p^{\nu+t}] \cap G_{\nu+t+1}[p^\nu] = G_{\nu+t} \cap G_{\nu+t+1}[p^\nu] = G_{\nu+t}[p^\nu] = G_\nu.$$

□

A homomorphism of p-divisible groups is a homomorphism

$$f : \varinjlim G_\nu \to \varinjlim H_\omega$$

compatible with the inductive system, that is a compatible collection of maps

$$f_\nu : G_\nu \to H_\nu.$$

Thus we obtain a fairly simple description

$$\mathrm{Hom}(\varinjlim_\nu G_\nu, \varinjlim_\omega H_\omega) = \varprojlim_\nu \mathrm{Hom}(G_\nu, H_\nu)$$

The image of $[p^\nu] : G_{\nu+t} \to G_{\nu+t}$ is killed by $[p^t]$ and hence factors uniquely over a map

$$j_{\nu,t} : G_{\nu+t} \to G_t.$$

Proposition 2.9. *The sequence*

$$0 \to G_\nu \overset{i_{\nu,t}}{\to} G_{\nu+t} \overset{j_{\nu,t}}{\to} G_t \to 0$$

is exact.

Proof. We note that $j_{\nu,t} \circ i_{\nu,t} = [p^\nu]$ and so it is left exact. We get the following induced injective map

$$G_{\nu+t}/i_{\nu,t}(G_\nu) \hookrightarrow G_t$$

of R-group schemes of order p^{ht} and so must be an isomorphism thanks to the following lemma. □

Lemma 2.10. *Let R be a noetherian ring and M a finitely generated R-module. Then any surjective R-endomorphism of M is bijective.*

Proof. Let $\phi : M \twoheadrightarrow M$ be an R-surjective map. Denote by $M_i = \ker(\phi^i)$. The family (M_i) is increasing and thus stabilises. We can choose $j \gg 0$ such that $M_j = M_{2j}$. Then the map $\phi^j : M_j = M_{2j} \to M_j$ is surjective but is also the zero map. Thus $M_j = 0$ and we see that ϕ is injective. □

Corollary 2.11. *Let $G = (G_\nu)$ be a p-divisible group. The map multiplication by p*

$$[p] : G \to G$$

is surjective on the fpqc sheaf defined by G.

Proof. The map is induced by the $j_{1,\nu} : G_{\nu+1} \to G_\nu$ each of which is surjective. □

Remark 2.12. The name p-divisible group comes from the above result.

Definition 2.13. *An isogeny is a morphism of p-divisible groups $f : G \to H$, which has finite flat kernel and is surjective as a map of fpqc sheaves. The degree of an isogeny is the order of its kernel.*

Remark 2.14. $[p] : G \to G$ is an isogeny of degree p^h.

Example 2.15. The constant p-divisible group

$$\mathbb{Q}_p/\mathbb{Z}_p$$

is given by $G_\nu = \mathbb{Z}/p^\nu\mathbb{Z}$ with height $h = 1$.

We next define the dimension of a finite group scheme, however this requires some work and in particular the introduction of formal lie groups. For a more complete treatment of formal lie groups we refer the reader to [2].

Definition 2.16. *Let R be a complete noetherian topological ring. We say R is linearly topologized if zero has a base consisting of ideals. An ideal of definition $\mathcal{J}$ is an open ideal such that for every open neighbourhood V of 0, there exists a positive integer n such that $\mathcal{J}^n \subseteq V$. Let $\mathcal{J}_\lambda$ be a neighbourhood basis for zero consisting of ideals of definition. The formal spectrum of R has structure sheaf*

$$\mathrm{Spf}(R) := \varprojlim_{\lambda} \mathcal{O}_{\mathrm{Spec}\, R/\mathcal{J}_\lambda}.$$

Finally an R-formal scheme is the glueing of the formal spectrum of R-algebras.

We can now define what an R-formal lie group is.

Definition 2.17. *An R-formal group is a group object in the category of R-formal schemes. An R-formal Lie group is a smooth connected R-formal group.*

Remark 2.18. The conditions of smoothness and connectedness for $\mathcal{G} = \mathrm{Spf}(A)$ imply that $A \cong R[[T_1, T_2, ..., T_n]]$. The number n is called the dimension of $\mathcal{G}$.

For what follows we omit the ring R if there is no ambiguity and we assume it is of mixed characteristic $(0,p)$. Assume also that $\mathcal{G}$ is of the form $\mathrm{Spf}(A)$. A formal Lie group $\mathcal{G} = \mathrm{Spf}(R[[T_1, T_2, ..., T_n]])$ is called p-divisible if the multiplication by p map, $[p]$, is an isogeny. Denote by ϕ the corresponding map between the $R[[T_1, T_2, ..., T_n]]$. We have that $\ker[p] = \mathrm{Spec}(R[[T_1, T_2, ..., T_n]]/\mathcal{I})$ where $\mathcal{I}$ is the ideal generated by $(\phi(T_i))$. Hence it is a finite connected group scheme of order a power of p, say p^h. In general the kernels $G_\nu = \ker[p^\nu] =: \mathcal{G}[p^\nu]$ form an inductive system of connected finite group schemes of order $p^{h\nu}$. Moreover the following sequence

$$0 \to G_\nu \xrightarrow{i_\nu} G_{\nu+1} \xrightarrow{[p^\nu]} G_{\nu+1}$$

is exact. Thus $\mathcal{G}_{p^\infty} := (G_\nu)$ is a connected p-divisible group.

Theorem 2.19 (Serre-Tate)**.** *Let R be a local complete noetherian ring with mixed characteristic $(0,p)$. The functor*

$$F : \mathcal{G} \rightsquigarrow \mathcal{G}_{p^\infty}$$

is an equivalence of categories between p-divisible formal Lie groups over R and connected p-divisible groups over R.

Proof. We first prove F is fully faithful. Fixing notation let $A = R[[T_1, T_2, ..., T_n]]$, $\mathcal{G} = \mathrm{Spf}(A)$, $\mathfrak{m}$ the maximal ideal of R and $\mathcal{I} = (T_1, T_2, ..., T_n)$. Then the maximal ideal of A is $M = \mathfrak{m}A + \mathcal{I}$. It is not difficult to see that $G_\nu = \mathrm{Spec}(A_\nu)$, where $A_\nu = A/\mathcal{J}_\nu$ and $\mathcal{J}_\nu$ is the ideal generated by the $\phi^\nu(T_i)$. Since A is $\mathfrak{m}$-adically complete, so is A_ν. We also have that $A/(\mathcal{J}_\nu + \mathfrak{m}^i) = A_\nu/\mathfrak{m}^i A_\nu$ is local artinian and so for some $\omega \gg 0$, $M^i \subset \mathcal{J}_\nu + \mathfrak{m}^i$. Now $\mathcal{J}_1 \subseteq p\mathcal{I} + \mathcal{I}^2 \subseteq M\mathcal{I}$ and so by induction $\mathcal{J}_\nu \subseteq M^\nu \mathcal{I}$ and so $\mathcal{J}_\nu \to 0$ in the M-adic topology. Thus we have the following:

$$A = \varprojlim_{\omega} A/M^\omega = \varprojlim_{\nu,i} A/(\mathcal{J}^\nu + \mathfrak{m}^i) = \varprojlim_{\nu,i} A_\nu/\mathfrak{m}^i A_\nu = \varprojlim_{\nu} A_\nu$$

and so we obtain that F is fully faithful:

$$\mathrm{Hom}_R(\mathcal{G}, \mathcal{G}') = \mathrm{Hom}_R(A, A') = \varprojlim_{\nu} \mathrm{Hom}_R(A_\nu, A'_\nu) = \mathrm{Hom}_R(\mathcal{G}_{p^\infty}, \mathcal{G}'_{p^\infty}).$$

It remains to prove that F is essentially surjective. Given a connected p-divisible group $G = (G_\nu)$, we have to construct a p-divisible formal Lie group $\mathcal{G}$ with $\mathcal{G}_{p^\infty} = (G_\nu)$. Locally we write $G_\nu = \mathrm{Spec}(A_\nu)$ and $A = \varprojlim A_\nu$. The comultiplication of the A_ν define a comultiplication $\Delta : A \to A \widehat{\otimes} A$, and this gives us an associated R-formal group $\mathcal{G} = \mathrm{Spf}(A)$. It suffices to show that $\mathcal{G}$ is smooth, connected and the map $[p] : \mathcal{G} \to \mathcal{G}$ is isogenic. Indeed $[p] : \mathcal{G} \to \mathcal{G}$ is the direct limit of the maps

$$G_{\nu+1} \xrightarrow{j_{1,\nu}} G_\nu,$$

hence surjective with finite order kernel $\mathcal{G}[p] = G_1$. Hence $[p]$ is an isogeny. Furthermore $\mathcal{G}[p^\nu] = G_\nu$ so that $\mathcal{G}_{p^\infty} = (G_\nu)$, which is promising. We need to construct a continuous isomorphism

$$f : R[[T_1, T_2, ..., T_n]] \to A. \tag{1}$$

Since $A = \varprojlim A_\nu \cong \prod_{n\in\mathbb{N}} R \cong R[T]$ as rings, A is regular and so by the Cohen structure theorem, we have a map

$$\bar{f} : (R/\mathfrak{m})[[T_1, T_2, ..., T_n]] \to A/\mathfrak{m}A$$

which by Nakayama's lemma lifts to a map (1). So we can assume $R = k$ is a field of characteristic $p > 0$. To determine what the value of n should be, we note for inspiration that the kernel of the frobenius endomorophisim of $k[[T_1, T_2, ..., T_n]]$ has rank p^n. This motivates us to make the following definition:

$$H_{\nu,t} = \ker(\mathrm{Fr}^\nu : G_t \to G_t^{(p^\nu)}), t \gg 0$$

where the Frobenius map:

$$\mathrm{Fr} : G \to G^{(p)}$$

of an affine algebraic R-group, is the homomorphism defined by raising to pth power on coordinates. Here $G^{(p)} = G \otimes_k k$, the k-ring obtained by scalar extension of the Frobenius. By induction we set

$$(G^{(p^i)})^p := G^{(p^{i+1})}.$$

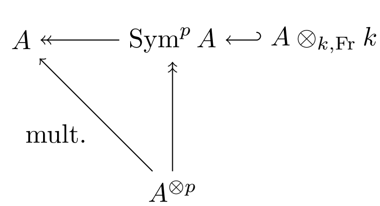
In order to help us discover the properties of $H_{\nu,t}$, we define the auxiliary map (*Verschiebung*):

$$V : G^{(p)} \to G.$$

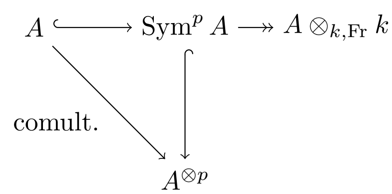
To obtain it, one takes the Cartier dual of the Frobenius of the Cartier dual group and makes identifications $((G')^{(p)})' \cong G^{(p)}$ and $(G')' \cong G$. Charged with these maps we need the following result.

Lemma 2.20. $V^\nu \circ \mathrm{Fr}^\nu = [p^\nu]$.

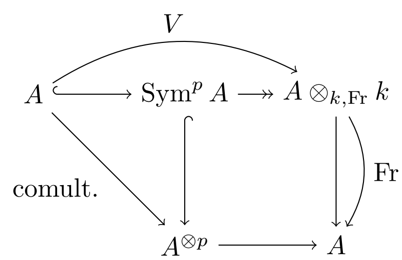
Proof. It is enough to prove $V \circ \mathrm{Fr} = [p]$. Write $G = \mathrm{Spec}(A)$ and let $\mathrm{Sym}^p A$ denote the p-th symmetric power of A over k. We can expand the definition of Fr on coordinate rings as the composite in the top line of the commutative diagram



where the top line is $a \otimes x \mapsto [x(a \otimes a \otimes \ldots \otimes a)] \mapsto x \cdot a^p$. It is not difficult to check that the formula in the upper right defines a k-linear homomorphism. We can take the above diagram for A' instead of A and dualize it over k to represent V as the composite in the commutative diagram



By the above constructions, we obtain the following diagram



Finally the associated diagram of group schemes gives us the result.

□

We see immediately that $H_{\nu,t} \subseteq G_\nu$ as a closed subgroup independent of t. Declare G_ν to have *Frobenius height* $n \in \mathbb{N}$, if it is the smallest integer such that Fr^n is the zero homomorphism, assuming of course it exists.

Lemma 2.21. *The Frobenius height of G_ν is finite.*

Proof. Denote $\operatorname{krdim} A_\nu = d$. Since $\operatorname{Spec}(A_\nu \otimes_k \overline{k})$ is noetherian we see that the coordinates T_i in $A_\nu \otimes_k \overline{k}$ are nilpotent. Thus for some n, $T_i^{p^n} = 0$ for all i. Thus in $\operatorname{Spec}(A)$

$$\dim \operatorname{Supp} T_i^{p^n} < d.$$

Finally we conclude the result by noetherian induction on the support. □

Returning to the proof of the theorem we see that conversely, $G_\nu \subset H_{w,t} =: H_w$ for $w \gg 0$ (depending on ν) and so we conclude $A = \varprojlim A_\nu = \varprojlim B_\nu$, where $H_\nu = \operatorname{Spec} B_\nu$. Let I_ν be the augmentation ideal of B_ν and fix elements $X_1, ..., X_n \in I = \varprojlim I_\nu$ such that the images form a basis of I_1/I_1^2. As

$$0 \to H_1 \to H_\nu \xrightarrow{\mathrm{Fr}} H_\nu^{(p)}$$

is exact for $\nu \geq 1$, we find (from the translation of group schemes and commutative Hopf algebras) $I_1 = I_\nu \bmod I_\nu^p$ and thus, by counting dimensions,

$I_\nu/I_\nu^2] \to I_1/I_1^2$ is an isomorphism. We deduce that the images of the $X_1, ..., X_n$ in I_ν generate it for each ν. The map of interest

$$k[[X_1, ..., X_n]] \to \varprojlim B_\nu = A$$

defined by the X_i is the projective limit of the corresponding surjective maps

$$k[[X_1, ..., X_n]]/\left(X_i^{p^\nu};\ 1 \le i \le n\right) \twoheadrightarrow B_\nu.$$

It remains to show that the order of H_ν is precisely:

$$p^{n\nu} = \dim_k k[[X_1, ..., X_n]]/\left(X_i^{p^\nu};\ 1 \le i \le n\right).$$

We proceed by induction on n. For $n = 1$ this follows from the structure theory of finite flat groups of Frobenius height 1, which we refer the reader to [2]. For the inductive step, we only need to show that

$$0 \to H_1 \to H_\nu \xrightarrow{\mathrm{Fr}} H_{\nu-1}^{(p)} \to 0$$

is exact and note that H_ν and $H_\nu^{(p)}$ have the same order. The multiplication by p map $[p] : G^{(p)} \to G^{(p)}$ for the Frobenius twisted p-divisible group $G^{(p)}$ is surjective as a map of fpqc sheaves. Now similar to Lemma 2.20, one can prove $[p] = \mathrm{Fr} \circ V$, we deduce that $\mathrm{Fr} : G \to G^{(p)}$ is surjective as a map of fpqc sheaves. Consequently, the map $\mathrm{Fr} : H_\nu \to H_{\nu-1}^{(p)}$ is surjective as a map of fpqc sheaves, which concludes the proof of the theorem. □

Corollary 2.22. *A p-divisible group G, has two important invariants, the height h and the dimension n (of its connected component).*

For completeness we include a result relating these two invariants.

