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Faisceaux monodromiques, théorie de Deligne-Lusztig, et cohomologie des champs de chtoucas en profondeur 0.

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Abstract. Let G be a reductive group over a finite field $\mathbb{F}_q$ of characteristic $p > 0$. In this thesis we first discuss some links between the free monodromic Hecke categories and Deligne-Lusztig theory. We start by giving a new construction of the free monodromic categories of Z. Yun [BY13] using twisted equivariant sheaves. We then use this to construct a $\overline{\mathbb{Z}}_\ell$-lift of the free monodromic Hecke categories studied by Bezrukavnikov, Riche, Yun and by Gouttard. In a second direction, we discuss how the monodromic formalism interacts with the (twisted) horocycle correspondence introduced by Lusztig and how to use it to recover some key facts of the theory. We then proceed to compute the trace of Frobenius on the monodromic Hecke category and show that we recover the category of representations of the finite group G^{F}. We apply this formalism to study the endomorphism algebra of the Gelfand-Graev representation of G^{F} and recover a result of Li expressing this algebra in terms of the dual torus.

Finally, assume that G is a quasi-split unramified group defined over local field of equal characteristic F. In this setting Lafforgue and Genestier have constructed a semisimple local Langlands correspondence. We show two expected properties for the depth 0 part of this correspondence. Namely, we show that the Langlands parameter associated to a depth 0 representation of $G(F)$ is tame and we describe the semisimple part of the image of a generator of the tame monodromy.

Résumé. Soit G un groupe réductif sur un corps fini $\mathbb{F}_q$ de caractéristique $p > 0$. Dans cette thèse, nous discutons tout d'abord des liens entre les catégories de Hecke monodromiques libres et la théorie de Deligne-Lusztig. Nous commençons par donner une nouvelle construction des catégories de faisceaux monodromiques libres de Z. Yun [BY13] en utilisant des faisceaux équivariants tordus. Nous utilisons ensuite cela pour construire un relèvement $\overline{\mathbb{Z}}_\ell$-linéaire des catégories de Hecke monodromiques libres étudiées par Bezrukavnikov, Riche, Yun et Gouttard. Dans une deuxième direction, nous discutons de l'interaction du formalisme monodromique avec la correspondance horocyclique (tordue) introduite par Lusztig dont nous nous servons pour donner de nouvelles preuves de certain résultats de la théorie de Deligne-Lusztig. Nous procédons ensuite au calcul de la trace de Frobenius sur la catégorie de Hecke monodromique et montrons que cette dernière est équivalente à la catégorie des représentations du groupe fini G^{F}. Nous appliquons ce formalisme à l'étude de l'algèbre d'endomorphismes de la représentation de Gelfand-Graev de G^{F} et retrouvons un résultat de Li exprimant cette algèbre en termes du tore dual.

Enfin, supposons que G soit un groupe quasi-déployé non ramifié défini sur un corps local d'égale caractéristique F. Dans ce contexte, Lafforgue et Genestier ont construit une correspondance de Langlands locale semi-simple. Nous montrons deux propriétés attendues pour la partie de profondeur 0 de cette correspondance. Plus précisément, nous montrons que le paramètre de Langlands associé à une représentation de profondeur 0 de $G(F)$ est modéré, et nous décrivons la partie semi-simple de l'image d'un générateur de la monodromie modéré.

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Chapter 1

Introduction en Français

1.1 Catégories de Hecke monodromiques libres

Soit $p > 0$ un nombre premier et k un corps algébriquement clos de caractéristique p.

1.1.1 Faisceaux monodromiques en tant que faisceaux équivariants

Soit T un tore sur k. Soit $\pi_1(T)$ le groupe fondamental étale de T au point géométrique 1, et soit $\pi_1^t(T)$ le quotient premier à p (ou quotient modéré). On sait que $\pi_1^t(T) = X_*(T) \otimes \hat{\mathbb{Z}}^{(p)}(1)$, où $X_*(T)$ désigne l'ensemble des cocaractères de T.

Soit X un schéma muni d'une action de T. Dans [Ver83], Verdier définit (pour $T = \mathbb{G}_m$) la notion de faisceaux monodromiques de la manière suivante. Soit $\ell \neq p$ un nombre premier, notons par $\mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)$ la catégorie dérivée des faisceaux constructibles ℓ-adiques sur X.

Définition 1.1.1 ([Ver83])**.** Un faisceau $A \in \mathrm{D}_{\mathrm{cons}}(T, \overline{\mathbb{Z}}_\ell)$ est monodromique si, pour tout j, le faisceau de cohomologie $H^j(A)$ est lisse sur T et que la représentation correspondante de $\pi_1(T)$ est modérée, c'est-à-dire qu'elle se factorise par par $\pi_1^t(T)$.

Définition 1.1.2 ([Ver83])**.** Un faisceau $A \in \mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)$ est monodromique si, pour tout $x \in X$, le faisceau $a_x^* A \in \mathrm{D}_{\mathrm{cons}}(T, \overline{\mathbb{Z}}_\ell)$ est monodromique, où $a_x : T \times X \to X$ est l'application orbite de x. Nous notons $\mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\mathrm{mon}}$ la sous-catégorie pleine des faisceaux monodromiques.

Théorème 1.1.3 ([Ver83])**.** *La catégorie $\mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\mathrm{mon}}$ est une catégorie triangulée (ou stable si nous travaillons avec des ∞-catégories). De plus, tout objet $A \in \mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\mathrm{mon}}$ possède une action canonique de $\pi_1^t(T)$ appelée la monodromie canonique. Cette action commute avec tous les morphismes de faisceaux.*

Notons $\mathrm{CH}(T)$ l'ensemble de tous les caractères continus $\pi_1^t(T) \to \overline{\mathbb{Z}}_\ell^\times$ d'ordre fini premier à ℓ. Pour chaque $\chi \in \mathrm{CH}(T)$, il existe un faisceau de Kummer $\mathcal{L}_\chi$ sur T. Nous dirons qu'un faisceau $A \in \mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)$ est χ-monodromique si sa monodromie canonique $\phi_A : \mathbb{Z}_\ell[\pi_1^t(T)] \to \mathrm{End}(A)$ se factorise à travers la complétion de $\mathbb{Z}_\ell[\pi_1^t(T)]$ le long du noyau de l'homomorphisme défini par χ. Nous notons $\mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\chi,\mathrm{mon}}$ la sous-catégorie pleine des faisceaux χ-monodromiques. Si χ est le caractère trivial, les faisceaux χ-monodromiques sont également appelés faisceaux monodromiques unipotents.

Proposition 1.1.4. *La catégorie* $\mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\mathrm{mon}}$ *est la somme directe*

$$\mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\mathrm{mon}} = \oplus_{\chi \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\chi,\mathrm{mon}}. \tag{1.1}$$

Nous notons $\Omega_{T,\mathbb{Z}_\ell}$ l'anneau

$$\Omega_{T,\mathbb{Z}_\ell} = \varprojlim_{n,m} \mathbb{Z}_\ell/\ell^n\mathbb{Z}_\ell[T[\ell^m]]. \tag{1.2}$$

Après avoir choisi une trivialisation $\pi_1^t(\mathbb{G}_m) \simeq \hat{\mathbb{Z}}^{(p)}$ et une base de $X_*(T)$, cet anneau devient isomorphe à $\mathbb{Z}_\ell[\![t_1, \ldots, t_n]\!]$, un anneau de séries formelles. Nous notons $\Omega_T = \Omega_{T,\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Z}}_\ell$.

Exemple 1.1.5. En prenant la fibre en $1 \in T$, on obtient une équivalence de catégories $\mathrm{D}_{\mathrm{cons}}(T, \overline{\mathbb{Z}}_\ell)_{\mathrm{unip,mon}} \simeq \mathrm{D}_{\mathrm{coh},\mathfrak{m}}(\Omega_T)$ entre la catégorie des faisceaux monodromiques unipotents sur T et la catégorie dérivée des faisceaux cohérents sur Ω_T supportés sur l'idéal d'augmentation de Ω_T.

L'anneau Ω_T n'est pas un anneau régulier, mais il est tout de même cohérent, c'est-à-dire que tout module de type fini est finiment présenté, la catégorie $\mathrm{D}_{\mathrm{coh}}(\Omega_T)$ se comporte ainsi comme si Ω_T était noéthérien. Cependant, la catégorie $\mathrm{D}_{\mathrm{coh},\mathfrak{m}}(\Omega_T)$ n'est pas aussi raisonnable. Pour remédier à cela, Z. Yun introduit dans [BY13], Appendice A, la notion de faisceaux monodromiques libres. Plus précisément, il construit une sous-catégorie pleine $\mathrm{D}_{\mathrm{cons}}(X \,/\!\!/\, T, \mathbb{Z}_\ell) \subset \mathrm{ProD}_{\mathrm{cons}}(X, \mathbb{Z}_\ell)_{\mathrm{mon}}$ functorielle en X et compatible avec les 6-foncteurs telle que lorsque $X = T$, nous obtenons une équivalence $\mathrm{D}_{\mathrm{cons}}(T \,/\!\!/\, T, \mathbb{Z}_\ell) = \mathrm{D}_{\mathrm{coh}}(\Omega_{T,\mathbb{Z}_\ell})$.

La difficulté de la construciton de *loc. cit.* est qu'il n'y a a priori pas de structure triangulée sur la catégorie des pro-objets dans une catégorie dérivée. L'un des points techniques est alors de construire la structure triangulée et les t-structures sur cette catégorie. Nous souhaitons donner une construction différente de cette catégorie. Jusqu'à présent, nos résultats fonctionnent bien pour les versions sur $\mathbb{Z}_\ell$ ou $\mathbb{F}_\ell$ de ces catégories, mais la version $\mathbb{Q}_\ell$ nécessite davantage de travail. L'idée principale est de réaliser cette catégorie comme une certaine catégorie de faisceaux équivariants tordus. Nous exposons maintenant les grandes lignes de cette construction.

Le formalisme des faisceaux adiques, ainsi que sa généralisation en utilisant le topos proétale de [BS15], nous permettent, en utilisant le formalisme de [HRS21], de définir pour tous les schémas de type fini X sur k deux catégories

$$\mathrm{D}_{\mathrm{cons}}(X, \Omega_T) \subset \mathrm{D}_{\mathrm{indcons}}(X, \Omega_T), \tag{1.3}$$

de faisceaux constructibles et ind-constructibles sur X respectivement. Ce sont naturellement des ∞-catégories.

Nous ajoutons une autre idée provenant de [GL96]. Il existe une application canonique

$$\mathrm{can} : \pi_1(T) \to \Omega_T^\times. \tag{1.4}$$

Cette application définit un Ω_T-système local de rang un sur T, que nous notons L_T. Cet objet est un faisceau multiplicatif sur T, c'est-à-dire qu'il existe un isomorphisme

$$m^* L_T = L_T \boxtimes_{\Omega_T} L_T \tag{1.5}$$

équipé de certaines compatibilités.

La catégorie $\mathrm{D}_{\mathrm{indcons}}(T, \Omega_T)$ est munie d'une structure monoïdale provenant de la convolution. Plus précisément, pour $A, B \in \mathrm{D}_{\mathrm{indcons}}(T, \Omega_T)$, nous définissons

$$A * B = m_!(A \boxtimes B), \tag{1.6}$$

où m est la multiplication. De même, si X est un schéma avec une action de T, la catégorie $\mathrm{D}_{\mathrm{indcons}}(T, \Omega_T)$ agit sur $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$. L'action est donnée par

$$A * B = a_!(A \boxtimes B), \tag{1.7}$$

où $a : T \times X \to X$ est l'action.

Nous suivons maintenant la construction de Gaitsgory [Gai20]. Nous pouvons tordre l'action de $\mathrm{D}_{\mathrm{indcons}}(T, \Omega_T)$ sur $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ par le faisceau multiplicatif L_T. En d'autres termes, nous définissons :

$$A *^{\mathrm{new}} B = (A \otimes L_T) * B \tag{1.8}$$

où A et B sont comme précédemment.

Définition 1.1.6. La catégorie des faisceaux équivariants tordus (T, L_T) sur X est la catégorie des invariants, au sens de *loc. cit.*, de la catégorie $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ pour l'action tordue de $\mathrm{D}_{\mathrm{indcons}}(T, \Omega_T)$. Nous la notons $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)_{\mathrm{unip}}$.

Remarque 1.1.7. D'une manière très imprécise, on peut considérer les objets de $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)_{\mathrm{unip}}$ comme des objets $A \in \mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ équipés d'un isomorphisme

$$a^* A \simeq L_T \boxtimes \Omega_T A. \tag{1.9}$$

De même, si $\chi \in \mathrm{CH}(T)$, le faisceau $L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi$ est également un faisceau multiplicatif sur T. Ainsi, nous pouvons reproduire la même construction et définir $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)_\chi$ comme la catégorie des faisceaux $(T, L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi)$-équivariants sur X. Nous pouvons également effectuer la même construction sur $\mathbb{F}_\ell, \overline{\mathbb{F}}_\ell$ ou $\mathbb{Z}_\ell$ au lieu de $\overline{\mathbb{Z}}_\ell$.

Théorème 1.1.8 (3.2.46)**.** *Il existe une équivalence naturelle de catégories*

$$\mathrm{ho}(\mathrm{D}_{\mathrm{cons}}(X, \Omega_{\mathbb{F}_\ell, T})) \simeq \mathrm{D}_{\mathrm{cons}}(X /\!\!/ T, \mathbb{F}_\ell). \tag{1.10}$$

Le théorème affirme non seulement que nous avons produit une nouvelle construction de la catégorie des faisceaux monodromiques libres sur X, mais nous avons également produit un relevé ∞-catégorique. Notre construction présente plusieurs avantages, les plus importants étant que nous n'avons pas à manipuler des pro-objets et que les six foncteurs s'étendent naturellement à ce cadre.

Remarque 1.1.9. Ce qui rend cette construction possible est la remarque vague suivante. La construction de [BY13] consiste à compléter la catégorie $\mathrm{D}_{\mathrm{cons}}(X, \mathbb{Z}_\ell)_{\mathrm{mon}}$ le long de la monodromie de Verdier. Alors que notre construction considère la catégorie de tous les faisceaux à coefficients dans Ω_T et impose ensuite que la Ω_T-structure soit la même que celle de la monodromie de Verdier.

1.1.2 Catégories de Hecke

Soit G un groupe réductif sur k, $B = TU$ une paire de Borel et W le groupe de Weyl de (G, T). L'étude des catégories de Hecke a une longue histoire. Pour les applications à la théorie de Deligne-Lusztig que nous souhaitons discuter dans cette thèse, nous nous intéresserons aux variantes monodromiques libres de ces catégories [BY13], [BR22b], [Gou21]. Plus précisément, nous souhaitons discuter des versions $\overline{\mathbb{Z}}_\ell$ de ces catégories de Hecke monodromiques libres. Elles ont été étudiées dans [BY13] pour le cas unipotent sur $\overline{\mathbb{Q}}_\ell$, dans [BR22b] pour le cas unipotent sur $\overline{\mathbb{F}}_\ell$ et dans

[Gou21] pour le cas non unipotent sur $\overline{\mathbb{F}}_\ell$. Le travail de Gouttard est lui-même une généralisation de [BR22b] et [LY20].

Considérons le champ $U\backslash G/U$ équipé de sa stratification de Bruhat. Il y a deux actions de T sur ce champ induites par translations à gauche et à droite. Ainsi, il existe trois versions de la catégorie des faisceaux monodromiques libres que nous pouvons définir.

(*i*). $\mathbb{H}^{\mathrm{left}} = \bigoplus_{\chi \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{indcons}}(U\backslash G/U, \Omega_T)^{\mathrm{left}}_\chi$ où l'équivariance est relative à l'action de T à gauche.

(*ii*). $\mathbb{H}^{\mathrm{right}} = \bigoplus_{\chi \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{indcons}}(U\backslash G/U, \Omega_T)^{\mathrm{right}}_\chi$ où l'équivariance est relative à l'action de T à droite.

(*iii*). $\mathbb{H}^{\mathrm{left,right}} = \bigoplus_{\chi,\chi' \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{indcons}}(U\backslash G/U, \Omega_{T\times T})^{\mathrm{left,right}}_{\chi,\chi'}$ où l'équivariance est relative à l'action de $T \times T$ et l'indice (χ, χ') fait référence aux faisceaux qui sont équivariants pour $L_{T\times T} \otimes_{\overline{\mathbb{Z}}_\ell} (\mathcal{L}_\chi \boxtimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_{\chi'})$.

Nous montrons qu'il existe des foncteurs d'oubli naturels

$$\mathbb{H}^{\mathrm{left}} \xleftarrow{\mathrm{For}^{\mathrm{left}}} \mathbb{H}^{\mathrm{left,right}} \xrightarrow{\mathrm{For}^{\mathrm{right}}} \mathbb{H}^{\mathrm{right}}, \tag{1.11}$$

et que ces foncteurs sont des équivalences, voir le lemme 3.4.4. Nous notons $\mathbb{H}$ l'une de ces catégories équivalentes et nous l'appelons la catégorie de Hecke universelle, empruntant la terminologie de [LNY23]

Nous suivons ensuite la construction de [BR22b] et [Gou21] pour étudier cette catégorie. Nous commençons par équiper cette catégorie d'une structure monoidale donnée par la convolution. Considérons le diagramme

$$\begin{tikzcd}
 & U\backslash G \times^U G/U \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[r, "m"] & U\backslash G/U \\
U\backslash G/U & & U\backslash G/U
\end{tikzcd}$$

où m est induit par l'application de multiplication. Ensuite, nous définissons pour $A, B \in \mathbb{H}^{\mathrm{left,right}}$

$$A * B = \mathrm{For}\, m_!(A \widehat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B)[\dim T]. \tag{1.12}$$

Décrivons maintenant ce foncteur. Tout d'abord, $(A \boxtimes_{\overline{\mathbb{Z}}_\ell} B)$ est naturellement un faisceau $\Omega_{T\times T} \otimes \Omega_{T\times T}$ sur $U\backslash G \times^U G/U$. Étant donné que les six foncteurs que nous utilisons se comportent mieux pour les anneaux complets, nous complétons d'abord ce faisceau pour passer à un $\Omega_{T\times T\times T\times T}$-faisceau, ce qui explique le symbole $\widehat{\boxtimes}$. Après avoir appliqué $m_!$, nous obtenons un faisceau $\Omega_{T\times T\times T\times T}$ sur $U\backslash G/U$, le foncteur For est le foncteur d'oubli induit par l'inclusion $\Omega_{T\times T} \to \Omega_{T\times T\times T\times T}$ induite par les deux inclusions extérieures.

Une fois cette structure monoidale construite, nous procédons à l'étude de la catégorie $\mathbb{H}$ en reprenant certaines des constructions principales de [Gou21]. Fixons $(\dot{w})$ un ensemble compatible de relèvements des éléments $w \in W$. Le choix de $\dot{w}$ donne un morphisme T-équivariant pour les actions par translation à droite $k_w : U\backslash BwB/U \to T$. Comme c'est standard en théorie de Soergel, nous définissons les faisceaux standard et costandard de la manière suivante.

Définition 1.1.10. Soit $w \in W$ et $\chi \in \mathrm{CH}(T)$.

(*i*). Le faisceau standard indexé par (w,χ) est $\Delta_{w,\chi} = i_{w,!}k_w^*(L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi)[\dim T + \ell(w)]$.

(*ii*). Le faisceau costandard indexé par (w,χ) est $\nabla_{w,\chi} = i_{w,*}k_w^*(L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi)[\dim T + \ell(w)]$.

Comme les inclusions i_w sont affines, tous ces faisceaux sont pervers.
Nous pouvons maintenant introduire les objets d'intérêt principal pour la théorie de Soergel.

Définition 1.1.11. Soit A un faisceau pervers dans $\mathbb{H}$. Une Δ-filtration (resp. une ∇-filtration) pour A est une filtration telle que toutes les gradués sont isomorphes à des faisceaux standards (resp. costandards). Le faisceau A est tilting s'il possède à la fois une Δ-filtration et une ∇-filtration.

Théorème 1.1.12 (3.B.12)**.** *Pour tous χ et w comme ci-dessus, il existe un unique faisceau tilting indécomposable qui est supporté sur la fermeture de $U\backslash BwB/U$ et tel que la multiplicité de $\Delta_{w,\chi}$ dans toute Δ-filtration (resp. la multiplicité de $\nabla_{w,\chi}$ dans toute ∇-filtration) soit égale à un.*

La démonstration de ce théorème consiste essentiellement à relever à $\overline{\mathbb{Z}}_\ell$ la preuve déjà connue dans la littérature. Nous désignons par $\mathrm{Tilt}(U\backslash G/U)$ la catégorie des faisceaux tilting pervers.

Supposons maintenant, pour simplifier, que le centre de G est connexe. Sinon, nous devrons introduire les notions de blocs, voir la section 3.4.2. Soient $\chi, \chi' \in \mathrm{CH}(T)$ dans la même orbite sous l'action de W. Nous notons

$$ {}_{\chi'}W_\chi = \{w \in W, w\chi = \chi'\}. \tag{1.13}$$

En particulier, ${}_{\chi'}W_\chi = W_\chi$ est le stabilisateur de χ. Nous équipons W de l'ordre de Bruhat, c'est-à-dire l'ordre induit par les spécialisations dans $U\backslash G/U$.

Lemme 1.1.13 ([LY20])**.** *L'ensemble ${}_{\chi'}W_\chi$, muni de l'ordre induit par W, possède un unique élément maximal. Nous notons cet élément $w^{\max}_{\chi',\chi}$.*

Remarque 1.1.14. Si $\chi = \chi'$ est trivial, alors cet élément maximal est w_0, l'élément le plus long du groupe de Weyl.

Nous notons $T_{\chi',\chi}$ le tilting correspondant à χ et $w^{\max}_{\chi',\chi}$. Nous pouvons maintenant énoncer les principaux théorèmes de la théorie.

Théorème 1.1.15 (Endomorphismensatz, 3.4.34)**.** *Pour tous les couples (χ',χ) comme précédemment, il existe un isomorphisme*

$$\mathrm{End}(T_{\chi',\chi}) = \Omega_T \otimes_{\Omega_T^{W_\chi}} \Omega_T. \tag{1.14}$$

Définition 1.1.16. Nous définissons

$$\mathbb{T} = \bigoplus_{\chi',\chi} T_{\chi',\chi}, \tag{1.15}$$

et nous l'appelons le grand faisceau tilting.

Définissons le schéma $\mathcal{C}(T)$ comme suit :

$$\mathcal{C}(T) = \bigsqcup_{\chi} \mathrm{Spec}(\Omega_T) \times \chi. \tag{1.16}$$

Cet espace a été introduit pour la première fois dans [GL96] en tant qu'espace de modules des faisceaux multiplicatifs sur T. Nous considérons le schéma suivant :

$$\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T). \tag{1.17}$$

Ce schéma est l'union de tous les graphes des actions des éléments $w \in W$. Ses composantes connexes sont indexées par les paires (χ', χ) dans la même orbite sous l'action de W. Pour une telle paire, la composante connexe correspondante est isomorphe à

$$\mathrm{Spec}(\Omega_T \otimes_{\Omega_T^{W_\chi}} \Omega_T). \tag{1.18}$$

Nous introduisons le foncteur global $\mathbb{V}$ qui est défini comme suit :

$$\begin{aligned} \mathrm{Tilt}(U\backslash G/U) &\to \mathrm{Coh}(\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T)) \\ T &\mapsto \mathrm{Hom}(\mathbb{T}, T). \end{aligned}$$

La catégorie cible est la catégorie abélienne des faisceaux cohérents sur $\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T)$. Elle est équipée d'une structure monoidale provenant de la convolution.

Théorème 1.1.17 (3.4.57, 3.4.45). (i). *Le foncteur $\mathbb{V}$ est pleinement fidèle.*

(ii). *Le foncteur $\mathbb{V}$ est monoidal.*

La première affirmation dans le théorème précédent est appelée le Struktursatz.

1.2 Quelques résultats en théorie de Deligne-Lusztig

1.2.1 Une formulation champêtre de la théorie

Soit G un groupe réductif sur $\bar{\mathbb{F}}_q$ avec un endomorphisme de Frobenius $\mathrm{F} : G \to G$ provenant d'une $\mathbb{F}_q$-structure. Soit $B = TU$ une paire de Borel stable par F et soit $W = \mathrm{N}_G(T)/T$ le groupe de Weyl de (G, T). Soit $\Lambda \in \overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Z}}_\ell$ un anneau de coefficients avec $\ell \neq p$. La théorie de Deligne-Lusztig, qui tire son nom de l'article original [DL76], étudie les représentations du groupe fini $G^{\mathrm{F}} = G(\mathbb{F}_q)$ sur des Λ-modules. Notons également par

$$\begin{aligned} \mathcal{L} : G &\to G \\ g &\mapsto g^{-1}\mathrm{F}(g), \end{aligned}$$

l'application de Lang. Classiquement, les variétés de Deligne-Lusztig sont définies comme suit. Soit $w \in W$ et définissons

$$X(w) = \{gB, \mathcal{L}(g) \in BwB\} \subset G/B, \tag{1.19}$$

et étant donné un relèvement $\dot{w} \in \mathrm{N}_G(T)$ de w,

$$Y(\dot{w}) = \{gU, \mathcal{L}(g) \in U\dot{w}U\} \subset G/U. \tag{1.20}$$

L'application naturelle $G/U \to G/B$ induit une application $\pi : Y(\dot{w}) \to X(w)$. Les faits suivants sont connus :

(i). Le groupe fini G^{F} agit par translations à gauche sur $X(w)$ et $Y(\dot{w})$, et l'application π est G^{F}-équivariante.

(ii). Le groupe fini $T^{w\mathrm{F}}$ agit par translations à droite sur $Y(\dot{w})$, et l'application π est un $T^{w\mathrm{F}}$-torseur pour cette action. De plus, les deux actions de $T^{w\mathrm{F}}$ et G^{F} sur $Y(\dot{w})$ commutent.

Nous n'utiliserons pas $X(w)$ dans cette thèse et travaillerons exclusivement avec $Y(\dot{w})$. Considérons la cohomologie

$$\mathrm{R}\Gamma_c(Y(\dot{w}),\Lambda) \in \mathrm{D}^b(\mathrm{Rep}_\Lambda(G^{\mathrm{F}} \times T^{w\mathrm{F}})). \tag{1.21}$$

Ce complexe a deux actions, celle de G^{F} et celle de $T^{w\mathrm{F}}$, et est donc un $(\Lambda[G^{\mathrm{F}}],\Lambda[T^{w\mathrm{F}}])$-bimodule. Par abstract nonsense, nous obtenons une paire de foncteurs adjoints :

$$\mathcal{R}_w : \mathrm{D}(\mathrm{Rep}_\Lambda T^{w\mathrm{F}}) \to \mathrm{D}(\mathrm{Rep}_\Lambda G^{\mathrm{F}}) \; M \mapsto M \otimes_{T^{w\mathrm{F}}} \mathrm{R}\Gamma_c(Y(\dot{w}),\Lambda),$$

et

$${}^*\mathcal{R}_w : \mathrm{D}(\mathrm{Rep}_\Lambda G^{\mathrm{F}}) \to \mathrm{D}(\mathrm{Rep}_\Lambda T^{w\mathrm{F}}) \; N \mapsto \mathrm{RHom}_{G^{\mathrm{F}}}(\mathrm{R}\Gamma_c(Y(\dot{w}),\Lambda), N),$$

appelés respectivement foncteurs d'induction et de restriction de Deligne-Lusztig.

Le fait le plus important concernant ces foncteurs est le théorème suivant.

Théorème 1.2.1 ([DL76] pour le cas $\overline{\mathbb{Q}}_\ell$, [BR03] pour un Λ général)**.** *La collection des complexes* $\mathrm{R}\Gamma_c(Y(\dot{w}),\Lambda)$ *engendre la catégorie* $\mathrm{Perf}(\Lambda[G^{\mathrm{F}}])$*,la catégorie des complexes parfaits de représentations de* G^{F}*.*

Corollaire 1.2.2 ([BR03])**.** *Si* Λ *est un corps, alors pour toute représentation irréductible* ρ *de* G^{F}*, il existe* $w \in W$ *et* $j \in \mathbb{Z}$ *tels que* ρ *soit un sous-quotient de* $H^j_c(Y(\dot{w}),\Lambda)$*.*

L'étude des complexes $\mathrm{R}\Gamma_c(Y(\dot{w}),\Lambda)$ et des foncteurs correspondants revêt une importance primordiale pour la théorie des représentations de G^{F}. L'un de ses succès les plus impressionnants est la classification des représentations irréductibles de G^{F} par Lusztig [Lus84].

Nous introduisons maintenant la correspondance F-horocyclique. Notons par Ad_{F} l'action de G sur lui-même par conjugaison tordue, c'est-à-dire l'action donnée par $g.x = gx\mathrm{F}(g^{-1})$. Nous considérons la correspondance de champs algébriques :

$$\frac{G}{\mathrm{Ad}_{\mathrm{F}}G} \xleftarrow{q} \frac{G}{\mathrm{Ad}_{\mathrm{F}}B} \xrightarrow{r} \frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}}T}. \tag{1.22}$$

Cette correspondance a été introduite sous une forme non champêtre dans [Lus15], [Lus17] et également étudiée dans [BDR20]. Cette correspondance est une version tordue de la correspondance horocyclique utilisée par Lusztig pour construire les faisceaux caractères [Lus85].

D'après le théorème de Lang, il existe un isomorphisme de champs $\frac{G}{\mathrm{Ad}_{\mathrm{F}}G} = \mathrm{pt}/G^{\mathrm{F}}$. Le côté droit de la correspondance ci-dessus est stratifié à l'aide de la stratification de Bruhat. Soit $w \in W$ et $\dot{w}$ un relèvement de W. Il existe des isomorphismes de champs

$$\frac{U\backslash BwB/U}{\mathrm{Ad}_{\mathrm{F}}T} = \dot{w}T/\mathrm{Ad}_{\mathrm{F}}(T \ltimes (U \cap \mathrm{Ad}(\dot{w})U)) = \mathrm{pt}/(T^{w\mathrm{F}} \ltimes (U \cap \mathrm{Ad}(\dot{w})U)). \tag{1.23}$$

Cet isomorphisme dépend du choix de $\dot{w}$. L'application naturelle $(T^{w\mathrm{F}} \rtimes (U \cap \mathrm{Ad}(\dot{w})U)) \to T^{w\mathrm{F}}$ induit une application

$$k_w : \frac{U\backslash BwB/U}{\mathrm{Ad}_{\mathrm{F}}T} \to \mathrm{pt}/T^{w\mathrm{F}}. \tag{1.24}$$

Notons également par $i_w : \frac{U\backslash BwB/U}{\mathrm{Ad}_{\mathrm{F}}T} \to \frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}}T}$ l'inclusion. Considérons le foncteur

$$\begin{aligned} \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/T^{w\mathrm{F}},\Lambda) &\to \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}},\Lambda) \\ M &\mapsto q_! r^* i_{w,!} k_w^M \end{aligned}$$

Sous l'équivalence naturelle $\mathrm{D}(\mathrm{pt}/T^{w\mathrm{F}},\Lambda) = \mathrm{D}(\Lambda[G^{\mathrm{F}}])$, nous montrons :

Théorème 1.2.3 (3.3.8)**.** *Le foncteur* $q_!r^*i_{w,!}k_w^*$ *est isomorphe à* $\mathcal{R}_w$*.*

Nous désignons par $\mathrm{HC}_\mathrm{F} = r_!q^*$ et par $\mathrm{CH}_\mathrm{F} = q_*r^!$ son adjoint à droite, et nous les appelons respectivement la correspondance horocyclique et la correspondance des F-caractères. Nous donnons ensuite une nouvelle démonstration du théorème 1.2.1 en montrant que

Théorème 1.2.4 (3.5.30)**.** *Le foncteur* HC_F *est conservatif.*

La démonstration de ce théorème est essentiellement une variation d'un argument de [BBM04b] et [MV88].

1.2.2 Trace catégorique de Frobenius

La catégorie $\mathbb{H}$ est une catégorie monoidale, le morphisme $\mathrm{F} : G \to G$ induit un endofoncteur monoidal $\mathrm{F}^* : \mathbb{H} \to \mathbb{H}$ de $\mathbb{H}$. Dans la situation présente, nous pouvons définir le F-centre et la trace catégorique F de $\mathbb{H}$. Ce F-centre est un raffinement ∞-catégorique du F-centre de Drinfeld tordu. Rappelons d'abord la théorie classique. Soit $\mathcal{C}$ une 1-catégorie monoidale munie d'un endofoncteur monoidal $\mathrm{F} : \mathcal{C} \to \mathcal{C}$. Son centre de F-Drinfeld tordu est la catégorie $\mathscr{Z}^1_\mathrm{F}(\mathcal{C})$ définie comme suit (où le symbole $(-)^1$ fait référence à la version 1-catégorique de cette construction).

(i). Ses objets sont des paires (X, ψ_X) où X est un objet de $\mathcal{C}$ et $\psi_X : \mathrm{F}(-) * X \to X * -$ est un isomorphisme de foncteurs compatible avec la structure tensorielle (nous n'expliciterons pas ce point).

(ii). Ses morphismes sont des morphismes de paires $u : (X, \psi_X) \to (Y, \psi_Y)$ où $u : X \to Y$ est un morphisme dans $\mathcal{C}$ compatible avec ψ_X et ψ_Y.

Pour nos besoins, nous devons passer à la version ∞-catégorique de cette construction.

Définition 1.2.5. Soit $\mathcal{C}$ une Λ-catégorie stable présentable cocomplète Λ-linéaire ∞-catégorique munie d'un endomorphisme $\mathrm{F} : \mathcal{C} \to \mathcal{C}$. Alors son centre F est défini comme suit :

$$\mathscr{Z}_\mathrm{F}(\mathcal{C}) = \mathrm{Fun}^L_{\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}}}(\mathcal{C}, \mathcal{C}_\mathrm{F}), \tag{1.25}$$

où cette catégorie de foncteurs est la catégorie des foncteurs de $\mathcal{C}$ vers $\mathcal{C}$ qui sont linéaires pour l'action de $\mathcal{C} \otimes \mathcal{C}^{\mathrm{rev}}$, où $\mathcal{C}^{\mathrm{rev}}$ est la même catégorie $\mathcal{C}$ mais avec la structure tensorielle opposée, et l'indice $(-)_\mathrm{F}$ signifie que nous tordons l'action à droite de $\mathcal{C}$ sur elle-même par l'endomorphisme F.

De manière duale, dans le cadre ∞-catégorique, il existe une notion de trace catégorique qui est définie comme suit.

Définition 1.2.6. Supposons que $\mathcal{C}$ et F sont comme ci-dessus. Alors la trace catégorique est la catégorie :

$$\mathrm{Tr}(\mathrm{F}, \mathcal{C}) = \mathcal{C} \otimes_{\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}}} \mathcal{C}_\mathrm{F}. \tag{1.26}$$

Dans le contexte des catégories de Hecke, le centre et les traces ∞-catégoriques ont été étudiés par D. Ben-Zvi et D. Nadler [BZN09]. L'idée de prendre la trace catégorique de Frobenius est venue du programme de Langlands. Plus précisément, il y a eu une tentative de concilier la construction de V. Lafforgue [Laf18] avec le programme de Langlands géométrique. L'un des résultats les plus impressionnants dans cette direction est le travail de [AGK$^+$21]. Essentiellement, en suivant l'argument de Ben-Zvi et Nadler, nous calculons le F-centre et la F-trace sur la catégorie $\mathbb{H}$.

Théorème 1.2.7 (3.5.19)**.** *Le foncteur* CH_{F} *induit une équivalence :*

$$\mathrm{Tr}(\mathrm{F}, \mathbb{H}) = \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell). \tag{1.27}$$

Le foncteur HC_{F} *induit une équivalence :*

$$\mathcal{Z}_{\mathrm{F}}(\mathbb{H}) = \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell). \tag{1.28}$$

Comme application de ce calcul, nous donnons une autre construction des ℓ-séries géométriques de Lusztig.

Théorème 1.2.8 (3.5.31)**.** *Il existe une collection complète d'idempotents orthogonaux* $e_s \in \overline{\mathbb{Z}}_\ell[G^{\mathrm{F}}]$*, où* $s \in (T^\vee /\!\!/ W)(\overline{\mathbb{F}}_\ell)$*. Ils sont caractérisés par la propriété suivante : soit* ρ *une représentation* $\overline{\mathbb{F}}_\ell$*-irréductible de* G^{F}*, alors* $e_s\rho = \rho$ *si et seulement s'il existe* (w, χ) *où* $w \in W$ *et* $\chi \in \mathrm{CH}(T) = T^\vee(\overline{\mathbb{F}}_\ell)$ *dans l'orbite correspondant à* s *telle que* ${}^*\mathcal{R}_{w,\chi}(\rho) \neq 0$*.*

1.2.3 Endomorphismes de la représentation de Gelfand-Graev

Discutons maintenant d'une application du théorème du centre à la description de l'algèbre d'endomorphismes de la représentation de Gelfand-Graev. Notons $\bar{U}$ le radical unipotent du Borel opposé à B. Le choix d'un épinglage de G détermine un morphisme

$$\phi : \bar{U} \to \bar{U}^{\mathrm{ab}} \simeq \prod_{\mathbb{G}_a} \xrightarrow{\Sigma} \mathbb{G}_a, \tag{1.29}$$

où $(-)^{\mathrm{ab}}$ désigne l'abélianisé et l'isomorphisme provient du choix de l'épinglage.

Soit $\psi : \mathbb{F}_q \to \Lambda^\times$ un caractère additif de $\mathbb{F}_q$. Nous notons encore $\psi : \bar{U}^{\mathrm{F}} \to \Lambda^\times$ sa composition avec ϕ. La représentation de Gelfand-Graev est $\Gamma_\psi = \mathrm{ind}_{\bar{U}^{\mathrm{F}}}^{G^{\mathrm{F}}}(\psi)$. Cette représentation est centrale dans la théorie des représentations de G^{F} et est la version pour les groupes réductifs finis de la représentation de Whittaker des groupes p-adiques.

Exemple 1.2.9. Si $G = \mathrm{GL}_n$ et F est la $\mathbb{F}_q$-structure provenant de la forme déployée de GL_n sur $\mathbb{F}_q$, il est connu que tous les caractères ψ de $\bar{U}^{\mathrm{F}}$ sont conjugués, et donc la représentation Γ_ψ ne dépend pas du choix de ψ. De plus, si $\Lambda = \overline{\mathbb{Q}}_\ell$, toutes les représentations cuspidales irréductibles de G^{F} sont des facteurs directs de Γ_ψ.

Pour les groupes autres que GL_n, toutes les représentations cuspidales ne sont pas nécessairement facteur direct de la représentation Γ_ψ, mais cette représentation contrôle néanmoins une grande partie de la structure de la catégorie $\mathrm{Rep}_\Lambda G^{\mathrm{F}}$.

Théorème 1.2.10 ([Ste16])**.** *Si* $\Lambda = \overline{\mathbb{Q}}_\ell$*, alors la représentation* Γ_ψ *est sans multiplicité. Pour un* Λ *général, l'algèbre d'endomorphismes* $\mathrm{End}_{\Lambda[G^{\mathrm{F}}]}(\Gamma_\psi)$ *est commutative.*

Sur $\overline{\mathbb{Q}}_\ell$, étant donné que cette représentation est sans multiplicité, sa décomposition produit de nombreuses représentations irréductibles. Plus précisément, nous avons le théorème suivant.

Théorème 1.2.11 ([DL76])**.** *Si* $\Lambda = \overline{\mathbb{Q}}_\ell$*, alors* Γ_ψ *contient exactement un facteur direct irréductible dans chaque série de Lusztig.*

Ce théorème a été initialement démontré en calculant le caractère de Γ_ψ et en le décomposant par rapport aux caractères virtuels des complexes $\mathrm{R}\Gamma_c(Y(\dot{w}), \Lambda)$. Cependant, cela peut être déduit du théorème suivant.

Théorème 1.2.12 ([Dud09])**.** *Pour tout Λ et tout w, il existe un isomorphisme*

$$^*\mathcal{R}_w(\Gamma_\psi) = \Lambda[T^{w\mathrm{F}}][\ell(w)]. \tag{1.30}$$

Nous donnons une preuve différente de ce dernier théorème en généralisant un argument de [BT22].

Théorème 1.2.13. *Il existe un isomorphisme à décalage près*

$$\mathrm{HC}_{\mathrm{F}}(\Gamma_\psi) = p_!\mathbb{T}, \tag{1.31}$$

où $\mathbb{T}$ est le grand faisceau tilting.

Notre démonstration du théorème 1.2.12 découle alors du théorème 1.2.13 et de la connaissance des fibres et des cofibres de $\mathbb{T}$. Par fonctorialité, ce théorème définit une application canonique

$$\mathrm{Cur}_w : \mathrm{End}_{\Lambda[G^{\mathrm{F}}]}(\Gamma_\psi) \to \Lambda[T^{w\mathrm{F}}], \tag{1.32}$$

appelée morphisme de w-Curtis. Elle a été initialement construite par Curtis [Cur94], en utilisant le théorème 1.2.11. Notons

$$\mathrm{Cur} = \oplus_w \mathrm{Cur}_w : \mathrm{End}_{\Lambda[G^{\mathrm{F}}]}(\Gamma_\psi) \to \oplus_w \Lambda[T^{w\mathrm{F}}], \tag{1.33}$$

la somme directe de tous les morphismes de w-Curtis. Cette application est injective. Un problème clé est de calculer son image et de l'exprimer en termes du tore dual.

Du côté dual, soit $T^\vee$ le tore dual sur Λ et $\mathrm{F}^\vee : T^\vee \to T^\vee$ le morphisme dual à F. Soit $w \in W$ et considérons le schéma des points fixes $(T^\vee)^{w\mathrm{F}^\vee}$ sous $w\mathrm{F}^\vee$. C'est le schéma tel que le diagramme suivant soit cartésien.

$$\begin{tikzcd}
(T^\vee)^{w\mathrm{F}^\vee} \arrow[r] \arrow[d] & T^\vee \arrow[d, "\mathrm{id}\times\mathrm{F}^\vee"] \\
T^\vee \arrow[r, "\Delta"'] & T^\vee \times T^\vee
\end{tikzcd}$$

De même, nous considérons également le schéma quotient GIT : $T^\vee /\!\!/ W$ où $\mathrm{F}^\vee$ le morphisme induit par $\mathrm{F}^\vee$ de $T^\vee$. Par fonctorialité des invariants, nous avons un morphisme

$$\mathrm{Cur}_w^{\mathrm{Spec}} : \mathcal{O}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee}) \to \mathcal{O}((T^\vee)^{w\mathrm{F}^\vee})$$

que nous appelons le morphisme de w-Curtis spectral. En prenant la somme directe sur w, comme précédemment, nous obtenons le morphisme de Curtis spectral,

$$\mathrm{Cur}^{\mathrm{Spec}} = \oplus_w \mathrm{Cur}_w^{\mathrm{Spec}} : \mathcal{O}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee}) \to \oplus_w \mathcal{O}((T^\vee)^{w\mathrm{F}^\vee}). \tag{1.34}$$

Après avoir choisi un ensemble de trivialisations des racines de l'unité de $\bar{\mathbb{F}}_q$, nous obtenons des isomorphismes $\Lambda[T^{w\mathrm{F}}] = \mathcal{O}((T^\vee)^{w\mathrm{F}^\vee})$ pour tout $w \in W$.

Théorème 1.2.14 ([Li21], [LS22])**.** *Supposons que ℓ soit bon pour G et que G ait un centre connexe. Il existe un isomorphisme $\mathrm{End}_{G^{\mathrm{F}}}(\Gamma_\psi) = \mathcal{O}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee})$ qui rend le diagramme suivant commutatif.*

$$\begin{tikzcd}
\mathrm{End}_{G^{\mathrm{F}}}(\Gamma_\psi) \arrow[r, equal] \arrow[d, "\mathrm{Cur}_w"'] & \mathcal{O}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee}) \arrow[d, "\mathrm{Cur}_w^{\mathrm{Spec}}"] \\
\Lambda[T^{w\mathrm{F}}] \arrow[r, equal] & \mathcal{O}((T^\vee)^{w\mathrm{F}^\vee})
\end{tikzcd}$$

La preuve de ce théorème dans *loc. cit.* se déroule comme suit. Tout d'abord, si $\Lambda = \overline{\mathbb{Q}}_\ell$, les deux algèbres $\mathrm{End}_{\overline{\mathbb{Q}}_\ell[G^{\mathrm{F}}]}(\Gamma_\psi)$ et $\mathcal{O}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee})$ sont isomorphes à $\overline{\mathbb{Q}}_\ell^n$ pour un certain n et sont donc isomorphes. Maintenant, les $\overline{\mathbb{Z}}_\ell$-versions de ces algèbres sont des réseaux à l'intérieur de chacune d'elles, ils montrent ensuite que l'isomorphisme sur $\overline{\mathbb{Q}}_\ell$ préserve ces réseaux essentiellement en calculant la matrice de ce morphisme et en vérifiant qu'il n'y a pas de dénominateurs. Nous donnons une preuve différente de ce théorème.

Théorème 1.2.15 (3.6.3)**.** *Il existe une application canonique* $\mathcal{O}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee}) \to \mathrm{End}_{G^{\mathrm{F}}}(\Gamma_\psi)$ *compatible avec les morphismes de Curtis et de Curtis spectraux.*

Une fois cette application construite, le théorème 1.2.14 découle d'un argument de formes symétrisantes. Nous donnons une construction de cette application en utilisant le théorème de centre catégorique 1.2.7.

1.3 Cohomologie des champs de chtoucas

1.3.1 Chtoucas et le programme de Langlands

Soit X une courbe lisse, projective et géométriquement connexe sur $\mathbb{F}_q$, et soit F le corps des fonctions de X. Soit G un groupe réductif sur X et notons $\mathbb{A}$ l'anneau des adèles de F. Supposons pour simplifier que G est déployé et notons $\hat{G}$ le groupe dual de G sur $\overline{\mathbb{Q}}_\ell$. Choisissons $\bar{\eta} \to X$ un point générique géométrique de X et Weil_F le groupe de Weil absolu de X au point géométrique $\bar{\eta}$.

Définition 1.3.1. Une représentation automorphe lisse et irréductible de $G(\mathbb{A})$ est un sous-quotient irréductible de $\mathcal{C}_c(G(F)\backslash G(\mathbb{A}), \overline{\mathbb{Q}}_\ell)$, l'espace des formes automorphes à support compact.

Définition 1.3.2. Un paramètre de Langlands pour G et F est un morphisme $\mathrm{Weil}_F \to \hat{G}(\overline{\mathbb{Q}}_\ell)$ qui est continu, défini sur une extension finie de $\mathbb{Q}_\ell$ et presque partout non ramifié. On dit qu'un paramètre est semi-simple si chaque fois qu'il se factorise à travers un parabolique de $\hat{G}$, alors il se factorise à travers un Levi de ce parabolique.

Conjecturalement, la correspondance de Langlands globale est une application

$$\mathrm{GLC} : \mathrm{Irr}^{\mathrm{autom}}(G(\mathbb{A})) \to \underline{Z}^1(F, \hat{G})/\hat{G}(\overline{\mathbb{Q}}_\ell), \tag{1.35}$$

qui associe à chaque représentation automorphe lisse et irréductible une classe de conjugaison de paramètre de Langlands globaux.

Conjecture 1.3.3. *Il existe une application* GLC *qui est compatible avec l'induction parabolique, avec l'isomorphisme de Satake aux places non ramifiées et avec la théorie du corps de classes globale pour les tores.*

Cette conjecture a été démontrée pour $G = \mathrm{GL}_2$ par [Dri77], puis pour $G = \mathrm{GL}_n$ par [Laf02]. En général, nous avons le théorème suivant :

Théorème 1.3.4 ([Laf18])**.** *Il existe une application* $\mathrm{GLC}^{\mathrm{ss}} : \mathrm{Irr}^{\mathrm{autom}}(G(\mathbb{A})) \to \underline{Z}^1(F, \hat{G})^{\mathrm{ss}}/\hat{G}(\overline{\mathbb{Q}}_\ell)$ *qui associe à chaque représentation automorphe lisse et irréductible un élément de classe de conjugaison dans l'ensemble des paramètres de Langlands semi-simples.*

Théorème 1.3.5 ([Xue20b])**.** *L'application* $\mathrm{GLC}^{\mathrm{ss}}$ *de [Laf18] est compatible avec l'induction parabolique.*

La construction de cette application se fait en étudiant la cohomologie des champs de chtoucas. Nous rappelons quelques faits clés concernant cette construction, en ignorant pour le moment les problèmes posés par les troncatures de Harder-Narasimhan ou par le centre de G. Soit $N \subset X$ un diviseur effectif, et soit I un ensemble fini et $W \in \mathrm{Rep}_\Lambda \hat{G}^I$. Alors il existe un champ algébrique $\mathrm{Cht}_{N,I,W}$ sur $(X-N)^I$ et un faisceau ℓ-adique $\mathcal{F}_{N,I,W}$ provenant de la correspondence de Satake géométrique. Les faisceaux de cohomologie sont ensuite définis comme suit :

$$\mathcal{H}_{N,I,W} = \mathfrak{p}_! \mathcal{F}_{N,I,W}, \tag{1.36}$$

où $\mathfrak{p} : \mathrm{Cht}_{N,I,W} \to (X-N)^I$ est l'application des pattes. En général, $\mathrm{Cht}_{N,I,W}$ n'est pas quasi-compact, il faut donc le filtrer en utilisant les troncatures de Harder-Narasimhan. Une propriété importante qu'ils possèdent est qu'ils sont munis d'actions des endomorphismes de Frobenius partiels. Plus précisément, pour $I_0 \subset I$ un sous-ensemble fini de I, il existe un endomorphisme $\mathrm{F}_{I_0} : (X-N)^I \to (X-N)^I$ donné par $\mathrm{F}_{I_0}(x_i) = (y_i)$ avec $y_i = \mathrm{F}(x_i)$ si $i \in I_0$ et $y_i = x_i$ sinon. Soit $\Delta(\bar{\eta}) \to (X-N)^I$ le point géométrique obtenu en composant $\bar{\eta} \to X - N$ avec la diagonale. En utilisant le lemme de Drinfeld [Laf18] et [Xue20d], on montre que

$$H^j_{N,I,W} = (\mathcal{H}^j_{N,I,W})|\Delta(\bar{\eta}), \tag{1.37}$$

n'est pas seulement muni d'une action de $\mathrm{Weil}((X-N)^I, \Delta(\bar{\eta}))$, mais que cette action se factorise à travers $\mathrm{Weil}(X-N)^I$. En conséquence, nous obtenons une application

$$\begin{aligned} \mathrm{Rep}_\Lambda(\hat{G}^I) &\to \mathrm{Rep}_\Lambda \mathrm{Weil}(X-N)^I \\ W &\mapsto H^j_{N,I,W}. \end{aligned}$$

Cette application est fonctorielle à la fois en I et en W. Une fois ce foncteur construit, la machinerie des opérateurs d'excursion de [Laf18] entre en jeu. En particulier, il existe une algèbre $\mathrm{Exc}(F, \hat{G})$ telle que les Λ-points(pour Λ un corps) de $\mathrm{Spec}(\mathrm{Exc}(F, \hat{G}))$ sont en bijection avec les paramètres de Langlands globaux semi-simples. De plus, étant donné tout système de foncteurs $(I, W) \mapsto H_{I,W}$ comme décrit ci-dessus, l'espace vectoriel $H_{\{0\},1}$ où 1 est la représentation triviale de $\hat{G}$ est muni d'une action de cette algèbre d'excursion.

Nous nous tournons maintenant vers le cadre des corps locaux d'égale caractéristique. Soit K un corps local de caractéristique résiduelle fixée et soit H un groupe réductif sur K, que nous supposons dans cette section être un groupe déployé. La correspondance de Langlands locale est une application conjecturale

$$\mathrm{LLC} : \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(H(K)) \to \underline{Z}^1(\mathrm{Weil}_K, \hat{H})/\hat{H}(\overline{\mathbb{Q}}_\ell) \tag{1.38}$$

qui associe à chaque représentation lisse et irréductible de $H(K)$ un paramètre de Langlands local. L'existence de cette application est connue pour GL_n [LRS93]. Bien que cette application n'ait pas été construite, il existe une version semi-simple de la correspondance qui a été construite par [GL17] et [FS21].

Théorème 1.3.6. *Il existe une application*

$$\mathrm{LLC}^{\mathrm{ss}} : \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(H(K)) \to \underline{Z}^1(\mathrm{Weil}_K, \hat{H})^{\mathrm{ss}}/\hat{H}(\overline{\mathbb{Q}}_\ell) \tag{1.39}$$

compatible avec la théorie du corps de classe et l'induction parabolique.

D'après un résultat de Li-Huerta [LH23], nous savons que les deux correspondances de Fargues-Scholze et de Lafforgue-Genestier coïncident.

1.3.2 Langlands local en profondeur 0

Nous énonçons maintenant nos principaux théorèmes concernant la correspondance locale de Genestier-Lafforgue pour les représentations de profondeur 0. Plus précisément, nous montrons une certaine compatibilité entre différentes paramétrisations de type Langlands local. Nous rappelons d'abord le théorème sur la structure des représentations de profondeur 0.

Théorème 1.3.7 ([Lan18], [Lan21])**.** *La catégorie* $\mathrm{Rep}^0_\Lambda H(K)$ *des représentations de profondeur 0 de* $H(K)$ *se décompose en une somme directe*

$$\mathrm{Rep}^0_\Lambda H(K) = \bigoplus_{s\in(\hat{T}/\!\!/W)^{\hat{\mathrm{F}}}(\Lambda)} \mathrm{Rep}^s_\Lambda H(K), \tag{1.40}$$

où $\hat{T}$ *désigne le tore dual sur* Λ, $\hat{T}/\!\!/W$ *le quotient GIT par l'action de* W *et* $(-)^{\hat{\mathrm{F}}}$ *le schéma des invariants sous le morphisme dual de Frobenius.*

Cette décomposition du théorème 1.3.7 induit une application

$$\mathrm{LS} : \mathrm{Irr}^0_\Lambda(H(K)) \to (\hat{T}/\!\!/W)^{\hat{\mathrm{F}}}(\Lambda), \tag{1.41}$$

où $\mathrm{Irr}^0_\Lambda(H(K))$ est l'ensemble des représentations irréductibles de profondeur 0 de $H(K)$, caractérisées par $\mathrm{LS}(\pi) = s$ si et seulement si π appartient au sous-groupe direct indexé par s.

Théorème 1.3.8 (4.1.6)**.** *Soit* $\pi \in \mathrm{Irr}^0_\Lambda(H(K))$. *Alors* $\mathrm{LLC}^{\mathrm{GL}}(\pi)$ *est un paramètre de Langlands local modéré. De plus, le diagramme suivant est commutatif.*

$$\begin{tikzcd}
\mathrm{Irr}^0 H(K) \arrow[r, "\mathrm{LLC}^{\mathrm{GL}}"] \arrow[d, "\mathrm{LS}"'] & (\underline{Z}^{1,t}(K,\hat{H}) /\!\!/ \hat{H})(\Lambda) \arrow[d, "\mathrm{ev}_{\tau_K}"] \\
(\hat{T}/\!\!/W)^{\hat{\mathrm{F}}} \arrow[r, equal] & (\hat{T}/\!\!/W)^{\hat{\mathrm{F}}}
\end{tikzcd}$$

La démonstration de ce théorème repose sur l'étude de certains faisceaux de cohomologie des champs de chtoucas. Pour accéder aux représentations de profondeur 0, nous commençons par fixer un tore T déployé et maximal inclus dans H sur K. Cela détermine un appartement A dans l'immeuble de Bruhat-Tits de $H(K)$. Soit σ un polysimplexe dans A et $\mathcal{H}_\sigma$ le schéma en groupe parahorique correspondant sur $\mathcal{O}_K$. Nous choisissons un point x dans X et un isomorphisme entre K et la complétion de F en x. Soit $\mathcal{G}_\sigma$ un schéma de groupe affine et lisse sur X, réductif sur $X-x$, et isomorphe à $\mathcal{H}_\sigma$ sur $\mathcal{O}_K$. Soit $N = x + N^x$ une structure de niveau telle que $x \notin N^x$. Nous considérons les groupes de cohomologie $H^j_{I,N,W}$ pour le groupe $\mathcal{G}\sigma$. Notons V_σ le radical unipotent de la fibre spéciale de $\mathcal{H}_\sigma$ et M_σ son quotient réductif. Le groupe $\mathcal{H}_\sigma(\mathbb{F}_x)$ agit sur $H^j_{I,N,W}$.

Théorème 1.3.9 (4.1.7)**.** (*i*). *Pour tout* I, W, *le* $\mathrm{Weil}^I_{F_x}$*-module* $(H^j_{I,W,N})^{V_\sigma(\mathbb{F}_x)}$ *est modérément ramifié, c'est-à-dire que l'action se factorise à travers le quotient modéré* $(\mathrm{Weil}^t_{F_x})^I$.

(*ii*). *Soit* $s \in (\hat{T}/\!\!/W)^{\hat{\mathrm{F}}}(\Lambda)$, *alors en tant que* $\mathrm{Exc}(\mathrm{Weil}^t_{F_x}, \hat{G})$*-module,* $e_s(H^j_{I,W,N})^{V_\sigma(\mathbb{F}_x)}$ *est supporté sur* $\mathrm{ev}^{-1}_{\tau_{F_x}}(s)$, *où* e_s *est l'idempotent dans* $\Lambda[M_\sigma(\mathbb{F}_x)]$ *correspondant à la série de Lusztig associée à* s, *nous renvoyons à la Section 4.2 pour les notations.*

Enfin, la construction principale de [LZ18] produit un faisceau quasi-cohérent $\mathcal{M}_N^j$ sur le champ des paramètres de Langlands locaux, qui est canoniquement associé au système de foncteurs $(I, W) \mapsto H^j_{I,N,W}$.

Corollaire 1.3.10 (4.1.8)**.** (i). *Le faisceau quasi-cohérent* $(\mathcal{M}_N^j)^{V_\sigma(\mathbb{F}_x)}$ *est supporté sur* $\underline{Z}^{1,t}(F_x, \hat{G})$.

(ii). *En utilisant les mêmes notations que dans 4.1.7, soit* $s \in (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}}(\overline{\mathbb{Q}}_\ell)$*, alors le faisceau quasi-cohérent* $e_s(\mathcal{M}_N^j)^{V_\sigma(\mathbb{F}_x)}$ *est supporté sur* $\mathrm{ev}_{\tau_{F_x}}^{-1}(s)$.

Organisation

Cette thèse comporte deux chapitres indépendants. Le premier contient les résultats concernant les faisceaux monodromiques, la théorie de Soergel et le théorie de Deligne-Lusztig. Le deuxième contient les résultats concernant les champs de chtoucas et la correspondence de Langlands locale. Chacun des deux chapitres est précédé d'une introduction détaillée.

Chapter 2

Introduction in English

2.1 Free monodromic Hecke categories

Let $p > 0$ be a prime number and let k be an algebraically closed field of characteristic p.

2.1.1 Monodromic sheaves as equivariant sheaves

Let T be a torus over k. Let $\pi_1(T)$ be the étale fundamental group of T at the geometric point 1 and let $\pi_1^t(T)$ be the itd prime to p-quotient (or tame quotient). It is known that $\pi_1^t(T) = X_*(T) \otimes \hat{\mathbb{Z}}^{(p)}(1)$ where $X_*(T)$ denotes the set of cocharacters of T.

Let X be a scheme with an action of T. In [Ver83], Verdier defines (for $T = \mathbb{G}_m$) the notion of monodromic sheaves as follows. First let $\ell \neq p$ be a prime and let us denote by $\mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)$ the derived category of constructible ℓ-adic sheaves on X.

Definition 2.1.1 ([Ver83])**.** A sheaf $A \in \mathrm{D}_{\mathrm{cons}}(T, \overline{\mathbb{Z}}_\ell)$ is monodromic if for all j, the cohomology sheaf $H^j(A)$ is lisse on T and the corresponding representation of $\pi_1(T)$ is tame, that is, factors through $\pi_1^t(T)$.

Definition 2.1.2 ([Ver83])**.** A sheaf $A \in \mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)$ is monodromic if for all $x \in X$, the sheaf $a_x^* A \in \mathrm{D}_{\mathrm{cons}}(T, \overline{\mathbb{Z}}_\ell)$ is monodromic, where $a_x : T \times X \to X$ is the orbit map of x. We denote by $\mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\mathrm{mon}}$ the full subcategory of monodromic sheaves.

Theorem 2.1.3 ([Ver83])**.** *The category $\mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\mathrm{mon}}$ is a triangulated (or stable if we work with ∞-categories) category. Furthermore any object $A \in \mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\mathrm{mon}}$ has a canonical action of $\pi_1^t(T)$ called the canonical monodromy. This action commutes with all morphism of sheaves.*

Let us denote by $\mathrm{CH}(T)$ the set of all continuous characters $\pi_1^t(T) \to \overline{\mathbb{Z}}_\ell^\times$ of finite order prime to ℓ. For each $\chi \in \mathrm{CH}(T)$, there is a Kummer sheaf $\mathcal{L}_\chi$ on T. We say that a sheaf $A \in \mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)$ is χ-monodromic if its canonical monodromy $\phi_A : \mathbb{Z}_\ell[\pi_1^t(T)] \to \mathrm{End}(A)$ factors through the completion of $\mathbb{Z}_\ell[\pi_1^t(T)]$ along the kernel of the morphism defined by χ. We denote by $\mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\chi,\mathrm{mon}}$ the full subcategory of χ-monodromic sheaves. If χ is the trivial character, χ-monodromic sheaves are also called unipotent monodromic.

Proposition 2.1.4. *The category* $\mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\mathrm{mon}}$ *is the direct sum*

$$\mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\mathrm{mon}} = \oplus_{\chi \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\chi,\mathrm{mon}}. \tag{2.1}$$

We denote by $\Omega_{T,\mathbb{Z}_\ell}$ the ring

$$\Omega_{T,\mathbb{Z}_\ell} = \varprojlim_{n,m} \mathbb{Z}_\ell/\ell^n\mathbb{Z}_\ell[T[\ell^m]]. \tag{2.2}$$

After choosing a trivialization $\pi_1^t(\mathbb{G}_m) \simeq \hat{\mathbb{Z}}^{(p)}$ and a basis of $X_*(T)$, this ring becomes isomorphic to $\mathbb{Z}_\ell[\![t_1, \dots, t_n]\!]$ a ring of power series. We denote by $\Omega_T = \Omega_{T,\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Z}}_\ell$.

Example 2.1.5. Taking the fiber at $1 \in T$, yields an equivalence of categories $\mathrm{D}_{\mathrm{cons}}(T, \overline{\mathbb{Z}}_\ell)_{\mathrm{unip,mon}} \simeq \mathrm{D}_{\mathrm{coh},\mathfrak{m}}(\Omega_T)$ between the category of unipotent monodromic sheaves on T and the derived category of coherent sheaves on Ω_T supported on the augmentation ideal of Ω_T.

The ring Ω_T is not a regular ring but is still coherent, that is, every finite type module is finitely presented, hence the category $\mathrm{D}_{\mathrm{coh}}(\Omega_T)$ is well behaved. However the category $\mathrm{D}_{\mathrm{coh},\mathfrak{m}}(\Omega_T)$ is not as nice. To remedy this Z. Yun introduced the notion of free monodromic sheaves in [BY13], Appendix A. Namely, he constructs a full subcategory $\mathrm{D}_{\mathrm{cons}}(X /\!\!/ T, \mathbb{Z}_\ell) \subset \mathrm{ProD}_{\mathrm{cons}}(X, \mathbb{Z}_\ell)_{\mathrm{mon}}$ functorial in X and compatible with the 6-functors such that when $X = T$, we have an equivalence $\mathrm{D}_{\mathrm{cons}}(T /\!\!/ T, \mathbb{Z}_\ell) = \mathrm{D}_{\mathrm{coh}}(\Omega_{T,\mathbb{Z}_\ell})$.

The construction of *loc. cit.* is highly non trivial as there is a priori no triangulated structure on the category of pro-objects on a derived category. One of the difficult technical points is then to construct the triangulated structure and the t-structures on it. We want to give a different construction of this category. So far our results work well for the $\mathbb{Z}_\ell$ or $\mathbb{F}_\ell$ versions of these categories, the $\mathbb{Q}_\ell$-version requires more work. The main idea is to realize this category as a certain category of twisted equivariant sheaves. We now outline this construction.

The formalism of adic sheaves, and its generalization using the proétale topos of [BS15] allows us, using the formalism of [HRS21], to define for all finite type schemes X over k two categories

$$\mathrm{D}_{\mathrm{cons}}(X, \Omega_T) \subset \mathrm{D}_{\mathrm{indcons}}(X, \Omega_T), \tag{2.3}$$

of constructible and ind-constructible sheaves on X respectively. These are naturally ∞-categories.

We add in another idea coming from [GL96]. There is a canonical map

$$\mathrm{can} : \pi_1(T) \to \Omega_T^\times. \tag{2.4}$$

This map defines a rank one Ω_T-local system L_T on T. This object is multiplicative sheaf on T, that is, there is an isomorphism

$$m^* L_T = L_T \boxtimes_{\Omega_T} L_T \tag{2.5}$$

equipped with certain compatibilities.

The category $\mathrm{D}_{\mathrm{indcons}}(T, \Omega_T)$ is equipped with a monoidal structure coming from convolution. Namely, for $A, B \in \mathrm{D}_{\mathrm{indcons}}(T, \Omega_T)$, we define

$$A * B = m_!(A \boxtimes B), \tag{2.6}$$

where m is the multiplication map. Similarly, if X is a scheme with an action of T, the category $\mathrm{D}_{\mathrm{indcons}}(T, \Omega_T)$ acts on $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$. The action is given for $A \in \mathrm{D}_{\mathrm{indcons}}(T, \Omega_T)$ and $B \in \mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ by

$$A * B = a_!(A \boxtimes B), \tag{2.7}$$

where a is the action map.

Now we follow a construction of Gaitsgory [Gai20]. We can twist the action of $\mathrm{D}_{\mathrm{indcons}}(T, \Omega_T)$ on $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ by the multiplicative sheaf L_T. Namely, we set

$$A *^{\mathrm{new}} B = (A \otimes L_T) * B \tag{2.8}$$

where A and B are as before.

Definition 2.1.6. The category of (T, L_T)-twisted equivariant sheaves on X is the category of invariants, in the sense of *loc. cit.* of the category $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ for the twisted action of $\mathrm{D}_{\mathrm{indcons}}(T, \Omega_T)$. We denote it by $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)_{\mathrm{unip}}$.

Remark 2.1.7. In some very imprecise way, we can think of objects in $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)_{\mathrm{unip}}$ as objects $A \in \mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ equipped with an isomorphism

$$a^* A \simeq L_T \boxtimes_{\Omega_T} A. \tag{2.9}$$

Similarly, if $\chi \in \mathrm{CH}(T)$, the sheaf $L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi$ is also a multiplicative sheaf on T. Hence we can reproduce the same construction and define $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)_\chi$ to be the category $(T, L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi)$-equivariant sheaves on X. We can also do the same over $\mathbb{F}_\ell, \overline{\mathbb{F}}_\ell$ or $\mathbb{Z}_\ell$ in place of $\overline{\mathbb{Z}}_\ell$.

Theorem 2.1.8 (3.2.46)**.** *There is a natural equivalence of categories*

$$\mathrm{ho}(\mathrm{D}_{\mathrm{cons}}(X, \Omega_{\mathbb{F}_\ell, T})) \simeq \mathrm{D}_{\mathrm{cons}}(X /\!\!/ T, \mathbb{F}_\ell). \tag{2.10}$$

The theorem not only states that we have produced a new construction of the category of free monodromic sheaves on X but we have also produced an ∞-categorical enhancement of this category. Our construction has has several advantages, the most important ones are that we do not have to manipulate pro-objects and that the six functors naturally extend to this setting.

Remark 2.1.9. What makes this construction work is the following vague remark. The construction of [BY13] consists in completing the category $\mathrm{D}_{\mathrm{cons}}(X, \mathbb{Z}_\ell)_{\mathrm{mon}}$ along Verdier's monodromy. While our construction consider the category of all sheaves with Ω_T-coefficients and then enforces the Ω_T-structure to be the same structure as Verdier's monodromy.

2.1.2 Hecke categories

Let G be a reductive group over k, let $B = TU$ be a Borel pair and W be the Weyl group of (G, T). The study of Hecke categories has a long story. For the applications to Deligne-Lusztig theory we want to discuss in this thesis, we will be interested in the free monodromic variants of these categories [BY13], [BR22b], [Gou21]. More specifically, we want to discuss $\overline{\mathbb{Z}}_\ell$-versions of these free monodromic Hecke categories. They were studied in [BY13] for the unipotent $\overline{\mathbb{Q}}_\ell$-case, [BR22b] for the $\overline{\mathbb{F}}_\ell$ unipotent case and in [Gou21] for the non-unipotent $\overline{\mathbb{F}}_\ell$-case. The work of Gouttard is itself a generalization of [BR22b] and [LY20].

Consider the stack $U \backslash G / U$ equipped with its Bruhat stratification. There are two actions of T on this stack induced by left and right translations. Hence there are three versions of the category of free monodromic sheaves that we can define.

(i). $\mathbb{H}^{\mathrm{left}} = \bigoplus_{\chi \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{indcons}}(U \backslash G / U, \Omega_T)^{\mathrm{left}}_\chi$ where the equivariance is relative to the action of T on the left.

(*ii*). $\mathbb{H}^{\mathrm{right}} = \bigoplus_{\chi \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{indcons}}(U\backslash G/U, \Omega_T)_{\chi}^{\mathrm{right}}$ where the equivariance is relative to the action of T on the right.

(*iii*). $\mathbb{H}^{\mathrm{left,right}} = \bigoplus_{\chi,\chi' \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{indcons}}(U\backslash G/U, \Omega_{T\times T})_{\chi,\chi'}^{\mathrm{left,right}}$ where the equivariance is relative to the action of $T \times T$ on the right and the index (χ, χ') refer to sheaves that are equivariant for $L_{T\times T} \otimes_{\overline{\mathbb{Z}}_\ell} (\mathcal{L}_\chi \boxtimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_{\chi'})$.

We show that there are natural forgetful functors

$$\mathbb{H}^{\mathrm{left}} \xleftarrow{\mathrm{For}^{\mathrm{left}}} \mathbb{H}^{\mathrm{left,right}} \xrightarrow{\mathrm{For}^{\mathrm{right}}} \mathbb{H}^{\mathrm{right}}. \tag{2.11}$$

And that these are are equivalences, see lemma 3.4.4. We denote by $\mathbb{H}$ either of these equivalent categories and we call it the universal Hecke category, borrowing the terminology from [LNY23].

We then follow most of the construction of [BR22b] and [Gou21] to study this category. We start by equipping this category with a monoidal structure given by convolution. Namely, consider the diagram

$$\begin{tikzcd} & U\backslash G \times^U G/U \arrow[r, "m"] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & U\backslash G/U \\ U\backslash G/U & & U\backslash G/U \end{tikzcd}$$

where m is induced by the multiplication map. Then we define for $A, B \in \mathbb{H}^{\mathrm{left,right}}$

$$A * B = \mathrm{For}\ m_!(A \widehat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B)[\dim T]. \tag{2.12}$$

Let us describe this functor. Firstly, $(A \boxtimes_{\overline{\mathbb{Z}}_\ell} B)$ is naturally an $\Omega_{T\times T} \otimes \Omega_{T\times T}$ sheaf on $U\backslash G \times^U G/U$. Since the six functors we are using behave best for complete rings, we first complete this sheaf to pass to an $\Omega_{T\times T\times T\times T}$, this explains the $\widehat{\boxtimes}$. After applying $m_!$ we get an $\Omega_{T\times T\times T\times T}$-sheaf on $U\backslash G/U$, the functor For is the forgetful functor induced by the inclusion $\Omega_{T\times T} \to \Omega_{T\times T\times T\times T}$ induced by the two outer inclusions.

Once this monoidal structure is constructed we proceed with the study of the category $\mathbb{H}$ by lifting some of the main constructions of [Gou21]. Let us fix $(\dot{w})$ a compatible set of lifts of the elements $w \in W$. The choice of $\dot{w}$ gives a T-equivariant morphism for the actions by right translations $k_w : U\backslash BwB/U \to T$. As it is standard in Soergel theory, we define the standard and costandard sheaves as follows.

Definition 2.1.10. Let $w \in W$ and $\chi \in \mathrm{CH}(T)$.

(*i*). The standard sheaf indexed by (w, χ) is $\Delta_{w,\chi} = i_{w,!} k_w^* (L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi)[\dim T + \ell(w)]$.

(*ii*). The costandard sheaf indexed by (w, χ) is $\nabla_{w,\chi} = i_{w,*} k_w^* (L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi)[\dim T + \ell(w)]$.

Since the inclusions i_w are affine all those sheaves are perverse. We can now introduce the objects of main interest for Soergel theory.

Definition 2.1.11. Let A be a perverse sheaf in $\mathbb{H}$. A Δ-filtration (resp. a ∇-filtration) for A is a filtration such that all graded pieces are isomorphic to standard sheaves (resp. costandard sheaves). The sheaf A is tilting if A has both a Δ and a ∇-filtration.

Theorem 2.1.12 (3.B.12)**.** *For all χ and w as above, there exists a unique indecomposable tilting sheaf that is supported on the closure of $U\backslash BwB/U$ and such that the multiplicity of $\Delta_{w,\chi}$ in any Δ-filtration (reps the multiplicity of $\nabla_{w,\chi}$ in any ∇-filtration) is one.*

The proof of this theorem consists essentially in lifting to $\overline{\mathbb{Z}}_\ell$ the already known proof from the literature. We denote by $\mathrm{Tilt}(U\backslash G/U)$ the category of perverse tilting sheaves.

Let us now assume that G has connected center for simplicity, otherwise we will have to introduce the notions of blocks, see section 3.4.2. Let $\chi, \chi' \in \mathrm{CH}(T)$ be in the same W-orbit. We denote by

$$ {}_{\chi'}W_\chi = \{w \in W, w\chi = \chi'\}. \tag{2.13} $$

In particular ${}_{\chi'}W_\chi = W_\chi$ is the stabilizer of χ. We equip W with the Bruhat order, that is, the order induced by specializations in $U\backslash G/U$.

Lemma 2.1.13 ([LY20])**.** *The set ${}_{\chi'}W_\chi$ equipped with the order induced from W has a unique maximal element. We denote this element by $w^{\max}_{\chi',\chi}$.*

Remark 2.1.14. If $\chi = \chi'$ is trivial, then this maximal element is w_0, the longest element in the Weyl group.

We denote by $T_{\chi',\chi}$ the tilting corresponding to χ and $w^{\max}_{\chi',\chi}$. We can now state the main theorems of the theory.

Theorem 2.1.15 (Endomorphismensatz, 3.4.34)**.** *For all pairs (χ', χ) as above. There is an isomorphism*

$$ \mathrm{End}(T_{\chi',\chi}) = \Omega_T \otimes_{\Omega_T^{W_\chi}} \Omega_T. \tag{2.14} $$

Definition 2.1.16. We define

$$ \mathbb{T} = \bigoplus_{\chi',\chi} T_{\chi',\chi}, \tag{2.15} $$

and we call it the big tilting sheaf.

Define the scheme $\mathcal{C}(T)$ to be

$$ \mathcal{C}(T) = \sqcup_\chi \mathrm{Spec}(\Omega_T) \times \{\chi\}. \tag{2.16} $$

This space was first introduced in [GL96] as the moduli space of multiplicative sheaves on T. We consider the scheme

$$ \mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T). \tag{2.17} $$

This scheme is the union of all the graphs of the actions of the elements $w \in W$. Its connected components are indexed by pairs (χ', χ) in the same W-orbit. For such a pair the corresponding connected component is isomorphic to

$$ \mathrm{Spec}(\Omega_T \otimes_{\Omega_T^{W_\chi}} \Omega_T). \tag{2.18} $$

We introduce the global $\mathbb{V}$-functor which is defined as

$$ \begin{aligned} \mathrm{Tilt}(U\backslash G/U) &\to \mathrm{Coh}(\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T)) \\ T &\mapsto \mathrm{Hom}(\mathbb{T}, T). \end{aligned} $$

The target category is the abelian category of coherent sheaves on $\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T)$. It is equipped with a monoidal structure coming from convolution.

Theorem 2.1.17 (3.4.57, 3.4.45)**.** *(i). The functor $\mathbb{V}$ is fully faithful.*

(ii). The functor $\mathbb{V}$ is monoidal.

The first statement in the previous theorem is called the Struktursatz.

2.2 Topics in Deligne-Lusztig theory

2.2.1 Stacky formulation of the theory

Let G be a reductive group over $\bar{\mathbb{F}}_q$ with a Frobenius endomorphism $\mathrm{F} : G \to G$ coming from some $\mathbb{F}_q$-structure. Let $B = TU$ be a F-stable Borel pair and let $W = \mathrm{N}_G(T)/T$ be of the Weyl group of (G, T). Let $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Z}}_\ell\}$ be a coefficient ring with $\ell \neq p$. Deligne-Lusztig theory, taking its name from the original paper [DL76], is a theory that studies the theory of representations of the finite group $G^{\mathrm{F}} = G(\mathbb{F}_q)$ on Λ-modules. Let us also denotes by

$$\begin{aligned} \mathcal{L} : G &\to G \\ g &\mapsto g^{-1}\mathrm{F}(g), \end{aligned}$$

the Lang map. Classically, the Deligne-Lusztig varieties are defined as follows. Let $w \in W$ and define

$$X(w) = \{gB, \mathcal{L}(g) \in BwB\} \subset G/B, \tag{2.19}$$

and given a lift $\dot{w} \in \mathrm{N}_G(T)$ of w,

$$Y(\dot{w}) = \{gU, \mathcal{L}(g) \in U\dot{w}U\} \subset G/U. \tag{2.20}$$

The natural map $G/U \to G/B$ induces a map $\pi : Y(\dot{w}) \to X(w)$. The following facts are known to hold

(i). The finite group G^{F} acts by left translations on $X(w)$ and $Y(\dot{w})$ and the map π is G^{F}-equivariant.

(ii). The finite group $T^{w\mathrm{F}}$ acts by right translations on $Y(\dot{w})$ and the map π is a $T^{w\mathrm{F}}$-torsor for this action. Moreover the two actions of $T^{w\mathrm{F}}$ and G^{F} on $Y(\dot{w})$ commute.