Theorem 2.23. *Let G be a p-divisible group over a local noetherian ring R with residue characteristic $p > 0$. Then we have*

$$\dim(G) + \dim(G') = \mathrm{height}(G).$$

Proof. The invariants under consideration only depend on the closed fibre of G. We may thus assume that $R = k$ is a field of characteristic $p > 0$. Since $V \circ \mathrm{Fr} = [p]$ and Fr is surjective, we have a short exact sequence of finite flat k-group schemes

$$0 \to \ker \mathrm{Fr} \to \ker[p] \xrightarrow{\mathrm{Fr}} \ker V \to 0.$$

The order of $\ker[p] = G$ is p^h, where $h = \mathrm{height}(G)$. Now Fr is injective on $G^{\text{ét}}$, so the kernel of Fr in G is the same as that of Fr in the connected component G^o. Viewing G^o as a formal Lie group on $n = \dim(G)$ parameters, we see that the order of the $\ker \mathrm{Fr}$ is p^n. Since Fr and V are dual with respect to Cartier duality one sees that $\ker V$ is the Cartier dual of the cokernel of the map $\mathrm{Fr} : G' \to (G')^{(p)}$, and consequently $\ker V$ has order $p^{n'}$. The assertion now follows from the multiplicativity of orders in an exact sequence. □

We now fix R to be a complete discrete valuation ring, with perfect residue field $k = R/\mathfrak{m}$ of characteristic $p > 0$, and let $K = \mathrm{Frac}(R)$ of characteristic 0. Choose an algebraic (possibly infinite) extension of K and equip it with the unique valuation extending the one on R. Let L be the completion and S its valuation ring. We will be especially interested in the case when L is the completion of the algebraic closure $\overline{K}$. Thus S is a complete rank 1 valuation ring, but the valuation on S may not be discrete. Let G be a p-divisible group over R. We would like to define the S-points of G, in a sense that takes into account the $\mathfrak{m}$-adic topology on S. The following definitions are therefore natural

$$G(S/\mathfrak{m}^i S) := \varinjlim_{\nu} G_\nu(S/\mathfrak{m}^i S)$$

and

$$G(S) := \varprojlim_{i} G(S/\mathfrak{m}^i S).$$

Clearly each of the terms in the limits can be considered as a $\mathbb{Z}$-module and since p is killed by $\mathfrak{m}$, $G(S)$ is endowed with a natural $\mathbb{Z}_p$-action. One cannot interchange the limits as

$$0 \to G_\nu(S/\mathfrak{m}^i S) \to \varinjlim_{\omega} G_\omega(S/\mathfrak{m}^i S) \overset{[p^\nu]}{\to} \varinjlim_{\omega} G_\omega(S/\mathfrak{m}^i S)$$

is exact and leads to

$$0 \to \varprojlim_{i} G_\nu(S/\mathfrak{m}^i S) \to G(S) \overset{[p^\nu]}{\to} G(S)$$

and thus

$$G(S)_{\mathrm{tors}} = \varinjlim_{\nu} \varprojlim_{i} G_\nu(S/\mathfrak{m}^i S).$$

Remark 2.24. As the Hom-functor commutes with inverse limits, $G(S)$ is in fact the functor of points in the category of formal schemes. So if $\mathcal{G} = \varinjlim_\nu G_\nu$ is the formal group associated to G we have

$$G(S) = \mathrm{Hom}(\mathrm{Spf}(S), \mathcal{G}).$$

It is constructive to look at the étale and connected cases separately.

Example 2.25. For an étale p-divisible group G we have a unique lifting of points, and so the maps $G_\nu(S/\mathfrak{m}^{i+1}S) \to G_\nu(S/\mathfrak{m}^i S)$ are bijective, cf. [5], Chapter 1, Theorem 5.5. In particular $G_\nu(S/\mathfrak{m}^i S) \cong G_\nu(S/\mathfrak{m}_S)$, hence

$$G(S) = \varprojlim_{i} \varinjlim_{\nu} G_\nu(S/\mathfrak{m}^i S) = \varinjlim_{\nu} \varprojlim_{i} G_\nu(S/\mathfrak{m}^i S) = G(S)_{\mathrm{tors}}.$$

Example 2.26. For a connected p-divisible group G of dimension n and $\mathcal{G} = \mathrm{Spf}\, R[[X_1, ..., X_d]]$ the associated p-divisible formal Lie group we have

$$G(S) = \mathrm{Hom}(\mathrm{Spf}\, S, \mathcal{G}) = \mathrm{Hom}_{\mathrm{cont}}(R[[X_1, ..., X_d]], S).$$

As $(\mathfrak{m}_S^i)$ constitutes a basis of neighbourhoods around 0, each of the coordinates X_i must get sent to the maximal ideal and so it is evident that $G(S) = \mathfrak{m}_S^d$ as topological spaces.

Next we introduce the p-adic Tate module and following Tate, the Tate comodule. Throughout we fix $\Gamma_K := \mathrm{Gal}(\overline{K}, K)$.

Definition 2.27. *The p-adic Tate module of G is the $\mathbb{Z}_p[\Gamma_K]$-module*

$$T_p(G) := \varprojlim_{\nu} G_\nu(\overline{K})$$

with transfer maps induced by multiplication by $[p] = j_{1,\nu} : G_{\nu+1} \to G_\nu$. The p-adic Tate comodule of G is the $\mathbb{Z}_p[\Gamma_K]$-module

$$\Phi_p(G) := \varinjlim_{\nu} G_\nu(\overline{K})$$

with transfer maps induced by $i_\nu : G_\nu \hookrightarrow G_{\nu+1}$.

Remark 2.28. The maps $G_\nu(S) \to G_\nu(L)$ are bijective, so that if $L = \widehat{\overline{K}}$, $\Phi_p(G) = G(S)_{\mathrm{tors}}$.

Example 2.29. If $G = \mathbb{G}_m(p)$, then $\Phi_p(G)$ is the group of roots of unity of p power order in $\overline{K}$, i.e., μ_{p^∞}.

Locally as a functor of points

$$G_\nu = \mathrm{Hom}\left(\prod_{p^{\nu h}} R, -\right)$$

and so locally as G_1 is killed by p and G_i is identified with G_{i+1}/pG_{i+1} as Hopf algebras, we see that $G_\nu \cong (\mathbb{Z}/p\mathbb{Z})^h$. Thus as $\mathbb{Z}_p$-modules $T_p(G) = \mathbb{Z}_p^h$ and $\Phi_p(G) = (\mathbb{Q}_p/\mathbb{Z}_p)^h$, on which Γ_K acts continuously, cf. [7], Proposition 3.3.1, §3.3.

Lemma 2.30. *We have the following isomorphisms as galois modules:*

$$T_p(G) = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, \Phi_p(G)) \quad \textit{and} \quad \Phi_p(G) = T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$$

Proof. We use the fact that the tensor product commutes with direct limits:

$$\begin{aligned} T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p &= T_p(G) \otimes_{\mathbb{Z}_p} \varinjlim_{\nu} \mathbb{Z}/p^\nu\mathbb{Z} \\ &= \varinjlim_{\nu} T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^\nu\mathbb{Z} \\ &= \varinjlim_{\nu} T_p(G)/p^\nu T_p(G) \\ &= \varinjlim_{\nu} G_\nu(\overline{K}) \\ &= \Phi_p(G). \end{aligned}$$

For the other equality we have:

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, \Phi_p(G)) &= \mathrm{Hom}_{\mathbb{Z}_p}(\varinjlim_{\nu} \mathbb{Z}/p^{\nu}\mathbb{Z}, \Phi_p(G)) \\
&= \varprojlim_{\nu} \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}/p^{\nu}\mathbb{Z}, \Phi_p(G)) \\
&= \varprojlim_{\nu} G_{\nu}(\overline{K}) \\
&= T_p(G).
\end{aligned}$$

□

Remark 2.31. Knowledge of $T_p(G)$ or $\Phi_p(G)$ determines the other.

We next introduce the logarithm map of G, recalling first the definition of the tangent space.

Definition 2.32. *Let $G = (G_\nu)$ be a p-divisible group over R and $A^o = R[[X_1, ..., X_d]]$ with augmentation ideal $I = (X_1, ..., X_d)$ be the R-algebra representing the p-divisible formal Lie group associated to G^o (the maximal connected component of G). The tangent space of G at* 0 *wih values in an R-module M is the R-module of continuous R-derivations:*

$$t_G(M) := \mathrm{Der}_R(A^o, M) = \mathrm{Hom}_R(I/I^2, M).$$

Remark 2.33. We will see that this is the right definition of the tangent space soon, but it already looks promising as it takes into account the formal completion of the neighbourhood around 0. In particular $t_G(L) = \mathrm{Der}_R(A^o, L) = \mathrm{Hom}_R(I/I^2, L)$ is an L-vector space of dimension $d = \dim(G)$.

By taking the dual, the cotangent space of G at 0 with values in an R-module M is

$$t_G^*(M) = I/I^2 \otimes_R M$$

and in particular $t_G^*(L) = \mathrm{Hom}_L(t_G(L), L)$. We need a preliminary result:

Lemma 2.34. *Let G be a finite group scheme. There is an exact sequence of finite group schemes*

$$0 \to G^o \to G \to G^{\text{ét}} \to 0$$

with G^o, the connected component of G and $G^{\text{ét}}$ finite étale over R.

Proof. We sketch a proof. Topologically we take G^o to be the maximal connected component of G, cf. [12], §6.7. Next to construct the étale component, we first work over a field k. Let A be the Hopf algebra associated to G. There is a unique maximal étale k-subalgebra $A^{\text{ét}}$, in the sense that any étale k-algebra equipped with a map to A uniquely factors through $A^{\text{ét}}$. To construct $A^{\text{ét}}$, we note that A can be written as a finite product of local factor rings. Uniquely lift the separable closure of k in the residue field up into each local factor ring via Hensel's lemma. In geometric terms, for a finite group k-scheme G, we have constructed a finite étale group k-scheme $G^{\text{ét}}$ and a map $f : G \to G^{\text{ét}}$. We relax the field condition and work over R with residue field k. So let G be a finite group R-scheme. By Hensel's lemma, to give a map from a finite étale R-algebra A to a finite R-algebra B is the same as to give a map $A_s \to B_s$ between their special fibers. In particular, finite étale k-algebras uniquely and functorially lift to finite étale R-algebras, and so $G_s^{\text{ét}}$ uniquely lifts to a finite étale R-scheme $G^{\text{ét}}$ and there is a unique R-map $f : G \to G^{\text{ét}}$ lifting $f_s : G_s \to G_s^{\text{ét}}$. f is faithfully flat since f_s was. We call $G^{\text{ét}}$ the maximal étale quotient of G. Now we can put it all together and obtain the connected-étale sequence of G. We have to show that the faithfully flat R-homomorphism $f : G \to G^{\text{ét}}$ to the maximal étale quotient has scheme-theoretic kernel G^o. The kernel $H = \ker f$ is a finite flat R-group. To show $G^o \subseteq H$ we have to check that the composite map $G^o \to G \to G^{\text{ét}}$ vanishes. Being a map from a finite R-scheme to a finite étale R-scheme, the map is determined by what it does on the special fiber, so it suffices to show $G_k^o \to G_k^{\text{ét}}$ vanishes. This is a map from a finite k-scheme to a finite étale k-scheme which carries the unique k-point to the identity point. Thus it factors through the identity section of $G_k^{\text{ét}}$. Now that $G^o \subseteq H$, to prove the resulting closed immersion $G^o \hookrightarrow H$ between finite flat R-schemes is an isomorphism it suffices to do so on special fibers. So we work over the residue field. We can increase it to be algebraically closed and so the problem is to show that if G is a finite flat group scheme over an algebraically closed field k then $G \to G^{\text{ét}}$ has kernel exactly G^o. But $G^{\text{ét}}$ is a constant k-scheme since it is étale and k is algebraically closed, so by construction $G^{\text{ét}}$ is just the disjoint union of the k-points of the connected components of G. The result now follows. □

Corollary 2.35. *If $G = (G_\nu)$ is now any p-divisible group, we have an exact sequence of p-divisible groups*

$$0 \to G^o \to G \to G^{\text{ét}} \to 0$$

with G^o, the connected component of G and $G^{\text{ét}}$ finite étale over R.

Proof. From the previous lemma we have the exact sequences

$$0 \to G_\nu^o \to G_\nu \to G_\nu^{\text{ét}} \to 0$$

and noting that $\varinjlim$ is an exact functor on a filtered index set, one arrives at the result. □

Proposition 2.36. *Let G be a p-divisible group. The map $G \to G^{\text{ét}}$ has a formal section, i.e., the associated map $\mathcal{G} \to \mathcal{G}^{\text{ét}}$ between formal groups admits a section as maps of formal schemes. Consequently, the sequence*

$$0 \to G^o(S) \to G(S) \to G^{\text{ét}}(S) \to 0$$

is exact.

Proof. Let A, A^o and $A^{\text{ét}}$ represent the algebras of the formal groups $\mathcal{G}$, $\mathcal{G}^o$ and $\mathcal{G}^{\text{ét}}$. Then

$$A^o = R[[X_1, ..., X_d]]$$

and

$$A \otimes_R k = (A^{\text{ét}} \otimes_R k)[[X_1, ..., X_d]]$$

since over k (perfect), reduced algebras are separable and so $G^{\text{ét}} \cong G_{\text{red}}$ and hence the sequence

$$0 \to G^o \to G \to G^{\text{ét}} \to 0$$

splits canonically module $\mathfrak{m}$. Lifting the variables by Nakayama, we obtain

$$A = A^{\text{ét}} \widehat{\otimes}_R A^o,$$

and so the sequence splits. □

Corollary 2.37. *If L is algebraically closed, then $G(S)$ is a divisible group.*

Proof. From the proposition it suffices to consider the connected and étale cases separately. For $G^{\text{ét}}(S) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^h$, as it's a constant group. For G^o, $G^o \cong \mathfrak{m}_S^d$ and the map $[p]$ is faithfully flat on A^o. Since L is algebraically closed the fibers are nonempty. □

By Example 2.24, $G^{\text{ét}}(S)$ is torsion and by Example 2.25, $G^o(S)$ can be identified with the set $\mathfrak{m}_S^d$. The formal group law on A^o, induces a group law on $\mathfrak{m}_S^d$, making it an analytic group. We can now multiply an $x \in G^o(S)$ by p^i in the sense of this law and we obtain another element $[p^i]x \in G^o(S)$. For a function $f \in A^o$, converges on all elements of absolute value < 1. We can thus evaluate f at $[p^i]x$. For general $x \in G(S)$, it is torsion in $G^{\text{ét}}(S)$, so for i big enough $[p^i]x \in G^o(S)$.

Definition 2.38. *The logarithm of G is the map*

$$\log_G : G(S) \to t_G(L)$$

defined as follows:

$$(\log_G(x))\,(f) = \lim_{i\to\infty} \left(\frac{f([p^i]x) - f(0)}{p^i} \right)$$

for $x \in G(S)$ and $f \in A^o$.

Remark 2.39. From the above discussion, the fraction in the limit is well defined. We just need to check that the limit exists. The filtration $F^\delta G^o(S) = \mathfrak{m}_{S,\delta}^d$ with

$$\mathfrak{m}_{S,\delta} = \{x \in S :\ \nu(x) \geq \delta\}$$

has the property

$$[p]\left(F^\delta G^o(S)\right) \subseteq F^{\delta + \min\{\delta, \nu(p)\}} G^o(S),$$

because

$$[p](X_i) = pX_i + \text{higher order terms}.$$

The sequence $\left([p^i]x\right)_i$ converges to 0, so we can assume that $x \in F^\delta G^o(S)$ for some $\delta \gg 0$. In particular we choose $\delta \geq \nu(p) = \varepsilon$, so that $[p^i]x \in F^{\delta+i\varepsilon}G^o(S)$. Putting it all together we conclude

$$\begin{aligned} \nu\left(\frac{f([p^{i+1}]x) - f(0)}{p^{i+1}} - \frac{f([p^i]x) - f(0)}{p^i}\right) &\geq -(i+1)\varepsilon + 2(\delta + i\varepsilon) \\ &= 2\delta + (i-1)\varepsilon. \end{aligned}$$

The estimate shows that the $\frac{f([p^i]x)-f(0)}{p^i}$ form a Cauchy sequence in L and by completeness the limit exists.

For the sake of brevity we write $I = I^o$ unless otherwise stated. By remark 2.30, the map $f \mapsto (\log_G(x))(f)$ is R-linear, vanishes on I^2 and we have the formula

$$(\log_G)(x)(fg) = f(0)(\log_G)(x)(g) + g(0)(\log_G)(x)(f).$$

We also have

$$f(x \cdot_G y) = f(x) + f(y) + \text{higher order terms}$$

and so

$$\log_G(x \cdot_G y) = \log_G(x) + \log_G(y).$$

Hence $\log_G$ is $\mathbb{Z}_p$-linear.