We will not use $X(w)$ in this thesis and work exclusively with $Y(\dot{w})$. Consider the cohomology

$$\mathrm{R}\Gamma_c(Y(\dot{w}), \Lambda) \in \mathrm{D}^b(\mathrm{Rep}_\Lambda(G^{\mathrm{F}} \times T^{w\mathrm{F}})). \tag{2.21}$$

This complex has two actions of G^{F} and $T^{w\mathrm{F}}$ and is thus a $(\Lambda[G^{\mathrm{F}}], \Lambda[T^{w\mathrm{F}}])$-bimodule. By general nonsense, we get a pair of adjoint functors

$$\begin{aligned} \mathcal{R}_w : \mathrm{D}(\mathrm{Rep}_\Lambda T^{w\mathrm{F}}) &\to \mathrm{D}(\mathrm{Rep}_\Lambda G^{\mathrm{F}}) \\ M &\mapsto M \otimes_{T^{w\mathrm{F}}} \mathrm{R}\Gamma_c(Y(\dot{w}), \Lambda), \end{aligned}$$

and

$$\begin{aligned} {}^*\mathcal{R}_w : \mathrm{D}(\mathrm{Rep}_\Lambda G^{\mathrm{F}}) &\to \mathrm{D}(\mathrm{Rep}_\Lambda T^{w\mathrm{F}}) \\ N &\mapsto \mathrm{RHom}_{G^{\mathrm{F}}}(\mathrm{R}\Gamma_c(Y(\dot{w}), \Lambda), N), \end{aligned}$$

called respectively Deligne-Lusztig, or Lusztig, induction and restriction functors.

The most important fact concerning these functors is the following theorem.

Theorem 2.2.1 ([DL76] for the $\overline{\mathbb{Q}}_\ell$-case, [BR03] for a general Λ)**.** *The collection of complexes* $\mathrm{R}\Gamma_c(Y(\dot{w}), \Lambda)$ *generate the category* $\mathrm{Perf}(\Lambda[G^{\mathrm{F}}])$ *or perfect complexes of representations of* G^{F}*.*

Corollary 2.2.2 ([BR03])**.** *If* Λ *is a field, then for all irreducible representations* ρ *of* G^{F}*, there exists* $w \in W$ *and* $j \in \mathbb{Z}$ *such that* ρ *is a subquotient of* $H^j_c(Y(\dot{w}), \Lambda)$*.*

The study of the complexes $\mathrm{R}\Gamma_x(Y(\dot{w}), \Lambda)$ and the corresponding functors is of prime importance for the theory of representations of G^{F}. One of its most impressive success is the classification of irreducible representations of G^{F} by Lusztig [Lus84].

We now introduce the F-horocycle correspondence. Denote by Ad_{F} the action of G on itself by twisted conjugation, that is, the action given by $g.x = gx\mathrm{F}(g^{-1})$. We consider the correspondence of stacks

$$\frac{G}{\mathrm{Ad}_{\mathrm{F}} G} \xleftarrow{q} \frac{G}{\mathrm{Ad}_{\mathrm{F}} B} \xrightarrow{r} \frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T}. \tag{2.22}$$

This correspondence was introduced in a somewhat non stacky form in [Lus15], [Lus17] and also studied in [BDR20]. This correspondence is a twisted version of the horocycle correspondence used by Lusztig to construct character sheaves [Lus85].

By Lang's theorem there is an isomorphism of stacks $\frac{G}{\mathrm{Ad}_{\mathrm{F}} G} = \mathrm{pt}/G^{\mathrm{F}}$. The right hand side stack is stratified using the Bruhat stratification. Let $w \in W$ and $\dot{w}$ be a lift of W. There are isomorphisms of stacks

$$\frac{U\backslash BwB/U}{\mathrm{Ad}_{\mathrm{F}} T} = \dot{w}T/\mathrm{Ad}_{\mathrm{F}}(T \rtimes (U \cap \mathrm{Ad}(\dot{w})U)) = \mathrm{pt}/(T^{w\mathrm{F}} \rtimes (U \cap \mathrm{Ad}(\dot{w})U)). \tag{2.23}$$

This isomorphism depends on the choice of $\dot{w}$. The natural map $(T^{w\mathrm{F}} \rtimes (U \cap \mathrm{Ad}(\dot{w})U)) \to T^{w\mathrm{F}}$ induces a map

$$k_w : \frac{U\backslash BwB/U}{\mathrm{Ad}_{\mathrm{F}} T} \to \mathrm{pt}/T^{w\mathrm{F}}. \tag{2.24}$$

Let us also denote by $i_w : \frac{U\backslash BwB/U}{\mathrm{Ad}_{\mathrm{F}} T} \to \frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T}$ the inclusion. Consider the functor

$$\begin{aligned} \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/T^{w\mathrm{F}}, \Lambda) &\to \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \Lambda) \\ M &\mapsto q_! r^* i_{w,!} k_w^* M \end{aligned}$$

Under the natural equivalence $\mathrm{D}(\mathrm{pt}/T^{w\mathrm{F}}, \Lambda) = \mathrm{D}(\Lambda[G^{\mathrm{F}}])$ we show.

Theorem 2.2.3 (3.3.8)**.** *The functor* $q_! r^* i_{w,!} k_w^*$ *is isomorphic to* $\mathcal{R}_w$*.*

We denote by $\mathrm{HC}_{\mathrm{F}} = r_! q^*$ and by $\mathrm{CH}_{\mathrm{F}} = q_* r^!$ its right adjoint and we call them the F-horocycle and F-character correspondence. We then give a new proof of theorem 2.2.1 by showing that

Theorem 2.2.4 (3.5.30)**.** *The functor* HC_{F} *is conservative.*

The proof of this theorem is essentially a variation on an argument of [BBM04b] and [MV88].

2.2.2 Categorical traces of Frobenius

The category $\mathbb{H}$ is a monoidal category, the morphism $\mathrm{F} : G \to G$ induces a monoidal endofunctor $\mathrm{F}^* : \mathbb{H} \to \mathbb{H}$ of $\mathbb{H}$. In the present situation, we can define the F-center and the F categorical trace of $\mathbb{H}$. This F-center is an ∞-categorical refinement of the F-twisted Drinfeld center. Let us first recall the classical theory. Let $\mathcal{C}$ be a monoidal 1-category equipped with a monoidal endofunctor $\mathrm{F} : \mathcal{C} \to \mathcal{C}$. Its F-twisted Drinfeld center is the category $\mathcal{Z}^1_{\mathrm{F}}(\mathcal{C})$ defines as follows (where the $(-)^1$ refers to the 1-categorical version of this construction).

(i). Its objects are pairs (X, ψ_X) where X is an object of $\mathcal{C}$ and $\psi : \mathrm{F}(-) * X \to X * -$ is an isomorphism of functors compatible with the tensor structure (we do not make this point explicit).

(ii). Its morphisms are morphisms of pairs $u : (X, \psi_X) \to (Y, \psi_Y)$ where $u : X \to Y$ is a morphism in $\mathcal{C}$ compatible with ψ_X and ψ_Y.

For our purposes, we need to move up to the ∞-categorical version of this construction.

Definition 2.2.5. Let $\mathcal{C}$ be monoidal stable presentable cocomplete Λ-linear ∞-category equipped with an endomorphism $\mathrm{F} : \mathcal{C} \to \mathcal{C}$. Then its F-center is defined as

$$\mathcal{Z}_{\mathrm{F}}(\mathcal{C}) = \mathrm{Fun}^L_{\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}}}(\mathcal{C}, \mathcal{C}_{\mathrm{F}}), \tag{2.25}$$

where this category of functors is the category of functors from $\mathcal{C}$ to $\mathcal{C}$ that are linear for the action of $\mathcal{C} \otimes \mathcal{C}^{\mathrm{rev}}$ where $\mathcal{C}^{\mathrm{rev}}$ is the same category $\mathcal{C}$ but with the opposite tensor structure and the index $(-)_{\mathrm{F}}$ means that we twist the right action of $\mathcal{C}$ on itself by the endomorphism F.

Dually, in the ∞-categorical setting there is a notion of categorical trace which is defined as follows.

Definition 2.2.6. Assume $\mathcal{C}$ and F are as above. Then the categorical trace is the category

$$\mathrm{Tr}(\mathrm{F}, \mathcal{C}) = \mathcal{C} \otimes_{\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}}} \mathcal{C}_{\mathrm{F}}. \tag{2.26}$$

In the context of Hecke categories, the ∞-categorical center and traces were studied by D. Ben-Zvi and D. Nadler [BZN09]. The idea to take the categorical trace of Frobenius came from the Langlands program. Namely, there was an attempt to reconcile V. Lafforgue's construction [Laf18] with the geometric Langlands program. One of the most impressive results in this direction is the work of [AGK+21]. Essentially by following the argument of Ben-Zvi and Nadler, we compute the F-center and F-trace on the category $\mathbb{H}$.

Theorem 2.2.7 (3.5.19)**.** *The functor* CH_{F} *induces an equivalence*

$$\mathrm{Tr}(\mathrm{F}, \mathbb{H}) = \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell). \tag{2.27}$$

The functor HC_{F} *induces an equivalence*

$$\mathcal{Z}_{\mathrm{F}}(\mathbb{H}) = \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell). \tag{2.28}$$

As an application of this computation. We give another construction of Lusztig's geometric ℓ-series.

Theorem 2.2.8 (3.5.31)**.** *There is a complete collection of orthogonal idempotent* $e_s \in \overline{\mathbb{Z}}_\ell[G^{\mathrm{F}}]$ *where* $s \in (T^\vee /\!\!/ W)(\overline{\mathbb{F}}_\ell)$. *They are characterized by the following property, let* ρ *be an irreducible* $\overline{\mathbb{F}}_\ell$*-representation of* G^{F} *then* $e_s\rho = \rho$ *if and only if there exists* (w, χ) *where* $w \in W$ *and* $\chi \in \mathrm{CH}(T) = T^\vee(\overline{\mathbb{F}}_\ell)$ *in the orbit corresponding to* s *such that,* ${}^*\mathcal{R}_{w,\chi}(\rho) \neq 0$.

2.2.3 Endomorphism of the Gelfand-Graev representation

Let us discuss an application of the center theorem to the endomorphism of the Gelfand-Graev representation. Denote by $\bar{U}$ the unipotent radical of the Borel opposite to B. The choice of a pinning of G determines a morphism

$$\phi : \bar{U} \to \bar{U}^{\mathrm{ab}} \simeq \prod_{\mathbb{G}_a} \xrightarrow{\Sigma} \mathbb{G}_a, \tag{2.29}$$

where $(-)^{\mathrm{ab}}$ denotes the abelianization and the isomorphism comes from the choice of the pinning.

Let $\psi : \mathbb{F}_q \to \Lambda^\times$ be an additive character of $\mathbb{F}_q$. We still denote by $\psi : \bar{U}^{\mathrm{F}} \to \Lambda^\times$ its composition with ϕ. The Gelfand-Graev representation is $\Gamma_\psi = \mathrm{ind}_{\bar{U}^{\mathrm{F}}}^{G^{\mathrm{F}}}(\psi)$. This representation is central to the representation theory of G^{F} and is the finite group version of the Whittaker representation of p-adic groups.

Example 2.2.9. If $G = \mathrm{GL}_n$ and F is the $\mathbb{F}_q$-structure coming from the split form of GL_n over $\mathbb{F}_q$. It is known that all characters ψ of $\bar{U}^{\mathrm{F}}$ are conjugate and therefore representation Γ_ψ does not depend on the choice of ψ. Moreover, if $\Lambda = \overline{\mathbb{Q}}_\ell$, all irreducible cuspidal representations of G^{F} are direct summands of Γ_ψ.

For groups other than GL_n, not all cuspidal representations need to appear in the representation Γ_ψ, but this representation still sees a lot of the structure of the category $\mathrm{Rep}_\Lambda G^{\mathrm{F}}$.

Theorem 2.2.10 ([Ste16])**.** *If $\Lambda = \overline{\mathbb{Q}}_\ell$, then the representation Γ_ψ is multiplicity free. For a general Λ, the endomorphism algebra $\mathrm{End}_{\Lambda[G^{\mathrm{F}}]}(\Gamma_\psi)$ is commutative.*

Over $\overline{\mathbb{Q}}_\ell$, since this representation is multiplicity free decomposing it produces many irreducible representations. More precisely, we have the following theorem.

Theorem 2.2.11 ([DL76])**.** *If $\Lambda = \overline{\mathbb{Q}}_\ell$, then Γ_ψ contains exactly one irreducible direct factor for each Lusztig series.*

This theorem was shown originally by computing the character of Γ_ψ and decomposing it with respect to the virtual characters of the complexes $\mathrm{R}\Gamma_c(Y(\dot{w}), \Lambda)$. This can however be deduced out of the following theorem.

Theorem 2.2.12 ([Dud09])**.** *For all Λ and all w, there is an isomorphism*

$${}^*\mathcal{R}_w(\Gamma_\psi) = \Lambda[T^{w\mathrm{F}}][\ell(w)]. \tag{2.30}$$

We give a different proof of this theorem generalizing an argument of [BT22].

Theorem 2.2.13. *There is an isomorphism up to shifts*

$$\mathrm{HC}_{\mathrm{F}}(\Gamma_\psi) = p_! \mathbb{T}, \tag{2.31}$$

where $\mathbb{T}$ is the big tilting sheaf.

Our proof of theorem 2.2.12 then follows from theorem 2.2.13 and the knowledge of the stalks and costalks of $\mathbb{T}$. By functoriality, this theorem defines a canonical map

$$\mathrm{Cur}_w : \mathrm{End}_{\Lambda[G^{\mathrm{F}}]}(\Gamma_\psi) \to \Lambda[T^{w\mathrm{F}}], \tag{2.32}$$

called the w-Curtis morphism. It was first constructed by Curtis [Cur94], using theorem 2.2.11. Let us denote by

$$\mathrm{Cur} = \oplus_w \mathrm{Cur}_w : \mathrm{End}_{\Lambda[G^{\mathrm{F}}]}(\Gamma_\psi) \to \oplus_w \Lambda[T^{w\mathrm{F}}], \tag{2.33}$$

the direct sum of all w-Curtis morphisms. This map is injective. A key problem is computing its image and express it in terms of the dual torus.

On the dual side let $T^\vee$ be the dual torus over Λ and $\mathrm{F}^\vee : T^\vee \to T^\vee$ be the morphism dual to F. Let $w \in W$ and consider the scheme of fixed points $(T^\vee)^{w\mathrm{F}^\vee}$ under $w\mathrm{F}^\vee$. This is the scheme making the following diagram cartesian.

$$\begin{tikzcd}
(T^\vee)^{w\mathrm{F}^\vee} \arrow[r] \arrow[d] & T^\vee \arrow[d, "\mathrm{id}\times\mathrm{F}^\vee"] \\
T^\vee \arrow[r, "\Delta"'] & T^\vee \times T^\vee
\end{tikzcd}$$

Similarly, we also consider the GIT quotient scheme $T^\vee /\!\!/ W$ and $\mathrm{F}^\vee$ the morphism induced by $\mathrm{F}^\vee$. By functoriality of taking the scheme of invariants, we have a morphism

$$\mathrm{Cur}_w^{\mathrm{spec}} : \mathcal{O}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee}) \to \mathcal{O}((T^\vee)^{w\mathrm{F}^\vee})$$

which we call the w-spectral Curtis morphism. Taking the direct sum over w, as before, we get the spectral Curtis morphism,

$$\mathrm{Cur}^{\mathrm{spec}} = \oplus_w \mathrm{Cur}_w^{\mathrm{Spec}} : \mathcal{O}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee}) \to \oplus_w \mathcal{O}((T^\vee)^{w\mathrm{F}^\vee}). \tag{2.34}$$

After choosing a set of trivializations of roots of unity of $\bar{\mathbb{F}}_q$, we get isomorphisms $\Lambda[T^{w\mathrm{F}}] = \mathcal{O}((T^\vee)^{w\mathrm{F}^\vee})$ for all $w \in W$.

Theorem 2.2.14 ([Li21], [LS22])**.** *Assume that ℓ is good for G and that G has connected center. There is an isomorphism $\mathrm{End}_{G^{\mathrm{F}}}(\Gamma_\psi) = \mathcal{O}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee}$ making the following diagram commutative.*

$$\begin{tikzcd}
\mathrm{End}_{G^{\mathrm{F}}}(\Gamma_\psi) \arrow[r, equal] \arrow[d, "\mathrm{Cur}_w"'] & \mathcal{O}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee}) \arrow[d, "\mathrm{Cur}_w^{\mathrm{spec}}"] \\
\Lambda[T^{w\mathrm{F}}] \arrow[r, equal] & \mathcal{O}((T^\vee)^{w\mathrm{F}^\vee})
\end{tikzcd}$$

The proof of this theorem in *loc. cit.* proceeds as follows. Firstly, if $\Lambda = \overline{\mathbb{Q}}_\ell$, both algebras $\mathrm{End}_{\overline{\mathbb{Q}}_\ell[G^{\mathrm{F}}]}(\Gamma_\psi)$ and $\mathcal{O}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee})$ are isomorphic to $\overline{\mathbb{Q}}_\ell^n$ for some n and thus are isomorphic. Now the $\overline{\mathbb{Z}}_\ell$-versions of these algebras are lattices inside each of them, they then show that the isomorphism over $\overline{\mathbb{Q}}_\ell$ preserve these lattices essentially by computing the matrix of this morphism and checking that there are no denominators. We give a different proof of this theorem.

Theorem 2.2.15 (3.6.3)**.** *There is a canonical map $\mathcal{O}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee}) \to \mathrm{End}_{G^{\mathrm{F}}}(\Gamma_\psi)$ compatible with the Curtis and spectral Curtis morphisms.*

Once this map is constructed theorem 2.2.14 follows from some symmetrizing form argument. We give a construction of this map using the center theorem 2.2.7.

2.3 Cohomology of stacks of chtoucas

2.3.1 Chtoucas and the Langlands correspondence

Let X be a smooth projective geometrically connected curve over $\mathbb{F}_q$ and let F be the function field of X. Let G be a reductive group of X and denote by $\mathbb{A}$ the ring of adèles of F. Let us assume for simplicity that G is split and denote by $\hat{G}$ the dual group of G over $\overline{\mathbb{Q}}_\ell$. Choose $\bar{\eta} \to X$ a generic geometric point of X and Weil_F the absolute Weil group of X at the geometric point $\bar{\eta}$.

Definition 2.3.1. A smooth irreducible automorphic representation of $G(\mathbb{A})$ is an irreducible subquotient of $\mathcal{C}_c(G(F)\backslash G(\mathbb{A}), \overline{\mathbb{Q}}_\ell)$ of the space of compactly supported automorphic forms.

Definition 2.3.2. A Langlands parameter for G and F is a morphism $\mathrm{Weil}_F \to \hat{G}(\overline{\mathbb{Q}}_\ell)$ that is continuous, defined over a finite extension of $\mathbb{Q}_\ell$ and almost everywhere unramified. We say that a parameter is semisimple, if whenever it factors through a parabolic of $\hat{G}$ then it factors trough a Levi of this parabolic.

Classically, the global Langlands correspondence is a map

$$\mathrm{GLC} : \mathrm{Irr}^{\mathrm{autom}}(G(\mathbb{A})) \to \underline{Z}^1(F, \hat{G})/\hat{G}(\overline{\mathbb{Q}}_\ell), \tag{2.35}$$

from the set of smooth irreducible automorphic representation to the set of conjugacy classes of global parameters.

Conjecture 2.3.3. *There exists a map* GLC *that is compatible with parabolic induction, compatible with the Satake isomorphism at the unramified places and with global class field theory for tori.*

This conjecture was first shown for $G = \mathrm{GL}_2$ by [Dri77], then $G = \mathrm{GL}_n$ [Laf02]. In general we have the following theorem

Theorem 2.3.4 ([Laf18])**.** *There exists a map* $\mathrm{GLC}^{\mathrm{ss}} : \mathrm{GLC} : \mathrm{Irr}^{\mathrm{autom}}(G(\mathbb{A})) \to \underline{Z}^1(F, \hat{G})^{\mathrm{ss}}/\hat{G}(\overline{\mathbb{Q}}_\ell)$ *to the set of conjugacy classes of semisimple Langlands parameters.*

Theorem 2.3.5 ([Xue20b])**.** *The map* $\mathrm{GLC}^{\mathrm{ss}}$ *of [Laf18] is compatible with parabolic induction.*

The construction of this map is done by studying the cohomology of stacks of chtoucas. We recall some key facts about this construction, for now let us ignore the problem coming from the Harder-Narasimhan truncations or from the center. Let $N \subset X$ be an effective divisor and let I be a finite set and $W \in \mathrm{Rep}_\Lambda \hat{G}^I$. Then there is an algebraic stack $\mathrm{Cht}_{N,I,W}$ over $(X - N)^I$ and an ℓ-adic sheaf $\mathcal{F}_{N,I,W}$ coming from geometric Satake over it. The cohomology sheaves are then defined as

$$\mathcal{H}_{N,I,W} = \mathfrak{p}_! \mathcal{F}_{N,I,W}, \tag{2.36}$$

where $\mathfrak{p} : \mathrm{Cht}_{N,I,W} \to (X - N)^I$ is the leg map. In general $\mathrm{Cht}_{N,I,W}$ is not quasi-compact so we have to filter it using Harder-Narasimhan truncations.

Theorem 2.3.6 ([Xue20d])**.** *The sheaves* $\mathcal{H}^j_{N,I,W}$ *are ind-lisse on* $(X - N)^I$.

One important piece of structure that they carry is that they are equipped with actions of the partial Frobenius endomorphisms. Namely for $I_0 \subset I$ a finite subset of I, there is an endomorphism $\mathrm{F}_{I_0} : (X - N)^I \to (X - N)^I$ given by $\mathrm{F}_{I_0}(x_i) = (y_i)$ with $y_i = \mathrm{F}(x_i)$ if $i \in I_0$ and $y_i = x_i$ otherwise.

Let $\Delta(\bar{\eta}) \to (X-N)^I$ be the geometric point obtained by composing $\bar{\eta} \to X-N$ with the diagonal. Then using Drinfeld's lemma [Laf18] and [Xue20d] show that

$$H^j_{N,I,W} = (\mathcal{H}^j_{N,I,W})_{|\Delta(\bar{\eta})}, \tag{2.37}$$

is not only equipped with an action of $\mathrm{Weil}((X-N)^I, \Delta(\bar{\eta}))$ but that this action factors through $\mathrm{Weil}(X-N)^I$. In turn we get a map

$$\begin{aligned} \mathrm{Rep}_\Lambda(\hat{G}^I) &\to \mathrm{Rep}_\Lambda \mathrm{Weil}(X-N)^I \\ W &\mapsto H^j_{N,I,W}. \end{aligned}$$

This map is functorial in both I and W. Once this functor is constructed the machinery of excursion operators of [Laf18] takes over. Namely, there is an algebra $\mathrm{Exc}(F, \hat{G})$ such that the Λ-points (for Λ a field) of $\mathrm{Spec}(\mathrm{Exc}(F, \hat{G}))$ are in bijection with semisimple global Langlands parameters. Moreover given any system of functors $(I, W) \mapsto H_{I,W}$ as above, the vector space $H_{\{0\},1}$ where 1 is the trivial representation of $\hat{G}$ is equipped with an action of this excursion algebra.

We now turn towards the local function field setting. Let K be a local field of equal characteristic and let H be a reductive group over K, which we assume for this section to be split. The local Langlands correspondence is a conjectural map

$$\mathrm{LLC} : \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(H(K)) \to \underline{Z}^1(\mathrm{Weil}_K, \hat{H})/\hat{H}(\overline{\mathbb{Q}}_\ell) \tag{2.38}$$

from smooth irreducible representations of $H(K)$ to local Langlands parameters. The existence of this map is known for GL_n [LRS93]. While this map has not been constructed, there is a semisimple version of the correspondence that has been constructed by [GL17] and [FS21].

Theorem 2.3.7. *There exists a map*

$$\mathrm{LLC} : \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(H(K)) \to \underline{Z}^1(\mathrm{Weil}_K, \hat{H})^{\mathrm{ss}}/\hat{H}(\overline{\mathbb{Q}}_\ell) \tag{2.39}$$

compatible with class field theory and parabolic induction.

By a result of Li-Huerta [LH23], we know that the two correspondences of Fargues-Scholze and of Lafforgue-Genestier agree.

2.3.2 Depth 0 local Langlands

We know state our main theorems concerning the Genestier Lafforgue local Langlands correspondence for depth 0 representations. More specifically, we show some compatibility between different local Langlands type parametrizations. We first recall the structure theorem of depth 0 representations.

Theorem 2.3.8 ([Lan18], [Lan21])**.** *The category $\mathrm{Rep}^0_\Lambda H(K)$ of depth 0 representations of $H(K)$ decomposes as a direct sum*

$$\mathrm{Rep}^0_\Lambda H(K) = \bigoplus_{s \in (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}}(\Lambda)} \mathrm{Rep}^s_\Lambda H(K), \tag{2.40}$$

where $\hat{T}$ denotes the dual torus over Λ, $\hat{T} /\!\!/ W$ the GIT-quotient by the action of W and $(-)^{\hat{\mathrm{F}}}$ the scheme of invariants under the morphism dual to the Frobenius.

This decomposition of theorem 2.3.8, yields a map

$$\mathrm{LS} : \mathrm{Irr}^0_\Lambda(H(K)) \to (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}}(\Lambda), \tag{2.41}$$

where $\mathrm{Irr}^0_\Lambda(H(K))$ is the set of irreducible depth 0 representations of $H(K)$, characterized by $\mathrm{LS}(\pi) = s$ if and only if π lies in the direct summand indexed by s.

Theorem 2.3.9 (4.1.6)**.** *Let $\pi \in \mathrm{Irr}^0_\Lambda(H(K))$ then $\mathrm{LLC}^{\mathrm{GL}}(\pi)$ is a tame local Langlands parameter. Furthermore the following diagram is commutative.*

$$\begin{tikzcd} \mathrm{Irr}^0 H(K) \arrow[r, "\mathrm{LLC}^{\mathrm{GL}}"] \arrow[d, "\mathrm{LS}"'] & (\underline{Z}^{1,t}(K, \hat{H}) /\!\!/ \hat{H})(\Lambda) \arrow[d, "\mathrm{ev}_{\tau_K}"] \\ (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}} \arrow[r, equal] & (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}} \end{tikzcd}$$

Our proof of this theorem will follow from the study of certain cohomology sheaves of stacks of chtoucas. To access depth 0 representations, we first fix a maximally split maximally unramified torus $T \subset H$ over K. This determines an apartment A in the Bruhat-Tits building of $H(K)$, we let σ be a polysimplex in A and $\mathcal{H}_\sigma$ be the corresponding parahoric group scheme over $\mathcal{O}_K$. We choose $x \in X$ a point and an isomorphism between K and the completion of F at x. Let $\mathcal{G}_\sigma$ be a smooth affine group scheme over X that is reductive over $X - x$ and such that over $\mathcal{O}_K$ it is isomorphic to $\mathcal{H}_\sigma$. Let $N = x + N^x$ be a level structure such that $x \notin N^x$. We consider the cohomology groups $H^j_{I,N,W}$ for the group $\mathcal{G}_\sigma$. Denote by V_σ the unipotent radical of the special fiber of $\mathcal{H}_\sigma$ and M_σ its reductive quotient. The group $\mathcal{H}_\sigma(\mathbb{F}_x)$ acts on $H^j_{I,N,W}$.

Theorem 2.3.10 (4.1.7)**.** *(i). For all I, W the $\mathrm{Weil}^I_{F_x}$-module $(H^j_{I,W,N})^{V_\sigma(\mathbb{F}_x)}$ is tamely ramified, that is, the action factors through the tame quotient $(\mathrm{Weil}^t_{F_x})^I$.*

(ii). Let $s \in (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}}(\Lambda)$, then as an $\mathrm{Exc}(\mathrm{Weil}^t_{F_x}, \hat{G})$-module $e_s(H^j_{I,W,N})^{V_\sigma(\mathbb{F}_x)}$ is supported on $\mathrm{ev}^{-1}_{\tau_{F_x}}(s)$, where e_s is the idempotent in $\Lambda[M_\sigma(\mathbb{F}_x)]$ corresponding to the Lusztig series attached to s, we refer to Section 4.2 for the notations.

Finally, following a construction of [LZ18], we construct a quasi-coherent sheaf $\mathcal{M}_N$ on the stack of local Langlands parameters that is canonically attached to the system of functors $(I, W) \mapsto H_{I,N,W}$.

Corollary 2.3.11 (4.1.8)**.** *(i). The quasi-coherent sheaf $(\mathcal{M}^j_N)^{V_\sigma(\mathbb{F}_x)}$ is supported on $\underline{Z}^{1,t}(F_x, \hat{G})$.*

(ii). Using the same notations as in 4.1.7, let $s \in (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}}(\overline{\mathbb{Q}}_\ell)$ then the quasi-coherent sheaf $e_s(\mathcal{M}^j_N)^{V_\sigma(\mathbb{F}_x)}$ is supported on $\mathrm{ev}^{-1}_{\tau_{F_x}}(s)$.

Organization

This thesis is composed of two independent chapters. The first one contains the results on monodromic sheaves, Soergel theory and Deligne-Lusztig theory. The second one contains the results on stacks of chtoucas and the local Langlands correspondence. Both chapters have their own detailed introduction.

Chapter 3

Soergel theory and Deligne-Lusztig theory

Contents

3.1 Introduction

Let p be a prime number and $\mathbb{F}_q$ be the finite field with q elements and characteristic p and let $\bar{\mathbb{F}}_q$ be an algebraic closure. Let G be a reductive group over $\bar{\mathbb{F}}_q$ equipped with an endomorphism F coming from a $\mathbb{F}_q$-structure, fix a Borel pair $B = TU$ which is F-stable. In this paper we are interested in understanding some links between the theory of representations of $G(\mathbb{F}_q) = G^{\mathrm{F}}$ and the theory of monodromic sheaves on $U\backslash G/U$. We also fix ℓ a prime number different from p.

The modern way to study the theory of representations of G^{F} is via the cohomology of the Deligne-Lusztig variety [DL76]. Let W denote the Weyl group of G and fix multiplicative liftings $(\dot{w})$ of W inside of $N(T)$. For any $w \in W$ there is a pair of varieties $\pi : Y(\dot{w}) \to X(w)$ with G^{F}-actions and the map π is a $T^{w\mathrm{F}}$-torsor, where F is the Frobenius of T, induced from the one of G. The cohomology $\mathrm{R}\Gamma_c(Y(\dot{w}), \overline{\mathbb{Z}}_\ell)$ is then equipped with two commuting actions of G^{F} and of $T^{w\mathrm{F}}$ and yields a pair of adjoint functors

$$R_w : \mathcal{D}^b(\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(T^{w\mathrm{F}})) \leftrightarrows \mathcal{D}^b(\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(G^{\mathrm{F}})) : {}^*R_w$$

called the Deligne-Lusztig induction and restriction functors, given by $R_w(M) = M \otimes_{T^{w\mathrm{F}}} \mathrm{R}\Gamma_c(Y(\dot{w}, \overline{\mathbb{Z}}_\ell)$ and ${}^*R_w(N) = \mathrm{RHom}_{G^\mathrm{F}}(\mathrm{R}\Gamma_c(Y(\dot{w}, \overline{\mathbb{Z}}_\ell), N)$. While in the original article, Deligne and Lusztig worked mostly with characters of representations and $\overline{\mathbb{Q}}_\ell$-cohomology, a functorial treatment for the integral setting has been worked out, see for instance [BR03] and [BDR17]. One of the output of the present article will be to recover these functors in a way that does not involve the Deligne-Lusztig varieties.

On the other side, consider the category of ℓ-adic sheaves on $B\backslash G/B$ or $U\backslash G/B$. These categories are well understood and are now described in terms of purely algebraic datum called Soergel bimodules. When passing from $U\backslash G/B$ to $U\backslash G/U$, we loose the key property that there are finitely many irreducible objects, this is partially regained when considering the full subcategory of unipotent monodromic sheaves, that is the category generated by pullback of objects from $U\backslash G/B$. Then, to get a category with good properties [BY13] have introduced a 'completed category' and a corresponding description of the completed category in terms of these Soergel bimodules. When passing from the ℓ-adic setting to the modℓ setting, a corresponding description has been worked out by [BR22b]. In his thesis Gouttard [Gou21] has gotten a similar description for the non-unipotent case.

The two theories are related in via the following diagram

$$\begin{tikzcd} U\backslash G/U \arrow[d, "p"'] & & \frac{G}{\mathrm{Ad}_\mathrm{F} B} \arrow[dll, "r"'] \arrow[dr, "q"] & \\ \frac{U\backslash G/U}{\mathrm{Ad}_\mathrm{F} T} & & & \frac{G}{\mathrm{Ad}_\mathrm{F} G} \end{tikzcd} \tag{3.1}$$

where the quotient always denote the quotient stack and the Ad_F refers to the action by Frobenius conjugation, that is $(g, x) \mapsto gx\mathrm{F}(g)^{-1}$ and the maps are the quotient maps for the various groups and actions, see also section 3.3.1. Lusztig had previously introduced the functors $q_! r^*$ and $r_! q^*$ in [Lus15] and [Lus17]. Lang's theorem yields an isomorphism $\frac{G}{\mathrm{Ad}_\mathrm{F} G} \simeq \mathrm{pt}/G^\mathrm{F}$, while on each of the Bruhat strata we also have an isomorphism of stacks $\frac{U\backslash BwB/U}{\mathrm{Ad}_\mathrm{F}(T)} \simeq \mathrm{pt}/(U_w \times T^{w\mathrm{F}})$ where $U_w \subset U$ is a closed connected subgroup. In particular considering the functor $q_! r^* i_{w,!}$, where i_w is induced by the inclusion $BwB \subset G$, gives a functor between the representations of $T^{w\mathrm{F}}$ and that of G^F. We will compare this functor with the Deligne-Lusztig induction functor.

The initial motivation of this paper was to gain a geometric insight into the following theorem. Let $\overline{U}$ denote the unipotent radical of the opposite Borel, and let $\overline{U} \to \mathbb{G}_a$ be a generic morphism, denote by $\mathcal{L}_\psi$ the pullback along this morphism of an Artin-Schreier sheaf. The trace of Frobenius function of this sheaf produces a character of $\overline{U}^\mathrm{F}$ and an idempotent $e_\psi \in \overline{\mathbb{Z}}_\ell[\overline{U}^\mathrm{F}]$. Denote by $\Gamma_\psi = \mathrm{ind}_{\overline{U}^\mathrm{F}}^{G^\mathrm{F}}(\psi)$ the Gelfand-Graev representation.

Theorem 3.1.1 ([Li21], [LS22]). *Assume that ℓ is good for G and that G has connected center, then we have an isomorphism*

$$\mathrm{End}_{G^\mathrm{F}}(\Gamma_\psi) \simeq \mathcal{O}_{(T^\vee /\!\!/ W)^{\mathrm{F}^\vee}},$$

where $T^\vee$ is the dual torus of T defined over $\overline{\mathbb{Z}}_\ell$, $\mathrm{F}^\vee : T^\vee \to T^\vee$ is the morphism dual to F_T and $(T^\vee /\!\!/ W)^{\mathrm{F}^\vee}$ is the scheme of invariant, that is the one deduced by intersection of the diagonal and the graph of $\mathrm{F}^\vee$.

We give a proof of this theorem in section 3.6.2 by using a twist of a method of [BT22] and relating the representation Γ_ψ to Soergel theory. More precisely, we show that

$$r_! q^* \Gamma_\psi \simeq p_! \mathbb{T}, \tag{3.2}$$

where $\mathbb{T}$ is an explicit object coming from Soergel theory. We then use the Endomorphismensatz 3.4.34, to produce a map

$$\mathrm{End}(\Gamma_\psi) \to \mathcal{O}_{(T^\vee /\!\!/ W)^{\mathrm{F}^\vee}}. \tag{3.3}$$

We now describe our main results in more details.

3.1.1 Monodromic categories

We want to work with a $\mathbb{Z}_\ell$ version of the completed categories of [BY13] and [BR22b]. We could follow their approach and define this completed category in terms of pro-objects of $\overline{\mathbb{Z}}_\ell$-sheaves, we instead want to rebuild them using the proétale topology of [BS15].

Let T be a torus over $\bar{\mathbb{F}}_q$ and denote by

$$\Omega_{T,\mathbb{Z}_\ell} = \varprojlim_{m,n} \mathbb{Z}/\ell^m\mathbb{Z}[T[\ell^n]],$$

and by $\Omega_T = \Omega_T \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Z}}_\ell$. We first discuss the existence of a good sheaf theory with coefficients in Ω_T. Namely for all schemes X of finite type over $\bar{\mathbb{F}}_q$ there is a stable $(\infty, 1)$-category $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)$ of constructible sheaves of Ω_T-modules. The formalism of [BS15] and [HRS21] yields a family of categories

$$X \to \mathrm{D}_{\mathrm{cons}}(X, \Omega_T)$$

equipped with a 6-functor formalism. They are furthermore equipped with a pair of t-structures, one classical and one perverse.

Denote by $\pi_1(T)$ the étale fundamental group of T at the geometric point 1 and $\pi_1^t(T)$ the tame fundamental group, that is its largest prime to p quotient. It is isomorphic to $\varprojlim_{(n,p)=1} T[n]$. Given a $\overline{\mathbb{Z}}_\ell$-character of order prime to ℓ of $\pi_1(T)$ we get a Kummer sheaf $\mathcal{L}_\chi$ on T.

With the above setup, we give a new definition of the completed monodromic categories. The ring Ω_T was introduced in [GL96], and there they define a rank one Ω_T character sheaf on T which we denote by L_T, which is given by the canonical map $\pi_1(T) \to \Omega_T$. Let $\pi : X \to Y$ be a T-torsor, we then define the completed category of unipotent monodromic sheaves on X as the category of *equivariant* sheaves for (T, L_T) and we denote this category by $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_{\mathrm{unip}}$.

We then show in section 3.2.7 that when one performs the same construction for the ring $\Omega_{T,\mathbb{F}_\ell} = \Omega_{T,\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell$, there is a natural equivalence with the construction of [BR22b]. We do not however compare with the $\overline{\mathbb{Q}}_\ell$-version of [BY13] but we plan to return to this case in a later work. We also discuss a non-unipotent version, but here this is simply requiring equivariance with respect to a character sheaf of the form $L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi$ where χ is as above a character of $\pi_1^t(T)$ of order prime to ℓ and we denote this category by $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_\chi$.

Finally, in the Appendix 3.B, we discuss how to set up the basics of Soergel theory, namely we place ourselves in the following situation : there is a stratification of $Y = \bigcup_s Y_s$ into strata that are affine spaces and we consider sheaves on X such that their pushforward to Y are constant along the strata. The results of this section are very easy generalizations of [BR22b] Section 1-5 and of [Gou21] Section 7., since we have to rebuild some of the theory we also have to discuss some variants of results of [RSW13] and [AR16].

3.1.2 Deligne-Lusztig theory

Let us now explain how to recover the Deligne-Lusztig induction and restriction functors. We will use the notations of (3.1). We first fix an element $w \in W$ and consider the category $\mathrm{D}_{\mathrm{cons}}(\frac{U\backslash UTwU/U}{\mathrm{Ad}_{\mathrm{F}}T}, \overline{\mathbb{Z}}_\ell)$. The choice of a lifting $\dot{w}$ determines an isomorphism of stacks

$$\frac{U\backslash UTwU/U}{\mathrm{Ad}_{\mathrm{F}}T} = \mathrm{pt}/(T^{w\mathrm{F}} \rtimes U_w),$$

where U_w is a closed unipotent connected subgroup of U. Hence the category $\mathrm{D}_{\mathrm{cons}}(\frac{U\backslash UTwU/U}{\mathrm{Ad}_{\mathrm{F}}T}, \overline{\mathbb{Z}}_\ell)$ is equivalent to the category $\mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \overline{\mathbb{Z}}_\ell)$, which is nothing else than the category of representations of $T^{w\mathrm{F}}$ on $\overline{\mathbb{Z}}_\ell$-modules. Consequently, the category $\mathrm{D}_{\mathrm{cons}}(\frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}}T}, \overline{\mathbb{Z}}_\ell)$ is obtained as the gluing of all the categories of representations of the finite groups $T^{w\mathrm{F}}$. Consider now the functor $q_! r^* i_{w,!} : \mathrm{D}_{\mathrm{cons}}(\frac{U\backslash UTwU/U}{\mathrm{Ad}_{\mathrm{F}}T}, \overline{\mathbb{Z}}_\ell) \to \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell)$, we first show the following theorem.

Theorem 3.1.2 (Theorem 3.3.8)**.** *Under the equivalence $\mathrm{D}^b(\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell} T^{w\mathrm{F}}) \simeq \mathrm{D}_{\mathrm{cons}}(\frac{U\backslash UTwU/U}{\mathrm{Ad}_{\mathrm{F}}T}, \overline{\mathbb{Z}}_\ell)$, the functor $q_! r^* i_{w,!}$ is isomorphic up to a shift to the Deligne-Lusztig induction functor.*

Remark 3.1.3. Passing to right adjoints yields the Deligne-Lusztig restriction functor.

We then want to relate the Deligne-Lusztig induction and restriction functors with Soergel theory. Since we reformulate Soergel theory in terms of L_T-equivariant sheaves, we first produce a second functor $\mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \overline{\mathbb{Z}}_\ell) \to \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell)$ that is defined using monodromic sheaves. On the stratum $UTwU$ there are two commuting actions : one of T by right translations and one of T by Frobenius conjugation, consider now $\mathrm{D}_{\mathrm{cons}}(\frac{U\backslash UTwU/U}{\mathrm{Ad}_{\mathrm{F}}T}, \Omega_T)_\chi$ the same category but with an equivariance condition for the right action of T.

Lemma 3.1.4 (Lemma 3.3.9)**.** *There is an equivalence*

$$\mathrm{D}_{\mathrm{cons}}(\frac{U\backslash UTwU/U}{\mathrm{Ad}_{\mathrm{F}}T}, \Omega_T)_{\mathrm{unip}} \simeq e_1 \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \overline{\mathbb{Z}}_\ell),$$

where e_1 denote the projector onto the principal block. Similarly replacing unip *by χ we get the projection onto the corresponding block provided that $w\mathrm{F}(\chi) = \chi$.*

This results follows from the computation of the averaging of L_T under Frobenius conjugation of T. We are now able to construct a functor from the category of $\overline{\mathbb{Z}}_\ell$ representations of $T^{w\mathrm{F}}$ to the category of G^{F}-representations. Namely consider the following composition

$$\begin{aligned}\mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \overline{\mathbb{Z}}_\ell) &\xrightarrow{\simeq} \bigoplus_\chi \mathrm{D}_{\mathrm{cons}}(\frac{U\backslash UTwU/U}{\mathrm{Ad}_{\mathrm{F}}T}, \Omega_T)_\chi \\ &\to \mathrm{D}_{\mathrm{cons}}(\frac{U\backslash UTwU/U}{\mathrm{Ad}_{\mathrm{F}}T}, \Omega_T) \\ &\xrightarrow{i_{w,!}} \mathrm{D}_{\mathrm{cons}}(\frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}}T}, \Omega_T) \\ &\xrightarrow{q_! r^*} \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \Omega_T) \\ &\xrightarrow{\mathrm{For}_{\Omega_T}} \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell),\end{aligned}$$

where the first functor is the one of 3.1.4, the second one is a forgetful functor (we refer to 4.2.1 for the precise definition of this forgetful functor) and the last one is simply forgetting the Ω_T-structure down to a $\overline{\mathbb{Z}}_\ell$-structure. In particular we show the following theorem

Theorem 3.1.5 (Theorem 3.3.13). *There is an isomorphism of functors between the composition of these functors and R_w.*

Remark 3.1.6. For $w \in W$, we have outlined the construction of two functors $\mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \overline{\mathbb{Z}}_\ell) \to \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell)$ and that they are both isomorphic with the Deligne-Lusztig induction. They are in particular isomorphic. Going the other way, we have two functors

$$r_! q^* \text{ and } \mathrm{For} \oplus_\chi \mathrm{Av}_\chi r_! q^*(- \otimes_{\overline{\mathbb{Z}}_\ell} \Omega_T) : \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell) \to \mathrm{D}_{\mathrm{cons}}(\frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T}, \overline{\mathbb{Z}}_\ell).$$

We refer to lemma 3.3.15 for their definition. We show that these two functors are isomorphic.

3.1.3 Soergel theory

The fourth section of our paper deals with the integral version of the results of [BR22b] and [Gou21]. All the results are generalization of *loc. cit.*. The category of equivariant sheaves with respect to a non trivial character χ had already been studied in [LY20]. Let us fix some notations, let $w \in W$ and χ be as before. We then have at our disposal

(i). The standard and costandard sheaves $\Delta_{w,\chi}$ and $\nabla_{w,\chi}$ which are the ! and $*$-extension of the sheaf a Bruhat stratum corresponding to $L_T \otimes \mathcal{L}_\chi$, they are in particular perverse.

(ii). The tilting sheaves $T_{w,\chi}$ that are perverse sheaves on $U\backslash G/U$ with both a Δ-filtration and a ∇-filtration, where a Δ-filtration (resp. ∇-filtration) is a filtration with graded pieces belonging to the set $\{\Delta_{w,\chi}, w \in W, \chi \in \mathrm{CH}(T)\}$ (reps. $\{\nabla_{w,\chi}, w \in W, \chi \in \mathrm{CH}(T)\}$).

We then carry out the study of the category $\mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_T)_\chi$. We equip this category with a convolution structure, it is built to ensure compatibility with the modular version of [BR22b] and [Gou21]. Assuming that G has connected center, we get a decomposition of the category

$$\mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_T)_\chi = \bigoplus_{\chi'} \mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_T)_{[\chi',\chi]},$$

where χ' ranges through the orbit of χ, and the category $\mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_T)_{[\chi',\chi]}$ is the category generated by all $\Delta_{w,\chi}$ for w such that $w\chi = \chi'$.

The next step is to show the Endomorphismensatz in our setup. Let us fix χ as before and χ' in the W-orbit of χ, there is a distinguished tilting object in $\mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_T)_{[\chi',\chi]}$ which we denote by $T_{\chi,\chi'}$. Denote by W_χ the stabilizer of χ.

Theorem 3.1.7 (Endomorphismensatz, 3.4.34). *Assume that ℓ is good for G. There is an isomorphism*

$$\Omega_T \otimes_{\Omega_T^{W_\chi}} \Omega_T \simeq \mathrm{End}(T_{\chi,\chi'}).$$

With this theorem in place we can define the $\mathbb{V}_{\chi,\chi'}$ functor which is nothing else than $\mathrm{Hom}(T_{\chi,\chi'}, -)$. The last result we need out of Soergel's theory is that this functor is fully faithful on tilting objects, which is done in section 3.4.7. We finally consider the direct sum category

$$\bigoplus_\chi \mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_T)_\chi$$

and we take the direct sum of all the $\mathbb{V}$ functors on all the blocks at once. Following [GL96], we introduce $\mathcal{C}(T) = \bigsqcup_\chi \mathrm{Spec}(\Omega_T) \times \{\chi\}$, which is the space of rank one character sheaves on T. With this notation the global $\mathbb{V}$ functor takes values in the category of coherent sheaves on $\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T)$ and the structure sheaf is the image of the direct sum of all the tilting sheaves $T_{\chi,\chi'}$, which we denote by $\mathbb{T}$. Only in defining the global $\mathbb{V}$-functor we require that G has connected center, all the other results extend to the non-connected center case.

3.1.4 Categorical centers

In the fifth section of this paper we consider the problem of realizing the category of representations of G^{F} as a twisted categorical center. This notion of categorical center is well behaved in the infini-categorical world and generalizes the more classical notion of Drinfeld centers. We refer to Appendix 3.C for some basic definitions about categorical centers.

Consider the functor

$$p^!\mathrm{HC}_{\mathrm{F}} : \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell) \to \bigoplus_{\chi \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{indcons}}(U\backslash G/U, \Omega_T)_\chi. \tag{3.4}$$

Theorem 3.1.8 (Theorem 3.5.19)**.** *The functor $p^!\mathrm{HC}_{\mathrm{F}}$ is equipped with a canonical F-central structure and induces a isomorphism*

$$\mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell) \simeq \mathcal{Z}_{\mathrm{F}}(\bigoplus_{\chi \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{indcons}}(U\backslash G/U, \Omega_T)_\chi). \tag{3.5}$$

In [Lus15] and [Lus17], Lusztig has shown a similar statement using abelian categories. There are two advantages to our method. The first one is that it works over $\overline{\mathbb{Z}}_\ell$, whereas Lusztig uses the theory of weights which is only valid over $\overline{\mathbb{Q}}_\ell$. Secondly our proof is almost formal, it is a variation (with Frobenius) of an argument of [BZN09], whereas Lusztig's construction inputs some knowledge about the classification of irreducible representations of G^{F}.

The only nonformal input in the proof of this theorem is the celebrated theorem of Deligne-Lusztig.

Theorem 3.1.9 ([DL76])**.** *Let $\rho \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G^{\mathrm{F}})$, then there exists, $w \in W$ and j an integer such that* $\mathrm{Hom}(H^j_c(Y(\dot{w}, \overline{\mathbb{Q}}_\ell), \rho) \neq 0$.

The proof of this theorem in *loc. cit.* relies on some character computations. We give a geometric proof of this fact using a technique from [BBM04b]. Namely, we compute the functor $\mathrm{CH}_{\mathrm{F}}\mathrm{HC}_{\mathrm{F}}$. A first remark is that the category $\mathrm{D}_{\mathrm{cons}}(\frac{G}{\mathrm{Ad}(G)}, \Lambda)$, where Ad denotes the adjoint action acts on $\mathrm{D}_{\mathrm{cons}}(\frac{G}{\mathrm{Ad}_{\mathrm{F}} G}, \Lambda)$ via convolution, we refer to section 3.5.5 for the definition of this action. We then show the following lemma, which is a F-twisted version of a theorem of [MV88].

Lemma 3.1.10. *There is an isomorphism of functors*

$$\mathrm{CH}_{\mathrm{F}}\mathrm{HC}_{\mathrm{F}} \simeq \mathrm{Spr} * -, \tag{3.6}$$

where $\mathrm{Spr} \in \mathrm{D}_{\mathrm{cons}}(\frac{G}{\mathrm{Ad}(G)}, \Lambda)$ *denotes the Springer sheaf.*

In characteristic 0, the sheaf δ_1, is a direct summand of the Springer sheaf. This is enough to imply the conservativity of the functor HC_{F} and then the generation statement. This proof also works with modular coefficients, in particular we get a new proof of the following theorem.

Theorem 3.1.11 ([BR03])**.** *The complexes* $\mathrm{R}\Gamma_c(Y(\dot{w}), \Lambda)$ *span* $\mathrm{Perf}(\Lambda[G^{\mathrm{F}}])$.

3.1.5 Endomorphism of the Gelfand Graev representation

We finally come to our last section and we want to illustrate how to recover the theorem of [Li21] in geometric terms. From now on we assume that G has connected center and that ℓ is good for G. The idea here is a twisted version of [BT22]. The main novelty is that we construct a map

$$\mathrm{End}_{G^{\mathrm{F}}}(\Gamma_\psi) \to \mathcal{O}_{(T^\vee /\!\!/ W)^{\mathrm{F}}}.$$

Once this map is constructed, the theorem will follow by standard arguments of symmetrizing forms. Let us explain how to construct this map.

Firstly, we compute $r_! q^* \Gamma_\psi$. We show the following lemma, we refer to lemma 3.6.8 for the normalization.

Lemma 3.1.12. *There is an isomorphism, up to shift,*

$$r_! q^* \Gamma_\psi = p_! \mathbb{T}.$$

Once this is in place we consider

$$\mathbb{V}(p^! r_! q^* \Gamma_\psi) = \mathrm{Hom}(p_! \mathbb{T}, p_! \mathbb{T}).$$

This is a coherent sheaf on $\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T)$. We then show that this sheaf is the structure sheaf of $Z = (\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T)) \times_{\mathcal{C}(T)\times\mathcal{C}(T)} \mathcal{C}(T)$, where the $\mathcal{C}(T)$ is embedded via the graph of $\mathrm{F}^\vee$. The closed subscheme $Z \subset \mathcal{C}(T) \times \mathcal{C}(T)$ is stable under the W-action given $w.(x, y) = (wx, \mathrm{F}(w)y)$. We then show that $Z /\!\!/ W = (T^\vee /\!\!/ W)^{\mathrm{F}}$. With this presentation we get two actions of $\mathrm{End}(\Gamma_\psi)$ and W on $\mathcal{O}_Z$, we show that they commute using the centrality statement of the previous section. This induces the desired map

$$\mathrm{End}(\Gamma_\psi) \to \mathrm{End}(\mathcal{O}_Z^W) = \mathcal{O}_{T^\vee /\!\!/ W}.$$

3.1.6 Conventions and notations

∞-categories.

We will use the language of $(\infty, 1)$-categories of Lurie [Lur09] [Lur]. We introduce the following notations concerning them. Given an $(\infty, 1)$-category $\mathcal{C}$, we denote by $\mathrm{ho}(\mathcal{C})$ the homotopy category. Given a stable category $\mathcal{C}$ with a t-structure, we denote by $\mathcal{C}^\heartsuit$ its heart, the notion of stable ∞ is defined in [Lur] 1.1.1.9. Recall that if $\mathcal{C}$ is a stable ∞-category, the category $\mathrm{ho}(\mathcal{C})$ is naturally a triangulated category, and the data of a t-structure on $\mathcal{C}$ is the data of a t-structure on $\mathrm{ho}(\mathcal{C})$, [Lur] 1.2.1.4. Let Λ be ring, we denote by $\mathrm{D}(\Lambda)$ the ∞-derived category of Λ, constructed in [Lur] 1.3.5.8., its homotopy category is naturally identified with the usual derived category of Λ. If Λ is commutative, then this category is a closed monoidal symmetric category. We will consider Λ-linear categories, that is categories that are modules over $\mathrm{D}(\Lambda)$. Given a Λ-linear stable category $\mathcal{C}$, we denote for $x, y \in \mathcal{C}$ by $\mathrm{Hom}_{\mathcal{C}}(x, y) \in \mathrm{D}(\Lambda)$ the mapping space between x and y. Its image in $\mathrm{ho}(\mathrm{D}(\Lambda))$ is a complex such that $H^i(\mathrm{Hom}_{\mathcal{C}}(x, y)) = \mathrm{Hom}_{\mathcal{C}}(x, y[i]) = \mathrm{Hom}^i_{\mathcal{C}}(x, y)$, if the context is clear, we will drop the $\mathcal{C}$. In the usual cases of categories of étale sheaves, this complex is simply given by the functor RHom. If $\mathcal{A} \subset \mathcal{C}$ is the heart of a t-structure on $\mathcal{C}$, then we will denote by $\mathrm{Hom}_{\mathcal{A}}(x, y)$ the Hom-set in the abelian category of $\mathcal{A}$, for $x, y \in \mathcal{A}$. Note that $\mathrm{Hom}_{\mathcal{A}}(x, y) = \mathrm{Hom}^0_{\mathcal{C}}(x, y)$.

Bar resolutions.

We denote by Δ the simplex category. In a category with products $\mathcal{C}$ (either ∞ or 1), given a group object G acting on an object X. We denote by $X \times G^{\bullet+1} \to X$ the augmented Bar simplicial objects. That is the simplicial object $\Delta^{\mathrm{op}} \to \mathcal{C}, [n] \mapsto X \times G^{n+1}$ and whose degeneracy maps are given by the action map and partial multiplications.

Categories of étale sheaves.

We fix p a prime number. We will denote by Λ a coefficient ring which is either $\overline{\mathbb{Z}}_\ell, \overline{\mathbb{F}}_\ell$ or Ω_T. We will denote by $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ the bounded category of constructible Λ sheaves on a stack X. For a morphism $f : X \to Y$, we denote by $f_*, f^*, f_!, f^!$ the corresponding derived functors, they are recalled in section 3.2. In general all functors will be understood in the derived sense. We fix once and for all a trivialization of the Tate twist $(1) = \mathrm{id}$.

Categories of coherent sheaves.

Given a ring A or more generally for X a(n underived) scheme, we denote by $\mathrm{D}(A), \mathrm{D}_{\mathrm{qcoh}}(X)$, the derived category of A-modules and the derived category of quasicoherent sheaves on X. We denote by $\mathrm{Perf}(A), \mathrm{Perf}(X)$ the full subcategories of $\mathrm{D}(A)$ and $\mathrm{D}(X)$ respectively of perfect complexes. We denote by $\mathrm{D}_{\mathrm{coh}}(A), \mathrm{D}_{\mathrm{coh}}(X)$ the full subcategories of $\mathrm{D}(A), \mathrm{D}_{\mathrm{qcoh}}(X)$ respectively of complexes which are cohomologically bounded and with coherent cohomologies. We denote by $\mathrm{Coh}(A)$ and $\mathrm{Coh}(X)$ the category of finite type A-modules and coherent $\mathcal{O}_X$-modules respectively. The categories $\mathrm{D}(A), \mathrm{D}_{\mathrm{qcoh}}(X)$ and $\mathrm{D}_{\mathrm{coh}}(A), \mathrm{D}_{\mathrm{coh}}(X)$ all carry standard t-structures. The categories $\mathrm{Coh}(A)$ and $\mathrm{Coh}(X)$ are then identified with the hearts of $\mathrm{D}_{\mathrm{coh}}(A)$ and $\mathrm{D}_{\mathrm{coh}}(X)$ respectively.

Reductive groups.

We let G be a reductive group over k, we fix a Borel pair $B = TU$ and we let $\bar{U}$ be the unipotent radical of the opposite Borel. We let W be the Weyl group of (G, T) and we denote by $\Delta \subset \Phi^+ \subset \Phi$ the simple, positive and roots associated to (G, B), similarly we denote by $\Phi^\vee$ the set of coroots corresponding to (G, T). For a root $\alpha \in \Phi$ we denote by $s_\alpha \in W$ the corresponding reflection and by $\alpha^\vee$ the corresponding coroots. We also fix a multiplicative family of liftings $(\dot{w})$ of the elements of W into $N_G(T)$, that is we impose that

$$\dot{w}\dot{w}' = \dot{\overline{ww'}}$$

whenever $\ell(w) + \ell(w') = \ell(ww')$. This choice produces natural trivializations $BwB \simeq U_w \times T \times U$, where U_w is $U \cap w^{-1}Uw$ and in particular is an affine space and we denote $\nu_w : BwB/U \to T$ the corresponding projection.

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3.2 Free monodromic categories

3.2.1 Setting up the 6-functors

Let k be an algebraically closed field of characteristic $p > 0$, and T be a torus over k. We denote by $X = X(T)$ and $Y = Y(T)$ its character and cocharacter lattices. We denote by $\pi_1(T)$ the étale fundamental group of T at the base point $1 \in T$. We denote by $\pi_1^t(T)$ the tame quotient, that is the largest quotient of pro-order prime to p. It is known that

$$\pi_1^t(T) = \varprojlim_{(n,p)=1} T[n] = Y(T) \otimes \pi_1^t(\mathbb{G}_m).$$

We also denote by $\pi_1(T)^{\mathrm{wild}}$ the kernel of the projection $\pi_1(T) \to \pi_1^t(T)$.

Let ℓ be a prime different from p, as in [GL96] we define $\pi_1(T)_\ell$ the largest pro-ℓ quotient of $\pi_1^t(T)$ and by

$$\Omega_{T,\mathbb{Z}_\ell} = \mathbb{Z}_\ell[\![\pi_1(T)_\ell]\!] = \varprojlim_{n,m} \mathbb{Z}/\ell^n\mathbb{Z}[T[\ell^m]].$$

Remark 3.2.1. We fix once and for all a topological generator $\gamma \in \pi_1(\mathbb{G}_m)_\ell$.

Remark 3.2.2. Note that we also have $\pi_1^t(T) = \prod_{(\ell',p)=1} \pi_1(T)_{\ell'}$ hence we can also see $\pi_1(T)_\ell$ as a subgroup of $\pi_1^t(T)$.

If R is a $\mathbb{Z}_\ell$-algebra we denote by $\Omega_{T,R} = \Omega_T \otimes_{\mathbb{Z}_\ell} R$, and by $\Omega_T = \Omega_{T,\overline{\mathbb{Z}}_\ell}$.

Definition 3.2.3. Let X be a k-scheme of finite type and let $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \Omega_T\}$. We set $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$, resp. $\mathrm{D}_{\mathrm{lis}}(X, \Lambda)$, resp $\mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$ to be the $(\infty, 1)$ stable category of constructible, resp. lisse, resp ind-constructible sheaves of Λ-modules as defined in [HRS21].

We recall a modern presentation of the 6-functors formalism. This presentation is due to Mann [Man22] based on work of [LZ17], we also refer to the lectures notes of P. Scholze [Sch23]. We consider the category $\mathrm{Sch}_k^{\mathrm{ft}}$ of separated finite type k-schemes. As explained in [Man22], Liu and Zheng construct an ∞-symmetric monoidal category $\mathrm{Corr}(\mathrm{Sch}_k^{\mathrm{ft}})$ whose objects are the same as the objects of $\mathrm{Sch}_k^{\mathrm{ft}}$ and whose morphisms $X \dashrightarrow Y$ in $\mathrm{Corr}(\mathrm{Sch}_k^{\mathrm{ft}})$ are given by correspondences $X \leftarrow Z \rightarrow Y$.

Definition 3.2.4. An abstract 3-functors formalism on $\mathrm{Sch}_k^{\mathrm{ft}}$ is the data of lax symmetric monoidal functor

$$\mathrm{D} : \mathrm{Corr}(\mathrm{Sch}_k^{\mathrm{ft}}) \to \infty - \mathrm{Cat}, \tag{3.7}$$

to the symmetric monoidal ∞-category of ∞-categories equipped with symmetric monoidal structure given by cartesian product. Given such a lax-monoidal functor, we define for $f : X \to Y$ a morphism of schemes

(*i*). $f^* : \mathrm{D}(Y) \to \mathrm{D}(X)$ as the image of the correspondence $Y \xleftarrow{f} X = X$,

(*ii*). $f_! : \mathrm{D}(X) \to \mathrm{D}(Y)$ as the image of the correspondence $X = X \xrightarrow{f} Y$.

For $X \in \mathrm{Sch}_k^{\mathrm{ft}}$, we define a symmetric monoidal structure on $\mathrm{D}(X)$ by

$$\mathrm{D}(X) \times \mathrm{D}(X) \to \mathrm{D}(X \times X) \xrightarrow{\Delta^*} \mathrm{D}(X), \tag{3.8}$$

where the first map is the data of the lax-monoidality of the functor D and the second one is the pullback along the diagonal.

Definition 3.2.5. A 6-functors formalism on $\mathrm{Sch}_k^{\mathrm{ft}}$ is the data of a 3-functors formalism such that for all maps f in $\mathrm{Sch}_k^{\mathrm{ft}}$ and all objects $A \in \mathrm{D}(X)$, the functors f^*, $f_!$ and $A \otimes -$ admit right adjoints. These right adjoints are then called f_*, $f^!$ and $\mathcal{H}\mathrm{om}$ respectively.

Remark 3.2.6. As explained in [Sch23], the data of a 6-functors formalism encodes all the base change maps, the Kunneth maps, the adjunctions maps as well as the compatibilities between them.

Theorem 3.2.7 (6 functors, [BS15] 6.7, [HRS21] 3.44)**.** *There exists* 6*-functors formalisms* $\mathrm{D}_{\mathrm{cons}}(-, \Omega_T)$ *and* $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ *on* $\mathrm{Sch}_k^{\mathrm{ft}}$ *such that for all* $X \in \mathrm{Sch}_k^{\mathrm{ft}}$

$$\mathrm{D}_{\mathrm{cons}}(-, \Omega_T)(X) = \mathrm{D}_{\mathrm{cons}}(X, \Omega_T), \tag{3.9}$$

and

$$\mathrm{D}_{\mathrm{indcons}}(-, \Omega_T)(X) = \mathrm{D}_{\mathrm{indcons}}(X, \Omega_T). \tag{3.10}$$

Proof. As explained in Lecture IV of [Sch23], to construct a 6-functors formalism, it is enough to construct a functor

$$\mathrm{Sch}_k^{\mathrm{ft,op}} \to \mathrm{CMon}(\infty - \mathrm{Cat}), \tag{3.11}$$

to the category of symmetric monoidal ∞-categories and two collections of morphisms I and P such that (see the conditions $(1) - (4)$ of *loc. cit.*) :

(i). The class I and P are stable under composition, pullbacks and contain all isomorphisms. Moreover any morphism f can be decomposed into a composition $p \circ j$ where $p \in P$ and $j \in I$.

(ii). For all $f \in I$, the functor f^* has a left adjoint denoted by $f_!$ satisfying the base change formula and the projection formula,

(iii). For all $p \in P$, the functor f^* has a right adjoint denoted by f_* satisfying the base change formula and the projection formula,

(iv). For any cartesian diagram with $j \in I$ and $p \in P$,

$$\begin{tikzcd} Y' \arrow[r, "j'"] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ X' \arrow[r, "j"'] & X \end{tikzcd}$$

the natural map $j_! p'_* \to p_* j'_!$, adjoint to the base change map, is an isomorphism.

Let us now apply this to our context, firstly the construction of [HRS21] produces the two desired functor $\mathrm{D}_{\mathrm{cons}} : \mathrm{Sch}_k^{\mathrm{ft,op}} \to \mathrm{CMon}(\infty - \mathrm{Cat})$. Indeed, they define the categories $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)$ and $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ as certain monoidal symmetric subcategories of $\mathrm{D}(X_{\mathrm{proet}}, \Omega_T)$ which depends functorially upon X. Next, we take P the class of proper morphisms and I the class of étale maps. The four statements above follow from classical statements about proper and smooth base change. They are checked (using the proétale setting) in [BS15] 6.7 and [HRS21]. $\square$

We now explain how this construction extends to stacks. First, we need the following descent properties.

Definition 3.2.8. Let D be a 6-functors formalism on $\mathrm{Sch}_k^{\mathrm{ft}}$. Let $f : X \to Y$ be a morphism of schemes, we say that D satisfies $*$ or !-descent along f if

$$(f_n^*) : \mathrm{D}(Y) \to \lim_{\Delta} \mathrm{D}(X^{n/Y}) \tag{3.12}$$

or

$$(f_n^!) : \mathrm{D}(Y) \to \lim_{\Delta} \mathrm{D}(X^{n/Y}) \tag{3.13}$$

are isomorphisms respectively, where $X^{n/Y}$ denotes the n-fold product of X over Y and $f_n : X^{n/Y} \to Y$ is the projection.

Theorem 3.2.9. *For all $X \in \mathrm{Sch}_k^{\mathrm{ft}}$ the category $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ is presentable. Furthermore the functor $X \mapsto \mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ satisfies smooth $*$ and ! descent.*

Proof. The presentability statement is done in [HRS21] 3.49. The $*$-descent property holds for v-cover by a result of [BM21] Theorem 1.8, see also the introduction of [HS23]. The !-descent is a consequence of Proposition 6.18 and Theorem 7.19 of [Sch23]. □

Notation 3.2.10. Denote by Stk, the category of Artin stacks locally of finite type over k.

Proposition 3.2.11 ([Man22] A.5.16)**.** *There is a canonical extension of the 6-functors formalism $\mathrm{D}_{\mathrm{indcons}}(-, \Omega_T)$ to $\mathrm{Corr(Stk)}$, the category of correspondences on Artin stacks locally of finite type over k. For X an Artin stack locally of finite type, the category $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ is given by*

$$\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T) = \varprojlim_{S \to X} \mathrm{D}_{\mathrm{indcons}}(S, \Omega_T), \tag{3.14}$$

where the limit is taken over all schemes $S \in \mathrm{Sch}_k^{\mathrm{ft}}$ and all maps $S \to X$ and transition maps are given by $*$-pullbacks.

Definition 3.2.12. For a stack X, denote by $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T) \subset \mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ the full subcategory of $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ of objects A such that for all $f : S \to X$, f^*A is in $\mathrm{D}_{\mathrm{cons}}(S, \Omega_T)$.

Remark 3.2.13. Contrary to schemes, the natural map $\mathrm{Ind}(\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)) \to \mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ is not necessarily an isomorphism.

Remark 3.2.14. Since we allow non representable maps in Stk_k, the 6-functors formalism $\mathrm{D}_{\mathrm{cons}}(-, \Omega_T)$ a priori does not extend to Stk_k. Consider the subcategory $\mathrm{Stk}_k^{\mathrm{repr}}$ of Stk_k composed of the same objects but with only representable maps. Consider the restriction of the 6-functors $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ to $\mathrm{Stk}_k^{\mathrm{repr}}$, then $X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \Omega_T) \subset \mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ defines a 6-functors formalism. This follows from the fact that constructible sheaves are preserved under !-pushforward along all representable maps.

Remark 3.2.15. Recall that for a scheme X, all objects in $\mathrm{D}_{\mathrm{cons}}(X, \Omega_{T,\mathbb{Z}_\ell})$ are derived complete, we refer to [BS15] section 3.5 for a discussion about derived completions. By [BS15] 3.5.1. a sheaf $K \in \mathrm{D}_{\mathrm{cons}}(X, \Omega_{T,\mathbb{Z}_\ell})$ is derived complete if the natural map

$$K \to \varprojlim_{n,m} K \otimes_{\Omega_{T,\mathbb{Z}_\ell}} \mathbb{Z}/\ell^n\mathbb{Z}[T[\ell^m]]$$

is an isomorphism. Moreover a sheaf $M \in \mathrm{D}(X_{\mathrm{proet}}, \Omega_{T,\mathbb{Z}_\ell})$ is constructible if and only is $M/\mathfrak{m} = M \otimes_{\Omega_{T,\mathbb{Z}_\ell}} \Omega_{T,\mathbb{Z}_\ell}/\mathfrak{m}$ is a constructible $\mathbb{F}_\ell$-sheaf where $\mathfrak{m}$ denotes the maximal ideal of $\Omega_{T,\mathbb{Z}_\ell}$.

Lemma 3.2.16 (Derived Nakayama)**.** *Let $M \in \mathrm{D}_{\mathrm{cons}}(X, \Omega_{T,\mathbb{Z}_\ell})$, if $M/\mathfrak{m} = 0$ then $M = 0$.*

Proof. Choose a stratification $X = \sqcup X_i$ for which M is constructible. After pulling back to each strata we can assume that M is lisse. Let $X' \to X$ be a proétale cover such that $M_{|X'}$ is perfect constant. Such a cover exists by [HRS21] 3.26. Then $M_{|X'} \simeq N_{X'}$ is isomorphic to a constant complex. The statement then reduces down to the derived Nakayama lemma for rings which holds by [Aut] Tag 0G1U. □

Remark 3.2.17. By [HRS21] 3.29, an object $A \in \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ is in $\mathrm{D}_{\mathrm{lis}}(X, \Lambda)$ if and only if all cohomology sheaves are lisse in the classical sense.

3.2.2 Setting up the perverse t-structures

Theorem 3.2.18. *There is a t-structure on $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)$ such that for all geometric points $x : \mathrm{Spec}(k) \to X$ the functor $x^* : \mathrm{D}_{\mathrm{cons}}(X, \Omega_T) \to \mathrm{Perf}_{\Omega_T}$ is t-exact.*

Definition 3.2.19. We call the t-structure of theorem 3.2.18 the standard t-structure.

Proof. We first check that $\mathrm{D}_{\mathrm{cons}}(X, \Omega_{T,\mathbb{Z}_\ell})$ has a natural t-structure. It is enough to show that the ring Ω_T is t-admissible in the sense of [HRS21] 3.27. The ring $\Omega_{T,\mathbb{Z}_\ell}$ is clearly regular and noetherian, in particular coherent. Let S be an extremally disconnected set and write it $S = \varprojlim_i S_i$ as a limit of finite sets. Then we have

$$\Gamma(S, \Omega_{T,\mathbb{Z}_\ell}) = \varprojlim_{n,m} \varinjlim_{i} \Gamma(S_i, \mathbb{Z}/\ell^n\mathbb{Z}[T[m]]).$$

Now each $\Gamma(S_i, \mathbb{Z}/\ell^n\mathbb{Z}[T[m]])$ is flat over $\mathbb{Z}/\ell^n\mathbb{Z}[T[m]]$ and therefore so is $\varinjlim_i \Gamma(S_i, \mathbb{Z}/\ell^n\mathbb{Z}[T[m]])$, we can now apply [Aut] Tag 0912 and get that $\Omega_{T,\mathbb{Z}_\ell} \to \Gamma(S, \Omega_{T,\mathbb{Z}_\ell})$ is flat. The same argument holds for $\Omega_{T,\mathcal{O}_E})$ for any finite extension $E/\mathbb{Q}_\ell$, then $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T) = \varinjlim_E \mathrm{D}_{\mathrm{cons}}(X, \Omega_{T,\mathcal{O}_E})$ and it is easy to see that the transition are t-exact and therefore induce a t-structure on the colimit. □

Theorem 3.2.20. *Let X be a k-scheme of finite type, there is a unique t-structure on $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)$ which we call the middle perversity t-structure such that A is in $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)^{\geq^p 0}$ if and only if*

$$\forall i \ \dim \mathit{supp} H^{-i} \leq i.$$

Sketch of proof. We only sketch the construction to convince the reader that the argument of [BBD82] 3.4 applies to our situation. Consider pairs $(\mathcal{S}, \mathcal{L})$ where $\mathcal{S}$ is a stratification $X = \bigsqcup_{s \in \mathcal{S}} X_s$ and $\mathcal{L}$ is a collection of $\mathbb{F}_\ell$-local systems $\mathcal{L}_s$ on each strata. We consider now the category $\mathrm{D}_{\mathrm{cons},(\mathcal{S},\mathcal{L})}(X, \Omega_T)$ defined as the full subcategory of $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)$ of $(\mathcal{S}, \mathcal{L})$-constructible sheaves, that is sheaves A that are constructible and such that for all $i \in \mathbb{Z}$ and $s \in \mathcal{S}$, we have $H^i(i_s^* A \otimes_{\Omega_{T,\mathbb{Z}_\ell}} \mathbb{F}_\ell)$ and $H^i(i_s^! A \otimes_{\Omega_{T,\mathbb{Z}_\ell}} \mathbb{F}_\ell)$ are successive extension of local systems in $\mathcal{L}_s$ where $i_s : X_s \to X$ is the inclusion of the stratum corresponding to s. Using the constructibility results of [BS15], we can always refine the pair $(\mathcal{S}, \mathcal{L})$ such that if A is an $(X_s, \mathcal{L}_s)$-constructible sheaf on X_s then $i_{s,!}A$ and $i_{s,*}A$ are $(\mathcal{S}, \mathcal{L})$-constructible. Once this is in place, we can apply the gluing formalism of [BBD82]. □

Remark 3.2.21. Just as in the case of $\mathbb{Z}_\ell$-sheaves, this perverse t-structure is not stable under Verdier duality.

3.2.3 The canonical sheaf and the unipotent categories

There is a canonical morphism

$$can : \pi_1(T) \to (\Omega_T)^\times \tag{3.15}$$

which defines an Ω_T-rank one local system on T which we denote by L_T.

Lemma 3.2.22 ([GL96] 3.1)**.** *Let $p : T \to T'$ be a morphism of tori, it induces a morphism $p_* : \Omega_T \to \Omega_{T'}$.*

(i). $p^* L_{T'} = L_T \otimes_{\Omega_T} \Omega_{T'}$.

(ii). Assume p is a quotient map of relative dimension d then we have $p_! L_T = L_{T'}[-2d](-d)$.

Remark 3.2.23. In *loc. cit.*, a quotient map means the projection on a direct factor.

Corollary 3.2.24. *The sheaf L_T is multiplicative, that is we have an isomorphism $m^* L_T \simeq L_T \boxtimes_{\Omega_T} L_T$ where $m : T \times T \to T$ is the multiplication.*

Let X be a scheme with an action of T. In [BY13], the authors have defined the free monodromic completion of sheaves on X. We define an integral version using the categories $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)$.

Remark 3.2.25. It should be possible to use a similar strategy as in [BY13] and [BR22b] using categories of monodromic sheaves and then completing the category, our method has the advantage to not have to use pro-objects.

Definition 3.2.26. We define $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_{\mathrm{unip}}$ to be the category of (T, L_T)-equivariant sheaves on X. We refer to Appendix 3.A for the definition of equivariant sheaves.

Remark 3.2.27. In appendix 3.A, we also define $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)_{\mathrm{unip}}$. The category $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_{\mathrm{unip}}$ is then the full subcategory of $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)_{\mathrm{unip}}$ of objects such that their image in $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ is constructible.

Lemma 3.2.28. *There is a perverse t-exact equivalence*

$$\begin{aligned} \mathrm{D}^b(\Omega_T) &\simeq \mathrm{D}_{\mathrm{cons}}(T, \Omega_T)_{\mathrm{unip}} \\ M &\mapsto M \otimes_{\Omega_T} L_T[\dim T]. \end{aligned}$$

Proof. A sheaf A on T is (T, L_T)-equivariant if and only if $A \otimes L_T^\vee$ is T-equivariant and therefore descends to the point, which yields the equivalence, the t-exactness is immediate. □

Proposition 3.2.29. *Consider the category $\mathrm{Sch}^{T,\mathrm{tf}}$ of schemes with a T-action. There exists a six functors formalism $\mathrm{D}_{\mathrm{cons}}(-, \Omega_T)_{\mathrm{unip}}$ (resp $\mathrm{D}_{\mathrm{indcons}}(-, \Omega_T)_{\mathrm{unip}}$) given by*

$$\mathrm{D}_{\mathrm{cons}}(-, \Omega_T)_{\mathrm{unip}}(X) = \mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_{\mathrm{unip}}. \tag{3.16}$$

(resp $\mathrm{D}_{\mathrm{indcons}}(-, \Omega_T)_{\mathrm{unip}}(X) = \mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)_{\mathrm{unip}}$), for all $X \in \mathrm{Sch}^{T,\mathrm{tf}}$.

Proof. We show it for $\mathrm{D}_{\mathrm{indcons}}$, the case of $\mathrm{D}_{\mathrm{cons}}$ follows by taking constructible objects in $\mathrm{D}_{\mathrm{indcons}}$. First consider the 6-functors formalism constructed previously $X \in \mathrm{Sch}^{\mathrm{tf}} \mapsto \mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$. Consider the functor $\mathrm{Sch}_k^{\mathrm{ft}} \to \mathrm{Corr}(\mathrm{Sch}_k^{\mathrm{ft}})$ sending $X \mapsto X$ and $f : X \to Y$ to the correspondence $X = X \to Y$. This functor is lax monoidal. Composing it with the lax monoidal functor $\mathrm{D}_{\mathrm{indcons}}(-, \Omega_T)$ yields a lax monoidal functor $\mathrm{Sch}_k^{\mathrm{ft}} \to \infty - \mathrm{Cat}$. By lax monoidality, the group object T is sent to the category $\mathrm{D}_{\mathrm{indcons}}(T, \Omega_T)$ which is then canonically equipped with a monoidal structure. This monoidal structure is nothing else than the !-convolution of Appendix 3.A.

Similarly, if X is a scheme with an action of T, the category $\mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ has a natural action of $\mathrm{D}_{\mathrm{indcons}}(T, \Omega_T)$. More generally, the functor $X \to \mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)$ yields a functor $\mathrm{Corr}(\mathrm{Sch}^{T,\mathrm{tf}}) \to \mathrm{D}_{\mathrm{indcons}}(T, \Omega_T)-\mathrm{Mod}$, the ∞-category of categories with an action of $\mathrm{D}_{\mathrm{indcons}}(T, \Omega_T)$. Taking (twisted)-invariants yields a lax monoidal functor $X \mapsto \mathrm{D}_{\mathrm{indcons}}(X, \Omega_T)_{\mathrm{unip}}$. This defines a 3-functor formalism, all categories in sight are presentable and cocomplete and all three functors commutes with all colimits, the existence of right adjoint follows from the adjoint functor theorem 5.5.2.9. ☐

3.2.4 Twisted variants

We keep the scheme X with a T-action. We introduce the following notational convention, we denote by χ a finite order character of $\pi_1^t(T)$ of order prime to ℓ defined over $\mathcal{O}_E$ where $E/\mathbb{Q}_\ell$ is a finite extension. We use the notation of [LY20] and denote by $\mathrm{CH}(T)$ the set of all such character, note however that we do not allow characters of order ℓ contrary to *loc. cit.* and we consider them as defined over $\overline{\mathbb{Z}}_\ell$. Then the sheaf $L_T \otimes \mathcal{L}_\chi$ is an Ω_T rank one character sheaf on T. We define $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_\chi$ to be the category of $(L_T \otimes \mathcal{L}_\chi)$-equivariant sheaves on X and extend the 6-functors as in the previous section to the twisted variants.

3.2.5 Functoriality with respect to finite étale isogenies

Lemma 3.2.30. *Let $[\ell] : T \to T$ be the ℓ-power map. It induces a map of rings $[\ell]_* : \Omega_T \to \Omega_T$ for which the target is free over the source of rank $|T[\ell]|$.*

Proof. We only need to prove the corresponding assertions for $\Omega_{T,\mathbb{Z}_\ell}$ since the rest follows after tensoring with $\overline{\mathbb{Z}}_\ell$. We first choose an isomorphism $T \simeq \mathbb{G}_m^r$ to generators $(\gamma_1, \dots, \gamma_r)$ of $\pi_1(T)_\ell$ so that $\Omega_{T,\mathbb{Z}_\ell} \simeq \mathbb{Z}_\ell[\![t_1, \dots t_r]\!]$ where $t_i + 1 = \gamma_i$. The map $[\ell]_*$ is given by

$$[\ell]_*(t_i) = (t_i + 1)^\ell - 1.$$

Reducing everything mod ℓ yields $[\ell]_*(t_i) = t_i^\ell \mathrm{mod} \ell$ and the induced map $[\ell]_* \mathrm{mod} \ell$ clearly defines a structure of a free module for the target, we can now apply [Aut] Tag 00NS, which implies the freeness hypothesis and the rank. ☐

Let $f : T' \to T$ be a finite étale isogeny. Consider $f_* L_{T'}$, this is a lisse $\Omega_{T'}$-sheaf of rank the degree of f. Note that since $L_T \otimes_{\Omega_T} \overline{\mathbb{Z}}_\ell = \overline{\mathbb{Z}}_\ell$, where $\Omega_T \to \overline{\mathbb{Z}}_\ell$ is the augmentation, we have

$$f_* L_{T'} \otimes_{\Omega_{T'}} \overline{\mathbb{Z}}_\ell = f_* \overline{\mathbb{Z}}_\ell.$$

We denote by $\mathrm{CH}(\ker(f)) \subset \mathrm{CH}(T)$ the set of characters of $\pi_1^t(T)$ of order prime to ℓ that factor through $\ker(f)$.

We denote by $\mathrm{For}_{T'}^T$ the forgetful functor $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T) \to \mathrm{D}_{\mathrm{cons}}(X, \Omega_{T'})$.

Lemma 3.2.31. *There is an isomorphism of $\Omega_{T'}$-sheaves on T.*

$$f_* L_{T'} = \bigoplus_{\chi \in \mathrm{CH}(\ker(f))} \mathrm{For}_{T'}^T(L_T) \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi.$$

Proof. The sheaf $f_*L_{T'}$ is lisse and the corresponding representation of $\pi_1(T)$ is the induction along the embedding $\pi_1(T') \to \pi_1(T)$ of the character can. We can factor f into $f_1 : T' \to T_1$ and $f_2 : T_1 \to T$ such that f_1 is a finite isogeny of degree prime to ℓ and f_2 is of degree a power of ℓ. Since the induced map $\Omega_{T'} \to \Omega_{T_1}$ is an isomorphism and the map $\Omega_{T_1} \to \Omega_T$ is a finite extension of degree $|\ker f_2|$, it is enough to show the lemma for f_1 and f_2 separately.

For f_1, we have $f_{1,*}L_T = \oplus_{\chi\in \mathrm{CH}(\ker f_1)} L_{T_1} \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi$. To show the statement for f_2, by choosing isomorphism between T, T' and $\mathbb{G}_m^{\dim T}$, we can reduce to $T = T' = \mathbb{G}_m$ and $f_2 = [\ell]$. Now note that $L_T = \varprojlim_n [\ell^n]\overline{\mathbb{Z}}_\ell$, now we have $[\ell]_* L_T = \varprojlim_n [\ell]_*[\ell^n]_*\overline{\mathbb{Z}}_\ell = \varprojlim_n [\ell^{n+1}]_*\overline{\mathbb{Z}}_\ell = L_T$. Note however that the Ω_T-structure is given by For_T^T along the map $[\ell]_*$. □

3.2.6 Monodromic sheaves

We keep X with a T-action and let $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Z}}_\ell, \Omega_T\}$ be a coefficient ring.

Definition 3.2.32. Let Y be a connected scheme and $\bar{y}$ a geometric point of Y. We define $\mathrm{Rep}_\Lambda(\pi_1(Y, \bar{y})) = \mathrm{D}_{\mathrm{lis}}(Y, \Lambda)^\heartsuit$. We say Y is a categorical $K(\pi, 1)$ if the realization functor

$$\mathrm{D}^b(\mathrm{Rep}_\Lambda(\pi_1(Y, \bar{y}))) \to \mathrm{D}_{\mathrm{lis}}(Y, \Lambda),$$

is an equivalence.

Remark 3.2.33. The category $\mathrm{Rep}_\Lambda(\pi_1(Y))$ is the usual abelian category of continuous representations of $\pi_1(Y)$ on Λ-modules of finite type.

Lemma 3.2.34 ([Ach17])**.** *The torus T is a categorical $K(\pi, 1)$.*

Let A in $\mathrm{Rep}_\Lambda(\pi_1(T))$. Since $\pi_1(T)^{\mathrm{wild}}$ is normal in $\pi_1(T)$ and of pro-order prime to ℓ, the sheaf A splits as a direct sum $A^{\mathrm{tame}} \oplus A^{\mathrm{wild}}$. The two summands are characterized by the fact that $\pi_1(T)^{\mathrm{wild}}$ acts trivially on A^{tame} and non trivially on all subquotients of A^{wild}.

Lemma 3.2.35. *The category $\mathrm{D}_{\mathrm{lis}}(T, \Lambda)$ splits as a direct sum*

$$\mathrm{D}_{\mathrm{lis}}(T, \Lambda) = \mathrm{D}_{\mathrm{lis}}(T, \Lambda)^{\mathrm{tame}} \oplus \mathrm{D}_{\mathrm{lis}}(T, \Lambda)^{\mathrm{wild}},$$

such that $A \in \mathrm{D}_{\mathrm{lis}}(T, \Lambda)^{\mathrm{tame}}$, resp. $\mathrm{D}_{\mathrm{lis}}(T, \Lambda)^{\mathrm{wild}}$ if and only if for all i, $H^i(A) = H^i(A)^{\mathrm{tame}}$ resp. $H^i(A) = H^i(A)^{\mathrm{wild}}$.

Definition 3.2.36. Let $A \in \mathrm{D}_{\mathrm{lis}}(T, \Lambda)$. We denote by A^{tame} and A^{wild} the two direct factors of lemma 3.2.35 and we call the objects of $\mathrm{D}_{\mathrm{lis}}(T, \Lambda)^{\mathrm{tame}}$ tame sheaves.

Proof of lemma 3.2.35. The abelian category $\mathrm{Rep}_\Lambda(\pi_1(T))$ splits as $\mathrm{Rep}_\Lambda(\pi_1(T))^{\mathrm{tame}} \oplus \mathrm{Rep}_\Lambda(\pi_1(T))^{\mathrm{wild}}$ hence so does the derived category. This induces the splitting on $\mathrm{D}_{\mathrm{lis}}(T, \Lambda)$ in view of lemma 3.2.34. □

Definition 3.2.37. Let $A \in \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ be a sheaf on X. The sheaf A is monodromic if for all $x \in X$, the pullback along $a_x : T \to X, t \mapsto t.x$ is lisse and tame on T.

Theorem 3.2.38 ([Ver83])**.** *Let A be a sheaf of Λ-modules on X, suppose A is monodromic then there is a canonical action of $\pi_1^t(T)$ on A. This action is functorial on monodromic sheaves.*

Proof. We recall its construction. Let $\tilde{T} = \varprojlim_n T$ where n is prime to p and the transition maps are given by $T \mapsto T, x \mapsto x^n$. Since all transition maps are affine, $\tilde{T}$ is an affine group scheme. The map $\tau : \tilde{T} \to T$ is pro-étale covering with group $\pi_1^t(T)$. Consider the action of $\tilde{T}$ on X given by the composition of τ with the action map. We prove that for all monodromic sheaves $A \in \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ there is a unique isomorphism

$$\phi : a^*A \to p^*A$$

on $\tilde{T} \times X$, where a is the action map and p is the projection, such that the induced map along $1 \times X$ is the identity. By [Ver83], we have $\mathrm{R}\Gamma(\tilde{T}, \Lambda) = \Lambda$. The Kunneth formula then implies that a^* is fully faithful, that is the map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(a^*A, a^*B)$ is an isomorphism for all $A, B \in \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ monodromic. This in particular implies the unicity of the map ϕ.

By [Ver83] 5.1, if Λ is a torsion ring, there exists n prime to p, such that $a_n^*A \simeq p^*A$ for a_n the n-dilated action map, that is, the action of T given by $(t, x) \mapsto t^n x$. Pulling back to $\tilde{T}$ produces the map ϕ. To pass from Λ a torsion ring to a general Λ, we present Λ as a limit and pass to the limit. Since $\pi_1^t(T) = \ker(\tilde{T} \to T)$ acts trivially on X, the map ϕ defines an action of $\pi_1(T)$ on A. $\square$

Remark 3.2.39. Verdier works a priori only with $T = \mathbb{G}_m$ but as explained in [BY13] Appendix A, the construction naturally extends to any torus T.

Definition 3.2.40. For a monodromic sheaf A, we denote by $\phi_A : \Lambda[\pi_1^t(T)] \to \mathrm{End}(A)$ the corresponding monodromy action. We say A is unipotent monodromic is ϕ_A factors through $\Lambda[\pi_1(T)_\ell]$, we denote by $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)_{\mathrm{mon,unip}}$ the full subcategory of $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ of unipotent monodromic sheaves.

Lemma 3.2.41. *Consider the category of* $\mathrm{Sch}^{T,\mathrm{ft}}$ *of schemes with a* T*-action. The functor* $X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \Lambda)_{\mathrm{mon,unip}}$ *satisfies descent in the smooth topology of* $\mathrm{Sch}^{T,\mathrm{ft}}$.

Proof. Let $Y \to X$ be a smooth T-equivariant cover. Firstly since $\mathrm{D}_{\mathrm{cons}}(-, \Lambda)$ satisfies descent along smooth cover, we have an isomorphism

$$(f_n^*) : \mathrm{D}_{\mathrm{cons}}(X, \Lambda) \simeq \varprojlim_{\Delta} \mathrm{D}_{\mathrm{cons}}(Y^{X/n}, \Lambda), \tag{3.17}$$

where as before $Y^{n/X}$ denotes the n-fold product of Y over X. As such $Y^{n/X}$ is equipped with the diagonal T-action and we consider $\mathrm{D}_{\mathrm{cons}}(Y^{n/X}, \Lambda)_{\mathrm{mon,unip}}$, since pullbacks along T-equivariants maps preserve the category $\mathrm{D}_{\mathrm{cons}}(-, \Lambda)_{\mathrm{mon,unip}}$. We have an induced cosimplicial category $n \mapsto \mathrm{D}_{\mathrm{cons}}(Y^{n/X}, \Lambda)_{\mathrm{mon,unip}}$. Taking limits yields a commutative diagram of categories

$$\begin{tikzcd}
\mathrm{D}_{\mathrm{cons}}(X, \Lambda) \arrow[r] & \varprojlim_{\Delta} \mathrm{D}_{\mathrm{cons}}(Y^{n/X}, \Lambda) \\
\mathrm{D}_{\mathrm{cons}}(X, \Lambda)_{\mathrm{mon,unip}} \arrow[r] \arrow[u, hook] & \varprojlim_{\Delta} \mathrm{D}_{\mathrm{cons}}(Y^{n/X}, \Lambda)_{\mathrm{mon,unip}} \arrow[u, hook]
\end{tikzcd}$$

Note that the map $\varprojlim_{\Delta} \mathrm{D}_{\mathrm{cons}}(Y^{n/X}, \Lambda)_{\mathrm{mon,unip}} \to \varprojlim_{\Delta} \mathrm{D}_{\mathrm{cons}}(Y^{n/X}, \Lambda)$ is fully faithful since taking limits preserves fully faithfulness. We want to show that the bottom map is an equivalence. Since all three other maps are fully faithful, the bottom map is fully faithful. It remains to control the essential surjectivity. If (A_n) is an object of $\varprojlim_{\Delta} \mathrm{D}_{\mathrm{cons}}(Y^{n/X}, \Lambda)_{\mathrm{mon,unip}}$, we can descend it to an object $A \in \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$. We only need to check that this object is unipotent monodromic. Let

$x \in X$ and $y \in Y$ over x. Let $\mathcal{O}_x$ be the T-orbit of x and $\mathcal{O}_y$ the one of y. We then have a commutative diagram

$$\begin{tikzcd} T \arrow[r, "a_x"] \arrow[dr, "a_y"'] & \mathcal{O}_x \\ & \mathcal{O}_y \arrow[u] \end{tikzcd}$$

induced by the orbit maps of x and y. By construction $a_x^* A = a_y^* A_1$ which is unipotent monodromic hence A is unipotent monodromic. $\square$

Consider now the full subcategory $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)^{\overline{\mathbb{Z}}_\ell-\mathrm{cons}}$ of $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)$ of objects such that their image under the forgetful functor $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T) \to \mathrm{D}(X_{\mathrm{proet}}, \overline{\mathbb{Z}}_\ell)$ is in $\mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)$. This then induces a well defined functor $\mathrm{For} : \mathrm{D}_{\mathrm{cons}}(X, \Omega_T)^{\overline{\mathbb{Z}}_\ell-\mathrm{cons}}_{\mathrm{unip}} \to \mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)$.

Lemma 3.2.42. *The forgetful functor induces an equivalence* $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)^{\overline{\mathbb{Z}}_\ell-\mathrm{cons}}_{\mathrm{unip}} \to \mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\mathrm{mon,unip}}$.

Proof. Consider the Bar resolution $X \times T^{n+1} \to X$ of X, since both sides satisfy smooth descent, we have a commutative diagram

$$\begin{tikzcd} \mathrm{D}_{\mathrm{cons}}(X, \Omega_T)^{\overline{\mathbb{Z}}_\ell-\mathrm{cons}}_{\mathrm{unip}} \arrow[r, equal] \arrow[d] & \varprojlim_\Delta \mathrm{D}_{\mathrm{cons}}(X \times T^{n+1}, \Omega_T)^{\overline{\mathbb{Z}}_\ell-\mathrm{cons}}_{\mathrm{unip}} \arrow[d] \\ \mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\mathrm{mon,unip}} \arrow[r, equal] & \varprojlim_\Delta \mathrm{D}_{\mathrm{cons}}(X \times T^{n+1}, \overline{\mathbb{Z}}_\ell)_{\mathrm{mon,unip}} \end{tikzcd}$$

Hence, it is enough to show the statement for $X \times T^{n+1}$. More generally assume that $X = Y \times T$ splits T-equivariantly as a product where T-acts trivially on Y.

We first show that the forgetful functor is fully faithful. Let $A' = A \boxtimes_{\overline{\mathbb{Z}}_\ell} (\overline{\mathbb{Z}}_\ell)_T \in \mathrm{D}_{\mathrm{cons}}(Y \times T, \overline{\mathbb{Z}}_\ell)_{\mathrm{mon,unip}}$, since $(\overline{\mathbb{Z}}_\ell)_T = L_T/\mathfrak{m}$, where $\mathfrak{m}$ denotes the augmentation ideal of Ω_T, we have $A' = \mathrm{For}(A_0)$, where $A_0 = (A \otimes_{\overline{\mathbb{Z}}_\ell} \Omega_T) \boxtimes_{\Omega_T} L_T/\mathfrak{m}$. Let $A' = A \boxtimes_{\overline{\mathbb{Z}}_\ell} (\overline{\mathbb{Z}}_\ell)_T, B' = B \boxtimes_{\overline{\mathbb{Z}}_\ell} (\overline{\mathbb{Z}}_\ell)_T$ be two such objects and denote by A_0 and $B_0 \in \mathrm{D}_{\mathrm{cons}}(X, \Omega_T)^{\overline{\mathbb{Z}}_\ell-\mathrm{cons}}_{\mathrm{unip}}$ the corresponding lifts, then the Kunneth formula implies that

$$\mathrm{Hom}(A_0, B_0) = \mathrm{Hom}_{\overline{\mathbb{Z}}_\ell}(A, B) \otimes_{\overline{\mathbb{Z}}_\ell} \mathrm{Hom}_{\mathrm{D}_{\mathrm{cons}}(T, \Omega_T)_{\mathrm{unip}}}((\overline{\mathbb{Z}}_\ell)_T, (\overline{\mathbb{Z}}_\ell)_T). \tag{3.18}$$

Let us evaluate $\mathrm{Hom}_{\mathrm{D}_{\mathrm{cons}}(T, \Omega_T)_{\mathrm{unip}}}((\overline{\mathbb{Z}}_\ell)_T, (\overline{\mathbb{Z}}_\ell)_T)$. We have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{D}_{\mathrm{cons}}(T, \Omega_T)_{\mathrm{unip}}}((\overline{\mathbb{Z}}_\ell)_T, (\overline{\mathbb{Z}}_\ell)_T) &= \mathrm{Hom}_{\Omega_T}(\Omega_T/\mathfrak{m}, \Omega_T) \otimes_{\Omega_T} \mathrm{Hom}_{\mathrm{D}_{\mathrm{cons}}(T, \Omega_T)_{\mathrm{unip}}}(L_T, L_T) \otimes_{\Omega_T} \Omega_T/\mathfrak{m} \\ &= \mathrm{Hom}_{\Omega_T}(\Omega_T/\mathfrak{m}, \Omega_T) \otimes_{\Omega_T} \Omega_T \otimes_{\Omega_T} \Omega_T/\mathfrak{m} \\ &= \mathrm{End}_{\Omega_T}(\Omega_T/\mathfrak{m}) \\ &= \mathrm{R}\Gamma(T, \overline{\mathbb{Z}}_\ell). \end{aligned}$$

The second line comes from the fact that $\mathrm{Hom}_{\mathrm{D}_{\mathrm{cons}}(T, \Omega_T)_{\mathrm{unip}}}(L_T, L_T) = \Omega_T$ which can be seen through the equivalence $\mathrm{D}_{\mathrm{cons}}(T, \Omega_T)_{\mathrm{unip}} = \mathrm{D}^b_{\mathrm{coh}}(\Omega_T)$. The last line comes from lemma 3.2.43. On the other hand after applying the forgetful functor, we get that

$$\mathrm{Hom}_{\mathrm{D}_{\mathrm{cons}}(Y \times T, \overline{\mathbb{Z}}_\ell)_{\mathrm{mon-unip}}}(A', B') = \mathrm{Hom}_{\overline{\mathbb{Z}}_\ell}(A, B) \otimes_{\overline{\mathbb{Z}}_\ell} \mathrm{R}\Gamma(T, \overline{\mathbb{Z}}_\ell), \tag{3.19}$$

as $\mathrm{R}\Gamma(T,\overline{\mathbb{Z}}_\ell) = \mathrm{End}((\overline{\mathbb{Z}}_\ell)_T)$. Therefore the forgetful functor is fully faithful on objects of the form A_0. We show that these objects generate the category $\mathrm{D}_{\mathrm{cons}}(Y\times T,\Omega_T)^{\overline{\mathbb{Z}}_\ell-\mathrm{cons}}_{\mathrm{unip}}$. Let $A_0 \in \mathrm{D}_{\mathrm{cons}}(Y\times T,\Omega_T)^{\overline{\mathbb{Z}}_\ell-\mathrm{cons}}_{\mathrm{unip}}$, as A_0 is (T,L_T)-equivariant A_0 can be written as $A_0 = A'\boxtimes L_T$, where A' is an Ω_T-constructible sheaf on Y. But as A_0 is also $\overline{\mathbb{Z}}_\ell$-constructible, the Ω_T-structure of A' factors is of $\mathfrak{m}^\infty$-torsion. As such A' is in the full subcategory of $\mathrm{D}_{\mathrm{cons}}(Y,\Omega_T)$ generated by the essential image of the functor $\mathrm{D}_{\mathrm{cons}}(Y,\overline{\mathbb{Z}}_\ell)\to\mathrm{D}_{\mathrm{cons}}(Y,\Omega_T)$ induced by the forgetful functor along the augmentation $\Omega_T\to\overline{\mathbb{Z}}_\ell$. Let $A'\in\mathrm{D}_{\mathrm{cons}}(Y,\Omega_T)$ be in the essential image of $\mathrm{D}_{\mathrm{cons}}(Y,\overline{\mathbb{Z}}_\ell)$, then we have

$$A'\boxtimes_{\Omega_T} L_T = A'\boxtimes_{\overline{\mathbb{Z}}_\ell} L_T/\mathfrak{m} = A'\boxtimes_{\overline{\mathbb{Z}}_\ell}(\overline{\mathbb{Z}}_\ell)_T. \tag{3.20}$$

Let $C,D\in\mathrm{D}_{\mathrm{cons}}(X,\Omega_T)^{\overline{\mathbb{Z}}_\ell-\mathrm{cons}}_{\mathrm{unip}}$, since objects of the form $A'\boxtimes(\overline{\mathbb{Z}}_\ell)_T$ generate the category we can write $C = \varinjlim_i (A'_i\boxtimes(\overline{\mathbb{Z}}_\ell)_T)$ and $D = \varinjlim_j (B'_j\boxtimes(\overline{\mathbb{Z}}_\ell)_T)$ where both colimits are finite. As the forgetful functor is a right adjoint it commutes with limits hence it also commutes with finite colimits [Lur] 1.1.4.1. We then have

$$\begin{aligned}\mathrm{Hom}(C,D) &= \varprojlim_i \varinjlim_j \mathrm{Hom}((A'_i\boxtimes(\overline{\mathbb{Z}}_\ell)_T),(B'_j\boxtimes(\overline{\mathbb{Z}}_\ell)_T))\\ &= \varprojlim_i \varinjlim_j \mathrm{Hom}(\mathrm{For}(A'_i\boxtimes(\overline{\mathbb{Z}}_\ell)_T),\mathrm{For}(B'_j\boxtimes(\overline{\mathbb{Z}}_\ell)_T))\\ &= \mathrm{Hom}(\mathrm{For}(A),\mathrm{For}(B)).\end{aligned}$$

Hence For is fully faithful. Since the objects of the form $\mathrm{For}(A'\boxtimes_{\overline{\mathbb{Z}}_\ell}(\overline{\mathbb{Z}}_\ell)_T)$ generate the category $\mathrm{D}_{\mathrm{cons}}(Y\times T,\overline{\mathbb{Z}}_\ell)_{\mathrm{mon,unip}}$ under finite colimits, the essential surjectivity is clear. Indeed let $A\in\mathrm{D}_{\mathrm{cons}}(Y\times T,\overline{\mathbb{Z}}_\ell)_{\mathrm{mon,unip}}$, we can then write $A = \varinjlim_i \mathrm{For}(A'\boxtimes_{\overline{\mathbb{Z}}_\ell}(\overline{\mathbb{Z}}_\ell)_T) = \mathrm{For}(\varinjlim_i A'\boxtimes_{\overline{\mathbb{Z}}_\ell}(\overline{\mathbb{Z}}_\ell)_T)$. ☐

Lemma 3.2.43. *There is a canonical isomorphism* $\mathrm{End}_{\Omega_T}(\Omega_T/\mathfrak{m}) = \mathrm{R}\Gamma(T,\overline{\mathbb{Z}}_\ell)$.

Proof. Consider the following functors $\mathrm{D}_{\mathrm{cons}}(\mathrm{pt},\Omega_T)^{\overline{\mathbb{Z}}_\ell-\mathrm{cons}} \xrightarrow{p^*} \mathrm{D}_{\mathrm{cons}}(T,\Omega_T)^{\overline{\mathbb{Z}}_\ell-\mathrm{cons}} \xrightarrow{\mathrm{For}} \mathrm{D}_{\mathrm{cons}}(T,\overline{\mathbb{Z}}_\ell)$, where $p: T\to\mathrm{pt}$ is the structure map. By fonctoriality we get a map

$$\mathrm{End}_{\Omega_T}(\Omega_T/\mathfrak{m}) \to \mathrm{End}_T((\overline{\mathbb{Z}}_\ell)_T) = \mathrm{R}\Gamma(T,\overline{\mathbb{Z}}_\ell). \tag{3.21}$$

It remains to check that this is an isomorphism. This can be done after taking cohomology, namely, we want that the induced map

$$\mathrm{Ext}^*_{\Omega_T}(\Omega_T/\mathfrak{m},\Omega_T/\mathfrak{m}) \to H^*(T,\overline{\mathbb{Z}}_\ell), \tag{3.22}$$

is an isomorphism. Since it is deduced from functoriality, this map is a map of algebras. It is known that both sides are exterior algebras on their degree one parts. For the left hand this is $(\mathfrak{m}/\mathfrak{m}^2)^\vee$ (where the $(-)^\vee$ is the $\overline{\mathbb{Z}}_\ell$-linear dual) and for the right hand side this is $H^1(T,\overline{\mathbb{Z}}_\ell)$. But those two are canonically isomorphic to $\mathrm{Hom}(\pi_1^\ell(T),\mathbb{Z}_\ell))\otimes_{\mathbb{Z}_\ell}\overline{\mathbb{Z}}_\ell$ where this Hom denotes the set of morphism in the 1-category of profinite groups. ☐

Let $A\in\mathrm{D}_{\mathrm{cons}}(X,\Omega_T)^{\overline{\mathbb{Z}}_\ell-\mathrm{cons}}_{\mathrm{unip}}$ and consider the object $A'\in\mathrm{D}_{\mathrm{cons}}(X,\overline{\mathbb{Z}}_\ell)_{\mathrm{unip,mon}}$ be the image of A under the forgetful functor. Consider the $\overline{\mathbb{Z}}_\ell[\pi_1^t(T)]$-module structure on A', since A' is unipotent monodromic the morphism $\overline{\mathbb{Z}}_\ell[\pi_1^t(T)]\to\mathrm{End}(A')$ factors through Ω_T.

Lemma 3.2.44. *The two Ω_T-structures on A', one coming from Verdier's monodromy and one coming from the forgetful functor, coincide.*

Proof. The object $A \in \mathrm{D}_{\mathrm{cons}}(X, \Omega_T)^{\overline{\mathbb{Z}}_\ell-\mathrm{cons}}$ is an Ω_T-unipotent monodromic, its canonical monodromy is a morphism $\Omega_T[\pi_1^t(T)] \to \mathrm{End}(A)$. As A comes from an equivariant sheaf this morphism factors through $\Omega_T[\pi_1^t(T)]/I$ where I is the ideal generated by elements $(t - \mathrm{can}(t))$ for $t \in \pi_1^t(T)$ and can is the canonical map (3.15). where the second map is the forgetful map and is the inverse functor to $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)^{\overline{\mathbb{Z}}_\ell-\mathrm{cons}}_{\mathrm{unip}} \to \mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\mathrm{mon,unip}}$. Consider now the following diagram

$$\begin{tikzcd} & \overline{\mathbb{Z}}_\ell[\pi_1^t(T)] \arrow[dl] \arrow[d] \\ \Omega_T \arrow[r] & \Omega_T[\pi_1^t(T)] \arrow[d] \\ & \Omega_T[\pi_1^t(T)]/I \arrow[d] \\ & \mathrm{End}(A') \end{tikzcd}$$

where the map $\overline{\mathbb{Z}}_\ell[\pi_1^t(T)] \to \Omega_T$ is induced by the morphism can and the other morphisms are the natural ones. The triangle does not commute but it commutes after projecting in $\Omega_T[\pi_1^t(T)]/I$. The canonical monodromy is the Ω_T-structure coming from the vertical composition while the Ω_T-structure on the forgetful functor is the map $\Omega_T \to \mathrm{End}(A')$. □

We now denote by $\Phi_{\mathrm{unip}} : \mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\mathrm{unip,mon}} \to \mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_{\mathrm{unip}}$ the inverse of the forgetful functor.

Remark 3.2.45. The previous construction naturally extends to the non unipotent setting, for $\chi \in \mathrm{CH}(T)$ we get a fully faithful functor

$$\Phi_\chi : \mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\mathrm{mon},\chi} \to \mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_\chi.$$

3.2.7 Reduction mod ℓ and the completed categories of [BY13]

Denote by $\Omega_{T,\mathbb{F}_\ell} = \Omega_T \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell$, this is the ring denoted by R_T in [BR22b] defined for the ring $k = \mathbb{F}_\ell$. We can work as previously using this ring instead of Ω_T and define the category $\mathrm{D}_{\mathrm{cons}}(X, \Omega_{T,\mathbb{F}_\ell})$, the sheaf $L_{T,\mathbb{F}_\ell}$ and the monodromic categories $\mathrm{D}_{\mathrm{cons}}(X, \Omega_{T,\mathbb{F}_\ell})_\chi$ as before for X with a T action. We now give a comparison between our categories of monodromic sheaves and the completion of the categories of monodromic sheaves of [BR22b] and [Gou21]. In this section, let Y be a scheme and let $X \to Y$ be a T-torsor.

Theorem 3.2.46. *We have a natural equivalence*

$$\mathrm{ho}(\mathrm{D}_{\mathrm{cons}}(X, \Omega_{T,\mathbb{F}_\ell})_{\mathrm{unip}}) \simeq \hat{\mathrm{D}}^b_c(X \mathbin{/\!\!/} T)$$

where the category on the right is the completed monodromic category of [BR22b]. In the non-unipotent case there is an equivalence

$$\mathrm{ho}(\mathrm{D}_{\mathrm{cons}}(X, \Omega_{T,\mathbb{F}_{\ell^n}})_\chi) \simeq \hat{\mathrm{D}}^b_c(X \mathbin{/\!\!/} T)_{\mathcal{L}_\chi}, \tag{3.23}$$

which holds after passing from $\mathbb{F}_\ell$ to a finite extension $\mathbb{F}_{\ell^n}$ where χ is defined.

Proof. We only show the version for $\chi = 1$, the generalization to other χ is straightforward. To define the desired functor, first note that the category $\hat{\mathrm{D}}^b_c(X /\!\!/ T)$ is a full subcategory of the category $\mathrm{ProD}^b_c(X /\!\!/ T)$ of monodromic objects on X, see [BR22b] 3.1 and 10.1. We first define a functor

$$\Psi : \mathrm{hoD}_{\mathrm{cons}}(X, \Omega_{T,\mathbb{F}_\ell})_{\mathrm{unip}} \to \mathrm{ProD}^b_c(X /\!\!/ T)$$

by $A \mapsto "\varprojlim" A \otimes_{\Omega_{T,\mathbb{F}_\ell}} \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n$ where $\mathfrak{m}$ is the maximal ideal of $\Omega_{T,\mathbb{F}_\ell}$. The ring $\Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n$ is an Artin ring over $\mathbb{F}_\ell$ so in particular it is finite dimension and $A \otimes_{\Omega_{T,\mathbb{F}_\ell}} \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n$ lives in $\mathrm{D}_{\mathrm{cons}}(X, \mathbb{F}_\ell)$ after forgetting the $\Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n$-structure. For any $y \in Y$ the restriction to the fiber $X_y = \pi^{-1}(y)$ of A is isomorphic to $M \otimes_{\Omega_{T,\mathbb{F}_\ell}} L_T$ for some $\Omega_{T,\mathbb{F}_\ell}$-module M and therefore the restriction of $A \otimes_{\Omega_{T,\mathbb{F}_\ell}} \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n$ is isomorphic to the sheaf denoted by $M \otimes_{\Omega_{T,\mathbb{F}_\ell}} \mathcal{L}_{T,n}$ in [BR22b] 3.2 and in particular is monodromic on T. This implies that $A \otimes_{\Omega_{T,\mathbb{F}_\ell}} \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n$ is monodromic as an $\mathbb{F}_\ell$-sheaf on X and that Ψ is well defined.

We first check that it factors through the category $\hat{\mathrm{D}}^b_c(X /\!\!/ T)$, which means checking the two properties of [BR22b] definition 3.1. The pro-object $\Psi(A)$ is π-constant, indeed we have the following computation

$$\begin{aligned}\varprojlim \pi_!(A \otimes_{\Omega_{T,\mathbb{F}_\ell}} \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n) &= \varprojlim(\pi_! A) \otimes_{\Omega_{T,\mathbb{F}_\ell}} \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n \\ &= \pi_! A.\end{aligned}$$

The first line follows from the compatibility of the formation of $\pi_!$ with change of coefficients and the second comes from the completeness property of the objects of $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)$ by 3.2.15. The object $\pi_! A$ a priori lives in $\mathrm{D}_{\mathrm{cons}}(Y, \Omega_T)$, but fiberwise it is isomorphic to $M \otimes_{\Omega_{T,\mathbb{F}_\ell}} \mathrm{R}\Gamma_c(T, L_{T,\mathbb{F}_\ell}) \simeq M \otimes_{\Omega_{T,\mathbb{F}_\ell}} \mathbb{F}_\ell[2\dim T]$ and therefore the $\Omega_{T,\mathbb{F}_\ell}$-structure factors trough an $\mathbb{F}_\ell$-structure via $\Omega_{T,\mathbb{F}_\ell} \to \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m} \simeq \mathbb{F}_\ell$, which is a well defined constructible $\mathbb{F}_\ell$-sheaf on Y.

The pro-object $\Psi(A)$ is also uniformly bounded in degrees. There exists $a \leq b$ two integers such that $F \in \mathrm{D}_{\mathrm{cons}}(X)^{[a,b]}$, the functor $- \otimes_{\Omega_{T,\mathbb{F}_\ell}} \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n$ is of cohomological dimension $[-\dim T, 0]$ and the forgetful functor $\mathrm{D}_{\mathrm{cons}}(X, \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n) \to \mathrm{D}_{\mathrm{cons}}(X, \mathbb{F}_\ell)$ is t-exact. This implies that $A \otimes_{\Omega_{T,\mathbb{F}_\ell}} \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n$ lives in cohomological degree $[a - \dim T, b]$ and the functor Ψ factors through $\hat{\mathrm{D}}^b_c(X /\!\!/ T)$.

The functor Ψ is fully faithful. Let $A, B \in \mathrm{D}_{\mathrm{cons}}(X, \Omega_{T,\mathbb{F}_\ell})_{\mathrm{unip}}$, we have

$$\begin{aligned}\mathrm{Hom}(A, B) &= \varprojlim_n \mathrm{Hom}(A, B \otimes_{\Omega_{T,\mathbb{F}_\ell}} \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n) \\ &= \varprojlim_n \varinjlim_m \mathrm{Hom}(A \otimes_{\Omega_{T,\mathbb{F}_\ell}} \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^m, B \otimes_{\Omega_{T,\mathbb{F}_\ell}} \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n).\end{aligned}$$

The first equality comes from the isomorphism $B = \varprojlim_n B \otimes_{\Omega_{T,\mathbb{F}_\ell}} \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n$ and the second from the same isomorphism for A and the fact that each $B \otimes_{\Omega_{T,\mathbb{F}_\ell}} \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n$ is discrete and thus a morphism from A factors through one of its quotients. We apply H^0 to this isomorphism, there is a Milnor short exact sequence

$$\begin{aligned}0 \to \mathrm{R}^1 \varprojlim H^{-1}(\mathrm{Hom}(A, B \otimes_{\Omega_{T,\mathbb{F}_\ell}} \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n) &\to H^0 \varprojlim_n \mathrm{Hom}(A, B \otimes_{\Omega_{T,\mathbb{F}_\ell}} \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n)) \\ &\to \varprojlim_n H^0(\mathrm{Hom}(A, B \otimes_{\Omega_{T,\mathbb{F}_\ell}} \Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n))) \to 0.\end{aligned}$$

Note that as a complex $\mathrm{Hom}(A,B) \in \mathrm{D}(\Omega_{T,\mathbb{F}_\ell})$ is perfect and thus derived complete, indeed the category of derived complete objects is stable and contains $\Omega_{T,\mathbb{F}_\ell}$ hence all perfect complexes. By [Aut] Tag 091P, all the cohomologies of $\mathrm{Hom}(A,B)$ are derived complete hence $H^{-1}(\mathrm{Hom}(A,B))$ is derived complete. Since it is an $\Omega_{T,\mathbb{F}_\ell}$-module of finite type, by Nakayama's lemma it is also $\mathfrak{m}$-adically separated and therefore $\mathfrak{m}$-adically complete by [Aut] Tag 091T hence $\varprojlim_n H^{-1}(\mathrm{Hom}(A,B))/\mathfrak{m}^n = H^{-1}(\mathrm{Hom}(A,B))$ and the first term of the above exact sequence vanishes. Hence H^0 commutes with the limit, since the colimit filtered it is exact and commutes with H^0. The fully faithfulness now follows from the description of the morphisms in $\hat{\mathrm{D}}^b_c(X \sslash T)$ [BR22b] 3.1 and [BY13] Section A.2.

It remains to show that Ψ is essentially surjective, note that we have a compatibility between free monodromic local systems as $\Psi(L_T) = \mathcal{L}_T$ where the second sheaf is the free monodromic local system of [BR22b] 3.2. Let $A = \text{"}\varprojlim_n\text{"} A_n$ be an object in $\hat{\mathrm{D}}^b_c(X \sslash T)$, we can assume that for each A_n, the Verdier's monodromy $\phi_{A_n} : \Omega_{T,\mathbb{F}_\ell} \to \mathrm{End}(A_n)$-factors through $\Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n$. Consider now $\tilde{A} = \varprojlim_n \Phi_{\mathrm{unip}}(A_n) \in \mathrm{D}_{\mathrm{cons}}(X, \Omega_{T,\mathbb{F}_\ell})_{\mathrm{unip}}$. By construction $\tilde{A}/\mathfrak{m}^n = \Phi_{\mathrm{unip}}(A_n)$ and forgetting the $\Omega_{T,\mathbb{F}_\ell}/\mathfrak{m}^n$-structure yields back A_n hence $\Psi(\tilde{A}) = A$ and Ψ is essentially surjective. □

3.2.8 The functor $\pi_\dagger$

We define two functors

$$\pi_\dagger : \mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_\chi \to \mathrm{D}_{\mathrm{cons}}(X/(T, \mathcal{L}_\chi), \overline{\mathbb{Z}}_\ell)$$

and

$$\pi_{\dagger,\overline{\mathbb{F}}_\ell} : \mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_\chi \to \mathrm{D}_{\mathrm{cons}}(X/(T, \mathcal{L}_\chi), \overline{\mathbb{F}}_\ell).$$

by

(*i*). $\pi_\dagger = - \otimes_{\Omega_T} \overline{\mathbb{Z}}_\ell$ and

(*ii*). $\pi_{\dagger,\overline{\mathbb{F}}_\ell} = - \otimes_{\Omega_T} \overline{\mathbb{F}}_\ell$, where $\Omega_T \to \overline{\mathbb{Z}}_\ell \to \overline{\mathbb{F}}_\ell$ is the augmentation.

The equivariant structure is clear since $L_T \otimes_{\Omega_T} \overline{\mathbb{Z}}_\ell = \overline{\mathbb{Z}}_\ell$.

Remark 3.2.47. Under the equivalence of theorem 3.2.46, the functors $\pi_\dagger$ and $\pi_{\dagger,\overline{\mathbb{F}}_\ell}$ correspond to the same functors $\pi_\dagger$ of [BR22b] and [Gou21].

Remark 3.2.48. These functors have a canonical right adjoint which we denote by $\pi^\dagger$ and $\pi^\dagger_{\overline{\mathbb{F}}_\ell}$.

Remark 3.2.49. In the unipotent case, the functor $\pi_\dagger$ is isomorphic to the functor $\pi^*\pi_![2\dim T]$.

3.2.9 Pushforward to the base on unipotent monodromic categories.

Consider a Cartesian diagram

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "p"'] & X' \arrow[d, "p'"] \\ Y \arrow[r, "f"'] & Y' \end{tikzcd}$$

where the stacks X and X' are equipped with an action of T, f' is T-equivariant and p and p' are T-torsors. Let $\chi \in \mathrm{CH}(T)$ and consider the categories $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_{\mathrm{unip}}$ and $\mathrm{D}_{\mathrm{cons}}(X', \Omega_T)_{\mathrm{unip}}$.

Lemma 3.2.50 (Analog of [BY13] A.3.4)**.** *The canonical base change maps*

(i). $p'_! f'_! \to f_! p_!$,

(ii). $p'_! f'_* \to f'_* p_!$,

(iii). $p_! f'^* \to f^* p'_!$,

(iv). $p_! f'^! \to f^! p'_!$

are isomorphisms of functors $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_{\mathrm{unip}} \to \mathrm{D}_{\mathrm{cons}}(Y', \overline{\mathbb{Z}}_\ell)$ *and* $\mathrm{D}_{\mathrm{cons}}(X', \Omega_T)_{\mathrm{unip}} \to \mathrm{D}_{\mathrm{cons}}(Y, \overline{\mathbb{Z}}_\ell)$.

Proof. The cases of $f_!$ and f^* are immediate by the compatibility of the composition of lower shrieks and proper base change. For the two other maps, both statements are Zariski local on Y and Y' respectively. We may assume that the T-torsors X and X' are trivial and we fix trivializations $X = Y \times T$ and $X' = Y' \times T$. We do the case for $f'^!$, the case of f'_* is similar. Let $A \in \mathrm{D}_{\mathrm{cons}}(X', \Omega_T)_{\mathrm{unip}}$ then $A \simeq A' \boxtimes L_T$ for some sheaf A' on Y'. Then we have

$$\begin{aligned} p_! f'^! A &= p_!(f \times \mathrm{id})^! A \\ &= p_!(f^! A' \boxtimes L_T) \\ &= f^! A' \otimes \mathrm{R}\Gamma_c(T, L_T) \\ &= f^! p'_!(A' \otimes L_T) \\ &= f^! p'_! A. \end{aligned}$$

□

3.3 Deligne-Lusztig theory and the F-horocycle space

3.3.1 The F-twisted horocycle transform.

We consider a variant of the horocycle transform that was used by Lusztig to define character sheaves. This variant was already considered by Lusztig in [Lus15] and [Lus17]. We also refer to [BDR20]. Consider the following diagram.

$$\begin{tikzcd} & (G \times G)/\Delta_{\mathrm{F}} U \arrow[dl] \arrow[d] \arrow[ddr] & \\ G/U \times G/U \arrow[d] & (G \times G)/\Delta_{\mathrm{F}} B \arrow[dl] \arrow[dr] & \\ (G/U \times G/U)/\Delta_{\mathrm{F}} T & & (G \times G)/\Delta_{\mathrm{F}} G \end{tikzcd}$$

Here Δ_{F} refers to the action on the right given by $(x, y).g = (xg, y\mathrm{F}(g))$. All maps are the obvious quotient maps. They are equivariant for the diagonal left action of G on all objects. Passing to the quotient and using the isomorphism $a : \Delta(G)\backslash(G \times G) \simeq G, (x, y) \mapsto x^{-1}y$ yields the following commutative diagram of algebraic stacks over $\overline{\mathbb{F}}_q$.

$$\begin{tikzcd}
& \frac{G}{\mathrm{Ad}_{\mathrm{F}} U} \arrow[dl, "\tilde{r}"'] \arrow[d, "\tilde{p}"] \arrow[ddr, "\tilde{q}"] & \\
U\backslash G/U \arrow[d, "p"'] & \frac{G}{\mathrm{Ad}_{\mathrm{F}}(B)} \arrow[dl, "r"] \arrow[dr, "q"'] & \\
\frac{(U\backslash G/U)}{\mathrm{Ad}_{\mathrm{F}} T} & & \frac{G}{\mathrm{Ad}_{\mathrm{F}}(G)}
\end{tikzcd}$$

where Ad_{F} refers to the action given by $g.x = gx\mathrm{F}(g)^{-1}$.

Remark 3.3.1. Consider the isomorphism $(G\times G)/\Delta_{\mathrm{F}} B \simeq G\times G/B$ given by $(x, y) \mapsto (\mathrm{F}(x)y^{-1}, xB)$. This morphism is equivariant for the diagonal action of G on the source and the action of G on the target given by $g.(x, yB) = (\mathrm{F}(g)xg^{-1}, gyB)$. Passing to the quotient yields the following isomorphism

$$\frac{G}{\mathrm{Ad}_{\mathrm{F}} B} \simeq G^{\mathrm{F}}\backslash G/B. \tag{3.24}$$

Note that we can freely replace B by any F-stable subgroup of G.

We call the space $\frac{(U\backslash G/U)}{\mathrm{Ad}_{\mathrm{F}} T}$ the F-twisted horocycle space. We also define the three functors F-character, F-horocycle correspondence and the $*-$ F-horocycle correspondence as

$$\begin{aligned}
\mathrm{HC}_F &= r_! q^* \\
\mathrm{CH}_F &= q_! r^! \\
\mathrm{HC}^*_{\mathrm{F}} &= r_* q^!.
\end{aligned}$$

The $*$ in $\mathrm{HC}^*_{\mathrm{F}}$ refers to the $*$-pushforward. Since q is proper we have an adjunction $(\mathrm{HC}_{\mathrm{F}}, \mathrm{CH}_{\mathrm{F}})$ and since r is smooth of relative dimension $\dim U$, we have an adjunction $(\mathrm{CH}_{\mathrm{F}}[2\dim U], \mathrm{HC}^*_{\mathrm{F}})$. These functors were introduced in [Lus15], [Lus17] and [BDR20] but in a non stacky form.

Consider the category $\mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \Lambda)$, $s : \mathrm{pt} \to \mathrm{pt}/G^{\mathrm{F}}$ and the adjunction $s_! : \mathrm{D}(\Lambda) \leftrightarrows \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \Lambda) : s^!$. By general nonsense, the functor $s^! s_! : \mathrm{D}(\Lambda) \to \mathrm{D}(\Lambda)$ is a monad acting on $\mathrm{D}(\Lambda)$. Recall that we say that the functor $s^!$ is monadic if it identifies $\mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \Lambda)$ with the category of algebras over the monad $s^! s_!$.

Lemma 3.3.2. *The functor $s^!$ is monadic and the monad $s^! s_!$ is canonically identified with the functor $\Lambda[G^{\mathrm{F}}] \otimes -$ seen as monad using the group algebra structure of $\Lambda[G^{\mathrm{F}}]$.*

Proof. To check the monadicity assertion, we apply the Barr-Beck-Lurie theorem [Lur] 4.7.0.3. First note that both categories are cocomplete, it is then enough to check that $s^!$ is conservative and commutes with geometric realization (i.e. Δ^{op}-shaped colimits). But since s is surjective and étale $s^! = s^*$ is conservative and commutes with all colimits this concludes the first part of the lemma.

For the second, since the functor $s^! s_!$ is a continuous endofunctor of $\mathrm{D}(\Lambda)$, by proper base change it is given by $\otimes_\Lambda \Lambda[G^{\mathrm{F}}]$ together with its group algebra structure. $\square$

Corollary 3.3.3. *There is a canonical equivalence of ∞-categories*

$$\mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \Lambda) = \mathrm{D}(\Lambda[G^{\mathrm{F}}]). \tag{3.25}$$

Proof. By lemma 3.3.2, there is a canonical equivalence $\mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}},\Lambda) = \Lambda[G^{\mathrm{F}}]-\mathrm{Mod}(\mathrm{D}(\Lambda))$ where the right hand side category is the category of $\Lambda[G^{\mathrm{F}}]$-modules in $\mathrm{D}(\Lambda)$. Identifying $\mathrm{D}(\Lambda[G^{\mathrm{F}}])$ with $\Lambda[G^{\mathrm{F}}]-\mathrm{Mod}(\mathrm{D}(\Lambda))$ is standard and follows from the Barr-Beck-Lurie theorem. Let us recall the argument, consider the adjunction $\Lambda[G^{\mathrm{F}}]\otimes_{\Lambda} - : \mathrm{D}(\Lambda) \leftrightarrows \mathrm{D}(\Lambda[G^{\mathrm{F}}]) : \mathrm{For}$. The forgetful functor is conservative and commutes with all colimits (since it has itself a right adjoint given by $\mathrm{Hom}_{\Lambda}(\Lambda[G^{\mathrm{F}}],-)$), hence For is monadic and the monad $\mathrm{For}\circ(\Lambda[G^{\mathrm{F}}]\otimes_{\Lambda}-)$ is simply given by tensoring with the algebra $\Lambda[G^{\mathrm{F}}]$. □

Remark 3.3.4. The subcategory $\mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^{\mathrm{F}},\Lambda)$ is then identified with the full subcategory of $\mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}},\Lambda)$ of objects such that their pullback to pt is constructible, i.e., is a perfect complex of Λ-modules. In classical terms, this is nothing else than (the ∞-enhancement of) the derived category $\mathrm{D}(\mathrm{Rep}^{\mathrm{ft}}_{\Lambda}G^{\mathrm{F}})$ of modules of $\Lambda[G^{\mathrm{F}}]$-modules of finite type.

Lemma 3.3.5. *We have a canonical isomorphism of functors.*

$$\begin{gathered}\mathrm{D}_{\mathrm{cons}}(U\backslash G/U,\Lambda)\to D(\Lambda)\\ \mathrm{R}\Gamma_c(G/U,(\mathrm{id}\times \mathrm{F}_{G/U})^*a^*-) = 1^*\mathrm{CH}_{\mathrm{F}}(p_!-)[-2\dim U],\end{gathered}$$

where 1 *is the map* $\mathrm{pt}\to\mathrm{pt}/G^{\mathrm{F}}$.

Proof. Consider the diagram

$$\begin{tikzcd}
G/U\times G/U \arrow[d] & G/U \arrow[l, "(\mathrm{id}\times\mathrm{F})"'] \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\
\Delta(G)\backslash(G/U\times G/U) \arrow[d] & G^{\mathrm{F}}\backslash G/U \arrow[l] \arrow[r, "a"'] \arrow[d] & G^{\mathrm{F}}\backslash\mathrm{pt} \arrow[d] \\
U\backslash G/U & \frac{G}{\mathrm{Ad}_{\mathrm{F}}U} \arrow[l, "\tilde{r}"] \arrow[r, "\tilde{q}"'] & \frac{G}{\mathrm{Ad}_{\mathrm{F}}G}
\end{tikzcd}$$

where the bottom vertical maps are isomorphims by the map $a:(x,y)\mapsto x^{-1}y$, the top vertical maps are the quotient maps and the top squares are Cartesian. We now have by proper and smooth base change

$$\begin{aligned}\mathrm{R}\Gamma_c(G/U,(\mathrm{id}\times\mathrm{F}_{G/U})^*a^*-) &= 1^*\tilde{q}_!\tilde{r}^*\\ &= 1^*q_!p_!\tilde{r}^![-2\dim U]\\ &= 1^*q_!r^!p_![-2\dim U] = 1^*\mathrm{CH}_{\mathrm{F}}(p_!-).\end{aligned}$$

□

Finally we denote by $\mathrm{For}_{\Omega_T}:\mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}},\Omega_T)\to\mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}},\overline{\mathbb{Z}}_\ell)$ the forgetful functor along the inclusion $\overline{\mathbb{Z}}_\ell\to\Omega_T$ and by $-\otimes_{\overline{\mathbb{Z}}_\ell}\Omega_T$ its left adjoint.

3.3.2 Sheaves on the F-twisted horocycle space and Deligne-Lusztig theory.

In this section we discuss some links between Deligne-Lusztig theory and twisted horocycle transform. In the next subsection, we will discuss a monodromic variant of the construction here. We first consider the space $\frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}}(T)}$ together with the stratification induced from the Bruhat stratification. Let $w \in W$ and consider the corresponding stratum $\frac{U\backslash UwTU/U}{\mathrm{Ad}_{\mathrm{F}}(T)}$.

We first note that we have an isomorphism of stacks, after choosing a lift $\dot{w}$ of w,

$$\frac{U\backslash UwTU/U}{\mathrm{Ad}_{\mathrm{F}}(T)} \simeq \mathrm{pt}/(U_w \rtimes T^{w\mathrm{F}}).$$

In particular after passing to categories of sheaves, we get $\mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/(U_w \rtimes T^{w\mathrm{F}}), \Lambda) \simeq \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \Lambda)$. We will also denote by $i_w : \frac{U\backslash UwTU/U}{\mathrm{Ad}_{\mathrm{F}}(T)} \to \frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}}(T)}$ the inclusion.

Remark 3.3.6. The category $\mathrm{D}_{\mathrm{cons}}(\frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T}, \Lambda)$ is obtained by gluing all the categories $\mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \Lambda)$ in a nontrival way.

Remark 3.3.7. Let us also highlight the dependency on the lift $\dot{w}$. We have an isomorphism of stacks

$$\frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}}(T)} = \Delta(G)\backslash(G/U \times G/U)/\Delta_{\mathrm{F}}(T),$$

induced by the map $a : G/U \times G/U \to U\backslash G/U$. Let $\mathcal{O}_w$ be the $\Delta(G) \times \Delta_{\mathrm{F}}(T)$-orbit of the point $(1, \dot{w})$ in $G/U \times G/U$, as a locally closed subscheme of $G/U \times G/U$ it is idenpendant of the choice of $\dot{w}$. Under this isomorphism $\Delta(G)\backslash\mathcal{O}_w/\Delta_{\mathrm{F}}(T)$ is sent to $\frac{U\backslash UwTU/U}{\mathrm{Ad}_{\mathrm{F}}(T)}$. Now $U_w \rtimes T^{w\mathrm{F}}$ is identified with the stabilizer of the point $(1, \dot{w})$ which yield the desired isomorphism.

We recall that to w and $\dot{w}$, Deligne and Lusztig [DL76] have attached a pair of varieties

$$X(w) = \{gB, \mathcal{L}(g) \in BwB\} \subset G/B,$$

and

$$Y(\dot{w}) = \{gU, \mathcal{L}(g) \in U\dot{w}U\} \subset G/U.$$

The following facts hold, we refer to *loc. cit.*

(i). Both varieties $X(w)$ and $Y(\dot{w})$ are G^{F}-stable in G/B and G/U respectively.

(ii). Consider the right action of T on G/U. The variety $Y(\dot{w})$ is $T^{w\mathrm{F}}$-stable. The $T^{w\mathrm{F}}$ and G^{F}-actions on $Y(\dot{w})$ commute.

(iii). The map induced by the projection $G/U \to G/B$ induces a map $Y(\dot{w}) \to X(w)$. This map is a $T^{w\mathrm{F}}$-torsor.

In particular the cohomology $\mathrm{R}\Gamma_c(Y(\dot{w}), \Lambda)$ has two commuting actions of $T^{w\mathrm{F}}$ and G^{F}. We can now define the Deligne-Lusztig induction functor as

$$R_w : \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \Lambda) \to \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \Lambda), M \mapsto M \otimes_{T^{w\mathrm{F}}} \mathrm{R}\Gamma_c(Y(\dot{w}), \Lambda).$$

Consider now the following diagram

$$\begin{tikzcd}
\frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T} & \frac{G}{\mathrm{Ad}_{\mathrm{F}} B} \arrow[l] \arrow[r] & \frac{G}{\mathrm{Ad}_{\mathrm{F}} G} \\
\Delta(G)\backslash(G/U \times G/U)/\Delta_{\mathrm{F}} T \arrow[u] & G^{\mathrm{F}}\backslash G/B \arrow[l] \arrow[r] \arrow[u] & G^{\mathrm{F}}\backslash \mathrm{pt} \arrow[u] \\
\Delta(G)\backslash \mathcal{O}_w/\Delta_{\mathrm{F}}(T) \arrow[u] & G^{\mathrm{F}}\backslash X(w) \arrow[l] \arrow[r] \arrow[u] & G^{\mathrm{F}}\backslash \mathrm{pt} \arrow[u, equal]
\end{tikzcd}$$

where the first line is the F-twisted horocycle correspondence, the second line is isomorphic to the first via the two isomorphisms $\Delta(G)\backslash(G/U \times G/U)/\Delta_{\mathrm{F}}(T) = \frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T}$ and $\frac{G}{\mathrm{Ad}_{\mathrm{F}} B} = \Delta(G)\backslash(G \times G)/\Delta_{\mathrm{F}}(B) = G^{\mathrm{F}}\backslash G/B$ induced by the map a. The bottom line is induced by the inclusion of the orbit $\mathcal{O}_w$ and the fact that $X(w)$ is the intersection of BwB/B with the graph of Frobenius.

Denote by $k_w : \frac{U\backslash BwB/U}{\mathrm{Ad}_{\mathrm{F}} T} \to \mathrm{pt}/T^{w\mathrm{F}}$ the map induced by the quotient map $T^{w\mathrm{F}} \rtimes U_w \to T^{w\mathrm{F}}$. Under the isomorphism $\Delta(G)\backslash \mathcal{O}_w/\Delta_{\mathrm{F}}(T) = \mathrm{pt}/(T^{w\mathrm{F}} \rtimes U_w)$, the map $G^{\mathrm{F}}\backslash X(w) \to \mathrm{pt}/(T^{w\mathrm{F}} \ltimes U_w) \xrightarrow{k_w} \mathrm{pt}/T^{w\mathrm{F}}$ corresponds to a $T^{w\mathrm{F}}$-torsor over $G^{\mathrm{F}}\backslash X(w)$. It is nothing else than $G^{\mathrm{F}}\backslash Y(\dot{w})$. To sum up we have the following diagram

$$\begin{tikzcd}
\frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T} & \frac{G}{\mathrm{Ad}_{\mathrm{F}} B} \arrow[l, "r"'] \arrow[r, "q"] & \frac{G}{\mathrm{Ad}_{\mathrm{F}} G} \\
\frac{U\backslash BwB/U}{\mathrm{Ad}_{\mathrm{F}} T} \arrow[u, "i_w"] & G^{\mathrm{F}}\backslash X(w) \arrow[l, "r"] \arrow[r, "q"'] \arrow[u, "i_w"'] & G^{\mathrm{F}}\backslash \mathrm{pt} \arrow[u, equal]
\end{tikzcd}$$

Let M be a sheaf on $\frac{U\backslash BwB/U}{\mathrm{Ad}_{\mathrm{F}} T}$, proper base change implies that

$$q_! r^* i_{w,!} M = q_! r^* M.$$

We then deduce the following theorem.

Theorem 3.3.8. *There are canonical isomorphisms of functors* $\mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \Lambda) \to \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \Lambda)$ *and* $\mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \Lambda) \to \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \Lambda)$ *respectively,*

$$q_! r^* i_{w,!} k_w^* \simeq \mathrm{R}\Gamma_c(Y(\dot{w}), \Lambda) \otimes_{T^{w\mathrm{F}}} -$$
$$k_{w,*} i_w^* r_! q^* \simeq \mathrm{R}\Gamma_c(Y(\dot{w}), \Lambda) \otimes_{G^{\mathrm{F}}} -.$$

Proof. Using the above diagram, both functors can be summarized with the following correspondence

$$\mathrm{pt}/T^{w\mathrm{F}} \xleftarrow{k_w} \mathrm{pt}/(T^{\mathrm{F}} \rtimes U_w) \xleftarrow{r} G^{\mathrm{F}}\backslash Y(\dot{w})/T^{w\mathrm{F}} \xrightarrow{q} \mathrm{pt}/G^{\mathrm{F}}. \tag{3.26}$$

The result now follows from the Kunneth formula. □

3.3.3 Sheaves on a torus

Before reconstructing Deligne-Lusztig theory we first study the case of a torus. Let us now introduce some notations for the representations of $T^{w\mathrm{F}}$. Note that $T^{w\mathrm{F}}$ is naturally a quotient of $\pi_1^t(T)$, induced by the Lang map $\mathcal{L}_{w\mathrm{F}} : T \to T$ which is a $T^{w\mathrm{F}}$-covering. We denote by $\mathrm{CH}(T^{w\mathrm{F}})$ the set of $\overline{\mathbb{Z}}_\ell$-characters of $T^{w\mathrm{F}}$ of order prime to ℓ, the quotient map $\pi_1^t(T) \to T^{w\mathrm{F}}$ defines an inclusion $\mathrm{CH}(T^{w\mathrm{F}}) \hookrightarrow \mathrm{CH}(T)$. For $\chi \in \mathrm{CH}(T^{w\mathrm{F}})$ there is a corresponding block of $\overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}]$ and we denote the corresponding idempotent by $e_\chi \in \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}]$. We want to relate the two equivariance conditions on T : the $\mathrm{Ad}_{w\mathrm{F}}$-equivariance and the (T, L_T)-equivariance.

Lemma 3.3.9. *We have an equivalence of categories*

$$\mathrm{D}_{\mathrm{cons}}(\frac{T}{\mathrm{Ad}_{w\mathrm{F}}T}, \Omega_T)_{\mathrm{unip}} \simeq e_1 \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \overline{\mathbb{Z}}_\ell). \tag{3.27}$$

Proof. Consider the adjunction

$$\mathrm{For} : \mathrm{D}_{\mathrm{cons}}(\frac{T}{\mathrm{Ad}_{w\mathrm{F}}(T)}, \Omega_T)_{\mathrm{unip}} \leftrightarrows \mathrm{D}_{\mathrm{cons}}(T, \Omega_T)_{\mathrm{unip}} : \mathrm{Av}^{\mathrm{Ad}_{w\mathrm{F}}},$$

where the functor $\mathrm{Av}^{\mathrm{Ad}_{w\mathrm{F}}} = a_!(- \boxtimes_{\Omega_T} \Omega_T[2\dim T])$ for $a : T \times T \to T, (x, y) \mapsto x\mathcal{L}_{w\mathrm{F}}(y)$ and For is the forgetful functor which is right adjoint to $\mathrm{Av}^{\mathrm{Ad}_{w\mathrm{F}}}$.

We compute the monad $\mathrm{For}\mathrm{Av}^{\mathrm{Ad}_{w\mathrm{F}}}$ acting on $\mathrm{D}_{\mathrm{cons}}(T, \Omega_T)_{\mathrm{unip}} \simeq \mathrm{Perf}_{\Omega_T}$. It is enough to compute $\mathrm{Av}^{\mathrm{Ad}_{w\mathrm{F}}}(L_T) = a_!(p^* L_T)$ where $p : T \times T \to T$ is the first projection. By lemma 3.2.22, we have

$$p^* L_T = L_{T\times T} \otimes_{\Omega_{T\times T}, p_*} \Omega_T.$$

We have $a = m \circ (\mathrm{id} \times \mathcal{L}_{w\mathrm{F}})$ and by using lemma 3.2.31 we have $(\mathrm{id} \times \mathcal{L}_{w\mathrm{F}})_! L_{T\times T} = \bigoplus_{\chi \in \mathrm{CH}(T^{w\mathrm{F}})} L_{T\times T} \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_{\chi \times 1}$. Then $m_! L_{T\times T} \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_{\chi\times 1}$ is 0 for non trivial χ and $m_! L_{T\times T} = L_T[-2\dim T]$ by lemma 3.2.22 and the Ω_T-module structure is the one obtained from $m_* : \Omega_{T\times T} \to \Omega_T$. Putting everything together we have $a_! p^* L_T[2\dim T] = L_T \otimes_{\Omega_{T\times T}} \Omega_T$. Taking the fiber at 1 yields $\Omega_T \otimes_{\Omega_{T\times T}} \Omega_T$ where the left Ω_T is an $\Omega_{T\times T}$-modules via the composition $m_*(\mathrm{id} \times \mathcal{L}_{w\mathrm{F}})_* = a_*$ and the right one via the first projection. But now we have

$$\Omega_T \otimes_{a_* \Omega_{T\times T}, p_*} \Omega_T \simeq \Omega_T \otimes_{\mathcal{L}_{w\mathrm{F},*}\Omega_T} \overline{\mathbb{Z}}_\ell \simeq \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}[\ell^\infty]].$$

Now it follows from the Barr-Beck-Lurie theorem as in lemma 3.3.3 that the category of algebras over this monad is equivalent to $\mathrm{D}^b(\overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}[\ell^\infty]])$, that is $e_1 \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \overline{\mathbb{Z}}_\ell)$. ☐

Remark 3.3.10. Let us comment on the right hand side of 3.27, the category $\mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \mathbb{Z}_\ell)$ is equivalent to the bounded derived category of modules over $\overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}]$ which has a natural direct sum decomposition according to the idempotent e_χ. For the idempotent e_1, first note that $e_1 \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}] \simeq \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}[\ell^\infty]]$ hence the category $e_1 \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \overline{\mathbb{Z}}_\ell)$ is equivalent to the bounded derived category of $\overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}[\ell^\infty]]$ modules. Consider now the composition of functors

$$e_1 \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \overline{\mathbb{Z}}_\ell) \simeq \mathrm{D}_{\mathrm{cons}}(\frac{T}{\mathrm{Ad}_{w\mathrm{F}}T}, \Omega_T)_{\mathrm{unip}} \xrightarrow{\mathrm{For}} \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \Omega_T)$$
$$\xrightarrow{1^*} \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}, \Omega_T) \simeq \mathrm{Perf}_{\Omega_T}$$

where $1 : \mathrm{pt} \to \mathrm{pt}/T^{w\mathrm{F}}$ is the universal torsor. Then for $A \in e_1\mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \overline{\mathbb{Z}}_\ell)$ the Ω_T-module obtained by applying all these functors is the following. Consider the Lang map $\mathcal{L}_{w\mathrm{F}} : T \to T$ by functoriality if induces a map of rings $\mathcal{L}_{w\mathrm{F},*} : \Omega_T \to \Omega_T$, then by lemma 3.2.31 we have a natural isomorphism

$$\overline{\mathbb{Z}}_\ell \otimes_{\Omega_T, \mathcal{L}_{w\mathrm{F},*}} \Omega_T \simeq \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}[\ell^\infty]]$$

and that the composite of the functors is simply the forgetful functor corresponding to the map $\Omega_T \to \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}[\ell^\infty]]$.

Similarly we have the following in the non unipotent setting.

Lemma 3.3.11. *Let $\chi \in \mathrm{CH}(T^{w\mathrm{F}})$ we have a canonical equivalence*

$$\mathrm{D}_{\mathrm{cons}}(\frac{T}{\mathrm{Ad}_{w\mathrm{F}}(T)}, \Omega_T)_\chi \simeq e_\chi \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}).$$

Corollary 3.3.12. *Combining all previous equivalences, we get the following equivalence*

$$\mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \overline{\mathbb{Z}}_\ell) \simeq \bigoplus_{\chi \in \mathrm{CH}(T^{w\mathrm{F}})} \mathrm{D}_{\mathrm{cons}}(\frac{T}{\mathrm{Ad}_{w\mathrm{F}}(T)}, \Omega_T)_\chi.$$

3.3.4 Monodromic variant

We now give a second construction of the Deligne-Lusztig induction functors but this time using monodromic sheaves. We will also show that the two constructions are equivalent. This second construction will make it much easier to compare the Deligne-Lusztig theory with Soergel theory and will be relevant in the last section to compute the image under HC_{F} of a Gelfand-Graev representation.

Denote by $\mathrm{For}_{T,\chi}$ the functor

$$\begin{aligned}
\mathrm{D}_{\mathrm{cons}}(\frac{T}{\mathrm{Ad}_{w\mathrm{F}}(T)}, \Omega_T)_\chi &\to \mathrm{D}_{\mathrm{cons}}(\frac{T}{\mathrm{Ad}_{w\mathrm{F}}(T)}, \Omega_T) \\
&\xrightarrow{k_w^*} \mathrm{D}_{\mathrm{cons}}(\frac{T}{\mathrm{Ad}_{w\mathrm{F}}(T) \ltimes U_w}, \Omega_T) \\
&\simeq \mathrm{D}_{\mathrm{cons}}(\frac{U\backslash BwB/U}{\mathrm{Ad}_{\mathrm{F}}(T)}, \Omega_T)
\end{aligned}$$

where the first functor is the forgetful functor of the $(T, L_T \otimes \mathcal{L}_\chi)$-equivariance, the second is a pullback under $k_w : \frac{T}{\mathrm{Ad}_{w\mathrm{F}}(T) \ltimes U_w} \to \frac{T}{\mathrm{Ad}_{w\mathrm{F}}(T)}$ and the last one is induced by the isomorphism $\frac{U\backslash BwB/U}{\mathrm{Ad}_{\mathrm{F}}(T)} \simeq \frac{T}{\mathrm{Ad}_{w\mathrm{F}}(T) \ltimes U_w}$. Consider now the following composition of functors

$$\begin{aligned}
\mathrm{D_{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \overline{\mathbb{Z}}_\ell) &\simeq \bigoplus_{\chi \in \mathrm{CH}(T^{w\mathrm{F}})} \mathrm{D_{cons}}(\frac{T}{\mathrm{Ad}_{w\mathrm{F}}(T)}, \Omega_T)_\chi \\
&\xrightarrow{\bigoplus_{\chi \in \mathrm{CH}(T^{w\mathrm{F}})} \mathrm{For}_{T,\chi}} \mathrm{D_{cons}}(\frac{U\backslash BwB/U}{\mathrm{Ad_F}(T)}, \Omega_T) \\
&\xrightarrow{i_{w,!}} \mathrm{D_{cons}}(\frac{U\backslash G/U}{\mathrm{Ad_F}(T)}, \Omega_T) \\
&\xrightarrow{q_! r^*} \mathrm{D_{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \Omega_T) \\
&\xrightarrow{\mathrm{For}_{\Omega_T}} \mathrm{D_{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell).
\end{aligned}$$

Lemma 3.3.13. *The previous composition of functors is isomorphic to the functor*

$$\begin{aligned}
\mathrm{D_{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \overline{\mathbb{Z}}_\ell) &\to \mathrm{D_{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell) \\
A &\mapsto A \otimes_{T^{w\mathrm{F}}} \mathrm{R}\Gamma_c(Y(\dot{w}), \overline{\mathbb{Z}}_\ell),
\end{aligned}$$

where $Y(\dot{w})$ is the Deligne-Lusztig variety associated to $\dot{w}$.

Proof. Consider the diagram

$$\begin{tikzcd}
T \arrow[d, "p_{w,3}"'] & T \times \mathrm{pt}/U_w \arrow[l, "k'_w"'] \arrow[r, "\sim"] \arrow[d, "p_{w,2}"'] & U\backslash BwB/U \arrow[r, "i_w"] \arrow[d, "p_{w,1}"'] & U\backslash G/U \arrow[d, "p"] \\
\mathrm{pt}/T^{w\mathrm{F}} & \mathrm{pt}/(T^{w\mathrm{F}} \ltimes U_w) \arrow[l, "k_w"] \arrow[r, "\sim"] & \frac{U\backslash BwB/U}{\mathrm{Ad_F}(T)} \arrow[r, "i_w"'] & \frac{U\backslash G/U}{\mathrm{Ad_F}(T)}
\end{tikzcd}$$

where all squares are Cartesian and the vertical arrows are the quotient maps for the $\mathrm{Ad}_{w\mathrm{F}}$-action. In the proof of lemma 3.3.9, we showed that $p_{w,3,!}(L_T \otimes \mathcal{L}_\chi) = e_\chi \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}][-2\dim T]$. Proper base change implies

$$p_! i_{w,!}(k'_w)^*(L_T \otimes \mathcal{L}_\chi) = i_{w,!}\mathrm{For}_{T,\chi}(e_\chi \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}])[-2\dim T].$$

We can now apply lemma 3.3.5, to get that

$$\mathrm{For}_{\Omega_T} q_! r^* i_{w,!}\mathrm{For}_{T,\chi}(e_\chi \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}]) = \mathrm{R}\Gamma_c(G/U, (\mathrm{id} \times \mathrm{F}_{G/U})^* a^* i_{w,!}(k'_w)^*(L_T \otimes \mathcal{L}_\chi))[2\dim T].$$

We now have to identify the right hand side with the cohomology of the Deligne-Lusztig varieties, which is lemma 3.3.14. ∎

Lemma 3.3.14. *There is a canonical isomorphism, compatible with the actions of G^{F} and $T^{w\mathrm{F}}$*

$$\mathrm{R}\Gamma_c(G/U, (\mathrm{id} \times \mathrm{F}_{G/U})^* a^* i_{w,!}\nu_w^*(L_T \otimes \mathcal{L}_\chi))[2\dim T] \simeq e_\chi \mathrm{R}\Gamma_c(Y(w), \overline{\mathbb{Z}}_\ell),$$

where $Y(\dot{w})$ is the Deligne-Luszig variety associated to $\dot{w}$ and the map $\nu_{\dot{w}} : BwB/B \to T$ is the projection onto T using the splitting $BwB = U \times T \times U_w$ induced by the choice of $\dot{w}$.

Proof. Denote by $\tilde{Y}(w) \subset G/U$ the subvariety of $\{gU, g^{-1}\mathrm{F}(g) \in BwB\}$. Note that $\tilde{Y}(w)$ is independent of the choice of $\dot{w}$. The map $\nu_{\dot{w}} : BwB \to T$ induces a map $\nu'_{\dot{w}} : \tilde{Y}(w) \to T$ defined as $\nu'_w(gU) = \nu_{\dot{w}}(g^{-1}\mathrm{F}(g))$. Denote by $X(w) \subset G/B$ the Deligne-Lusztig variety associated to w, that is $X(w) = \{gB, g^{-1}\mathrm{F}(g) \in BwB\}$. The variety $Y(\dot{w}) \subset G/U$ is $\{gU, g^{-1}\mathrm{F}(g) \in U\dot{w}U\}$, it is therefore identified with the fiber at 1 of $\nu'_{\dot{w}}$. Let us summarize the situation via the following diagram

$$\begin{tikzcd}
 & Y(\dot{w}) \arrow[r] \arrow[dl, "q'"'] \arrow[d] & 1 \arrow[d] \\
X(w) & \tilde{Y}(w) \arrow[l, "q"] \arrow[r, "\nu'_{\dot{w}}"] \arrow[d] & T \arrow[d, equal] \\
 & U\backslash BwB/U \arrow[r, "\nu_{\dot{w}}"] & T.
\end{tikzcd}$$

The subscheme $\tilde{Y}(w) \subset G/U$ is T-stable for the action of T by right translation and the map $\nu'_{\dot{w}}$ is equivariant for the action of T on itself via the map $\mathcal{L}_{w\mathrm{F}}$. The map q is a T-torsor and the map q' is a $T^{w\mathrm{F}}$-torsor. The action of T on $\tilde{Y}(w)$ then induces an isomorphism

$$Y(\dot{w}) \times^{T^{w\mathrm{F}}} T \simeq \tilde{Y}(w).$$

Where $\times^{T^{w\mathrm{F}}}$ denotes the quotient by the action of $T^{w\mathrm{F}}$ given by $t.(x, y) = (xt^{-1}, ty)$. In particular this induces a canonical isomorphism

$$\nu'^*_{\dot{w}}(L_T \otimes \mathcal{L}_\chi) = \Omega_T \boxtimes^{T^{w\mathrm{F}}}_{\Omega_T} \mathcal{L}^*_{w\mathrm{F}}(L_T \otimes \mathcal{L}_\chi).$$

where $\boxtimes^{T^{w\mathrm{F}}}_{\Omega_T}$ denotes the sheaf on $Y(\dot{w}) \times^{T^{w\mathrm{F}}} T$ descended from the sheaf $\boxtimes_{\Omega_T}$ on $Y(\dot{w}) \times T$. The Kunneth formula now yields

$$q_!\nu'^*_{\dot{w}}(L_T \otimes \mathcal{L}_\chi) = (q'_!\Omega_T \otimes_{\Omega_T} \mathrm{R}\Gamma_c(T, \mathcal{L}^*_{w\mathrm{F}}(L_T \otimes \mathcal{L}_\chi)))^{T^{w\mathrm{F}}}.$$

We claim that $\mathrm{R}\Gamma_c(T, \mathcal{L}^*_{w\mathrm{F}}(L_T \otimes \mathcal{L}_\chi)) = e_\chi \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}][-2\dim T]$. Let us first show that this implies the result,

$$\begin{aligned}
q_!\nu'^*_{\dot{w}}(L_T \otimes \mathcal{L}_\chi) &= (q'_!\Omega_T \otimes_{\Omega_T} \mathrm{R}\Gamma_c(T, \mathcal{L}^*_{w\mathrm{F}}(L_T \otimes \mathcal{L}_\chi)))^{T^{w\mathrm{F}}} \\
&= (q'_!\Omega_T \otimes_{\Omega_T} e_\chi \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}][-2\dim T])^{T^{w\mathrm{F}}} \\
&= e_\chi q'_! \overline{\mathbb{Z}}_\ell[-2\dim T].
\end{aligned}$$

We now prove the claim. Decompose $T^{w\mathrm{F}} = T_1 \times T_2$ where T_2 is of ℓ-torsion and T_1 is of prime to ℓ-torsion. We factor $\mathcal{L}_{w\mathrm{F}} : T \to T$ into $T \xrightarrow{f_1} T \xrightarrow{f_2} T$ where f_1 is the quotient by T_1 and f_2 the quotient by T_2.

As T_2 is of ℓ-torsion, we have an isomorphism $f_2^*\mathcal{L}_\chi = \mathcal{L}_\chi$. On the other hand, by lemma 3.2.22, $f_2^*L_T = L_T \otimes_{\Omega_T, f_{2,*}} \Omega_T$ where $f_{2,*} : \Omega_T \to \Omega_T$ is the map induced by f_2. Combining both of those facts, we have $f_2^*(L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi) = (L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi) \otimes_{\Omega_T, f_{2,*}} \Omega_T$.