Lemma 2.40. *The logarithm $\log_G : G(S) \to t_G(L)$ is a local homeomorphism. More precisely, for each $\delta > \frac{v(p)}{p-1}$ in $\mathbb{Q}$ we have a homeomorphism*

$$\log_G : F^\delta G(S) \xrightarrow{\sim} \{\tau \in t_G(L) :\ \nu(\tau(X_i)) \geq \delta,\ \forall i \in [d]\}.$$

Proof. Identifying $G^o(S)$ with $\mathfrak{m}_S^d$ it is easy to check that the inverse is given by $\tau \mapsto (\tau(X_1), ..., \tau(X_d))$ The map is linear in $\mathbb{Z}_p$, so it is continuous as well. □

Corollary 2.41. *The kernel of $\log_G$ is the torsion subgroup $G(S)_{\text{tors}}$ of $G(S)$ and the logarithm induces an isomorphism*

$$\log_G : G(S) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} t_G(L).$$

Proof. As $t_G(L)$ is torsion free,

$$G(S)_{\text{tors}} \subseteq \ker(\log_G).$$

To show the other inclusion, note that x and $[p^i]x$ have the same image. Choosing i large enough we can assume $[p^i]x$ lies in the domain where $\log_G$ is a local homeomorphism and so

$$[p^i]x = 0$$

. Thus

$$\ker(\log_G) \subseteq G(S)_{\text{tors}}.$$

For the last part it suffices to note that

$$\{\tau \in t_G(L) :\ \nu(\tau(X_i)) \geq \delta, \forall i \in [d]\} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong t_G(L).$$

□

In Corollary 2.36, we saw that if L is algebraically closed then $G(S)$ is divisible. In that case $\log(G(S))$ is also divisible and so $\log(G(S)) \cong t_G(L)$.

Example 2.42. If $G = \mathbb{G}_m(p)$, then $G(S)$ is the group of units congruent to 1 in S. It's of dimension 1, so $t_G(L)$ can be identified with just L and the logarithm map becomes the ordinary p-adic logarithm map.

The field of p-adic complex numbers is the field

$$\mathbb{C}_K = \widehat{\overline{K}},$$

the p-adic completion of $\overline{K}$, which by Krasner's Lemma is again algebraically closed. We denote the ring of integers by D, which is the p-adic completion of $\overline{R}$. The galois group Γ_K acts on $\overline{K}$ fixing $\overline{R}$ so that the action extends uniquely to a continuous action of Γ_K on $\mathbb{C}_K$ that fixes D. We have the exact sequence of Γ_K-modules

$$0 \to \Phi(G) \to G(D) \overset{\log_G}{\to} t_G(\mathbb{C}_K) \to 0.$$

By Examples 2.28 and 2.41, in the case of $\mathbb{G}_m(p)$, the exact sequence is

$$0 \to \mu_{p^\infty} \to U \to \mathbb{C}_K \to 0,$$

where $U = 1 + \mathfrak{m}_D$.

Proposition 2.43. *We have a natural isomorphism of $\mathbb{Z}_p[\Gamma_K]$-modules*

$$T_p(G') \xrightarrow{\sim} \mathrm{Hom}_D(G\widehat{\otimes}_R D, \mathbb{G}_m(p))$$

Proof. Recall from an earlier discussion that the Cartier dual is given by

$$G'_\nu(D) = \mathrm{Hom}_D(G_\nu \otimes_R D, \mathbb{G}_m).$$

Since G_ν is killed by p^ν, we get naturally

$$G'_\nu(D) = \mathrm{Hom}_D(G_\nu \otimes_R D, \mathbb{G}_m[p^\nu]).$$

The multiplication by p map $G'_{\nu+1} \to G'_\nu$ is dual to the inclusion $i_\nu : G_\nu \to G_{\nu+1}$. In other words, in the diagram,

$$\begin{array}{ccc} G'_{\nu+1}(D) & \xrightarrow{\sim} & \mathrm{Hom}_D(G_{\nu+1} \otimes_R D, \mathbb{G}_m) \\ \downarrow p & & \downarrow h \\ G'_\nu(D) & \xrightarrow{\sim} & \mathrm{Hom}_D(G_\nu \otimes_R D, \mathbb{G}_m) \end{array}$$

the map h is composition with $i_\nu : G_\nu \to G_{\nu+1}$.

Thus a system of elements $x_\nu \in G'_\nu(D)$ such that $px_{\nu+1} = x_\nu$ is equivalent to a sequence of the maps $\phi_\nu \in \mathrm{Hom}_D(G_\nu \otimes_R D, \mathbb{G}_m) = \mathrm{Hom}_D(G_\nu \otimes_R D, \mathbb{G}_m[p^\nu])$ such that $\phi_{\nu+1} \circ i_\nu = \phi_\nu$. The (ϕ_ν) is exactly a homomorphism from G_D to $\mathbb{G}_m(p)$. □

Our aim is to use Proposition 2.42 to find a previously unknown structure on $T_p(G')$. It is useful to rewrite the above result in terms of pairings. We obtain

$$T_p(G') \times G(D) \to (\mathbb{G}_m(p)(D)) \cong U$$

and

$$T_p(G') \times t_G(\mathbb{C}_K) \to t_{\mathbb{G}_m(p)}(\mathbb{C}_K) \cong \mathbb{C}_K.$$

These pairings are compatible with the logarithm map $\log_G : G(D) \to t_G(\mathbb{C}_K)$ and the ordinary p-adic logarithm $U \to \mathbb{C}_K$. Note also that all the pairings are Γ_K-equivariant. We discussed earlier that the kernels of these logarithm maps are respectively $\Phi_p(G)$ and μ_{p^∞}. Thus we get an exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Phi_p(G) & \longrightarrow & G(D) & \xrightarrow{\log_G} & t_G(\mathbb{C}_K) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle d\alpha} & & \\
0 & \to & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G'), \mu_{p^\infty}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G'), U) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G'), \mathbb{C}_K) & \to & 0
\end{array}$$

The vertical rows are Γ_K-equivariant and keep in mind that Γ_K acts on $\mathrm{Hom}(M,N)$ by $(\sigma f)(m) = \sigma(f(\sigma^{-1}m))$. The commutativity of the right square comes from the functorial nature of $\log_G$.

It turns out that there is a fully faithful functor $G \to T_p(G)$ from the category of p-divisible groups over R to the category of Tate-modules (with Galois action). Precisely we have $\mathrm{Hom}(G,H) \hookrightarrow \mathrm{Hom}(T_p(G), T_p(H))$. We will not prove this though, however we will do enough to obtain an appropriate Hodge-Tate representation. This will aide us in reading of the dimension of our p-divisible group G from its Galois representation $T_p(G)$. The key will be understanding the map $d\alpha$, which will turn out to be injective. We will also make use of the following surprising fact:

$$(\mathbb{C}_K)^{\Gamma_K} = K. \qquad (2)$$

Lemma 2.44. *We have $t_G(\mathbb{C}_K)^{\Gamma_K} = t_G(K)$. Furthermore let W be a vector space over $\mathbb{C}_K$ on which Γ_K acts semi-linearly (that is $\sigma(cw) = \sigma(c)\sigma(w)$), then*

$$W^{\Gamma_K} \otimes_K \mathbb{C}_K \to W$$

is injective.

Proof. We have $t_G(K) \cong K^{\dim(G)}$ and $t_G(\mathbb{C}_K) \cong \mathbb{C}_K^{\dim(G)}$ and so the result follows from (2). For the second part, it suffices to show that any $w_i \in W^{\Gamma_K}$ of K-linearly independent vectors remain linearly independent over $\mathbb{C}_K$ in W. We argue by contradication and suppose not. Choose the minimal length relation $\Sigma c_i w_i = 0$ and scale so that $c_1 = 1$. For any $\sigma \in \Gamma_K$, we have

$$\Sigma(\sigma(c_i) - c_i)w_i = 0.$$

But since $c_1 = 1$, we have $\sigma(c_1) - c_1 = 0$ so we have a shorter relation and thus $\sigma(c_i) = c_i$ for all σ. Using (2), we get $c_i \in K$ contradicting K-independence of the w_i. □

Proposition 2.45. *α_0 is bijective; α and $d\alpha$ are injective.*

Proof. We follow Tate's original proof. The proof will be in a series of steps.

Step 1 (*α_0 is bijective*). At the finite level, we have a perfect duality thanks to Cartier duality

$$G_\nu(\mathbb{C}_K) \times G'_\nu(\mathbb{C}_K) \to \mu_{p^\infty}$$

for each ν. The result follows on passage to the limit as $\nu \to \infty$, inductively with the G_ν and projective with the G'_ν. Incidentally, if we pass to the limit projectively with both, we find a Γ_K-isomorphism

$$T_p(G) \cong \mathrm{Hom}(T_p(G'), \mathbb{Z}_p(1)).$$

Step 2 ($\ker\alpha$ *and* $\operatorname{coker}\alpha$ *are vector spaces over* $\mathbb{Q}_p$). A priori, they are only $\mathbb{Z}_p$-modules since the vertical arrows are only $\mathbb{Z}_p$-homomorphisms. However by the previous step and the snake lemma, the kernel and cokernel of α are isomorphic to the kernel and cokernel of $d\alpha$. On the other hand $d\alpha$ is a $\mathbb{Q}_p$-linear map and the result follows.

Step 3 (*We have $G(R) = G(D)^{\Gamma_K}$*). By corollary 2.40 we have $G(D) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong t_G(\mathbb{C}_K)$. Taking Γ_K-invariants and using the previous Lemma gives the result.

Step 4 (*α is injective on $G(R)$*). Indeed the kernel of the restriction of α to $G(R)$ is $(\ker\alpha)^{\Gamma_K}$ by **Step 3** and is therefore uniquely divisible by p, by **Step 2**. If G is connected we have that

$$\bigcap_{\nu \geq 1} p^\nu G(R) = 0$$

because G can be viewed as a formal group and multiplication by p looks like

$$pX + \mathcal{O}(X^2).$$

Thus if G is connected it follows that $\ker(\alpha) \cap G(R) = 0$. Using the fact that the commutative diagram is functorial with respect to $G \mapsto G^o$ and the fact that $T_p(G') \to T_p((G^o)')$ is surjective we see that in general

$$\ker(\alpha) \cap G^o(R) = 0.$$

Finally since $\ker\alpha$ is torsion-free and $G(R)/G^o(R)$ is a torsion group, it follows that

$$\ker(\alpha)\cap G(R)=0$$

as claimed.

Step 5 (*The map $d\alpha$ is injective on $t_G(K)$*). From **Steps 1-4**, we conclude $d\alpha$ is injective on $\log_G(G(R))$ which spans $t_G(K)$ as a $\mathbb{Q}_p$-vector space.

Step 6 (*The map $d\alpha$ is injective*). The arrow $d\alpha$ can be factored as follows

$$t_G(\mathbb{C}_K)\cong t_G(K)\otimes_K\mathbb{C}_K\to \mathrm{Hom}(T_p(G'),\mathbb{C}_K)^{\Gamma_K}\otimes_K\mathbb{C}_K\to \mathrm{Hom}(T_p(G'),\mathbb{C}_K).$$

The first arrow is injective by **Step 5**. The second arrow is injective by the previous Lemma.

Step 7 (*The map α is injective*). This follows from **Step 2** and **Step 6**. □

Theorem 2.46. *The maps*

$$\alpha_R: G(R)\to \mathrm{Hom}_{\Gamma_K}(T_p(G'),U)$$

and

$$d\alpha_R: t_G(K)\to \mathrm{Hom}_{\Gamma_K}(T_p(G'),\mathbb{C}_K)$$

induced by α and $d\alpha$ are bijective.

Proof. We already know that α_R and $d\alpha_R$ are bijective from the previous Proposition. Taking Γ_K-invariants of the exact sequence

$$0\to G(D)\to \mathrm{Hom}(T_p(G'),U)\to \operatorname{coker}\alpha\to 0$$

yields a left-exact sequence

$$0\to G(R)\to \mathrm{Hom}_{\Gamma_K}(T_p(G'),U)\to (\operatorname{coker}\alpha)^{\Gamma_K}.$$

Exactness in the middle tells us that

$$\operatorname{coker}(\alpha_R)\hookrightarrow \operatorname{coker}(\alpha)^{\Gamma_K}.$$

Similarly we obtain

$$\operatorname{coker}(d\alpha_R)\hookrightarrow \operatorname{coker}(d\alpha)^{\Gamma_K}.$$

Since $\operatorname{coker}\alpha\to\operatorname{coker}d\alpha$ is bijective, it follows that the map

$$\operatorname{coker}\alpha_R\to\operatorname{coker}d\alpha_R$$

is injective. Thus it suffices to show that $d\alpha_R$ is surjective. Since $d\alpha_R$ is K-linear and injective, we are reduced to counting dimensions. We define now a Γ_K-semilinear pairing on $\mathbb{C}_K$-vector spaces. We set

$$W=\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G),\mathbb{C}_K)$$

and

$$W'=\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G'),\mathbb{C}_K).$$

Note that W and W' are spaces of dimension the height of G $(= h)$ and so in particular they are finite dimensional. As Galois modules, we saw that

$$T_p(G) = \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G'), \mathbb{Z}_p(1)) = \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G'), \mathbb{Z}_p)(1)$$

and so after twisting by $\mathbb{Z}_p(-1)$ and tensoring by $\mathbb{C}_K$ we obtain

$$W' = T_p(G) \otimes \mathbb{C}_K(-1).$$

Using the relation $\mathrm{Hom}_{\mathbb{C}_K}(W, \mathbb{C}_K(-1)) = W^* \otimes_{\mathbb{C}_K} \mathbb{C}_K(-1)$, the non-degnerate pairing

$$W \times W' \to \mathbb{C}_K(-1)$$

is immediate. Fixing notation, set $d = \dim_K W^{\Gamma_K}$, $d' = \dim_K (W')^{\Gamma_K}$, $n = \dim G$ and $n' = \dim G'$. By injectivity of $d\alpha_R$ we already know that $n \leq d'$ and $n' \leq d$, and we wish to show that inequality holds. By Theorem 2.23, $n + n' = h$, so it suffices to show that $d + d' \leq h$. Taking Γ_K-invariants we obtain a pairing

$$W^{\Gamma_K} \times (W')^{\Gamma_K} \to \mathbb{C}_{\mathbb{K}}(-1)^{\Gamma_K}.$$

By the Tate-Sen Theorem $\mathbb{C}_{\mathbb{K}}(-1)^{\Gamma_K} = 0$. It follows by Lemma 2.44, that $W^{\Gamma_K} \otimes_K \mathbb{C}_K$ and $(W')^{\Gamma_K} \otimes_K \mathbb{C}_K$ are orthogonal $\mathbb{C}_K$-subspaces of W and W' respectively. Hence $d + d' \leq h = \dim_{\mathbb{C}_K} W$, as desired. □

Corollary 2.47. *The dimension of G is given in terms of the Tate-module by*

$$\dim G = \dim_K(\mathrm{Hom}_{\Gamma_K}(T_p(G'), \mathbb{C}_K)) = \dim_K(T_p(G) \otimes \mathbb{C}_K(-1))^{\Gamma_K}.$$

Finally we are in a position to address the Hode-Tate decomposition associated to a p-divisible group G.

Corollary 2.48. *There is a canonical isomorphism of $\mathbb{C}_K[\Gamma_K]$-modules*

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K) \cong t_{G'}(\mathbb{C}_K) \oplus t_G^*(\mathbb{C}_K)(-1).$$

Proof. The proof of Theorem 2.46, shows that under the pairing $W \times W' \to \mathbb{C}_K(-1)$, the subspaces $t_G(\mathbb{C}_K)$ and $t_{G'}(\mathbb{C}_K)$ were orthogonal complements. Thus we obtain the exact sequence

$$0 \to t_{G'}(\mathbb{C}_K) \xrightarrow{d\alpha'} \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K) \to \mathrm{Hom}_{\mathbb{C}_K}(t_G(\mathbb{C}_K), \mathbb{C}_K(-1)) \to 0$$

where $\mathrm{Hom}_{\mathbb{C}_K}(t_G(\mathbb{C}_K), \mathbb{C}_K(-1)) = t_G^*(\mathbb{C}_K)(-1)$. The exact sequence is of the form

$$0 \to \mathbb{C}_K^{\dim G'} \to W \to \mathbb{C}_K(-1)^{\dim G} \to 0$$

and the existence of splitting follows from $H^1(\Gamma_K, \mathbb{C}_K(1)) = 0$ and uniqueness from $H^0(\Gamma_K, \mathbb{C}_K(1)) = 0$. □

Now let A be an abelian variety such that A/K has good reduction over R. The p-adic Tate module $T_p(A)$ is the p-adic Tate module for the p-divisible group $A(p)$ over R. Hence the Γ_K-representation

$$W = \mathrm{Hom}_{\mathbb{Z}_p}(T_p(A), \mathbb{C}_K) = H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K$$

has a Hodge-Tate decomposition

$$W = (t^*_G(\mathbb{C}_K) \otimes_K \mathbb{C}_K(-1)) \oplus (t_{G'}(\mathbb{C}_K) \otimes_K \mathbb{C}_K).$$

The space $t^*_G(\mathbb{C}_K)$ can be identified with the space of A-invariant differentials on A, or $H^0(A, \Omega^1_{A/K})$, whereas the space $t_{G'}(\mathbb{C}_K)$ is the tangent space at 0 for the dual abelian variety which by deformation theory equals $H^1(A, \mathcal{O}_A)$. Putting all the results together, finally leads to a Hodge-Tate decomposition for an abelian scheme A

$$H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K = (H^0(A, \Omega^1_{A/K}) \otimes_K \mathbb{C}_K(-1)) \oplus (H^1(A, \mathcal{O}_A) \otimes_K \mathbb{C}_K).$$

We can do better and note that the above result holds for X/K a smooth, proper and geometrically connected variety thanks to the Albanese map. More generally, we know now by the work of Fontaine, the following result conjectured by Tate.