As f_1 is the quotient by T_1 which is of prime to ℓ-torsion, the induced map $f_{1,*} : \Omega_T \to \Omega_T$ is an isomorphism, hence $f_1^*L_T = L_T$. We have

$$\begin{aligned}
f_1^*f_2^*(L_T \otimes \mathcal{L}_\chi) &= f_1^*(L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi) \otimes_{\Omega_T, f_{2,*}} \Omega_T \\
&= f_1^*(L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi) \otimes_{\Omega_T, f_{2,*}} \Omega_T \\
&= L_T \otimes_{\overline{\mathbb{Z}}_\ell} f_1^*\mathcal{L}_\chi \otimes_{\Omega_T, f_{2,*}} \Omega_T.
\end{aligned}$$

To compute the cohomology, we first apply $f_{1,*}$. This yields

$$f_{1,*}(L_T \otimes_{\overline{\mathbb{Z}}_\ell} f_1^*\mathcal{L}_\chi) \otimes_{\Omega_T, f_{2,*}} \Omega_T = f_{1,*}L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi \otimes_{\Omega_T, f_{2,*}} \Omega_T$$

by the projection formula. The sheaf $f_{1,*}L_T$ decompose as $\bigoplus_{\chi" \in \mathrm{CH}(T^{w\mathrm{F}})} \mathcal{L}_{\chi"} \otimes_{\overline{\mathbb{Z}}_\ell} L_T$. We have

$$\begin{aligned}
\mathrm{R}\Gamma_c(T, f_{1,*}(L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi) \otimes_{\Omega_T, f_{2,*}} \Omega_T) &= \bigoplus_{\chi"} \mathrm{R}\Gamma_c(T, (L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_{\chi"} \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi) \otimes_{\Omega_T, f_{2,*}} \Omega_T) \\
&= \mathrm{R}\Gamma_c(T, L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_{\chi^{-1}} \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi) \otimes_{\Omega_T, f_{2,*}} \Omega_T \\
&= \chi \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Z}}_\ell[-2\dim T] \otimes_{\Omega_T, f_{2,*}} \Omega_T \\
&= \chi \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}][\ell^\infty][-2\dim T] \\
&= e_\chi \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}][-2\dim T]
\end{aligned}$$

where the second line comes from the fact that the cohomology of $L_T \otimes \mathcal{L}_\chi$ is 0 if χ is non trivial, the third one comes from the identification of the action of T_1 on the cohomology and χ denotes the one dimensional representation of T_1 corresponding to χ. □

3.3.5 Comparison of the two functors

Consider the two functors $\mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell) \to \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/T^{w\mathrm{F}}, \overline{\mathbb{Z}}_\ell)$,

- (i). $i_w^! \mathrm{HC}_{\mathrm{F}}^*(-)$, where $\mathrm{HC}_{\mathrm{F}}^* = r_* q^!$ as before,
- (ii). $i_w^! \oplus_\chi \mathrm{Av}_\chi \mathrm{HC}_{\mathrm{F}}^*(- \otimes_{\overline{\mathbb{Z}}_\ell} \Omega_T)$, where $\mathrm{Av}_\chi = a_!(- \boxtimes_{\Omega_T} (L_T \otimes \mathcal{L}_\chi[2\dim T)])$ and a is the action map of T acting by right translations.

These functors are the right adjoints of the functors of theorem 3.3.8 and lemma 3.3.13, which we have shown to be isomorphic to the Deligne-Lusztig induction functors. By the unicity of right adjoints, these functors are isomorphic.

Let $M \in \mathrm{D}_{\mathrm{cons}}(\frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}}T}, \overline{\mathbb{Z}}_\ell)$ and consider the action of T by right translation. On each T-orbit, the sheaf M is monodromic for this action. The inverse of the forgetful functor of lemma 3.2.44 is a functor

$$\Phi = \oplus_\chi \Phi_\chi : \mathrm{D}_{\mathrm{cons}}(\frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}}T}, \overline{\mathbb{Z}}_\ell) \to \bigoplus_\chi \mathrm{D}_{\mathrm{cons}}(\frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}}T}, \Omega_T)_\chi.$$

Lemma 3.3.15. *The functor Φ is an equivalence and the following diagram commutes.*

$$\begin{tikzcd}
\mathrm{D}_{\mathrm{cons}}(\frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}}T}, \overline{\mathbb{Z}}_\ell) \arrow[rr, "\Phi"] & & \bigoplus_\chi \mathrm{D}_{\mathrm{cons}}(\frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}}T}, \Omega_T)_\chi \\
& \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell) \arrow[ul, "\mathrm{HC}_{\mathrm{F}}^?"] \arrow[ur, "\oplus_\chi \mathrm{Av}_\chi \mathrm{HC}_{\mathrm{F}}^?(-\otimes_{\overline{\mathbb{Z}}_\ell} \Omega_T)"'] &
\end{tikzcd}$$

where $\mathrm{HC}_{\mathrm{F}}^?$ denotes either the functor HC_{F} or the functor $\mathrm{HC}_{\mathrm{F}}^*$.

Proof. We show the statement for $\mathrm{HC}_{\mathrm{F}}^*$ the same proof applies to the functor HC_{F}. By construction, the functor Φ is an equivalence onto its image. It is therefore enough to see that Φ is essentially surjective. Let $M \in \mathrm{D}_{\mathrm{cons}}(\frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}}T}, \Omega_T)_\chi$, then on all strata the Ω_T-structure factors through a quotient Ω_T/I^n where I is the augmentation ideal. Hence this happens globally, therefore M is in the category of constructible $\overline{\mathbb{Z}}_\ell$-sheaves, which lies in the essential image of Φ.

To show the commutativity, note that $\mathrm{Av}_\chi \mathrm{HC}^*_{\mathrm{F}}(- \otimes_{\overline{\mathbb{Z}}_\ell} \Omega_T) = \mathrm{Av}_\chi(\mathrm{HC}^*_{\mathrm{F}}(-) \otimes_{\overline{\mathbb{Z}}_\ell} \Omega_T)$. Now

$$\begin{aligned}\mathrm{Av}_\chi((- \otimes_{\overline{\mathbb{Z}}_\ell} \Omega_T) &= a_!((L_T \otimes \mathcal{L}_\chi)[2\dim T] \boxtimes_{\Omega_T} (- \otimes_{\overline{\mathbb{Z}}_\ell} \Omega_T)) \\ &= a_!((L_T \otimes \mathcal{L}_\chi)[2\dim T] \boxtimes_{\overline{\mathbb{Z}}_\ell} (-)).\end{aligned}$$

But on χ-monodromic sheaves the functor $a_!((L_T \otimes \mathcal{L}_\chi)[2\dim T] \boxtimes_{\overline{\mathbb{Z}}_\ell} (-))$ is nothing else than the functor Φ. $\square$

3.3.6 Parabolic variants and parabolic Deligne-Lusztig functors

In the previous section we have discussed the link between the Deligne-Lusztig induction functors from a torus. In this section we give a a more direct construction of the Deligne-Lusztig induction and restriction functor but which also recovers the parabolic induction/restriction functors.

Consider the following situation. Let $L \subset G$ be an F-stable Levi subgroup and $P = LV$ a parabolic with Levi L and unipotent radical V, we do not require P to be F-stable. We will denote by $\mathrm{F}(V)$ the image of V under F.

Consider now the correspondence

$$\frac{V\backslash G/\mathrm{F}(V)}{\mathrm{Ad}_{\mathrm{F}}L} \xleftarrow{r_P} \frac{G}{\mathrm{Ad}_{\mathrm{F}}P} \xrightarrow{q_P} \frac{G}{\mathrm{Ad}_{\mathrm{F}}G}.$$

Consider the inclusion $i : P\mathrm{F}(P) \subset G$. We have $\frac{V\backslash P\mathrm{F}(P)/\mathrm{F}(V)}{\mathrm{Ad}_{\mathrm{F}}L} = \mathrm{pt}/(L^{\mathrm{F}} \rtimes (V \cap \mathrm{F}^{-1}(V))$. Denote by $k_L : \mathrm{pt}/(L^{\mathrm{F}} \rtimes (V \cap \mathrm{F}^{-1}(V)) \to \mathrm{pt}/L^{\mathrm{F}}$ the map induced by the projection $L^{\mathrm{F}} \rtimes (V \cap \mathrm{F}^{-1}(V) \to \mathrm{pt}/L^{\mathrm{F}}$. We consider the functor $q_{P,!}r_P^* i_!$. Attached to the data of (P, L, F), there are parabolic Deligne-Lusztig varieties, see for instance [DM14],

$$X_P = \{gP, \mathcal{L}(g) \in P\mathrm{F}(P)\} \subset G/P$$

and

$$Y_P = \{gV, \mathcal{L}(g) \in V\mathrm{F}(V)\} \subset G/V.$$

The map $Y_P \to X_P$ is an L^{F}-torsor and the cohomology $\mathrm{R}\Gamma_c(Y_P, \Lambda)$ is equipped with commuting actions of G^{F} and L^{F}.

Theorem 3.3.16. *We have an isomorphism of functors* $\mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/L^{\mathrm{F}}, \Lambda) \to \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \Lambda)$,

$$q_{P,!}r_P^* i_! k_L^* \simeq \mathrm{R}\Gamma_c(Y_P, \Lambda) \otimes_{L^{\mathrm{F}}} -.$$

Definition 3.3.17. We denote this functor by $i_{L\subset P}$. It is usually called the parabolic Deligne-Lusztig induction functor. Its right adjoint is denoted by $r_{L\subset P}$ and is usually called the parabolic Deligne-Lusztig restriction functor.

Proof. The proof is very similar to the proof of theorem 3.3.8. Consider the diagram

$$\begin{tikzcd}
\frac{V\backslash G/\mathrm{F}(V)}{\mathrm{Ad}_{\mathrm{F}}L} & \frac{G}{\mathrm{Ad}_{\mathrm{F}}P} \arrow[l, "r_P"'] \arrow[r, "q_P"] & \frac{G}{\mathrm{Ad}_{\mathrm{F}}G} \\
\frac{V\backslash P\mathrm{F}(P)/V}{\mathrm{Ad}_{\mathrm{F}}L} \arrow[u, "i"] & G^{\mathrm{F}}\backslash X_P \arrow[l, "r_P"] \arrow[r, "q_P"'] \arrow[u, "i"'] & G^{\mathrm{F}}\backslash \mathrm{pt} \arrow[u, equal] \\
\mathrm{pt} \arrow[u, "1"] & G^{\mathrm{F}}\backslash \tilde{Y}_P \arrow[l] \arrow[u, "\pi"] \arrow[ur] &
\end{tikzcd}$$

where the two left squares are Cartesian and $\tilde{Y}_P = \{g \in G, \mathcal{L}(g) \in V\}$. Note that the map $\tilde{Y}_P \to Y_P$ is a $V \cap \mathrm{F}^{-1}(V)$-torsor. Then we have by proper base change for $M \in \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/L^{\mathrm{F}}, \Lambda)$

$$\begin{aligned}
q_{P,!}r_P^* i_! M &= q_{P,!}r_P^*(1_! 1^* M)_{L^{\mathrm{F}}}[2\dim(V \cap \mathrm{F}^{-1}(V))] \\
&= (q_{P,!}i_! r_P^* 1_! 1^* M)_{L^{\mathrm{F}}}[2\dim(V \cap \mathrm{F}^{-1}(V))] \\
&= (q_{P,!}r_P^* 1_! 1^* M)_{L^{\mathrm{F}}}[2\dim(V \cap \mathrm{F}^{-1}(V))] \\
&= 1^* M \otimes_{L^{\mathrm{F}}} \mathrm{R}\Gamma_c(\tilde{Y}_P, \Lambda)[2\dim(V \cap \mathrm{F}^{-1}(V))] \\
&= \mathrm{R}\Gamma_c(Y_P, \Lambda) \otimes_{L^{\mathrm{F}}} -.
\end{aligned}$$

The first line follows from the fact that the map $1 : \mathrm{pt} \to \mathrm{pt}/(L^{\mathrm{F}} \rtimes (V \cap \mathrm{F}^{-1}(V)))$ is a $L^{\mathrm{F}} \rtimes (V \cap \mathrm{F}^{-1}(V))$-torsor and the last one by the fact that $\mathrm{R}\Gamma_c(\tilde{Y}_P, \Lambda)[2\dim(V \cap \mathrm{F}^{-1}(V))] = \mathrm{R}\Gamma_c(Y_P, \Lambda)$. ☐

3.3.7 Compatibility with parabolic induction

Finally we want to discuss some compatibilities with parabolic induction. There are two statements we will be interested in

(*i*). compatibility of the F-horocycle transform with parabolic induction,

(*ii*). transitivity of the Deligne-Lusztig induction.

The second point is already well known but we give a proof with the stacky formalism for completeness.

Let us consider the first situation. Let $B \subset P \subset G$ be a F-stable standard parabolic with F-stable Levi F. Denote by V the unipotent radical of P, by $B_L = L \cap B$, this is a Borel of L and by U_L the unipotent radical of B_L. In particular, we have $B = B_L V$. We define the parabolic induction functors

(*i*). For the horocycle correspondence, consider the correspondence

$$\frac{U_L\backslash L/U_L}{\mathrm{Ad}_{\mathrm{F}}T} \xleftarrow{s} \frac{U\backslash P/U}{\mathrm{Ad}_{\mathrm{F}}T} \xrightarrow{t} \frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}}T}, \tag{3.28}$$

where the left map is induced by projection $P \to L$ and the quotient $U \to U_L$ and the right map is induced by the inclusion $P \to G$. The parabolic induction functor is $i_{L\subset P}^{\mathrm{HC}} = t_! s^*$.

(ii). On the representation side, consider the correspondence

$$\mathrm{pt}/L^{\mathrm{F}} \xleftarrow{s'} \mathrm{pt}/P^{\mathrm{F}} \xrightarrow{t'} \mathrm{pt}/G^{\mathrm{F}}, \tag{3.29}$$

the parabolic induction functor is $i_{L\subset P} = t'_! s'^*$.

Lemma 3.3.18. *There is an isomorphism of functors* $\mathrm{D}_{\mathrm{cons}}(\frac{U_L\backslash L/U_L}{\mathrm{Ad}_{\mathrm{F}} T}, \Lambda) \to \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \Lambda)$,

$$\mathrm{HC}^G_{\mathrm{F}} i^{\mathrm{HC}}_{L\subset P} = i_{L\subset P} \mathrm{HC}^L_{\mathrm{F}}, \tag{3.30}$$

where $\mathrm{HC}^G_{\mathrm{F}}$ *and* $\mathrm{HC}^L_{\mathrm{F}}$ *denote the horocycle transform for* G *and* L *respectively.*

Proof. Consider the following commutative diagram of stacks.

$$\begin{tikzcd}
\frac{U_L\backslash L/U_L}{\mathrm{Ad}_{\mathrm{F}} T} & \frac{U\backslash P/U}{\mathrm{Ad}_{\mathrm{F}} T} \arrow[l, "s"'] \arrow[r, "t"] & \frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T} \\
\frac{L}{\mathrm{Ad}_{\mathrm{F}} B_L} \arrow[u, "r_L"] \arrow[d, "q_L"'] & \frac{P}{\mathrm{Ad}_{\mathrm{F}} B} \arrow[l, "s_1"'] \arrow[r, "t_1"] \arrow[u, "r_P"] \arrow[d, "q_P"'] & \frac{G}{\mathrm{Ad}_{\mathrm{F}} B} \arrow[u, "r"'] \arrow[d, "q"] \\
\mathrm{pt}/L^{\mathrm{F}} & \mathrm{pt}/P^{\mathrm{F}} \arrow[l, "s'"] \arrow[r, "t'"'] & \mathrm{pt}/G^{\mathrm{F}}
\end{tikzcd}$$

where the bottom and top lines are the maps defining the functors $i_{L\subset P}$ and $i^{\mathrm{HC}}_{L\subset P}$, the middle horizontal line is induced by the projections $P \to L$ and $B \to B_L$ and by the inclusion $P \to G$. The exterior vertical lines are the maps defining the functors $\mathrm{HC}^L_{\mathrm{F}}$ and $\mathrm{HC}^G_{\mathrm{F}}$. The map r_P is the quotient by U acting on the left and the map q_P is the map $\frac{P}{\mathrm{Ad}_{\mathrm{F}} B} \to \frac{P}{\mathrm{Ad}_{\mathrm{F}} P}$ induced by the inclusion $B \subset P$. It follows from these descriptions that the top right square is Cartesian and the bottom left one is Cartesian as well. The lemma is a proper base change exercise

$$\begin{aligned} i_{L\subset P} \mathrm{HC}^L_{\mathrm{F}} &= t'_! s'^* q_{L,!} r_L^* \\ &= t'_! q_{P,!} s_1^* r_L^* \\ &= q_! t_{1,!} r_P^* s^* \\ &= q_! r^* t_! s^* \\ &= \mathrm{HC}^G_{\mathrm{F}} i^{\mathrm{HC}}_{L\subset P}. \end{aligned}$$

□

We now pass to the second statement. Consider now the following tower of groups

$$M \subset Q \subset L \subset P \subset G, \tag{3.31}$$

where P is a parabolic of G with Levi L and Q is a parabolic of L with Levi M. Denote by U_P and U_Q the unipotent radicals of Q and P and by $Q' = QU_P \subset G$. This is a parabolic of G with Levi M. The following lemma is well known, see for instance [CE04] 7.1.9 for a different proof.

Lemma 3.3.19. *There is an isomorphism of functors* $\mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/M^{\mathrm{F}}, \Lambda) \to \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \Lambda)$ *a*

$$i_{L\subset P} i_{M\subset Q} = i_{M\subset Q'}. \tag{3.32}$$

Proof. Consider the following diagram.

$$\begin{tikzcd}
 & \frac{G}{\mathrm{Ad}_{\mathrm{F}} G} & & \\
\frac{G}{\mathrm{Ad}_{\mathrm{F}} Q'} \arrow[r] & \frac{G}{\mathrm{Ad}_{\mathrm{F}} P} \arrow[u] & & \\
\frac{U_P \backslash G / \mathrm{F}(U_P)}{\mathrm{Ad}_{\mathrm{F}} Q} \arrow[u] \arrow[d] & \frac{Q'\mathrm{F}(Q')}{\mathrm{Ad}_{\mathrm{F}} Q'} \arrow[ul] \arrow[r] \arrow[d] & \frac{P\mathrm{F}(P)}{\mathrm{Ad}_{\mathrm{F}} P} \arrow[ul] \arrow[d] & \\
\frac{U_{Q'} \backslash G / \mathrm{F}(U_{Q'})}{\mathrm{Ad}_{\mathrm{F}} M} & \frac{U_P \backslash (Q'\mathrm{F}(Q'))/\mathrm{F}(U_P)}{\mathrm{Ad}_{\mathrm{F}} Q} \arrow[ul] \arrow[r] \arrow[d] \arrow[dr] & \frac{U_P \backslash (P\mathrm{F}(P))/\mathrm{F}(U_P)}{\mathrm{Ad}_{\mathrm{F}} L} \arrow[dr] & \\
 & \frac{U_{Q'} \backslash (Q'\mathrm{F}(Q'))/\mathrm{F}(U_{Q'})}{\mathrm{Ad}_{\mathrm{F}} M} \arrow[ul] \arrow[dr] \arrow[ddr] & \frac{Q\mathrm{F}(Q)}{\mathrm{Ad}_{\mathrm{F}} Q} \arrow[r] \arrow[d] \arrow[dr] & \frac{L}{\mathrm{Ad}_{\mathrm{F}} L} \\
 & & \frac{U_Q \backslash (Q\mathrm{F}(Q))/\mathrm{F}(U_Q)}{\mathrm{Ad}_{\mathrm{F}} M} \arrow[d] \arrow[dr] & \frac{L}{\mathrm{Ad}_{\mathrm{F}} Q} \arrow[u] \arrow[d] \\
 & & \frac{M}{\mathrm{Ad}_{\mathrm{F}} M} & \frac{U_Q \backslash L / \mathrm{F}(U_Q)}{\mathrm{Ad}_{\mathrm{F}} M}
\end{tikzcd}$$

We will detail this diagram while doing the proof of the lemma. Consider the two functors $\mathrm{D}_{\mathrm{cons}}(\frac{M}{\mathrm{Ad}_{\mathrm{F}} M}, \Lambda) \to \mathrm{D}_{\mathrm{cons}}(\frac{G}{\mathrm{Ad}_{\mathrm{F}} G}, \Lambda)$ obtained by doing $*$-pullbacks and !-pushforwards along two exterior paths $\frac{M}{\mathrm{Ad}_{\mathrm{F}} M} \dashrightarrow \frac{G}{\mathrm{Ad}_{\mathrm{F}} G}$, doing so along the leftmost exterior path computes $i_{M \subset Q'}$ and doing so along the right exterior path computes the composition $i_{L \subset P} i_{M \subset Q}$. Therefore once we show that this diagram commutes and that enough squares are cartesian the statement will follow from the proper base change theorem.

Firstly, the square

$$\begin{tikzcd}
\frac{G}{\mathrm{Ad}_{\mathrm{F}} Q'} \arrow[r] & \frac{G}{\mathrm{Ad}_{\mathrm{F}} P} & \\
 & \frac{Q'\mathrm{F}(Q')}{\mathrm{Ad}_{\mathrm{F}} Q'} \arrow[ul] \arrow[r] & \frac{P\mathrm{F}(P)}{\mathrm{Ad}_{\mathrm{F}} P} \arrow[ul]
\end{tikzcd}$$

is induced by the inclusion of $Q' \subset P \subset G$ hence it is commutative.

The squares

$$\begin{tikzcd}
\frac{G}{\mathrm{Ad}_{\mathrm{F}} Q'} \arrow[d] & \\
\frac{U_P\backslash G/\mathrm{F}(U_P)}{\mathrm{Ad}_{\mathrm{F}} Q} \arrow[d] & \frac{Q'\mathrm{F}(Q')}{\mathrm{Ad}_{\mathrm{F}} Q'} \arrow[ul] \arrow[d] \\
\frac{U_{Q'}\backslash G/\mathrm{F}(U_{Q'})}{\mathrm{Ad}_{\mathrm{F}} M} & \frac{U_P\backslash(Q'\mathrm{F}(Q'))/\mathrm{F}(U_P)}{\mathrm{Ad}_{\mathrm{F}} Q} \arrow[ul] \arrow[d] \\
 & \frac{U_{Q'}\backslash(Q'\mathrm{F}(Q'))/\mathrm{F}(U_{Q'})}{\mathrm{Ad}_{\mathrm{F}} Q} \arrow[ul]
\end{tikzcd}$$

are horizontally induced by the inclusion $Q'\mathrm{F}(Q') \subset G$ and by the quotient by U_P and U_Q vertically hence these commute and are cartesian.

The square

$$\begin{tikzcd}
\frac{Q'\mathrm{F}(Q')}{\mathrm{Ad}_{\mathrm{F}} Q'} \arrow[r] \arrow[d] & \frac{P\mathrm{F}(P)}{\mathrm{Ad}_{\mathrm{F}} P} \arrow[d] \\
\frac{U_P\backslash(Q'\mathrm{F}(Q'))/\mathrm{F}(U_P)}{\mathrm{Ad}_{\mathrm{F}} Q} \arrow[r] & \frac{U_P\backslash(P\mathrm{F}(P))/\mathrm{F}(U_P)}{\mathrm{Ad}_{\mathrm{F}} L}
\end{tikzcd}$$

is horizontally induced by the inclusion $Q' \subset P$ and vertically taking quotients by U_P acting on by left translations. In particular, both vertical maps are U_P-torsors hence the square commutes and is cartesian.

The square

$$\begin{tikzcd}
\frac{U_P\backslash(Q'\mathrm{F}(Q'))/\mathrm{F}(U_P)}{\mathrm{Ad}_{\mathrm{F}} Q} \arrow[r] \arrow[dr] & \frac{U_P\backslash(P\mathrm{F}(P))/\mathrm{F}(U_P)}{\mathrm{Ad}_{\mathrm{F}} L} \arrow[dr] & \\
 & \frac{Q\mathrm{F}(Q)}{\mathrm{Ad}_{\mathrm{F}} Q} \arrow[r] & \frac{L}{\mathrm{Ad}_{\mathrm{F}} L}
\end{tikzcd}$$

is induced by the commutative diagram of groups

$$\begin{tikzcd}
P \arrow[r] & L \\
Q' \arrow[r] \arrow[u] & Q \arrow[u]
\end{tikzcd}$$

where both horizontal maps are the quotients by U_P. There is a canonical isomorphism of stack $\frac{U_P\backslash(Q'\mathrm{F}(Q'))/\mathrm{F}(U_P)}{\mathrm{Ad}_{\mathrm{F}} Q} = \frac{Q\mathrm{F}(Q)}{\mathrm{Ad}_{\mathrm{F}} Q \ltimes (U_P \cap \mathrm{F}^{-1}(U_P))}$ induced by the inclusion $Q \subset Q'$ and the left vertical map of the former square is isomorphic to the projection $\frac{Q\mathrm{F}(Q)}{\mathrm{Ad}_{\mathrm{F}} Q \rtimes (U_P \cap \mathrm{F}(U_P))} \to \frac{Q\mathrm{F}(Q)}{\mathrm{Ad}_{\mathrm{F}} Q}$, in particular the left vertical map is a trivial $U_P \cap \mathrm{F}^{-1}(U_P)$-gerbe. The same applies the the right vertical map and both trivializations of these gerbes are compatible. Indeed these trivializations are induced by the splittings of $P \to L$ and $Q' \to Q$ which are by hypothesis compatible, hence the square we are interested in is commutative and cartesian.

The square

$$\begin{tikzcd}
\frac{U_P\backslash(Q'\mathrm{F}(Q'))/\mathrm{F}(U_P)}{\mathrm{Ad}_{\mathrm{F}}Q} \arrow[d] \arrow[dr] & \\
\frac{U_{Q'}\backslash(Q'\mathrm{F}(Q'))/\mathrm{F}(U_{Q'})}{\mathrm{Ad}_{\mathrm{F}}M} \arrow[dr] & \frac{Q\mathrm{F}(Q)}{\mathrm{Ad}_{\mathrm{F}}Q} \arrow[d] \\
& \frac{U_Q\backslash(Q\mathrm{F}(Q))/\mathrm{F}(U_Q)}{\mathrm{Ad}_{\mathrm{F}}M}
\end{tikzcd}$$

is induced by the maps $Q' \to Q \to M$ and both vertical maps are U_Q-torsor hence is it commutative and cartesian.

The triangle

$$\begin{tikzcd}
\frac{U_{Q'}\backslash(Q'\mathrm{F}(Q'))/\mathrm{F}(U_{Q'})}{\mathrm{Ad}_{\mathrm{F}}Q} \arrow[dr] \arrow[ddr] & \\
& \frac{U_Q\backslash(Q\mathrm{F}(Q))/\mathrm{F}(U_Q)}{\mathrm{Ad}_{\mathrm{F}}M} \arrow[d] \\
& \frac{M}{\mathrm{Ad}_{\mathrm{F}}M}
\end{tikzcd}$$

is isomorphic to the triangle

$$\begin{tikzcd}
\frac{M}{\mathrm{Ad}_{\mathrm{F}}M \ltimes (U_{Q'} \cap \mathrm{F}^{-1}(U_{Q'}))} \arrow[dr] \arrow[ddr] & \\
& \frac{M}{\mathrm{Ad}_{\mathrm{F}}M \ltimes (U_Q \cap \mathrm{F}^{-1}(U_Q))} \arrow[d] \\
& \frac{M}{\mathrm{Ad}_{\mathrm{F}}M}
\end{tikzcd}$$

and the maps are induced by the quotient $U_{Q'} \to U_Q$, hence the triangle is commutative.

The remaining right hand side of the diagram is clear commutative and the remaining square is cartesian. The lemma now follows from iterated applications of the proper base change theorem and the commutativity of the diagram. $\square$

3.4 Integral Soergel theory

In this section we set up an integral version of the results of [BR22b] and [Gou21]. Most of our arguments will reduce to their statements.

3.4.1 Setting up the geometry

Consider the stack $U\backslash G/U$. There are three actions of tori that we can consider :

(*i*). the action of T induced by left translations $U\backslash G/U$,

(*ii*). the action of T on $U\backslash G/U$ induced by the action of T on G given by $t.x = xt$, we will refer to this action as the right action of T,

(*iii*). the action of $T \times T$ induced by left and right translations.

The quotients by these actions are the stacks $B\backslash G/U, U\backslash G/B$ and $B\backslash G/B$ respectively. There are therefore three version of the free monodromic categories that we can attach to this stack, we will soon see that they are isomorphic. We refer to Appendix 3.A for the definition of the equivariant categories. Denote by

(*i*). $\mathbb{H}^{\mathrm{left}} = \bigoplus_{\chi \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{indcons}}(U\backslash G/U, \Omega_T)^{\mathrm{left}}_{\chi}$ where the equivariance is relative to the action of T on the left.

(*ii*). $\mathbb{H}^{\mathrm{right}} = \bigoplus_{\chi \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{indcons}}(U\backslash G/U, \Omega_T)^{\mathrm{right}}_{\chi}$ where the equivariance is relative to the action of T on the right.

(*iii*). $\mathbb{H}^{\mathrm{left,right}} = \bigoplus_{\chi,\chi' \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{indcons}}(U\backslash G/U, \Omega_T)^{\mathrm{left,right}}_{\chi,\chi'}$ where the equivariance is relative to the action of $T \times T$ on the right and the index (χ, χ') refer to sheaves that are equivariant for $L_{T\times T} \otimes_{\overline{\mathbb{Z}}_\ell} (\mathcal{L}_\chi \boxtimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_{\chi'})$.

We equip the space $U\backslash G/U$ with its Bruhat stratification. The strata are indexed by the Weyl group W, and the stratum corresponding to $w \in W$ is $U\backslash BwB/U$. We denote by $i_w : U\backslash BwB/U \to U\backslash G/U$. We first need a lemma to deal with the case of a torus.

Lemma 3.4.1. *The category* $\mathrm{D}_{\mathrm{indcons}}(T, \Omega_{T\times T})^{\mathrm{left,right}}_{\chi,\chi'}$ *is zero unless* $\chi' = \chi$. *In this case, this category is equivalent to* $\mathrm{D}(\Omega_T)$.

Proof. Consider the multiplication map $m : T \times T \to T$. Consider the monad $m^! m_!$ acting on $\mathrm{D}_{\mathrm{indcons}}(T \times T, \Omega_{T\times T})_{\chi,\chi'}$. Since the map m is surjective, $m^!$ is conservative as it is continuous, we can apply the Barr-Beck-Lurie theorem [Lur] 4.7.0.3 and identify $\mathrm{D}_{\mathrm{indcons}}(T, \Omega_T)^{\mathrm{left,right}}_{\chi,\chi'}$ with the category of $m^! m_!$-algebras in $\mathrm{D}_{\mathrm{indcons}}(T \times T, \Omega_{T\times T})_{\chi,\chi'} = \mathrm{D}(\Omega_{T\times T})$. The sheaf $m_!(L_{T\times T} \otimes_{\overline{\mathbb{Z}}_\ell} (\mathcal{L}_\chi \boxtimes \mathcal{L}_{\chi'})$ is 0 if $\chi \neq \chi'$ and isomorphic to $L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi[2\dim T]$ by lemma 3.2.22 if $\chi = \chi'$. Hence if $\chi = \chi'$, we have $1^* m^! m_!(L_{T\times T} \otimes_{\overline{\mathbb{Z}}_\ell} (\mathcal{L}_\chi \boxtimes \mathcal{L}_{\chi'})) = \Omega_T$. The algebra $\Omega_T \in \mathrm{D}(\Omega_{T\times T})$ is the quotient of $\Omega_{T\times T} \to \Omega_T$ induced by the map m. Hence $\mathrm{D}_{\mathrm{indcons}}(T, \Omega_T)^{\mathrm{left,right}}_{\chi,\chi'}$ is equivalent to $\Omega_T - \mathrm{Mod}(\mathrm{D}(\Omega_{T\times T})) = \mathrm{D}(\Omega_T)$. □

Lemma 3.4.2. *All three categories* $\mathbb{H}^{\mathrm{left}}, \mathbb{H}^{\mathrm{right}}$ *and* $\mathbb{H}^{\mathrm{left,right}}$ *are compactly generated. The categories of compact objects are the categories*

(*i*). $\bigoplus_{\chi \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_T)^{\mathrm{left}}_{\chi}$,

(*ii*). $\bigoplus_{\chi \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_T)^{\mathrm{right}}_{\chi}$,

(*iii*). *and* $\bigoplus_{\chi,\chi' \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_T)^{\mathrm{left,right}}_{\chi,\chi'}$,

respectively.

Proof. First since the inclusion i_w are quasi-compact and schematic all functors $i_{w,!}, i_w^!, i_{w,*}, i_w^*$ between the categories $\mathbb{H}^?$ and $\bigoplus_{\chi\in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{indcons}}(U\backslash BwB/U, \Omega_T)^?_\chi$ where $? \in \{\text{left}, \text{right}, (\text{left}, \text{right})\}$ commute with arbitrary direct sums. For $i_{w,!}$ and i_w^* this is clear since they are left adjoints. We show it for $i_{w,*}$, the case $i_w^!$ can be deduced from the case of $i_{w,*}$ using excision triangles. We only need to check that the canonical map $\oplus_i i_{w,*}A_i \to i_{w,*} \oplus_i A_i$ for $A_i \in \mathbb{H}^?$ is an isomorphism, since the functor $i_{w,*}$ commutes with the forgetful functor and smooth pullbacks this can checked after pulling back to G/U, where this now follows from the statement on schemes by [BS15] 6.4.5.

Since $i_w^!$ and $i_{w,*}$ are continuous, their left adjoints preserve compact objects. We now show the lemma by induction on the strata. Denote by $V \xrightarrow{j} U\backslash G/U \xleftarrow{i} Z$ the inclusion of the open stratum U and Z its closed complement. Using the exicision triangles for $A \in \mathbb{H}^?$,

$$j_! j^* A \to A \to i_* i^* A, \tag{3.33}$$

we see that A is a colimit of compact objects if and only if j^*A and i^*A are so. By induction this reduces to showing that $\bigoplus_{\chi\in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{indcons}}(U\backslash BwB/U, \Omega_T)^?_\chi$ is compactly generated. But this category is equivalent to the category $\bigoplus_\chi \mathrm{D}(\Omega_T)$. This is clear for $? \in \{\text{left}, \text{right}\}$ and by lemma 3.4.1 for the case of the action of $T\times T$. This proves the compact generation statement.

We now identify the compact objects. Again by induction on the strata, and using the same triangle, we see that an object A is compact if and only if for all w, i_w^*A is compact. Hence the category of compact objects is the stable category generated by all objects of the form $i_{w,!}A$ for varying w and $A \in \bigoplus_{\chi\in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{indcons}}(U\backslash BwB/U, \Omega_T)^?_\chi$ a compact object. On $U\backslash BwB/U$ the category of compact objects is $\bigoplus_{\chi\in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{cons}}(U\backslash BwB/U, \Omega_T)^?_\chi$. And the category generated by all $i_{w,!}A$ for varying w and A compact is then $\bigoplus_\chi \mathrm{D}_{\mathrm{cons}}(U\backslash BwB/U, \Omega_T)^?_\chi$. □

The inclusions of $T \xrightarrow{i_{\mathrm{left}}} T\times T \xleftarrow{i_{\mathrm{right}}} T$ given by $i_{\mathrm{left}}(t) = (t,1)$ and $i_{\mathrm{right}}(t) = (1,t)$ induce inclusions $\Omega_T \xrightarrow{i_{\mathrm{left},*}} \Omega_{T\times T} \xleftarrow{i_{\mathrm{right},*}} \Omega_T$.

Lemma 3.4.3. *There are well defined functors*

$$\mathbb{H}^{\mathrm{left}} \xleftarrow{\mathrm{For}^{\mathrm{left}}} \mathbb{H}^{\mathrm{left,right}} \xrightarrow{\mathrm{For}^{\mathrm{right}}} \mathbb{H}^{\mathrm{right}} \tag{3.34}$$

induced by forgetting the $(T\times T, L_{T\times T} \otimes_{\overline{\mathbb{Z}}_\ell} (\mathcal{L}_\chi \boxtimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_{\chi'}))$-equivariance along $i_{\mathrm{left},*}$ and $i_{\mathrm{right},*}$ respectively.

Proof. To check that these functors are well defined we have to check that the functors $\mathrm{For}^{\mathrm{left}}$ and $\mathrm{For}^{\mathrm{right}}$ preserve constructibility. Let $A \in \mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_{T\times T})^{\mathrm{left,right}}_{\chi,\chi'}$. As in the previous lemma, we can assume $A = i_{w,!}A_0$ for some object

$$A_0 \in \bigoplus_{\chi,\chi'\in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{cons}}(U\backslash BwB/U, \Omega_{T\times T})_{\chi,\chi'}.$$

We can further assume that $A_0 \in \mathrm{D}_{\mathrm{cons}}(U\backslash BwB/U, \Omega_{T\times T})_{\chi,\chi'}$ and that $w\chi' = \chi$ otherwise this category is zero. By lemma 3.4.1, the category $\mathrm{D}_{\mathrm{cons}}(U\backslash BwB/U, \Omega_{T\times T})_{\chi,\chi'}$ is equivalent to $\mathrm{D}_{\mathrm{coh}}(\Omega_T)$. We can assume that A_0 corresponds to Ω_T as this objects generates the stable category $\mathrm{D}_{\mathrm{coh}}(\Omega_T)$. Therefore $A_0 \simeq \nu_w^*((L_{T\times T} \otimes_{\Omega_{T\times T}} \Omega_T) \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi)[\ell(w) + \dim T]$ as an $\Omega_{T\times T}$-sheaf. Since $L_{T\times T} \otimes_{\Omega_{T\times T}} \Omega_T \simeq L_T$ as an Ω_T-sheaf after forgetting along either the left of right inclusion, we get the desired constructibility statement. □

Lemma 3.4.4. *Both functors* $\mathrm{For}^{\mathrm{left}}$ *and* $\mathrm{For}^{\mathrm{right}}$ *are equivalences.*

Proof. We first show that $\mathrm{For}^{\mathrm{left}}$ and $\mathrm{For}^{\mathrm{right}}$ induce equivalence on each strata.

$$\bigoplus_{\chi,\chi' \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{cons}}(U\backslash BwB/U, \Omega_{T\times T})_{\chi,\chi'} \simeq \bigoplus_{\chi \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{cons}}(U\backslash BwB/U, \Omega_T)^{\mathrm{left}}_{\chi}, \tag{3.35}$$

and

$$\bigoplus_{\chi,\chi' \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{cons}}(U\backslash BwB/U, \Omega_{T\times T})_{\chi,\chi'} \simeq \bigoplus_{\chi' \in \mathrm{CH}(T)} \mathrm{D}_{\mathrm{cons}}(U\backslash BwB/U, \Omega_T)^{\mathrm{right}}_{\chi'}. \tag{3.36}$$

We do it for the for the first one. Note that we have an equivalence of categories $\mathrm{D}_{\mathrm{cons}}(U\backslash BwB/U, \Omega_{T\times T})_{\chi,\chi'} = \mathrm{D}_{\mathrm{cons}}(wT, \Omega_{T\times T})_{\chi,\chi'}$ where $wT \subset N(T)$ is the closed subscheme of $N(T)$, the normalizer of T in G, over the element $w \in W$. Let $\dot{w} \in wT$, this choice determines a map $m_w : T \times T \times T, (t,h) \mapsto \dot{w}(\mathrm{Ad}(w)(t)h)$ where $\mathrm{Ad}(w)$ denotes the adjoint action of W on T. Arguing as in lemma 3.4.1 with the map m_w instead of m, we get that $\mathrm{D}_{\mathrm{cons}}(U\backslash BwB/U, \Omega_{T\times T})_{\chi,\chi'} = 0$ if $w\chi' \neq \chi$ and if $\chi = w\chi'$, then $\mathrm{D}_{\mathrm{cons}}(U\backslash BwB/U, \Omega_{T\times T})_{\chi,\chi'} = \Omega_T - \mathrm{Mod}(\mathrm{D}_{\mathrm{cons}}(\Omega_{T\times T})$ where Ω_T is the $\Omega_{T\times T}$-algebra induced by the map m_w. In particular, if $w\chi' = \chi$ then $\mathrm{D}_{\mathrm{cons}}(U\backslash BwB/U, \Omega_{T\times T})_{\chi,\chi'} \simeq \mathrm{D}_{\mathrm{coh}}(\Omega_T)$. Moreover we have $\mathrm{D}_{\mathrm{cons}}(U\backslash BwB/U, \Omega_T)^{\mathrm{left}}_{\chi} \simeq \mathrm{D}_{\mathrm{coh}}(\Omega_T)$ and the left forgetful functor therefore induces a functor $\mathrm{D}_{\mathrm{coh}}(\Omega_T) \to \mathrm{D}_{\mathrm{coh}}(\Omega_T)$ which sends Ω_T to itself and is therefore an equivalence. To conclude that the functor $\mathrm{For}^{\mathrm{left}}$ is an equivalence, we proceed by induction on the strata. Let $V \subset U\backslash G/U$ be a stratum and let $Z = \overline{V} - V$ be the closed complementary of the closure of V. Denote by i and j the inclusions $Z \subset \overline{V}$ and $V \subset \overline{V}$ respectively. Assume by induction that $\mathrm{For}^{\mathrm{left}}$ induces an equivalence on the full subcategory of $\mathbb{H}^{\mathrm{left}}$ and $\mathbb{H}^{\mathrm{left,right}}$ supported on Z. Let $A, B \in \mathbb{H}^{\mathrm{left,right}}$ be supported on $\overline{V}$. Using excision triangles, we can assume that $A = i_*A_0$ or $A = j_!A_0$ and that $B = j_!B_0$ or $B = i_*B_0$. We now have

(i). if $A = i_*A_0$ and $B = i_*B_0$, then $\mathrm{Hom}(A,B) = \mathrm{Hom}(\mathrm{For}^{\mathrm{left}}(A), \mathrm{For}^{\mathrm{left}}(B))$ by induction,

(ii). if $A = j_!A_0$ and $B = j_!B_0$, then $\mathrm{Hom}(A,B) = \mathrm{Hom}(\mathrm{For}^{\mathrm{left}}(A), \mathrm{For}^{\mathrm{left}}(B))$ using the stratum case,

(iii). if $A = j_!A_0$ and $B = i_*B_0$, then $\mathrm{Hom}(A,B) = 0$ and $\mathrm{Hom}(\mathrm{For}^{\mathrm{left}}(A), \mathrm{For}^{\mathrm{right}}(B)) = 0$,

(iv). finally if $A = i_*A_0$ and $B = j_!B_0$, then as the forgetful functor commutes with $i^!$ and $j_!$, we have

$$\begin{aligned}\mathrm{Hom}(A,B) = \mathrm{Hom}(A_0, i^! j_! B_0) &= \mathrm{Hom}(\mathrm{For}^{\mathrm{left}} A_0, \mathrm{For}^{\mathrm{left}} i^! j_! B_0) \\ &= \mathrm{Hom}(\mathrm{For}^{\mathrm{left}} A_0, i^! j_! \mathrm{For}^{\mathrm{left}} B_0) \\ &= \mathrm{Hom}(\mathrm{For}^{\mathrm{left}} A, \mathrm{For}^{\mathrm{left}} B).\end{aligned}$$

This establishes that $\mathrm{For}^{\mathrm{left}}$ is fully faithful, as the subcategories of $\mathbb{H}^{\mathrm{left,right}}$ and $\mathbb{H}^{\mathrm{left}}$ of sheaves supported on $\overline{V}$ are generated by the sheaves of the form i_*A_0 and $j_!A_0$, we get the essential surjectivity.

∎

Remark 3.4.5. Note that the functor $\mathrm{For}^{\mathrm{right},-1}\mathrm{For}^{\mathrm{left}}$ is an equivalence that is *not* Ω_T-linear.

From now on we denote by $\mathbb{H}$ either of the categories $\mathbb{H}^{\mathrm{left}}, \mathbb{H}^{\mathrm{right}}$ or $\mathbb{H}^{\mathrm{left,right}}$ which are identified through $\mathrm{For}^{\mathrm{left}}$ and $\mathrm{For}^{\mathrm{right}}$. This category is equipped with its perverse t-structure. For most constructions, we will work with $\mathbb{H}^{\mathrm{right}}$. We denote by $\mathbb{H}^{\omega}$ the full subcategory of compact objects and by $\mathbb{H}^{\omega}_{\chi}$ the category $\mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_T)_{\chi}$.

Remark 3.4.6. Since $\mathrm{For}^{\mathrm{right}}$ is an equivalence, the category $\mathbb{H}$ is equipped with an $\Omega_{T\times T}$-linear structure.

We will need to apply some classical results of Soergel theory, which are recalled in Appendix 3.B. Consider the schemes

$$G/B = \sqcup_{w\in W} BwB/B \text{ and } G/U = \sqcup_{w\in W} BwB/U.$$

equipped with their Bruhat stratification. The map $G/U \to G/B$ is a T-torsor. The pullback along $G/B \to U\backslash G/B$ yields an equivalence $\mathrm{D}_{\mathrm{cons}}(U\backslash G/B, \overline{\mathbb{Z}}_\ell) \simeq \mathrm{D}'_{\mathrm{cons}}(G/B, \overline{\mathbb{Z}}_\ell)$ where the category is defined in Appendix 3.B. Recall that we have chosen a compatible system $(\dot{w})$ of lifts of the elements of the Weyl group W. Each $\dot{w}$ induces a T-equivariant splitting $BwB = U \times T \times U_w$. For $w \in W$, recall that we denote by $\nu_w : U\backslash BwB/U \to T$ the induced projection onto T. Furthermore, it is known that the schemes BwB/B are affine spaces and that the inclusions $BwB/B \to G/B$ are affine hence we can apply the results of Appendix 3.B.

We thus have

- (i). the standard and costandard sheaves $\Delta_{w,\chi} = i_{w,!}\nu_w^*(L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi)[\dim T + \ell(w)]$ and $\nabla_{w,\chi} = i_{w,*}\nu_w^*(L_T \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{L}_\chi)[\dim T + \ell(w)]$. These sheaves are perverse sheaves on $U\backslash G/U$.
- (ii). For all pairs (w, χ), there exists an indecomposable perverse tilting sheaf $T_{w,\chi}$ by theorem 3.B.12. This is a sheaf that is supported on the closure of $U\backslash BwB/U$ and which admits a Δ and a ∇-filtration in $\mathrm{Perv}(U\backslash G/U, \Omega_T)$. We refer to Appendix 3.B for the definitions.

Since we used the torsor $G/U \to G/B$ for the construction of the tilting sheaves all these sheaves are *a priori* considered as (twisted) equivariant sheaves for the right action of T. By lemma 3.4.4, they also carry an equivariant structure with respect to the left action of T. Note that, while the categories $\mathbb{H}^{\mathrm{right}}$ and $\mathbb{H}^{\mathrm{left,right}}$ are equivalent, we cannot apply the construction of Appendix 3.B to the $T \times T$-torsor $U\backslash G/U \to B\backslash G/B$ as the strata of the target are the stacks $B\backslash BwB/B$ which are not cohomologically contractible.

Definition 3.4.7. We define $\mathrm{Tilt}(U\backslash G/U) \subset \mathbb{H}^{\omega,\heartsuit}$ the full subcategory of tilting objects. We will also denote by $\mathrm{Tilt}(U\backslash G/U)_\chi$ the corresponding full subcategory of $\mathbb{H}^{\omega}_{\chi}$.

Definition 3.4.8 (Weyl groups)**.** Let $\chi, \chi' \in \mathrm{CH}(T)$ be two characters, we set

- (i). $W_\chi = \mathrm{Stab}_W(\chi)$,
- (ii). W_χ° the subgroup of W_χ generated by all s_α such that $\alpha^{\vee,*}\mathcal{L}_\chi$ is trivial,
- (iii). ${}_{\chi}W_{\chi'} = \{w \in W, \chi = w\chi'\}$.

Definition 3.4.9. Let $\chi, \chi' \in \mathrm{CH}(T)$ we denote by $\mathbb{H}^{\omega}_{[\chi',\chi]}$ the full subcategory of $\mathbb{H}^{\omega}_{\chi}$ generated by the $\Delta_{w,\chi}$ such that $w\chi = \chi'$.

Lemma 3.4.10. *Let $\chi \in \mathrm{CH}(T)$ and $w, w' \in W$ we have $\mathrm{Hom}_{\mathbb{H}^{\omega}_{\chi}}(\Delta_{w,\chi}, \Delta_{w',\chi}) = 0$ if $w\chi \neq w'\chi$.*

Proof. We have

$$\begin{aligned}\mathrm{Hom}_{\mathbb{H}^\omega_\chi}(\Delta_{w,\chi},\Delta_{w',\chi}) &= \mathrm{Hom}(i^!_{w'}\Delta_{w,\chi},\nu^*_w(L_T\otimes\mathcal{L}_\chi[\dim T+\ell(w')])) \\ &= \mathrm{Hom}_{\mathrm{D}_{\mathrm{cons}}(T,\Omega_T)_\chi}(\nu_{w',!}i^!_{w'}\Delta_{w,\chi},(L_T\otimes\mathcal{L}_\chi[\dim T+\ell(w')-2N])).\end{aligned}$$

where N is the relative dimension of the smooth morphism ν_w, hence we have $\nu^*_w=\nu^!_w[-2N]$. The sheaf $\nu_{w',!}i^!_{w'}\Delta_{w,\chi}$ is by construction $(T,L_T\otimes\mathcal{L}_\chi)$-equivariant on the right and $(T,(L_T\otimes\mathcal{L}_{w'^{-1}w\chi})$-equivariant on the left. As $\mathrm{Hom}(\mathcal{L}_{w'^{-1}w\chi},\mathcal{L}_\chi)=0$ if $w'^{-1}w\chi\neq\chi$, we get the desired vanishing. □

Lemma 3.4.11. *We have*

$$\mathbb{H}^\omega_\chi=\bigoplus_{\chi'}\mathbb{H}^\omega_{[\chi',\chi]}.$$

Proof. First note that $\mathbb{H}^\omega_{[\chi',\chi]}$ is zero if χ' is not in the W-orbit of χ. The result then follows from Lemma 3.4.10 □

3.4.2 Blocks

For this whole section we refer to [LY20] Section 4. We equip the group W with the Bruhat order, this is the order induced by the closure relations of the Bruhat stratification, ie $w\leq w'$ if and only if $BwB\subset\overline{Bw'B}$.

Definition 3.4.12. We call the elements of

$${}_\chi\underline{W}_{\chi'}={}_\chi W_{\chi'}/W^\circ_{\chi'}=W^\circ_{\chi'}\backslash{}_\chi W_{\chi'}$$

the blocks for $[\chi,\chi']$. Given a block $\beta\in{}_\chi\underline{W}_{\chi'}$, we denote by $\mathbb{H}^{\omega,\beta}_{[\chi',\chi]}$ the full subcategory of $\mathbb{H}^\omega_{[\chi',\chi]}$ generated by the objects $\Delta_{w,\chi}$ for $w\in\beta$.

Definition 3.4.13. Let $\alpha\in{}_\chi\underline{W}_{\chi'},\beta\in{}_{\chi'}\underline{W}_{\chi''}$ and $w\in\alpha,v\in\beta$ then wv lie in a block called $\alpha\beta\in{}_\chi\underline{W}_{\chi''}$. The formation $(\alpha,\beta)\mapsto\alpha\beta$ does not depend on the chosen representative and is associative.

Definition 3.4.14. Let $\chi\in\mathrm{CH}(T)$ and denote by

(i). $\Phi^\vee_\chi=\{\alpha^\vee\in\Phi^\vee,(\alpha^\vee)^*\mathcal{L}_\chi\simeq(\overline{\mathbb{Z}}_\ell)_T\}$,

(ii). $\Phi_\chi=\{\alpha\in\Phi,\alpha^\vee\in\Phi^\vee_\chi\}$,

(iii). $\Phi^{\vee,+}_\chi=\Phi^\vee_\chi\cap\Phi^{\vee,+}$.

(iv). Denote by $S_\chi=\{s_\alpha\in W^\circ_\chi$, such that $\alpha^\vee$ is indecomposable in $\Phi^\vee_{\chi,+}\}$.

It is known that (W°_χ,S_χ) is a Coxeter group and that $(\Phi_\chi,\Phi^\vee_\chi)$ is a subroot system of $(\Phi,\Phi^\vee)$ with Coxeter group (W°_χ,S_χ).

Definition 3.4.15. Let $\beta\in\chi'\underline{W}\chi$ be a block, and consider it as a poset with order induced by W, then there is a unique maximal element and a unique minimal element $w^{\max}_\beta$ and $w^{\min}_\beta$.

Lemma 3.4.16 ([LY20] 4.2, 4.3)**.** *Let $\alpha\in{}_\chi\underline{W}_{\chi'}$ and $\beta\in{}_{\chi'}\underline{W}_{\chi''}$ then the following holds*

(i). $w^{\min}_\alpha w^{\min}_\beta=w^{\min}_{\alpha\beta}$,

(ii). $w^{\max}_\alpha w^{\max}_\beta=w^{\max}_{\alpha\beta}$.

3.4.3 Tilting objects

Let $\chi \in \mathrm{CH}(T)$ and consider the ring $\Omega_T \otimes_{\Omega_T^{W_\chi^\circ}} \Omega_T$, where $\Omega_T^{W_\chi^\circ} \subset \Omega_T$ is the ring of W_χ°-invariants. This ring is naturally a quotient of $\Omega_T \otimes_{\overline{\mathbb{Z}}_\ell} \Omega_T$.

Lemma 3.4.17 ([BR22b] 8.4)**.** *The map $\Omega_T \otimes_{\overline{\mathbb{Z}}_\ell} \Omega_T \to \Omega_T \otimes_{\Omega_T^{W_\chi^\circ}} \Omega_T$ factors through $\Omega_{T\times T}$ making the following diagram commutative*

$$\begin{tikzcd} \Omega_T \otimes_{\overline{\mathbb{Z}}_\ell} \Omega_T \arrow[r] \arrow[dr] & \Omega_{T\times T} \arrow[d] \\ & \Omega_T \otimes_{\Omega_T^{W_\chi^\circ}} \Omega_T. \end{tikzcd}$$

Proof. The proof of [BR22b] is done for the mod ℓ variant but the same argument applies. □

Lemma 3.4.18. *We have $\mathrm{End}_{\mathbb{H}^\omega}(\Delta_{w,\chi}) = \Omega_T$ and the $\Omega_{T\times T}$-module structure is given by the map $\Omega_{T\times T} \to \Omega_T$ induced from $T \times T \mapsto T, (t,h) \to w(t)h$. One can replace $\Delta_{w,\chi}$ by $\nabla_{w,\chi}$. And for all $A \in \mathrm{Perv}(U\backslash G/U, \Omega_T)_\chi$ the $\Omega_{T\times T}$-structure factors through $\Omega_T \otimes_{\Omega_T^{W_\chi^\circ}} \Omega_T$.*

Proof. Since $i_w^! \Delta_{w,\chi} = \nu^*(L_T \otimes \mathcal{L}_\chi)[\dim T + \ell(w)]$ as $U\backslash BwB/U$ is open in the support of $\Delta_{w,\chi}$, we have $\mathrm{End}_{\mathbb{H}^\omega}(\Delta_{w,\chi}) = \mathrm{End}(\nu^*(L_T \otimes \mathcal{L}_\chi)[\dim T + \ell(w)]) \simeq \Omega_T$. The statement about the factorization follows from the proof of lemma 3.4.4. The last statement follows from the fact that the sheaves $\Delta_{w,\chi}$ generate $\mathbb{H}^\omega_\chi$. □

Lemma 3.4.19. *Let $\chi \in W$ and $w \neq w' \in W$. Then we have*

$$\mathrm{Hom}_{\mathrm{Perv}(U\backslash G/U,\Omega_T)_\chi}(\Delta_{w,\chi}, \Delta_{w',\chi}) = 0.$$

Proof. The proof of [BR22b] 6.2 extends verbatim after replacing $R_T^\vee \otimes R_T^\vee$ by $\Omega_{T\times T}$ and inputing lemma 3.4.18. □

As in [BR22b] 6.3, we define the graded functor on tilting sheaves. We fix a total ordering on W extending the Bruhat order. For $w \in W$ denote by $(U\backslash G/U)_{<w}$ the union of all Bruhat strata corresponding to $w' < w$ and $j_{<w} : (U\backslash G/U)_{<w} \subset G$. For a tilting sheaf T denote by $T_{\geq w}$ the kernel of the adjunction map $T \to j_{<w,*}j_{<w}^* T$. The set of subobjects $(T_{\geq w})_w$ forms an exhaustive decreasing filtration of T, the corresponding graded parts $\mathrm{gr}_w(T) = T_{\geq w}/T_{>w}$ is a direct sum of copies of $\Delta_{w,\chi}$.

Lemma 3.4.20. *The filtration $(T_{\geq w})_w$ is functorial and the associated graded functor*

$$\bigoplus_w \mathrm{gr}_w = \mathrm{gr} : \mathrm{Tilt}(U\backslash G/U)_\chi \to \mathrm{Perv}(U\backslash G/U, \Omega_T)_\chi$$

is faithful.

Proof. The functoriality of the filtration follows from the fact that $\mathrm{Hom}(\Delta_{w,\chi}, \Delta_{w',\chi}) = 0$ if $w > w'$. The faithfulness follows from lemma 3.4.19. □

Lemma 3.4.21. *Let $T \in \mathrm{Tilt}(U\backslash G/U)$ then there is canonical isomorphism $\mathrm{gr}_w(T) = i_{w,!}i_w^* T$.*

Proof. Denote by $j_{\geq w} : (U\backslash G/U)_{\geq w} \to U\backslash G/U$ and $j_{>w} : (U\backslash G/U)_{>w} \to U\backslash G/U$ the inclusions of all strata indexed by $y \geq w$ and $y > w$ respectively. Both of these maps are open immersions. We then have an excision triangle,

$$j_{\geq w,!}j^!_{\geq w}T \to T \to j_{<w,*}j^*_{<w}T. \tag{3.37}$$

Since the adjunction map $T \to j_{<w,*}j^*_{<w}T$ is surjective, this triangle is a short exact sequence of perverse sheaves and $T_{\geq w} = j_{\geq w,!}j^!_{\geq w}T$. Let $i_{>w} : (U\backslash G/U)_{>w} \to (U\backslash G/U)_{\geq w}$ denote the inclusion and $i'_w : U\backslash BwB/U \to (U\backslash G/U)_{\geq w}$ the inclusion of the stratum w. We then have an excision triangle in $(U\backslash G/U)_{\geq w}$

$$i_{>w,!}i^!_{>w}j^!_{\geq w}T \to j^!_{\geq w}T \to i'_{w,*}i'^*_w j^!_{\geq w}T. \tag{3.38}$$

We apply $j_{\geq,!}$ to this triangle and using the fact that $i_{>w}$ is an open immersion and that i'_w is a closed immersion we get a triangle

$$j_{>w,!}j^!_{>w}T \to j_{\geq w,!}j^!_{\geq w}T \to i_{w,!}i^*_wT. \tag{3.39}$$

By definition the first map is an injection of perverse sheaves and as T is tilting i^*_w is perverse hence $i_{w,!}i^*_wT$ is perverse and this triangle is a short exact sequence of perverse sheaves. □

3.4.4 Convolution structure

We define the convolution structure on $\mathbb{H}^\omega$. We do it in several steps. We will use the model of $\mathbb{H}$ given by $\mathbb{H}^{\text{left,right}}$.

First let X be a stack. We define a functor

$$\begin{aligned} \hat{\otimes}_{\overline{\mathbb{Z}}_\ell} : \mathrm{D}_{\text{cons}}(X, \Omega_T) \times \mathrm{D}_{\text{cons}}(X, \Omega_T) &\to \mathrm{D}_{\text{cons}}(X, \Omega_{T\times T}) \\ (A, B) &\mapsto (A\hat{\otimes}_{\overline{\mathbb{Z}}_\ell}B). \end{aligned}$$

For $A_0, B_0 \in \mathrm{D}_{\text{cons}}(X, \Omega_{T,\mathbb{Z}_\ell})$ we first construct a sheaf $(A_0\hat{\otimes}_{\mathbb{Z}_\ell}B_0) \in \mathrm{D}_{\text{cons}}(X, \Omega_{T\times T,\mathbb{Z}_\ell})$. First consider $A_0 \otimes_{\mathbb{Z}_\ell} B_0$, this is naturally an $\Omega_{T,\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \Omega_{T,\mathbb{Z}_\ell}$-sheaf on X_{proet}. Let $\mathfrak{m}_{T\times T}$ be the ideal of $\Omega_{T,\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \Omega_{T,\mathbb{Z}_\ell}$ given by $\Omega_{T,\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathfrak{m} + \mathfrak{m} \otimes_{\mathbb{Z}_\ell} \Omega_{T,\mathbb{Z}_\ell}$. The ring $\Omega_{T\times T}$ is then the completion of $\Omega_{T,\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \Omega_{T,\mathbb{Z}_\ell}$ along I. We then denote by $(A_0\hat{\otimes}_{\mathbb{Z}_\ell}B_0)$ the derived completion of $A_0 \otimes_{\mathbb{Z}_\ell} B_0$ along the ideal I in $\mathrm{D}(X_{\text{proet}}, \Omega_{T,\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \Omega_{T,\mathbb{Z}_\ell})$ in the sense of [BS15] Section 3.5. This derived completion is the functor $\mathrm{D}(X_{\text{proet}}, \Omega_{T,\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \Omega_{T,\mathbb{Z}_\ell}) \to \mathrm{D}(X_{\text{proet}}, \Omega_{T\times T,\mathbb{Z}_\ell})$ given by

$$C_0 \mapsto \lim_n (C_0 \otimes_{(\Omega_{T,\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \Omega_{T,\mathbb{Z}_\ell})} (\Omega_{T,\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \Omega_{T,\mathbb{Z}_\ell})/I^n). \tag{3.40}$$

Lemma 3.4.22. *For $A_0, B_0 \in \mathrm{D}_{\text{cons}}(X, \Omega_{T,\mathbb{Z}_\ell})$ the $\Omega_{T\times T,\mathbb{Z}_\ell}$-sheaf $(A_0\hat{\otimes}_{\mathbb{Z}_\ell}B_0)$ is constructible.*

Proof. Recall that an $\Omega_{T\times T,\mathbb{Z}_\ell}$-complete sheaf A_0 is constructible if $A_0 \otimes_{\Omega_{T\times T,\mathbb{Z}_\ell}} \Omega_{T\times T,\mathbb{Z}_\ell}/\mathfrak{m}_{T\times T}$ is constructible, where $\mathfrak{m}_{T\times T}$ is the maximal ideal of $\Omega_{T\times T,\mathbb{Z}_\ell}$. But we have

$$(A_0\hat{\otimes}_{\mathbb{Z}_\ell}B_0) \otimes_{\Omega_{T\times T,\mathbb{Z}_\ell}} \Omega_{T\times T,\mathbb{Z}_\ell}/\mathfrak{m}_{T\times T} = (A_0 \otimes_{\Omega_{T,\mathbb{Z}_\ell}} \Omega_{T,\mathbb{Z}_\ell}/\mathfrak{m}) \otimes_{\mathbb{F}_\ell} (B_0 \otimes_{\Omega_{T,\mathbb{Z}_\ell}} \Omega_{T,\mathbb{Z}_\ell}/\mathfrak{m}). \tag{3.41}$$

But $(A_0 \otimes_{\Omega_{T,\mathbb{Z}_\ell}} \Omega_{T,\mathbb{Z}_\ell}/\mathfrak{m})$ and $(B_0 \otimes_{\Omega_{T,\mathbb{Z}_\ell}} \Omega_{T,\mathbb{Z}_\ell}/\mathfrak{m})$ are constructible $\mathbb{F}_\ell$-sheaf therefore this tensor product is also constructible. □

Definition 3.4.23. The functor

$$\hat{\otimes}_{\overline{\mathbb{Z}}_\ell} : \mathrm{D}_{\mathrm{cons}}(X, \Omega_T) \times \mathrm{D}_{\mathrm{cons}}(X, \Omega_T) \to \mathrm{D}_{\mathrm{cons}}(X, \Omega_{T\times T})$$
$$(A, B) \mapsto (A \hat{\otimes}_{\overline{\mathbb{Z}}_\ell} B).$$

is defined as the $\overline{\mathbb{Z}}_\ell$-extension of the functor

$$\hat{\otimes}_{\mathbb{Z}_\ell} : \mathrm{D}_{\mathrm{cons}}(X, \Omega_{T,\mathbb{Z}_\ell}) \times \mathrm{D}_{\mathrm{cons}}(X, \Omega_{T,\mathbb{Z}_\ell}) \to \mathrm{D}_{\mathrm{cons}}(X, \Omega_{T\times T,\mathbb{Z}_\ell}). \tag{3.42}$$

Then we define a functor

$$\mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_{T\times T}) \times \mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_{T\times T}) \to \mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_{T\times T\times T\times T})$$

as follows. Consider the following diagram

$$\begin{tikzcd} & U\backslash G \times^U G/U \arrow[r, "m"] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & U\backslash G/U \\ U\backslash G/U & & U\backslash G/U \end{tikzcd}$$

where m is induced by the multiplication map. Then we set for $A, B \in \mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_{T\times T})$

$$A * B = m_!(A \widehat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B)[\dim T].$$

Lemma 3.4.24. *Assume that $A \in \mathbb{H}^\omega_{[\chi_1,\chi_2]}$ and $B \in \mathbb{H}^\omega_{[\chi_3,\chi_4]}$, if $\chi_3 \neq \chi_2$ then $A * B = 0$ and in general the $\Omega_{T\times T\times T\times T}$-structure on $A * B$ is constructible as an $\Omega_{T\times T}$-sheaf after forgetting along the inclusion $\Omega_{T\times T} \to \Omega_{T\times T\times T\times T}$ induced by the outer inclusions.*

Proof. We argue as in [BY13] 4.3. We decompose the map m in two steps

$$U\backslash G \times^U G/U \xrightarrow{q} U\backslash G \times^B G/U \xrightarrow{\tilde{m}} U\backslash G/U,$$

where $\tilde{m}$ is the map induced by the multiplication in G and q is the quotient by the T-action $t.(g, g') = (gt^{-1}, tg)$, in particular the map q is a T-torsor. It is enough to check the triviality of $q_!(A \widehat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B)$ if $\chi_2 \neq \chi_3$. The triviality can be checked after reducing modulo $\mathfrak{m}_{T\times T\times T\times T}$. We have an isomorphism

$$q_!(A \hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B)/\mathfrak{m}_{T\times T\times T\times T} = q_!((A/\mathfrak{m}_{T\times T}) \boxtimes_{\overline{\mathbb{F}}_\ell} (B/\mathfrak{m}_{T\times T})). \tag{3.43}$$

The sheaf $A/\mathfrak{m}_{T\times T}$ is $(T, \mathcal{L}_{\chi_2,\overline{\mathbb{F}}_\ell})$-equivariant for the right action of T and $B/\mathfrak{m}_{T\times T}$ is $(T, \mathcal{L}_{\chi_3,\overline{\mathbb{F}}_\ell})$-equivariant for the left action of T. Hence their tensor product is $(T, \mathcal{L}_{\chi_2^{-1}\chi_3,\overline{\mathbb{F}}_\ell})$-equivariant for the action of T given by $t.(x, y) = (xt^{-1}, ty)$. The pushforward along $q_!$ is therefore 0 if $\chi_2 \neq \chi_3$. The constructibility assertion follows from the fact that $A \hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B$ is already $\Omega_{T\times T}$-constructible after forgetting along the outer inclusions. This follows from lemma 3.4.4, indeed A is Ω_T-constructible after forgetting the right action of Ω_T and B is constructible after forgetting the left action of Ω_T. ◻

We can now define the convolution functor

$$\mathbb{H}^\omega \times \mathbb{H}^\omega \to \mathbb{H}^\omega$$
$$(A, B) \mapsto \mathrm{For}_{\mathrm{ext}} m_!(A \hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B),$$

where $\mathrm{For}_{\mathrm{ext}}$ is the forgetful functor induced by the map $\Omega_{T\times T} \to \Omega_{T\times T\times T\times T}$ induced by the outer inclusions.

Remark 3.4.25. Replacing Ω_T by $\Omega_{T,\overline{\mathbb{F}}_\ell}$ in all this construction yields the same structure as in [BR22b]. Moreover if we denote by $A*_{\overline{\mathbb{F}}_\ell}B$ the convolution of two sheaves $A, B \in \mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_{T,\overline{\mathbb{F}}_\ell})_{\mathrm{unip}}$. Then for all $C, D \in \mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_T)_{\mathrm{unip}}$ we have

$$(C * D) \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell = (C \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell) *_{\overline{\mathbb{F}}_\ell} (D \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell).$$

Corollary 3.4.26. *The convolution defines a monoidal structure on $\mathbb{H}^\omega$ and $\mathbb{H}^\omega_{[\chi,\chi]}$.*

As $\mathbb{H} = \mathrm{Ind}(\mathbb{H}^\omega)$, we can extend the monoidal structure to $\mathbb{H}$ using the universal property of ind-completions. In particular we have a monoidal structure on $\mathbb{H}$ that is continuous on both variables.

The following lemma is standard, we refer to [BR22b] 7.7 and [Gou21] 8.4.2.

Lemma 3.4.27. *We have isomorphism*

(i). $\Delta_{w,w'\chi} * \Delta_{w',\chi} = \Delta_{ww',\chi}$ *if* $\ell(ww') = \ell(w) + \ell(w')$.

(ii). $\nabla_{w,w'\chi} * \nabla_{w',\chi} = \nabla_{ww',\chi}$ *if* $\ell(ww') = \ell(w) + \ell(w')$.

(iii). $\Delta_{w^{-1},w\chi} * \nabla_{w,\chi} = \Delta_{e,\chi} = \nabla_{w,\chi} * \Delta_{w^{-1},w\chi}$.

Proof. We only prove that Δ_e is the unit in $\mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_T)_{\mathrm{unip}}$ as the rest is standard. It is a computation done on T, consider $L_T[\dim T] * L_T[\dim T]$ on T where $*$ is defined as previously, by 3.2.22 we have $L_T[\dim T] * L_T[\dim T] = m_! L_{T\times T}[3\dim T] = L_T[\dim T]$. $\square$

Lemma 3.4.28. *Let $\alpha, \beta \in {}_{\chi'}\underline{W}_\chi$ be two distinct blocks then for all $w \in \alpha, v \in \beta$ we have* $\mathrm{Hom}(\Delta_{w,\chi}, \Delta_{v,\chi}) = 0$. *In particular we get a direct sum decomposition of*

$$\mathbb{H}^\omega_{[\chi',\chi]} = \bigoplus_\beta \mathbb{H}^{\omega,\beta}_{[\chi',\chi]}.$$

Proof. Using the presentation of both objects as pro-objects, we reduce to the case $\pi^{\dagger,\overline{\mathbb{F}}_\ell}\pi_{\dagger,\overline{\mathbb{F}}_\ell}\Delta_{w,\chi}$ and $\pi^{\dagger,\overline{\mathbb{F}}_\ell}\pi_{\dagger,\overline{\mathbb{F}}_\ell}\Delta_{v,\chi}$ which is done in [Gou21] 8.5.6. $\square$

Definition 3.4.29. For each character $\chi \in \mathrm{CH}(T)$ and each block β we denote by $T_{\beta,\chi}$ the tilting sheaf corresponding to $w_\beta^{\max}$.

Lemma 3.4.30. *Let $\chi \in \mathrm{CH}(T)$ and β be a block. The multiplicity of $\Delta_{w,\chi}$ in a Δ-filtration of $T_{\beta,\chi}$ is one. Similarly the multiplicity of $\nabla_{w,\chi}$ in a ∇-filtration of $T_{\beta,\chi}$ is one.*

Proof. Reducing mod ℓ preserves the multiplicities of Δ-filtrations and ∇-filtrations, hence the statement follows from [Gou21] 9.3.3. $\square$

Lemma 3.4.31. *The convolution is compatible with the block decomposition. Let $\chi_1, \chi_2, \chi_3 \in \mathrm{CH}(T)$ in the same W-orbit and fix two blocks $\alpha \in {}_{\chi_3}\underline{W}_{\chi_2}$ and $\beta \in {}_{\chi_2}\underline{W}_{\chi_1}$, then for all $A \in \mathbb{H}^\alpha_{[\chi_3,\chi_2]}$ and $B \in \mathbb{H}^\beta_{[\chi_2,\chi_1]}$ we have $A * B \in \mathbb{H}^{\alpha\beta}_{[\chi_3,\chi_1]}$.*

Proof. The statement follows from [Gou21] 8.5.1 after reducing mod ℓ. $\square$

Lemma 3.4.32 ([BR22b] 7.10, [Gou21] 9.7.6). *Let $\chi_1, \chi_2, \chi_3, \chi_4 \in \mathrm{CH}(T)$ in the same W-orbit, fix three blocks $\delta \in {}_{\chi_4}\underline{W}_{\chi_3}$, $\beta \in {}_{\chi_3}\underline{W}_{\chi_2}$ and $\gamma \in {}_{\chi_2}\underline{W}_{\chi_1}$. Let $\alpha \in {}_{\chi_4}\underline{W}_{\chi_1}$ be the block $\delta\beta\gamma$. For any $w \in \delta$ and $v \in \gamma$ we have an isomorphism*

$$\Delta_{w,\chi_3} * T_{\chi_2,\beta} * \Delta_{v,\chi_1} = T_{\chi_1,\alpha}.$$

Sketch of proof. We will indicate the main reduction. It is enough to prove both isomorphisms

$$\Delta_{w,\chi_3} * T_{\chi_2,\beta} \simeq T_{\chi_2,\delta\beta}, T_{\chi_2,\beta} * \Delta_{v,\chi_1} = T_{\chi_1,\beta\gamma}.$$

For both isomorphism, it is enough to identify $\pi_{\dagger,\overline{\mathbb{F}}_\ell}(\Delta_{w,\chi_3} * T_{\chi_2,\beta})$ by theorem 3.B.12 but this is checked to be the corresponding tilting in *loc. cit.*. $\square$

3.4.5 The Endomorphismensatz

Definition 3.4.33. A torsion prime for G is either a prime dividing the order of the fundamental group of G_{der} the derived subgroup of G or a prime in one of the table of [SS68] 4.3. for each quasi-simple quotient of G. A bad prime is a prime in one of the tables of *loc. cit.* for each quasi-simple quotient of G.

The goal of the section is to formulate a version of the Endomorphismensatz in our setting.

Theorem 3.4.34 (Endomorphismensatz). *Assume that ℓ is not a torsion prime for $\hat{G}$. Let $\chi_1, \chi_2 \in$ $\mathrm{CH}(T)$ and $\beta \in {}_{\chi_2}\underline{W}_{\chi_1}$. There is a canonical isomorphism*

$$\Omega_T \otimes_{\Omega_T^{W_{\chi_1}^\circ}} \Omega_T \to \mathrm{End}(T_{\chi_1,\beta}).$$

Remark 3.4.35. If we assume further that G has connected center, then the hypothesis is satisfied if ℓ is good for G.

Remark 3.4.36. The condition on ℓ comes from [Gou21], it is here to control the problems of the non connectedness of the center of G or of its endoscopic groups. If we assume that G has connected center and that $\chi = 1$, then the Endomorphismensatz holds without the hypothesis on ℓ. This is almost the case that appears in [BR22b] as they assume that G is adjoint.

We will explain the key steps of the proof which is in essence the proof from [BR22b] Section 9. and [Gou21] 10.4. The key input is theorem 3.4.38. This last theorem is a technical result which is an immediate generalization of the corresponding statements in *loc. cit.*, their proof extends verbatim, note that the only external input in their proofs is a result of [KK90] which is valid over $\mathbb{Z}$.

Theorem 3.4.37 (Completed Steinberg Pitie, [Gou21] 10.2.3). *Assume that ℓ is not a torsion prime for $\hat{G}$. Let $\chi \in \mathrm{CH}(T)$, the module Ω_T is free of rank $|W_\chi^\circ|$ over $\Omega_T^{W_\chi^\circ}$.*

Recall that we have fixed a generator γ of $\pi_1^t(\mathbb{G}_m)$. For $\alpha \in \Phi_\chi$ we denote by $e^{\alpha^\vee} \in \Omega_T$ the element obtained by image of the generator of $\pi_1(\mathbb{G}_m)_\ell$ along the map $\alpha^\vee : \mathbb{G}_m \to T$.

Theorem 3.4.38 ([BR22b] 8.4,[Gou21] 10.3.4). *Assume that ℓ is not a torsion prime for $\hat{G}$. Consider the map $\tau : \Omega_T \otimes_{\Omega_T^{W_\chi^\circ}} \Omega_T \to \mathrm{Fun}(W_\chi^\circ, \Omega_T)$ defined by*

$$a \otimes b \mapsto (w \mapsto a.w^{-1}(b)).$$

Then this map is injective and its image is the space of functions f such that $f(w) = f(ws_{\alpha^\vee}) \bmod (1-e^{\alpha^\vee})$ for $w \in W_\chi^\circ$ and $\alpha \in \Phi_\chi$.

Sktech of proof of theorem 3.4.34. We only sketch the proof for the block $\beta = 1$, the other blocks are deduced from this one as in [Gou21] 10.4.3. We have already gotten a morphism

$$\Omega_T \otimes_{\Omega_T^{W_\chi^\circ}} \Omega_T \to \mathrm{End}(T_{\chi,1}),$$

consider the composition

$$\Omega_T \otimes_{\Omega_T^{W_\chi^\circ}} \Omega_T \to \Omega_T \otimes_{\Omega_T^{W_\chi^\circ}} \Omega_T \to \mathrm{End}(T_{\chi,1}) \xrightarrow{\mathrm{gr}_w} \bigoplus_{w \in W_\chi^\circ} \mathrm{End}(\Delta_{w,\chi}) \simeq \mathrm{Fun}(W_\chi^\circ, \Omega_T)$$

where the first map is induced by $a \otimes b \mapsto b \otimes a$. The key remark is that this composite is the map τ of theorem 3.4.38. This yields the injectivity of the map $\Omega_T \otimes_{\Omega_T^{W_\chi^\circ}} \Omega_T \to \mathrm{End}(T_{\chi,1})$. To get the surjectivity, by lemma 3.4.20, it is enough to show that any tuple (a_w) in the image of this map satisfies the condition of theorem 3.4.38, that is for all $\alpha \in \Phi_\chi^{\vee,+}$, we have $a_{ws_\alpha^\vee} = a_w \bmod (1-e^{\alpha^\vee})$.

The proof of [BR22b] and [Gou21] is split in two key steps.

(i). First build a $W_\chi^\circ \times W_\chi^\circ$-action on $\mathrm{End}(T_{\chi,1})$ such that all maps in the previous composition are $W_\chi^\circ \times W_\chi^\circ$-equivariant. This allows one to reduce to showing that $a_{s_{\alpha^\vee}} = a_1 \mathrm{mod} 1 - e^{\alpha^\vee}$ for all α.

(ii). Then show the claim of (i), which is then reduced to a computation in rank one.

For the first point the action on $\mathrm{End}(T_{\chi,1})$ is defined as follows. For $w, v \in W_\chi^\circ$, set

$$\mathrm{End}(T_{\chi,1}) \to \mathrm{End}(\Delta_{w,\chi} * T_{\chi,1} * \Delta_{v,\chi}) \to \mathrm{End}(T_{\chi,1}),$$

where the first morphism is induced by the functor $\Delta_{w,\chi} * (-) * \Delta_{v,\chi}$ and the second by the isomorphisms of 3.4.32. We then refer to [BR22b] 9.6 and [Gou21] 10.4.5 for the argument why this does not depend on the chosen isomorphisms and the compatibility with the action.

For the second point, the first step is to set $T_s = j_{s,*} j_s^* T_{\chi,1}$ where $s \in S_\chi$ and j_s is the inclusion of both strata corresponding to s and e. This sheaf is the tilting sheaf corresponding to s as it is the case after applying $\pi_{\dagger,\overline{\mathbb{F}}_\ell}$ by [BR22b] 6.10 and [Gou21] 10.4.4. This implies that as in [BR22b] 9.4, the map

$$\mathrm{gr}_w(T_{\chi,1}) \to \mathrm{gr}_w(T_s)$$

is an isomorphism for $w \in \{e, s\}$. We then have a commutative diagram

$$\begin{tikzcd} \mathrm{End}(T_{\chi,1}) \arrow[r] & \mathrm{End}(T_s) \arrow[r] & \Omega_T \oplus \Omega_T \\ \Omega_T \otimes_{\Omega_T^{W_\chi^\circ}} \Omega_T. \arrow[u] \arrow[urr] & & \end{tikzcd}$$

The diagonal map is simply the composite $\Omega_T \otimes_{\Omega_T^{W_\chi^\circ}} \Omega_T \to \mathrm{Fun}(W, \Omega_T) \to \mathrm{Fun}(\{e,s\}, \Omega_T)$ where the second map is the restriction. The map $\Omega_T \otimes_{\Omega_T^{W_\chi^\circ}} \Omega_T \to \mathrm{End}(T_s)$ is surjective. This is a direct application of Nakayama's lemma once we see that after reducing mod the maximal ideal of Ω_T is becomes surjective, but this follows from using [BR22b] 6.6 after applying the functor $\pi_{\dagger,\overline{\mathbb{F}}_\ell}$. The rest follows as in [Gou21] 10.4.7. $\square$

3.4.6 Whittaker model

We define $\mathcal{L}_\psi$ a rank one character sheaf on $\bar{U}$ as follows, consider the following composition

$$\phi : \bar{U} \to \bar{U}/[\bar{U}, \bar{U}] \simeq \prod_{\alpha\in\Delta} U_{-\alpha} \simeq \prod_{\alpha\in\Delta} \mathbb{G}_a \xrightarrow{\sum} \mathbb{G}_a,$$

where the third map comes from the chosen pinning of G. We fix a character $\psi : \mathbb{F}_p \to \overline{\mathbb{Z}}_\ell^\times$ and denote by $AS : \mathbb{G}_a \to \mathbb{G}_a$ the Artin-Schreier covering. The choice of ψ determines a direct summand of $AS_!\overline{\mathbb{Z}}_\ell$ called the Artin-Schreier sheaf, denoted by $\mathcal{L}_\psi$. We then still denote by $\mathcal{L}_\psi = \phi^*\mathcal{L}_\psi$.

Remark 3.4.39. The Artin-Schreier sheaf we have chosen takes its values in $\overline{\mathbb{Z}}_\ell$ but we can also see it as an Ω_T-sheaf if we need to.

We consider sheaves on G/U that are $\mathcal{L}_\chi \otimes L_T$ equivariant on the right and $\mathcal{L}_\psi$-equivariant on the left. We denote the category of those sheaves by

$$\mathrm{D}_{\mathrm{cons}}((\bar{U}, \psi)\backslash G/U, \Omega_T)_\chi.$$

We have averaging functors $\mathrm{Av}_{\psi,!}$ and $\mathrm{Av}_{\psi,*} : \mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_T)_\chi \to \mathrm{D}_{\mathrm{cons}}((\bar{U}, \psi)\backslash G/U, \Omega_T)_\chi$ defined by

$$a_?(\mathcal{L}_\psi \boxtimes -)[\dim U],$$

where $? \in \{!, *\}$ and $a : \bar{U} \times G/U \to G/U$ is the multiplication. By standard arguments [BBM04b], the natural map $\mathrm{Av}_{\psi,!} \to \mathrm{Av}_{\psi,*}$ is an isomorphism, and from now on we drop the ! and $*$ in its definition. Note that as a is affine $a_!$ right perverse exact and a_* is left perverse exact hence Ad_ψ is perverse exact. The functor Av_ψ has both a left and right adjoint given by $\mathrm{Av}_{U,!}$ and $\mathrm{Av}_{U,*}$ defined similarly.

Lemma 3.4.40. *Any sheaf in* $\mathrm{D}_{\mathrm{cons}}((\bar{U}, \psi)\backslash G/U, \Omega_T)_\chi$ *is supported on* $\bar{U}B$.

Proof. This follows immediatly from the genericity of ψ and [BBM04b]. □

Corollary 3.4.41. *The functor* $i_T^* : \mathrm{D}_{\mathrm{cons}}((\bar{U}, \psi)\backslash G/U, \Omega_T)_\chi \to \mathrm{D}_{\mathrm{cons}}(T)_\chi$ *realizes a perverse t-exact equivalence.*

We denote the image of $L_T \otimes \mathcal{L}_\chi$ under the inverse equivalence by $\delta_{\chi,\psi}$.

Lemma 3.4.42. *There is an isomorphism* $\mathrm{Av}_{U,!}(\delta_{\chi,\psi}) = \bigoplus_{\chi'} \bigoplus_{\beta\in_{\chi'}\underline{W}_\chi} T_{\chi,\beta}$.

Proof. The proof is the same as in *loc. cit.*. Using the characterization of theorem 3.B.12 of tilting perverse sheaves and their classification, it is enough to show the corresponding statement for $\pi_{\dagger,\mathbb{F}_\ell}\mathrm{Av}_U(\delta_{\chi,\psi})$, which is done in [Gou21] 12.9.3.(ii). □

Corollary 3.4.43. *The sheaves* $T_{\chi,\beta}$ *for varying* χ *and* β *are projective in* $\mathrm{Perv}(U\backslash G/U, \Omega_T)_\chi$.

Proof. It is enough to show that $\mathrm{Av}_U(\delta_{\chi,\psi})$ is projective since $T_{\chi,\beta}$ is a direct factor of it. But by adjunction we have

$$\mathrm{Hom}(\mathrm{Av}_U(\delta_{\chi,\psi}), -) = \mathrm{Hom}((\delta_{\chi,\psi}), \mathrm{Av}_\psi -),$$

the result now follows from the exactness of Av_ψ and the projectivity of $\delta_{\chi,\psi}$ in $\mathrm{Perv}((\bar{U}, \psi)\backslash G/U, \Omega_T)_\chi$. □

Notation 3.4.44. We define $T_\chi = \mathrm{Av}_{U,!}\mathrm{Av}_\psi\Delta_{1,\chi} = \mathrm{Av}_{U,!}\delta_{\chi,\psi}$. We will also denote by $\mathbb{H}_\psi^\omega$ the category

$$\bigoplus_\chi \mathrm{D}_{\mathrm{cons}}((\overline{U}, \psi)\backslash G/U, \Omega_T)_\chi. \tag{3.44}$$

3.4.7 The functors $\mathbb{V}$

Following [Gou21] 9.10, given a pair of characters (χ', χ) in the same W-orbit and a block $\beta \in {}_{\chi'}\underline{W}_{\chi}$ we define the functor

$$\begin{aligned}\mathbb{V}_{\chi,\beta} : \mathrm{Tilt}(U\backslash G/U)_{\chi,\beta} &\to \mathrm{End}(T_{\chi,\beta}) - \mathrm{mod} \\ T &\mapsto \mathrm{Hom}_{\mathrm{Perv}(U\backslash G/U)}(T_{\chi,\beta}, T).\end{aligned}$$

Theorem 3.4.45 (Struktursatz)**.** *The functor $\mathbb{V}_{\chi,\beta}$ is fully faithful.*

Proof. The proof is adapted from [BY13] 4.7, [BR22b] 11.2 and [Gou21] 9.10.2. The functor $\mathbb{V}_{\chi,\beta}$, defined on the whole category $\mathrm{Perv}(U\backslash G/U, \Omega_T)^{\beta}_{[\chi',\chi]}$ with values in the abelian category $\mathrm{End}(T_{\chi,\beta}) - \mathrm{mod}$ has a left adjoint given by $M \mapsto M \otimes_{\mathrm{End}(T_{\chi,\beta})} T_{\chi,\beta}$. This is well defined because $T_{\chi,\beta}$ is projective. This then reduces to showing that for $T \in \mathrm{Tilt}(U\backslash G/U)_{\chi,\beta}$ the adjunction map $adj_T : T_{\chi,\beta} \otimes_{\mathrm{End}(T_{\chi,\beta})} \mathbb{V}_{\chi,\beta}(T) \to T$ is an isomorphism. Consider the exact sequence

$$0 \to A \to T_{\chi,\beta} \otimes_{\mathrm{End}(T_{\chi,\beta})} \mathbb{V}_{\chi,\beta}(T) \xrightarrow{adj_T} T \to B \to 0.$$

We have to show that both A and B vanish. There is a corresponding statement after applying $\pi_{\dagger,\overline{\mathbb{F}}_\ell}$ which is shown in the first part of *loc. cit.* and as in *loc. cit.* we will reduce to it. Applying $\pi_{\dagger,\overline{\mathbb{F}}_\ell}$ the arrow adj_T produces an isomorphism by *loc. cit.* and as $\pi_{\dagger,\overline{\mathbb{F}}_\ell}$ is right perverse t-exact this implies that ${}^pH^0(\pi_{\dagger,\overline{\mathbb{F}}_\ell} B) = 0$ and hence that $B = 0$ by lemma 3.B.10 which yields the surjectivity of the arrow adj_T. For the injectivity, it is enough to show that $\mathrm{Hom}_{\mathrm{Perv}}(A, T') = 0$ for all tilting sheaves and then that $\mathrm{Hom}_{\mathrm{Perv}}(A, \Delta_{w,\chi}) = 0$. As in *loc. cit.*, writing the object $\Delta_{w,\chi}$ as an inverse limit and then each term as a successive extension of $\pi^{\dagger}_{\overline{\mathbb{F}}_\ell} \Delta_{w,\chi}$, it is enough to have $\mathrm{Hom}_{\mathrm{Perv}}(A, \pi^{\dagger}_{\overline{\mathbb{F}}_\ell} \pi_{\dagger,\overline{\mathbb{F}}_\ell} \Delta_{w,\chi}) = \mathrm{Hom}_{\mathrm{Perv}}(\pi_{\dagger,\overline{\mathbb{F}}_\ell} A, \pi_{\dagger,\overline{\mathbb{F}}_\ell} \Delta_{w,\chi}) = \mathrm{Hom}_{\mathrm{Perv}}({}^pH^0 \pi_{\dagger,\overline{\mathbb{F}}_\ell} A, \pi_{\dagger,\overline{\mathbb{F}}_\ell} \Delta_{w,\chi})$ vanish. But this garanteed by the fact that the map $\pi_{\dagger,\overline{\mathbb{F}}_\ell}(adj_T)$ is an isomorphism. $\square$

3.4.8 The global $\mathbb{V}$ functor

From now on we assume that G has connected center so that there is a unique block in each $\mathbb{H}^{\omega}_{[\chi,\chi']}$. Let us consider the scheme

$$\mathcal{C}(T) = \bigsqcup_{\chi \in \mathrm{CH}(T)} \mathrm{Spec}(\Omega_T) \times \{\chi\} \tag{3.45}$$

in a similar fashion as the scheme of characters of [GL96] 3.2. This is a scheme that is not of finite type over $\overline{\mathbb{Z}}_\ell$ because it has infinitely many connected components.

Lemma 3.4.46. *The scheme $\mathcal{C}(T)$ is canonically isomorphic to $\bigsqcup_{x \in T^{\vee}_{\mathbb{Z}_\ell,\mathrm{cl}}} T^{\vee,\wedge}_{\mathbb{Z}_\ell,x} \times_{\mathbb{Z}_\ell} \overline{\mathbb{Z}}_\ell$, where $T^{\vee}_{\mathbb{Z}_\ell}$ is the torus dual to T defined over $\mathbb{Z}_\ell$, the set $\{x \in T^{\vee}_{\mathbb{Z}_\ell,\mathrm{cl}}\}$ is the set of all closed points of $T^{\vee}_{\mathbb{Z}_\ell}$ (in particular they are all points of the special fiber of $T^{\vee}_{\mathbb{Z}_\ell}$) and $T^{\vee,\wedge}_{\mathbb{Z}_\ell,x} \times_{\mathbb{Z}_\ell} \overline{\mathbb{Z}}_\ell$ denotes the completion of $T^{\vee}_{\mathbb{Z}_\ell}$ at the point x then base changed to $\overline{\mathbb{Z}}_\ell$.*

Remark 3.4.47. The statement of the previous lemma is made to avoid introducing the field of definition of a given character $\chi \in \mathrm{CH}(T)$.

Proof. First note that there is a bijection $\mathrm{CH}(T) = T^\vee(\overline{\mathbb{F}}_\ell)$. Let x be a closed point of $T^\vee_{\mathbb{F}_\ell}$ and let $\mathbb{F}_\ell(x)$ be the residue field of $T^\vee_{\mathbb{F}_\ell}$ at x. Let $\mathfrak{o}_x \subset \mathrm{CH}(T)$ be the $\mathrm{Gal}(\overline{\mathbb{F}}_\ell/\mathbb{F}_\ell)$-orbit corresponding to x. And let $E/\mathbb{Q}_\ell$ be the unramified extension of $\mathbb{Q}_\ell$ correspoding to the extension $\mathbb{F}_\ell(x)/\mathbb{F}_\ell$. We then have

$$T^{\vee,\wedge}_{\mathbb{Z}_\ell,x} \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Z}}_\ell = \bigsqcup_{y \in \mathfrak{o}_x} T^{\vee,\wedge}_{\mathcal{O}_E,y} \otimes_{\mathcal{O}_E} \overline{\mathbb{Z}}_\ell. \tag{3.46}$$

We further make the identification $T^{\vee,\wedge}_{\mathcal{O}_E,y} \otimes_{\mathcal{O}_E} \overline{\mathbb{Z}}_\ell = \mathrm{Spec}(\Omega_T) \times \{y\}$. Since $\mathrm{CH}(T) = \sqcup_{x \in T^\vee_{\mathbb{Z}_\ell,\mathrm{cl}}} \mathfrak{o}_x$, taking a disjoint union yields the lemma. □

We now consider the action of W on $\mathcal{C}(T)$ and we consider the GIT quotient $\mathcal{C}(T) /\!\!/ W$. For all W-orbits $\mathfrak{o} \subset \mathrm{CH}(T)$ of characters, we choose a representative $\chi_\mathfrak{o} \in \mathfrak{o}$.

Lemma 3.4.48. *There is an isomorphism*

$$\mathcal{C}(T) /\!\!/ W = \bigsqcup_{\mathfrak{o}} \mathrm{Spec}(\Omega_T^{W_{\chi_\mathfrak{o}}}). \tag{3.47}$$

Proof. We have $\mathcal{C}(T) = \sqcup_{\chi \in \mathrm{CH}(T)} \mathrm{Spec}(\Omega_T) \times \{\chi\} = \sqcup_\mathfrak{o} \sqcup_{\chi \in \mathfrak{o}} \mathrm{Spec}(\Omega_T) \times \{\chi\}$ and each $\sqcup_{\chi \in \mathfrak{o}} \mathrm{Spec}(\Omega_T) \times \{\chi\}$ is stable under the action of W. The inclusion of the component $\chi_\mathfrak{o}$ then induces an isomorphism $(\sqcup_{\chi \in \mathfrak{o}} \mathrm{Spec}(\Omega_T) \times \{\chi\}) /\!\!/ W = \mathrm{Spec}(\Omega_T) /\!\!/ W_{\chi_\mathfrak{o}} = \mathrm{Spec}(\Omega_T^{W_{\chi_\mathfrak{o}}})$. □

We finally consider the scheme $\mathcal{C}(T) \times_{\mathcal{C}(T) /\!\!/ W} \mathcal{C}(T)$. As before it has infinitely many connected components. By definition it is the closed subscheme of $\mathcal{C}(T) \times \mathcal{C}(T)$ obtained as the union of the graphs $\Gamma_w \subset \mathcal{C}(T) \times \mathcal{C}(T)$ of the actions of the elements of W. Let us now describe its connected components.

Lemma 3.4.49. *There is a canonical isomorphism*

$$\mathcal{C}(T) \times_{\mathcal{C}(T) /\!\!/ W} \mathcal{C}(T) = \bigsqcup_{\mathfrak{o}, \chi, \chi' \in \mathfrak{o}} \mathrm{Spec}(\Omega_T \otimes_{\Omega_T^{W_\chi}} \Omega_T) \times \{\chi' \times \chi\}, \tag{3.48}$$

where the union is indexed over all W-orbits in $\mathrm{CH}(T)$.

Proof. We have

$$\begin{aligned}
\mathcal{C}(T) \times_{\mathcal{C}(T) /\!\!/ W} \mathcal{C}(T) &= \bigsqcup_{\chi, \chi'} (\mathrm{Spec}(\Omega_T) \times \{\chi\}) \times_{\mathcal{C}(T) /\!\!/ W} (\mathrm{Spec}(\Omega_T) \times \{\chi'\}) \\
&= \bigsqcup_{\mathfrak{o}, \chi, \chi' \in \mathfrak{o}} (\mathrm{Spec}(\Omega_T) \times \{\chi\}) \times_{(\sqcup_{\chi'' \in \mathfrak{o}} \mathrm{Spec}(\Omega_T) \times \{\chi\}) /\!\!/ W} (\mathrm{Spec}(\Omega_T) \times \{\chi'\}) \\
&= \bigsqcup_{\mathfrak{o}, \chi, \chi' \in \mathfrak{o}} (\mathrm{Spec}(\Omega_T) \times \{\chi\}) \times_{\mathrm{Spec}(\Omega_T^{W_\chi})} (\mathrm{Spec}(\Omega_T) \times \{\chi'\}).
\end{aligned}$$

□

Lemma 3.4.50. *There is a unique fully faithful functor*

$$\mathbb{V} : \mathrm{Tilt}(U \backslash G / U) \to \mathrm{Coh}(\mathcal{C}(T) \times_{\mathcal{C}(T) /\!\!/ W} \mathcal{C}(T)),$$

that restricts to the functor $\mathbb{V}_\chi$ on each $\mathrm{Tilt}(U \backslash G / U)_\chi$.

Proof. The existence and unicity are immediate from the fact that both categories on the left and right hand side are direct sums and we have prescibed the functor on each direct factor. $\square$

Definition 3.4.51. The objects in the essential image of $\mathbb{V}$ are called Soergel bimodules.

Remark 3.4.52. Note that the functor $\mathbb{V}$ induces a well defined functor $\mathbb{H}^\omega \to \mathrm{D}_{\mathrm{coh}}(\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T)$. It is however not fully faithful on all of $\mathbb{H}^\omega$.

Definition 3.4.53. We define $\mathbb{T} = \bigoplus_{\chi \in \mathrm{CH}(T)} T_\chi \in \mathbb{H}$ and we call it the big tilting sheaf.

Remark 3.4.54. Note that the object $\mathbb{T}$ is not compact.

Lemma 3.4.55. *Under the functor $\mathbb{V}$, the sheaf $\mathbb{T}$ corresponds to $\mathcal{O}_{\mathcal{C}(T)\times_{\mathcal{C}(T)/\!\!/W}\mathcal{C}(T)}$.*

Proof. This is clear since it holds on all connected component of $\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T)$. $\square$

3.4.9 Monoidality of the global $\mathbb{V}$ functor

We still assume that G has connected center.

Lemma 3.4.56. *The category* $\mathrm{Tilt}(U\backslash G/U)$ *is monoidal.*

Proof. Let $T, T' \in \mathrm{Tilt}(U\backslash G/U)$. The statement breaks down in two steps :

(i). the convolution $T * T'$ is perverse,

(ii). the convolution $T * T'$ is tilting.

Both properties can be checked after reducing mod ℓ by theorem 3.B.12. For the mod ℓ version, this is then [Gou21] 9.7.5. $\square$

Consider the category $\mathrm{D}_{\mathrm{qcoh}}(\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T))$, it is equipped with a convolution monoidal structure as explained in Appendix 3.C.

Theorem 3.4.57 (Analog of [Gou21] 12.10.1 and [BR22b] 11.5)**.** *The functor $\mathbb{V}$ is equipped with a canonical monoidal structure.*

Proof. Consider the category $\mathbb{H}_\psi^\omega$. By corollary 3.4.41, the functor

$$\mathrm{Hom}(\oplus\delta_{\chi,\psi}, -) : \mathbb{H}_\psi^\omega \to \bigoplus_\chi \mathrm{D}_{\mathrm{cons}}(T, \Omega_T)_\chi = \mathrm{Perf}(\mathcal{C}(T)),$$

is an equivalence. The category $\mathbb{H}^\omega$ acts on $\mathbb{H}_\psi$ and therefore on $\mathrm{Perf}(\mathcal{C}(T))$. By lemma 3.C.11, we have a monoidal equivalence

$$\begin{aligned}&\mathrm{End}(\mathrm{Perf}(\mathcal{C}(T)) \simeq \mathrm{Perf}(\mathcal{C}(T) \times \mathcal{C}(T))\\ &F \mapsto F(\mathcal{O}_{\mathcal{C}(T)}),\end{aligned}$$

where $\mathcal{O}_{\mathcal{C}(T)}$ is a considered as a $\mathcal{O}_{\mathcal{C}(T)}$-bimodule and therefore $F(\mathcal{O}_{\mathcal{C}(T)})$ has two $\mathcal{O}_{\mathcal{C}(T)}$ actions : one coming from functoriality and one from the fact that F takes its values in the category

$\mathrm{Perf}(\mathcal{O}_{\mathcal{C}(T)})$. The target category is equipped with the convolution of bimodules structure. We therefore have a monoidal functor

$$\mathbb{V}' : \mathbb{H}^{\omega} \to \mathrm{Perf}(\mathcal{C}(T) \times \mathcal{C}(T)).$$

It is then enough to check that this functor is naturally isomorphic to $\mathbb{V}$. By the above description, we can compute the functor $\mathbb{V}'$ as follows

$$\begin{aligned}
\mathbb{V}'(T)(\mathcal{O}_{\mathcal{C}(T)}) &= \mathrm{Hom}(\oplus_{\chi}\delta_{\chi,\psi}, \oplus_{\chi}\delta_{\chi,\psi} * T) \\
&= \mathrm{Hom}(\oplus_{\chi}\delta_{\chi,\psi}, \mathrm{Av}_{\psi}T) \\
&= \mathrm{Hom}(\mathrm{Av}_U \oplus_{\chi} \delta_{\chi,\psi}, T) \\
&= \mathrm{Hom}(\mathbb{T}, T) \\
&= \mathbb{V}(T).
\end{aligned}$$

Hence we have a commutative diagram

$$\begin{tikzcd}
\mathbb{H}^{\omega} \arrow[r, "\mathbb{V}"] \arrow[dr, "\mathbb{V}'"'] & \mathrm{Perf}(\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T)) \arrow[d] \\
& \mathrm{Perf}(\mathcal{C}(T) \times \mathcal{C}(T)).
\end{tikzcd}$$

After restricting to tilting objects, we get a natural commutative diagram

$$\begin{tikzcd}
\mathrm{Tilt}(U\backslash G/U) \arrow[r] \arrow[dr] & \mathrm{Coh}(\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T)) \arrow[d] \\
& \mathrm{Coh}(\mathcal{C}(T) \times \mathcal{C}(T)).
\end{tikzcd}$$

On the abelian category of coherent sheaves the vertical functor is fully faithful and monoidal. Hence as $\mathbb{V}'$ is monoidal, there exists a unique lift of this monoidal structure to $\mathbb{V}$. $\square$

Consider the full subcategory $\langle \mathbb{T} \rangle \subset \mathbb{H}^{\omega}$ generated by $\mathbb{T}$ and all its direct summands.

Lemma 3.4.58. *The category $\langle \mathbb{T} \rangle$ is a monoidal subcategory of $\oplus_{\chi}\mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_T)_{\chi}$ and the functor $\mathbb{V}$ induces a monoidal equivalence*

$$\mathrm{Perf}(\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T)) \simeq \langle \mathbb{T} \rangle. \tag{3.49}$$

Proof. Since $\mathbb{T}$ satisfies $\mathrm{Hom}(\mathbb{T}, \mathbb{T}) = \mathcal{O}_{\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T)}$, it is enough to check that $\langle \mathbb{T} \rangle$ is a monoidal subcategory. For this we compute $\mathbb{V}(\mathbb{T} * \mathbb{T}) = \mathbb{V}(\mathbb{T}) \otimes_{\mathcal{O}_{\mathcal{C}(T)}} \mathbb{V}(\mathbb{T})$. Let

$$p_{13} : \mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T) \to \mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T), \tag{3.50}$$

be the map given by the projection on the outer factors. Then we have,

$$\mathbb{V}(\mathbb{T} * \mathbb{T}) = p_{13,*}\mathcal{O}. \tag{3.51}$$

The map p_{13} is finite and faithfully flat. We argue component by component on $\mathcal{C}(T)$ and we show that the restriction to a component (χ', χ) of $p_{13,*}\mathcal{O}$ is free. We do it for the case $(\chi', \chi) = (1, 1)$ the other are similar. Consider the map

$$p'_{13} : \mathrm{Spec}(\Omega_T) \times_{\mathrm{Spec}(\Omega_T^W)} \mathrm{Spec}(\Omega_T) \times_{\mathrm{Spec}(\Omega_T^W)} \mathrm{Spec}(\Omega_T) \to \mathrm{Spec}(\Omega_T) \times_{\mathrm{Spec}(\Omega_T^W)} \mathrm{Spec}(\Omega_T), \tag{3.52}$$

given by the projection on the outer coordinates. The restriction to the component $(1, 1)$ of $p_{13,*}\mathcal{O}$ is given by $p'_{13}\mathcal{O}$. The ring $\Omega_T \otimes_{\Omega_T^W} \Omega_T$ is local by [BR22b] 8.5. Since the map p'_{13} is finite and faithfully flat so is $p'_{13}\mathcal{O}$. A finite flat module over a local ring is free, see [Aut] Tag 00NZ. □

3.5 $\mathbb{F}$-Categorical center of the Hecke category

In this section, we compute the F-categorical center for the category $\mathbb{H}$. We consider all relevant categories to live in $\mathrm{D}(\overline{\mathbb{Z}}_\ell)$-Mod. This is a variation on [BZN09] [GKRV22]. In the abelian setting [Lus15], [Lus17] has shown a similar statement, the key difference is that Lusztig inputs the classification of representations of G^{F} whereas our construction is formal. We have to work with $\mathbb{H}$ and not $\mathbb{H}^\omega$ to be able to dualize the relevant categories.

3.5.1 Duality on monodromic categories

In [BT22], the authors define a duality functor on completed unipotent monodromic categories extending the usual Verdier duality on $\overline{\mathbb{Q}}_\ell$-constructible monodromic sheaves. We give a construction here that does not involve pro-objects, works for all schemes X equipped with an action of T and is valid in the non-unipotent setting.

Lemma 3.5.1. *Let Λ be a coefficient ring. Let X be a stack with a T action and let A be a Λ constructible T-monodromic sheaf. The Verdier dual $\mathbb{D}_\Lambda(A)$ is monodromic and its canonical monodromy is given by*

$$\phi^\vee : \Lambda[\pi_1^t(T)] \to \mathrm{End}(A), \tag{3.53}$$

where ϕ is the canonical monodromy of A and $\phi^\vee = \phi \circ \mathrm{inv}^*$ with $\mathrm{inv} : \Lambda[\pi_1^t(T)] \to \Lambda[\pi_1^t(T)]$ induced by $t \mapsto t^{-1}$.

Proof. We only need to check this on the fibers of $X \to X/T$ which are all isomorphic to T. This now follows from the fact that since T is smooth, the Verdier dual of a lisse sheaf is lisse and corresponds to the dual representation of $\pi_1(T)$. □

Definition 3.5.2. The map $\mathrm{inv} : T \to T$ induces a map $\mathrm{inv}_* : \Omega_T \to \Omega_T$. Given an Ω_T module M, we denote by $M(\varepsilon) = M \otimes_{\Omega_T, \mathrm{inv}_*} \Omega_T$.

Remark 3.5.3. Note that $L_T(\varepsilon) = L_T^\vee$ is the Ω_T-linear dual of L_T.

Definition 3.5.4. Let X be a stack with an action of T. We define

$$\begin{aligned} \mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_\chi &\to \mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_{\chi^{-1}} \\ \mathbb{D}' &= \mathbb{D}_{\Omega_T}(-)(\varepsilon), \end{aligned}$$

where $\mathbb{D}_{\Omega_T}$ is the Ω_T-linear Verdier duality functor.

Lemma 3.5.5. *The functor* $\mathbb{D}'$ *satisfies*

(*i*). *On the full subcategory* $\mathrm{D}_{\mathrm{cons}}(X, \overline{\mathbb{Z}}_\ell)_{\chi,\mathrm{mon}}$*, we have a canonical isomorphism of functors*

$$\mathbb{D}' = \mathbb{D}_{\overline{\mathbb{Z}}_\ell}[-\dim T], \tag{3.54}$$

(*ii*). $\mathbb{D}'\mathbb{D}' = \mathrm{id}$,

(*iii*). *For* $A, B \in \mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_\chi$ *we have* $\mathrm{Hom}(A, B) = \mathrm{Hom}(\mathbb{D}'(B), \mathbb{D}'(A))$.

(*iv*). *Given* $f : X \to Y$ *a morphism of* T*-scheme, we have* $\mathbb{D}' f_! = f_* \mathbb{D}'$ *and* $\mathbb{D}' f^! = f^* \mathbb{D}'$.

Proof. The last three points follow from the definition. We discuss the first one. We can work locally in the lisse topology and assume that we have a T-equivariant splitting $X = Y \times T$. We can then further assume that Y is a point. Let A be a χ-monodromic sheaf on T, we can write $A = M \otimes_{\Omega_T} (L_T \otimes \mathcal{L}_\chi)[\dim T]$ for the Ω_T-module $M = 1^*[-\dim T]A$. Then by definition $\mathbb{D}'(A) = \mathrm{Hom}_{\Omega_T}(M, \Omega_T) \otimes_{\Omega_T} (L_T \otimes \mathcal{L}_{\chi^{-1}})[\dim T]$. On the other hand, we have $1^*[-\dim T]\mathrm{D}_{\overline{\mathbb{Z}}_\ell}(A) = \mathrm{Hom}_{\overline{\mathbb{Z}}_\ell}(M, \overline{\mathbb{Z}}_\ell)$, where M is the $\overline{\mathbb{Z}}_\ell$-module obtained by forgetting the Ω_T-stucture along the inclusion $\overline{\mathbb{Z}}_\ell \to \Omega_T$.

We claim, that there is a natural Ω_T-linear isomorphism

$$\mathrm{Hom}_{\overline{\mathbb{Z}}_\ell}(M, \overline{\mathbb{Z}}_\ell) = \mathrm{Hom}_{\Omega_T}(M, \Omega_T)(\varepsilon)[\dim T], \tag{3.55}$$

which is induced by local Serre duality for the pushforward along the map $\mathrm{Spec}(\Omega_T) \to \mathrm{Spec}(\overline{\mathbb{Z}}_\ell)$.

Let us show this claim. Let $I \subset \Omega_T$ be the augmentation ideal. Since A is $\overline{\mathbb{Z}}_\ell$-constructible, M is of I-power torsion. Then we have

$$\begin{aligned}
\mathrm{Hom}_{\overline{\mathbb{Z}}_\ell}(M, \overline{\mathbb{Z}}_\ell) &= \mathrm{Hom}_{\Omega_T}(M, \mathrm{Hom}_{\overline{\mathbb{Z}}_\ell}(\Omega_T, \overline{\mathbb{Z}}_\ell)) \\
&= \mathrm{Hom}_{\Omega_T}(M, \Gamma_I(\mathrm{Hom}_{\overline{\mathbb{Z}}_\ell}(\Omega_T, \overline{\mathbb{Z}}_\ell)) \\
&= \mathrm{Hom}_{\Omega_T}(M, \varinjlim_n(\mathrm{Hom}_{\Omega_T}(\Omega_T/I^n, \mathrm{Hom}_{\overline{\mathbb{Z}}_\ell}(\Omega_T, \overline{\mathbb{Z}}_\ell)) \\
&= \mathrm{Hom}_{\Omega_T}(M, \varinjlim_n(\mathrm{Hom}_{\overline{\mathbb{Z}}_\ell}(\Omega_T/I^n, \overline{\mathbb{Z}}_\ell)) \\
&= \varinjlim_n \mathrm{Hom}_{\Omega_T}(M, (\mathrm{Hom}_{\overline{\mathbb{Z}}_\ell}(\Omega_T/I^n, \overline{\mathbb{Z}}_\ell))
\end{aligned}$$

where the first line comes from the adjunction between forgetful and Hom, the second one from the fact that M is of I-power torsion, the third one from the definition of local cohomology, the fourth one again from the adjunction and the last one from the compacity of M as an Ω_T-module.

On the other side, we have

$$\begin{aligned}
\mathrm{Hom}_{\Omega_T}(M, \Omega_T) &= \mathrm{Hom}_{\Omega_T}(M, \Gamma_I(\Omega_T)) \\
&= \mathrm{Hom}_{\Omega_T}(M, \varinjlim_n \mathrm{Hom}_{\Omega_T}(\Omega_T/I^n, \Omega_T)) \\
&= \varinjlim_n \mathrm{Hom}_{\Omega_T}(M, \mathrm{Hom}_{\Omega_T}(\Omega_T/I^n, \Omega_T)).
\end{aligned}$$

Let T' be the torus dual to T defined over $\mathrm{Spec}(\mathbb{Z})$ and let $R = \mathcal{O}(T')$ and let $I' \subset R$ be the augmentation ideal. There is a natural flat map $R \to \Omega_T$ and such that $I'\Omega_T = I$. This implies in particular that $\mathrm{Hom}_{\Omega_T}(\Omega_T/I^n, \Omega_T) = \Omega_T \otimes \mathrm{Hom}_R(R/I'^n, R)$. Let $f : T' \to \mathrm{Spec}(\mathbb{Z})$ be the structure map. Embedding the categories of R-modules and $\mathbb{Z}$-modules into the categories of solid R-modules and solid $\mathbb{Z}$-modules, built in [CS18]. We get a pair of adjoint functors

$$f_! : \mathrm{D}(R_\blacksquare) \leftrightarrows \mathrm{D}(\mathbb{Z}_\blacksquare) : f^!. \tag{3.56}$$

Moreover by [CS18] Observation 8.12, $f^!\mathbb{Z} = R[\dim T]$. A priori $f_!$ can be difficult to compute, but for the R-module R/I', we have $f_!R/I'^n = f_*R/I'^n$ since $R/I'^n = i_*R/I'^n$ where $i : \mathrm{Spec}(R/I'^n) \to \mathrm{Spec}(R)$ is the closed embedding. Indeed, the formation of $i_!$ is compatible with composition and $i_! = i_*$ for proper maps,see [CS18] Theorem 11.1 and the following discussion, but the map $\mathrm{Spec}(R/I'^n) \to \mathrm{Spec}(\mathbb{Z})$ is finite hence proper. The adjunction and base change therefore provide a canonical $\overline{\mathbb{Z}}_\ell$-linear isomorphism

$$\mathrm{Hom}_{\overline{\mathbb{Z}}_\ell}(M, \overline{\mathbb{Z}}_\ell) \to \mathrm{Hom}_{\Omega_T}(M, \Omega_T)[\dim T]. \tag{3.57}$$

We still need to promote this to an Ω_T-linear isomorphism. Note that by our construction, it is enough to do so for the objects Ω_T/I^n. We further note that for $M = \Omega_T/I^n$ both complexes lie in degree 0. We now check the Ω_T-linearity on H^0, it amounts to the Ω_T-linearity of the isomorphism,

$$\mathrm{Hom}^0_{\overline{\mathbb{Z}}_\ell}(M, \overline{\mathbb{Z}}_\ell) \to \mathrm{Ext}^{\dim T}_{\Omega_T}(M, \Omega_T)(\varepsilon). \tag{3.58}$$

On the LHS, the Ω_T-structure comes from Verdier's monodromy which is obtained by twisting by (ε) by lemma 3.5.1. We now have a commutative diagram

$$\begin{tikzcd}
\mathrm{D}(\Omega_T) \arrow[r] & \mathrm{D}(\overline{\mathbb{Z}}_\ell) \\
\mathrm{D}^\heartsuit(\Omega_T) \arrow[r] \arrow[u] & \mathrm{D}^\heartsuit(\overline{\mathbb{Z}}_\ell) \arrow[u]
\end{tikzcd}$$

where the horizontal maps are the forgetful functors and the vertical ones the inclusions. The map we consider therefore lives in $\mathrm{D}^\heartsuit(\Omega_T)$ and thus we get the desired Ω_T-linearity. $\square$

Remark 3.5.6. The setup of [CS18] requires to consider rings that are of finite type over $\mathbb{Z}$ which is why we reduced everything to the ring R.

3.5.2 Rigidity of $\mathbb{H}$

Definition 3.5.7 (Rigid category, [GR17])**.** Let $\mathcal{C}$ be a monoidal compactly generated category. Then $\mathcal{C}$ is quasi-rigid if it is generated by left and right compact dualizable objects. The category $\mathcal{C}$ is rigid if the unit object is also compact.

Lemma 3.5.8. *The category* $\mathbb{H}$ *is quasi-rigid.*

Proof. The $\Delta_{w,\chi}$ generate it and are compact and dualizable. $\square$

Remark 3.5.9. The only failure to rigidity here is the fact that the unit is not compact since it is an infinite direct sum. If we consider the category $\mathbb{H}_\mathfrak{o} = \oplus_{\chi\in\mathfrak{o}}\mathrm{D}_{\mathrm{indcons}}(U\backslash G/U, \Omega_T)_\chi$ where $\mathfrak{o} \subset \mathrm{CH}(T)$ is a W-orbit then this category is rigid.

3.5.3 Duality in $\mathbb{H}$

We now work out the left and right duals in $\mathbb{H}^\omega$.

Definition 3.5.10. We define the following duality functor

$$\mathbb{D}^- : \mathbb{H}^\omega \to \mathbb{H}^\omega, M \mapsto \mathrm{inv}^*\mathbb{D}'(M)(\varepsilon)[-\dim T],$$

where $\mathrm{inv} : G \to G$ denotes the inversion map.

Lemma 3.5.11. *There is a canonical isomorphism*

$$\mathbb{D}'(- * -) = (\mathbb{D}'(-) * \mathbb{D}'(-))[-2\dim T]. \tag{3.59}$$

Proof. Recall that the convolution 3.4.4 was defined as

$$A * B = \mathrm{For}^{\Omega_T}_{\Omega_{T\times T}} m_!(A \hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B)[\dim T], \tag{3.60}$$

where the forgetful functor is induced by the second inclusion $\Omega_T \to \Omega_{T\times T}$. Consider the $\Omega_{T\times T}$-module $\Omega_{T\times T}(\epsilon_{T\times T}) \otimes_{\Omega_T} \Omega_T(\epsilon_T)$ where $\Omega_T \to \Omega_{T\times T}$ is induced via the second inclusion. Tensoring by this module defines a twist $M \mapsto M(\varepsilon_{T\times T/T})$ for $M \in \mathrm{D}(\Omega_T)$. We claim that there are natural $\Omega_{T\times T}$-linear isomorphism of functors

(*i*). $\mathbb{D}'_{\Omega_T}(\mathrm{For}^{\Omega_T}_{\Omega_{T\times T}}(-)) = \mathrm{For}^{\Omega_T}_{\Omega_{T\times T}}(\mathbb{D}'_{\Omega_{T\times T}}(-))[-\dim T](\varepsilon_{T\times T/T})$, where the index Ω_T or $\Omega_{T\times T}$ specifies where we use the version of the duality $\mathbb{D}'$ we use.

(*ii*). $m_! = m_*[\dim T]$ on objects that are $L_{T\times T} \otimes \mathcal{L}_{\chi,\chi'}$-equivariant.

Let us assume both claims. And let us show how this implies the theorem

$$\begin{aligned}
\mathbb{D}'_{\Omega_T}(A * B) &= \mathrm{For}^{\Omega_T}_{\Omega_{T\times T}} \mathbb{D}'_{\Omega_{T\times T}} m_!(A \hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B)(\varepsilon_{T\times T/T}) \\
&= \mathrm{For}^{\Omega_T}_{\Omega_{T\times T}} m_* \mathbb{D}'_{\Omega_{T\times T}}(A \hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B)(\varepsilon_{T\times T/T}) \\
&= \mathrm{For}^{\Omega_T}_{\Omega_{T\times T}} m_!(\mathbb{D}'_{\Omega_T}(A) \hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} \mathbb{D}'_{\Omega_T}(B))(\varepsilon_{T\times T/T})[-\dim T].
\end{aligned}$$

The first line follow from the first point and the last one from the compatibility between $\otimes, \mathrm{Hom}$ and the Kunneth formula. Note that once we forget along $\Omega_T \to \Omega_{T\times T}$ the second inclusion, the twist $(\varepsilon_{T\times T/T})$ becomes trivial.

To prove the first claim, we show more a general statement. Let X be a scheme with a T-action, and let $T = T_1 \times T_2$ be a decomposition into a product of tori. Define the twist $M \mapsto M(\varepsilon_{T/T_1})$ in a similar way. There is Ω_T-linear isomorphism of functors $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)^{\mathrm{op}}_\chi \to \mathrm{D}_{\mathrm{cons}}(X, \Omega_T)$

$$\mathrm{For}^{\Omega_{T_1}}_{\Omega_T} \mathbb{D}'_{\Omega_T} = \mathbb{D}'_{\Omega_{T_1}} \mathrm{For}^{\Omega_{T_1}}_{\Omega_T}[\dim T_2](\varepsilon_{T/T_1}) \tag{3.61}$$

This isomorphism can be constructed locally in the smooth topology of X/T, hence we can assume that $X = Y \times T$. We can furthermore assume that Y is a point. The compatibility with the duality follows from the same argument of the proof of lemma 3.5.1 (*i*) with the pair $(\overline{\mathbb{Z}}_\ell, \Omega_T)$ replaced by (Ω_{T_1}, Ω_T).

For the second point, we show that there is an Ω_T-linear isomorphism of functors $\mathrm{D}_{\mathrm{cons}}(U\backslash G \times^U G/U, \Omega_{T\times T})_{\chi',\chi} \to \mathrm{D}_{\mathrm{cons}}(U\backslash G/U, \Omega_T)_\chi$

$$m_![\dim T] = m_*. \tag{3.62}$$

Recall that m is the composition of a T-torsor and a proper map. We show that for a T-torsor $f : X \to Y$ there is an isomorphism of functors $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_{\mathrm{unip}} \to \mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_{\mathrm{unip}}, f_! = f_*[-\dim T]$. This isomorphism can be constructed locally in the smooth topology and as before we can assume that $X = T$ and Y is a point. The statement then follows from the computation of $\mathrm{R}\Gamma(T, L_T) = \overline{\mathbb{Z}}_\ell[-\dim T]$ which is done in [GL96] 3.1.1. $\square$

Lemma 3.5.12. *Let X be a T-scheme, there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{D}(X,\Omega_T)_\chi}(A, B)[\dim T] = \mathrm{R}\Gamma(X, \mathbb{D}'(A) \hat{\otimes}^!_{\overline{\mathbb{Z}}_\ell} B) \tag{3.63}$$

where $\otimes^!_{\overline{\mathbb{Z}}_\ell}$ denotes $\Delta^!(- \boxtimes_{\overline{\mathbb{Z}}_\ell} -)$ and the completion as in section 3.4.4.

Proof. By descent, we can assume that $X = Y \times T$. We assume that $\chi = 1$. Let $A, B \in \mathrm{D}(X, \Omega_T)_{\mathrm{unip}}$. Then $A = A' \boxtimes_{\Omega_T} L_T$ and $B = B' \boxtimes_{\Omega_T} L_T$. We can thus compute

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}(X,\Omega_T)_{\mathrm{unip}}}(A, B) &= \mathrm{Hom}_{\mathrm{D}_{\mathrm{cons}}(Y,\Omega_T)}(A', B') \\ &= \mathrm{R}\Gamma(Y, \mathbb{D}'_{\Omega_T}(A) \otimes^!_{\Omega_T} B).\end{aligned}$$

On the other side, we have

$$\mathbb{D}'(A) \hat{\otimes}^!_{\overline{\mathbb{Z}}_\ell} B = (\mathbb{D}_{\Omega_T}(A) \hat{\otimes}^!_{\overline{\mathbb{Z}}_\ell} B) \boxtimes_{\Omega_{T\times T}} \Delta^* L_{T\times T}. \tag{3.64}$$

Applying the functor $\mathrm{R}\Gamma(X, -)$ we get

$$\begin{aligned}\mathrm{R}\Gamma(X, \mathbb{D}'(A) \hat{\otimes}^!_{\overline{\mathbb{Z}}_\ell} B) &= \mathrm{R}\Gamma(Y, \mathbb{D}_{\Omega_T}(A) \hat{\otimes}^!_{\overline{\mathbb{Z}}_\ell} B) \otimes_{\Omega_{T\times T}} \mathrm{R}\Gamma(T, \Delta^* L_{T\times T}) \\ &= \mathrm{R}\Gamma(Y, \mathbb{D}_{\Omega_T}(A) \hat{\otimes}^!_{\overline{\mathbb{Z}}_\ell} B) \otimes_{\Omega_{T\times T}} \Omega_T[\dim T] \\ &= \mathrm{R}\Gamma(Y, \mathbb{D}_{\Omega_T}(A) \otimes_{\Omega_T} B)[\dim T].\end{aligned}$$

$\square$

Theorem 3.5.13. *All objects $A \in \mathbb{H}^\omega$ are left and right dualizable with left and right duals canonically identified with $\mathbb{D}^-(A)$.*

Proof. We want to show that there are canonical isomorphisms for all $A, B, C \in \mathbb{H}^\omega$,

$$\mathrm{Hom}(A * B, C) = \mathrm{Hom}(A, C * \mathbb{D}^-(B)) = \mathrm{Hom}(B, \mathbb{D}^-(A) * C). \tag{3.65}$$

By symmetry we will only show the first one. We follow the construction of [BZN09]. Assume that $A, B, C \in \mathbb{H}^\omega$. Then we have by lemma 3.5.12

$$\mathrm{Hom}(A * B, C) = \mathrm{R}\Gamma(U\backslash G/U, \mathbb{D}'(A * B) \hat{\otimes}^!_{\overline{\mathbb{Z}}_\ell} C) = \mathrm{R}\Gamma(U\backslash G/U, \mathbb{D}'(A) * \mathbb{D}'(B) \otimes^! C) \tag{3.66}$$

and

$$\mathrm{Hom}(A, C * \mathbb{D}^-(B)) = \mathrm{R}\Gamma(U\backslash G/U, \mathbb{D}'(A) \hat{\otimes}^!_{\overline{\mathbb{Z}}_\ell} (C * \mathbb{D}^-(B)). \tag{3.67}$$

Replacing A, B by $\mathbb{D}'(A)$ and $\mathbb{D}'(B)$, it is enough to show that

$$\mathrm{R}\Gamma(U\backslash G/U, A \hat{\otimes}^!_{\overline{\mathbb{Z}}_\ell} (C * \mathrm{inv}^* B)) = \mathrm{R}\Gamma(U\backslash G/U, (A * B) \hat{\otimes}^!_{\overline{\mathbb{Z}}_\ell} C). \tag{3.68}$$

Consider the following diagram

$$\begin{tikzcd} U\backslash G \times^U G/U \arrow[r, "\Delta_1"] \arrow[d, "m"'] & U\backslash G \times^U G/U \times U\backslash G/U \arrow[r, "q_1"] \arrow[d, "\mathrm{id}\times m"] & U\backslash G/U \times U\backslash G/U \times U\backslash G/U \\ U\backslash G/U \arrow[r, "\Delta"'] & U\backslash G/U \times U\backslash G/U & \end{tikzcd}$$

where m is the multiplication and $\Delta_1 = \mathrm{id} \times m$ and q_1 is simply the projection. The square in this diagram is Cartesian. We have

$$\begin{aligned}(A * B)\hat{\otimes}^!_{\overline{\mathbb{Z}}_\ell} C &= \Delta^!(m \times \mathrm{id})_! q_1^*(A\hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B\hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{Z}}_\ell C)) \\ &= m_! \Delta_1^! q_1^*(A\hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B\hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} C).\end{aligned}$$

This follows from the fact that $m_!(A \boxtimes B) \simeq m_*(A \boxtimes B)[\dim T]$ as in the proof of lemma 3.5.11.

Similarly, we have

$$\begin{tikzcd} U\backslash G \times^U G/U \arrow[r, "\Delta_2"] \arrow[d, "m"'] & U\backslash G \times^U G/U \times U\backslash G/U \arrow[r, "q_2"] \arrow[d, "\mathrm{id}\times m"] & U\backslash G/U \times U\backslash G/U \times U\backslash G/U \\ U\backslash G/U \arrow[r, "\Delta"'] & U\backslash G/U \times U\backslash G/U & \end{tikzcd}$$

where $\Delta_2(x, y) = (m \times \mathrm{id})$ and q_2 is induced by the maps $G^3 \to G^3, (a, b, c) \mapsto (a, \mathrm{inv}(c), b)$ and the square is cartesian. Then we have

$$\begin{aligned}A\hat{\otimes}^!_{\overline{\mathbb{Z}}_\ell}(C * \mathrm{inv}^* B) &= \Delta^!(\mathrm{id} \times m)_! q_2^*(A\hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B\hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} C) \\ &= m_! \Delta_2^! q_2^*(A\hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B\hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} C)\end{aligned}$$

Consider now the diagram

$$\begin{tikzcd} U\backslash G \times^U G/U \arrow[r, "q_2\Delta_2"] \arrow[d, "m"'] \arrow[dr, "r"] & U\backslash G/U \times U\backslash G/U \times U\backslash G/U \\ U\backslash G/U & U\backslash G \times^U G/U \arrow[l, "m"] \arrow[u, "q_1\Delta_1"'] \end{tikzcd}$$

where r is the map induced by the map $G \times G \to G \times G, (x, y) \mapsto (xy, \mathrm{inv}(y))$. This diagram is commutative. We therefore have

$$\begin{aligned}\mathrm{R}\Gamma(U\backslash G/U, m_! \Delta_2^! q_2^*(A\hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B\hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} C)) &= \mathrm{R}\Gamma(U\backslash G/U, m_! r^! \Delta_1^! q_1^*(A\hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B\hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} C)) \\ &= \mathrm{R}\Gamma(U\backslash G/U, m_! r_! r^! \Delta_1^! q_1^*(A\hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B\hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} C)) \\ &= \mathrm{R}\Gamma(U\backslash G/U, m_! \Delta_1^! q_1^*(A\hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} B\hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} C)).\end{aligned}$$

The first line comes from the remark that q_1 and q_2 are smooth of relative dimension $\dim U$ hence since $\Delta_2^! q_2^! = r^! \Delta_1^! q_1^!$ after shifting by $[-2\dim U]$ we get $\Delta_2^! q_2^* = r^! \Delta_1^! q_1^*$. The passage from the third to the fourth line follows from the fact that r is an isomorphism. □

The proof of this theorem yields a finer information. Since all objects of $\mathbb{H}^\omega$ are dualizable, there are well defined functors $L, R : \mathbb{H}^{\omega,\mathrm{op}} \to \mathbb{H}$ such that for all $x \in \mathbb{H}^\omega$, the sheaf $L(x)$ (resp. $R(x)$) is the left dual of x (reps. the right dual of x). The next corollary is then also a consequence of the proof theorem 3.5.13.

Corollary 3.5.14. *There is a monoidal isomorphism of functors* $\mathbb{H}^{\omega,\mathrm{op}} \to \mathbb{H}^\omega, L \to R$.

We now pass to the Ind-extensions. Recall that we extended the convolution product to all of $\mathbb{H}$ by continuity. Recall also the following facts from Appendix 3.C. Since the category $\mathbb{H}$ is compactly generated, it is dualizable and its dual is canonically identified with $\mathbb{H}^\vee = \mathrm{Ind}(\mathbb{H}^{\omega,\mathrm{op}})$. By extending by continuity the functors L and R, we get continuous functor $L, R : \mathbb{H}^\vee \to \mathbb{H}$ defined on compact objects by taking left and right duals.

Definition 3.5.15. A pivotal structure on $\mathbb{H}$ is the data of a monoidal isomorphism $L \to R$.

Remark 3.5.16. We refer to [BZN09] Section 3.4 for a discussion about pivotal structures in the context on quasi-rigid categories.

Theorem 3.5.17 ([BZN09], 3.13)**.** *Assume that $\mathcal{C}$ is a monoidal quasi-rigid category, and let M be a $\mathcal{C}$-bimodule. Assume further that $\mathcal{C}$ is equipped with a pivotal structure. Then it induces an isomorphism*

$$\mathcal{C} \otimes_{\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}}} M = \mathrm{Hom}_{\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}}}(\mathcal{C}, M). \tag{3.69}$$

We can now also reformulate theorem 3.5.13 as follows.

Corollary 3.5.18. *The category $\mathbb{H}$ is equipped with a canonical pivotal structure.*

3.5.4 F-center and F-trace of $\mathbb{H}$

We refer to Appendix 3.C for the notion of F-central functors and of F-trace functors. We consider the functor

$$\mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell) \xrightarrow{\mathrm{HC}^*_{\mathrm{F}}} \mathrm{D}_{\mathrm{cons}}(\frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T}, \overline{\mathbb{Z}}_\ell) \xrightarrow{p^!\Phi} \mathbb{H}^\omega \tag{3.70}$$

where Φ is the functor defined in section 3.3.5. We will abbreviate this functor to $p^!\mathrm{HC}^*_{\mathrm{F}}$. Similarly we denote by $\mathrm{CH}_{\mathrm{F}} p_!$ its left adjoint.

Consider the functor $\mathrm{F}^* : \mathbb{H}^\omega \to \mathbb{H}^\omega$. Note that this functor is monoidal.

Theorem 3.5.19. *The functor $p^!\mathrm{HC}^*_{\mathrm{F}}$ is equipped with a canonical F^*-central structure and the functor $\mathrm{CH}_{\mathrm{F}} p_!$ is equipped with a canonical F-trace structure. That is, we have a diagram with two commuting triangle.*

$$\begin{tikzcd}
& \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell) \arrow[dl] \arrow[d, shift right, "p^! r_* q^!"'] & \\
\mathcal{Z}_{\mathrm{F}}(\mathbb{H}) \arrow[r] & \mathbb{H} \arrow[u, shift right, "q_* r^* p_!"'] \arrow[r] & \mathrm{Tr}(\mathrm{F}, \mathbb{H}) \arrow[ul]
\end{tikzcd}$$

where the functors $\mathcal{Z}_{\mathrm{F}}(\mathbb{H}) \to \mathbb{H}$ and $\mathbb{H} \to \mathrm{Tr}(\mathrm{F}, \mathbb{H})$ are the canonical functors. Moreover, both functors $\mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell) \to \mathcal{Z}_{\mathrm{F}}(\mathbb{H})$ and $\mathrm{Tr}(\mathrm{F}, \mathbb{H}) \to \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell)$ are equivalences.

Corollary 3.5.20. *The functor HC_{F} is equipped with a canonical F^*-central structure.*

Proof. By lemma 3.5.11, the functor $\mathbb{D}'$ is equipped with a monoidal structure hence induces an isomorphism $\mathcal{Z}_{\mathrm{F}}(\mathbb{H}) \to \mathcal{Z}_{\mathrm{F}}(\mathbb{H})$ since $\mathbb{D}'\mathrm{F}^*\mathbb{D}' = \mathrm{F}^! = \mathrm{F}^*$. The functor $\mathbb{D}'\mathrm{HC}_{\mathrm{F}}\mathbb{D} : \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell) \to \mathbb{H}$ is isomorphic up to a shift to $\mathrm{HC}_{\mathrm{F}}^*$ hence is equipped with a canonical central structure. Therefore $\mathrm{HC}_{\mathrm{F}}\mathbb{D} : \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell)^\vee \to \mathbb{H}$ is equipped with a central structure. Since the functor $\mathbb{D}$ induces an equivalence $\mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell)^\vee \simeq \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell)$ we get a F-central structure on HC_{F}. ☐

Proof of theorem 3.5.19. The argument closely follows the one of [BZN09]. First recall that $\mathbb{H}$ is a $\mathbb{H}_T$-bimodule, consider the following augmented simplicial object $\mathbb{H}^{\otimes_{\mathbb{H}_T} \bullet+2}$,

$$\mathbb{H} \longleftarrow \mathbb{H} \otimes_{\mathbb{H}_T} \mathbb{H} \;\overleftleftarrows\; \mathbb{H} \otimes_{\mathbb{H}_T} \mathbb{H} \otimes_{\mathbb{H}_T} \mathbb{H} \;\overleftleftleftarrows\; \dots$$

where the maps are given by the partial convolutions. This gives a resolution of $\mathbb{H}$ as an $\mathbb{H}$-bimodule, therefore we have

$$\mathbb{H} = \varinjlim_{\Delta^{\mathrm{op}}} \mathbb{H}^{\otimes_{\mathbb{H}_T} \bullet+2}. \tag{3.71}$$

We first build the F-central structure. The F-center of $\mathbb{H}$ is

$$\mathcal{Z}_{\mathrm{F}}(\mathbb{H}) = \mathrm{End}_{\mathbb{H}\otimes\mathbb{H}^{\mathrm{rev}}}(\mathbb{H}, \mathbb{H}_{\mathrm{F}}), \tag{3.72}$$

where $\mathbb{H}_{\mathrm{F}}$ denotes $\mathbb{H}$ with its right $\mathbb{H}$-module structure twisted by F^*. We use the previous resolution to compute it.

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{H}\otimes\mathbb{H}^{\mathrm{rev}}}(\mathbb{H}, \mathbb{H}_{\mathrm{F}}) &= \mathrm{Hom}_{\mathbb{H}\otimes\mathbb{H}^{\mathrm{rev}}}(\varinjlim \mathbb{H}^{\otimes_{\mathbb{H}_T} \bullet+2}, \mathbb{H}_{\mathrm{F}}) \\
&= \varprojlim \mathrm{Hom}_{\mathbb{H}\otimes\mathbb{H}^{\mathrm{rev}}}(\mathbb{H}^{\otimes_{\mathbb{H}_T} \bullet+2}, \mathbb{H}_{\mathrm{F}}) \\
&= \varprojlim \mathrm{Hom}_{\mathbb{H}_T\otimes\mathbb{H}_T}(\mathbb{H}^{\otimes_{\mathbb{H}_T} \bullet}, \mathbb{H}_{\mathrm{F}})
\end{aligned}$$

Let us identify the object $\mathrm{Hom}_{\mathbb{H}_T\otimes\mathbb{H}_T}(\mathbb{H}^{\otimes_{\mathbb{H}_T} \bullet}, \mathbb{H}_{\mathrm{F}})$. By lemma 3.5.21, we have

$$\mathbb{H}^{\otimes_{\mathbb{H}_T} n} = \bigoplus_{\chi} \mathrm{D}_{\mathrm{indcons}}(U\backslash G/U \times^T U\backslash G/U \times^T \dots \times^T U\backslash G/U, \Omega_T)_\chi, \tag{3.73}$$

where there are n-copies of G and $T \times T$-acts on the left of the first copy of G and on right of the last copy of G. By lemma 3.5.22, we have

$$\mathrm{Hom}_{\mathbb{H}_T\otimes\mathbb{H}_T}(\mathbb{H}^{\otimes_{\mathbb{H}_T} n}, \mathbb{H}_{\mathrm{F}}) = \bigoplus_{\chi} \mathrm{D}_{\mathrm{indcons}}(\frac{U\backslash G/U \times^T U\backslash G/U \times^T \dots \times^T U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T}, \Omega_T)_\chi, \tag{3.74}$$

where there are $(n+1)$-copies of G. The maps in the simplicial diagram

$$\bigoplus_{\chi} \mathrm{D}_{\mathrm{indcons}}(\frac{U\backslash G/U \times^T U\backslash G/U \times^T \dots \times^T U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T}, \Omega_T)_\chi \tag{3.75}$$

are given by the right adjoints of the partial convolutions. The functor Φ of section 3.3.5 induces an equivalence

$$\bigoplus_{\chi} \mathrm{D}_{\mathrm{indcons}}(\frac{U\backslash G/U \times^T \dots \times^T U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T}, \Omega_T)_\chi \simeq \mathrm{D}_{\mathrm{indcons}}(\frac{U\backslash G/U \times^T \dots \times^T U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T}, \overline{\mathbb{Z}}_\ell). \tag{3.76}$$

Consider the following map of simplicial stacks

$$p_n : \frac{G \times^B G \times^B \cdots \times^B G}{\mathrm{Ad_F} B} \to \frac{U\backslash G/U \times^T \cdots \times^T U\backslash G/U}{\mathrm{Ad_F} T}, \tag{3.77}$$

given in degree n by the quotient map for the action of U^{n+1} on the left of each copies of G. In particular, at level n, this map is a torsor over a unipotent group scheme. We claim that the pushforward map

$$p_{n,*} : \mathrm{D_{indcons}}(\frac{G \times^B G \times^B \cdots \times^B G}{\mathrm{Ad_F} B}, \overline{\mathbb{Z}}_\ell) \to \mathrm{D_{indcons}}(\frac{U\backslash G/U \times^T U\backslash G/U \times^T \cdots \times^T U\backslash G/U}{\mathrm{Ad_F} T}, \overline{\mathbb{Z}}_\ell), \tag{3.78}$$

defines a morphism of cosimplicial objects. The source cosimplicial object is nothing else than the category of sheaves on the simplicial stack obtained as the Čech nerve of the map $\frac{G}{\mathrm{Ad_F} B} \to \frac{G}{\mathrm{Ad_F} G}$. And the morphism are given by the !-pullbacks along partial multiplication maps. Denote by $m'_i : \frac{G \times^B G \times^B \cdots \times^B G}{\mathrm{Ad_F} B} \to \frac{G \times^B G \times^B \cdots \times^B G}{\mathrm{Ad_F} B}$ the partial multiplication map. We now prove the claim. After passing to left adjoints, we need to show that there is a canonical isomorphism

$$p_i^*[c_i^j] = m'_{i,!} p_{i+1}^*, \tag{3.79}$$

where $[c_i^j] : \mathrm{D_{indcons}}(\frac{U\backslash G/U^{\times^T(i+1)}}{\mathrm{Ad_F} T}, \overline{\mathbb{Z}}_\ell) \to \mathrm{D_{indcons}}(\frac{U\backslash G/U^{\times^T(i)}}{\mathrm{Ad_F} T}, \overline{\mathbb{Z}}_\ell)$ denotes the j-th convolution. For clarity of exposition, we show it in the first degree. Consider the diagram

$$\begin{tikzcd}
\frac{U\backslash G/U \times^T U\backslash G/U}{\mathrm{Ad_F} T} & \frac{U\backslash G \times^B G/U}{\mathrm{Ad_F} T} \arrow[l, "\mathrm{pr}"'] \arrow[r, "m"] & \frac{U\backslash G/U}{\mathrm{Ad_F} T} \\
\frac{G \times^B G}{\mathrm{Ad_F} B} \arrow[u, "p_1"] \arrow[ur, "\tilde{p}_1"] \arrow[rr, "m'"'] & & \frac{G}{\mathrm{Ad_F} B} \arrow[u, "p_0"']
\end{tikzcd}$$

where pr is the quotient map and m induced by the multiplication. The right slanted square is Cartesian and the left triangle is commutative. Then we have

$$\begin{aligned} p_0^*[c] &= p_0^* m_! \mathrm{pr}^* \\ &= m'_! \tilde{p}_1^* \mathrm{pr}^* \\ &= m'_! p_1^*, \end{aligned}$$

as desired.

By !-descent theorem 3.2.9, this defines a map

$$Z : \mathrm{D_{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \Omega_T) \to \mathcal{Z}_{\mathrm{F}}(\mathbb{H}). \tag{3.80}$$

Furthermore the following diagram commutes

$$\begin{tikzcd}
\mathrm{D_{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \Omega_T) \arrow[r] \arrow[dr, "p^! \mathrm{HC}_{\mathrm{F}}^*"'] & \mathcal{Z}_{\mathrm{F}}(\mathbb{H}) \arrow[d] \\
& \mathbb{H}.
\end{tikzcd}$$

By theorems 3.5.17 and 3.5.13, there is an identification $\mathcal{Z}_{\mathrm{F}}(\mathbb{H})$ with $\mathrm{Tr}(\mathrm{F}, \mathbb{H})$. Under this identification the two canonical maps

$$\mathcal{Z}_{\mathrm{F}}(\mathbb{H}) \to \mathbb{H} \to \mathrm{Tr}(\mathrm{F}, \mathbb{H}) \tag{3.81}$$

are adjoints, with $\mathbb{H} \to \mathrm{Tr}(\mathrm{F}, \mathbb{H})$ being the left adjoint. We now build the map $\mathrm{Tr}(\mathrm{F}, \mathbb{H}) \to \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell)$. This is roughly the same argument but with left adjoints instead. Using the previous resolution we get

$$\mathrm{Tr}(\mathrm{F}, \mathbb{H}) = \mathbb{H} \otimes_{\mathbb{H} \otimes \mathbb{H}^{\mathrm{rev}}} \mathbb{H}_{\mathrm{F}} = \varinjlim_{\Delta^{\mathrm{op}}} \mathbb{H}^{\otimes_{\mathbb{H}_T} \bullet} \otimes_{\mathbb{H}_T \otimes \mathbb{H}_T} \mathbb{H}_{\mathrm{F}}. \tag{3.82}$$

As before we identify the terms

$$\mathbb{H}^{\otimes_{\mathbb{H}_T} n} \otimes_{\mathbb{H}_T \otimes \mathbb{H}_T} \mathbb{H}_{\mathrm{F}} = \mathrm{D}_{\mathrm{indcons}}(\frac{U\backslash G/U \times^T \cdots \times^T U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T}, \overline{\mathbb{Z}}_\ell) \tag{3.83}$$

with $(n+1)$-copies of G. The pullback along the maps p_n defines a morphism of simplicial objects

$$p_\bullet^* : \mathbb{H}^{\otimes_{\mathbb{H}_T} n} \otimes_{\mathbb{H}_T \otimes \mathbb{H}_T} \mathbb{H}_{\mathrm{F}} \to \mathrm{D}_{\mathrm{indcons}}(\frac{G \times^B G \times^B \cdots \times^B G}{\mathrm{Ad}_{\mathrm{F}} B}, \overline{\mathbb{Z}}_\ell), \tag{3.84}$$

where the right simplicial object is, as before, the category of sheaves on the simplicial stack obtained as the Čech nerve of the map
$fracG\mathrm{Ad}_{\mathrm{F}}B \to \frac{G}{\mathrm{Ad}_{\mathrm{F}} G}$. This in turn induces a map

$$T : \mathrm{Tr}(\mathrm{F}, \mathbb{H}) \to \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell). \tag{3.85}$$

Moreover the following diagram commutes

$$\begin{tikzcd}
\mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell) & \\
\mathbb{H} \arrow[u, "q_* r^* p_!"] \arrow[r] & \mathrm{Tr}(\mathrm{F}, \mathbb{H}) \arrow[ul, "T"']
\end{tikzcd}$$

We want to show that T is fully faithful. For this we compute its right adjoint and verify that $\mathrm{id} \to T^R T$ is an isomorphism. We consider now the morphisms

$$p_{\bullet,*} : \mathrm{D}_{\mathrm{indcons}}(\frac{G \times^B G \times^B \cdots \times^B G}{\mathrm{Ad}_{\mathrm{F}} B}, \overline{\mathbb{Z}}_\ell) \to \mathbb{H}^{\otimes_{\mathbb{H}_T} n} \otimes_{\mathbb{H}_T \otimes \mathbb{H}_T} \mathbb{H}_{\mathrm{F}}, \tag{3.86}$$

and we claim that this induces a morphism of simplicial objects. We check it as before on the first term. Consider the following diagram

$$\begin{tikzcd}
\frac{U\backslash G/U \times^T U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T} & \frac{U\backslash G \times^B G/U}{\mathrm{Ad}_{\mathrm{F}} T} \arrow[l, "\mathrm{pr}"'] \arrow[r, "m"] & \frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T} \\
 & \mathcal{Z} \arrow[u, "q_2"'] \arrow[dl, "q_1"] & \\
\frac{G \times^B G}{\mathrm{Ad}_{\mathrm{F}} B} \arrow[uu, "p_2"] \arrow[uur, "\tilde{p}_2"] \arrow[rr, "m'"'] & & \frac{G}{\mathrm{Ad}_{\mathrm{F}} B} \arrow[uu, "p_1"']
\end{tikzcd}$$

where the maps are the same as before and $\mathcal{Z}$ is the fiber product of both maps p_2 and pr. Note that the triangle $(q_1, \tilde{p}_2, q_2)$ is commutative. We then have

$$\begin{aligned} m_! \mathrm{pr}_* p_{2,*} &= m_! q_{2,*} q_1^* \\ &= m_! \tilde{p}_{2,*} q_{1,*} q_1^* \\ &= m_* \tilde{p}_{2,*} \\ &= p_{1,*} m'_* \\ &= p_{1,*} m'_!, \end{aligned}$$

where the first line follows from smooth base change, the second one from the commutativity of the tringle, the third one from the fact that q_1 is a U-torsor and thus the map $\mathrm{id} \to q_{1,*} q_1^*$ is an isomorphism and the fact that m is proper, the fourth one from the commutativity of the square $(\tilde{p}_2, m, p_1, m')$ and the last one from the properness of the map m'.

This in turns induces a morphism

$$T^R : \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell) \to \mathrm{Tr}(\mathrm{F}, \mathbb{H}), \tag{3.87}$$

which is right adjoint to 3.85. The composition $\mathrm{id} \to T^R T$ is computed as the colimit of the corresponding $\mathrm{id} \to p_{\bullet,*} p_\bullet^*$ all of which are isomorphism. This implies that T is fully faithful. Its essential image contains all the complexes $\mathrm{R}\Gamma_c(Y(\dot{w}), \overline{\mathbb{Z}}_\ell)$. By [BR03], these are known to generate the category $\mathrm{Perf}(\overline{\mathbb{Z}}_\ell[G^{\mathrm{F}}])$ and hence all of $\mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell)$. The functor T is therefore also essentially surjective and thus an equivalence. □

Lemma 3.5.21. *The exterior tensor product induces a natural equivalence*

$$\mathbb{H}^{\otimes_{\mathbb{H}_T} n} = \bigoplus_{\chi} \mathrm{D}_{\mathrm{indcons}}(U\backslash G/U \times^T U\backslash G/U \times^T \cdots \times^T U\backslash G/U, \Omega_T)_\chi. \tag{3.88}$$

Proof. First consider the functor

$$\begin{aligned} \mathbb{H} \otimes_{\mathrm{D}(\overline{\mathbb{Z}}_\ell)} \cdots \otimes_{\mathrm{D}(\overline{\mathbb{Z}}_\ell)} \mathbb{H} &\to \bigoplus_{\chi_1, \dots, \chi_n} \mathrm{D}_{\mathrm{indcons}}(U\backslash G/U \times \cdots \times U\backslash G/U, \Omega_{T^n})_{\chi_1, \dots, \chi_n} \\ A_1 \otimes \cdots \otimes A_n &\mapsto A_1 \hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} \dots \hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell} A_n, \end{aligned}$$

where the χ_i refers to the right action of T on the i-th factor of $U\backslash G/U$. Denote by p the projection

$$U\backslash G/U \times \cdots \times U\backslash G/U \to U\backslash G/U \times^T \cdots \times^T U\backslash G/U. \tag{3.89}$$

By the universal property of tensor products, there is a commutative diagram

$$\begin{CD} \mathbb{H} \otimes_{\mathrm{D}(\overline{\mathbb{Z}}_\ell)} \cdots \otimes_{\mathrm{D}(\overline{\mathbb{Z}}_\ell)} \mathbb{H} @>>> \bigoplus_{\chi_1, \dots, \chi_n} \mathrm{D}_{\mathrm{indcons}}(U\backslash G/U \times \cdots \times U\backslash G/U, \Omega_{T^n})_{\chi_1, \dots, \chi_n} \\ @VVV @VV{p_!}V \\ \mathbb{H} \otimes_{\mathbb{H}_T} \cdots \otimes_{\mathbb{H}_T} \mathbb{H} @>>> \bigoplus_{\chi_1, \dots, \chi_n} \mathrm{D}_{\mathrm{indcons}}(U\backslash G/U \times^T \cdots \times^T U\backslash G/U, \Omega_{T^n})_{\chi_1, \dots, \chi_n}. \end{CD}$$

We want to show that the bottom functor is an equivalence. Proceeding as in lemma 3.4.4, both sides are stratified with strata indexed by tuples $(w_1, \dots, w_n) \in W^n$. As the functor $\hat{\boxtimes}_{\overline{\mathbb{Z}}_\ell}$ commutes with the functors $i_*, i_!, i^*$ and $i^!$ for i the inclusion of a stratum, the gluing is immediate, it is then enough to show that the bottom map induces an equivalence on each strata. Let $(w_1, \dots, w_n) \in W^n$ be tuple and let $\chi_1, \dots, \chi_n \in \mathrm{CH}(T)$. Note that the category

$$\mathrm{D}_{\mathrm{indcons}}(U\backslash Bw_1B/U, \Omega_T)_{\chi_1} \otimes_{\mathbb{H}_T} \cdots \otimes_{\mathbb{H}_T} \mathrm{D}_{\mathrm{indcons}}(U\backslash Bw_nB/U, \Omega_T)_{\chi_n}, \tag{3.90}$$

is zero unless $\chi_{i-1} = w_i\chi_i$ for $i = 2, \dots, n$. Clearly the same applies to the category $\mathrm{D}_{\mathrm{indcons}}(U\backslash Bw_1B/U \times^T \cdots \times^T U\backslash Bw_nB/U, \Omega_{T^n})_{\chi_1,\dots,\chi_n}$. We now assume that $\chi_{i-1} = w_i\chi_i$ for $i = 2, \dots, n$. We can simplify the tensors as follows

$$\begin{aligned}&\mathrm{D}_{\mathrm{indcons}}(U\backslash Bw_1B/U, \Omega_T)_{\chi_1} \otimes_{\mathbb{H}_T} \cdots \otimes_{\mathbb{H}_T} \mathrm{D}_{\mathrm{indcons}}(U\backslash Bw_1B/U, \Omega_T)_{\chi_n} \\ &= \mathrm{D}_{\mathrm{indcons}}(U\backslash Bw_1B/U, \Omega_T)_{\chi_1} \otimes_{\mathbb{H}_{T,\chi_1}} \cdots \otimes_{\mathbb{H}_{T,\chi_{n-1}}} \mathrm{D}_{\mathrm{indcons}}(U\backslash Bw_nB/U, \Omega_T)_{\chi_n}.\end{aligned}$$

The right hand side is then equivalent to

$$\mathrm{D}(\Omega_T) \otimes_{\mathrm{D}(\Omega_T)} \cdots \otimes_{\mathrm{D}(\Omega_T)} \mathrm{D}(\Omega_T) \simeq \mathrm{D}(\Omega_T). \tag{3.91}$$

On the other hand after forgetting along the last inclusion, proceeding as in lemma 3.4.4, we have equivalences

$$\mathrm{D}_{\mathrm{indcons}}(U\backslash G/U \times^T \cdots \times^T U\backslash G/U, \Omega_{T^n})_{\chi_1,\dots,\chi_n} = \mathrm{D}(U\backslash G/U \times^T \cdots \times^T U\backslash G/U, \Omega_T)_{\chi_n}, \tag{3.92}$$

and

$$\begin{aligned}&\mathrm{D}_{\mathrm{indcons}}(U\backslash Bw_1B/U \times^T \cdots \times^T U\backslash Bw_nB/U, \Omega_{T^n})_{\chi_1,\dots,\chi_n} \\ &= \mathrm{D}_{\mathrm{indcons}}(U\backslash Bw_1B/U \times^T \cdots \times^T U\backslash Bw_nB/U, \Omega_T)_{\chi_n} = \mathrm{D}(\Omega_T).\end{aligned}$$

Using this second equivalence, we see that the bottom functor of the above diagram is an equivalence. Using the first equivalence, we get the desired equivalence of the lemma. □

Lemma 3.5.22. *There is a natural equivalence*

$$\mathrm{Hom}_{\mathbb{H}_T \otimes \mathbb{H}_T}(\mathbb{H}^{\otimes_{\mathbb{H}_T} n}, \mathbb{H}_{\mathrm{F}}) = \bigoplus_{\chi} \mathrm{D}_{\mathrm{indcons}}\Big(\frac{U\backslash G/U \times^T U\backslash G/U \times^T \cdots \times^T U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T}, \Omega_T\Big)_{\chi}. \tag{3.93}$$

Proof. Since the category $\mathbb{H}^{\otimes_{\mathbb{H}_T} n}$ is compactly generated, it is dualizable. The duality functor $\mathbb{D}^-$ defines a self duality on $\mathbb{H}^{\otimes_{\mathbb{H}_T} n}$. Furthermore, since the category $\mathbb{H}_T \otimes \mathbb{H}_T$ is quasi-rigid, by [BZN09] 3.14 the category $\mathbb{H}$ and thus $\mathbb{H}^{\otimes_{\mathbb{H}_T} n}$ is dualizable as an $\mathbb{H}_T \otimes \mathbb{H}_T$-module. Hence we get that

$$\mathrm{Hom}_{\mathbb{H}_T \otimes \mathbb{H}_T}(\mathbb{H}^{\otimes_{\mathbb{H}_T} n}, \mathbb{H}_{\mathrm{F}}) = \mathbb{H}^{\otimes_{\mathbb{H}_T} n} \otimes_{\mathbb{H}_T \otimes \mathbb{H}_T} \mathbb{H}_{\mathrm{F}}. \tag{3.94}$$

The duality $\mathbb{D}^-$ exchanges the left and right actions of T on the first factor and also twists them by $t \mapsto t^{-1}$. As previously, the category on the right hand side is identifies with the category of sheaves on the product of the spaces, that is sheaves on $U\backslash G/U \times^T U\backslash G/U \times^T \cdots \times^T U\backslash G/U$. By the same argument of lemma 3.5.21, taking invariants by $T \times T$ firstly contract above T for one of the action of T and for the second one takes the Ad_{F} invariants. □

3.5.5 Generation of the category $\mathrm{Perf}(\overline{\mathbb{Z}}_\ell[G^{\mathrm{F}}])$

The last argument of theorem 3.5.19 requires the input of the following theorem.

Theorem 3.5.23 ([BR03])**.** *The category* $\mathrm{Perf}(\overline{\mathbb{Z}}_\ell[G^{\mathrm{F}}])$ *is generated by the complexes* $\mathrm{R}\Gamma_c(Y(\dot{w}), \overline{\mathbb{Z}}_\ell)$*.*

The proof of *loc. cit.* reduces to the following statement of [DL76].