Theorem 2.49. *Let X/K be a smooth, proper and geometrically connected variety. Then the Γ_K-representations $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ of p-adic étale cohomology are Hodge-Tate and thus*

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K = \bigoplus_{a+b=n} H^a(X, \Omega^b_{X/K}) \otimes_K \mathbb{C}_K(-b).$$

Tate in fact proved the above theorem, as a consequence of his work on p-divisible groups, for abelian varieties A/K of good reduction as

$$H^*(A_{\overline{K}}, \mathbb{Q}_p) = \bigwedge H^1(A_{\overline{K}}, \mathbb{Q}_p).$$

3 Hodge-Tate Representations

In this section we introduce the theory and method of approach, as given by Fontaine, in order to approach questions such as Theorem 2.49. The theory of Hodge-Tate representations was inspired by Tate's study of $T_p(A)$ for abelian varieties A/K with good reductions over p-adic fields and in particular by Tate's question as to how the p-adic representation $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$ arising from a smooth proper K-scheme X is related to the Hodge cohomology $\oplus_{p+q=n} H^p(X, \Omega^q_{X/K})$. This question concerns finding a p-adic analogue of the classical Hodge decomposition

$$H^n(Z(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{p+q=n} H^p(Z, \Omega^q_Z)$$

for a smooth proper $\mathbb{C}$-scheme Z. It was only around 15 years later until a rigorous approach was developed. We mainly follow the material as given in [4].

Definition 3.1. *A p-adic representation of a profinite group* Γ *is a representation* $\rho : \Gamma \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$ *of* Γ *on a finite-dimensional* $\mathbb{Q}_p$*-vector space* V *such that* ρ *is continuous (viewing* $\mathrm{Aut}_{\mathbb{Q}_p}(V)$ *as* $\mathrm{GL}_n(\mathbb{Q}_p)$ *upon choosing a basis of* V*). The category of such representations is denoted* $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma)$*.*

For the rest of this section we assume K is a complete local field of characteristic 0 (for example a finite extension of $\mathbb{Q}_p$) of which the residue field k is perfect of characteristic $p > 0$. We fix Γ_K to be the absolute Galois group of K and $\mathbb{C}_K$ to be the completion of the algebraic closure of K.

The goal is to identify and study various good classes of p-adic representations of Γ_K, especially motivated by properties of p-adic representations arising from algebraic geometry over p-adic fields. More often than not, we get agreement at the level of cohomology. The form that this study takes place is to provide faithful functors from these nice classes of p-adic representations to various categories of semilinear algebraic objects. It is often easier to work in terms of semilinear algebra than with Galois representations.

The theory of Hodge-Tate representations will involve studying the Γ_K-action on $V \otimes_{\mathbb{Q}_p} \mathbb{C}_K$ for a p-adic representation V of Γ_K (where the action $\sigma(c \otimes v) = \sigma(c) \otimes \sigma(v)$ is semilinear). This motivates the following definition.

Definition 3.2. *A* $\mathbb{C}_K$*-representation of* Γ_K *is a finite-dimensional* $\mathbb{C}_K$*-vector space* W *equipped with a continuous* Γ_K*-action map* $\Gamma_K \times W \to W$ *that is semilinear (i.e.,* $\sigma(cw) = \sigma(c)\sigma(w)$ *for all* $c \in \mathbb{C}_K$ *and* $w \in W$*). The category of such objects is denoted* $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$*.*

We now state a fundamental theorem of Tate, without proof cf. [11], §3.3.

Theorem 3.3 (Tate-Sen)**.** *We have canonical isomorphisms (where the cohomology groups are continuous)*

$$H^i(\Gamma_K, \mathbb{C}_K(j)) = \begin{cases} 0 & \text{for } i \geq 2 \text{ or } j \neq 0, \\ K & \text{for } i = 0, 1 \text{ and } j = 0. \end{cases}$$

More generally, if $\eta : \Gamma_K \to \mathcal{O}_K^\times$ *is a continuous character such that* $\eta(\Gamma_K)$ *is either finite or contains* $\mathbb{Z}_p$ *as an open subgroup, then* $H^i(\Gamma_K, \mathbb{C}_K(\eta)) = 0$ *for* $i = 0, 1$ *when* $\eta(I_K)$ *is infinite and these cohomologies are 1-dimensional over* K *when* $\eta(I_K)$ *is finite.*

For $W \in \mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$ and $q \in \mathbb{Z}$, consider the K-vector space

$$W\{q\} := W(q)^{\Gamma_K} \cong \{w \in W | g(w) = \chi(g)^{-q} w \quad \text{for all}\, g \in \Gamma_K\},$$

where the isomorphism depends on the choice of basis of $\mathbb{Z}_p(1)$. In particular, this isomorphism is not canonical when $q \neq 0$ and $W\{q\} \neq 0$, so $W\{q\}$ is canonically a K-subspace of $W(q)$ but is only non-canonically a K-subspace of W when $q \neq 0$ and $W\{q\} \neq 0$. More importantly, $W\{q\}$ is not a $\mathbb{C}_K$-subspace of $W(q)$ when it is nonzero.

We have a natural $K[\Gamma_K]$-equivariant multiplication map

$$K(-q) \otimes_K W\{q\} \hookrightarrow K(-q) \otimes_K W(q) \cong W,$$

so extending scalars defines maps

$$\mathbb{C}_K(-q) \otimes_K W\{q\} \to W$$

in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$ for all $q \in \mathbb{Z}$.

We are now in a position to state a certain lemma of Serre and Tate that gets p-adic Hodge Theory off the ground. It generalizes Lemma 2.44.

Lemma 3.4 (Serre-Tate)**.** *For $W \in \mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$, the natural $\mathbb{C}_K[\Gamma_K]$-equivariant map*

$$\xi_W : \bigoplus_q (\mathbb{C}_K(-q) \otimes_K W\{q\}) \to W$$

is injective. In particular, $W\{q\} = 0$ for all but finitely many q and $\dim_K(W\{q\}) < \infty$ for all q, with

$$\sum_q \dim_K W\{q\} \leq \dim_{\mathbb{C}_K}(W);$$

equality holds here if and only if ξ_W is an isomorphism.

Proof. We proceed by contradiction. Choose a nonzero $v = (v_q)_q \in \ker \xi_W$ of minimal length. Choose $q_0 \in \mathbb{Z}$, such that v_{q_0} is nonzero. By applying a $\mathbb{C}_K^*$-scaling we can arrange a minimal-length expression

$$v_{q_0} = \sum_j c_j \otimes y_j$$

with $c_j \in \mathbb{C}_K^*$, $y_j \in W\{q_0\}$ and $c_{j_0} = 1$. For $\sigma \in \Gamma_K$, $\sigma(v) \in \ker \xi_W$ and hence $\sigma(v) - \chi(g)^{-q_0}(v) \in \ker \xi_W$. For each $q \in \mathbb{Z}$, the qth component of $\sigma(v) - \chi(g)^{-q_0}(v)$ is $\sigma(v_q) - \chi(g)^{-q_0}(v_q)$. If $\Sigma c_{j,q} \otimes y_{j,q}$ is a minimal-length expression for v_q then since

$$\sigma(v_q) - \chi(\sigma)^{-q_0}(v_q) = \sum (\chi(\sigma)^{-q}\sigma(c_{j,q}) - \chi(\sigma)^{-q_0}c_{j,q}) \otimes y_{j,q}.$$

By Theorem 3.3, for v_{q_0}, this has strictly smaller length because $c_{j_0} = 1$. So the minimal length of v is 1, so $v = c \otimes w$ for some $c \in \mathbb{C}_K^*$ and nonzero $w \in W\{q_0\}$. But then $\xi_W(v) = cw \neq 0$. □

Definition 3.5. *A representation $W \in \mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$ is Hodge-Tate if ξ_W is an isomorphism. We say that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is Hodge-Tate if $\mathbb{C}_K \otimes_{\mathbb{Q}_p} V \in \mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$ is Hodge-Tate.*

If W is Hodge-Tate, then we have $W \cong \oplus \mathbb{C}_K(-q)^{h_q}$ in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$ with $h_q = \dim_K W\{q\}$. Conversely suppose $W \in \mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$ admits a finite direct sum decomposition

$$W \cong \bigoplus_q \mathbb{C}_K(-q)^{h_q}$$

with $h_q \geq 0$ for all q and $h_q = 0$ for all but finitely many q. Theorem 3.3 gives that $W\{q\}$ has dimension h_q for all q, so that W is in fact Hodge-Tate. Hence W is Hodge-Tate if and only if it decomposes into a sum of Tate twists.

Definition 3.6. *For any Hodge-Tate object W in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$ we define the Hodge-Tate weights of W to be those $q \in \mathbb{Z}$ such that $W\{q\}$ is nonzero, and then we call $h_q := \dim_K W\{q\} \geq 1$ the multiplicity of q as a Hodge-Tate weight of W.*

Remark 3.7. $\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K)$ from Corollary 2.48 is Hodge-Tate with Hodge-Tate weights 0 and 1.

We next define the categories Gr_K and $\mathrm{Gr}_{K,f}$.

Definition 3.8. *A $\mathbb{Z}$-graded vector space over a field K is a K-vector space*

$$D = \bigoplus_{q \in \mathbb{Z}} D_q$$

for K-subspaces $D_q \subseteq D$ (and we define the qth graded piece of D to be $\mathrm{gr}^q(D) := D_q$). Morphisms between graded K-vector spaces are K-linear maps that preserve the grading. We denote this category by Gr_K. We denote also by $\mathrm{Gr}_{K,f}$ the full subcategory of Gr_K of finite dimensional K-vector spaces.

Remark 3.9. Gr_K is an abelian category with evident notions of kernel, cokernel and exact sequences.

We write $K\langle r\rangle$ for $r \in \mathbb{Z}$ to denote the K-vector space K endowed with the grading for which the unique non-vanishing graded piece is in degree r. If V, $W \in \mathrm{Gr}_K$, then $V \otimes_K W \in \mathrm{Gr}_K$, with

$$\mathrm{gr}^i(V \otimes_K W) = \bigoplus_{p+q=i} \mathrm{gr}^p V \otimes_K \mathrm{gr}^q W.$$

Similarly if $D \in \mathrm{Gr}_{K,f}$, then the dual D^* has its qth graded piece D^*_{-q}.

We now define the first period ring, whose importance will only be appreciated after some later developments.

Definition 3.10. *The* Hodge-Tate *ring of K is the $\mathbb{C}_K$-algebra*

$$B_{\mathrm{HT}} = \bigoplus_{q \in \mathbb{Z}} \mathbb{C}_K(q)$$

in which multiplication is defined via the natural maps $\mathbb{C}_K(q) \otimes_{\mathbb{C}_K} \mathbb{C}_K(q') \cong \mathbb{C}_K(q+q')$.

If we choose a basis T of $\mathbb{Z}_p(1)$ then we can identify B_{HT} with $\mathbb{C}_K[T, T^{-1}]$ and Γ_K-action $\sigma(T^i) = \chi(g)^i T^i$ for $i \in \mathbb{Z}$ and $\sigma \in \Gamma_K$.

Next we define the first of a series of functors associated to these period rings.

Definition 3.11. *The covariant functor* $D_{\mathrm{HT}} : \mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K) \to \mathrm{Gr}_K$ *is*

$$D_{\mathrm{HT}}(V) := (B_{\mathrm{HT}} \otimes_{\mathbb{C}_K} V)^{\Gamma_K}.$$

Remark 3.12. The letter D stands for Dieudonné, who introduced the theory of Dieudonné modules that provides a categorical equivalence between certain categories of group schemes and certain categories of structures in semilinear algebra.

Note $D_{\mathrm{HT}}(V) = (B_{\mathrm{HT}} \otimes_{\mathbb{C}_K} V)^{\Gamma_K} = \oplus_q (\mathbb{C}_K(q) \otimes_{\mathbb{C}_K} V)^{\Gamma_K} = \oplus_q V\{q\}$. Thus

$$\dim_K(D_{\mathrm{HT}})(V) \leq \dim_{\mathbb{C}_K}(V)$$

with equality if and only if V is Hodge-Tate.

Example 3.13. $D_{\mathrm{HT}}(\mathbb{C}_K(r)) = K\langle -r \rangle$ for all $r \in \mathbb{Z}$.

The functor D_{HT} is evidentally left-exact, however we have the following useful exactness property on Hodge-Tate objects.

Proposition 3.14. *If* $0 \to W' \to W \to W'' \to 0$ *is a short exact sequence in* $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$ *and* W *is Hodge-Tate then so are* W' *and* W''*, in which case the sequence*

$$0 \to D_{\mathrm{HT}}(W') \to D_{\mathrm{HT}}(W) \to D_{\mathrm{HT}}(W'') \to 0$$

in $\mathrm{Gr}_{K,f}$ *is short exact.*

Proof. We have a left-exact sequence

$$0 \to D_{\mathrm{HT}}(W') \to D_{\mathrm{HT}}(W) \to D_{\mathrm{HT}}(W'') \tag{3}$$

with $\dim_K D_{\mathrm{HT}}(W') \leq \dim_{\mathbb{C}_K}(W')$ and similarly for W and W''. But equality holds for W by the Hodge-Tate property, so

$$\begin{aligned} \dim_{\mathbb{C}_K}(W) = \dim_K D_{\mathrm{HT}}(W) &\leq \dim_K D_{\mathrm{HT}}(W') + \dim_K D_{\mathrm{HT}}(W'') \\ &\leq \dim_{\mathbb{C}_K}(W') + \dim_{\mathbb{C}_K}(W'') \\ &= \dim_{\mathbb{C}_K}(W), \end{aligned}$$

forcing equality throughout. In particular, W' and W'' are Hodge-Tate and so for K-dimension reasons the left-exact sequence (3) is right-exact too. ☐

Remark 3.15. The converse of the above proposition is false, in the sense that if W' and W'' are Hodge-Tate then W may not be. To see this recall from the Tate-Sen Theorem $H^1(\Gamma_K, \mathbb{C}_K) \neq 0$. This gives a non-split exact sequence

$$0 \to \mathbb{C}_K \to W \to \mathbb{C}_K \to 0 \tag{4}$$

in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$ and we claim that such a W cannot be Hodge-Tate. Indeed, applying the left-exact functor D_{HT} to the exact sequence above gives a left-exact sequence

$$0 \to K\langle 0 \rangle \to D_{\mathrm{HT}}(W) \to K\langle 0 \rangle$$

of graded K-vector spaces, so in particular $D_{\mathrm{HT}}(W) = W\{0\} = W^{\Gamma_K}$. If W were Hodge-Tate then by the above proposition, this left-exact sequence would be short exact, so there would exist some $w \in W^{\Gamma_K}$ with nonzero image in $K\langle 0 \rangle$. We would then get a $\mathbb{C}_K$-linear Γ_K-equivariant section

$$\begin{aligned} \mathbb{C}_K &\to W \\ c &\mapsto cw \end{aligned}$$

This splits (4) in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$, contradicting the non-split property.

This next theorem highlights the insensitivity of the Hodge-Tate property when replacing K with a finite extension or restricting to the inertia group (i.e., replacing K with $\widehat{K^{\mathrm{ur}}}$). The insensitivity to finite extensions is a bad feature, indicating that the Hodge-Tate property is not sufficiently fine (e.g., to distinguish between good reduction and potentially good reduction for elliptic curves, cf. [10], Ch. VII, §5). For clarity we write $D_{\mathrm{HT}} = D_{\mathrm{HT},K}$.

Theorem 3.16. *For any $W \in \mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$, the natural map $K' \otimes_K D_{\mathrm{HT},K}(W) \to D_{\mathrm{HT},K'}(W)$ in $\mathrm{Gr}_{K',f}$ is an isomorphism for all finite extensions K'/K contained in $\overline{K} \subset \mathbb{C}_K$. Likewise, the natural map $\widehat{K^{\mathrm{ur}}} \otimes_K D_{\mathrm{HT},K}(W) \to D_{\mathrm{HT},\widehat{K^{\mathrm{ur}}}}(W)$ in $\mathrm{Gr}_{\widehat{K^{\mathrm{ur}}},f}$ is an isomorphism.*

In particular, for any finite extension K'/K inside of $\overline{K}$, an object W in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$ is Hodge-Tate if and only if it is Hodge-Tate when viewed in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_{K'})$, and similarly W is Hodge-Tate in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$ if and only if it is Hodge-Tate when viewed in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_{\widehat{K^{\mathrm{ur}}}}) = \mathrm{Rep}_{\mathbb{C}_K}(I_K)$, where $I_K \subseteq \Gamma_K$ is the inertia subgroup.

Proof. In the case of finite extensions, it is not difficult to reduce to the case when K'/K is Galois. Observe that $\mathrm{Gal}(K'/K)$ naturally acts semilinearly on the finite-dimensional K'-vector space $D_{\mathrm{HT},K'}(W)$ with invariant subspace $D_{\mathrm{HT},K}(W)$ over K. We need a lemma:

Lemma 3.17. *Let V' be a K'-vector space and assume that $\mathrm{Gal}(K'/K)$ acts continuously semilinearly on V'. Let $V = (V')^{\mathrm{Gal}(K'/K)}$. Then*

$$V' = K' \otimes_K V.$$

Proof. Fixing notation let $v \in V'$, $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis for K'/K and $\{\sigma_1, \sigma_2, \ldots, \sigma_n\} = \mathrm{Gal}(K'/K)$. For each $1 \le i \le n$, consider the vector

$$w_i = \sum_{j=1}^{n} (\alpha_i v)^{\sigma_j} = \mathrm{Tr}_{K'/K}(\alpha_i v).$$

It is clear that $w_i \in V$. The matrix $(\alpha_i^{\sigma_j})_{1 \le i,j \le n}$ is nonsingular, so each v^{σ_j} and in particular v, is a K'-linear combination of the w_i's. □

Returning to the proof it is evident that the Lemma solves the finite case.

For the case of $\widehat{K^{\mathrm{ur}}}$, we have to modify the preceding argument since $\widehat{K^{\mathrm{ur}}}/K$ is generally not algebraic. First we check that the natural semilinear action on $D' := D_{\mathrm{HT},\widehat{K^{\mathrm{ur}}}}(W)$ by the profinite group $\Gamma_k = \Gamma_K/I_K$ is continuous relative to the natural topology on D' as a finite-dimensional $\widehat{K^{\mathrm{ur}}}$-vector space. It suffices to check such continuity on the finitely many nonzero graded pieces $D_q^{'}$ separately. Note that the natural topology on $D_q^{'}$ coincides with its subspace topology from naturally sitting in the $\mathbb{C}_K$-vector space $\mathbb{C}_K(-q) \otimes_{\widehat{K^{\mathrm{ur}}}} D_q^{'}$ and we get the asserted continuity property for the action of Γ_k on $D_q^{'}$. Before we continue we need another lemma:

Lemma 3.18. *For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma)$, there exists a Γ-stable $\mathbb{Z}_p$-lattice $\Lambda \subseteq V$.*

Proof. Let $\rho : \Gamma \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$ be the continuous action map. Choose a $\mathbb{Z}_p$-lattice $\Lambda_0 \subseteq V$. Since $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda_0$, we naturally have $\mathrm{Aut}_{\mathbb{Z}_p}(\Lambda_0) \subseteq \mathrm{Aut}_{\mathbb{Q}_p}(V)$ and this is an open subgroup. Hence, the preimage $\Gamma_0 = \rho^{-1}(\mathrm{Aut}_{\mathbb{Z}_p}(\Lambda_0))$ is open in Γ. Such an open subgroup has finite index since Γ is compact, so Γ/Γ_0 has a finite set of coset representatives $\{\gamma_i\}$. Thus, the finite sum

$$\Lambda := \sum_i \rho(\gamma_i)\Lambda_0$$

is a $\mathbb{Z}_p$-lattice in V, that is Γ-stable since Λ_0 is Γ_0-stable and $\Gamma = \amalg \gamma_i \Gamma_0$. $\square$

Returning to the proof, although Γ_k acts $\widehat{K^{\mathrm{ur}}}$-semilinearly rather than $\widehat{K^{\mathrm{ur}}}$-linearly on D', since $\widehat{K^{\mathrm{ur}}}$ is the fraction field of a complete discrete valuation ring $\mathcal{O} := \mathcal{O}_{\widehat{K^{\mathrm{ur}}}}$, the proof of the above Lemma adapts to construct a Γ_k-stable $\mathcal{O}$-lattice $\Lambda \subseteq D'$. Consider the natural $\mathcal{O}$-linear Γ_k-equivariant map

$$\mathcal{O} \otimes_{\mathcal{O}_K} \Lambda^{\Gamma_k} \to \Lambda.$$

We shall prove that this is an isomorphism with Λ^{Γ_k} a finite free $\mathcal{O}_K$-module. Once this is proved, inverting p on both sides will give the desired isomorphism $\widehat{K^{\mathrm{ur}}} \otimes_K D_{HT,K}(W) \cong D' = D_{\mathrm{HT},\widehat{K^{\mathrm{ur}}}}(W)$.

Let $\pi \in \mathcal{O}_K$ be a uniformizer, so it is also a uniformizer of $\mathcal{O}$ and Γ_k acts trivially on π. The quotient $\Lambda/\pi\Lambda$ is a vector space over $\overline{k}$ with dimension equal to $d = \mathrm{rank}_{\mathcal{O}} \Lambda = \dim_{\widehat{K^{\mathrm{ur}}}} D'$ and it is endowed with a natural semilinear action by $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$. Hence, repeating the argument (Galois descent) of the finite case, applied to $\overline{k}/k$, gives that $\Lambda/\pi\Lambda = \overline{k} \otimes_k \Delta$ in $\mathrm{Rep}_{\overline{k}}(\Gamma_k)$ for the d-dimensional k-vector space $\Delta = (\Lambda/\pi\Lambda)^{\Gamma_k}$. In particular, $\Lambda/\pi\Lambda \cong \overline{k}^d$ compatible with the Γ_k-action, so $H^1(\Gamma_k, \Lambda/\pi\Lambda)$ vanishes since $H^1(\Gamma_k, \overline{k}) = 0$ by Hilbert 90. By looking at continuous 1-cocycles $c : \Gamma_k \to \Lambda \to \Lambda/\pi\Lambda$ and performing a successive approximation argument we arrive at $H^1(\Gamma_k, \Lambda) = 0$. Hence, passing to Γ_k-invariants on the exact sequence

$$0 \to \Lambda \xrightarrow{\pi} \Lambda \to \Lambda/\pi\Lambda \to 0$$

gives an exact sequence

$$0 \to \Lambda^{\Gamma_k} \xrightarrow{\pi} \Lambda^{\Gamma_k} \to (\Lambda/\pi\Lambda)^{\Gamma_k} \to 0.$$

That is, we have $\Lambda^{\Gamma_k}/\pi\Lambda^{\Gamma_k} \cong (\Lambda/\pi\Lambda)^{\Gamma_k}$ as k-vector spaces.

Since Λ^{Γ_k} is a closed $\mathcal{O}_K$-submodule of the finite free $\mathcal{O}$-module Λ of rank d and we have just proved that $\Lambda^{\Gamma_k}/\pi\Lambda^{\Gamma_k}$ is finite-dimensional of dimension d over $k = \mathcal{O}/(\pi)$, any lift of a k-basis of $\Lambda^{\Gamma_k}/\pi\Lambda^{\Gamma_k}$ to a subset of Λ^{Γ_k} is an $\mathcal{O}_K$-spanning set of Λ^{Γ_k} of size d. Thus Λ^{Γ_k} is a f.g. torsion free $\mathcal{O}_K$-module, so it is free of rank d, since its reduction module π is d-dimensional over k. □

Remark 3.19. Most good properties of p-adic representations of Γ_K for a p-adic field K will turn out to be detected on the inertia group I_K, so replacing K with $\widehat{K^{\mathrm{ur}}}$ is quite a useful device in the theory.

We next define a functor which almost an inverse to D_{HT}.

Definition 3.20. *The covariant functor* $V_{\mathrm{HT}} : \mathrm{Gr}_{K,f} \to \mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$ *is*

$$V_{\mathrm{HT}}(D) := \bigoplus_q \mathbb{C}_K(-q) \otimes_K D_q.$$

Remark 3.21. $V_{\mathrm{HT}}(D)$ is a Hodge-Tate representation and V_{HT} is evidently an exact functor.

Example 3.22. We have that $V_{\mathrm{HT}}(K\langle r\rangle) = \mathbb{C}_K(-r)$.

For any $W \in \mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$, the multiplicative structure on B_{HT} defines a natural B_{HT}-linear composite comparison morphism

$$\gamma_W : B_{\mathrm{HT}} \otimes_K D_{\mathrm{HT}}(W) \hookrightarrow B_{\mathrm{HT}} \otimes_K (B_{\mathrm{HT}} \otimes_{\mathbb{C}_K} W) \to B_{\mathrm{HT}} \otimes_{\mathbb{C}_K} W$$

that respects the Γ_K-action and grading. The Serre-Tate Lemma admits the following powerful reformulation:

Lemma 3.23. *For* $W \in \mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$*, the comparison morphism* γ_W *is injective. It is an isomorphism if and only if* W *is Hodge-Tate, in which case there is a natural isomorphism*

$$V_{\mathrm{HT}}(D_{\mathrm{HT}}(W)) \cong W$$

in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$.

Proof. The first statement follows directly from

$$\sum_q \dim_K W\{q\} \leq \dim_{\mathbb{C}_K} W$$

with equality iff W is Hodge-Tate. For the second statement we note that

$$V_{\mathrm{HT}}(D_{\mathrm{HT}}(W)) = \mathrm{gr}^0(B_{\mathrm{HT}} \otimes_K D_{\mathrm{HT}}(W)) \xrightarrow[\sim]{\gamma_W} \mathrm{gr}^0(B_{\mathrm{HT}} \otimes_{\mathbb{C}_K} W) = W$$

in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$. □

If $D \in \mathrm{Gr}_{K,f}$, $V_{\mathrm{HT}}(D)$ is a Hodge-Tate object in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$, so

$$\gamma_{V_{\mathrm{HT}}(D)} : B_{\mathrm{HT}} \otimes_K D_{\mathrm{HT}}(V_{\mathrm{HT}}(D)) \to B_{\mathrm{HT}} \otimes_{\mathbb{C}_K} V_{\mathrm{HT}}(D)$$

is an isomorphism respecting Γ_K-actions and gradings. Passing to Γ_K-invariants, we get an isomorphism

$$D_{\mathrm{HT}}(V_{\mathrm{HT}}(D)) \cong \bigoplus_r V_{\mathrm{HT}}(D)(r)^{\Gamma_K}$$

in Gr_K with $V_{\mathrm{HT}}(D)(r) = \oplus_q \mathbb{C}_K(r-q) \otimes_K D_q$. Hence $V_{\mathrm{HT}}(D)(r)^{\Gamma_K} = D_r$ by the Tate-Sen theorem, so we get an isomorphism

$$D_{\mathrm{HT}}(V_{\mathrm{HT}}(D)) \cong \bigoplus_r D_r = D$$

in Gr_K.

Thus the covariant functors D_{HT} and V_{HT} are quasi-inverse equivalences between the categories of Hodge-Tate representations in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$ and $\mathrm{Gr}_{K,f}$.

Theorem 3.24. *For any W, W' in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$ the natural map*

$$D_{\mathrm{HT}}(W) \otimes_K D_{\mathrm{HT}}(W') \to D_{\mathrm{HT}}(W \otimes W')$$

in Gr_K induced by the Γ_K-equivariant map

$$(B_{\mathrm{HT}} \otimes_{\mathbb{C}_K} W) \otimes_{\mathbb{C}_K} (B_{\mathrm{HT}} \otimes_{\mathbb{C}_K} W') \to B_{\mathrm{HT}} \otimes_{\mathbb{C}_K} (W \otimes_{\mathbb{C}_K} W')$$

defined by multiplication in B_{HT} is an isomorphism when W and W' are Hodge-Tate. Likewise, if W is Hodge-Tate then the natural map

$$D_{\mathrm{HT}}(W) \otimes_K D_{\mathrm{HT}}(W^*) \to D_{\mathrm{HT}}(W \otimes W^*) \to D_{\mathrm{HT}}(\mathbb{C}_K) \to K\langle 0 \rangle$$

in Gr_K is perfect duality, so the induced map $D_{\mathrm{HT}}(W^*) \to D_{\mathrm{HT}}(W)^*$ is an isomorphism in $\mathrm{Gr}_{K,f}$. In other words, D_{HT} is compatible with tensor products and duality on Hodge-Tate objects.

Similar compatibilities hold for V_{HT} with respect to tensor products and duality.

Proof. For the tensor product and duality claims for D_{HT}, one first checks that both sides have compatible evident functorial behaviour with respect to direct sums in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$. Hence we immediately reduce to the special case $W = \mathbb{C}_K(q)$ and $W' = \mathbb{C}_K(q')$ for some q, $q' \in \mathbb{Z}$, and this case is a straightforward calculation. Likewise, to analyze the natural map $V_{\mathrm{HT}}(D) \otimes_{\mathbb{C}_K} V_{\mathrm{HT}}(D') \to V_{\mathrm{HT}}(D \otimes D')$ we can reduce to the special case of the graded objects $D = K\langle r \rangle$ and $D' = K\langle r' \rangle$ for some r, $r' \in \mathbb{Z}$. Similarly for the case of duality. □

The following definition is now overdue.

Definition 3.25. *Let* $\mathrm{Rep}_{\mathrm{HT}}(\Gamma_K) \subseteq \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ *be the full subcategory of objects* V *that are Hodge-Tate (i.e.,* $\mathbb{C}_K \otimes_{\mathbb{Q}_p} V$ *is Hodge-Tate in* $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$*) and define the functor* $D_{\mathrm{HT}} : \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K) \to \mathrm{Gr}_{K,f}$ *by*

$$D_{\mathrm{HT}}(V) = D_{\mathrm{HT}}(\mathbb{C}_K \otimes_{\mathbb{Q}_p} V) = (B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V)^{\Gamma_K}$$

with grading induced by that on B_{HT}.

Our results in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$ show that $\mathrm{Rep}_{\mathrm{HT}}(\Gamma_K)$ is stable under tensor products, duality, subrepresentations and quotients (but not extensions) in $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$, and D_{HT} naturally commutes with finite extensions on K as well as with scalar extensions to $\widehat{K^{\mathrm{ur}}}$. Also, our proceding results show that on $\mathrm{Rep}_{\mathrm{HT}}(\Gamma_K)$ the functor D_{HT} is exact and is compatible with tensor products and duality. The comparison morphism

$$\gamma_V : B_{\mathrm{HT}} \otimes_K D_{\mathrm{HT}}(V) \to B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V$$

for $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is an isomorphism when V is Hodge-Tate and hence $D_{\mathrm{HT}} : \mathrm{Rep}_{\mathrm{HT}}(\Gamma_K) \to \mathrm{Gr}_{K,f}$ is a faithful functor.

Example 3.26. Theorem 2.49 can be written in the following more appealing form. If X/K is a smooth, proper and geometrically connected variety, then for $n \geq 0$ the representation $V := H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is in $\mathrm{Rep}_{\mathrm{HT}}(\Gamma_K)$ with $D_{\mathrm{HT}}(V) \cong H^n_{\mathrm{Hodge}}(X/K) := \oplus_q H^{n-q}(X, \Omega^q_{X/K})$. Thus, the comparison morphism γ_V takes the form of a B_{HT}-linear Γ_K-equivariant isomorphism

$$B_{\mathrm{HT}} \otimes_K H^n_{\mathrm{Hodge}}(X/K) \cong B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} H^n(X_{\overline{K}}, \mathbb{Q}_p)$$

in $\mathrm{Gr}_{K,f}$.