Theorem 3.5.24 ([DL76])**.** *For all irreducible* $\overline{\mathbb{Q}}_\ell$*-representations* ρ *of* G^{F}*, there exists* $w \in W$ *and* i *an integer such that*

$$\mathrm{Hom}^0(\rho, H^i_c(Y(\dot{w}), \overline{\mathbb{Q}}_\ell)) \neq 0. \tag{3.95}$$

We want to give a second proof of both these statements using a geometric argument. The strategy is to compute the functor $\mathrm{CH}_{\mathrm{F}}\mathrm{HC}_{\mathrm{F}}$ and show as in [MV88] that it is given by convolution against the Springer sheaf. We consider the category $\mathrm{D}_{\mathrm{cons}}(G, \Lambda)$. This category is equipped with the following convolution product. Consider the diagram

$$\begin{tikzcd} & G \times G \arrow[r, "m"] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & G \\ G & & G \end{tikzcd}$$

where m is the multiplication and p_i are the projections. Then for $A, B \in \mathrm{D}_{\mathrm{cons}}(G, \Lambda)$, the convolution is defined as

$$A * B = m_!(A \boxtimes B). \tag{3.96}$$

Consider the two categories $\mathrm{D}_{\mathrm{cons}}(\frac{G}{\mathrm{Ad}G}, \Lambda)$ and $\mathrm{D}_{\mathrm{cons}}(\frac{G}{\mathrm{Ad}_{\mathrm{F}}G}, \Lambda)$ where Ad and Ad_{F} denote the adjoint and F-adjoint action as before. The convolution structure induces a convolution structure on $\mathrm{D}_{\mathrm{cons}}(\frac{G}{\mathrm{Ad}G}, \Lambda)$ and a module structure over it on $\mathrm{D}_{\mathrm{cons}}(\frac{G}{\mathrm{Ad}_{\mathrm{F}}G}, \Lambda)$.

Definition 3.5.25. The (multiplicative) Springer resolution is the space $\tilde{\mathcal{U}} = \{(g, B_0), g \in U_{B_0}\} \subset G \times G/B$ where B_0 is a Borel subgroup and U_{B_0} is its unipotent radical. Let $s : \tilde{\mathcal{U}} \to G$ be the projection. The space $\tilde{\mathcal{U}}$ is equipped with the G-action induced by the adjoint action on G and the natural action on G/B. The Springer sheaf is defined as

$$\mathrm{Spr} = s_!\Lambda[\dim \tilde{\mathcal{U}}]. \tag{3.97}$$

The following lemma is well known, [BM83].

Lemma 3.5.26. *The sheaf* Spr *is a perverse sheaf equipped with a* W*-action and over* $\overline{\mathbb{Q}}_\ell$ *the sheaf* δ_1 *is a direct summand of* Spr*.*

Remark 3.5.27. There are two possible normalization of the Springer action on Spr. They differ by a twist by the sign representation of W. For one of them the irreducible representation yielding δ_1 is the trivial representation, for the other one, it is the sign representation.

Lemma 3.5.28. *There is an isomorphism of functors*

$$q_!r^*r_!q^* \simeq \mathrm{Spr}[-2\dim U] * -. \tag{3.98}$$

where r *and* q *are the maps defining the horocycle correspondence.*

Proof. The proof follows closely the argument of [MV88]. Consider the following diagram

$$\begin{tikzcd}
 & (G\times G)/\Delta_{\mathrm{F}}G & \\
(G\times G)/\Delta_{\mathrm{F}}G & Z \arrow[l, "p_1"'] \arrow[u, "p_2"'] \arrow[r] \arrow[d] & (G\times G)/\Delta_{\mathrm{F}}B \arrow[d] \\
 & (G\times G)/\Delta_{\mathrm{F}}B \arrow[r] & (G/U\times G/U)/\Delta_{\mathrm{F}}T
\end{tikzcd}$$

where Z is defined such that the bottom right square is Cartesian. Note that it is equipped with a G action such that the two maps p_1 and p_2 are equivariant for the diagonal left action of G on their target. Under the identification $\frac{G}{\mathrm{Ad}_{\mathrm{F}}G} \simeq \Delta G\backslash(G\times G)/\Delta_{\mathrm{F}}G$, the functor $q_!r^*r_!q^* = p_{2,!}p_1^*$. We identify $(G\times G)/\Delta_{\mathrm{F}}G \simeq G$ via $(x,y)\mapsto \mathrm{F}(x)y^{-1}$.

We compute the space Z, let $(g,g'),(h,h')\in(G\times G)/\Delta_{\mathrm{F}}B$ such that $(gU,g'U)=(hU,h'U)\mathrm{mod}\Delta_{\mathrm{F}}T$. Then there exists $t\in T, u,u'\in U$ such that $g=hut, g'=h'u'\mathrm{F}(t)$. Since the pair (h,h') is only considered up to $\Delta_{\mathrm{F}}B$, we can assume $g=h$ and $g'=h'u$ for some $u\in U$. There is an isomorphism $Z=Y/B$ where $Y=\{(g,g',h),h'^{-1}g\in U\}\subset G\times G\times G$ and B acts by $\mathrm{id}\times\mathrm{F}\times\mathrm{F}$. Under the identification $(G\times G)/\Delta_{\mathrm{F}}G\simeq G$, the maps p_1 and p_2 are then given as $p_1(g,g',h')=\mathrm{F}(g)g'^{-1}$ and $p_2(g,g',h')=\mathrm{F}(g)h'^{-1}$.

Consider $(p_1,p_2): Z\to G\times G$ be the product of the two maps. And let $K=(p_1,p_2)_!\Lambda$, this sheaf is G-equivariant on $G\times G$ for the diagonal action of G (not twisted by Frobenius) and we have a canonical isomorphism

$$p_{2,!}p_1^* = \mathrm{pr}_{2,!}(K\otimes \mathrm{pr}_1^*-), \tag{3.99}$$

where $\mathrm{pr}_i: G\times G\to G$ are the two projections. Recall $a: G\times G\to G$ is the map $(x,y)\mapsto x^{-1}y$. Using the $\Delta(G)$ equivariance of the sheaf K, there exists a sheaf K° and isomorphisms $K=a^*K^\circ$ and

$$\mathrm{pr}_{2,!}(K\otimes\mathrm{pr}_1^*-) = K^\circ * -. \tag{3.100}$$

We now compute the sheaf K°. Let $1\times G\subset G\times G$ be the inclusion of the second factor. Then $K^\circ=(1\times\mathrm{id})^*K$. Let Z° be the pullback of Z to $1\times G$.

$$\begin{tikzcd}
Z^\circ \arrow[r] \arrow[d, "q"'] & Z \arrow[d, "{(p_1,p_2)}"] \\
G \arrow[r, "{(1,\mathrm{id})}"'] & G\times G
\end{tikzcd}$$

Then Z° is the space of pairs $(g,h')\in(G\times G)/\Delta_{\mathrm{F}}B$ such that $h'^{-1}\mathrm{F}(g)\in U$ and the map $q(g,h)=\mathrm{F}(g)h'^{-1}$. Let $z_0=\mathrm{F}(g)h'^{-1}$, then $h'^{-1}\mathrm{F}(g)=\mathrm{F}(g^{-1})z_0\mathrm{F}(g)$. Let $Z'^\circ=\{(gB,h),h\in\mathrm{Ad}(\mathrm{F}(g))(U))$ and let $q': Z'^\circ\to G$ be the second projection. Then the map $Z^\circ\to Z'^\circ,(g,h')\mapsto(gB,\mathrm{F}(g)h'^{-1})\}\subset G/B\times G$ is an isomorphism over G. Finally the map $\tilde{\mathcal{U}}\to Z'^\circ$ given by $(gB,h)\mapsto(\mathrm{F}(g)B,h)$ is a universal homeomorphism, hence $K^\circ=q_!\Lambda\simeq\mathrm{Spr}[-2\dim U]$.

$$\begin{tikzcd}
Z^\circ \arrow[r] \arrow[dr, "q"'] & Z'^\circ \arrow[d, "q'"] & \tilde{\mathcal{U}} \arrow[l] \arrow[dl] \\
 & G &
\end{tikzcd}$$

□

Lemma 3.5.29. *Over $\overline{\mathbb{F}}_\ell$, the simple perverse sheaf δ_1 is a subquotient of* Spr.

Proof. The Sringer sheaf is a perverse sheaf on the stack $\frac{\mathcal{U}}{\mathrm{Ad}(G)}$, where $\mathcal{U}$ denotes the unipotent variety of G. It is known that there are finitely many irreducible perverse sheaves on this stack up to isomorphism with coefficients in either $\overline{\mathbb{Q}}_\ell$ or $\overline{\mathbb{F}}_\ell$. We are interested in the multiplicity $[\mathrm{Spr}_{\overline{\mathbb{F}}_\ell} : \delta_{1,\overline{\mathbb{F}}_\ell}]$ where the indexed $\overline{\mathbb{F}}_\ell$ is here to indicate over which coefficients we consider this multiplicity. Let

$$d : K_0(\mathrm{Perv}(\frac{\mathcal{U}}{\mathrm{Ad}(G)}, \overline{\mathbb{Q}}_\ell)) \to K_0(\mathrm{Perv}(\frac{\mathcal{U}}{\mathrm{Ad}(G)}, \overline{\mathbb{F}}_\ell)), \tag{3.101}$$

be the decomposition morphism, we refer to [Jut09] for a discussion about this morphism. We write $[\mathrm{Spr}_{\overline{\mathbb{Q}}_\ell}]$ (resp. $[\mathrm{Spr}_{\overline{\mathbb{F}}_\ell}]$) for the image of $\mathrm{Spr}_{\overline{\mathbb{Q}}_\ell}$ (resp. $\mathrm{Spr}_{\overline{\mathbb{F}}_\ell}$) in his K_0. It is then enough to check that in the basis of $K_0(\mathrm{Perv}(\frac{\mathcal{U}}{\mathrm{Ad}(G)}, \overline{\mathbb{F}}_\ell))$ indexed by irreducible objects, we have $[\mathrm{Spr}_{\overline{\mathbb{F}}_\ell}] = a[\delta_{1,\overline{\mathbb{F}}_\ell}] + \sum_{\rho \neq \delta_{1,\overline{\mathbb{F}}_\ell}} a_\rho \rho$, where ρ ranges through the irreducible objects in $\mathrm{Perv}(\frac{\mathcal{U}}{\mathrm{Ad}(G)}, \overline{\mathbb{F}}_\ell)$ different from $\delta_{1,\overline{\mathbb{F}}_\ell}$, that a is a positive integer. We have $[\mathrm{Spr}_{\overline{\mathbb{F}}_\ell}] = d([\mathrm{Spr}_{\overline{\mathbb{Q}}_\ell}])$ and in the basis composed of irreducible objects, the matrix of d has nonnegative entries. □

Proof of theorems 3.5.24 and 3.5.23. The following argument already appeared in [BBM04b]. By lemmas 3.5.28 and 3.5.29, the identity functor is subquotient functor of $\mathrm{CH}_\mathrm{F}\mathrm{HC}_\mathrm{F}$ over both $\overline{\mathbb{Q}}_\ell$ and $\overline{\mathbb{F}}_\ell$. This implies in particular that the functor HC_F is conservative, over any coefficient ring. We want to show that the category $\mathrm{Perf}(\Lambda[G^\mathrm{F}])$ is generated by the complexes $\mathrm{R}\Gamma_c(Y(\dot{w}), \Lambda)$. Arguing as in [BR03], it is enough to show that for all irreducible representations ρ either over $\overline{\mathbb{Q}}_\ell$ or over $\overline{\mathbb{F}}_\ell$, there exists $w \in W$ such that $\mathrm{Hom}(\mathrm{R}\Gamma_c(Y(\dot{w}, \Lambda), \rho) \neq 0$. But we have by adjunctions

$$\begin{aligned}\mathrm{Hom}(\mathrm{R}\Gamma_c(Y(\dot{w}, \Lambda), \rho) &= \mathrm{Hom}(q_! r^* i_{w,!} k_w^* \Lambda[T^{w\mathrm{F}}], \rho) \\ &= \mathrm{Hom}(\Lambda[T^{w\mathrm{F}}], k_{w,*} i_w^! r_* q^! \rho).\end{aligned}$$

By Verdier duality the functor $r_* q^!$ is conservative. There is thus one of the costalks of $r_* q^! \rho$ that is nonzero. For such a costalk, $k_{w,*} i_w^! r_* q^! \rho$ is nonzero and therefore $\mathrm{Hom}(\Lambda[T^{w\mathrm{F}}], k_{w,*} i_w^! r_* q^! \rho)$ is nonzero. □

The following corollary is a consequence of the above proof of theorem 3.5.23.

Corollary 3.5.30. *The functor* $\mathrm{HC}_\mathrm{F} : \mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^\mathrm{F}, \overline{\mathbb{Z}}_\ell) \to \mathrm{D}_{\mathrm{cons}}(\frac{U\backslash G/U}{\mathrm{Ad}_\mathrm{F} T}, \overline{\mathbb{Z}}_\ell)$ *is conservative.*

3.5.6 Lusztig's ℓ-series

As a corollary of theorem 3.5.19, we explain how to recover Lusztig's geometric ℓ-series, see [BR03] Section 8. Denote by $\mathrm{CH}(T)/W$ the set of W-orbits of $\mathrm{CH}(T)$.

Theorem 3.5.31. *For each W-orbit $\mathfrak{o} \subset \mathrm{CH}(T)/W$ such that $\mathrm{F}(\mathfrak{o}) = \mathfrak{o}$, there exists a central idempotent $e_\mathfrak{o} \in \Lambda[G^\mathrm{F}]$ satisfying the following properties.*

- (i). *The collection of all $e_\mathfrak{o}$ for $\mathfrak{o}$ ranging trough the set of orbits such that $\mathrm{F}(\mathfrak{o}) = \mathfrak{o}$ is a complete set of orthogonal idempotents in $\overline{\mathbb{Z}}_\ell[G^\mathrm{F}]$.*
- (ii). *Let ρ be an irreducible $\overline{\mathbb{F}}_\ell$-representation of G^F, then $e_\mathfrak{o}\rho = \rho$ if and only if there exists a pair (w, χ) with $w \in W$ and $\chi \in \mathfrak{o}$, such that the Deligne-Lusztig restriction ${}^*\mathrm{R}_{w,\chi}(\rho) \neq 0$.*

Before giving a proof of the theorem we first need the following classical lemma.

Lemma 3.5.32. *Let A be a ring. There exists a natural isomorphism of rings*

$$Z(A) = \mathrm{End}^0(\mathrm{id}_{\mathrm{D}(A)}), \tag{3.102}$$

between the center of A and the algebra of endomorphisms of the identity functor of $\mathrm{D}(A)$.

Proof. Let us construct the bijection by hand. Let $a : \mathrm{id}_{\mathrm{D}(A)} \to \mathrm{id}_{\mathrm{D}(A)}$ be an endomorphism of the identity functor. Then evaluating it at $A \in \mathrm{D}(A)$ yields an endomorphism $z \in \mathrm{End}^0_{\mathrm{D}(A)}(A) = A$. It remains to check that it is central. Let $f \in A$ then the multiplication by f is an endomorphism of A, as a is a natural transformation the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "z"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ A \arrow[r, "z"'] & A \end{tikzcd}$$

and therefore $fz = zf$ and thus z is central in A.

Conversely let $z \in Z(A)$, then the left multiplication by z defines an endomorphism of all A-modules. After passing to the derived category this defines an endomorphism of all object $M \in \mathrm{D}(A)$ functorial in M hence an endomorphism of the identity functor. It is clear that the two constructions are inverse of each other and that they are morphisms of algebras. $\square$

Remark 3.5.33. Let A be a ring and assume that we have a direct sum decomposition $\mathrm{D}(A) = \mathcal{C}_1 \oplus \mathcal{C}_2$. Then the identity functor decomposes as $\mathrm{id}_{\mathrm{D}(A)} = \mathrm{id}_{\mathcal{C}_1} \oplus \mathrm{id}_{\mathcal{C}_2}$ and the morphism $\mathrm{id}_{\mathrm{D}(A)} \to \mathrm{id}_{\mathcal{C}_1} \to \mathrm{id}_{\mathrm{D}(A)}$ induced by the projection and the inclusion yields an endomorphism of the identity functor $\mathrm{id}_{\mathrm{D}(A)}$. This endomorphism is idempotent hence the corresponding element of $Z(A)$ is a also idempotent.

Proof of theorem 3.5.31. First note that we can write $\mathbb{H}$ as a direct sum as follows

$$\mathbb{H} = \bigoplus_{\mathfrak{o} \in \mathrm{CH}(T)/W} \mathbb{H}_{\mathfrak{o}} \tag{3.103}$$

where $\mathbb{H}_{\mathfrak{o}} = \bigoplus_{\chi} \mathbb{H}_{\chi}$. The Frobenius acts on the set $\mathrm{CH}(T)/W$, let $A \subset \mathrm{CH}(T)/W$ be an orbit of the Frobenius and denote by $\mathbb{H}_A = \bigoplus_{\mathfrak{o} \in A} \mathbb{H}_{\mathfrak{o}}$. This category is a monoidal subcategory of $\mathbb{H}$ and F^* acts on $\mathbb{H}_A$ for all A. The theorem is now a consequence of the following facts

(i). there is a direct sum decomposition

$$\mathrm{Tr}(\mathrm{F}^*, \mathbb{H}) = \bigoplus_{A} \mathrm{Tr}(\mathrm{F}^*, \mathbb{H}_A), \tag{3.104}$$

where A ranges through the collection of all F-orbits in $\mathrm{CH}(T)/W$.

(ii). If A is a F-orbit in $\mathrm{CH}(T)/W$ that is not reduced to a single element then $\mathrm{Tr}(\mathrm{F}^*, \mathbb{H}_A) = 0$.

Assuming these two points, let us show the theorem. We have

$$\begin{aligned}\mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell) &= \mathrm{Tr}(\mathrm{F}^*, \mathbb{H}) \\ &= \bigoplus_A \mathrm{Tr}(\mathrm{F}^*, \mathbb{H}_A) \\ &= \bigoplus_{\mathfrak{o}, \mathrm{F}(\mathfrak{o}) = \mathfrak{o}} \mathrm{Tr}(\mathrm{F}^*, \mathbb{H}_{\mathfrak{o}}).\end{aligned}$$

By remark 3.5.33, corresponding to each orbit $\mathfrak{o}$ such that $\mathrm{F}(\mathfrak{o}) = \mathfrak{o}$ there is a central idempotent $e_{\mathrm{o}} \in Z(\overline{\mathbb{Z}}_\ell[G^{\mathrm{F}}])$ such that for all $M \in \mathrm{D}_{\mathrm{indcons}}(\mathrm{pt}/G^{\mathrm{F}}, \overline{\mathbb{Z}}_\ell)$, we have $e_{\mathfrak{o}} M = M$ if and only if $M \in \mathrm{Tr}(\mathrm{F}^*, \mathbb{H}_{\mathfrak{o}})$. Furthermore the collection of all $e_{\mathfrak{o}}$ is a complete collection of orthogonal idempotents of $\overline{\mathbb{Z}}_\ell[G^{\mathrm{F}}]$. It remains to check that this implies the desired property on irreducible representations. Let $\rho \in \mathrm{Tr}(\mathrm{F}^*, \mathbb{H}_{\mathfrak{o}})$ be an irreducible $\overline{\mathbb{F}}_\ell$-representation and consider $\mathrm{HC}_{\mathrm{F}}(\rho)$. By theorem 3.5.19, $p^! \mathrm{HC}_{\mathrm{F}}(\rho)$ is an object of $\mathbb{H}_{\mathfrak{o}}$. By corollary 3.5.30, the functor HC_{F} is conservative hence $\mathrm{HC}_{\mathrm{F}}(\rho)$ is non-zero, and therefore there exists (w, χ), with $\chi \in \mathfrak{o}$, such that $\mathrm{Hom}_{\overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}]}(\chi, k_{w,*} i_w^! \mathrm{HC}_{\mathrm{F}}(\rho)) \neq 0$. By lemma 3.3.8, this implies that ${}^*\mathrm{R}_{w,\chi}(\rho) \neq 0$.

Let us now prove the two claims. Firstly let us recall that Lurie's tensor product of categories commutes with colimits in both variables [Lur] 4.8.1.24. Therefore we have

$$\begin{aligned}\mathrm{Tr}(\mathrm{F}^*, \mathbb{H}) &= \mathbb{H} \otimes_{\mathbb{H} \otimes \mathbb{H}^{\mathrm{rev}}} \mathbb{H}_{\mathrm{F}} \\ &= \bigoplus_{A,B} \mathbb{H}_A \otimes_{\mathbb{H} \otimes \mathbb{H}^{\mathrm{rev}}} \mathbb{H}_{B,\mathrm{F}}.\end{aligned}$$

Futhermore if $A \neq B$ then $\mathbb{H}_A \otimes_{\mathbb{H} \otimes \mathbb{H}^{\mathrm{rev}}} \mathbb{H}_{B,\mathrm{F}} = 0$ and for $A = B$, we have $\mathbb{H}_A \otimes_{\mathbb{H} \otimes \mathbb{H}^{\mathrm{rev}}} \mathbb{H}_{B,\mathrm{F}} = \mathrm{Tr}(\mathrm{F}, \mathbb{H}_A)$. This yields (i).

For the second point, let A be an F-orbit in $\mathrm{CH}(T)/W$. Assume that A is not reduced to a single element and denote by $A = \{\mathfrak{o}_1, \dots, \mathfrak{o}_n\}$ the elements of A ordered such that $\mathrm{F}(\mathfrak{o}_i) = \mathfrak{o}_{i+1}$. We then have

$$\mathrm{Tr}(\mathrm{F}^*, \mathbb{H}_A) = \oplus_{i,j} \mathbb{H}_{\mathfrak{o}_i} \otimes_{\mathbb{H} \otimes \mathbb{H}^{\mathrm{rev}}} \mathbb{H}_{\mathfrak{o}_j, \mathrm{F}}.$$

But $\mathbb{H}_{\mathfrak{o}_i} \otimes_{\mathbb{H}} \mathbb{H}_{\mathfrak{o}_j} = 0$ with respect to the left action of $\mathbb{H}$ if $i \neq j$. Similarly, $\mathbb{H}_{\mathfrak{o}_i} \otimes_{\mathbb{H}^{\mathrm{rev}}} \mathbb{H}_{\mathfrak{o}_j, \mathrm{F}} = 0$ if $i+1 \neq j \bmod n$. As these two conditions are mutually exclusive if $n > 1$, we have $\mathrm{Tr}(\mathrm{F}, \mathbb{H}_A) = 0$. ☐

3.6 Endomorphism of the Gelfand-Graev representation

We keep the notations of the previous sections, we further assume that G has connected center and that ℓ is good so that we can use the global $\mathbb{V}$ functor. The sheaf $\mathcal{L}_\psi$ on $\bar{U}$ is equipped with an $\mathbb{F}_q$-structure and the trace of Frobenius function corresponding to it is a generic character of the group $\bar{U}^{\mathrm{F}}$ which we still denote by ψ. We let e_ψ be the idempotent

$$\frac{1}{|\bar{U}^{\mathrm{F}}|} \sum_{\bar{U}^{\mathrm{F}} \bar{U}} \psi(u^{-1}) u \in \mathbb{Z}[\zeta_p][\frac{1}{p}][\bar{U}^{\mathrm{F}}],$$

where ζ_p is a primitive p-th roof of 1.

We also denote by $\Gamma_\psi = \mathrm{ind}_{\bar{U}^{\mathrm{F}}}^{G^{\mathrm{F}}} \psi$ the corresponding Gelfand-Graev representation. On the dual side consider the dual torus $T^\vee$ over $\mathbb{Z}_\ell$. The Frobenius of T induces an endomorphism $\mathrm{F}^\vee$ which we still call the Frobenius. Consider the scheme $(T^\vee /\!/ W)^{\mathrm{F}^\vee}$ defined as the intersection of the diagonal and the graph of $\mathrm{F}^\vee$ in $T^\vee /\!/ W \times T^\vee /\!/ W$, that is the scheme fitting into the following cartesian diagram

$$\begin{tikzcd}
(T^\vee /\!/ W)^{\mathrm{F}^\vee} \arrow[r] \arrow[d] & T^\vee /\!/ W \arrow[d, "\Delta"] \\
T^\vee /\!/ W \arrow[r, "(\mathrm{id}\times\mathrm{F}^\vee)"'] & T^\vee /\!/ W \times T^\vee /\!/ W.
\end{tikzcd}$$

The main goal of this section is to give a new geometric proof of the following theorem.

Theorem 3.6.1 ([Li21] 0.3 and [LS22] Main theorem)**.** *Assume that G has connected center and that ℓ is good for G. There is an isomorphism of algebras*

$$\mathrm{End}_{G^{\mathrm{F}}}(\Gamma_\psi) = \mathcal{O}((T^\vee /\!/ W)^{\mathrm{F}^\vee}) \tag{3.105}$$

Let us outline our construction. We first recall the construction of the Curtis morphisms. The modern way to define them is through the computation of the Deligne-Lusztig restriction of the Gelfand-Graev representation.

Theorem 3.6.2 ([Dud09])**.** *Let $w \in W$, there is a $T^{w\mathrm{F}}$-equivariant isomorphism*

$$\mathrm{R}\Gamma_c(Y(\dot{w}), \overline{\mathbb{Z}}_\ell) \otimes_{G^{\mathrm{F}}} \Gamma_\psi \simeq \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}][-\ell(w)].$$

This isomorphism then induces a canonical map

$$\mathrm{Cur}_w : \mathrm{End}(\Gamma_\psi) \to \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}] = \mathrm{End}_{T^{w\mathrm{F}}}(\overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}][-\ell(w)]) \tag{3.106}$$

which we call the w-Curtis morphism. The full Curtis morphism is then defined as the direct sum of all w-Curtis morphisms,

$$\mathrm{Cur} : \mathrm{End}(\Gamma_\psi) \xrightarrow{\oplus \mathrm{Cur}_w} \oplus_w \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}]. \tag{3.107}$$

Consider the following commutative diagram

$$\begin{tikzcd}
T^\vee \arrow[r, "w\mathrm{F}^\vee"] \arrow[d] & T^\vee \arrow[d] \\
T^\vee /\!/ W \arrow[r, "\mathrm{F}^\vee"'] & T^\vee /\!/ W
\end{tikzcd}$$

where the vertical maps are the quotient maps. This diagram is commutative and thus induces a map on fixed points

$$(T^\vee)^{w\mathrm{F}^\vee} \to (T^\vee /\!/ W)^{\mathrm{F}^\vee}.$$

Taking rings of global sections, we have a map, which we call the spectral Curtis morphism

$$\mathrm{Cur}_w^{\mathrm{spec}} : \mathcal{O}_{(T^\vee /\!/ W)^{F^\vee}} \to \mathcal{O}_{(T^\vee)^{w\mathrm{F}^\vee}} = \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}]. \tag{3.108}$$

We show the following theorem

Theorem 3.6.3. *Assume that G has connected center and that ℓ is good for G. Then the map* Cur *factors through* $\oplus_w \mathrm{Cur}_w^{\mathrm{spec}}$ *and we have a commutative diagram*

$$\begin{tikzcd} & \mathrm{End}(\Gamma_\psi) \arrow[dl] \arrow[dr, "\mathrm{Cur}"] & \\ \mathcal{O}_{(T^\vee /\!\!/ W)^{F^\vee}} \arrow[rr, "\oplus_w \mathrm{Cur}_w^{\mathrm{spec}}"'] & & \oplus_w \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}]. \end{tikzcd}$$

Once this factorization is constructed, we show by a standard argument of symmetrizing forms that the map $\mathrm{End}(\Gamma_\psi) \to \mathcal{O}_{(T^\vee /\!\!/ W)^{F^\vee}}$ is an isomorphism.

Remark 3.6.4. Let us highlight the difference between our construction and the constructions of [Li21] and [LS22]. It is known that after inverting ℓ both algebras are isomorphic, the problem then lies in comparing two $\overline{\mathbb{Z}}_\ell$ -lattices inside. The key idea in *loc. cit.* is to show that for suitable bases, the matrix giving the isomorphism is in fact defined over $\overline{\mathbb{Z}}_\ell$. The proof is then a difficult computation to check that this property holds, in particular it uses non trivial facts about almost characters. Our construction on the other hand is purely geometric and free of Lusztig's classification.

Remark 3.6.5. In the proof of 3.6.3, we will also deduce a new proof of theorem 3.6.2.

3.6.1 Intersection with the graph of Frobenius on Soergel bimodules.

We have defined Soergel bimodules as certain coherent sheaves on $\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T)$. Consider the dual torus $T^\vee$ over $\mathbb{Z}_\ell$. Recall that there is a canonical isomorphism between $\mathcal{C}(T)$ and a disjoint union of completions of $T^\vee$, hence there is a canonical map :

$$\mathrm{can} : \mathcal{C}(T) \to T^\vee_{\overline{\mathbb{Z}}_\ell}.$$

Consider now the space $T^\vee \times T^\vee$ over $\overline{\mathbb{Z}}_\ell$ and denote by $\Gamma_{\mathrm{F}^\vee}$ the closed subscheme equal to the graph of $\mathrm{F}^\vee$. This is the image of $T^\vee$ under $\mathrm{id} \times \mathrm{F}^\vee$. We also denote by $\tilde{\Gamma}_{\mathrm{F}^\vee} \subset \mathcal{C}(T) \times \mathcal{C}(T)$ the graph of $\mathrm{F}^\vee$. Note that we have

$$\mathcal{O}_{\tilde{\Gamma}_{\mathrm{F}^\vee}} = (\mathrm{can} \times \mathrm{can})^* \mathcal{O}_{\Gamma_{\mathrm{F}^\vee}}.$$

Lemma 3.6.6. *We have an isomorphism of functors* $\mathbb{H}^\omega \to \mathrm{D}_{\mathrm{coh}}(\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T))$,

$$\mathbb{V}(-) \otimes_{\mathcal{O}_{\mathcal{C}(T) \times \mathcal{C}(T)}} \mathcal{O}_{\tilde{\Gamma}_{\mathrm{F}^\vee}} = \mathbb{V}(p^! p_! -).$$

Proof. The statement is an analog of [BT22] 4.4. Let $A \in \mathrm{D}_{\mathrm{cons}}(U \backslash G/U, \Omega_T)_\chi$. All the sheaves $\Delta_{w,\chi}$ are T-monodromic for the action of T given by Ad_{F} hence so is A. Suppose A is unipotent monodromic for this action then $p^! p_! A = A \otimes_{\Omega_T} \overline{\mathbb{Z}}_\ell$ by remark 3.2.49. In general A splits as $A = \oplus_\chi A_\chi$ such that each A_χ is χ-monodromic. Applying $p_!$ to a non-unipotent monodromic sheaf kills it hence $p^! p_! A = A \otimes_{\mathcal{O}_{\mathcal{C}(T)}} \overline{\mathbb{Z}}_\ell$ where $\mathcal{O}_{\mathcal{C}(T)} \to \overline{\mathbb{Z}}_\ell$ is the augmentation of the component corresponding to the trivial character. Consider the following Cartesian diagram,

$$\begin{tikzcd} T^\vee & T^\vee \times T^\vee \arrow[l, "t"'] \\ 1 \arrow[u] & T^\vee \arrow[l] \arrow[u, "\Gamma_{\mathrm{F}^\vee}"'] \end{tikzcd}$$

where $t(x,y) = x^{-1}\mathrm{F}^\vee(y)$. The functor $\mathbb{V}$ is $\Omega_T \otimes \Omega_T$-linear. We then have

$$\begin{aligned}
\mathbb{V}(-) \otimes_{\Omega_T \otimes \Omega_T} \mathcal{O}_{\tilde{\Gamma}_{\mathrm{F}^\vee}} &= \mathbb{V}(- \otimes_{\mathcal{O}_{\mathcal{C}(T)\times\mathcal{C}(T)}} \mathcal{O}_{\tilde{\Gamma}_{\mathrm{F}}}) \\
&= \mathbb{V}(- \otimes_{\mathcal{O}_{\mathcal{C}(T)}} \overline{\mathbb{Z}}_\ell) \\
&= \mathbb{V}(p^! p_! -),
\end{aligned}$$

where the third line come from the cartesianity of the above diagram. $\square$

Lemma 3.6.7. *There is a canonical W-action on $\mathcal{O}_{\mathcal{C}(T)\times_{\mathcal{C}(T)/\!\!/W}\mathcal{C}(T)} \otimes_{\mathcal{O}_{\mathcal{C}(T)\times\mathcal{C}(T)}} \mathcal{O}_{\tilde{\Gamma}_{\mathrm{F}^\vee}}$ and we have an isomorphism*

$$(\mathcal{O}_{\mathcal{C}(T)\times_{\mathcal{C}(T)/\!\!/W}\mathcal{C}(T)} \otimes_{\mathcal{O}_{\mathcal{C}(T)\times\mathcal{C}(T)}} \mathcal{O}_{\tilde{\Gamma}_{\mathrm{F}}})^W \simeq \mathcal{O}_{(T^\vee/\!\!/W)^{\mathrm{F}^\vee}}.$$

Proof. We will do the proof in two steps, firstly we will prove an analog statement for $T^\vee$ in place of $\mathcal{C}(T)$ and then use the flatness of can and an analog of [BR22b] 8.5 to pass from $T^\vee$ to $\mathcal{C}(T)$.

Let Z the derived scheme obtained as the derived intersection of $T^\vee \times_{T^\vee/\!\!/W} T^\vee$ and the graph of $\mathrm{F}^\vee$,

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & T^\vee \arrow[d, "(\mathrm{id}\times\mathrm{F}^\vee)"] \\
T^\vee \times_{T^\vee/\!\!/W} T^\vee \arrow[r, "i"'] & T^\vee \times T^\vee
\end{tikzcd}$$

where i is the closed immersion. By definition of $\mathcal{O}_{\Gamma_{\mathrm{F}^\vee}}$ we have an isomorphism of $\mathcal{O}_{T^\vee\times T^\vee}$-modules

$$\mathcal{O}_Z = \mathcal{O}_{T^\vee\times_{T^\vee/\!\!/W}T^\vee} \otimes_{\mathcal{O}_{T^\vee\times T^\vee}} \mathcal{O}_{\Gamma_{\mathrm{F}^\vee}}.$$

Consider the natural action of $W \times W$ on $T^\vee \times T^\vee$, the closed subscheme $T^\vee \times_{T^\vee/\!\!/W} T^\vee$ is stable under this action. The closed subspace $T^\vee$ embedded via the graph of $\mathrm{F}^\vee$ is stable under the action of W obtained by restriction along $(\mathrm{id}\times\mathrm{F}) : W \to W \times W$. Consider further the diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] \arrow[ddr] & T^\vee \arrow[d, dashed, "(\mathrm{id}\times\mathrm{F}^\vee)"] \arrow[ddr] & \\
T^\vee \times_{T^\vee/\!\!/W} T^\vee \arrow[r] \arrow[ddr] & T^\vee \times T^\vee \arrow[ddr] & \\
& (T^\vee /\!\!/ W)^{\mathrm{F}^\vee} \arrow[r] \arrow[d] & T^\vee /\!\!/ W \arrow[d, dashed, "(\mathrm{id}\times\mathrm{F})"] \\
& T^\vee /\!\!/ W \arrow[r, "\Delta"'] & T^\vee /\!\!/ W \times T^\vee /\!\!/ W
\end{tikzcd}$$

where Δ is the diagonal and the long arrows are induced by the natural map $T^\vee \to T^\vee /\!\!/ W$. We claim that the top face is Cartesian. By construction the back, front and bottom are Cartesian. In particular the composition of the top and front faces is Cartesian, to deduce that the top face is Cartesian it is enough to know that the map $(T^\vee /\!\!/ W)^{\mathrm{F}^\vee} \to T^\vee /\!\!/ W$ is a monomorphism, but this map is a closed immersion as it is obtained via the pullback of one and is therefore a monomorphism.

We furthermore want to identify $Z /\!\!/ W$ with $(T^\vee /\!\!/ W)^{\mathrm{F}^\vee}$. First note that $T^\vee \to T^\vee /\!\!/ W$ is faithfully flat and finite of rank $|W|$ then so is $Z \to (T^\vee /\!\!/ W)^{\mathrm{F}^\vee}$, in particular Z is a finite flat $\overline{\mathbb{Z}}_\ell$-scheme. The claim is clear after inverting ℓ, we now have the following diagram

$$\begin{tikzcd}
\mathcal{O}_{(T^\vee /\!\!/ W)^{\mathrm{F}}} \arrow[rr] \arrow[d] & & \mathcal{O}_{(T^\vee /\!\!/ W)^{\mathrm{F}}}[\frac{1}{\ell}] \arrow[d] \\
(\mathcal{O}_{T^\vee} \otimes_{\mathcal{O}_{T^\vee /\!\!/ W}} \mathcal{O}_{(T^\vee /\!\!/ W)^{\mathrm{F}}})^W \arrow[rr] & & (\mathcal{O}_{T^\vee} \otimes_{\mathcal{O}_{T^\vee /\!\!/ W}} \mathcal{O}_{(T^\vee /\!\!/ W)^{\mathrm{F}}})^W[\frac{1}{\ell}].
\end{tikzcd}$$

We first claim that the scheme $(T^\vee /\!\!/ W)^{\mathrm{F}^\vee}$ is underived. This intersection over $\overline{\mathbb{Z}}_\ell$ is the base change of the same intersection over $\mathbb{Z}_\ell$. Over $\mathbb{Z}_\ell$, both copies of $(T^\vee /\!\!/ W)$ are regularly immersed in $(T^\vee /\!\!/ W) \times (T^\vee /\!\!/ W)$ via the diagonal and the graph of $\mathrm{F}^\vee$ since $T^\vee /\!\!/ W$ is regular by the Pittie-Steinberg theorem. By [Li21] 3.16, we know that the classical part of $(T^\vee /\!\!/ W)^{\mathrm{F}^\vee}$ is finite free over $\overline{\mathbb{Z}}_\ell$, hence the underived intersection has the expected codimension. This is enough to imply that the intersection is itself underived. Indeed any regular sequence that determines locally $\Gamma_{\mathrm{F}^\vee}$ is still regular when restricted to the diagonal by the Cohen-Macaulay property [Aut] Tag 02JN. As the map $Z \to (T^\vee /\!\!/ W)^{\mathrm{F}^\vee}$ is faithfully flat, the scheme Z is also underived. We are dealing with classical finite free $\overline{\mathbb{Z}}_\ell$-algebra.

After inverting ℓ, the map $Z /\!\!/ W \to (T^\vee /\!\!/ W)^{\mathrm{F}^\vee}$ is an isomorphism by the exactness of the functor of W-invariants. This implies that over $\overline{\mathbb{Z}}_\ell$ both algebras have the same rank. It is then enough to show that the map $\mathcal{O}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee}) \to \mathcal{O}(Z /\!\!/ W)$ is ℓ-saturated. Since we have a commutative diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[dr] & Z /\!\!/ W \arrow[d] \\
& (T^\vee /\!\!/ W)^{\mathrm{F}^\vee}
\end{tikzcd}$$

it is enough to show that the map $\mathcal{O}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee}) \to \mathcal{O}(Z)$ is ℓ-saturated. Over $\overline{\mathbb{F}}_\ell$, the map $\mathcal{O}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee}) \to \mathcal{O}(Z)$ is faithfully flat hence injective and thus our original map is ℓ-saturated.

We now pass to $\mathcal{C}(T)$, the statement will follow from the fact that the following diagram is Cartesian.

$$\begin{tikzcd}
\mathcal{C}(T) \times_{\mathcal{C}(T) /\!\!/ W} \mathcal{C}(T) \arrow[r] \arrow[d] & \mathcal{C}(T) \times \mathcal{C}(T) \arrow[d] \\
T^\vee \times_{T^\vee /\!\!/ W} T^\vee \arrow[r] & T^\vee \times T^\vee
\end{tikzcd}$$

Let us assume this and show the lemma. Firstly this implies that

$$(\mathrm{can} \times \mathrm{can})^* \mathcal{O}_Z = \mathcal{O}_{\mathcal{C}(T) \times_{\mathcal{C}(T) /\!\!/ W} \mathcal{C}(T)} \otimes_{\mathcal{O}_{\mathcal{C}(T) \times \mathcal{C}(T)}} \mathcal{O}_{\tilde{\Gamma}_{\mathrm{F}^\vee}}.$$

Since the map can $\times$ can is compatible with the $W \times W$ action on both the source and target, we get the desired action on $\mathcal{O}_{\mathcal{C}(T) \times_{\mathcal{C}(T) /\!\!/ W} \mathcal{C}(T)} \otimes_{\mathcal{O}_{\mathcal{C}(T) \times \mathcal{C}(T)}} \mathcal{O}_{\tilde{\Gamma}_{\mathrm{F}^\vee}}$. Moreover since can is flat it commutes with taking invariants under W, hence after taking invariants we get an isomorphism

$$(\mathcal{O}_{\mathcal{C}(T) \times_{\mathcal{C}(T) /\!\!/ W} \mathcal{C}(T)} \otimes_{\mathcal{O}_{\mathcal{C}(T) \times \mathcal{C}(T)}} \mathcal{O}_{\tilde{\Gamma}_{\mathrm{F}^\vee}})^W \simeq \mathcal{O}_{(\mathcal{C}(T) /\!\!/ W)^{\mathrm{F}}}.$$

But now we have a natural morphism $(\mathcal{C}(T) /\!\!/ W)^{\mathrm{F}} \to (T^\vee /\!\!/ W)^{\mathrm{F}}$, and we show that it is an isomorphism, consider the following diagram

$$\begin{tikzcd}
 & \mathcal{C}(T) \times_{\mathcal{C}/\!\!/W} \mathcal{C}(T) \arrow[rr] \arrow[d] & & \mathcal{C}(T) \times \mathcal{C}(T) \arrow[d] \\
 & T^\vee \times_{T^\vee/\!\!/W} T^\vee \arrow[rr] & & T^\vee \times T^\vee \\
Z' \arrow[rr] \arrow[d] \arrow[uur] & & \mathcal{C}(T) \arrow[d] \arrow[uur] & \\
Z \arrow[rr] \arrow[uur] & & T^\vee \arrow[uur] &
\end{tikzcd}$$

where Z' is defined to make the top square Cartesian. By hypothesis the back square is Cartesian, and so are the right, bottom and front ones. Hence the diagonal square

$$\begin{tikzcd}
Z' \arrow[r] \arrow[d] & \mathcal{C}(T) \times \mathcal{C}(T) \arrow[d] \\
Z \arrow[r] & T^\vee \times T^\vee
\end{tikzcd}$$

is Cartesian. It is enough to check that $Z \simeq Z'$ since we want to show that $Z /\!\!/ W \simeq Z' /\!\!/ W$. But Z is just a collection of copies of $\overline{\mathbb{Z}}_\ell$ and all of them factor through torsion points in $T^\vee \times T^\vee$. We choose one of them for instance the point $(1,1)$ but then the fiber over $(1,1)$ in Z' is $\mathrm{Spec}((\Omega_T \otimes \Omega_T)) \times_{T^\vee \times T^\vee} \mathrm{Spec}(\overline{\mathbb{Z}}_\ell) = \mathrm{Spec}\overline{\mathbb{Z}}_\ell$.

It remains to check the claim about the Cartesianity of the above diagram. This can be checked one pair of torsion points at a time in $T^\vee \times T^\vee$, the proof is similar for all pairs of torsion points, we do it for the point $(1,1)$, but then this is lemma 8.4 of [BR22b] (again suitably lifted to $\overline{\mathbb{Z}}_\ell$). ☐

3.6.2 Isomorphism of the two functors and proof of theorem 3.6.1.

The representation Γ_ψ defines a sheaf on $\mathrm{pt}/G^{\mathrm{F}}$ and we denote by Ξ_ψ the sheaf $\Gamma_\psi \otimes_{\overline{\mathbb{Z}}_\ell} \Omega_T$. We now want to relate $\mathrm{HC}_{\mathrm{F}}(\Xi_\psi)$ with $p_!\mathbb{T}$.

Lemma 3.6.8. *We have an isomorphim*

$$\Phi\mathrm{HC}_{\mathrm{F}}(\Gamma_\psi) = p_!\mathbb{T}[\dim T]. \tag{3.109}$$

Proof. We show the following.

$$\bigoplus_{\chi \in \mathrm{CH}(T)} \mathrm{Av}_\chi \mathrm{HC}_{\mathrm{F}}(\Xi_\psi) \simeq p_!\mathbb{T}[\dim T]$$

where Av_χ is as before $a_!(L_T \otimes \mathcal{L}_\chi[2\dim T] \boxtimes_{\Omega_T} -)$. In view of the equivalence of lemma 3.3.15 this is equivalent to proving the lemma. The proof follows the argument of [BT22] 5.5.1, but we have to bypass the use of the vanishing conjecture of [Che21] which is not available in the integral setting.

Recall that we have a sheaf $\mathcal{L}_\psi$ on $\overline{U}$ which we interpret as an Ω_T sheaf. As a sheaf on $\overline{U}$ it is Ad_F-equivariant and the corresponding object in $\mathrm{D}_{\mathrm{cons}}(\frac{\overline{U}}{\mathrm{Ad}_\mathrm{F}\overline{U}}, \Omega_T)$ is nothing else than the representation ψ. Denote by $v : \overline{U} \to G$ the embedding, and consider the sheaf $v_!\mathcal{L}_\psi$ on $\frac{G}{\mathrm{Ad}_\mathrm{F}\overline{U}}$. Now denote by c the map $\frac{G}{\mathrm{Ad}_\mathrm{F}\overline{U}} \to \frac{G}{\mathrm{Ad}_\mathrm{F}G}$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{D}_{\mathrm{cons}}(\frac{\overline{U}}{\mathrm{Ad}_\mathrm{F}\overline{U}}, \Omega_T) \arrow[r, "v_!"] \arrow[d, equal] & \mathrm{D}_{\mathrm{cons}}(\frac{G}{\mathrm{Ad}_\mathrm{F}\overline{U}}, \Omega_T) \arrow[r, "c_!"] & \mathrm{D}_{\mathrm{cons}}(\frac{G}{\mathrm{Ad}_\mathrm{F}G}, \Omega_T) \arrow[d, equal] \\
\mathrm{D}^b(\Omega_T[\overline{U}^\mathrm{F}]) \arrow[rr, "\mathrm{ind}_{U^\mathrm{F}}^{G^\mathrm{F}}"'] & & \mathrm{D}^b(\Omega_T[G^\mathrm{F}])
\end{tikzcd}$$

where the vertical arrows are the usual identification with categories of representations and the bottom horizontal one is the usual induction for finite groups. In particular we get that $c_!v_!\mathcal{L}_\psi = \Xi_\psi$.

Consider now the diagram

$$\begin{tikzcd}
\frac{G}{\mathrm{Ad}_\mathrm{F}\overline{U}} \arrow[d, "c"'] & \Delta G\backslash(G \times G)/\Delta_\mathrm{F}\overline{U} \arrow[l, "i_1"'] \arrow[d, "c_1"'] & (G \times G)/\Delta_\mathrm{F}\overline{U} \arrow[l, "\pi_1"'] \arrow[d, "c_2"] & \\
\frac{G}{\mathrm{Ad}_\mathrm{F}G} & \Delta G\backslash(G \times G)/\Delta_\mathrm{F}G \arrow[l, "i_2"'] & (G \times G)/\Delta_\mathrm{F}G \arrow[l, "\pi_2"'] & \\
\frac{G}{\mathrm{Ad}_\mathrm{F}B} \arrow[u, "q"] \arrow[d, "r"'] & \Delta G\backslash(G \times G)/\Delta_\mathrm{F}B \arrow[l, "i_3"'] \arrow[u, "q_1"'] \arrow[d, "r_1"'] & (G \times G)/\Delta_\mathrm{F}B \arrow[l, "\pi_3"'] \arrow[u, "q_2"'] \arrow[d, "r_2"] & (G \times G)/\Delta_\mathrm{F}U \arrow[l, "p_2"'] \arrow[d, "r_3"] \\
\frac{U\backslash G/U}{\mathrm{Ad}_\mathrm{F}T} & \Delta G\backslash(G/U \times G/U)/\Delta_\mathrm{F}T \arrow[l, "i_4"] & (G/U \times G/U)/\Delta_\mathrm{F}T \arrow[l, "\pi_4"] & G/U \times G/U \arrow[l, "p_1"]
\end{tikzcd}$$

where the maps are as follow

(*i*). The maps i_j for $j = 1, \dots, 4$ are isomorphisms induced by $G \times G \mapsto G, (x, y) \mapsto x^{-1}y$.

(*ii*). The maps π are the quotient maps for the diagonal action of G on the left.

(*iii*). The maps p_1 and p_2 are the quotient maps with respect to the right Δ_F-action of T.

(*iv*). The maps c, r and q have already been defined and the other maps are defined so that all squares are Cartesian, they are all quotient maps for the obvious groups.

Consider the sheaf $\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi$ on $(\overline{U} \times \overline{U})/\Delta_\mathrm{F}(\overline{U})$ and the diagram

$$\begin{tikzcd}
(\overline{U} \times \overline{U})/\Delta_\mathrm{F}\overline{U} \arrow[r, "v_2"] \arrow[d, "\pi_0"'] & (G \times G)/\Delta_\mathrm{F}\overline{U} \arrow[d, "\tilde{\pi}_1"'] \arrow[dr, "\pi_1"] & \\
\Delta\overline{U}\backslash(\overline{U} \times \overline{U})/\Delta_\mathrm{F}\overline{U} \arrow[r, "v_1"'] \arrow[d, "i_0"'] & \Delta\overline{U}\backslash(G \times G)/\Delta_\mathrm{F}\overline{U} \arrow[r, "\tilde{c}_1"'] & \Delta G\backslash(G \times G)/\Delta_\mathrm{F}\overline{U} \arrow[d, "i_1"] \\
\frac{\overline{U}}{\mathrm{Ad}_\mathrm{F}\overline{U}} \arrow[rr, "v"] & & \frac{G}{\mathrm{Ad}_\mathrm{F}(\overline{U})}
\end{tikzcd}$$

where all maps are either the obvious quotient maps or induced by the inclusion $\overline{U} \to G$. Note that the two squares are Cartesian and the maps i_1 and i_0 isomorphisms. There is an isomorphism

$$\pi_0^* i_0^* \mathcal{L}_\psi = \mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi.$$

We then deduce

$$\begin{aligned} i_1^* v_! \mathcal{L}_\psi &= \tilde{c}_{1,!} v_{1,!} i_0^* \mathcal{L}_\psi \\ &= \tilde{c}_{1,!} v_{1,!} \pi_{0,!} \pi_0^* i_0^* \mathcal{L}_\psi [2 \dim \overline{U}] \\ &= \pi_{1,!} v_{2,!} (\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi)[2 \dim \overline{U}] \end{aligned}$$

where, in the second line, we have used the fact that $\pi_{0,!}\pi_0^* \simeq \mathrm{id}[-2\dim U]$ since π_0 is a $\overline{U}$-torsor. We now want to compute $i_4^* \mathrm{HC}_{\mathrm{F}}(\Xi_\psi)$ as follows.

$$\begin{aligned} i_4^* \mathrm{HC}_{\mathrm{F}}(\Xi_\psi) &= i_4^* r_! q^* c_! v_! \mathcal{L}_\psi \\ &= r_{1,!} q_1^* c_{1,!} i_1^* v_! \mathcal{L}_\psi \\ &= r_{1,!} q_1^* c_{1,!} \pi_{1,!} v_{2,!} (\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi)[2 \dim \overline{U}] \\ &= \pi_{4,!} r_{2,!} q_2^* c_{2,!} v_{2,!} (\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi)[2 \dim \overline{U}] \end{aligned}$$

We now want to discuss $p_1^* r_{2,!} q_2^* c_{2,!} v_{2,!} (\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi)$. First note that $v_{2,!}\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi$ was $(\overline{U} \times \overline{U}, \psi^{-1} \times \psi)$-equivariant on the left and remains so after applying all the functors $p_1^* r_{2,!} q_2^* c_{2,!}$ the resulting sheaf is $\psi^{-1} \times \psi$-equivariant on the left on $G/U \times G/U$ hence has to be supported on the open cell, which is $\overline{U}TU/U \times \overline{U}TU/U$. We now compute its pullback to $T \times T$. Consider the following diagram

$$\begin{tikzcd} & T \arrow[r, "\gamma"] \arrow[ddl, "\mathrm{triv}"'] & (TU \times TU)/\Delta_{\mathrm{F}} U \arrow[r, "r_4"] \arrow[d, "j_2"] \arrow[ddl, "j_3"'] & TU/U \times TU/U \arrow[d, "j_1"] \\ & & (G \times G)/\Delta_{\mathrm{F}} U \arrow[r, "r_3"'] \arrow[d, "p_2"] & G/U \times G/U \arrow[d, "p_1"] \\ (\overline{U} \times \overline{U})/\Delta_{\mathrm{F}} \overline{U} \arrow[r, "c_2 v_2"'] & (G \times G)/\Delta_{\mathrm{F}} G & (G \times G)/\Delta_{\mathrm{F}} B \arrow[l, "q_2"] \arrow[r, "r_2"'] & (G/U \times G/U)/\Delta_{\mathrm{F}} T \end{tikzcd}$$

where the maps j_1 and j_2 are induced by the inclusion $TU \subset G$, the map $j_3 = q_2 p_2 j_2$, the map triv is the map with constant value 1. Both of the right squares are Cartesian. We identify $(\overline{U} \times \overline{U})/\Delta_{\mathrm{F}} \overline{U} \simeq \overline{U}$ via the inclusion of the second factor, in particular the map $\overline{U} \times \overline{U} \to (\overline{U} \times \overline{U})/\Delta_{\mathrm{F}} \overline{U} \simeq \overline{U}$ is nothing else than the map $(x, y) \mapsto (y\mathrm{F}(x^{-1}))$. Similarly we identify $(G \times G)/\Delta_{\mathrm{F}} G$ with G and $(TU \times TU)/\Delta_{\mathrm{F}} U \simeq T \times T \times U$. Under these identification, we set the map γ to be given by the graph of F on $T \times T$ and the map with constant value 1 on U. With these choices, the left square is also Cartesian. We have

$$\begin{aligned} j_1^* p_1^* r_{2,!} q_2^* c_{2,!} v_{2,!} (\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi)[2 \dim \overline{U}] &= r_{4,!} j_2^* p_2^* q_2^* c_{2,!} v_{2,!} (\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi)[2 \dim \overline{U}] \\ &= r_{4,!} j_3^* c_{2,!} v_{2,!} (\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi)[2 \dim \overline{U}] \\ &= r_{4,!} \gamma_! \mathrm{triv}^* (\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi)[2 \dim \overline{U}] \\ &= r_{4,!} \gamma_! \Omega_T [2 \dim \overline{U}]. \end{aligned}$$

Under the identification $TU/U \times TU/U \simeq T \times T$, the sheaf $j_1^* p_1^* r_{2,!} q_2^* c_{2,!} v_{2,!}(\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi)[2\dim \overline{U}]$ is therefore nothing else than the sheaf $(\mathrm{id} \times \mathrm{F})_! \Omega_T$.

Consider the following cartesian diagram

$$\begin{tikzcd} G/U \times G/U \arrow[r, "a"] \arrow[d, "p_1"'] & U\backslash G/U \arrow[d, "p"] \\ (G/U \times G/U)/\Delta_{\mathrm{F}} T \arrow[r, "\pi_4"'] & \frac{U\backslash G/U}{\mathrm{Ad}_{\mathrm{F}} T} \end{tikzcd}$$

We now compute

$$a_! p_1^* r_{2,!} q_2^* c_{2,!} v_{2,!}(\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi)[2\dim \overline{U}].$$

By the preceding discussion, the sheaf $p_1^* r_{2,!} q_2^* c_{2,!} v_{2,!}(\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi)[2\dim \overline{U}]$ is nothing else than the sheaf $p_1^* p_{1,!}(\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi)[2\dim \overline{U}]$ where $\mathcal{L}_\psi$ is considered as a sheaf on G/U via the inclusion $\bar{U} \hookrightarrow G/U$. Note that the maps a, p_1, p and π_4 are equivariant for the action of T by translation either on the right or on the second copy of G/U, hence Av_χ commutes with the functors $p_!, p^*, \ldots$. We now compute

$$\begin{aligned} \mathrm{Av}_\chi a_! p_1^* p_{1,!}(\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi)[2\dim \bar{U}] &= \mathrm{Av}_\chi p^* \pi_{4,!} p_{1,!}(\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi)[2\dim \bar{U}] \\ &= p^* p_! a_! \mathrm{Av}_\chi^2(\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi)[2\dim \bar{U}]), \end{aligned}$$

where Av_χ^2 refers to the averaging functor on the second copy of G/U. Let us now evaluate $a_! \mathrm{Av}_\chi^2(\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi)[2\dim \bar{U}])$. Consider the following diagram

$$\begin{tikzcd} & G \times G/U \arrow[r, "b"] \arrow[d, "\tilde{z}"'] & G/U \arrow[d, "z"] \\ \bar{U}U/U \times \bar{U}TU/U \arrow[ur, "\tilde{j}"] \arrow[d, "\tilde{z}"'] & G/U \times G/U \arrow[r, "a"'] & U\backslash G/U \\ \bar{U}U/U \times \bar{U}TU/U \arrow[ur, "j"'] & & \end{tikzcd}$$

where z and $\tilde{z}$ are the quotient maps for the action of U, the maps j and $\tilde{j}$ are the obvious inclusions and the map b is $(x, yU) \mapsto x^{-1}yU$. Note that both squares are Cartesian. Since U is a unipotent group the map $\mathrm{id} \to z_! z^!$ is an isomorphism.

$$\begin{aligned} a_! \mathrm{Av}_\chi^2(\mathcal{L}_{\psi^{-1}} \boxtimes \mathcal{L}_\psi)[2\dim \bar{U}]) &= a_! j_!(\mathcal{L}_{\psi^{-1}} \boxtimes (\mathrm{Av}_\chi^2 \mathcal{L}_\psi))[2\dim \bar{U}]) \\ &= z_! z^! a_! j_!(\mathcal{L}_{\psi^{-1}} \boxtimes (\mathrm{Av}_\chi^2 \mathcal{L}_\psi))[2\dim \bar{U}]) \\ &= z_! b_! \tilde{z}^! j_!(\mathcal{L}_{\psi^{-1}} \boxtimes (\mathrm{Av}_\chi^2 \mathcal{L}_\psi))[2\dim \bar{U}]) \\ &= z_! b_! \tilde{j}_!(\mathcal{L}_{\psi^{-1}} \boxtimes (\mathrm{Av}_\chi^2 \mathcal{L}_\psi))[4\dim \bar{U}]) \\ &= z_!(\delta_{\chi,\psi})[\dim T + \dim \bar{U}] \\ &= \mathrm{Av}_U(\delta_{\chi,\psi})[\dim T] = T_\chi[\dim T]. \end{aligned}$$

Putting everything together, we have,

$$\begin{aligned}\mathrm{Av}_\chi p^*\mathrm{HC}_{\mathrm{F}}(\Xi_\psi) &= p^*\pi_{4,!}\mathrm{Av}^2_\chi r_{2,!}q_2^*c_{2,!}v_{2,!}(\mathcal{L}_{\psi^{-1}}\boxtimes\mathcal{L}_\psi)[2\dim\overline{U}]\\ &= a_!p_1^*\mathrm{Av}^2_\chi r_{2,!}q_2^*c_{2,!}v_{2,!}(\mathcal{L}_{\psi^{-1}}\boxtimes\mathcal{L}_\psi)[2\dim\overline{U}]\\ &= p^*p_!T_\chi[\dim T].\end{aligned}$$

The sheaf $p_!T_\chi[\dim T]$ is concentrated in a single perverse degree. Indeed, since T_χ is tilting, both the stalks and costalks of T_χ are direct sums of copies of $\nu_w^*L_T\otimes\mathcal{L}_\chi[\dim T+\ell(w)]$. Since the map p is a T-torsor, by lemma 3.2.50, the stalks and costalks of $p_!T_\chi[\dim T]$ are direct sums of copies of $p_!\nu_w^*L_T\otimes\mathcal{L}_\chi[2\dim T+\ell(w)]$ which is concentrated in a single degree.

We can now apply [BBD82] 4.2.5, to deduce that

$$p_!T_\chi[\dim T] = \mathrm{Av}_\chi\mathrm{HC}_{\mathrm{F}}(\Xi_\psi).$$

Taking a direct sum over all χ yields the lemma. $\square$

Proof of theorem 3.6.2. Let $w\in W$. Using theorem 3.3.8, we have that

$$k_{w,*}i_w^*\mathrm{HC}_{\mathrm{F}}(\Gamma_\psi) = \Gamma_\psi\otimes_{G^{\mathrm{F}}}\mathrm{R}\Gamma_c(Y(\dot w),\overline{\mathbb{Z}}_\ell).$$

Applying the functor $k_{w,*}i_w^*$ to both sides of 3.6.8, we deduce that

$$\Gamma_\psi\otimes_{G^{\mathrm{F}}}\mathrm{R}\Gamma_c(Y(\dot w),\overline{\mathbb{Z}}_\ell) = k_{w,*}i_w^*p_!\mathbb{T} = k_{w,*}p_!i_w^*\mathbb{T}[\dim T]. \tag{3.110}$$

Since $\mathbb{T}$ is tilting and by lemma 3.4.30, we get

$$i_w^*\mathbb{T} = \bigoplus_{\chi\in\mathrm{CH}(T)}\nu_w^*(L_T\otimes\mathcal{L}_\chi)[\dim T+\ell(w)].$$

By lemma 3.3.9, the right hand side of 3.110 is isomorphic to the regular representation of $T^{w\mathrm{F}}$. $\square$

Proof of theorem 3.6.3. By functoriality the isomorphism $\Phi\mathrm{HC}_{\mathrm{F}}\Gamma_\psi\simeq p_!\mathbb{T}[\dim T]$ defines a map

$$\mathrm{End}(\Gamma_\psi)\to\mathrm{End}(p_!\mathbb{T}). \tag{3.111}$$

The right hand side is isomorphic to $\mathbb{V}(p^!p_!\mathbb{T})\simeq\mathcal{O}_Z$. By lemma 3.6.7, there is a W-action on Z such that $Z/\!\!/W\simeq(T^\vee/\!\!/W)^{\mathrm{F}^\vee}$. In particular the statement is equivalent to the commutation of the action of $\mathrm{End}(\Gamma_\psi)$ with the W-action. We will use the F^*-central structure to obtain a factorization by the W-invariants.

Recall that $\langle\mathbb{T}\rangle$ denotes the stable subcategory of $\mathbb{H}$ generated by $\mathbb{T}$. By lemma 3.4.58, this is monoidal subcategory of $\mathbb{H}^\omega$. We have $p^!p_!\mathbb{T}\in\langle\mathbb{T}\rangle$. Indeed, choosing a regular sequence determining the graph of Frobenius in $\mathcal{C}(T)\times\mathcal{C}(T)$ and then restricting it to $\mathcal{C}(T)\times_{\mathcal{C}(T)/\!\!/W}\mathcal{C}(T)$ is still a regular sequence by the proof of lemma 3.6.7. The object $p^!p_!\mathbb{T}$ is then isomorphic to the totalization of the complex of $\mathbb{T}\otimes_{\mathcal{C}(T)\times_{\mathcal{C}(T)/\!\!/W}\mathcal{C}(T)}\mathrm{Kos}$ where Kos denotes the Koszul complex corresponding to our chosen regular sequence.

Consider the full subcategory $\langle\Gamma_\psi\rangle\subset\mathrm{D}_{\mathrm{cons}}(\mathrm{pt}/G^{\mathrm{F}},\overline{\mathbb{Z}}_\ell)$ generated by Γ_ψ. The F^*-central functor $p^!\mathrm{HC}_{\mathrm{F}}$ then defines an F^*-central functor

$$\langle\Gamma_\psi\rangle\to\langle\mathbb{T}\rangle. \tag{3.112}$$

By lemma 3.4.58, the functor $\mathbb{V}$ induces a monoidal equivalence $\langle \mathbb{T} \rangle \simeq \mathrm{D}_{\mathrm{qcoh}}(\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T))$. Therefore $\mathbb{V}(p^! \mathrm{HC}_{\mathrm{F}}(-))$ then defines a $\mathrm{F}^\vee$-central functor $\langle \Gamma_\psi \rangle \to \mathrm{D}_{\mathrm{qcoh}}(\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T))$. By theorem 3.C.12, we have an equivalence

$$\mathcal{Z}_{\mathrm{F}^\vee}(\mathrm{D}_{\mathrm{qcoh}}(\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T))) \simeq \mathrm{D}_{\mathrm{qcoh}}((\mathcal{C}(T) /\!\!/ W)^{\mathrm{F}^\vee}) \simeq \mathrm{D}_{\mathrm{qcoh}}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee}). \tag{3.113}$$

We then get a well defined functor

$$\langle \Gamma_\psi \rangle \to \mathrm{D}_{\mathrm{qcoh}}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee}) \tag{3.114}$$

making the following diagram commute

$$\begin{tikzcd}
\langle \Gamma_\psi \rangle \arrow[rr] \arrow[dr] & & \mathrm{D}_{\mathrm{qcoh}}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee}) \arrow[dl] \\
& \mathrm{D}_{\mathrm{qcoh}}(\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T)) &
\end{tikzcd}$$

where the map $\mathrm{D}_{\mathrm{qcoh}}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee}) \to \mathrm{D}_{\mathrm{qcoh}}(\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T))$ is given by $i_* z^*$ where i and z are the following maps

$$\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T) \xleftarrow{i} Z \xrightarrow{z} (T^\vee /\!\!/ W)^{\mathrm{F}^\vee}. \tag{3.115}$$

The image of Γ_ψ is then an $\mathcal{O}((T^\vee /\!\!/ W)^{\mathrm{F}^\vee})$-module M whose image in $\mathrm{D}_{\mathrm{qcoh}}(\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T))$ is $\mathbb{V}(p^! p_! \mathbb{T}) = i_* \mathcal{O}_Z$. Hence we have an isomorphism $z^* M = \mathcal{O}_Z$. As $Z \to (T^\vee /\!\!/ W)^{\mathrm{F}^\vee}$ is faithfully flat, we see that M is locally free of rank one. Moreover, since $(T^\vee /\!\!/ W)^{\mathrm{F}^\vee}$ is a finite disjoint of (Artinian) local schemes, M is free of rank one. We therefore have a map

$$\phi : \mathrm{End}(\Gamma_\psi) \to \mathcal{O}_{(T^\vee /\!\!/ W)^{\mathrm{F}^\vee}} = \mathrm{End}(M), \tag{3.116}$$

compatible with the map $\mathrm{End}(\Gamma_\psi) \to \mathrm{End}(p_! \mathbb{T})$.

To conclude the proof of the theorem, we still need to prove that $\mathrm{Cur}_w^{\mathrm{spec}} \phi = \mathrm{Cur}_w$. Firstly recall from the proof of the Endomorphismensatz 3.4.34, that there is an embedding $\Omega_T \otimes_{\Omega_T^{W_\chi}} \Omega_T \to \oplus_w \Omega_T$ and that the following diagram commutes

$$\begin{tikzcd}
\mathrm{End}(T_\chi) \arrow[r] \arrow[d, equal] & \mathrm{End}(\mathrm{gr}_w T_\chi) \arrow[d, equal] \\
\Omega_T \otimes_{\Omega_T^{W_\chi}} \Omega_T \arrow[r, "\mathrm{pr}_w"'] & \Omega_T
\end{tikzcd}$$

where pr_w is the map induced by the projection $\oplus_w \Omega_T \to \Omega_T$ onto the w-th component. Recall from lemma 3.4.21 that $\mathrm{gr}_w T_\chi = i_{w,!} i_w^* T_\chi$ and thus $\mathrm{End}(\mathrm{gr}_w T_\chi) = \mathrm{Hom}(i_w^* T_\chi, i_w^* T_\chi)$. We take a direct sum over all χ and then apply the functor $\mathcal{O}_Z \otimes_{\mathcal{O}_{\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T)}} -$ to the previous diagram. Using lemma 3.6.6 and the fact that $\mathcal{O}_Z \otimes_{\mathcal{O}_{\mathcal{C}(T) \times_{\mathcal{C}(T)/\!\!/W} \mathcal{C}(T)}} \mathcal{O}_{\mathcal{C}(T)} = \mathcal{O}_{(T^\vee)^{w\mathrm{F}^\vee}}$, we have a commutative diagram.

$$\begin{tikzcd}
\mathrm{End}(\Gamma_\psi) \arrow[r] \arrow[d] & \mathrm{End}(p_! \mathbb{T}) \arrow[r] \arrow[d, equal] & \mathrm{End}(p_! i_w^* \mathbb{T}) \arrow[d, equal] \\
\mathcal{O}_{(T^\vee /\!\!/ W)^{\mathrm{F}^\vee}} \arrow[r] & \mathcal{O}_Z \arrow[r] & \mathcal{O}_{(T^\vee)^{w\mathrm{F}^\vee}}
\end{tikzcd}$$

By lemma 3.3.8, and since $\mathrm{End}(p_! i_w^* \mathbb{T}) = \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}]$ the top line is the w-Curtis morphism and the bottom one is the w-spectral Curtis morphism.

□

Proof of theorem 3.6.1. To conclude we use a symmetrizing form argument. Firstly consider the map $(T^\vee)^{w\mathrm{F}} \to (T^\vee /\!\!/ W)^{\mathrm{F}^\vee}$. We have $\mathcal{O}_{(T^\vee)^{w\mathrm{F}}} = \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}]$, and we therefore have a map $\mathcal{O}_{(T^\vee /\!\!/ W)^{\mathrm{F}^\vee}} \to \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}]$.

Consider the composition $\mathrm{End}(\Gamma_\psi) \to \mathcal{O}_{(T^\vee /\!\!/ W)^{\mathrm{F}^\vee}} \to \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}]$. By theorem 3.6.3, this map is the Curtis morphism. By [BK08], the map $\mathrm{End}(\Gamma_\psi) \to \bigoplus_{w\in W} \overline{\mathbb{Z}}_\ell[T^{w\mathrm{F}}]$ is injective and both sides are equipped with compatible symmetrizing forms. As $\overline{\mathbb{Q}}_\ell \otimes \mathrm{End}(\Gamma_\psi) \simeq \mathcal{O}_{(T^\vee /\!\!/ W)^{\mathrm{F}^\vee}} \otimes \overline{\mathbb{Q}}_\ell$ the map $\mathrm{End}(\Gamma_\psi) \to \mathcal{O}_{(T^\vee /\!\!/ W)^{\mathrm{F}^\vee}}$ is an isomorphism by *loc. cit.* Lemma 3.8. □

3.A Equivariant sheaves

We will recall a construction of [Gai20]. We let T be a torus over k and X be a scheme with a T-action and Λ be a coefficient ring. We first equip the category $\mathrm{D}_{\mathrm{cons}}(T,\Lambda)$ with the $*$-convolution structure defined as follow. Consider the convolution diagram

$$\begin{tikzcd} & T\times T \arrow[r, "m"'] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & T \\ T & & T \end{tikzcd}$$

where p_i are the projections and m is the multiplication. The $*$ convolution is defined as

$$A *^* B = m_*(A \boxtimes B)$$

where $A, B \in \mathrm{D}_{\mathrm{cons}}(T,\Lambda)$.

Remark 3.A.1. This monoidal structure extends to the category $\mathrm{D}_{\mathrm{indcons}}(T,\Lambda)$ by the continuity of m_*.

Similarly the category $\mathrm{D}_{\mathrm{cons}}(X,\Lambda)$ is a module over $\mathrm{D}_{\mathrm{cons}}(T,\Lambda)$, namely there is an action

$$A \star B = a_*(A \boxtimes B),$$

where $a : T\times X \to X$ is the action map and $A \in \mathrm{D}_{\mathrm{cons}}(T,\Lambda)$ and $B \in \mathrm{D}_{\mathrm{cons}}(X,\Lambda)$. As before this action extends to an action on the ind-completions. Consider the category $\mathrm{D}(\Lambda)$ as a $\mathrm{D}_{\mathrm{indcons}}(T,\Lambda)$-module with the trivial action, that is the action given by $(A, M) \mapsto \mathrm{R}\Gamma(T, A) \otimes_\Lambda M$, we denote it $\mathrm{D}(\Lambda)_{\mathrm{triv}}$.

Definition 3.A.2. Let $\mathcal{C}$ be a stable cocomplete $\mathrm{D}_{\mathrm{indcons}}(T,\Lambda)$-module. We define the categorical invariants and coinvariants as

(*i*). $\mathcal{C}^T = \mathrm{Hom}_{\mathrm{D}_{\mathrm{indcons}}(T,\Lambda)}(\mathrm{D}(\Lambda)_{\mathrm{triv}}, \mathcal{C})$,

(*ii*). $\mathcal{C}_T = \mathcal{C} \otimes_{\mathrm{D}_{\mathrm{indcons}}(T,\Lambda)} \mathrm{D}(\Lambda)_{\mathrm{triv}}$.

The evaluation at $\Lambda \in \mathrm{D}(\Lambda)_{\mathrm{triv}}$ defines the forgetful functor $\mathrm{For}_T : \mathcal{C}^T \to \mathcal{C}$. Its right adjoint is denoted by $\mathrm{Av}_{T,*}$. The functor $\mathrm{Av}_{T,*} : \mathcal{C} \to \mathcal{C}^T$ factors through a functor

$$\mathcal{C}_T \to \mathcal{C}^T.$$

Theorem 3.A.3 ([Gai20], B.1.2)**.** *The functor $\mathcal{C}_T \to \mathcal{C}^T$ is an equivalence.*

Remark 3.A.4. By [Gai20] 1.4.5, the category $\mathcal{C}^T$ is identified with the category of comodules over the comonad $\mathrm{Av}_{T,*}\mathrm{For}_T$.

Definition 3.A.5. For the category $\mathcal{C} = \mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$, the categorical invariants/coinvariants are identified with the category $\mathrm{D}_{\mathrm{indcons}}(X/T, \Lambda)$ of sheaves on the quotient stack by [Gai20] 1.4.6. The category $\mathrm{D}_{\mathrm{cons}}(X/T, \Lambda)$ is the full subcategory of $\mathrm{D}_{\mathrm{indcons}}(X/T, \Lambda)$ of sheaves such that their pullback to $\mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$ is in $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$.

Remark 3.A.6. In the rest of the paper, we have used the !-convolution instead. As Verdier duality exchanges ! and $*$-convolution, we get variants of the previous results. In particular, the category $\mathrm{D}_{\mathrm{cons}}(X/T, \Lambda)$ is identified with the modules over the monad $\mathrm{For}_T\mathrm{Av}_{T,!}$.

The twisted case.

Definition 3.A.7. A multiplicative sheaf $\mathcal{L} \in \mathrm{D}_{\mathrm{cons}}(T, \Lambda)$ is a sheaf equipped with the following data

(*i*). a trivialization at $1 \in T$, i.e., an isomorphism $1^*\mathcal{L} = \Lambda$,

(*ii*). an isomorphism $m^*\mathcal{L} \simeq \mathcal{L} \boxtimes \mathcal{L}$ where $m : T \times T \to T$ is the multiplication map such that the restriction at $(1, 1) \in T \times T$ of this isomorphism is compatible with the trivialization of (i).

For this section we refer to [Gai20] 1.5. Let $\mathcal{L}$ be a multiplicative sheaf on T of Λ-modules. Let $\mathcal{C}$ be a category with a $\mathrm{D}_{\mathrm{indcons}}(T, \Lambda)$-action, denote this action by $A, c \mapsto A \star c$ for $A \in \mathrm{D}_{\mathrm{indcons}}(T, \Lambda)$ and $c \in \mathcal{C}$. We twist the action and define a new action $\star^{\mathrm{new}}$

$$A \star^{\mathrm{new}} c = (A \otimes_\Lambda \mathcal{L}) \star c$$

where $A \in \mathrm{D}_{\mathrm{indcons}}(T, \Lambda)$ and $c \in \mathcal{C}$.

Definition 3.A.8. The category of $(T, \mathcal{L})$-equivariant sheaves on X is defined to be the category of invariants $\mathrm{D}_{\mathrm{indcons}}(X/(T, \mathcal{L}), \Lambda) = \mathrm{D}_{\mathrm{indcons}}(X, \Lambda)^T$ for the twisted action of $\mathrm{D}_{\mathrm{indcons}}(T, \Lambda)$. Similarly we define $\mathrm{D}_{\mathrm{cons}}(X/(T, \mathcal{L}), \Lambda)$ to be the full subcategory of $\mathrm{D}_{\mathrm{indcons}}(X/(T, \mathcal{L}), \Lambda)$ of sheaves such that their pullback to $\mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$ is constructible.

Locally constant actions and monodromic sheaves

Denote by $\mathrm{D}_{\mathrm{indcons}}(T, \Lambda)^0$ the subcategory generated by the constant sheaf. This is nothing else than the ind-completion of the category of unipotent monodromic sheaves $\mathrm{D}_{\mathrm{cons}}(T, \Lambda)_{\mathrm{mon,unip}}$. The inclusion $\mathrm{D}_{\mathrm{indcons}}(T, \Lambda)^0 \subset \mathrm{D}_{\mathrm{indcons}}(T, \Lambda)$ has a right adjoint, which is monoidal for the $*$-convolution.

Definition 3.A.9. Let $\mathcal{C}$ be a category with an action of $\mathrm{D}_{\mathrm{indcons}}(T, \Lambda)$. We set

$$\mathcal{C}^0 = \mathrm{D}_{\mathrm{indcons}}(T, \Lambda)^0 \otimes_{\mathrm{D}_{\mathrm{indcons}}(T,\Lambda)} \mathcal{C}.$$

The adjunction $\mathrm{D}_{\mathrm{indcons}}(T, \Lambda)^0 \leftrightarrows \mathrm{D}_{\mathrm{indcons}}(T, \Lambda)$ induces an adjunction $\mathcal{C}^0 \leftrightarrows \mathcal{C}$.

Lemma 3.A.10 ([Gai20] B.5)**.** *The arrow $\mathcal{C}^0 \to \mathcal{C}$ is fully faithful and induces an equivalence $(\mathcal{C}^0)^T \simeq \mathcal{C}^T$. Moreover the subcategory $\mathcal{C}^0$ is the one generated by the image of the forgetful functor from $(\mathcal{C}^0)^T$.*

Remark 3.A.11. Lemma 3.A.10 implies that for $\mathcal{L}_\chi$ a tame character sheaf on T and for the corresponding twisted action of $\mathrm{D}_{\mathrm{indcons}}(T, \Lambda)$ on $\mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$, the category $\mathrm{D}_{\mathrm{indcons}}(X, \Lambda)^0$ is identified with the category of $\mathcal{L}_\chi$-monodromic sheaves on X.

3.B Monodromic Tilting sheaves

3.B.1 Stratified spaces and tilting objects in the unipotent case

We consider a T torsor $X \to Y$ and we assume that the scheme $Y = \sqcup_{s \in \mathcal{S}} Y_s$ is equipped with a finite stratification such that

(i). For each s the scheme Y_s is smooth and has trivial cohomology, that is $R\Gamma(Y_s, \mathbb{Z}_\ell) = \mathbb{Z}_\ell$.

(ii). For each s, the torsor $X_s = \pi^{-1}(Y_s) \to Y_s$ is trivial and we fix a trivialization $X_s = T \times Y_s$ and denote by $\nu_s : X_s \to T$ the projection.

(iii). The inclusion $i_s : X_s \to X$ are affine.

For $s \in \mathcal{S}$ denote by $i'_s : Y_s \to Y$ the inclusion. We denote by $\mathrm{D}'_{\mathrm{cons}}(Y, \overline{\mathbb{Z}}_\ell)$ the full subcategory of $\mathrm{D}_{\mathrm{cons},\mathcal{S}}(Y, \overline{\mathbb{Z}}_\ell)$, of $\mathcal{S}$-constructible sheaves on Y, generated by the sheaves $i_{s,!}(\overline{\mathbb{Z}}_\ell)_{Y_s}$. We will also make the following assumption

(C) For all s, t, all cohomology sheaves $H^i(i_t^* i_{s,*}(\overline{\mathbb{Z}}_\ell)_{Y_s})$ are constant.

We now adapt to our setup several of the definitions of [BY13] Appendix A. We first do it in the unipotent case.

Definition 3.B.1 (Unipotent standard and costandard objects)**.** We define the free monodromic standard and costandard objects as

(i). $\Delta_s = i_{s,!}\nu_s^* L_T[\dim X_s]$,

(ii). $\nabla_s = i_{s,*}\nu_s^* L_T[\dim X_s]$.

and similarly on Y as

(i). $\Delta_s^Y = i'_{s,!}(\overline{\mathbb{Z}}_\ell)_{Y_s}[\dim Y_s]$,

(ii). $\nabla_s^Y = i'_{s,*}(\overline{\mathbb{Z}}_\ell)_{Y_s}[\dim Y_s]$.

Remark 3.B.2. Reducing modulo ℓ the sheaves Δ_s and ∇_s gives $\Delta_s \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell$ and $\nabla_s \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell$ which are the pro-monodromic standard and costandard sheaves of [BR22b] 5.3.

Remark 3.B.3. The hypothesis (C) implies that the perverse t structure on the category $\mathrm{D}'_{\mathrm{cons}}(Y, \overline{\mathbb{Z}}_\ell)$ is obtained by gluing the perverse t-structures on the stratification $\mathcal{S}$ and for the constant local system on each Y_s.

Lemma 3.B.4. *We have* $\mathrm{Hom}_{\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_{\mathrm{unip}}}(\Delta_s, \nabla_t) = \Omega_T[0]$ *if* $s = t$ *and* 0 *otherwise.*

Proof. This is immediate. □

Definition 3.B.5. We define $\mathrm{D}'_{\mathrm{cons}}(X,\Omega_T)_{\mathrm{unip}}$ to be the full subcategory of $\mathrm{D}_{\mathrm{cons}}(X,\Omega_T)_{\mathrm{unip}}$ of sheaves A such that $\pi_\dagger(A) \in \mathrm{D}'_{\mathrm{cons}}(Y,\mathbb{Z}_\ell) \subset \mathrm{D}_{\mathrm{cons}}(Y,\mathbb{Z}_\ell)$.

Lemma 3.B.6. *(i). The perverse t-structure of $\mathrm{D}_{\mathrm{cons}}(X,\Omega_T)_{\mathrm{unip}}$ induces a well defined t-structure on $\mathrm{D}'_{\mathrm{cons}}(X,\Omega_T)_{\mathrm{unip}}$, with heart denoted by $\mathrm{Perv}'(X,\Omega_T)_{\mathrm{unip}}$.*

(ii). The category $\mathrm{D}'_{\mathrm{cons}}(X,\Omega_T)_{\mathrm{unip}}$ is generated as a triangulated category, either by the (Δ_s) or by the (∇_s).

Proof. For the first point, note that all sheaves in $\mathrm{D}'_{\mathrm{cons}}(X,\Omega_T)_{\mathrm{unip}}$ are constructible with respect to the (X_s)-stratification. We then do an induction on the number of strata. If there is only one strata the statement is obvious. Now let X_s be the open strata and let $A \in \mathrm{D}'_{\mathrm{cons}}(X,\Omega_T)$, denote by $i : X \setminus X_s \to X$ the inclusion of the other strata. Since the inclusion $X_s \to X$ is affine the functors $i_{s,!}$ and $i_{s,*}$ are perverse t-exact. The functors i^* and $i^!$ are respectively right and left t-exact. Consider the two fiber sequence

$$i_{s,!}i_s^*A \to A \to i_*i^*A$$
$$i_*i^!A \to A \to i_{s,*}i_s^*A$$

and apply the perverse truncation functors ${}^p\tau_{\geq n}$ and ${}^p\tau_{\leq n}$ respectively. We get two fiber sequence

$$i_{s,!}{}^p\tau_{\geq n}i_s^*A \to {}^p\tau_{\geq n}A \to i_*{}^p\tau_{\geq n}i^*A$$
$$i_*{}^p\tau_{\leq n}i^!A \to {}^p\tau_{\leq n}A \to i_{s,*}{}^p\tau_{\leq n}i_s^*A.$$

Now by induction and the one stratum case, all the sheaves ${}^p\tau_{\geq n}i_s^*A, {}^p\tau_{\geq n}i^*A, {}^p\tau_{\leq n}i^!A$ and ${}^p\tau_{\leq n}i_s^*A$ are in their corresponding $\mathrm{D}'_{\mathrm{cons}}$ and therefore so are ${}^p\tau_{\geq n}A$ and ${}^p\tau_{\leq n}A$. Hence the subcategory $\mathrm{D}'_{\mathrm{cons}}(X,\Omega_T)$ is stable under perverse truncations and we have a well defined induced t-structure.

For the second point, one argues again by induction on the number of strata, if there is only one stratum this is trivial. In general let $A \in \mathrm{D}'_{\mathrm{cons}}(X,\Omega_T)_{\mathrm{unip}}$, as before let $X_s \to X$ be the open stratum and $i : X \setminus X_s \to X$ be the inclusion of the other strata. Using the excisions triangles for (i, i_s) as before, we reduce to the case of a single stratum (using the one with $is,*$ for the ∇_s and $i_{s,!}$ for the Δ_s). □

Definition 3.B.7. A sheaf $A \in \mathrm{Perv}(X,\Omega_T)_{\mathrm{unip}}$ is said to have a Δ-filtration (reps. ∇-filtration), if it has a filtration whose graded parts are isomorphic to some Δ_s (resp. ∇_s). A perverse sheaf is tilting if it has both a Δ-filtration and a ∇-filtration. There is a corresponding definition on Y, we refer to [AR16].

Lemma 3.B.8. *Let $A \in \mathrm{Perv}'(X,\Omega_T)_{\mathrm{unip}}$ be a perverse sheaf with a Δ-filtration and B with a ∇-filtration then*

(i). $\mathrm{Hom}(A,B)$ is concentrated in degree 0 and is a free Ω_T-module of finite rank.

(ii). $\mathrm{Hom}(A,B) \otimes_{\Omega_T} \mathbb{F}_\ell \simeq \mathrm{Hom}(\pi_{\dagger,\mathbb{F}_\ell}A, \pi_{\dagger,\mathbb{F}_\ell}B)$.

(iii). An object in $A \in \mathrm{D}_{\mathrm{cons}}(X,\Omega_T)_{\mathrm{unip}}$ is a tilting perverse sheaf if and only if $\pi_{\dagger,\mathbb{F}_\ell}A$ is a tilting perverse sheaf on Y.

Proof. As explained in [BR22b], for (i), this is an immediate induction on the number of terms in the filtrations of A and B. For (ii) this is a consequence of the five lemma together with lemma 3.B.4. For the third point, the proof of [BR22b] 5.9 works verbatim in our setup after replacing $\pi_\dagger$ in *loc. cit.* by $\pi_{\dagger,\overline{\mathbb{F}}_\ell}$. $\square$

Remark 3.B.9. Let T be a tilting sheaf, then $T_{\mathbb{F}_\ell}$ is a tilting sheaf in $\mathrm{D}_{\mathrm{cons}}(X, \Omega_{T,\mathbb{F}_\ell})_{\mathrm{mon,unip}}$.

Lemma 3.B.10. *Let $A \in \mathrm{Perv}'(X, \Omega_T)_{\mathrm{unip}}$ be a perverse sheaf on X such that ${}^pH^0\pi_{\dagger,\overline{\mathbb{F}}_\ell}(A) = 0$ then $A = 0$.*

Proof. The same statement is proved for $\Omega_{T,\mathbb{F}_\ell}$ in place of Ω_T in [BR22b] 5.2. We can therefore reduce to their statement. First consider ${}^pH^0(A \otimes_{\Omega_T} \Omega_{T,\overline{\mathbb{F}}_\ell})$ then this is an $\Omega_{T,\overline{\mathbb{F}}_\ell}$-perverse sheaf and ${}^pH^0\pi_\dagger {}^pH^0(A \otimes_{\Omega_T} \Omega_{T,\overline{\mathbb{F}}_\ell}) = {}^pH^0\pi_{\dagger,\overline{\mathbb{F}}_\ell} A = 0$. By *loc. cit.*, $(A \otimes_{\Omega_T} \Omega_{T,\overline{\mathbb{F}}_\ell}) = 0$. Furthermore, the reduction mod ℓ is conservative on perverse sheaves by derived Nakayama. $\square$

Lemma 3.B.11. *The realization functor $\mathcal{D}^b(\mathrm{Perv}'(X, \Omega_T)_{\mathrm{unip}}) \to \mathrm{D}'_{\mathrm{cons}}(X, \Omega_T)_{\mathrm{unip}}$ is an equivalence.*

Proof. We will prove this in several steps, this is essentially the proof of [RSW13] 2.3.1 and 2.3.4. Since the perverse t-structure of $\mathrm{D}'_{\mathrm{cons}}(X, \Omega_T)$ is glued from the one on each stratum, for X_s a stratum, we have an intermediate extension $i_{s,!*} : \mathrm{Perv}(X_s, \Omega_T)_{\mathrm{unip}} \to \mathrm{Perv}(X, \Omega_T)_{\mathrm{unip}}$. Since the category $\mathrm{Perv}(X_s, \Omega_T)_{\mathrm{unip}}$ is equivalent to the abelian category $\Omega_T - \mathrm{mod}$, for $M \in \Omega_T - \mathrm{mod}$ we will denote by $\mathrm{IC}(X_s, M)$ the corresponding object in $\mathrm{Perv}(X, \Omega_T)_{\mathrm{unip}}$.

Step 1: Every object in $\mathrm{Perv}(X, \Omega_T)_{\mathrm{unip}}$ has a finite filtration with graded components of the form $\mathrm{IC}(X_s, M)$ for varying s and M. One can just apply the proof of lemma 2.1.4 of [RSW13] as the proof requires only the formalism of recollement of t-structures and our geometric setup.

Step 2: For all M and s there exists a projective object with a Δ-filtration in $\mathrm{Perv}(X, \Omega_T)_{\mathrm{unip}}$ that surjects onto $\mathrm{IC}(s, M)$. The proof essentially copies the one of [RSW13] 2.3.1 and of [BGS96]. The argument is an induction on the strata, if there is only one stratum then the statement is clear. Let $X_s \subset X$ be an open stratum and $i' : X' \subset X$ be the closed complement. The functor $i'_!$ is t-exact and induces an isomorphism $\mathrm{Ext}^1_{X'}(A, B) = \mathrm{Ext}^1(i'_!A, i'_!B)$ for $A, B \in \mathrm{Perv}(X')$. It is enough to show the statement for $\mathrm{IC}(X_t, \Omega_T) = \mathrm{IC}_t$ as $\mathrm{IC}(-,-)$ preserve surjections, see [Jut09] 2.27. Since $i^* = i^!$ is t-exact the sheaf Δ_s is projective and surjects onto IC_s. Let $t \neq s$ and let $P' \to \mathrm{IC}_t$ be a surjection on X'. Since $i'_!$ is exact the map $i'_!P' \to i'_!\mathrm{IC}_t = \mathrm{IC}_t$ is still surjective.

Let $E = \mathrm{Ext}^1(P', \Delta_s)$ and $E_f \to E$ be a free Ω_T-module of finite rank surjecting onto E and let $E_f^\vee$ be its Ω_T-dual. The map

$$\Omega_T \to E_f \otimes_{\Omega_T} E_f^\vee \to E \otimes_{\Omega_T} E_f^\vee \simeq \mathrm{Ext}^1_X(P', E_f^\vee \otimes_{\Omega_T} \Delta_s),$$

sends 1 to an element which corresponds to an extension

$$0 \to E_f^\vee \otimes_{\Omega_T} \Delta_s \to P \to P' \to 0. \tag{3.117}$$

The object P surjects onto IC_t and has a Δ-filtration. It remains to see that it is projective. This follows from the following points.

(i). For all $B \in \mathrm{Perv}(X')$, we have $\mathrm{Ext}^1(P, B) = 0$. This follows from the long exact sequence attached to 3.117 and the projectivity of Δ_s.

(ii). We have $\mathrm{Ext}^1(P, \Delta_s) = 0$. Again the long exact sequence from (3.117) yields

$$\mathrm{Hom}(E_f^\vee \otimes_{\Omega_T} \Delta_s, \Delta_s) \to \mathrm{Ext}^1(P', \Delta_s) = E \to \mathrm{Ext}^1(P, \Delta_s) \to \mathrm{Ext}^1(E_f^\vee \otimes_{\Omega_T} \Delta_s, \Delta_s) = 0.$$

The first map identifies with the map $E_f \to E$ and is then surjective, the third term is therefore 0.

(iii). We have $\mathrm{Ext}^2(P, B) = 0$ for all $B \in \mathrm{Perv}(X)$. It is enough to show it for all IC_u for varying u. For such u we have a short exact sequence $0 \to \mathrm{IC}_u \to \nabla_u \to K \to 0$ which induces a long exact sequence

$$\mathrm{Ext}^1(P, K) \to \mathrm{Hom}^2_{\mathrm{D}_{\mathrm{cons}}(X,\Omega_T)_{\mathrm{unip}}}(P, \mathrm{IC}_u) \to \mathrm{Hom}^2_{\mathrm{D}_{\mathrm{cons}}(X,\Omega_T)_{\mathrm{unip}}}(P, \nabla_u).$$

We can assume K lives on X' and then $\mathrm{Ext}^1(P, K) = 0$ and since P is Δ-filtered $\mathrm{Hom}^2(P, \nabla) = 0$.

(iv). Now use the exact sequence $0 \to K' \to \Delta_s \to \mathrm{IC}_s \to 0$ and the corresponding long exact sequence

$$\mathrm{Ext}^1(P, \Delta_s) \to \mathrm{Ext}^1(P, \mathrm{IC}_s) \to \mathrm{Hom}^2(P, K').$$

But we have already killed the first and last terms. Hence $\mathrm{Ext}^1(P, \mathrm{IC}_s) = 0$.

Step 3: For this step, we repeat the arguement of [RSW13] 2.3.4. Since $\mathrm{Perv}'(X, \Omega_T)$ generates both the categories $\mathrm{D}_{\mathrm{cons}}(X, \Omega_T)_{\mathrm{unip}}$ and $\mathcal{D}^b(\mathrm{Perv}'(X, \Omega_T)_{\mathrm{unip}})$, we only need to show that for all $A, B \in \mathrm{Perv}'(X, \Omega_T)_{\mathrm{unip}}$ the map

$$\mathrm{Ext}^i_{\mathrm{Perv}'}(A, B) \to \mathrm{Hom}^i_{\mathrm{D}_{\mathrm{cons}}(X,\Omega_T)_{\mathrm{mon}}}(A, B)$$

is an isomorphism. First assume that A is projective with a Δ filtration, we already know that for $i = 0, 1$ both side coincide, we want to show that for $i > 0$ both sides vanish. For the left hand side this is clear, for the right hand side, by the first step, we can reduce to the case $B = \mathrm{IC}(X_s, M)$, using the exact sequence

$$0 \to \mathrm{IC}(X_s, M) \to \nabla_s \otimes_{\Omega_T} M \to K \to 0$$

and the induced long exact sequence after applying Hom^*, using lemma 3.B.4 we get that $\mathrm{Hom}^{i-1}(A, K) = 0$ implies $\mathrm{Hom}^i(A, \mathrm{IC}(X_s, M)) = 0$. Now we use step 2, to find a surjection from a projective P with a Δ-filtration onto A, then one argues as in [RSW13] 2.3.4. to conclude. ☐

Theorem 3.B.12 (Structure of Tilting sheaves)**.** (i). *For all $s \in S$ there exists an indecomposable tilting sheaf T_s on $\overline{X}_s$ such that $i_s^* T_s \simeq \nu_s^* L_T[\dim X_s]$.*

(ii). *The isomorphism classes of indecomposable tilting sheaves are in bijection with $\mathcal{S}$ and the sheaf T_s corresponding to s is characterized by the fact that $\pi_{\dagger,\mathbb{F}_\ell} T_s$ is an indecomposable tilting in the sense of [BBM04a].*

(iii). *It T is a tilting sheaf then $T \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell$ is a free monodromic tilting sheaf in the sense of [BR22b].*

Proof. For the points (ii) and (iii) we refer to [BR22b] 5.12. We discuss the first point, which is the most technical, the proof is just essentially the same as the one of [AR16] B.2.

The proof is by induction on the number of strata in $\overline{X}_s$. If there is only one stratum this is clear. Otherwise let $X_t \subset \overline{X}_s$ be a closed stratum and $U = \overline{X}_s \setminus X_t \xrightarrow{j} \overline{X}_s$ be the open complement. By induction there exists T_U on U satisfying the assumption. We consider $j_! T_U$, it is a perverse sheaf since j is affine and it is equipped with a Δ-filtration. Let $E = \mathrm{Ext}^1(\Delta_t, j_! T_U)$ and $E_f \to E$ a surjection from a free Ω_T-module of finite type. Consider the composition $\Omega_T \to E_f^\vee \otimes E \simeq \mathrm{Ext}^1(\Delta_t \otimes_{\Omega_T} E_f^\vee, j_! T)$ it sends 1 to an extension

$$0 \to j_! T_U \to T \to \Delta_t \otimes_{\Omega_T} E_f^\vee \to 0 \tag{3.118}$$

where T is a perverse sheaf, which clearly has a Δ-filtration. We now claim that T is tilting. Firstly let $\tilde{i}_u : X_u \to U$ be a stratum and $i_u : X_u \to U \to \overline{X}_s$ be the composite, then $i_u^! = \tilde{i}_u^! j^!$ applying it to 3.118 yields $i_u^! T = \tilde{i}_u^! T_U$ which is a perverse sheaf made of direct sums of copies of $\nu_u^* L_T[\dim X_u]$. We now want to show $\mathrm{Ext}^i(\Delta_t, T)$ vanishes for all $i > 0$.

(i). First applying the functor $\mathrm{Hom}(\Delta_t, -)$ to 3.118, yields an exact sequence

$$E_f \to E \to \mathrm{Ext}^1(\Delta_t, T) \to \mathrm{Ext}^1(\Delta_t, \Delta_t \otimes_{\Omega_T} E_f^\vee) = 0$$

hence $\mathrm{Ext}^1(\Delta_t, T) = 0$.

(ii). Since T_U has a ∇-filtration, $j_! T$ has a filtration with graded of the form $j_! \nabla_u$, it is therefore enough to show that $\mathrm{Hom}^i(\Delta_t, j_! \nabla_u) = 0$ for $i > 1$.

(iii). There are two fiber sequence

$$M \to j_! \nabla_u \to j_{!*} \nabla_u, \text{ and } j_{!*} \nabla_u \to j_* \nabla_u \to N.$$

Both M and N are supported on X_t and in negative perverse degrees, hence $\mathrm{Hom}^k(\Delta_t, M) = \mathrm{Hom}^k(\Delta_t, N) = 0$ for $k > 0$.

(iv). We now apply the functor $\mathrm{Hom}(\Delta_t, -)$ to both triangles and get long exact sequences, inputing that $\mathrm{Hom}^k(\Delta_t, j_* \nabla_u) = 0$ for $k > 0$, we first get that $\mathrm{Ext}^i(\Delta_t, j_{!*} \nabla_u) = 0$ for $i > 1$ and then that $\mathrm{Hom}^i(\Delta_t, j_! \nabla_u) = 0$ for $i > 1$.

This yields that $i_t^! T$ is perverse and it remains to see that is a direct sum of copies of $\nu_t^* L_T$ (only the freeness is non trivial here). To check this we apply the functor $\pi_{\dagger, \overline{\mathbb{F}}_\ell}$. The stalks of $\pi_{\dagger, \overline{\mathbb{F}}_\ell} i_t^! T$ are the reductions mod $\mathfrak{m}$, the maximal ideal of Ω_T, and therefore the stalks of $\pi_{\dagger, \overline{\mathbb{F}}_\ell} i_t^! T$ are projective Ω_T-modules if and only if their mod $\mathfrak{m}$ reduction is concentrated in a single degree. But after reducing mod $\mathfrak{m}$, the construction we have done is nothing else than the construction of [AR16], Appendix B.

□

3.B.2 Tilting sheaves in the non unipotent case

We extend the definitions of the previous section to this one to cover the non unipotent case. We fix $\chi \in \mathrm{CH}(T)$.

Definition 3.B.13. We define

(*i*). the category $\mathrm{D}'_{\mathrm{cons}}(X/(T,\mathcal{L}_\chi),\overline{\mathbb{Z}}_\ell)$ to be the full subcategory of $\mathrm{D}_{\mathrm{cons}}(X/(T,\mathcal{L}_\chi),\overline{\mathbb{Z}}_\ell))$ of sheaves generated by the collection $i_s\nu_s^*(\mathcal{L}_\chi)$.

(*ii*). The standard and costandard sheaves as $\Delta_{s,\chi} = i_{s,!}\nu_s^*(L_T\otimes\mathcal{L}_\chi)[\dim(X_s)]$ and $\nabla_{s,\chi} = i_{s,*}\nu_s^*(L_T\otimes\mathcal{L}_\chi)[\dim(X_s)]$.

(*iii*). The category $\mathrm{D}'_{\mathrm{cons}}(X,\Omega_T)_\chi$ as a full subcategory of $\mathrm{D}_{\mathrm{cons}}(X,\Omega_T)_\chi$ of sheaves A such that $\pi_\dagger(A)\in\mathrm{D}'_{\mathrm{cons}}(X/(T,\mathcal{L}_\chi),\overline{\mathbb{Z}}_\ell)$.

The perverse sheaves with a Δ or ∇-filtration are defined accordingly and so are the tilting sheaves.

We also assume the following condition :

$(C)_\chi$ For all s,t, all cohomology sheaves $H^i(i_t^*i_{s,*}\nu_s^*\mathcal{L}_\chi)$ are of the form $M\otimes_{\Omega_T}\nu_t^*\mathcal{L}_\chi$ where M is an Ω_T-module.

Remark 3.B.14. The category of sheaves satisfying condition $(C)_\chi$ is independent of the choice of the trivializations ν_s.

Theorem 3.B.15. *All results of the previous section remain valid in the twisted setting.*

Proof. All statements, except lemma 3.B.10 do not require the fact that we deal with unipotent sheaves and the proof are very axiomatic. For the remaining statement we refer to [Gou21] 7.5.6. ☐

3.C Twisted categorical centers

In this appendix, we recall some known facts about categorical centers and traces. We will mostly follow [BZNF10]. We refer to [GKRV22] for the various twisted versions.

3.C.1 Monoidal structure and categorical centers

Let $\Lambda\in\{\overline{\mathbb{Q}}_\ell,\overline{\mathbb{Z}}_\ell,\overline{\mathbb{F}}_\ell\}$ and consider DGCat_Λ the category of all presentable stable cocomplete Λ-linear categories. It is equipped with the Lurie tensor product defined in [Lur] Section 4.8. We refer to [GR17] Chapter 1 for a construction of this category. In this section, we call a category an object of DGCat_Λ.

Definition 3.C.1 (F-categorical center)**.** Let $\mathcal{C}$ be a monoidal category in DGCat_Λ and let $\mathrm{F}:\mathcal{C}\to\mathcal{C}$ be a monoidal endofunctor of $\mathcal{C}$. We denote by $\mathcal{C}^{\mathrm{rev}}$ the category $\mathcal{C}$ equipped with the opposite monoidal structure.

The category $\mathcal{C}$ has the structure of a $\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}}$-module. We denote by $\mathcal{C}_\mathrm{F}$ the same category $\mathcal{C}$ but with its bimodule structure twisted on the right by F. The F-categorical center of $\mathcal{C}$ is then defined as

$$\mathcal{Z}_\mathrm{F}(\mathcal{C}) = \mathrm{Hom}_{\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}}}(\mathcal{C},\mathcal{C}_\mathrm{F}). \tag{3.119}$$

Definition 3.C.2 (F-categorical trace)**.** Let $\mathcal{C}$ be a monoidal category and $\mathrm{F}:\mathcal{C}\to\mathcal{C}$ be a monoidal endofunctor of $\mathcal{C}$. The (2)-categorical trace of F on $\mathcal{C}$ is defined as

$$\mathrm{Tr}_2(\mathrm{F},\mathcal{C}) = \mathcal{C}\otimes_{\mathcal{C}\otimes\mathcal{C}^{\mathrm{rev}}}\mathcal{C}_\mathrm{F}. \tag{3.120}$$

Definition 3.C.3. Let $\mathcal{C}$ be a monoidal category. An object $x \in \mathcal{C}$ is left dualizable (resp. right dualizable) if there exists an object ${}^{\vee}x$ and morphisms

$$\mathrm{ev} : x \otimes {}^{\vee}x \to 1_{\mathcal{C}}, \mathrm{coev} : 1_{\mathcal{C}} \to {}^{\vee}x \otimes x, \tag{3.121}$$

where $1_{\mathcal{C}}$ denotes the unit of $\mathcal{C}$, (resp. if there exists an object $x^{\vee}$ and morphisms,

$$\mathrm{ev} : x^{\vee} \otimes x \to 1_{\mathcal{C}}, \mathrm{coev} : 1_{\mathcal{C}} \to x \otimes x^{\vee}) \tag{3.122}$$

satisfying usual identities.

Remark 3.C.4. If $\mathcal{C}$ is symmetric monoidal, we will freely identify left and right adjoints.

Definition 3.C.5. A category $\mathcal{C}$ is called dualizable if it is dualizable as an object of DGCat_{Λ}.

Lemma 3.C.6 ([GR17], 1.7.3.2)**.** *If $\mathcal{C}$ is compactly generated, i.e. $\mathcal{C} = \mathrm{Ind}\mathcal{C}^{\omega}$, where $\mathcal{C}^{\omega}$ is the full subcategory of compact objects of $\mathcal{C}$, then $\mathcal{C}$ is dualizable. Its dual is identified with $\mathcal{C}^{\vee} = \mathrm{Ind}\mathcal{C}^{\omega,\mathrm{op}}$ and the evaluation is given by*

$$\mathrm{ev} : \mathcal{C} \otimes \mathcal{C}^{\vee} \to \mathrm{D}(\Lambda), \tag{3.123}$$

for $A, B \in \mathcal{C}^{\omega}, \mathrm{ev}(A \otimes B) = \mathrm{Hom}(B, A)$.

Remark 3.C.7 ([GR17]). Let $\mathcal{C}$ be a compactly generated category. Then an equivalence $\mathbb{D} : \mathcal{C}^{\omega,\mathrm{op}} \to \mathcal{C}^{\omega}$ induces a equivalence $\mathcal{C} \simeq \mathcal{C}^{\vee}$.

Remark 3.C.8. The cateogry $\mathrm{D}(\Lambda)$ is the unit object of DGCat_{Λ}. We have $\mathrm{Hom}(\mathrm{D}(\Lambda), \mathrm{D}(\Lambda)) = \mathrm{D}(\Lambda)$.

Lemma 3.C.9. *Let $\mathcal{C}$ be a dualizable category. Then for all categories $\mathcal{D}$, there is a natural isomorphism*

$$\mathcal{C}^{\vee} \otimes \mathcal{D} \simeq \mathrm{Hom}(\mathcal{C}, \mathcal{D}). \tag{3.124}$$

Lemma 3.C.10. *Let X be a quasi-compact Λ-scheme, then $\mathrm{D}_{\mathrm{qcoh}}(X)$ is compactly generated by the category of perfect complexes. The naive duality*

$$\mathbb{D} : \mathrm{Perf}(X)^{op} \to \mathrm{Perf}(X), M \mapsto \mathcal{H}om_X(M, \mathcal{O}_X) \tag{3.125}$$

induces a self duality on $\mathrm{D}_{\mathrm{qcoh}}(X)$.

Lemma 3.C.11. *There is a natural equivalence*

$$\mathrm{End}(\mathrm{D}_{\mathrm{qcoh}}(X)) \simeq \mathrm{D}_{\mathrm{qcoh}}(X \times X). \tag{3.126}$$

3.C.2 Twisted centers of Hecke categories

Let $f : X \to Y$ be a faithfully flat finite type morphism of schemes. We consider the category $\mathrm{QCoh}(X \times_Y X)$ which we call the Hecke category of f. The category $\mathrm{D}_{\mathrm{qcoh}}(X \times_Y X)$ is equipped with the convolution structure of [BZNF10]. On objects A, B it is given by

$$A * B = p_{13,*}(p_{12}^{*}A \otimes p_{23}^{*}B), \tag{3.127}$$

where the maps are the projections in the following diagram

$$\begin{tikzcd}
 & X \times_Y X \times_Y X \arrow[dl, "p_{12}"'] \arrow[d, "p_{13}"'] \arrow[dr, "p_{23}"] & \\
X \times_Y X & X \times_Y X & X \times_Y X.
\end{tikzcd}$$

Let $\mathrm{F}_X : X \to X$ and $\mathrm{F}_Y : Y \to Y$ be morphisms commuting with f. We denote by $\mathcal{L}_{\mathrm{F}}Y$ the intersection of the diagonal and the graph of F_Y. It fits into the following cartesian diagram.

$$\begin{tikzcd}
\mathcal{L}_{\mathrm{F}}Y \arrow[r] \arrow[d] & Y \arrow[d, "(\mathrm{id}\times\mathrm{F})"] \\
Y \arrow[r, "\Delta"'] & Y \times Y
\end{tikzcd}$$

We introduce as [BZNF10], the (twisted) horocycle transform

$$\mathcal{L}_{\mathrm{F}}Y \xleftarrow{q} X \times_Y Y \times_{Y\times Y} Y = X \times_{X\times Y} X \xrightarrow{r} X \times_Y X, \tag{3.128}$$

where in the fiber product $X \times_{X\times Y} X$, the two maps $X \to X \times Y$ are given by $(\mathrm{id} \times f)$ and $(\mathrm{F}_X \times f)$ respectively. The F-horocycle transform is then defined as the functor $r_* q^*$.

Theorem 3.C.12 (Twisted variant of [BZNF10] Theorem 5.3)**.** *There is an equivalence*

$$\mathcal{Z}_F(\mathrm{D}_{\mathrm{qcoh}}(X \times_Y X)) \simeq \mathrm{D}_{\mathrm{qcoh}}(\mathcal{L}_F Y), \tag{3.129}$$

such that following diagram commutes

$$\begin{tikzcd}
\mathcal{Z}_{\mathrm{F}}(\mathrm{D}_{\mathrm{qcoh}}(X \times_Y X)) \arrow[d] & \mathrm{D}_{\mathrm{qcoh}}(\mathcal{L}_{\mathrm{F}}Y) \arrow[l] \arrow[dl, "r_* q^*"] \\
\mathrm{D}_{\mathrm{qcoh}}(X \times_Y X). &
\end{tikzcd}$$

Remark 3.C.13. In [BZNF10], they show this statement for the categorical center and not its twisted version. The proof in the twisted version is shown in the same way.

Chapter 4

On depth 0 local Langlands and global Chtoucas

Contents

4.1 Introduction

Let $q = p^r$ be a power of prime number $p > 0$ and let $\mathbb{F}_q$ be the finite field with q elements. Let K be a local field of equal characteristic with ring of integers $\mathcal{O}_K$ and let X be a smooth projective curve over $\mathbb{F}_q$ with function field F. Let $\mathcal{G}$ be a reductive group over X with generic fiber G and let $\mathcal{H}$ be a reductive group over $\mathcal{O}_K$ with generic fiber H. Denote by Weil_F and Weil_K the absolute Weil groups of F and K and by ${}^LG = \hat{G} \rtimes \mathrm{Weil}_F$ and ${}^LH = \hat{H} \rtimes \mathrm{Weil}_K$ the L-groups of G and H respectively. Finally fix $\ell \neq p$ a prime and denote by $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell\}$ a coefficient field.

Theorem 4.1.1. *(i). [Laf18] There is a map* $\mathrm{GLC} : \pi \mapsto \sigma_\pi$ *from cuspidal irreducible automorphic Λ-linear representations of $G(\mathbb{A}_F)$ to conjugacy classes of semisimple global Langlands parameters, where $\mathbb{A}_F$ denotes the ring of adèles of F. They are morphisms* $\mathrm{Weil}_F \to {}^LG(\Lambda)$ *satisfying the hypothesis of* loc. cit.

(ii). [GL17] There is a map $\mathrm{LLC}^{\mathrm{GL}} : \pi \mapsto \sigma_\pi$ *from the set of smooth irreducible Λ-linear representation of $H(K)$ to conjugacy classes of semisimple local Langlands parameters. They are morphisms* $\mathrm{Weil}_K \to {}^LH(\Lambda)$ *satisfying the hypothesis of* loc. cit.

(*iii*). *[GL17] The two constructions are compatible in the following sense. Let $x \in X$ be a closed point and choose an isomorphism between F_x the completion of F at x with K and an isomorphism $G_{F_x} = H$. This defines an inclusion of* $\mathrm{Weil}_K \subset \mathrm{Weil}_F$ *and a morphism* ${}^L H \to {}^L G$. *Let* $\pi = \otimes'_{y \in X} \pi_y$ *be a cuspidal automorphic representation of* $G(\mathbb{A}_F)$. *Then the semisimplification of the restriction of* σ_π *to* Weil_K *is conjugate to* σ_{π_x}.

Remark 4.1.2. If the group G is split, then using the compatibility with parabolic induction established in [Xue20b], we can remove the words 'cuspidal' in (i).

The main goal of this paper is to discuss, in the local setting, the structure of local Langlands parameters associated to depth 0 representations of $H(K)$. In the global setting, we want to discuss the local structure of global parameters associated to automorphic representations whose local component at a place x has depth 0. In view of (iii) of theorem 4.1.1, those two questions are essentially equivalent.

Let us now formulate our main results. We consider the local situation. Let $T \subset H$ be a maximal torus and let W be the Weyl group of $T_{K^{\mathrm{alg}}}$, where K^{alg} is an algebraic closure of K. We first need to recall some properties of the structure of Λ-linear smooth representations of $H(K)$, we refer to section 4.3 for a more detailed account.

Theorem 4.1.3 ([Lan18], [Lan21])**.** *The category* $\mathrm{Rep}^0_\Lambda H(K)$ *of depth* 0 *representations of* $H(K)$ *decomposes as a direct sum*

$$\mathrm{Rep}^0_\Lambda H(K) = \bigoplus_{s \in (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}}(\Lambda)} \mathrm{Rep}^s_\Lambda H(K), \tag{4.1}$$

where $\hat{T}$ *denotes the dual torus over* Λ, $\hat{T} /\!\!/ W$ *the GIT-quotient by the action of* W *and* $(-)^{\hat{\mathrm{F}}}$ *the scheme of invariants under the morphism dual to the Frobenius of* T.

Remark 4.1.4. We consider the decomposition into geometric series of *loc. cit.*. There is however a finer decomposition into rational series, this will play no role in this paper.

The decomposition of theorem 4.1.3, yields a map

$$\mathrm{LS} : \mathrm{Irr}^0_\Lambda(H(K)) \to (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}}(\Lambda), \tag{4.2}$$

where $\mathrm{Irr}^0_\Lambda(H(K))$ is the set of irreducible depth 0 representations of $H(K)$, characterized by $\mathrm{LS}(\pi) = s$ if and only if π lies in the direct summand indexed by s.

Definition 4.1.5. Let $\phi : \mathrm{Weil}_K \to {}^L H(\Lambda)$ be a Langlands parameter. We say that this parameter is tame if it factors through $\mathrm{Weil}^t_K = \mathrm{Weil}_K / P_K$, the tame Weil group, where P_K denotes the wild inertia subgroup. We denote by $(\underline{Z}^{1,t}(K, \hat{H}) /\!\!/ \hat{H})(\Lambda)$ the set conjugacy classes of semisimple tame local Langlands parameters.

We fix τ_K a topological generator of the tame inertia. Given a tame local Langlands parameter ϕ, we denote by $\mathrm{ev}_{\tau_K}(\phi)$ the image in $\hat{H} /\!\!/ \hat{H} = \hat{T} /\!\!/ W$ of $\phi(\tau_K)$. We can now state our main theorem.

Theorem 4.1.6. *Let $\pi \in \mathrm{Irr}^0_\Lambda(H(K))$ then $\mathrm{LLC}^{\mathrm{GL}}(\pi)$ is a tame local Langlands parameter. Furthermore the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Irr}^0_\Lambda H(K) \arrow[r, "\mathrm{LLC}^{\mathrm{GL}}"] \arrow[d, "\mathrm{LS}"'] & (\underline{Z}^{1,t}(K,\hat{H}) /\!\!/ \hat{H})(\Lambda) \arrow[d, "\mathrm{ev}_{\tau_K}"] \\
(\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}} \arrow[r, equal] & (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}}
\end{tikzcd}$$

We fix a closed point $x \in X$, an isomorphism $F_x = K$ between the completion of F at x and K and an isomorphism $\mathcal{G}_{\mathcal{O}_x} = \mathcal{H}$. This last isomorphism yields an isomorphism $G_{F_x} = H$. Let σ be a polysimplex in the Bruhat-Tits building of $H(K)$ and let $\mathcal{H}_\sigma$ be the parahoric group scheme over $\mathcal{O}_K$ corresponding to σ. Using the isomorphism $G_{F_x} = H$, we glue $\mathcal{H}_\sigma$ with $\mathcal{G}_{|(X-x)}$ and we get a smooth group scheme $\mathcal{G}_\sigma$ over X whose restriction to $\mathcal{O}_K$ is $\mathcal{H}_\sigma$. Finally let $N = x + N^x$ be an effective divisor of X with $x \notin N^x$ and I be a finite set. The stack of chtoucas $\mathrm{Cht}_{\mathcal{G}_\sigma,I,N}$ with $|I|$-legs and level structure N is an algebraic stack over $(X-N)^I$, its definition is recalled in section 4.4. Let us for now ignore the Harder-Narasimhan truncations and the role of the center. Let $W \in \mathrm{Rep}_\Lambda({}^LG)^I$, there is a sheaf $\mathcal{F}_{I,W}$ on $\mathrm{Cht}_{\mathcal{G}_\sigma,I,N}$ coming from geometric Satake. Denote by $\mathfrak{p} : \mathrm{Cht}_{\mathcal{G}_\sigma,I} \to (X-N)^I$ the leg map and by $\mathcal{H}^j_{I,N,W} = R^j\mathfrak{p}_!\mathcal{F}_{I,W}$ the cohomology sheaf. This sheaf is an ind-lisse sheaf on $(X-N)^I$ by the main theorem of [Xue20d]. Let us further fix $\bar{\eta} \to X$ a generic geometric point, we take

$$H^j_{I,N,W} = (\mathcal{H}^j_{I,N,W})_{|\Delta(\bar{\eta})}, \tag{4.3}$$

the fiber of $\mathcal{H}^j_{I,N,W}$ at the geometric point $\Delta(\bar{\eta})$. As this sheaf is ind-lisse, taking its fiber at a geometric point yields a representation of $\mathrm{Weil}((X-N)^I, \Delta(\bar{\eta}))$. Using Drinfeld's lemma, Xue [Xue20d], generalizing an argument of [Laf18], shows that the representation of $\mathrm{Weil}((X-N)^I, \Delta(\bar{\eta}))$ factors through $\mathrm{Weil}(X-N, \bar{\eta})^I$. Hence we have a functor

$$\begin{aligned}
\mathrm{Rep}_\Lambda(({}^LG)^I) &\to \mathrm{Rep}_\Lambda(\mathrm{Weil}(X-N,\bar{\eta})^I) \\
W &\mapsto H^j_{I,N,W}.
\end{aligned}$$

The collection of these functors as I-varies carries a lot more structure. This structure is the data of morphism of cocartesian functors over FinSet, the category of finite sets. The definition of this structure is recalled in section 4.4. We will systematically restrict the action of $\mathrm{Weil}(X-N)^I$ to $\mathrm{Weil}^I_{F_x}$. The formalism of excursion of [Laf18] produces an algebra $\mathrm{Exc}(\mathrm{Weil}_{F_x}, \hat{G})$ whose Λ-points are in bijection with conjugacy classes of semisimple local Langlands parameters. There is also an evaluation morphism $\mathrm{Spec}(\mathrm{Exc}(\mathrm{Weil}_{F_x}, \hat{G})) \to \hat{G} /\!\!/ \hat{G}$, such that on Λ-points this map is identified with the evaluation and semisimplification on τ_{F_x}. Moreover given such a cocartesian functor, each of vector spaces $H^j_{I,N,W}$ is equipped with a canonical action of $\mathrm{Exc}(\mathrm{Weil}_{F_x}, \hat{G})$. Denote by V_σ the unipotent radical of the special fiber of $\mathcal{H}_\sigma$ and M_σ is reductive quotient. The vector space $H^j_{I,N,W}$ carries an action of the finite group $\mathcal{H}_{\sigma,\mathbb{F}_x}(\mathbb{F}_x)$. We consider the vector space $(H^j_{I,N,W})^{V_\sigma(\mathbb{F}_x)}$ which then also carries an action of $M_\sigma(\mathbb{F}_x)$.

Theorem 4.1.7. (*i*). *For all I, W the $\mathrm{Weil}^I_{F_x}$-module $(H^j_{I,W,N})^{V_\sigma(\mathbb{F}_x)}$ is tamely ramified, that is, the action of $\mathrm{Weil}^I_{F_x}$ factors through the tame quotient $(\mathrm{Weil}^t_{F_x})^I$.*

(ii). *Let $s \in (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}}(\Lambda)$, then as an $\mathrm{Exc}(\mathrm{Weil}^t_{F_x}, \hat{G})$-module $e_s(H^j_{I,W,N})^{V_\sigma(\mathbb{F}_x)}$ is supported on $\mathrm{ev}^{-1}_{\tau_{F_x}}(s)$, where e_s is the idempotent in $\Lambda[M_\sigma(\mathbb{F}_x)]$ corresponding to the Lusztig series attached to s, we refer to Section 4.2 for the notations.*

Finally let $\underline{Z}^1(F_x, \hat{G})$ be the moduli space of local Langlands parameters as constructed in [DHKM20], [Zhu21] and [FS21]. In this setting a construction of [LZ18] attaches to the cocartesian functor $(I, W) \mapsto H^j_{I,N,W}$ a canonical $\hat{G}$-equivariant quasi-coherent sheaf on $\underline{Z}^1(F_x, \hat{G})$. Let us denote this coherent sheaf $\mathcal{M}^j_N$, whose construction is recalled in section 4.4.5. Let us select $\underline{Z}^{1,t}(F_x, \hat{G})$ the closed subscheme of tame Langlands parameters, this is a union of connected components of $\underline{Z}^1(F_x, \hat{G})$. We still denotes by $\mathrm{ev}_{\tau_{F_x}} : \underline{Z}^{1,t}(F_x, \hat{G}) \to \hat{G} /\!\!/ \hat{G}$ the morphism induced by evaluation at τ_{F_x}. The quasi-coherent sheaf $\mathcal{M}^j_N$ still carries an action of $\mathcal{H}_\sigma(\mathbb{F}_x)$.

Corollary 4.1.8. (i). *The quasi-coherent sheaf $(\mathcal{M}^j_N)^{V_\sigma(\mathbb{F}_x)}$ is supported on $\underline{Z}^{1,t}(F_x, \hat{G})$.*

(ii). *Using the same notations as in theorem 4.1.7, let $s \in (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}}(\overline{\mathbb{Q}}_\ell)$, then the quasi-coherent sheaf $e_s(\mathcal{M}^j_N)^{V_\sigma(\mathbb{F}_x)}$ is supported on $\mathrm{ev}^{-1}_{\tau_{F_x}}(s)$.*

While preparing this paper, we learned that Andrew Salmon was working on a similar project, [Sal23a]. Let us highlight the overlap and differences between our work. In [Sal23a], Salmon shows theorem 4.1.7 assuming the following hypothesis, $\Lambda = \overline{\mathbb{Q}}_\ell$, $\mathcal{G}$ is constant and split, σ is a hyperspecial point and the degree of $x/\mathbb{F}_q$ is one. On the other hand, in this situation, he is able to describe the unipotent part of the parameters while we restrict ourselves to the semisimple part.

4.1.1 Outline of the proof

We now give an outline of the proof. For simplicity, we will keep on ignoring Harder-Narasimhan truncations and the role of the center in this introduction. These technicalities will be addressed in section 5 through 8.

There is a map recalled in section 4.4

$$\epsilon : \mathrm{Cht}_{\mathcal{G}_\sigma, I, N} \to L^+_I \mathcal{G}_\sigma \backslash \mathrm{Gr}_{\mathcal{G}_\sigma, I}.$$

Over $(X - N)^I$, the geometric Satake equivelence provides a fully-faithful functor $\mathrm{Rep}_\Lambda({}^L G)^I \to \mathrm{Perv}(L^+_I \mathcal{G}_\sigma \backslash \mathrm{Gr}_{\mathcal{G}_\sigma, I}, \Lambda)$. The cohomology sheaves of stacks of chtoucas are then defined as

$$\mathcal{H}_{I,N,W} = \mathfrak{p}_! \epsilon^* \mathrm{Sat}(W), \tag{4.4}$$

where $\mathfrak{p} : \mathrm{Cht}_{\mathcal{G}_\sigma, I, N} \to X^I$ is the leg map. By the main theorem of [Xue20d], these sheaves are ind-lisse on $(X - N)^I$.

The local Langlands correspondence of [GL17] relies on the formalism of excursion applied to the functor $(I, W) \mapsto \mathcal{H}_{I,N,W}$. The properties of the parameters we want to discuss can be read off the Galois representations corresponding to the sheaves $\mathcal{H}^j_{I,N,W}$. We choose a maximal torus $\mathcal{T} \subset \mathcal{H}$ over $\mathcal{O}_K$, this choice then defines a maximal torus $T_M \subset M_\sigma$, we denote by W_M the corresponding Weyl group of the pair $(M_{\sigma, \bar{\mathbb{F}}_q}, T_{M, \bar{\mathbb{F}}_q})$. Combining the formalism of excursion and of type theory for depth 0 representations reduces the problem to computing the local Weil representation of ${}^*\mathcal{R}_{w,\chi} \mathcal{H}^{V_\sigma(k_x)}_{I,N,W}$, where ${}^*\mathcal{R}_{w,\chi}$ denotes the χ-isotypic component of the Deligne-Lusztig restriction functor attached to an element $w \in W_M$ the Weil group of M_σ and χ is a character of $T^{w\mathrm{F}}_M$ the corresponding finite torus.

We compute the action of the inertia on this sheaf via a nearby cycles construction. In section 4.2, we reformulate the Deligne-Lusztig induction and restriction functors in a geometric way. We then introduce the following correspondence

$$(X-N)^I \times x \xleftarrow[\mathfrak{p}]{} \mathrm{Cht}_{\mathcal{G}_\sigma, I, N} \times^{M_\sigma(\mathbb{F}_x)} M_\sigma/B_M \xrightarrow[\epsilon]{} L_I^+\mathcal{G}_\sigma \backslash \mathrm{Gr}_{\mathcal{G}_\sigma, I} \times \frac{U_M \backslash M_\sigma / U_M}{\mathrm{Ad}_{\mathrm{F}} T}$$

and there is a sheaf $\nabla_w(\mathrm{Reg}_{T^{w\mathrm{F}}})$ on $\frac{U_M \backslash M_\sigma / U_M}{\mathrm{Ad}_{\mathrm{F}} T}$ such that ${}^*\mathrm{R}_{w,\chi}\mathcal{H}_{I,N,W}^{V_\sigma(\mathbb{F}_x)} = \mathfrak{p}_!\epsilon^*(\mathrm{Sat}(W) \boxtimes \nabla_w(\mathrm{Reg}_{T^{w\mathrm{F}}}))$. We refer to section 4.5 for the details of this construction.

The second step in this construction is to extend all three stacks to stacks over $(X - N^x)^I$ so that we can take some nearby cycles to x. For this, we construct a map emb from $\mathrm{Cht}_{\mathcal{G}_\sigma, I, N} \times^{M_\sigma(\mathbb{F}_x)} M_\sigma/B_M$ to a stack of chtoucas for a different group. This new group is $\mathcal{G}_C$ which is a group scheme over X that differs from $\mathcal{G}_\sigma$ only at x. The group $\mathcal{G}_C$ is the group that is obtained by dilating a Borel subgroup in the special fiber of $\mathcal{G}_\sigma$ and as such its group of $\mathcal{O}_x$-points is an Iwahori in $G(F_x)$. Similarly we also introduce $\mathcal{G}_{C^0}$, obtained by dilating the unipotent radical of the same Borel in the special fiber. On the right side of the above correspondence, there is also a map emb making the following diagram commute and the right square Cartesian.

$$\begin{tikzcd}
(X-N)^I \arrow[d] & \mathrm{Cht}_{\mathcal{G}_\sigma, I, N} \times^{M(\mathbb{F}_x)} M_\sigma/B_M \arrow[l, "\mathfrak{p}"'] \arrow[r, "\epsilon"'] \arrow[d, "\mathrm{emb}"] & L_I^+\mathcal{G}_\sigma \backslash \mathrm{Gr}_{\mathcal{G}_\sigma, I} \times \frac{U_M \backslash M_\sigma / U_M}{\mathrm{Ad}_{\mathrm{F}} T} \arrow[d] \\
(X-N^x)^I & \mathrm{Cht}_{\mathcal{G}_C, I \sqcup \{0\}, N^x} \arrow[l] \arrow[r] & \frac{L^+_{I \sqcup \{0\}}\mathcal{G}_{C^0} \backslash \mathrm{Gr}_{\mathcal{G}_{C^0}, I}}{\mathrm{Ad}_{\mathrm{F}} T}
\end{tikzcd}$$

Now that we have extended our stacks to stacks that live over $(X - N^x)^I$, we can take nearby cycles. Since we are over a power of a curve and not simply over one copy of the curve there are several way to take nearby cycles. The main construction of Salmon [Sal23b] shows that these different nearby cycles coincide and that the functor $\mathfrak{p}_!$ commutes with these nearby cycles. We extend his proof to our setting. In the end, we are interested in the local monodromy of $\mathfrak{p}_!\epsilon^*(\mathrm{Sat}(W) \boxtimes \nabla_w(\mathrm{Reg}_{T^{w\mathrm{F}}}))$, this local monodromy can be extracted from the nearby cycles (with respect to the identity map $(X-N)^I \to (X-N)^I$) applied the the cohomology sheaves. Since these nearby cycles commute with $\mathfrak{p}_!$, it is enough to show their property on stacks of chtoucas. Furthermore, the map ϵ is smooth and thus the nearby cycles commute with smooth pullbacks, hence, the control of the monodromy can be done on the side of affine Grassmannians. On the geometric side, we can compute the nearby cycles of the sheaf $\mathrm{Sat}(W) \boxtimes \nabla_w(\mathrm{Reg}_{T^{w\mathrm{F}}})$ in terms of (a variation of) Gaitsgory's central functors [Gai01]. We refer to section 4.7 for a discussion about these central functors. Finally, we use the Wakimoto filtration on these central functors constructed in [AB09] and [BFO09] to control the monodromy action on the nearby cycles.

4.1.2 Organization of the paper

In section 4.2, we recall some aspects of Deligne-Lusztig theory and the F-twisted horocycle transform. In section 4.3, we recall results of type theory of depth 0 representations of $H(K)$. In section 4.4, we recall some of the main player for the geometry of the problem, that is, affine Grassmannian, stacks of chtoucas, the excursion algebra and the space of parameters. We also recall the key structural properties of the cohomology of stacks of chtoucas which we will use. In section 4.5, we construct the morphism emb and discuss its main properties. In section 4.6, we discuss tame nearby cycles over a self product of a curve. In section 4.7, we recall the construction of (non unipotent) central functors and their Wakimoto filtration. Section 4.8 is the heart of this paper, we assemble the results of the previous sections and we show the main theorem. Finally, in the appendix 4.A, we extend the result of [AB09] and [BFO09] for the Wakimoto filtrations on the central functors to modular coefficients.

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4.1.4 Notations

Etale sheaves

We will use the following notations. We fix $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell\}$ a coefficient field. If X is a stack, we denote by $\mathrm{D}(X, \Lambda)$ the category of ind-constructible ℓ-adic sheaves on X. If X is a scheme, this category is defined in [HRS21] Section 3. If X is a stack, we define this category by descent. For all algebraic stacks X, the category $\mathrm{D}(X, \Lambda)$ is a closed symmetric monoidal Λ-linear triangulated category. We will denote by $\otimes$ the tensor product or $\otimes_\Lambda$ if we want to put some emphasis on the coefficients. We denote by $\mathcal{H}om$ the internal mapping spaces. If $f : X \to Y$, we denote by $f^*, f_*, f^!$ and $f_!$ the usual derived functors between the categories $\mathrm{D}(X, \Lambda)$ and $\mathrm{D}(Y, \Lambda)$.

Finite group actions

Let Γ be a finite group. By classical theory, we have $\mathrm{D}(\mathrm{pt}/\Gamma, \Lambda) = \mathrm{D}(\mathrm{Rep}_\Lambda \Gamma)$, we refer to section 3.3.1 for a proof. The same argument shows that for all algebraic stacks Y, we have $\mathrm{D}(Y \times \mathrm{pt}/\Gamma, \Lambda) = \mathrm{D}(Y, \Lambda[\Gamma])$. If X is a stack with an action of Γ and $f : X \to Y$ is a Γ-equivariant morphism for the trivial action of Γ on Y. Then for $A \in \mathrm{D}(X, \Lambda)$ a Γ-equivariant sheaf on X, the pushforward $f_! A$ canonically lifts to $\mathrm{D}(Y, \Lambda[\Gamma])$. This can be seen as follows, since A is Γ-equivariant it descends to X/Γ, and denote by $f' : X/\Gamma \to Y/\Gamma = Y \times \mathrm{pt}/\Gamma$ the map on quotients stacks. The lift is then provided by $f'_! A \in \mathrm{D}(Y, \Lambda) = \mathrm{D}(Y, \Lambda[\Gamma])$. It moreover clear that this construction is compatible with the 6-functors.

Geometric objects

We fix X a smooth projective geometrically connected curve over $\mathbb{F}_q$. We choose $x \in X$ a point (not necessarily of degree one) and $\bar{x} : \mathrm{Spec}(\bar{\mathbb{F}}_q) \to X$ a geometric point over x. We denote by

(i). $\eta = \mathrm{Spec}(\mathbb{F}_q(X)) \in X$ the generic point of X, we let $\eta^{\mathrm{nr}} = \mathrm{Spec}(\bar{\mathbb{F}}_q(X)) = \eta \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ denote the generic point of $X_{\bar{\mathbb{F}}_q}$.

(ii). We let F_x be the completion at x, $\mathcal{O}_x$ its ring of integers and $\mathbb{F}_x$ be the residue field of F_x. We denote by $\eta_x = \mathrm{Spec}(F_x)$ and $\eta_x^{\mathrm{nr}} = \mathrm{Spec}(F_x^{\mathrm{nr}})$.

Roots of unity

We fix a trivialization of the group of roots of unity $\bar{\mathbb{F}}_q^\times \simeq (\mathbb{Q}/\mathbb{Z})_{p'}$ where p' denotes the prime to p-part. We also fix a trivialization of the Tate twist $(1) = \mathrm{id}$.

Local field

Let K be a local field of equal characteristic. We introduce the following notations

(*i*). K^{nr} denotes the maximal unramified extension of K.

(*ii*). $\mathcal{O}_K$ is the ring of integers of K and k_K its residue field,

(*iii*). $\mathcal{O}_{K^{\mathrm{nr}}}$ the ring of integers of K^{nr} and $\bar{k}_K$ its residue field.

(*iv*). We fix an algebraic closure of $K \subset K^{\mathrm{nr}} \subset K^{\mathrm{alg}}$ and denote by $\Gamma_K = \mathrm{Gal}(K^{\mathrm{alg}}/K)$.

(*v*). We let $P_K \subset I_K \subset \Gamma_K$ denote the wild inertia and inertia subgroup respectively.

(*vi*). We identify Γ_K/I_K with $\mathrm{Gal}(\bar{k}_K/k_K)$ and denote by F the topological generator obtained as the Frobenius $x \mapsto x^{q_K}$. We denote by Weil_K the Weil group of K.

Adelic notations

We now fix some adelic notations.

(*i*). We denote by $\mathbb{A}$ the ring of adèles of X.

(*ii*). We fix a divisor $N \subset X$, which we assume to be of the form $N = x + N^x$ where $x \notin N^x$.

(*iii*). We denote by $\mathbb{O}$ the ring of integral adèles. We have a canonical map $\mathbb{O} \to \mathcal{O}_N$. The kernel of this map is denoted by $\mathbb{O}_N$ and is the compact open subgroup of principal adèles of level N.

(*iv*). Denote by Z the center of G. We fix $\Xi \subset Z(F)\backslash Z(\mathbb{A})$ a cocompact lattice of the center of G.

Dual groups

We fix some notations for the dual group. Let F'/F be a finite Galois extension of F such that $G_{|F'}$ is split and we let $Q = \mathrm{Gal}(F'/F)$. Similarly, we fix x' be a place of F' over x and denote by $Q_{\mathrm{loc}} = \mathrm{Gal}(F'_{x'}/F_x)$ the corresponding Galois group. Since G is the generic fiber of a reductive group over X, we can assume that F'/F is everywhere unramified.

(*i*). We denote by $\hat{G}$ the Langlands dual group of G over Λ.

(*ii*). We denote by ${}^L G = \hat{G} \rtimes Q$ the global L-group.

(*iii*). We denote by ${}^L G_{\mathrm{loc}} = \hat{G} \rtimes Q_{\mathrm{loc}}$ the local L-group at x.

4.2 Deligne-Lusztig theory.

4.2.1 Basics of Deligne-Lusztig theory

Let M be a reductive group over $\bar{\mathbb{F}}_q$ equipped with a Frobenius endomorphism $\mathrm{F}: M \to M$ coming from some $\mathbb{F}_q$ structure. We fix $B_M = TU_M$ a F-stable Borel pair and we let $W_M = N_M(T)/T$ be the corresponding Weyl group.

The Deligne-Lusztig varieties are defined as follows. Let $w \in W_M$ and fix a lift $\dot{w} \in N_M(T)$, then consider the variety

$$Y(\dot{w}) = \{mU_M, m^{-1}\mathrm{F}(m) \in U_M \dot{w} U_M\} \subset M/U_M. \tag{4.5}$$

It is equipped with two commuting actions of M^{F} and $T^{w\mathrm{F}}$ acting by left and right translations on $Y(\dot{w})$. The cohomology $\mathrm{R}_w = \mathrm{R}\Gamma_c(Y(\dot{w}), \Lambda)$ is then equipped with two commuting actions of the same finite groups. We can then introduce the Deligne-Lusztig induction and restriction functors

$$\begin{aligned}
\mathcal{R}_w : \mathrm{D}^b(\mathrm{Rep}_\Lambda(T^{w\mathrm{F}})) &\to \mathrm{D}^b(\mathrm{Rep}_\Lambda(M^{\mathrm{F}})) \\
A &\mapsto A \otimes_{T^{w\mathrm{F}}} \mathrm{R}_w \\
{}^*\mathcal{R}_w : \mathrm{D}^b(\mathrm{Rep}_\Lambda(M^{\mathrm{F}})) &\to \mathrm{D}^b(\mathrm{Rep}_\Lambda(T^{w\mathrm{F}})) \\
B &\mapsto \mathrm{RHom}_{M^{\mathrm{F}}}(\mathrm{R}_w, B).
\end{aligned}$$

The key theorem we will need is the following one.

Theorem 4.2.1 ([DL76] for $\overline{\mathbb{Q}}_\ell$-version, [BR03] for a general Λ)**.** *The collections of functors ${}^*\mathcal{R}_w$ is conservative.*

We also introduce the following notations. Let $\widehat{T^{w\mathrm{F}}}$ be the set of characters of $T^{w\mathrm{F}}$ of order invertible in Λ. For each $\theta \in \widehat{T^{w\mathrm{F}}}$, there is an idempotent $e_\theta \in \Lambda[T^{w\mathrm{F}}]$ projecting onto the θ-isotypic part. We denote by ${}^*\mathcal{R}_{w,\theta}$ the functor the θ-isotypic part of ${}^*\mathcal{R}_{w,\theta}$.

Consider the action of M on itself by F-conjugation. This is the action given by $m.x = mx\mathrm{F}(m^{-1})$ and we denote it by Ad_{F}. By Lang's theorem, the quotient stack for this action is

$$\frac{M}{\mathrm{Ad}_{\mathrm{F}} M} = \mathrm{pt}/M^{\mathrm{F}}.$$

The Lang map $\mathcal{L} : M \to M, m \mapsto m^{-1}\mathrm{F}(m)$ induces an isomorphism of quotient stacks

$$M^{\mathrm{F}} \backslash M / B_M = \frac{M}{\mathrm{Ad}_{\mathrm{F}} B_M}. \tag{4.6}$$

We introduce now the correspondence

$$M^{\mathrm{F}} \backslash \mathrm{pt} \xleftarrow{r} M^{\mathrm{F}} \backslash M / B_M = \frac{M}{\mathrm{Ad}_{\mathrm{F}} B_M} \xrightarrow{q} \frac{U_M \backslash M / U_M}{\mathrm{Ad}_{\mathrm{F}} T}, \tag{4.7}$$

where r is induced by the map $M/B_M \to \mathrm{pt}$ and q is the quotient map for the left (equivalently right) action of U_M acting by translations.

The stack $\frac{U_M \backslash M / U_M}{\mathrm{Ad}_{\mathrm{F}} T}$ is stratified using the Bruhat stratification. Let $w \in W_M$ and fix $\dot{w}$ a lift of w. We have an isomorphism $\frac{U_M \backslash B_M w B_M / U_M}{\mathrm{Ad}_{\mathrm{F}} T} = \frac{T\dot{w}}{\mathrm{Ad}_{\mathrm{F}} T \ltimes U_{M,w}} = \mathrm{pt}/(T^{w\mathrm{F}} \ltimes U_{M,w})$ where $U_{M,w} = U_M \cap \mathrm{Ad}(\dot{w}) U_M$.

Consider now the following diagram

$$\begin{tikzcd}
\frac{U_M\backslash B_M w B_M/U_M}{\mathrm{Ad}_{\mathrm{F}} T} \arrow[r, "j_w"'] \arrow[d, equal] & \frac{U_M\backslash M/U_M}{\mathrm{Ad}_{\mathrm{F}} T} \\
\mathrm{pt}/(T^{w\mathrm{F}} \ltimes U_{M,w}) \arrow[d, "k_w"] & \\
\mathrm{pt}/T^{w\mathrm{F}} &
\end{tikzcd}$$

Lemma 4.2.2 (3.3.1)**.** *There is an isomorphism of functor* $\mathrm{D}^b(\mathrm{Rep}_\Lambda T^{w\mathrm{F}}) \to \mathrm{D}^b(\mathrm{Rep}_\Lambda G^F)$

$$\mathcal{R}_w = r_! q^* j_{w,!} k_w^*. \tag{4.8}$$

Definition 4.2.3. Let $w \in W_M$. We define

$$\nabla_w : \mathrm{Rep}_\Lambda T^{w\mathrm{F}} \to \mathrm{D}(\frac{U_M\backslash M/U_M}{\mathrm{Ad}_{\mathrm{F}} T}, \Lambda)$$
$$M \mapsto j_{w,*} k_w^* M.$$

Note that we do not normalize it to be perverse.

4.2.2 Lusztig series

Let $\hat{T}$ be the torus dual to T defined over Λ. We recall the construction of the Lusztig series [DL76]. We denote by $\hat{\mathrm{F}}$ the isogeny dual to $\hat{T}$. Note that we have an isomorphism (depending on the fixed trivialization of roots of unity of $\bar{\mathbb{F}}_q$),

$$\Lambda[T^{w\mathrm{F}}] = \mathcal{O}(\hat{T}^{w\hat{\mathrm{F}}}), \tag{4.9}$$

as both side are isomorphic to $\Lambda[X_*(T)/(w\mathrm{F} - \mathrm{id})X_*(T)]$. We therefore have a bijection

$$\mathrm{Hom}(T^{w\mathrm{F}}, \Lambda^\times) = \hat{T}^{w\hat{\mathrm{F}}}(\Lambda). \tag{4.10}$$

Let θ be a character of $T^{w\mathrm{F}}$ and let π be an irreducible representation of M^{F}. We say that π lie in the Lusztig series of (w, θ) if π is a subquotient of a cohomology group of $\mathcal{R}_w(\theta)$. Conversely, by theorem 4.2.1, for every π there exist a pair (w, θ) such that π lie in the Lusztig series of (w, θ).

Consider the morphism $\xi : \hat{T} \to \hat{T} /\!\!/ W_M$. We still denote by $\hat{\mathrm{F}}$ the morphism induced by $\hat{\mathrm{F}}$ on $\hat{T} /\!\!/ W_M$.

Theorem 4.2.4 ([DL76] if $\Lambda = \overline{\mathbb{Q}}_\ell$, [BR03] if $\Lambda = \overline{\mathbb{F}}_\ell$)**.** *(i). Let π be an irreducible representation of M^{F}. Suppose that π belong to the Lusztig series (w, θ) and (w', θ') then $\xi(\theta) = \xi(\theta')$. Hence we have a well defined map*

$$\mathrm{LS} : \mathrm{Irr}_\Lambda M^{\mathrm{F}} \to (\hat{T} /\!\!/ W_M)^{\hat{\mathrm{F}}}(\Lambda). \tag{4.11}$$

(ii). There is a complete collection of central orthogonal idempotents $e_s \in \Lambda[M^{\mathrm{F}}]$ for $s \in (\hat{T} /\!\!/ W_M)^{\hat{\mathrm{F}}}(\Lambda)$ such that for all $\pi \in \mathrm{Irr}_\Lambda M^{\mathrm{F}}$, we have $e_s\pi = \pi$ if and only if $\mathrm{LS}(\pi) = s$.

4.3 Depth 0 representations of unramified groups

4.3.1 Parahoric subgroups and transfer of dual semisimple conjugacy classes

Let $\mathcal{H}$ be a reductive group over $\mathcal{O}_K$ with generic fiber H. We recall the setup of [Lan18]. The group H splits over an unramified extension. We introduce the following notations.

(i). Let $\mathcal{T} \subset \mathcal{H}$ be a maximally K-split maximal torus. We denote its generic fiber by $T \subset H$. Let $X_*(T)$ and $X^*(T)$ be the groups of cocharacters and characters respectively defined over K^{nr}. The group $\mathrm{Gal}(\bar{k}_K/k_K)$ naturally acts on them.

(ii). We denote by $\mathcal{B}(H)$ the reduced Bruhat-Tits building of $H(K)$ and by $\mathcal{B}(H_{K^{\mathrm{nr}}})$ the one $H(K^{\mathrm{nr}})$, by A the apartment corresponding to T and by $\tilde{A}$ the one corresponding to $T_{K^{\mathrm{nr}}}$.

(iii). For a point $x \in \mathcal{B}(H)$, we denote by $\mathcal{H}_x$ the corresponding parahoric group scheme over $\mathcal{O}_K$. We denote by M_x the reductive quotient of $\mathcal{H}_{x,k_K}$ the special fiber of $\mathcal{H}_x$.

The integral model $\mathcal{H}$ of H yields a hyperspecial point $x_0 \in A$. We choose $C \subset A$ a chamber containing x_0 in its closure. We denote by $\tilde{C} \subset \tilde{A}$ the chamber containing C.

The group $\mathcal{H}_x(\mathcal{O}_K)$ comes equipped with its Moy-Prasad filtration [MP94] denoted by $\mathcal{H}_x(\mathcal{O}_K)_r$ for $r \in \mathbb{R}_{\geq 0}$. We have a canonical identification $\mathcal{H}_x(\mathcal{O}_K)/\mathcal{H}_x(\mathcal{O}_K)_{>0} = M_x(k_K)$, where $\mathcal{H}_x(\mathcal{O}_K)_{>0} = \cup_{r>0}\mathcal{H}_x(\mathcal{O}_K)_r$.

Let $x \in \mathcal{B}(H)$. The choices of $\mathcal{T}$ and C determine a Borel pair $B_M = T_M U_M$ of M_x over k_K. Let W_M be the Weyl group of $T_{M,\bar{k}_K}$.

Let $\tilde{W}$ be the extended affine Weyl group of H and W_{aff} be the affine Weyl group, that is, the group generated by reflection along all hyperplanes in $\tilde{A}$. The Weyl group W_M of M_x is canonically identified with the subgroup of $\tilde{W}$ generated by reflections along hyperplanes in $\tilde{A}$ containing the image of the point x. Consider the composition $W_M \subset \tilde{W} \to W$, since the kernel $\tilde{W} \to W$ is torsion free, the morphism $W_M \to W$ is injective.

There are isomorphisms

$$X_*(T_M) \simeq X_*(\mathcal{T}_{\mathcal{O}_{K^{\mathrm{nr}}}}) \simeq X_*(T), \tag{4.12}$$

their composition is equivariant under $W_M \to W$ and under the action of $\mathrm{Gal}(\bar{k}_K/k_K)$. Subsequently taking group algebras over Λ, taking quotient under W_M and W and taking Frobenius fixed points we get a well defined transfer morphism

$$\xi_x : (\hat{T}_M /\!\!/ W_M)^{\hat{\mathrm{F}}} \to (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}}, \tag{4.13}$$

where $(\hat{-})$ always denote the dual torus over Λ.

4.3.2 Depth and inertial decomposition of the depth 0 part

We let $\mathrm{Rep}_\Lambda H(K)$ be the category of smooth representations of $H(K)$ on Λ-modules. We recall the definition of the depth of an irreducible representation.

Definition 4.3.1. Let π be an irreducible representation of $H(K)$. We say that π has depth r if

$$\pi = \bigcup_{x \in \mathcal{B}(H)} \pi^{\mathcal{H}_x(\mathcal{O}_K)_{>r}}, \tag{4.14}$$

where $\mathcal{H}_x(\mathcal{O}_K)_{>r} = \cup_{r'>r}\mathcal{H}_x(\mathcal{O}_K)_{r'}$.

Theorem 4.3.2 ([MP94])**.** *Every representation has a well defined depth.*

Theorem 4.3.3. *The category* $\mathrm{Rep}_\Lambda H(K)$ *canonically split as*

$$\mathrm{Rep}_\Lambda H(K) = \bigoplus_r \mathrm{Rep}^r_\Lambda H(K). \tag{4.15}$$

The direct summand $\mathrm{Rep}^r_\Lambda H(K)$ *is characterized by the fact that an irreducible representation* π *lies in* $\mathrm{Rep}^r_\Lambda H(K)$ *if an only if* π *has depth* r*.*

We refer to [Dat09], Appendix A for a proof of this theorem over any coefficient ring. We now recall how to construct types for depth 0 representations.

Definition 4.3.4. (i). An unrefined minimal depth 0 type is a pair (x,τ) where x is a point in $\mathcal{B}(H)$ and τ is an irreducible supercuspidal representation of $M_x(k_K)$.

(ii). Let π be an irreducible depth 0 representation of $H(K)$. A depth 0 type for π is an unrefined minimal depth 0 type (x,τ) such that there is exists a non zero map $\mathrm{c-ind}_{\mathcal{H}_x(\mathcal{O}_K)}^{H(K)}\tau \to \pi$ where τ is a representation of $\mathcal{H}_x(\mathcal{O}_K)$ obtained by inflation along $\mathcal{H}_x(\mathcal{O}_K) \to M_x(k_K)$.

Theorem 4.3.5 ([MP94])**.** *Every irreducible depth* 0 *representation admits a depth* 0 *type.*

Remark 4.3.6. Given a depth 0 irreducible representation π, a depth 0 type (x,τ) for it can be chosen such that x lies in the closure of the chamber C.

Theorem 4.3.7 ([Lan18], [Lan21])**.** (i)*. Let* π *be an irreducible depth* 0 *representation of* $H(K)$ *and let* $(x,\tau),(x',\tau')$ *be two depth* 0 *types for* π *such that* $x,x' \in A$*. We have* $\xi_x(\mathrm{LS}(\tau)) = \xi_{x'}(\mathrm{LS}(\tau'))$*. In particular, we have a well defined map*

$$\mathrm{LS}_H : \mathrm{Irr}(\mathrm{Rep}^0_\Lambda H(K)) \to (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}}(\Lambda). \tag{4.16}$$

(ii)*. Let* $\mathfrak{Z}^0_H$ *be the Bernstein center of* $\mathrm{Rep}^0_\Lambda H(K)$*. There is a complete collection* $(e_s)_{s\in(\hat{T}/\!\!/W)^{\hat{\mathrm{F}}}}$ *of orthogonal idempotents in* $\mathfrak{Z}^0_H$ *such that for all* $\pi \in \mathrm{Irr}(\mathrm{Rep}^0_\Lambda H(K))$*, we have* $e_s\pi = \pi$ *if and only if* $\mathrm{LS}_H(\pi) = s$*.*

4.4 Recollections about stacks of chtoucas and excursion

4.4.1 Stacks of chtoucas

Let $\mathcal{Q}$ be a smooth affine group scheme over X. We assume that $\mathcal{Q}$ is generically reductive.

Definition 4.4.1 (Loop groups)**.** Let $x = (x_i) : S \to X^I$ be a tuple of S-points in X, we denote by $\Gamma_x \subset X \times S$ the union $\bigcup_i \Gamma_{x_i}$ the graphs of the x_i, by $\Gamma_{x,n}$ the nth infinitesimal neighborhood of x, by $\widehat{\Gamma}_x$ the formal neighborhood of Γ_x and by $\mathring{\Gamma}_x = \widehat{\Gamma}_x \setminus \Gamma_x$.

We denote by

(i). $L_I^+\mathcal{Q}$ the functor over $\mathbb{F}_q$ defined as $S \mapsto \{(x,g)|x : S \to X^I, g \in \mathcal{Q}(\hat{\Gamma}_x)\}$, it is representable by a group scheme of infinite type, it is equipped with a map $L_I^+\mathcal{Q} \to X^I$ that induced by $(x,g) \mapsto x$,

(ii). $L_I\mathcal{Q}$ the functor over $\mathbb{F}_q$ defined as $S \mapsto \{(x,g)|x : S \to X^I, g \in \mathcal{Q}(\overset{\circ}{\Gamma}_x)\}$, it is representable by an group ind-scheme of infinite type and equipped with a map to X^I.

(iii). $L_{I,n}\mathcal{Q}$ the functor over $\mathbb{F}_q$ defined as $S \mapsto \{(x,g)|x : S \to X^I, g \in \mathcal{Q}(\Gamma_{x,n})\}$, it is representable by a group scheme of finite type and is equipped as before with an map to X^I.

The inclusion $\Gamma_{x,n} \to \widehat{\Gamma}_x$ induces a map of group schemes $L_I^+\mathcal{Q} \to L_{I,n}\mathcal{Q}$ whose kernel is a pro-unipotent group.

Definition 4.4.2 ($\mathrm{Bun}_\mathcal{Q}$ and Hecke stacks)**.** We denote by $\mathrm{Bun}_\mathcal{Q}$ the algebraic stack over $\mathbb{F}_q$ defined as $S \mapsto \{\mathcal{E}\}$ the groupoid of $\mathcal{Q}$-torsors over $X \times S$.

The Hecke stack Hecke_I is the stack whose S-point classifies the following data

(i). $x : S \to X^I$ a tuple of S-points of X,

(ii). A pair of $\mathcal{Q}$-torsors $(\mathcal{E}, \mathcal{E}')$ over $X \times S$,

(iii). A isomorphism of $\mathcal{Q}$-torsors $\phi : \mathcal{E} \to \mathcal{E}'$ over $X \times S - \Gamma_x$.

This stack is equipped with a map $\mathrm{Hecke}_I \xrightarrow{h} \mathrm{Bun}_\mathcal{Q} \times \mathrm{Bun}_\mathcal{Q}$ and a map to X^I.

Definition 4.4.3 (Principal level structure)**.** Given $N \subset X$ a divisor, we define $\mathrm{Bun}_{\mathcal{Q},N}$ to be the stack that classifies the same data as $\mathrm{Bun}_\mathcal{Q}$ plus a trivialization of the $\mathcal{Q}$-torsor $\mathcal{E}$ on N. We have $\mathrm{Hecke}_{I,N}$ the stack whose S-points is the groupoid of tuples $(x, (\mathcal{E}, \psi), (\mathcal{E}', \psi'), \phi)$ where

(i). $(x, (\mathcal{E}, \mathcal{E}'), \phi) \in \mathrm{Hecke}_I(S)$,

(ii). $x \in (X - N)^I(S)$,

(iii). ψ and ψ' are trivializations of $\mathcal{E}$ and $\mathcal{E}'$ respectively on $N \times S$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{E}_{|N\times S} \arrow[rr, "\phi_{|N\times S}"] \arrow[dr, "\phi"'] & & \mathcal{E}'_{|N\times S} \arrow[dl, "\phi'"] \\ & \mathcal{E}^0_{|N\times S} & \end{tikzcd}$$

where $\mathcal{E}^0$ denotes the trivial $\mathcal{Q}$-torsor over $X \times S$.

Definition 4.4.4 (Beilinson-Drinfeld Affine Grassmannian)**.** The Beilinson-Drinfeld affine grassmannian is defined as $\mathrm{Gr}_{I,\mathcal{Q}} = L_I\mathcal{Q}/L_I^+\mathcal{Q}$. It is an ind-projective ind-scheme of ind-finite type over U^I, where U is the open of X where $\mathcal{Q}$ is reductive. It represents the functor whose S-points classifies the following data

(i). $x : S \to X^I$ a tuple of S-points of X,

(ii). a $\mathcal{Q}$-torsor $\mathcal{E}$ over $X \times S$, (*iii*). a isomorphism of $\mathcal{Q}$-torsors $\phi : \mathcal{E} \to \mathcal{E}^0$ over $X \times S - \Gamma_x$, where $\mathcal{E}^0$ denotes the trivial $\mathcal{Q}$-torsor.