Whereas D_{HT} on the category of Hodge-Tate objects in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$ is fully faithful into $\mathrm{Gr}_{K,f}$, it is only faithful on the category $\mathrm{Rep}_{\mathrm{HT}}(\Gamma_K)$. To see that it is not full, let $\eta : \Gamma_K \to \mathbb{Z}_p^*$ have finite order, then $D_{\mathrm{HT}}(\mathbb{Q}_p(\eta)) \cong K\langle 0 \rangle = D_{\mathrm{HT}}(\mathbb{Q}_p)$ by the Tate-Sen theorem, but $\mathbb{Q}_p(\eta)$ and $\mathbb{Q}_p$ have no nonzero maps between them when $\eta \neq 1$. This is due to the fact that the operation

$$V \mapsto \mathbb{C}_K \otimes_{\mathbb{Q}_p} V$$

loses information about V and needs to be replaced by something more sophisticated.

To improve on D_{HT} so as to get a fully faithful functor from a nice category of p-adic representations of Γ_K into a category of semilinear algebra objects, we need to do two things: we must refine B_{HT} to a ring with more structure and we need to introduce a target semilinear algebra category that is richer than $\mathrm{Gr}_{K,f}$.

4 Construction of the field B_{dR}

The ring B_{HT} provides a convenient mechanism for working with Hodge-Tate representations, but the Hodge-Tate condition on a p-adic representation of the Galois group Γ_K is too weak to be really useful. What we seek is a class of p-adic representations that is broad enough to include the representations arising from algebraic geometry but also small enough to permit the existence of an equivalence of categories with a category of semilinear algebra objects.

Definition 4.1. *A filtered module over a commutative ring R is an R-module M endowed with a collection $\{\mathrm{Fil}^i(M)\}_{i\in\mathbb{Z}}$ of submodules that is decreasing in the sense that $\mathrm{Fil}^{i+1}(M) \subseteq \mathrm{Fil}^i(M)\ \forall i \in \mathbb{Z}$. If $\cup \mathrm{Fil}^i(M) = M$ then the filtration is exhaustive and if $\cap \mathrm{Fil}^i(M) = 0$ then the filtration is separated. For any filtered R-module M, the associated graded module is*

$$\mathrm{gr}^\bullet(M) = \bigoplus_i (\mathrm{Fil}^i(M)/\mathrm{Fil}^{i+1}(M)).$$

A filtered ring is a ring R equipped with an exhaustive and separated filtration $\{R^i\}$ by additive subgroups such that $1 \in R^0$ and $R^i \cdot R^j \subseteq R^{i+j}\ \forall i, j \in \mathbb{Z}$. (In particular, R^0 is a subring of R and each R^i is an R^0-submodule of R.) The associated graded ring is

$$\mathrm{gr}^\bullet(M) = \bigoplus_i R^i/R^{i+1}.$$

If k is a ring then a filtered k-algebra is a k-algebra A equipped with a structure of a filtered ring such that the filtered pieces A^i are k-submodules of A, and the associated graded k-algebra is

$$\mathrm{gr}^\bullet(A) = \bigoplus_i A^i/A^{i+1}.$$

Example 4.2. For an arbitrary field k of characteristic 0 and for any smooth proper k-scheme X, the k-vector space $H^n_{\mathrm{dR}}(X/k)$ has associated graded vector space $\mathrm{gr}^\bullet(H^n_{\mathrm{dR}}(X/k)) = H^n_{\mathrm{Hodge}}(X/k)$. It is important to note that this filtration generally does not admit a functorial splitting.

A natural idea for improving Theorem 2.49 between p-adic étale and graded Hodge cohomology via B_{HT} is to replace the graded $\mathbb{C}_K$-algebra B_{HT} with a filtered K-algebra B_{dR} endowed with a Γ_K-action respecting the filtration. Given such a B_{dR}, consider the associated functor $D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{\Gamma_K}$ on $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ with values in finite dimensional K-vector spaces. A serious test of a good definition for B_{dR} is that it should lead to a refinement of Theorem 2.49 by using de Rham cohomology instead. That is, for smooth proper X over K the p-adic representations $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ should be B_{dR}-admissible, that is to say a natural isomorphism

$$D_{\mathrm{dR}}(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^n_{\mathrm{dR}}(X/K)$$

whose induced isomorphism between associated graded K-vector spaces leads to the comparison isomorphism between p-adic étale and Hodge cohomologies.

Example 4.3. Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and residue field k, and let $A = \mathrm{Frac}(R)$. There is a natural structure of a filtered ring on A via $A^i = \mathfrak{m}^i$ for $i \in \mathbb{Z}$. In this case the associated graded ring $\mathrm{gr}^\bullet(A)$ is a k-algebra that is non-canonically isomorphic to the ring of Laurent polynomials $k[t, 1/t]$ upon choosing a k-basis of $\mathfrak{m}/\mathfrak{m}^2$.

Since we require $\mathrm{gr}^\bullet(B_{\mathrm{dR}}) = B_{\mathrm{HT}}$ and we have the description $B_{\mathrm{HT}} \cong \mathbb{C}_K[T, T^{-1}]$, the above example inspires us to seek a complete discrete valuation ring B^+_{dR} over K (with maximal ideal denoted $\mathfrak{m}$) endowed with a Γ_K-action such that the residue field is naturally Γ_K-equivariantly isomorphic to $\mathbb{C}_K$ and the Zariski cotangant space $\mathfrak{m}/\mathfrak{m}^2$ is naturally isomorphic to $\mathbb{C}_K(1)$ in $\mathrm{Rep}_{\mathbb{C}_K}(\Gamma_K)$.

A first guess might be to take

$$B^+_{\mathrm{dR}} = \prod_{q \geq 0} \mathbb{C}_K(q) \cong \mathbb{C}_K[[t]]$$

with Γ_K-action given by $\sigma(\Sigma a_n t^n) = \Sigma \sigma(a_n)\chi(\sigma)^n t^n$. Unfortunately this does not lead to new concepts refining the theory of Hodge-Tate representations since the filtration is too closely related to the grading to give anything interesting. More specifically, with such a definition we would get $D_{\mathrm{dR}} = D_{\mathrm{HT}}$.

A more promising idea is to imitate the procedure in commutative algebra whereby for perfect fields k of characteristic $p > 0$ there is a functorially associated complete discrete valuation ring $W(k)$ (of Witt vectors) that has uniformizer p and residue field k. The difference here is that we want to functorially build a complete discrete valuation ring with residue field $\mathbb{C}_K$ of characteristic 0. Thus, we cannot use a naive Witt construction, however, we shall see that Witt-style ideas are useful. For this reason, we recall some basic definitions and statements about Witt Vectors (without proof), cf. [9], Ch. II, §6 for a more complete treatment.

Theorem 4.4. *For every perfect field k of characteristic p, there exists a complete discrete valuation ring and only one (up to unique isomorphism) which is absolutely unramified and has k as its residue field.*

This ring will be denoted $W(k)$, the Witt ring of k. It is unique in the following sense: if A_1 and A_2 satisfy the conditions of the theorem, there is a unique isomorphism $g : A_1 \to A_2$ which commutes with the maps $A_i \to k$. We similarly define $W(A)$ for a perfect $\mathbb{F}_p$-algebra A.

Definition 4.5. *A p-ring is a ring B that is separated and complete for the topology defined by a specified decreasing collection of ideals $\mathfrak{b}_1 \supseteq \mathfrak{b}_2 \supseteq \dots$ such that $\mathfrak{b}_n\mathfrak{b}_m \subseteq \mathfrak{b}_{n+m}$ for all $n, m \geq 1$ and $B/\mathfrak{b}_1$ is a perfect $\mathbb{F}_p$-algebra (so $p \in \mathfrak{b}_1$). We say that B is a strict p-ring if moreover $\mathfrak{b}_i = p^i B$ for all $i \geq 1$ (i.e., B is p-adically separated and complete with B/pB a perfect $\mathbb{F}_p$-algebra) and $p : B \to B$ is injective.*

Lemma 4.6. *Let B be a p-ring. There is a unique set-theoretic section $r_B : B/\mathfrak{b}_1 \to B$ to the reduction map such that $r_B(x^p) = r_B(x)^p$ for all $x \in B/\mathfrak{b}_1$. Moreover, r_B is multiplicative and $r_B(1) = 1$.*

Remark 4.7. We adapt the Witt coordinatization $(r_0, r_1, \dots) = \Sigma p^n r_B[r_n^{p^{-n}}]$

An immediate consequence of this lemma is that in a strict p-ring B endowed with the p-adic topology, each element $b \in B$ has the unique form

$$b = \sum_{n \geq 0} r_B(b_n) p^n$$

with $b_n \in B/\mathfrak{b}_1 = B/pB$. This leads to the following useful universal property of certain Witt rings.

Proposition 4.8. *If A is a perfect $\mathbb{F}_p$-algebra and B is a p-ring, then the natural reduction map $\mathrm{Hom}(W(A), B) \to \mathrm{Hom}(A, B/\mathfrak{b}_1)$ (which makes sense since $A = W(A)/(p)$ and $p \in \mathfrak{b}_1$) is bijective. More generally, for any strict p-ring $\mathcal{B}$, the natural map*

$$\mathrm{Hom}(\mathcal{B}, B) \to \mathrm{Hom}(\mathcal{B}/(p), B/\mathfrak{b}_1)$$

is bijective for every p-ring B. In particular, since $\mathcal{B}$ and $W(\mathcal{B}/(p))$ satisfy the same universal property in the category of p-rings for any strict p-ring $\mathcal{B}$, strict p-rings are precisely the rings of the form $W(A)$ for perfect $\mathbb{F}_p$-algebras A.

Observe that the additive multiplication map $p : W(A) \to W(A)$ is given by $(a_i) \mapsto (0, a_0^p, a_1^p, \ldots)$, so it is injective and the subset $p^n W(A) \subseteq W(A)$ consists of Witt vectors (a_i) such that $a_0 = \ldots = a_{n-1} = 0$. So we naturally have $W(A)/(p^n) \cong W_n(A)$ by projection to the first n Witt components.

If A is any $\mathbb{F}_p$-algebra whatsoever then we can construct a canonically associated $\mathbb{F}_p$-algebra $A^\flat$ (called the tilt of A) as follows:

$$A^\flat := \varprojlim_{x \mapsto x^p} A = \{(x_0, x_1, \ldots) \in \prod_{n \geq 0} | x_{i+1}^p = x_i \ \forall i\}$$

with the product ring structure. This is perfect because the additive pth power map on $A^\flat$ is surjective by construction and is injective since if $(x_i) \in A^\flat$ satisfies $(x_i)^p = (0)$ then $x_{i-1} = x_i^p = 0$ for all $i \geq 1$ (so $(x_i) = 0$). In terms of universal properties, observe that the map $A^\flat \to A$ defined by $(x_i) \mapsto x_0$ is a map to A from a perfect $\mathbb{F}_p$-algebra and this is final among all maps to A from perfect $\mathbb{F}_p$-algebras.

Example 4.9. If A is a perfect $\mathbb{F}_p$-algebra then the canonical map $A^\flat \to A$ is an isomorphism, and the inverse map is explicitly given by $a \mapsto (a, a^{1/p}, a^{1/p^2}, \ldots)$.

Example 4.10. If F is a field of characteristic p, then $F^\flat$ is the largest perfect subfield of F. For example, $\mathbb{F}_p(x)^\flat = \mathbb{F}_p$.

We will be particularly interested in the perfect $\mathbb{F}_p$-algebra

$$R := (\mathcal{O}_{\overline{K}}/(p))^\flat = (\mathcal{O}_{\mathbb{C}_K}/(p))^\flat$$

endowed with its natural Γ_K-action.

Since $\mathcal{O}_{\overline{K}/(p)}$ is canonically an algebra over the perfect field $\overline{k}$, we have by functoriality a ring map

$$\overline{k} = \overline{k}^{\flat} \to (\mathcal{O}_{\overline{K}/(p)})^{\flat} = R$$

described concretely by

$$c \mapsto (j(c), j(c^{1/p}), j(c^{1/p^2}), \ldots)$$

where $j : \overline{k} \to \mathcal{O}_{\overline{K}/(p)}$ is the canonical k-algebra section to the reduction map $\mathcal{O}_{\overline{K}/(p)} \twoheadrightarrow \overline{k}$.

We would like to construct a ring map

$$W(R) \to \mathcal{O}_{\mathbb{C}_K}$$

however we cannot use Proposition 4.8, since $\mathcal{O}_{\mathbb{C}_K}/(p)$ is not perfect. Our route will thus be a bit more indirect. For now, it is convenient to work more generally with any p-adically separated and complete ring (e.g., $\mathcal{O}_{\mathbb{C}_K}$ but not $\mathcal{O}_{\overline{K}}$).

Proposition 4.11. *Let $\mathcal{O}$ be a p-adically separated and complete ring, and let $\mathfrak{a} \subseteq \mathcal{O}$ be an ideal containing $p\mathcal{O}$ such that $\mathfrak{a}^N \subseteq p\mathcal{O}$ for some $N \gg 0$ (i.e., the $\mathfrak{a}$-adic and p-adic topologies on $\mathcal{O}$ coincide). The multiplicative map of sets*

$$\varprojlim_{x \mapsto x^p} \mathcal{O} \to (\mathcal{O}/\mathfrak{a})^{\flat}$$

defined by $(x^{(n)})_{n \geq 0} \mapsto (x^{(n)} \bmod \mathfrak{a})$ is bijective. Also, for any $x = (x_n) \in (\mathcal{O}/\mathfrak{a})^{\flat}$ and arbitrary lifts $\widehat{x_r} \in \mathcal{O}$ of $x_r \in \mathcal{O}/\mathfrak{a}$ for all $r \geq 0$, the limit

$$l_n(x) = \lim_{m \to \infty} \widehat{x_{n+m}}^{p^m}$$

exists in $\mathcal{O}$ for all $n \geq 0$ and is independent of the choice of lifts $\widehat{x_r}$. Morevoer, the inverse is given by $x \mapsto (l_n(x))$.

In particular, $(\mathcal{O}/p\mathcal{O})^{\flat} \to (\mathcal{O}/\mathfrak{a})^{\flat}$ is an isomorphism and this common ring is a domain if $\mathcal{O}$ is a domain.

Proof. We note that for each $n \geq 0$ and $m' \geq m \geq 0$ we have

$$\widehat{x_{n+m'}}^{p^{m'-m}} \equiv \widehat{x_{n+m}} \bmod p\mathcal{O},$$

so $\widehat{x_{n+m'}}^{p^{m'}} \equiv \widehat{x_{n+m}}^{p^m} \bmod p^{m+1}\mathcal{O}$. Hence, there is a well-defined limit

$$\lim_{n \to \infty} \widehat{x_{n+m}}^{p^m}.$$

If we make another choice of lifting $\widetilde{x_r}$, then the congruence $\widetilde{x_r} \equiv \widehat{x_r} \bmod p\mathcal{O}$, shows that the limit is the same $\bmod\, p^k\mathcal{O}$ for any $k \geq 0$, so in fact they must be equal as $\mathcal{O}$ is complete. Thus $l_n(x)$ is independent of the choice of liftings $\widehat{x_r}$. The proposed inverse map $x \mapsto (l_n(x))$ is therfore well-defined, and in view of it being independent of the liftings we see that it is indeed an inverse map. ☐

In what follows, for any $x \in (\mathcal{O}/\mathfrak{a})^\flat = (\mathcal{O}/p\mathcal{O})^\flat$, we write $x^{(n)} \in \mathcal{O}$ to denote the limit $l_n(x)$ for all $n \geq 0$.

Remark 4.12. The bijection in Proposition 4.11, allows us to transfer the natural $\mathbb{F}_p$-algebra structure on $(\mathcal{O}/\mathfrak{a})^\flat$ over to such a structure on the inverse limit set $\varprojlim \mathcal{O}$ of p-power compatible sequences $x = (x^{(n)})_{n \geq 0}$ in $\mathcal{O}$. The multiplication structure translates through this bijection as $(xy)^{(n)} = x^{(n)}y^{(n)}$. For addition, the proof of the proposition gives

$$(x + y)^{(n)} = \lim_{m \to \infty} (x^{(n+m)} + y^{(n+m)})^{p^m}.$$

Fixing notation, an element $x \in R$ will be denoted $(x_n)_{n \geq 0}$ when we wish to view its p-power compatible components as elements of $\mathcal{O}_{\mathbb{C}_K}/(p)$ and we use the notation $(x^{(n)})_{n \geq 0}$ to denote its unique representation using a p-power compatible sequence of elements $x^{(n)} \in \mathcal{O}_{\mathbb{C}_K}$. An element $x \in R$ is a unit if and only if the component $x_0 \in \mathcal{O}_{\overline{K}}/(p)$ is a unit, so R is a local ring. An element $x \in R$ is a unit if and only if the component $x_0 \in \mathcal{O}_{\overline{K}}/(p)$ is a unit, so R is a local ring. Also every element of $\mathcal{O}_{\overline{K}}$ is a square, it follows that the nonzero maximal ideal $\mathfrak{m}$ of R satisfies $\mathfrak{m} = \mathfrak{m}^2$. In particular, R is not noetherian. We now state and prove some non-obvious properties of the ring R.

Proposition 4.13. *Let* $|\cdot|_p : \mathbb{C}_K \twoheadrightarrow p^{\mathbb{Q}} \cup \{0\}$ *be the normalized absolute value satisfying* $|p|_p = 1/p$. *The map* $|\cdot|_R : R \to p^{\mathbb{Q}} \cup \{0\}$ *defined by* $x = (x^{(n)}) \mapsto |x^{(0)}|_p$ *is a* Γ_K*-equivariant absolute value on* R *that makes* R *the valuation ring for the unique valuation* v_R *on* $\mathrm{Frac}(R)$ *extending* $-\log_p |\cdot|_R$ *on* R *(and having value group* $\mathbb{Q}$*).*

Also, R *is* v_R*-adically separated and complete, and the subfield* $\overline{k}$ *of* R *maps isomorphically onto the residue field of* R*.*

Proof. Obviously $x^{(0)} = 0$ if and only if $x = 0$, and $|xy|_R = |x|_R|y|_R$ since $(xy)^{(0)} = x^{(0)}y^{(0)}$. To show that $|x + y|_R \leq \max(|x|_R, |y|_R)$ we may assume $x, y \neq 0$, so $x^{(0)}, y^{(0)} \neq 0$. By symmetry we may assume $|x^{(0)}|_p \leq |y^{(0)}|_p$, so for all $n \geq 0$ we have

$$|x^{(n)}|_p = |x^{(0)}|_p^{p^{-n}} \leq |y^{(0)}|_p^{p^{-n}} = |y^{(n)}|_p.$$

The ratios $x^{(n)}/y^{(n)}$ therefore lie in $\mathcal{O}_{\mathbb{C}_K}$ for $n \geq 0$ and form a p-power compatible sequence. This sequence is therefore an element $z \in R$, and $yz = x$ in R. Hence,

$$|x + y|_R = |y(z + 1)|_R = |y|_R|z + 1|_R \leq |y|_R = \max(|x|_R, |y|_R).$$

The same argument shows that R is the valuation ring of v_R on $\mathrm{Frac}(R)$.

To prove $|\cdot|_R$-completeness of R, first note that if we let $v = -\log_p |\cdot|_p$ on $\mathbb{C}_K$ then $v_R(x) = v(x^{(0)}) = p^n v(x^{(n)})$ for $n \geq 0$. Thus, $v_R(x) \geq p^n$ if and only if $v(x^{(n)}) \geq 1$. Hence, if we let

$$\theta_n : R \to \mathcal{O}_{\mathbb{C}_K}/(p)$$

denote the ring homomorphism $x = (x_m)_{m\geq 0} \mapsto x_n$ then $\{x \in R|\ v_R(x) \geq p^n\} = \ker\theta_n$. In view of how the inverse limit R sits within the product space $\prod_{m\geq 0}(\mathcal{O}_{\mathbb{C}_K}/(p))$, or more specifically since $x_n = 0$ implies $x_m = 0$ for all $m \leq n$, we conclude that the v_R-adic topology on R coincides with its subspace topology within $\prod_{m\geq 0}(\mathcal{O}_{\mathbb{C}_K}/(p))$ where the factors are given the discrete topology, so the v_R-adic completeness follows (as R is closed in this product topology).

Finally $\theta_0 : R \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}/(p)$ is a $\overline{k}$-algebra map, which is local, and so induces an injection on residue fields. □

Example 4.14. An important example of an element R is

$$\varepsilon = (\varepsilon^{(n)})_{n\geq 0} = (1, \zeta_p, \zeta_{p^2}, \ldots).$$

Any two such elements are $\mathbb{Z}_p^\times$-powers of each other. It is easy to check that $v_R(\varepsilon - 1) = p/(p-1)$.

Theorem 4.15. *The field* $\mathrm{Frac}(R)$ *of characteristic* p *is algebraically closed.*

Proof. Since R is a valuation ring, it suffices to constuct a root in R for any monic polynomial $f \in R[X]$ with $d = \deg P > 0$. We assume $d \geq 2$.

For each $m \geq 1$, consider the ring map $\theta_m : R \to \mathcal{O}_{\mathbb{C}_K}/(p)$ defined by $x = (x_i) \mapsto x_m$ and let $f_m = \theta_m(f) \in (\mathcal{O}_{\mathbb{C}_K}/(p))[X]$. This is a monic polynomial of degree d, so it lifts to a monic polynomial $\widetilde{f_m} \in \mathcal{O}_{\mathbb{C}_K}[X]$ of degree d. Thus $\widetilde{f_m}$ admits a set of d roots (with multiplicity) $\{\rho_{1,m}, \ldots, \rho_{d,m}\}$ in $\mathcal{O}_{\mathbb{C}_K}$. The reductions $\overline{\rho_{i,m}}$ of these modulo p are roots of f_m, and if we could arrange a p-power compatible sequence of these we could get the desired root of f in R. The problem is that $\mathcal{O}_{\mathbb{C}_K}/(p)$ is not a domain and so f_m always has infinitely many roots.

Since $f_m(\overline{\rho_{i,m+1}}^p) = f_{m+1}(\overline{\rho_{i,m+1}})^p = 0$ in $\mathcal{O}_{\mathbb{C}_K}/(p)$, we have $\widetilde{f}_m(\rho_{i,m+1}^p) \in p\mathcal{O}_{\mathbb{C}_K}$ for each $1 \leq i \leq d$. But $\widetilde{f_m} = \prod_j (X - \rho_{j,m})$ so for each i we have $\prod_j(\rho_{i,m+1}^p - \rho_{j,m}) \in p\mathcal{O}_{\mathbb{C}_K}$. Since there are d terms in the product, for each $1 \leq i \leq d$, there exists $1 \leq j(i) \leq d$ such that

$$\rho_{i,m+1}^p \equiv \rho_{j(i),m} \bmod p^{1/d}\mathcal{O}_{\mathbb{C}_K}.$$

Hence we conclude that

$$\rho_{i,m+1}^{p^d} \equiv \rho_{j(i),m}^{p^{d-1}} \bmod p\mathcal{O}_{\mathbb{C}_K}$$

and the rest is now clear. □

Remark 4.16. Peter Scholze on his work on Perfectoid spaces, cf. [8], Proposition 3.8, proves a much stronger result.

We have now assembled enough work to carry out the first important refinement on the graded ring B_{HT}, namely the construction of the field of p-adic periods B_{dR}. Inspired by the universal property of Witt vectors as in Proposition 4.8 and the perfectness of the $\mathbb{F}_p$-algebra R, we seek to lift the Γ_K-equivariant surjective ring map $\theta_0 : R \to \mathcal{O}_{\mathbb{C}_K}/(p)$ defined by $(x_i) \mapsto x_0$ to a Γ_K-equivariant surjective ring map $\theta : W(R) \to \mathcal{O}_{\mathbb{C}_K}$. As we have already observed, $\mathcal{O}_{\mathbb{C}_K}/(p)$ is not perfect, nonetheless, we will construct such a θ in a canonical way.

Definition 4.17. *Our definition for $\theta : W(R) \to \mathcal{O}_{\mathbb{C}_K}$ as a set-theoretic map is simple and explicit:*

$$\theta(\sum_n [c_n]p^n) = \sum_n c_n^{(0)} p^n.$$

In terms of the Witt coordinatization

$$\theta : (r_0, r_1, \ldots) \mapsto \sum_n (r_n^{p^{-n}})^{(0)} p^n.$$

Note, however for $r \in R$, $(r^{p^{-n}})^{(0)} = r^{(n)}$. Thus we have the formula

$$\theta : (r_0, r_1, \ldots) \mapsto \sum_n r_n^{(n)} p^n.$$

Remark 4.18. By definition θ is Γ_K-equivariant.

We need to check that $\theta : W(R) \to \mathcal{O}_{\mathbb{C}_K}$ is a ring map.

Lemma 4.19. *The map $\theta : W(R) \to \mathcal{O}_{\mathbb{C}_K}$ is a ring homomorphism.*

Proof. It suffices to prove that

$$\theta_n := \theta \bmod p^n : W_n(R) = W(R)/p^n W(R) \to \mathcal{O}_{\mathbb{C}_K}/p^n\mathcal{O}_{\mathbb{C}_K}$$

is a ring map for all $n \geq 1$. We show first that θ_n is additive. Writing $w = (x_0, \ldots, x_{n-1})$ with $x_i \in R$, by definition

$$\theta_n(w) = \sum_{i=0}^{n-1} p^i x_i^{(i)} = \sum_{i=0}^{n} (x_i^{(n)})^{p^{n-i}} = \Phi(x_0^{(n)} \bmod p^n, \ldots, x_{n-1}^{(n)} \bmod p^n)$$

where $\Phi_n : W_n(\mathcal{O}_{\mathbb{C}_K}/p^n\mathcal{O}_{\mathbb{C}_K}) \to \mathcal{O}_{\mathbb{C}_K}/p^n\mathcal{O}_{\mathbb{C}_K}$ is the nth Witt polynomial map defined by

$$(z_0, \ldots, z_{n-1}) \mapsto \sum_{i=0}^{n-1} p^i z_i^{p^{n-i}}.$$

By definition Φ_n is additive. We note that $\Phi_n = \overline{\Phi_n} \circ \pi_n$ where $\pi_n : W_n(\mathcal{O}_{\mathbb{C}_K}/p^n\mathcal{O}_{\mathbb{C}_K}) \twoheadrightarrow W_n(\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K})$ is the natural quotient map and $\overline{\Phi_n} : W_n(\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}) \to \mathcal{O}_{\mathbb{C}_K}/p^n\mathcal{O}_{\mathbb{C}_K}$ is the map of sets

$$(\overline{z_0}, \ldots, \overline{z_{n-1}}) \mapsto \sum_{i=0}^{n-1} p^i z_i^{p^{n-i}},$$

where z_i is a lift of $\overline{z_i}$. Since π_n is surjective and additive and Φ_n is additive, $\overline{\Phi_n}$ is also additive. Finally let $f_n : R \to \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$ denote the projection $r \mapsto r^{(n)} \bmod p$ to the nth member of the p-power compatible system. Note that f_n is a ring homomorphism and so we conclude θ_n is additive.

It remains to prove θ_n is multiplicative. The identity $\theta_n(ww') = \theta_n(w)\theta_n(w')$ depends $\mathbb{Z}$-bilinearly on (w, w'). We have already established additivity, so we can assume via the teichmüller representations $w = [r]$ and $w' = [r']$:

$$\theta([r][r']) = \theta([rr']) = (rr')^{(0)} = r^{(0)} r'^{(0)} = \theta([r])\theta([r']).$$

This completes the proof. □

The explicit definition of θ makes it evident that θ is surjective (since $R \to \mathcal{O}_{\mathbb{C}_K}/(p)$ via $r \mapsto r^{(n)} \bmod p$ is surjective for each $n \geq 0$).

We now have a Γ_K-equivariant surjective ring homomorphism

$$\theta_{\mathbb{Q}} : W(R)[1/p] \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}[1/p] = \mathbb{C}_K$$

but the source ring is not a complete discrete valuation ring. We shall replace $W(R)[1/p]$ with its $\ker \theta_{\mathbb{Q}}$-adic completion, and the reason this works is that $\ker \theta_{\mathbb{Q}} = (\ker \theta)[1/p]$ turns out to be a principal ideal. We need a proposition.

Proposition 4.20. *Choose $\widetilde{p} \in R$ such that $\widetilde{p}^{(0)} = p$ (i.e., $\widetilde{p} = (p, p^{1/p}, p^{1/p^2}, \ldots) \in \varprojlim \mathcal{O}_{\mathbb{C}_K} = R$, so $v_R(\widetilde{p}) = 1$) and let $\xi = \xi_{\widetilde{p}} = [\widetilde{p}] - p = (\widetilde{p}, -1, 0, 0, \ldots) \in W(R)$. Then the ideal $\ker \theta \subseteq W(R)$ is the principal ideal generated by ξ. Furthermore an element $w = (r_0, r_1, \ldots) \in \ker \theta$ is a generator of $\ker \theta$ if and only if $r_1 \in R^\times$.*

Proof. Clearly $\theta(\xi) = \theta([\widetilde{p}]) - p = \widetilde{p}^{(0)} - p = 0$ and $\ker \theta \cap p^n W(R) = p^n \ker \theta$ since $W(R)/(\ker \theta) = \mathcal{O}_{\mathbb{C}_K}$ has no nonzero p-torsion. Since $W(R)$ is p-adically separated and complete (as R is a perfect domain, so the p-adic topology on $W(R)$ is just the product topology on $W(R)$ using the discrete topology of R), to prove that ξ is a principal generator of $\ker \theta$ we claim it suffices to show $\ker \theta \subseteq (\xi, p) = ([\widetilde{p}], p)$. Indeed assuming we have proved the latter, suppose $\ker \theta \neq (\xi)$. Then there exists $a\xi + bp \in \ker \theta \backslash (\xi)$ for some $a, b \in W(R)$. In particular $b \in \ker \theta \backslash (\xi)$ and so we can write $b = a_1 \xi + b_1 p$ for some $b_1 \in \ker \theta \backslash (\xi)$. Repeating the process we arrive at

$$\xi(a + a_1 p + \cdots + a_{n-1} p^{n-1}) + b_{n-1} p^n \in \ker \theta \backslash (\xi)$$

with $b_{n-1} \in \ker \theta \backslash (\xi)$. However $b_{n-1} p^n \to 0$ as $n \to 0$ and so we obtain a contradiction. So it remains to show $\ker \theta \subseteq (\xi, p) = ([\widetilde{p}], p)$. Observe if $w = (r_0, r_1, \ldots) \in \ker \theta$ then $r_0^{(0)} \equiv 0 \bmod p$, so $v_R(r_0) \geq 1 = v_R(\widetilde{p})$ and hence $r_0 \in \widetilde{p}R$. We conclude that $w \in ([r_0], p) \subseteq ([\widetilde{p}, p])$, as desired.

A general element $w = (r_0, r_1, \ldots) \in \ker \theta$ has the form

$$w = \xi(r_0', r_1', \ldots) = (\widetilde{p}, -1, \ldots)(r_0', r_1', \ldots) = (\widetilde{p} r_0', \widetilde{p}^p r_1' - r_0'^p),$$

so $r_1 = \widetilde{p}^p r_1' - r_0'^p$. Hence $r_1 \in R^\times$ if and only if $r_0' \in R^\times$, and this final unit condition is equivalent to the multiplier $(r_0', r_1', \ldots)$ being a unit in $W(R)$, which amounts to w being a principal generator of $\ker \theta$. ☐

Corollary 4.21. *For all $j \geq 1$,*

$$W(R) \cap (\ker(\theta_{\mathbb{Q}})^j) = (\ker \theta)^j$$

Also, $\cap (\ker \theta)^j = \cap (\ker \theta_{\mathbb{Q}})^j = 0$.

Proof. The first part follows from the fact that $\ker \theta_{\mathbb{Q}} = (\ker \theta)[1/p]$.

Since any element of $W(R)[1/p]$ admits a p-power multiple in $W(R)$, we conclude that

$$\cap (\ker \theta_{\mathbb{Q}})^j = (\cap (\ker \theta)^j)[1/p].$$

To prove this vanishes, it suffices to consider an arbitrary $w = (r_0, r_1, \ldots) \in W(R)$ lying in $\cap(\ker\theta)^j$. Thus, w is divisible by arbitrary high powers of $\xi = [\widetilde{p}] - p = (\widetilde{p}, -1, 0, 0, \ldots)$, so r_0 is divisible by arbitrary high powers of $\widetilde{p}$ in R. But $v_R(\widetilde{p}) = 1 > 0$, so by v_R-adic separatedness of R we see that $r_0 = 0$. This says that $w = pw'$ for some $w' \in W(R)$ since R is a perfect $\mathbb{F}_p$-algebra. Hence $w' \in (\cap(\ker\theta)^j)[1/p] = \cap(\ker\theta_{\mathbb{Q}})^j$. Thus $w' \in W(R) \cap (\ker(\theta_{\mathbb{Q}})^j) = (\ker\theta)^j$ for all j. This shows that each element of $\cap(\ker\theta)^j$ in $W(R)$ lies in $\cap p^n W(R)$ and this vanishes since $W(R)$ is a strict p-ring. □

We conclude that $W(R)[1/p]$ injects into the inverse limit

$$B_{\mathrm{dR}}^+ := \varprojlim_j W(R)[1/p]/(\ker\theta_{\mathbb{Q}})^j$$

whose transition maps are Γ_K-equivariant, so B_{dR}^+ has a natural Γ_K-action that is compatible with the action on its subring $W(R)[1/p]$. The inverse limit B_{dR}^+ maps Γ_K-equivariantly onto each quotient $W(R)[1/p]/(\ker\theta_{\mathbb{Q}})^j$ via the evident natural map, and in particular for $j = 1$ the map $\theta_{\mathbb{Q}}$ induces a natural Γ_K-equivariant surjective map $\theta_{\mathrm{dR}}^+ : B_{\mathrm{dR}}^+ \twoheadrightarrow \mathbb{C}_K$. Since θ_{dR}^+ restricts to $\theta_{\mathbb{Q}}$ on the subring $W(R)[1/p]$ we have

$$\ker\theta_{\mathrm{dR}}^+ \cap W(R) = \ker\theta$$

and

$$\ker\theta_{\mathrm{dR}}^+ \cap W(R)[1/p] = \ker\theta_{\mathbb{Q}}.$$

Proposition 4.22. *The ring B_{dR}^+ is a complete discrete valuation ring with residue field $\mathbb{C}_K$, and any generator of $\ker\theta_{\mathbb{Q}}$ in $W(R)[1/p]$ is a uniformizer of B_{dR}^+. Furthermore the natural map $B_{\mathrm{dR}}^+ \to W(R)[1/p]/(\ker\theta_{\mathbb{Q}})^j$ is identified with the projection to the quotient modulo the jth power of the maximal ideal for all $j \geq 1$.*

Proof. Since $\ker\theta_{\mathbb{Q}}$ is a nonzero principal maximal ideal (with residue field $\mathbb{C}_K$) in the domain $W(R)[1/p]$, for $j \geq 1$ we see that $W(R)[1/p]/(\ker\theta_{\mathbb{Q}})^j$ is an artinian local ring whose only ideals are $(\ker\theta_{\mathbb{Q}})^i/(\ker\theta_{\mathbb{Q}})^j$ for $0 \leq i \leq j$. In particular, an element of B_{dR}^+ is a unit if and only if it has nonzero image under θ_{dR}^+. Thus the maximal ideal $\ker\theta_{\mathrm{dR}}^+$ consists of precisely the non-units, so B_{dR}^+ is a local ring.

Consider a non-unit $b \in B_{\mathrm{dR}}^+$, so its image in each $W(R)[1/p]/(\ker\theta_{\mathbb{Q}})^j$ has the form $b_j\xi$ with b_j uniquely determined modulo $(\ker\theta_{\mathbb{Q}})^{j-1}$. In particular, the residue classes $b_j \bmod (\ker\theta_{\mathbb{Q}})^{j-1}$ are a compatible sequence and so define an element $b' \in B_{\mathrm{dR}}^+$ with $b = \xi b'$. Hence, the maximal ideal of B_{dR}^+ has the principal generator ξ.

It now follows that for each $j \geq 1$ the multiples of ξ^j in B_{dR}^+ are the elements killed by the surjective projection to $W(R)[1/p]/(\ker\theta_{\mathbb{Q}})^j$. In particular, B_{dR}^+ is ξ-adically separated, so it is a discrete valuation ring with uniformizer ξ. Since B_{dR}^+ is the inverse limit of these rings, it is also complete. □

Definition 4.23. *The field of p-adic periods (or the de Rham period ring) is* $B_{\mathrm{dR}} := \mathrm{Frac}(B_{\mathrm{dR}}^+)$ *equipped with its natural* Γ_K*-action and* Γ_K*-stable filtration via the* $\mathbb{Z}$*-powers of the maximal ideal of* B_{dR}^+*.*

In order to prove that the filtered field B_{dR} is an appropriate refinement of B_{HT}, we shall prove that B_{dR}^+ admits a uniformizer t, canonical up to $\mathbb{Z}_p^\times$-multiples, on which Γ_K acts by the cyclotomic character. Furthermore the set of such t's is naturally $\mathbb{Z}_p^\times$-equivariantly bijective with the set of $\mathbb{Z}_p$-bases of $\mathbb{Z}_p(1)$.

Choosing ε from Example 4.14, we have $\theta([\varepsilon]-1) = \varepsilon^{(0)} - 1 = 0$. Hence $[\varepsilon]-1 \in \ker\theta \subseteq \ker\theta_{\mathrm{dR}}^+$, so $[\varepsilon] = 1 + ([\varepsilon]-1)$ is a 1-unit in the complete discrete valuation ring B_{dR}^+ over K. We can therefore make sense of the logarithm

$$t := \log([\varepsilon]) = \log(1 + ([\varepsilon]-1)) = \sum_{n\geq 1} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n} \in B_{\mathrm{dR}}^+.$$

This lies in the maximal ideal of B_{dR}^+. Note that if we make another choice ε' then $\varepsilon' = \varepsilon^a$ for a unique $a \in \mathbb{Z}_p^\times$ and so $[\varepsilon'] = [\varepsilon]^a$. Thus, $t' = \log([\varepsilon']) = \log([\varepsilon]^a)$.

Remark 4.24. The identity $\log([\varepsilon]^a) = a\log([\varepsilon])$ which the reader expects requires an argument because the logarithm is defined as a convergent sum relative to a topology on B_{dR}^+ that ignores the v_R-adic topology on R. A good way to deal with this is to introduce a topological ring structure on B_{dR}^+ that is finer than its discrete valuation topology. We omit the details.

For any $\sigma \in \Gamma_K$ we have $\sigma(\varepsilon) = \varepsilon^{\chi(\sigma)}$ and so by Γ_K-equivariance of the logarithm on 1-units of B_{dR}^+,

$$\sigma(t) = \log(\sigma([\varepsilon])) = \log([\sigma(\varepsilon)]) = \log([\varepsilon^{\chi(\sigma)}]) = \log([\varepsilon]^{\chi(\sigma)}) = \chi(\sigma)t.$$

We conclude that $\mathbb{Z}_p t$ is a canonical copy of $\mathbb{Z}_p(1)$ as a Γ_K-stable line in B_{dR}^+. The key fact concerning such elements t is that they are uniformizers of B_{dR}^+, and hence we get a canonical isomorphism $\mathrm{gr}^\bullet(B_{\mathrm{dR}}) \cong B_{\mathrm{HT}}$. We now prove this uniformizer property.

Proposition 4.25. *The element* $t = \log([\varepsilon])$ *in* B_{dR}^+ *is a uniformizer.*

Proof. By construction of t, $\theta_{\mathrm{dR}}^+(t) = 0$ and so t is a non-unit. We have to prove that t is not in the sqaure of the maximal ideal. In view of its definition as an infinite series in powers $([\varepsilon]-1)^n/n$ with $[\varepsilon]-1$ in the maximal ideal, all such terms with $n \geq 2$ can be ignored. Thus, we just have to check that $[\varepsilon]-1$ is not in the square of the maximal ideal. Note that the projection from B_{dR}^+ onto the quotient modulo the square of its maximal ideal is the same as the natural map onto $W(R)[1/p]/(\ker\theta_{\mathbb{Q}})^2$, so we have to prove that $[\varepsilon]-1$ is not contained in $(\ker\theta_{\mathbb{Q}})^2$, or equivalently is not contained in $W(R) \cap (\ker\theta_{\mathbb{Q}})^2 = (\ker\theta)^2 = \xi^2 W(R)$.

To show that $[\varepsilon]-1$ is not a $W(R)$-multiple of ξ^2, it suffices to show that $\varepsilon - 1$ is not an R-multiple of $\widetilde{p}^2$. That is, it suffices to prove $v_R(\varepsilon - 1) < v_R(\widetilde{p}^2) = 2$.

From Example 4.14, we have that $v_R(\varepsilon - 1) = p/(p-1)$, so for $p > 2$ we have a contradiction. For $p = 2$, since $\xi^2 = [\widetilde{p}]^2 - 4[\widetilde{p}] + 4 = (\widetilde{p}^2, 0, \ldots)$ in $W(R)$, for any $w = (r_0, r_1, \ldots) \in W(R)$ we compute $\xi^2 w = (r_0\widetilde{p}^2, r_1\widetilde{p}^4, \ldots)$. However, for $p = 2$ we have $-1 = (1, 1, \ldots)$ and so $[\varepsilon] - 1 = (\varepsilon - 1, \varepsilon - 1, \ldots)$ in $W(R)$. Thus $\varepsilon - 1 = r_1\widetilde{p}^4$ for some $r_1 \in R$, but this implies $v_R(\varepsilon - 1) \geq v_R(\widetilde{p}^4) = 4$, a contradiction as $v_R(\varepsilon - 1) = 2$. $\square$

We end our discussion of B_{dR}^+ by recording one important property that is not easily seen from its explicit construction.

Since B_{dR}^+ is a complete discrete valuation ring, and $\overline{K}$ is a subfield of the residue field $\mathbb{C}_K$, it follows from Hensel's Lemma that $\overline{K}$ uniquely lifts to a subfield in B_{dR}^+. The uniqueness of the lifting ensures that this is a Γ_K-equivariant lifting. Thus $K \subseteq B_{\mathrm{dR}}^{\Gamma_K}$, and this inclusion is an equality, due to the Tate-Sen theorem.

Theorem 4.26. *The inclusion $K \subseteq B_{\mathrm{dR}}^{\Gamma_K}$ is an equality.*

Proof. Since the Γ_K-actions respect the (exhaustive and separated) filtration, the field extension $B_{\mathrm{dR}}^{\Gamma_K}$ of K with the subspace filtration has associated graded K-algebra $(\mathrm{gr}^\bullet(B_{\mathrm{dR}}))^{\Gamma_K} = B_{\mathrm{HT}}^{\Gamma_K}$. By the Tate-Sen theorem this latter space of invariants is K. $\square$

5 Formalism of Admissible Representations

We have seen various functors between Galois representations and semilinear algebra categories. In this section, we wish to axiomatize this kind of situation involving period rings.

Definition 5.1. *Let F be a field and G be a group. Let B be an F-algebra domain equipped with a G-action (as an F-algebra), and assume that the invariant F-subalgebra $E = B^G$ is a field. We let $C = \mathrm{Frac}(B)$, and observe that G also acts on C in a natural way. We say B is (F, G)-regular if $C^G = B^G$ and if every nonzero $b \in B$ whose F-linear span Fb is G-stable is a unit in B.*

Remark 5.2. Note if B is a field, then it is clearly (F, G)-regular.

Inspired by the theory of Hodge-Tate representations our goal is to use B to construct a functor from finite-dimensional F-linear representations of G to finite-dimensional E-vector spaces (endowed with extra structure, depending on B).

Example 5.3. Let K be a p-adic field with a fixed algebraic closure $\overline{K}$, and let $\mathbb{C}_K$ denote the completion of $\overline{K}$. Let $G = \Gamma_K = \mathrm{Gal}(\overline{K}, K)$. Let $B = B_{\mathrm{HT}} = \oplus_{n \in \mathbb{Z}} \mathbb{C}_K(n)$ endowed with its natural G-action. Non canonically, $B = \mathbb{C}_K[T, 1/T]$ and so in this case $C = \mathbb{C}_K(T)$. We claim that B is $(\mathbb{Q}_p, G)$-regular.

By the Tate-Sen theorem, $B^G = K$. To compute C^G, we consider the G-equivariant inclusion

$$C \hookrightarrow \mathbb{C}_K((T)).$$

It suffices to show that $\mathbb{C}_K((T))^G = K$. The action of $g \in G$ on a formal Laurent series $\Sigma c_n T^n$ is given by

$$\sum c_n T^n \mapsto \sum g(c_n)\chi(g)^n T^n,$$

so the result again follows from the Tate-Sen theorem.

We are left to verify the second property in $(\mathbb{Q}_p, G)$-regularity. If $b \in B\backslash\{0\}$ spans a G-stable $\mathbb{Q}_p$-line, then G acts on the line $\mathbb{Q}_p b$ by some character $\psi : G \to \mathbb{Q}_p^\times$. Note ψ must be continuous and so it takes values in $\mathbb{Z}_p^\times$. Writing the Laurent polynomial $b = \Sigma c_j T^j$, we have $\psi(g)b = g(b) = \Sigma g(c_j)\chi(g)^j T^j$, so for each j we have $(\psi^{-1}\chi^j)(g) \cdot g(c_j) = c_j$ for all $g \in G$. That is each c_j is G-equivariant in $\mathbb{C}_K(\psi^{-1}\chi^j)$. By the Tate-Sen theorem $(\psi^{-1}\chi^j)|_{I_K}$ has finite order whenever $c_j \neq 0$. Thus we cannot have $c_j, c_{j'} \neq 0$ for some $j \neq j'$, otherwise we conclude that $\chi|_{I_K}$ has finite order, a contradiction. Hence, $b = cT^j$ for some j and some $c \in \mathbb{C}_K^\times$, so $b \in B^\times$.

Example 5.4. Consider $B = B_{\mathrm{dR}}^+$ equipped with its natural action by $G = \Gamma_K$. This is a complete discrete valuation ring with uniformizer t on which G acts through χ and with fraction field $C = B_{\mathrm{dR}} = B[1/t]$. We have seen that $C^G = B^G = K$, however the second requirement in $(\mathbb{Q}_p, G)$-regularity fails: $t \in B$ spans a G-stable $\mathbb{Q}_p$-line but $t \notin B^\times$.

The most interesting examples of $(\mathbb{Q}_p, \Gamma_K)$-regular rings are Fontaine's rings B_{cris} and B_{st} (certain subrings of B_{dR}), which have subring of Γ_K-invariants equal to $K_0 = \mathrm{Frac}(W(k)) = W(k)[1/p]$ and K respectively.

In the general axiomatic setting, if B is an (F, G)-regular domain and E denotes the field $C^G = B^G$ then for any object V in the category $\mathrm{Rep}_F(G)$ of finite-dimensional F-linear representations of G we define

$$D_B(V) = (B \otimes_F V)^G$$

so $D_B(V)$ is an E-vector space equipped with a canonical map

$$\alpha_V : B \otimes_E D_B(V) \to B \otimes_E (B \otimes_F V) = (B \otimes_E B) \otimes_F V \to B \otimes_F V.$$

By inspection, this is a B-linear G-equivariant map.

We shall prove that $\dim_E D_B(V) \leq \dim_F V$. In case of equality we call V a B-admissible representation. For example if we fix a p-adic field K and let $F = \mathbb{Q}_p$ and $G = \Gamma_K$, then for $B = B_{\mathrm{HT}}$ this coincides with the concept of being a Hodge-Tate representation. For the ring B_{dR} and Fontaine's finer period rings B_{cris} and B_{st} the corresponding notions are called being a *de Rham*, *crystalline* and *semi-stable* representation respectively.

Theorem 5.5. *Fix V as above*

(1) *The map α_V is always injective and $\dim_E D_B(V) \leq \dim_F V$, with equality if and only if α_V is an isomorphism.*

(2) *Let* $\mathrm{Rep}_F^B(G) \subseteq \mathrm{Rep}_F(G)$ *be the full subcategory of* B*-admissible representations. The covariant functor* $D_B : \mathrm{Rep}_F^B(G) \to \mathrm{Vec}_E$ *to the category of finite-dimensional* E*-vector spaces is exact and faithful, and any subrepresentation or quotient of a* B*-admissible representation is* B*-admissible.*

(3) *If* $V_1, V_2 \in \mathrm{Rep}_F^B(G)$ *then there is a natural isomorphism*

$$D_B(V_1) \otimes_E D_B(V_2) \cong D_B(V_1 \otimes V_2),$$

so $V_1 \otimes_F V_2 \in \mathrm{Rep}_F^B(G)$*. If* $V \in \mathrm{Rep}_F^B(G)$ *then its dual representation* V^* *lies in* $\mathrm{Rep}_F^B(G)$ *and the natural map*

$$D_B(V) \otimes_E D_B(V^*) \cong D_B(V \otimes_F V^*) \to D_B(F) = E$$

is a perfect duality between $D_B(V)$ *and* $D_B(V^*)$*.*

In particular, $\mathrm{Rep}_F^B(G)$ *is stable under the formation of duals and tensor products in* $\mathrm{Rep}_F(G)$ *and* D_B *naturally commutes with the formation of these constructions in* $\mathrm{Rep}_F^B(G)$ *and in* Vec_E*.*

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