Remark 4.4.5. There is an action of $L_I^+\mathcal{Q}$ on $\mathrm{Gr}_{I,\mathcal{Q}}$ by left translation. The quotient stack $L_I^+\mathcal{Q}\backslash\mathrm{Gr}_{I,\mathcal{Q}}$ is the stack whose groupoid of S-points classifies the following data

(*i*). $x : S \to X^I$ a tuple of S-points of X,

(*ii*). a pair $\mathcal{E}_1, \mathcal{E}_2$ of $\mathcal{Q}$-torsors on $\widehat{\Gamma_x}$,

(*iii*). and an isomorphism of $\mathcal{Q}$-torsors

$$\mathcal{E}_{1,|\overset{\circ}{\Gamma}_x} \to \mathcal{E}_{2,|\overset{\circ}{\Gamma}_x}. \tag{4.17}$$

This stack is not algebraic. There is a map of stacks $\epsilon : \mathrm{Hecke}_{\mathcal{Q},I} \to L_I^+\mathcal{Q}\backslash\mathrm{Gr}_{I,\mathcal{Q}}$ such that the image of a modification $((x_i), \mathcal{E}_1, \mathcal{E}_2, \phi)$ is given by the restriction of both torsors and of ϕ to $\widehat{\Gamma_x}$.

Definition 4.4.6 (Stacks of Chtoucas)**.** For a $\mathcal{Q}$-torsor $\mathcal{E}$ over $X \times S$ denote by ${}^{\tau}\mathcal{E}$ the $\mathcal{Q}$-torsor $(\mathrm{id}_X \times \mathrm{F}_S)^*\mathcal{E}$, where F_S is the absolute q-power Frobenius of S. We define the stack of G-chtoucas with I-legs as the following pullback

$$\begin{tikzcd} \mathrm{Cht}_I \arrow[r] \arrow[d] & \mathrm{Hecke}_I \arrow[d] \\ \mathrm{Bun}_{\mathcal{Q}} \arrow[r, "(\mathrm{id},{}^{\tau})"'] & \mathrm{Bun}_{\mathcal{Q}} \times \mathrm{Bun}_{\mathcal{Q}} \end{tikzcd}$$

Replacing Hecke_I by $\mathrm{Hecke}_{I,N}$ in the previous diagram yields the stack $\mathrm{Cht}_{I,N}$, which is the stack of chtoucas with level structure N.

4.4.2 The setup for the main construction

Group schemes from Bruhat-Tits theory

Recall that we have fixed $\mathcal{G}$ be reductive group over X and that we denote by G its generic fiber. We consider the restriction of $\mathcal{G}$ to $\mathcal{O}_x$, we fix a maximally F_x-split maximal torus $\mathcal{T}_{\mathcal{O}_x} \subset \mathcal{G}_{\mathcal{O}_x}$ with generic fiber T_{F_x}. This choice determines an apartment $A_x \subset \mathcal{B}(G_{F_x})$ and the integral model $\mathcal{G}_{\mathcal{O}_x}$ determines a hyperspecial point $s_0 \in A$. Let C be a chamber in A containing s_0 in its closure and let σ be polysimplex in the closure of C. The choices of C and σ determine three smooth group schemes over $\mathcal{O}_x$.

(*i*). $\mathcal{G}_{\sigma,\mathcal{O}_x}$ the parahoric group scheme corresponding to σ,

(*ii*). $\mathcal{G}_{C,\mathcal{O}_x}$ the Iwahori group scheme corresponding to C. Its group of $\mathcal{O}_x$-points is an Iwahori subgroup of $G(F_x)$.

(*iii*). Finally let $\mathcal{G}_{C^0,\mathcal{O}_x}$ be the smooth group scheme over $\mathcal{O}_x$ whose group of $\mathcal{O}_x$-points is the pro-unipotent radical of the Iwahori $\mathcal{G}_{C,\mathcal{O}_x}(\mathcal{O}_x)$.

We also denote by $\mathcal{G}_\sigma, \mathcal{G}_C$ and $\mathcal{G}_{C^0}$ the smooth group schemes over X obtained by gluing $\mathcal{G}_{|(X-x)}$ with $\mathcal{G}_{\sigma,\mathcal{O}_x}, \mathcal{G}_{C,\mathcal{O}_x}$ and $\mathcal{G}_{C^0,\mathcal{O}_x}$ respectively. They are summed up in the following diagram of group schemes over X.

$$\begin{tikzcd} \mathcal{G} & & \mathcal{G}_\sigma \\ & \mathcal{G}_C \arrow[ul] \arrow[ur] & \\ & \mathcal{G}_{C^0} \arrow[u] & \end{tikzcd}$$

Note that all these groups schemes are isomorphic over $X - x$.

Special fibers.

We denote by $V_\sigma \subset \mathcal{G}_{\sigma,\mathbb{F}_x}$ the unipotent radical of the special fiber of $\mathcal{G}_{\sigma,\mathcal{O}_x}$ (equivalently the fiber at x of $\mathcal{G}_\sigma$). We denote by M_σ the reductive quotient of $\mathcal{G}_{\sigma,\mathbb{F}_x}$. In particular we have a short exact sequence of algebraic groups over $\mathbb{F}_x$,

$$1 \to V_\sigma \to \mathcal{G}_{\sigma,\mathbb{F}_x} \to M_\sigma \to 1. \tag{4.18}$$

The choice of C and $\mathcal{T}_{\mathcal{O}_x}$ determines a Borel pairs $B_M = T_M U_M$ of M_σ. We also denote their inverse image in $\mathcal{G}_{\sigma,\mathbb{F}_x}$ by B_σ and M_σ. Hence we have short exact sequences of algebraic groups over $\mathbb{F}_x$

$$\begin{tikzcd} 1 \arrow[r] & V_\sigma \arrow[r] & B_\sigma \arrow[r] & B_M \arrow[r] & 1 \\ 1 \arrow[r] & V_\sigma \arrow[r] \arrow[u, equal] & U_\sigma \arrow[r] \arrow[u] & U_M \arrow[r] \arrow[u] & 1. \end{tikzcd}$$

Let $\widetilde{\mathcal{G}_\sigma}^C$ (resp $\widetilde{\mathcal{G}_\sigma}^{C^0}$) be the dilatation of the subgroup B_σ of the fiber at x of $\mathcal{G}_\sigma$ (resp the dilatation of U_σ) in the sense of [MRR20].

Lemma 4.4.7. *The morphism $\mathcal{G}_C \to \mathcal{G}_\sigma$ (resp. $\mathcal{G}_{C^0} \to \mathcal{G}_\sigma$) induces an isomorphism $\mathcal{G}_C = \widetilde{\mathcal{G}_\sigma}^C$ (resp. $\mathcal{G}_{C^0} = \widetilde{\mathcal{G}_\sigma}^{C^0}$).*

Proof. We do it for $\mathcal{G}_C$, as the same argument holds for $\mathcal{G}_{C^0}$. The problem is local at x since all three group schemes $\mathcal{G}_\sigma, \mathcal{G}_C$ and $\widetilde{\mathcal{G}_\sigma}^C$ differ only over $\mathrm{Spec}(\mathcal{O}_x)$. Since $\widetilde{\mathcal{G}_\sigma}^C$ is smooth and has connected fibers over $\mathrm{Spec}(\mathcal{O}_x)$ by [MRR20], to identify it with the Bruhat-Tits group scheme, it is enough to identify the set of its $\mathcal{O}_x$-points. By [MRR20], lemma 4.1 and remark 4.2, there is a sequence

$$1 \to \widetilde{\mathcal{G}_\sigma}^C(\mathcal{O}_x) \to \mathcal{G}_\sigma(\mathcal{O}_x) \to \mathcal{G}_\sigma(\mathbb{F}_x)/B_\sigma(\mathbb{F}_x) \to 1, \tag{4.19}$$

where the last term is only a pointed set. This sequence is exact provided that the étale cohomology group $H^1(\mathbb{F}_x, B_\sigma)$ vanishes. This follows from Lang's theorem since $\mathbb{F}_x$ is finite and B_σ is connected. We deduce that the group of $\mathcal{O}_x$-points of $\widetilde{\mathcal{G}_\sigma}^C(\mathcal{O}_x)$ is the Iwahori subgroup defined by the chamber C, which then yields the desired isomorphism. $\square$

Harder Narasimhan truncations

Proceeding as in [Laf18] Section 12, we denote by $\mathcal{G}_\sigma^{\mathrm{ad}}$ and $\mathcal{G}_C^{\mathrm{ad}}$ the adjoint groups of $\mathcal{G}_\sigma$ and $\mathcal{G}_C$ repsectively. We fix two vector bundles $\mathcal{V}_\sigma$ and $\mathcal{V}_C$ together with trivialization of their determinants and faithful representations of $\mathcal{G}_\sigma^{\mathrm{ad}} \to \mathrm{GL}(\mathcal{V}_\sigma)$ and $\mathcal{G}_C^{\mathrm{ad}} \to \mathrm{GL}(\mathcal{V}_C)$. Since we have a map $\mathcal{G}_C \to \mathcal{G}_\sigma$, we get an action of $\mathcal{G}_C^{\mathrm{ad}}$ on $\mathcal{V}_\sigma$. We let Λ_σ and Λ_C denote the lattice of cocharacters of $\mathrm{GL}_{\mathrm{rk}(\mathcal{V}_\sigma)}$ and $\mathrm{GL}_{\mathrm{rk}(\mathcal{V}_C)}$. The representations we have fixed define maps $\mathrm{Bun}_{\mathcal{G}_\sigma} \to \mathrm{Bun}_{\mathrm{GL}_{\mathrm{rk}(\mathcal{V}_\sigma)}}$ and $\mathrm{Bun}_{\mathcal{G}_C} \to \mathrm{Bun}_{\mathrm{GL}_{\mathrm{rk}(\mathcal{V}_C \oplus \mathcal{V}_\sigma)}}$ which are representable, quasi-affine and of finite presentation. For every cocharacter $\mu \in \Lambda_\sigma$ (reps. in $\Lambda_\sigma \oplus \Lambda_C$), there is an open $\mathrm{Bun}^{\leq \mu}_{\mathrm{GL}_{\mathrm{rk}(\mathcal{V}_\sigma)}}$ of $\mathrm{Bun}_{\mathrm{GL}_{\mathrm{rk}(\mathcal{V}_\sigma)}}$ (reps. of $\mathrm{Bun}^{\leq \mu}_{\mathrm{GL}_{\mathrm{rk}(\mathcal{V}_C \oplus \mathcal{V}_\sigma)}}$ of $\mathrm{Bun}_{\mathrm{GL}_{\mathrm{rk}(\mathcal{V}_C \oplus \mathcal{V}_\sigma)}}$) whose inverse image in $\mathrm{Bun}_{\mathcal{G}_\sigma}$ is denoted by $\mathrm{Bun}^{\leq \mu}_{\mathcal{G}_\sigma}$ and similarly for $\mathrm{Bun}_{\mathcal{G}_C}$. This further defines an open $\mathrm{Cht}^{\leq \mu}_{\mathcal{G}_\sigma, I}$ for any finite set I and again similarly for $\mathcal{G}_C$.

Remark 4.4.8. We will need some compatibility between the Harder-Narasimhan truncations for $\mathcal{G}_\sigma$ and for $\mathcal{G}_C$, this is why we use the representation of $\mathcal{G}_C$ given by $\mathcal{V}_\sigma \oplus \mathcal{V}_C$. Ideally we would only use $\mathcal{V}_\sigma$ but then we cannot guarantee that the representation of $\mathcal{G}_C$ on $\mathcal{V}_\sigma$ is faithful.

4.4.3 Cocartesian Functors

We now recall the notion of cocartesian functors. This definition is taken from [Sal23b].

Definition 4.4.9 (Categories cofibered over FinSet)**.** Consider the category FinSet of finite sets, a category cofibered over FinSet is the data of a functor $F : \mathcal{C}^{\mathrm{tot}} \to \mathrm{FinSet}$ such that $\mathcal{C}^{\mathrm{tot,op}} \to \mathrm{FinSet}^{\mathrm{op}}$ is a fibered category. The functor $F : \mathcal{C}^{\mathrm{tot}} \to \mathrm{FinSet}$ is given by the following data

- (i). For all $I \in \mathrm{FinSet}$ a category $\mathcal{C}_I = F^{-1}(I)$,
- (ii). For all morphism $\xi : I \to J$ a functor $F(\xi) : \mathcal{C}_I \to \mathcal{C}_J$ compatible with composition.

Definition 4.4.10. A cocartesian functor is a 2-functor FinSet $\to$ Cat to the 2-category of categories.

It is well known that the Grothendieck construction exchanges categories cofibered over FinSet and cocartesian functors out of FinSet.

The examples of cofibered categories over FinSet we will use are the following. Let $\Lambda \in \{\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Z}}_\ell \, \overline{\mathbb{F}}_\ell\}$.

- (i). The category $\mathrm{Rep}_\Lambda({}^L G)^\bullet$, given by $I \mapsto \mathrm{Rep}_\Lambda({}^L G)^I$. For $\xi : I \to J$, we get a morphism $\xi : G^J \to G^I$ and for $W \in \mathrm{Rep}\hat{G}^I$ we get a representation $W^\xi = \xi^* W \in \mathrm{Rep}\hat{G}^J$.
- (ii). The category $\mathrm{D}(X^\bullet, \Lambda)$, for $I \to J$ we get a diagonal morphism $\Delta_\xi : X^J \to X^I$, and a pullback functor $\Delta_\xi^* \mathrm{D}(X^I, \Lambda) \to \mathrm{D}(X^J, \Lambda)$.
- (iii). The category $\mathrm{Rep}_\Lambda \mathrm{Weil}_F^\bullet$ defined in the same way as (i).

Definition 4.4.11 (Partial Frobenius)**.** Let I be a finite set and let I_0 be a finite subset. The partial Frobenius indexed by the set I_0 is the morphism $\mathrm{F}_{I_0} : X^I \to X^I$ defined by $(x_i) \mapsto (y_i)$ where $y_i = \mathrm{F}(x_i)$ is $i \in I_0$ and $y_i = x_i$ otherwise.

Remark 4.4.12. It is clear that the partial Frobenius endomorphims commute with each other and that F_I is the absolute Frobenius of X^I.

Definition 4.4.13 (Partial Frobenius and filtration)**.** Let Λ_0 an ordered free abelian group of finite rank, a Λ_0-filtered coCartesian functor over FinSet is the data

(i). for all $\mu \in \Lambda_0$ of a functor $H^{\leq \mu} : \mathrm{Rep}_\Lambda({}^LG)^\bullet \to \mathrm{D}(X^\bullet, \Lambda)$ over FinSet,

(ii). for all $\mu \leq \nu$ of a morphism $H^{\leq \mu} \to H^{\leq \nu}$ compatible with the order on Λ_0.

Given a filtered cocartesian functor $H^{\leq \mu} : \mathrm{Rep}_\Lambda({}^LG)^\bullet \to \mathrm{D}(X^\bullet, \Lambda)$, we say that $H^{\leq \mu}$ is filtered with respect to partial Frobenius if

(i). For all $W \in \mathrm{Rep}({}^LG)^I$, there exists $\kappa \in \Lambda$ and morphisms for all $I_0 \subset I$,

$$F_{I_0} : \mathrm{F}^*_{I_0} H^{\leq \mu}_{I,W} \to H^{\leq \mu+\kappa}_{I,W} \tag{4.20}$$

commuting with each other.

(ii). They are functorial in I.

Definition 4.4.14. Let Λ_1, Λ_2 be two ordered free abelian groups of finite types. And consider two cocartesian functor filtered with respect to partial Frobenius $H^{\leq \mu}$ and $H'^{\leq \mu'}$ for $\mu \in \Lambda_1$ and $\mu' \in \Lambda_2$. A compatibility datum between $(H^{\leq \mu})$ and $(H'^{\leq \mu'})$ is the datum of

(i). An increasing morphism $\varphi : \Lambda_1 \to \Lambda_2$,

(ii). For all $\mu \in \Lambda_1$, there exists $\mu'_0 \geq \varphi(\mu)$, and for all $\mu' \geq \mu'_0$, we are given a map $H^{\leq \mu}_{I,W} \to H^{\leq \mu'}_{I,W}$ functorial in (I, W) such that the following diagram commutes

$$\begin{tikzcd} H^{\leq \mu_1}_{I,W} \arrow[r] \arrow[d] & H^{\leq \mu_2}_{I,W} \arrow[d] \\ H'^{\leq \mu'_1}_{I,W} \arrow[r] & H^{\leq \mu'_2}_{I,W} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{F}^*_{I_0} H^{\leq \mu}_{I,W} \arrow[r] \arrow[d] & H^{\leq \mu+\kappa}_{I,W} \arrow[d] \\ \mathrm{F}^*_{I_0} H'^{\leq \mu'}_{I,W} \arrow[r] & H'^{\leq \mu'+\kappa'}_{I,W} \end{tikzcd}$$

where $\mu_1 \leq \mu_2 \in \Lambda_1$ and $\mu'_1 \geq \varphi(\mu_1), \mu'_2 \geq \varphi(\mu_2)$ are large enough with $\mu'_2 \geq \mu'_1$, and κ and κ' are chosen as in definition 4.4.13 for H and H' respectively.

4.4.4 Geometric Satake and cohomology of stacks of chtoucas

In this section, unless indicated, the stacks Gr and Cht are the ones for $\mathcal{G}_\sigma$ and the Harder-Narasimhan truncations refer to the ones induced by Λ_σ. We consider the Beilinson-Drinfeld affine Grassmannian for $\mathcal{G}_\sigma$. We recall the following version of geometric Satake due to [MV09] [Ric15], [Zhu15] and spelled out in [Laf18] 12.16.

Theorem 4.4.15 ([Ric15], [Zhu15])**.** *There is a cocartesian functor*

$$\mathrm{Rep}_\Lambda({}^LG)^I \to \mathrm{Perv}^{\mathrm{ULA}}(\mathrm{Gr}_{\mathcal{G},I}), W \mapsto \mathrm{Sat}(W). \tag{4.21}$$

Moreover, let $X_{(x)}$ denote the henselization of X at x. The restriction of Sat *to* $\mathrm{Gr}_{\mathcal{G},I})_{|X^I_{(x)}}$ *factors through* $\mathrm{Rep}_\Lambda({}^LG_{\mathrm{loc}})^I$ *and we have a commutative diagram*

$$\begin{tikzcd} \mathrm{Rep}_\Lambda({}^LG)^I \arrow[r] \arrow[d] & \mathrm{Perv}^{\mathrm{ULA}}(\mathrm{Gr}_{\mathcal{G},I}) \arrow[d] \\ \mathrm{Rep}_\Lambda({}^LG_{\mathrm{loc}})^I \arrow[r] & \mathrm{Perv}^{\mathrm{ULA}}(\mathrm{Gr}_{\mathcal{G},I})_{|X^I_{(x)}}) \end{tikzcd}$$

Let I be a finite set and $W \in \mathrm{Rep}({}^LG)^I$, we denote by $\mathrm{Gr}_{I,W}$ the support of $\mathrm{Sat}(W)$, it is a finite dimensional scheme that is a finite union of orbits of $L_I^+\mathcal{G}$, in particular the action of $L_I^+\mathcal{G}$ factors through a finite dimensional quotient $L_{I,n}^+\mathcal{G}$, hence $\mathrm{Sat}(W)$ descends to the quotient stack $L_{I,n}^+\mathcal{G}\backslash\mathrm{Gr}_{I,W}$ as explained in [Xue20b] 2.2, we can interpret those sheaves as sheaves on $L_I^+\mathcal{G}\backslash\mathrm{Gr}_{I,W}$. As explained in [Xue20b] 2.2.5, the action of $L^+Z(G)$ on Gr is trivial and we can also interpret $\mathrm{Sat}(W)$ as a sheaf on $L_I^+\mathcal{G}^{\mathrm{ad}}\backslash\mathrm{Gr}_I$.

Consider the diagram

$$\begin{tikzcd}
\mathrm{Cht}_{I,N}^{\leq\mu} \arrow[dr, "\epsilon_I^{\leq\mu}"'] & & \mathrm{Gr}_I \arrow[dl] \\
& L_I^+\mathcal{G}\backslash\mathrm{Gr}_I &
\end{tikzcd}$$

We denote by $\mathcal{F}_{I,N,W}^{\leq\mu} = (\epsilon_I^{\leq\mu})^*\mathrm{Sat}(W)$ and by $\mathrm{Cht}_{I,N,W}^{\leq\mu}$ its support. Similarly we have a map $\epsilon_{I,\mathrm{ad}}^{\leq\mu} : \mathrm{Cht}_{I,N}^{\mu}/\Xi \to L_I^+\mathcal{G}_{\mathrm{ad}}\backslash\mathrm{Gr}_I$, however μ is a cocharacter of G_{ad}, we refer to [Xue20b] 2.4 for a discussion. We denote similarly $\mathcal{F}_{I,N,W}^{\leq\mu,\mathrm{ad}} = \epsilon_{I,\mathrm{ad}}^{\leq\mu}\mathrm{Sat}(W)$. We also denote by $\mathfrak{p}_I : \mathrm{Cht}_{I,N}^{\leq\mu}/\Xi \to (X-N)^I$ the leg map.

Theorem 4.4.16 ([Laf18], [Xue20c])**.** *We denote by* $\mathcal{H}_{I,N,W}^{\leq\mu} = \mathfrak{p}_{I,!}\mathcal{F}_{I,N,W}^{\leq\mu,\mathrm{ad}}$ *the association*

$$(I,W) \mapsto \mathcal{H}_{I,N,W}^{\leq\mu} \in \mathrm{D}((X-N)^I,\Lambda) \tag{4.22}$$

defines a filtered cocartesian functor with respect to partial Frobenius.

Remark 4.4.17. As pointed out in [Xue20a], if $\Lambda = \overline{\mathbb{F}}_\ell$ the sheaves $\mathcal{H}_{I,N,W}^{\leq\mu}$ are a priori not contructible on $(X-N)^I$ as they may be unbounded. Indeed, the map $\mathfrak{p}$ is not representable and ℓ may divide the order of certain groups of automorphisms of points in $\mathrm{Cht}_{I,W,N}$.

Definition 4.4.18 (Cohomology sheaves)**.** Let $j \in \mathbb{Z}$, we set $\mathcal{H}_{I,N,W}^j = \varinjlim_\mu \mathcal{H}_{I,N,W}^{j,\leq\mu}$. Recall that $\overline{\eta} \to X$ is a generic geometric point, for all I denote by $\Delta(\overline{\eta}) \to X^I$ the corresponding diagonal point. We set $H_{I,N,W}^j = (\mathcal{H}_{I,N,W}^j)_{|\Delta(\overline{\eta})}$.

Theorem 4.4.19 ([Xue20d], Section 6 for the non-split case)**.** *(i). Each of the* Λ*-modules* $H_{I,N,W}^j$ *is equipped with an action of* $\mathrm{Weil}(\overline{\eta}/\eta)^I$.

(ii). The association $(I,W) \mapsto H_{I,N,W}^j$ *defines a cocartesian functor* $\mathrm{Rep}^LG^\bullet \to \mathrm{Rep}_\Lambda\mathrm{Weil}(\overline{\eta}/\eta)^\bullet$ *over* FinSet.

(iii). The sheaves $\mathcal{H}_{I,N,W}^j$ *are ind-lisse on* $(X-N)^I$.

Remark 4.4.20. As explained in [LZ18], taking $I = \{0\} \sqcup J$ in the previous theorem and the forgetting the action of Weil on the leg indexed by 0, the functor

$$\mathrm{Rep}_\Lambda({}^LG \times ({}^LG)^I) \to \mathrm{Rep}_\Lambda\mathrm{Weil}_F^J, V \boxtimes W \to H_{I,V\boxtimes W,N}$$

factors through $\mathrm{Rep}_\Lambda(\hat{G} \times ({}^LG)^I)$ hence we get a cocartesian functor

$$\mathrm{Rep}_\Lambda(\hat{G} \times ({}^LG)^I) \to \mathrm{Rep}_\Lambda\mathrm{Weil}_F^I.$$

4.4.5 Excursion algebra and the stack of local Langlands parameters

We consider the group G restricted to $\eta_x = \mathrm{Spec}(F_x)$. We will use the space of local Langlands parameters for G and F_x and we define the local excursion algebra. This construction was first defined in [Laf18].

Definition 4.4.21. An Λ-valued Langlands paramters is a morphism $\sigma : \mathrm{Weil}_{F_x} \to {}^L G_{\mathrm{loc}}(\Lambda)$ that satisfy the following hypothesis :

- (i). σ is defined over a finite extension of $\mathbb{Q}_\ell$ or $\mathbb{F}_\ell$ respectively and is continuous (for the ℓ-adic topology and the discrete topology) respectively).
- (ii). The following diagram commutes

$$\begin{tikzcd} \mathrm{Weil}_{F_x} \arrow[r] \arrow[dr] & {}^L G_{\mathrm{loc}}(\Lambda) \arrow[d] \\ & Q_{\mathrm{loc}} \end{tikzcd}$$

The set of all Λ-valued Langlands parameters is denoted by $\underline{Z}^1(F_x, \hat{G})(\Lambda)$. A tame parameter is a parameter σ which factors through the tame Weil group $\mathrm{Weil}^t_{F_x} \to {}^L G_{\mathrm{loc}}(\Lambda)$. We denote by $\underline{Z}^{1,t}(F_x, \hat{G})(\Lambda)$ the subset of tame parameters.

Theorem 4.4.22 ([FS21], [DHKM20], [Zhu21])**.** *There exists an ind-scheme $\underline{Z}^1(F_x, \hat{G})$ over $\mathbb{Z}_\ell$ whose Λ-points are $\underline{Z}^1(F_x, \hat{G})(\Lambda)$, it is locally of finite type over $\mathbb{Z}_\ell$ and lci. There is a closed subscheme $\underline{Z}^{1,t}(F_x, \hat{G})$ which is the modui space of tame parameters. It is both open and closed in $\underline{Z}^1(F_x, \hat{G})$*

We will now recall its construction as well as the construction of the local excursion algebra. Recall that $\mathrm{Weil}^t_{F_x}$ is the tame quotient of Weil_{F_x}. This group is topologically generated by the elements F_{F_x} and τ_{F_x} which are respectively a lift of the Frobenius of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_x)$ and a topological generator of the tame inertia. Let $\mathrm{Weil}^{t,\circ}_{F_x}$ be the discrete group generated by these two elements. We denote by $\mathrm{Weil}^{\circ}_{F_x}$ the inverse image of $\mathrm{Weil}^{t,\circ}_{F_x}$ in Weil_{F_x}. If $P^e_{F_x} \subset P_{F_x}$ is an open subgroup of the wild inertia, the quotient $\mathrm{Weil}^{\circ}_{F_x}/P^e_{F_x}$ is a discrete group.

For all discrete groups Γ with a map $\Gamma \to Q_{\mathrm{loc}}$ we construct $\underline{Z}^1(\Gamma, \hat{G})$ and $\mathrm{Exc}(\Gamma, \hat{G})$ which are the moduli space of Γ-cocycles in $\hat{G}$ and the excursion algebra of Γ and $\hat{G}$. We now proceed as in [Zhu21]. Let us now fix a discrete group Γ and a morphism $\Gamma \to Q_{\mathrm{loc}}$.

Definition 4.4.23. We denote by FFS the category whose objects are finite sets written as $\mathrm{FS}(I)$ and the morphisms $\mathrm{FS}(I) \to \mathrm{FS}(J)$ are morphisms of monoids from the free monoid generated by I to the free monoid generated by J. We denote by FFS/Γ the category of pairs $(\mathrm{FS}(I), \phi)$ where $\phi : \mathrm{FS}(I) \to \Gamma$ is a morphism of monoids and morphisms are morphisms in FFS compatible with the morphism to Γ.

Definition 4.4.24. Let H be an affine algebraic group over an affine base scheme S. We denote by $\mathrm{Hom}(\Gamma, H)$ the functor whose R points is the set of group morphisms $\mathrm{Hom}(\Gamma, H(R))$. It is easily seen that this functor is representable by an affine scheme over S.

The next lemma is an easy exercise.

Lemma 4.4.25. *The category* FFS *is generated by the maps*

(*i*). $\mathrm{FS}(I) \to \mathrm{FS}(J)$ *coming from a map* $I \to J$,

(*ii*). *the map* $\mathrm{FS}(\{1\}) \to \mathrm{FS}(\{1,2\})$ *sending the generator* $x_1 \in \mathrm{FS}(\{1\})$ *to* $x_1 x_2 \in \mathrm{FS}(\{1,2\})$ *the product of the two generators.*

Lemma 4.4.26. (*i*). *There is an isomorphism in the category of monoids*

$$\Gamma = \varinjlim_{\mathrm{FFS}/\Gamma} \mathrm{FS}(I). \tag{4.23}$$

(*ii*). *There is an isomorphism in the category of schemes over* S,

$$\mathrm{Hom}(\Gamma, H) = \varprojlim_{\mathrm{FFS}/\Gamma} H^I. \tag{4.24}$$

Proof. The first point is restatement of the fact that the $\mathrm{FS}(I)$ generate the category of monoids under colimits. The second point can be checked on R-points, namely we have

$$\mathrm{Hom}(\Gamma, H)(R) = \mathrm{Hom}(\Gamma, H(R)) = \varprojlim_{\mathrm{FFS}/\Gamma} \mathrm{Hom}(\mathrm{FS}(I), H(R)) = H^I(R). \tag{4.25}$$

□

Definition 4.4.27. (*i*). The excursion algebra $\mathrm{Exc}(\Gamma, \hat{G})$ is defined as

$$\mathrm{Exc}(\Gamma, \hat{G}) = \varinjlim_{\mathrm{FFS}/\Gamma} \mathcal{O}(({}^L G_{\mathrm{loc}})^I)^{\hat{G}}, \tag{4.26}$$

where $\hat{G}$ acts on $({}^L G_{\mathrm{loc}})^I$ by simultaneous conjugation.

(*ii*). The scheme $\underline{Z}^1(\Gamma, \hat{G}) \subset \mathrm{Hom}(\Gamma, {}^L G_{\mathrm{loc}})$ is defined as the fiber of the morphism the

$$\mathrm{Hom}(\Gamma, {}^L G_{\mathrm{loc}}) \to \mathrm{Hom}(\Gamma, Q_{\mathrm{loc}}) \tag{4.27}$$

over the point $\Gamma \to Q_{\mathrm{loc}}$.

It follows from the definition that there is a canonical map $\mathrm{Exc}(\Gamma, \hat{G}) \to \mathcal{O}(\underline{Z}^1(\Gamma, \hat{G}))$. The formation of $\mathrm{Hom}(\Gamma, H)$ is clearly functorial in Γ and H. By definition of this space of cocyles, the following diagram is cartesian,

$$\begin{tikzcd} \underline{Z}^1(\Gamma, \hat{G}) \arrow[r] \arrow[d] & \mathrm{Hom}(\Gamma, {}^L G_{\mathrm{loc}}) \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathrm{Hom}(\Gamma, Q_{\mathrm{loc}}) \end{tikzcd} \tag{4.28}$$

the point in $\mathrm{Hom}(\Gamma, Q_{\mathrm{loc}})$ is given by the projection $\Gamma \to Q_{\mathrm{loc}}$. In terms of the limit description given above, consider the following. Let $\varphi : \mathrm{FS}(I) \to \Gamma$ be a morphism and consider $Z_\varphi \subset {}^L G^I_{\mathrm{loc}}$ be the closed subset obtained as the fiber over the point $z_\varphi \in \mathrm{Hom}(\mathrm{FS}(I), Q_{\mathrm{loc}})$ corresponding to $\mathrm{FS}(I) \xrightarrow{\varphi} \Gamma \to Q_{\mathrm{loc}}$.

Lemma 4.4.28. *There is a canonical isomorphism*

$$\underline{Z}^1(\Gamma, \hat{G}) = \varprojlim_{\varphi \in \mathrm{FFS}/\Gamma} Z_\varphi. \tag{4.29}$$

Remark 4.4.29. Note that the group $\hat{G}$ acts on Z_φ for each $\varphi \in \mathrm{FFS}/\Gamma$ and the the isomorphism in lemma 4.4.28 is $\hat{G}$-equivariant.

Definition 4.4.30. Let $(P^e_{F_x})_{e \in \mathbb{N}}$ be an exhaustive decreasing filtration of P_{F_x}.

(i). The local excursion algebra is $\mathrm{Exc}(\mathrm{Weil}_{F_x}, \hat{G}) = \varprojlim_e \mathrm{Exc}(\mathrm{Weil}^\circ_{F_x}/P^e_{F_x}, \hat{G})$, the tame excursion algebra is $\mathrm{Exc}(\mathrm{Weil}^t_{F_x}, \hat{G}) = \mathrm{Exc}(\mathrm{Weil}^{t,\circ}_{F_x}, \hat{G})$.

(ii). The moduli of parameters is $\underline{Z}^{1,\circ}(\mathrm{Weil}_{F_x}, \hat{G}) = \varinjlim_e \underline{Z}^1(\mathrm{Weil}^\circ_{F_x}/P^e_{F_x}, \hat{G})$. The moduli of tame parameters is $\underline{Z}^{1,t,\circ}(\mathrm{Weil}_{F_x}, \hat{G}) = \underline{Z}^1(\mathrm{Weil}^{t,\circ}_{F_x}, \hat{G})$.

This definition depends on the chosen elements F_{F_x} and τ_{F_x}, but there is an isomorphism (depending on the same data) $\underline{Z}^{1,\circ}(\mathrm{Weil}_{F_x}, \hat{G}) = \underline{Z}^1(\mathrm{Weil}_{F_x}, \hat{G})$ with the moduli space defined in [FS21], [DHKM20] and [Zhu21].

Theorem 4.4.31. *Let $H_I : \mathrm{Rep}_\Lambda({}^L G_{\mathrm{loc}})^I \to \mathrm{Rep}_\Lambda(\mathrm{Weil}^\circ_{F_x}/P^e_{F_x})^I$ be $\mathrm{Rep}(Q_{\mathrm{loc}})^I$-linear cocartesian functor. Then there is a canonical $\mathrm{Exc}(\mathrm{Weil}^\circ_{F_x}/P^e_{F_x}, \hat{G})$-module structure on $H_{I,W}$ such that*

(i). *This structure is compatible with the cocartesian structure, that is for all maps $\zeta : I \to J$, the isomorphism $H_{I,W} = H_{J,W_\zeta)}$ is $\mathrm{Exc}(\mathrm{Weil}^\circ_{F_x}/P^e_{F_x}, \hat{G})$-linear.*

(ii). *This structure is also compatible with the action of $(\mathrm{Weil}^\circ_{F_x}/P^e_{F_x})^I$.*

(iii). *If $H_{I,W}$ is the cocartesian functor given by the cohomology of stacks of chtoucas, then this action is the same as the one constructed in [Laf18].*

(iv). *This structure is functorial in the collection of functors H.*

Proof. Assume that we have constructed a functorial action of $\mathrm{Exc}(\mathrm{Weil}^\circ_{F_x}/P^e_{F_x}, \hat{G})$ on $H_{\emptyset,1}$. Let (I, W) be as before, define $\tilde{H}_{J,V} = H_{I \sqcup J, W \boxtimes V}$, it is clear that this defines an cocartesian functor, hence by assumption there is an an $\mathrm{Exc}(\mathrm{Weil}^\circ_{F_x}/P^e_{F_x}, \hat{G})$-module structure on it. The compatibilities (i) and (ii) are then deduced from the functoriality of this action.

To construct the action and guarantee its compatibility with [Laf18], we follow the construction of *loc. cit.*. We then construct an action of $\mathrm{Exc}(\mathrm{Weil}^\circ_{F_x}/P^e_{F_x}, \hat{G})$ on $H_{\emptyset,1}$.

Let $V \in \mathrm{Rep}_\Lambda({}^L G_{\mathrm{loc}})^I$ and let $x \in V^{\hat{G}}$ and $\xi \in (V^*)^{\hat{G}}$ be fixed vectors and covectors for the diagonal action of $\hat{G}$. Let $(\gamma_i) \in (\mathrm{Weil}^\circ_{F_x}/P^e_{F_x})^I$. We then define the excursion operator $F_{V,x,\xi,(\gamma_i)} : H_{\emptyset,1} \to H_{\emptyset,1}$ as the composition

$$H_{\emptyset,1} = H_{\{0\},1} \xrightarrow{x} H_{\{0\},V} = H_{I,V} \xrightarrow{(\gamma_i)} H_{I,V} = H_{I,V} \xrightarrow{\xi} H_{\{0\},V} = H_{\emptyset,1}.$$

Let $f \in \mathcal{O}(\hat{G} \backslash ({}^L G_{\mathrm{loc}})^I / \hat{G})$ be the function $f(g) = \langle \xi, g.x \rangle$ and denote by $\phi_\gamma : \mathrm{FS}(I) \to \mathrm{Weil}^\circ_{F_x}/P^e_{F_x}$ the morphism induced by the choice of the (γ_i). Then the argument of [Laf18] Proposition 10.8 show that

(i). the map $F_{V,x,\xi,(\gamma_i)}$ depends only on the function f and not on V, x or ξ, we can therefore $F_{f,(\gamma_i)}$ the previous composition.

(ii). after identifying $\mathcal{O}(\hat{G}\backslash({}^L G_{\mathrm{loc}})^{\{0\}\sqcup I}/\hat{G}) \otimes_{\mathcal{O}((Q_{\mathrm{loc}})^{\{0\}\sqcup I})} \mathcal{O}((Q_{\mathrm{loc}})^I) = \mathcal{O}(({}^L G_{\mathrm{loc}})^I)^{\hat{G}}$, the morphism $F_{f,(1,\gamma_i)}$ defines an algebra morphism

$$\mathcal{O}(({}^L G_{\mathrm{loc}})^I)^{\hat{G}} \to \mathrm{End}(H_{\emptyset,1}). \tag{4.30}$$

(iii). This morphism is functorial in $\mathrm{FFS}/(\mathrm{Weil}^{\circ}_{F_x}/P^{e}_{F_x})$ hence defines an action of $\mathrm{Exc}(\mathrm{Weil}^{\circ}_{F_x}/P^{e}_{F_x}, \hat{G})$.

The functoriality in H is clear by construction. □

Lemma 4.4.32. *Let $H_{I,W}$ be a cocartesian functor as before. Assume that for all I, W the action of $(\mathrm{Weil}_{F_x})^I$ on $H_{I,W}$ is tame, that is, factors through $(\mathrm{Weil}^t_{F_x})^I$. Then all the $\mathrm{Exc}(\mathrm{Weil}_{F_x}, \hat{G})$-modules $H_{I,W}$ are supported on $\mathrm{Spec}(\mathrm{Exc}(\mathrm{Weil}^t_{F_x}, \hat{G}))$.*

Proof. Let $\pi : \mathrm{Weil}_{F_x} \to \mathrm{Weil}^t_{F_x}$ be the projection. It follows from the hypothesis and the definition of the excursion operators that for all V, x, ξ the morphism $F_{V,x,\xi,(\gamma_i)} = F_{V,x,\xi,(\pi_{\gamma_i})}$. Hence $H_{I,W}$ is killed by all the operators $F_{f,(\gamma_i)} - F_{f,(\pi(\gamma_i))}$ which generate the kernel of $\mathrm{Exc}(\mathrm{Weil}_{F_x}, \hat{G}) \to \mathrm{Exc}(\mathrm{Weil}^t_{F_x}, \hat{G})$. □

Theorem 4.4.33 ([LZ18]). *Assume that Λ is a field. Let H be a cocartesian functor $\mathrm{Rep}_\Lambda(\hat{G} \times {}^L G^I) \to \mathrm{Rep}_\Lambda(\mathrm{Weil}^{t,\circ}_{F_x})^I$ which is $\mathrm{Rep}_\Lambda(Q^I_{\mathrm{loc}})$-linear. Then the vector space $H_{\{0\},\mathrm{Reg}_{\hat{G}}}$ has a canonical structure of quasicoherent sheaf on $\underline{Z}^{1,t,\circ}(F_x, \hat{G})/\hat{G}$ that is compatible with the $\mathrm{Exc}(\mathrm{Weil}^t_{F_x}, \hat{G})$-module structure.*

Proof. We will follow the argument of [LZ18], Section 6. Let $\mathcal{M}_H = H_{\{0\},\mathrm{Reg}}$ where Reg denotes the regular representation of $\hat{G}$ which we consider as an ind-object in $\mathrm{Rep}_\Lambda(\hat{G})$. Let us first construct the structure of an $\mathrm{Hom}(\mathrm{Weil}^{t,\circ}_{F_x}, {}^L G_{\mathrm{loc}})$-module and then show that it is supported on $\underline{Z}^{1,t,\circ}$.

We first claim that, by the presentation we have given the spaces $\mathrm{Hom}(\mathrm{Weil}^{t,\circ}_{F_x}, {}^L G_{\mathrm{loc}})$ and $\underline{Z}^{1,t,\circ}(F_x, \hat{G})$ are generated by elements $F_{f,\gamma}$ where $f \in \mathcal{O}({}^L G_{\mathrm{loc}})$ and $\gamma \in \mathrm{Weil}^{t,\circ}_{F_x}$. These elements are defined as follow. The limit description of both of these spaces yields an isomorphism $\mathcal{O}(\mathrm{Hom}(\mathrm{Weil}^{t,\circ}_{F_x}, {}^L G_{\mathrm{loc}})) = \varinjlim_{\varphi \in \mathrm{FFS}/\mathrm{Weil}^{t,\circ}_{F_x}} \mathcal{O}({}^L G^I_{\mathrm{loc}})$. The data of $\gamma \in \mathrm{Weil}^{t,\circ}_{F_x}$ yields a morphism $\varphi : \mathrm{FS}(\{0\}) \to \mathrm{Weil}^{t,\circ}_{F_x}$, then the canonical inclusion $\mathrm{inc}_\varphi : \mathcal{O}({}^L G_{\mathrm{loc}}) \to \mathcal{O}(\mathrm{Hom}(\mathrm{Weil}^{t,\circ}_{F_x}, {}^L G_{\mathrm{loc}}))$ yields applied to f yields the element $F_{f,\gamma}$. To show that these elements generate the ring of function it is enough to do it for $\mathcal{O}(\mathrm{Hom})$ since $\mathcal{O}(\underline{Z}^1)$ is a quotient of the former. It is then enough to show that for any $\phi : \mathrm{FS}(I) \to \mathrm{Weil}^{t,\circ}_{F_x}$ the image of $\mathcal{O}({}^L G^I_{\mathrm{loc}})$ in $\mathcal{O}(\mathrm{Hom})$ is contained in the subring generated by the $F_{f,\gamma}$. Let $h \in \mathcal{O}({}^L G^I_{\mathrm{loc}})$, we can assume that $h = h_1 \boxtimes \cdots \boxtimes h_n$ where $n = |I|$. Then $\mathrm{inc}_\varphi(h) = \mathrm{inc}_{\varphi_1}(h_1) \dots \mathrm{inc}_{\varphi_n}(h_n)$ where $\varphi_i : \mathrm{FS}(\{i\}) \to \mathrm{Weil}^{t,\circ}_{F_x}$ is obtained via the restriction along $\{i\} \subset I$.

Let $V \in \mathrm{Rep}_\Lambda \hat{G}$ be an algebraic representation. Following [LZ18], there is an $\hat{G}$-equivariant isomorphism

$$\theta : \mathrm{Reg} \otimes \underline{V} \to \mathrm{Reg} \otimes V, \tag{4.31}$$

where $\underline{V}$ is the underlying vector space of V without the $\hat{G}$ action. The morphism θ is given by $f \otimes x \mapsto (g \mapsto f(g)g.x)$.

We now define an action of the function $F_{f,\gamma}$. The element f can be represented as a coefficient matrix (this is clear since we assumed that Λ is a field). Thus there exists a representation V and elements $x \in V$ and $\xi \in V^*$ such that $f(g) = \langle \xi, g.x\rangle$. The endomorphism of $H_{\{0\},\mathrm{Reg}}$ is then defined as the following composition.

$$\begin{aligned}
H_{\{0\},\mathrm{Reg}} &\xrightarrow{\mathrm{id}\otimes x} H_{\{0\},\mathrm{Reg}} \otimes \underline{V} \\
&\simeq H_{\{0\},\mathrm{Reg}\otimes \underline{V}} \\
&\simeq H_{\{0\},\mathrm{Reg}\otimes V} \\
&\simeq H_{\{0,1\},\mathrm{Reg}\boxtimes V} \\
&\xrightarrow{\gamma} H_{\{0,1\},\mathrm{Reg}\boxtimes V} \\
&\simeq H_{\{0\},\mathrm{Reg}} \otimes \underline{V} \\
&\xrightarrow{\mathrm{id}\otimes \xi} H_{\{0\},\mathrm{Reg}}.
\end{aligned}$$

Here the third line is induced by the morphism θ, the fourth one by the cocartesianity of H, the fifth one for the action of γ is on leg indexed by 1 and the sixth one is the inverse of the second, third and fourth ones. The argument of [LZ18] shows that this construction defines an action of $\mathcal{O}(\mathrm{Hom}(\mathrm{Weil}^{t,\circ}_{F_x}, {}^L G_{\mathrm{loc}}))$. Hence we get a quasi-coherent module on $\mathrm{Hom}(\mathrm{Weil}^{t,\circ}_{F_x}, {}^L G_{\mathrm{loc}})$.

Let us show that this quasi-coherent module is supported on $\underline{Z}^{1,t,\circ}$. Let I be the ideal of $\underline{Z}^{1,t,\circ}$ in Hom. We first describe this ideal. Let $\phi \in \mathrm{FFS}/\mathrm{Weil}^{t,\circ}$. Let I_ϕ be the ideal defining Z_ϕ in ${}^L G^I_{\mathrm{loc}}$. Since the diagram 4.28 is cartesian, the ideal I_ϕ is generated by functions of the form $(\lambda - \lambda(\pi\phi))f$ where $f \in \mathcal{O}({}^L G^I_{\mathrm{loc}})$ is any function and $\lambda \in \mathcal{O}(Q^I_{\mathrm{loc}})$ is any function and $\lambda(\pi\phi)$ is the value at the morphism $\mathrm{FS}(I) \xrightarrow{\phi} \mathrm{Weil}^{t,\circ} \xrightarrow{\pi} Q_{\mathrm{loc}}$. We show that the relation $F_{\lambda.f,\phi} = F_{\lambda(\pi\phi)f,\phi}$ holds in $\mathcal{O}(\underline{Z}^{1,t,\circ})$. By the presentation of $I = \varinjlim_\phi I_\phi$, it is clear that $H_{\{0\},\mathrm{Reg}}$ is then killed by all functions in I and thus that this module is supported on $\underline{Z}^{1,t,\circ}$. Let us choose $V, x \in V, \xi \in V^*$ representing f and $W, x_W \in W$ and ξ_W representing λ where W is a representation of Q^I_{loc}. Then the following diagram is commutative

$$\begin{tikzcd}
H_{\{0\},\mathrm{Reg}} \arrow[d, "\mathrm{id}\otimes x\otimes x_W"] & \\
H_{\{0\},\mathrm{Reg}\otimes V\otimes W} \arrow[d] \arrow[dr] & \\
H_{\{0,1\},\mathrm{Reg}\boxtimes(V\otimes W)} \arrow[d, dashed, "\gamma"] & H_{\{0,1\},\mathrm{Reg}\boxtimes V} \otimes W \arrow[l] \arrow[d, dashed, "\gamma"] \\
H_{\{0,1\},\mathrm{Reg}\boxtimes(V\otimes W)} \arrow[d] & H_{\{0,1\},\mathrm{Reg}\boxtimes V} \otimes W \arrow[l] \arrow[dl] \\
H_{\{0\},\mathrm{Reg}\otimes V\otimes W} \arrow[d, dashed, "\mathrm{id}\otimes\xi\otimes\xi_W"] & \\
H_{\{0\},\mathrm{Reg}} &
\end{tikzcd}$$

where the first and last vertical maps are induced by $x \otimes x_W$, θ and $\xi \otimes \xi_W$ as before. The two horizontal maps are the $\mathrm{Rep}Q^I_{\mathrm{loc}}$-linearity of the functor H, the action of γ on the left column is given by the action of Weil on the leg indexed by 1 while the action on the right column is induced by the action of Q_{loc} on W. Hence the composition along the left column is $F_{f\lambda,\phi}$ while the composition along the right column is $F_{f\lambda(\gamma),\phi}$.

We now show that it is equipped with an action of $\hat{G}$. Since we used the left regular representation, there is still a right action of $\hat{G}$ on Reg hence $H_{\{0\},\mathrm{Reg}}$ acquires the structure of a $\hat{G}$-module. Now, arguing as in [LZ18] this action is compatible with the action of $\hat{G}$ on $\mathrm{Hom}(\mathrm{Weil}^{t,\circ}_{F_x}, {}^L G_{\mathrm{loc}})$, in the following sense. Let $f \in \mathcal{O}({}^L G_{\mathrm{loc}})$ and denote by ${}^g f$ the function on ${}^L G_{\mathrm{loc}}$ defined as $h \mapsto f(g^{-1}hg)$. Then we have $g.(F_{f,\gamma}(\alpha)) = F_{{}^g f,\gamma}(g.\alpha)$ for $\alpha \in H_{\{0\},\mathrm{Reg}}$. $\square$

Let $V \in \mathrm{Rep}_\Lambda({}^L G)^I$ and consider its restriction V_δ to $\hat{G}$ along the diagonal embedding $\hat{G} \to ({}^L G)^I$. Let $\pi : \underline{Z}^{1,t,\circ}(F_x, \hat{G})/\hat{G} \to \mathrm{pt}/\hat{G}$, the morphism induced by the structure map of $\underline{Z}^{1,t,\circ}(F_x, \hat{G})$. Then $\pi^* V = \mathcal{E}_V$ is a vector bundle on $\underline{Z}^{1,t,\circ}(F_x, \hat{G})/\hat{G}$. Moreover it has an action of $(\mathrm{Weil}^{t,\circ}_{F_x})^I$, such that its fiber over a points $\sigma \in \underline{Z}^{1,t,\circ}(F_x, \hat{G})$ is the representation $(\mathrm{Weil}^{t,\circ}_{F_x})^I \xrightarrow{\sigma^I} ({}^L G)^I \to \mathrm{GL}(V)$.

We consider the functors $(I, W) \mapsto H^j_{I,N,W}$ given by the cohomology of stacks of chtoucas. For all j, there is a quasicoherent sheaf $\mathcal{M}^j_N$ corresponding to this functor given by theorem 4.4.33.

Lemma 4.4.34. *(i). If $\Lambda = \overline{\mathbb{Q}}_\ell$, then for all $W \in \mathrm{Rep}({}^L G)^I$, there is a $(\mathrm{Weil}^{t,\circ}_{F_x})^I$-linear isomorphism $(\mathcal{M}^j_N \otimes \mathcal{E}_W)^{\hat{G}} = H^j_{I,N,W}$.*

(ii). In general there is a spectral sequence

$$E_2^{p,q} = H^p(\hat{G}, \mathcal{M}^q_N \otimes \mathcal{E}_W) \Rightarrow H^{p+q}_{I,W,N}. \tag{4.32}$$

Proof. Firstly, note that if $\Lambda = \overline{\mathbb{Q}}_\ell$ then $H^i(\hat{G}, -) = 0$ for all $i > 0$ hence the first point follows from the second. In general, for $V \in \mathrm{Rep}_\Lambda({}^L G)^I$, consider the sheaf $\mathrm{Sat}(\mathrm{Reg}_{\hat{G}}) \otimes V$ on $L^+_{\{0\}}\mathcal{G}_\sigma \backslash \mathrm{Gr}_{\{0\}}$. Using the notations of [FS21] section VIII-5, the sheaf $\mathfrak{p}_!^{\leq \mu} \epsilon^*(\mathrm{Sat}(\mathrm{Reg}_{\hat{G}}) \otimes V)$ naturally lifts to $\mathrm{IndPerf}(B({}^L G)^I) \otimes \mathrm{D}(\bar{\eta}, \Lambda)$. After forgetting along the $\hat{G} \to ({}^L G)^I$ we get an object of $\mathrm{IndPerf}(B(\hat{G})) \otimes \mathrm{D}(\bar{\eta}, \Lambda)$. Taking $\mathrm{R}\Gamma(\hat{G}, -)$ yields $\mathrm{R}\Gamma(\hat{G}, \mathfrak{p}_!^{\leq \mu} \epsilon^*(\mathrm{Sat}(\mathrm{Reg}_{\hat{G}}) \otimes V)) = \mathfrak{p}_!^{\leq \mu} \epsilon^* \mathrm{R}\Gamma(\hat{G}, (\mathrm{Sat}(\mathrm{Reg}_{\hat{G}}) \otimes V)) = \mathfrak{p}_!^{\leq \mu} \epsilon^* \mathrm{Sat}(V_\delta)$. After taking colimits over μ, we get $\mathrm{R}\Gamma(\hat{G}, H_{\{0\},\mathrm{Reg}} \otimes V) = H_{\{0\},V_\delta}$. Taking cohomology yields $H^i(\mathrm{R}\Gamma(\hat{G}, H_{\{0\},\mathrm{Reg}} \otimes V) = H_{\{0\},V_\delta}) = H^i_{\{0\},V_\delta}$, the spectral sequence is then the hypercohomology spectral sequence for the functor $\mathrm{R}\Gamma(\hat{G}, -)$ applied to the complex $H_{\{0\},\mathrm{Reg}} \otimes V$. $\square$

Lemma 4.4.35. *Let $Z_{\mathrm{Exc}} \subset \mathrm{Spec}(\mathrm{Exc}(\mathrm{Weil}^t_{F_x}, \hat{G}))$ be a closed subscheme of $\mathrm{Spec}(\mathrm{Exc}(\mathrm{Weil}^t_{F_x}, \hat{G}))$ and let $Z \subset \underline{Z}^{1,t,\circ}(F_x, \hat{G})$ be its inverse image. Suppose that for all (I, W) the $\mathrm{Exc}(\mathrm{Weil}^t_{F_x}, \hat{G})$-module $H^j_{I,W,N}$ is supported on Z_{Exc} then M^j_N is supported on Z.*

Proof. This is immediate since $M^j_N = H_{\{0\},\mathrm{Reg}}$ and the $\mathrm{Exc}(\mathrm{Weil}^t_{F_x}, \hat{G})$-module is compatible with the $\mathcal{O}(\underline{Z}^{1,t,\circ}(F_x, \hat{G}))$-module structure. $\square$

4.5 Deligne-Lusztig theory and cohomology of stacks of chtoucas

The goal of this section of to explain how to geometrically compute the Deligne-Lusztig restriction of the cohomology of stacks of chtoucas. Consider the stack $\mathrm{Cht}_{\mathcal{G}_\sigma,N,I}$ for the level structure N as above. This is a stack over $(X-N)^I$, it is equipped with an action of the finite group $\mathcal{G}_\sigma(\mathbb{F}_x)$.

4.5.1 The morphism emb

Lemma 4.5.1. *Consider the stack*

$$\mathrm{Cht}_{\mathcal{G}_\sigma,N,I} \times^{\mathcal{G}_{\sigma,\mathbb{F}_x}(\mathbb{F}_x)} \mathcal{G}_{\sigma,\mathbb{F}_x}/B_\sigma. \tag{4.33}$$

This is a stack over $(X-N)^I \times_{\mathbb{F}_q} \mathbb{F}_x$. Let S be an $\mathbb{F}_q$-scheme, its groupoid of S-points classifies

(i). An I-tuple of points $z_i : S \to (X-N) \times x$ and denote by $y_i : S \to (X-N)$ the composition of z_i with the projection,

(ii). A $\mathcal{G}_\sigma$-chtouca $\phi : \mathcal{E} \to {}^\tau\mathcal{E}$ over S with legs at (y_i).

(iii). A principal level structure on $(N-x) \times_{\mathbb{F}_q} S$, we denote it by ψ^x.

(iv). A B_σ-reduction of the corresponding $\mathcal{G}_{\sigma,\mathbb{F}_x}$-torsor $\mathcal{E}_{x\times S}$, we denote it by ψ_x.

Proof. Let us denote by $\mathrm{Cht}^{B_\sigma-\mathrm{red}_x}_{\mathcal{G}_\sigma,I,N^x}$ the stack classifying the data (i)-(iv) of the lemma. Similarly denote by $\mathrm{Cht}^{B_\sigma-\mathrm{red}_x}_{\mathcal{G}_\sigma,I,N}$ the stack classfying both a $\mathcal{G}_\sigma$-chtouca with level structure N and a B_σ-reduction of the first torsor at x. We claim that there is a $\mathcal{G}_{\sigma,\mathbb{F}_x}(\mathbb{F}_x)$-equivariant isomorphism $\mathrm{Cht}^{B_\sigma-\mathrm{red}_x}_{\mathcal{G}_\sigma,I,N} \to \mathrm{Cht}_{\mathcal{G}_\sigma,I,N} \times \mathcal{G}_{\sigma,\mathbb{F}_x}/B_\sigma$ making the following diagram commute

$$\begin{tikzcd}
\mathrm{Cht}^{B_\sigma-\mathrm{red}_x}_{\mathcal{G}_\sigma,I,N} \arrow[r] \arrow[dr] & \mathrm{Cht}_{\mathcal{G}_\sigma,I,N} \times \mathcal{G}_{\sigma,\mathbb{F}_x}/B_\sigma \arrow[d] \\
& \mathrm{Cht}_{\mathcal{G}_\sigma,I,N^x}
\end{tikzcd}$$

where the vertical maps are given by forgetting the B_σ-reduction at x and the level structure and by the first projection respectively. Taking quotients by $\mathcal{G}_{\sigma,\mathbb{F}_x}(\mathbb{F}_x)$ yields the desired isomorphism

$$\mathrm{Cht}^{B_\sigma-\mathrm{red}_x}_{\mathcal{G}_\sigma,I,N^x} \simeq \mathrm{Cht}_{\mathcal{G}_\sigma,N,I} \times^{\mathcal{G}_{\sigma,\mathbb{F}_x}(\mathbb{F}_x)} \mathcal{G}_{\sigma,\mathbb{F}_x}/B_\sigma. \tag{4.34}$$

Let us now prove the claim. Since the level structures on N^x play no role we will ignore them. Let S be an $\mathbb{F}_q$-scheme and let $((z_i), \mathcal{E}, \phi, \psi_x, \psi_x^{B_\sigma})$ be an S-point of $\mathrm{Cht}^{B_\sigma-\mathrm{red}_x}_{\mathcal{G}_\sigma,I,N}$ where

(i). $z_i : S \to (X-N) \times_{\mathbb{F}_q} \mathbb{F}_x$ and $y_i : S \to X-N$ its composition with the projection,

(ii). ψ_x the principal level structure at x,

(iii). $\psi_x^{B_\sigma}$ the B_σ-reduction at $x \times S$.

The level structure ψ_x is an isomorphism of $\mathcal{G}_\sigma$-torsor over $x \times S$

$$\mathcal{E}_{|x\times S} \simeq \mathcal{E}^0_{|x\times S}, \tag{4.35}$$

where $\mathcal{E}^0_{|x\times S}$ is as before the trivial $\mathcal{G}_\sigma$-torsor. The B_σ-reduction is the data of a section of an $S \times x$-point of $\mathcal{E}_{|x\times S}/(B_\sigma)_{x\times S}$. The isomorphism ψ_x induces an isomorphism

$$\mathcal{E}_{|x\times S}/B_\sigma \simeq \mathcal{E}^0_{|x\times S}/B_\sigma. \tag{4.36}$$

Under this isomorphism the point $\psi_x^{B_\sigma}$ yields an $S \times x$ point of $\mathcal{G}_{\sigma,\mathbb{F}_x}/B_\sigma$ denoted by $\psi_x^{B_\sigma,\mathrm{triv}}$. It is clear the map sending $((z_i), \mathcal{E}, \phi, \psi_x, \psi_x^{B_\sigma}) \mapsto (((z_i), \mathcal{E}, \phi, \psi_x, \psi_x^{B_\sigma,\mathrm{triv}})$ is functorial in S and therefore defines a morphism of stacks

$$\mathrm{Cht}^{B_\sigma-\mathrm{red}_x}_{\mathcal{G}_\sigma,I,N} \to \mathrm{Cht}_{\mathcal{G}_\sigma,I,N} \times \mathcal{G}_{\sigma,\mathbb{F}_x}/B_\sigma. \tag{4.37}$$

It is clear that this map is an isomorphism on S-points hence an isomorphism. Moreover the commutativity of the above diagram is clear since we have not modified the level structure outside of x nor the underlying $\mathcal{G}_\sigma$-chtouca. Finally the $\mathcal{G}_{\sigma,\mathbb{F}_x}(\mathbb{F}_x)$-equivariance follows from the fact that the action on ψ_x is given by right translation and the action on $\mathcal{G}_{\sigma,\mathbb{F}_x}/B_\sigma$ is given by left translations. □

Remark 4.5.2. Note that we have $\mathcal{G}_{\sigma,\mathbb{F}_x}/B_\sigma = M_\sigma/B_\sigma$, hence we can also write

$$\mathrm{Cht}_{\mathcal{G}_\sigma,N,I} \times^{\mathcal{G}_{\sigma,\mathbb{F}_x}(\mathbb{F}_x)} \mathcal{G}_{\sigma,\mathbb{F}_x}/B_\sigma = \mathrm{Cht}_{\mathcal{G}_\sigma,N,I} \times^{\mathcal{G}_{\sigma,\mathbb{F}_x}(\mathbb{F}_x)} M_\sigma/B_\sigma. \tag{4.38}$$

By lemma 4.4.7, we identified $\mathcal{G}_C$ with the dilatation of B_σ in $\mathcal{G}_\sigma$, we can then apply the following theorem.

Theorem 4.5.3 ([MRR20])**.** *There is a functorial isomorphism of stacks over* $\mathbb{F}_q$.

$$\mathrm{Bun}_{\mathcal{G}_\sigma,B_\sigma} = \mathrm{Bun}_{\mathcal{G}_C}, \tag{4.39}$$

between the stacks of $\mathcal{G}_\sigma$-torsor with B_σ level structure at x and the stack of $\mathcal{G}_C$-torsor.

Using theorem 4.5.3, we define a map of stacks

$$\mathrm{emb} : \mathrm{Cht}_{\mathcal{G}_\sigma,I,N} \times^{\mathcal{G}\sigma(\mathbb{F}_x)} M_\sigma/B_M \to (\mathrm{Cht}_{\mathcal{G}_C,I\sqcup\{0\},N^x})_{(X-N)^I\times x} \tag{4.40}$$

over $(X-N)^I \times x$ as follows. On the right hand side this is the stack of chtoucas for the group scheme $\mathcal{G}_C$ with $|I|+1$ legs and the last one fixed at x. Let S be a scheme, and $((y_i), \mathcal{E}, \phi, \psi^x, \psi_x)$ be an S-point of the left hand side. From theorem 4.5.3, the data $(\mathcal{E}, \psi_x)$ yields a $\mathcal{G}_C$-torsor $\mathcal{E}_C$ and similarly for ${}^\tau\mathcal{E}, {}^\tau\psi_x$. We then define the following S-point of the right hand side

(i). The legs are $((y_i), x)$,

(ii). The chtoucas is $(\phi : \mathcal{E}_C \to {}^\tau\mathcal{E}_C)$,

(iii). The level structure on $N - x$ is given by ψ^x using the isomorphism $(\mathcal{E}_C)_{S\times(X-x)} = \mathcal{E}_{S\times(X-x)}$.

This construction is clearly functorial in S and therefore defines a morphism of stacks over $(X-N)^I \times x$.

4.5.2 A key diagram

We denote by $\mathrm{Iw}^0 = L_x^+\mathcal{G}_{C^0}$ and by $\mathrm{Iw} = L_x^+\mathcal{G}_C$. From now on, instead of the double coset $U_M\backslash M_\sigma/U_M$ we will consider $\mathrm{Iw}^0\backslash M_\sigma/U_M$, where Iw^0 is the pro-unipotent part of the Iwahori Iw which acts on M through its quotient $\mathrm{Iw}^0 \to U_M$. Since the kernel of this map is unipotent, we have an equivalence induced by pullback

$$\mathrm{D}(U_M\backslash M_\sigma/U_M, \Lambda) = \mathrm{D}(\mathrm{Iw}^0\backslash M_\sigma/U_M, \Lambda). \tag{4.41}$$

We have $L_x^+\mathcal{G}_\sigma/\mathrm{Iw}^0 = M_\sigma/U_M$. This induces an embedding $M_\sigma/U_M \subset L_xG/\mathrm{Iw}^0$.

Let S be a scheme and let $\mathcal{E}_{C^0}$ be a $\mathcal{G}_{C^0}$-torsor over $x \times S$. Let $t \in T_M(S)$ be a point in T_M, we define a $\mathcal{G}_{C^0}$-torsor $\mathcal{E}_{C^0}^t$ by twisting the action by $\mathrm{Ad}(t)$. That is, the underlying space of $\mathcal{E}_{C^0}^t$ is $\mathcal{E}_{C^0}$ and the action of $\mathcal{G}_{C^0}$ is given by

$$g.^t x = (tgt^{-1}).x \tag{4.42}$$

where $.^t$ denotes the action of $\mathcal{G}_{C^0}$ on $\mathcal{E}_{C^0}^t$.

Consider the map of stacks over $(X - N^x)^I \times x$

$$\mathrm{Cht}_{\mathcal{G}_{C^0}, I\sqcup\{0\}} \to \mathrm{Cht}_{\mathcal{G}_C, I\sqcup\{0\}} \tag{4.43}$$

induced by $\mathcal{G}_{C^0} \to \mathcal{G}_C$. Using the above action on $\mathcal{G}_{C^0}$ torsors over x, this map of stacks is a T-torsor. Consider now the map ϵ for the group $\mathcal{G}_{C^0}$,

$$\mathrm{Cht}_{\mathcal{G}_{C^0}, N^x, I\sqcup\{0\}})_{|(X-N^x)^I\times x} \to (L^+_{I\sqcup\{0\}}\mathcal{G}_{C^0}\backslash \mathrm{Gr}_{\mathcal{G}_{C^0}, I\sqcup\{0\}})_{|(X-N^x)^I\times x}. \tag{4.44}$$

The right hand side is equipped with two actions of T. Indeed the S-points of the right hand side classify

(*i*). Some points $(y_i) : S \to X$,

(*ii*). Two $\mathcal{G}_{C^0}$-torsors on $\widehat{\cup_i\Gamma_i \cup x}, \mathcal{E}, \mathcal{E}'$,

(*iii*). An isomorphism between $\mathcal{E}$ and $\mathcal{E}'$ on the punctured neighborhood of the graphs.

On the stack $(L^+_{I\sqcup\{0\}}\mathcal{G}_{C^0}\backslash \mathrm{Gr}_{\mathcal{G}_{C^0}, I\sqcup\{0\}})_{|(X-N^x)^I\times x}$, there are two actions of T obtained by rescaling both torsors $\mathcal{E}$ and $\mathcal{E}'$. The map ϵ is equivariant for the action of T on the source, and for the Ad_{F}-action on the target. That is, the action induced by restriction along the map $t \mapsto (t, \mathrm{F}(t^{-1}))$. We now get a cartesian diagram, where the vertical maps are T-torsors.

$$\begin{array}{ccc} (\mathrm{Cht}_{\mathcal{G}_{C^0}, N^x, I\sqcup\{0\}})_{|(X-N^x)^I\times x} & \xrightarrow{\epsilon} & (L^+_{I\sqcup\{0\}}\mathcal{G}_{C^0}\backslash \mathrm{Gr}_{\mathcal{G}_{C^0}, I\sqcup\{0\}})_{|(X-N^x)^I\times x} \\ \downarrow & & \downarrow \\ (\mathrm{Cht}_{\mathcal{G}_C, N^x, I\sqcup\{0\}})_{|(X-N^x)^I\times x} & \xrightarrow{\tilde{\epsilon}} & \dfrac{(L^+_{I\sqcup\{0\}}\mathcal{G}_{C^0}\backslash \mathrm{Gr}_{\mathcal{G}_{C^0}, I\sqcup\{0\}})}{\mathrm{Ad}_{\mathrm{F}}T_M}_{|(X-N^x)^I\times x} \end{array}$$

On $(X - N)^I \times x$, we have a decomposition

$$\frac{(L^+_{I\sqcup\{0\}}\mathcal{G}_{C^0}\backslash \mathrm{Gr}_{\mathcal{G}_{C^0}, I\sqcup\{0\}})}{\mathrm{Ad}_{\mathrm{F}}T_M} = L^+\mathcal{G}_{C^0}\backslash \mathrm{Gr}_{\mathcal{G}_{C^0}, I} \times \frac{\mathrm{Iw}^0\backslash LG/\mathrm{Iw}^0}{\mathrm{Ad}_{\mathrm{F}}T_M}. \tag{4.45}$$

Consider now the map $\mathrm{emb} : L^+\mathcal{G}_{C^0}\backslash \mathrm{Gr}_{\mathcal{G}_{C^0}, I} \times \frac{\mathrm{Iw}^0\backslash M_\sigma/U_M}{\mathrm{Ad}_{\mathrm{F}}T_M} \to L^+\mathcal{G}_{C^0}\backslash \mathrm{Gr}_{\mathcal{G}_{C^0}, I} \times \frac{\mathrm{Iw}^0\backslash LG/\mathrm{Iw}^0}{\mathrm{Ad}_{\mathrm{F}}T_M}$.

Lemma 4.5.4. *Assume that* $\deg(x/\mathbb{F}_q) = 1$. *The following diagram is Cartesian* $(X-N)^I \times x$

$$\begin{tikzcd}
\mathrm{Cht}_{\mathcal{G}_\sigma,N,I} \times^{\mathcal{G}_\sigma(\mathbb{F}_x)} M_\sigma/B_M \arrow[r, "\mathrm{emb}"] \arrow[d, "\epsilon"] & \mathrm{Cht}_{\mathcal{G}_C,N^x,I\sqcup\{0\}} \arrow[d, "\epsilon"] \\
L_I^+\mathcal{G}_{C^0}\backslash \mathrm{Gr}_{I,\mathcal{G}_{C^0}} \times \frac{\mathrm{Iw}^0\backslash M_\sigma/U_M}{\mathrm{Ad}_{\mathrm{F}}T_M} \arrow[r, "\mathrm{emb}"'] & L_I^+\mathcal{G}_{C^0}\backslash \mathrm{Gr}_{I,\mathcal{G}_{C^0}} \times \frac{\mathrm{Iw}^0\backslash LG/\mathrm{Iw}^0}{\mathrm{Ad}_{\mathrm{F}}T_M}
\end{tikzcd}$$

where the left vertical map is induced by the map $M_\sigma(\mathbb{F}_x)\backslash M_\sigma/B_M \simeq \frac{M_\sigma}{\mathrm{Ad}_{\mathrm{F}}B_M} \to \frac{U_M\backslash M_\sigma/U_M}{\mathrm{Ad}_{\mathrm{F}}T_M} \to \frac{\mathrm{Iw}^0\backslash M_\sigma/U_M}{\mathrm{Ad}_{\mathrm{F}}T_M}$ *where the first isomorphism comes from the Lang map of* M_σ.

Proof. Let $\alpha = ((y_i), \mathcal{E} \xrightarrow{\phi} {}^\tau\mathcal{E}, \psi_x, \psi^x)$ be an S-point of $\mathrm{Cht}_{\mathcal{G}_\sigma,N,I} \times^{\mathcal{G}_\sigma(\mathbb{F}_x)} M_\sigma/B_M$.

(i). The S-point $\epsilon \circ \mathrm{emb}(\alpha)$ is the S-point of $\mathrm{Cht}_{\mathcal{G}_\sigma,N,I} \times^{\mathcal{G}_\sigma(\mathbb{F}_x)} M_\sigma/B_M$ given by the legs (y_i), the restriction of the modification ϕ to Γ_y for the first projection and by the modification of the $\mathcal{G}_{C^0}$-torsors corresponding to $(\mathcal{E}_{|\Gamma_x}, \psi_x) \to {}^\tau(\mathcal{E}_{|\Gamma_x}, {}^\tau\psi_x)$ induced by ϕ where Γ_x is the formal neighborhood of $S \times x$ in $S \times X$.

(ii). Similarly, the S-point obtained as $\mathrm{emb} \circ \epsilon(\alpha)$ is given by the restriction of ϕ to Γ_y for the first factor. For the second factor, it is given by the $\mathcal{G}_{C^0}$-torsors corresponding to $(\mathcal{E}_{|\Gamma_x}, \psi_x) \to {}^\tau(\mathcal{E}_{|\Gamma_x}, \mathrm{F}_{M_\sigma}\psi_x)$ induced by ϕ.

The commutativity of the diagram follows from the fact that ${}^\tau\psi_x = \mathrm{F}_{M_\sigma}\psi_x$ since x is a degree one point hence F_{M_σ} is naturally the q-power absolute Frobenius.

To show the cartesianity of the diagram, let us describe the closed substack $L_I^+\mathcal{G}_{C^0}\backslash \mathrm{Gr}_{I,\mathcal{G}_{C^0}} \times \frac{\mathrm{Iw}^0\backslash M_\sigma/U_M}{\mathrm{Ad}_{\mathrm{F}}T_M} \subset L_I^+\mathcal{G}_{C^0}\backslash \mathrm{Gr}_{I,\mathcal{G}_{C^0}} \times \frac{\mathrm{Iw}^0\backslash LG/\mathrm{Iw}^0}{\mathrm{Ad}_{\mathrm{F}}T_M}$. Let $\alpha = ((y_i), \mathcal{E} \xrightarrow{\phi} \mathcal{E}')$ be an S-point of $L_I^+\mathcal{G}_{C^0}\backslash \mathrm{Gr}_{I,\mathcal{G}_{C^0}} \times \frac{\mathrm{Iw}^0\backslash LG/\mathrm{Iw}^0}{\mathrm{Ad}_{\mathrm{F}}T_M}$ then this substack the locus where the modification ϕ induces an isomorphism of $\mathcal{G}_\sigma$-torsors on $S \times x$. The pullback

$$Z = \mathrm{Cht}_{\mathcal{G}_C,N^x,I\sqcup\{0\}} \times_{L_I^+\mathcal{G}_{C^0}\backslash \mathrm{Gr}_{I,\mathcal{G}_{C^0}} \times \frac{\mathrm{Iw}^0\backslash LG/\mathrm{Iw}^0}{\mathrm{Ad}_{\mathrm{F}}T_M}} L_I^+\mathcal{G}_{C^0}\backslash \mathrm{Gr}_{I,\mathcal{G}_{C^0}} \times \frac{\mathrm{Iw}^0\backslash M_\sigma/U_M}{\mathrm{Ad}_{\mathrm{F}}T_M}$$

then classifies a $\mathcal{G}_{C^0}$-chtouca such that over $x \times S$ the modification of $\mathcal{G}_{C^0}$-torsors extends to an isomorphism of $\mathcal{G}_\sigma$-torsors and thus yields a point in $\mathrm{Cht}_{\mathcal{G}_\sigma,N,I} \times^{\mathcal{G}_\sigma(\mathbb{F}_x)} M_\sigma/B_M$. □

Corollary 4.5.5. *The map* emb *is a closed immersion.*

Lemma 4.5.6. *The map* $\epsilon : \mathrm{Cht}_{\mathcal{G}_C,N^x,I\sqcup\{0\}} \to \frac{(L_{I\sqcup\{0\}}^+\mathcal{G}_{C^0}\backslash \mathrm{Gr}_{\mathcal{G}_{C^0},I\sqcup\{0\}})}{\mathrm{Ad}_{\mathrm{F}}T_M}$ *is smooth over* $(X-N^x)^I \times x$.

Proof. The proof of [Laf18] 2.8 yields immediatly that ϵ for the group $\mathcal{G}_{C^0}$ is smooth, indeed it the argument of *loc. cit.* only requires that $\mathrm{Bun}_{\mathcal{G}_{C^0}}$ and $\mathrm{Bun}_{\mathcal{G}_{C^0},N}$ are smooth but both statements follow from the smoothness of the group scheme $\mathcal{G}_{C^0}$. Modding out by the action of T_M yields the result. □

4.5.3 The case of higher degree points

If $\deg(x) > 1$ the diagram of lemma 4.5.4 is a priori not commutative. To go around this issue, we modify the map $\epsilon_{\mathcal{G}_C}$. What we do here is a variation of the construction of the restriction morphism of [GL17].

Definition 4.5.7. Denote by $\epsilon_{d,\mathcal{G}_C} : \mathrm{Cht}_{\mathcal{G}_C, I\sqcup\{0\}, N^x} \to L_I^+\mathcal{G}_C\backslash \mathrm{Gr}_{I,\mathcal{G}_C} \times \frac{\mathrm{Iw}^0\backslash LG/\mathrm{Iw}^0}{\mathrm{Ad}_{\mathrm{F}} T_M}$ the map of stacks over $X^I \times x$ that sends a chtoucas $((y_i), \mathcal{E}, \phi)$ (we ignore the level structure on N^x) to the following datum. Firstly denote by ϕ_d the modification

$$\mathcal{E} \xrightarrow{\phi} {}^\tau\mathcal{E} \xrightarrow{{}^\tau\phi} \cdots \to {}^{\tau^d}\mathcal{E}, \tag{4.46}$$

between $\mathcal{E}$ and ${}^{\tau^d}\mathcal{E}$. The map ϵ_d is obtained by restricting the modification ϕ_d to $\cup_i \widehat{\Gamma_{y_i} \cup \Gamma_x}$.

Remark 4.5.8. Note that the modification ϕ_d happens at the legs $(y_i), ({}^\tau y_i), \ldots$ and x.

Definition 4.5.9. Let $U_d \subset X_x^I$ be the open subset corresponding to the equations $y_i \neq \tau^k y_{i'}$ and $y_i \neq \tau^k \overline{x}$ for $k = 1, \ldots, d-1$ and $i, i' \in I$. As in [GL17], this open contains the diagonal geometric point $\Delta(\overline{x})$.

Lemma 4.5.10. *Over $U_d \cap \mathring{X}^I$ the following diagram of stacks is cartesian*

$$\begin{tikzcd} \mathrm{Cht}_{\mathcal{G}_\sigma, N, I} \times^{(G_\sigma)(\mathbb{F}_x)} M_\sigma/B_M \arrow[r, "\mathrm{emb}"] \arrow[d, "\epsilon"] & \mathrm{Cht}_{\mathcal{G}_C, N^x, I\sqcup\{0\}} \arrow[d, "\epsilon_d"] \\ L_I^+\mathcal{G}_\sigma\backslash \mathrm{Gr}_{I,\mathcal{G}_\sigma} \times \frac{\mathrm{Iw}^0\backslash M_\sigma/U_M}{\mathrm{Ad}_{\mathrm{F}} T_M} \arrow[r, "\mathrm{emb}"'] & L_I^+\mathcal{G}_C\backslash \mathrm{Gr}_{I,\mathcal{G}_C} \times \frac{\mathrm{Iw}^0\backslash LG/\mathrm{Iw}^0}{\mathrm{Ad}_{\mathrm{F}} T_M} \end{tikzcd}$$

Proof. As in the proof of lemma 4.5.4, we can decouple the problem in checking what happens for the $\mathcal{G}_\sigma$-torsors and for the B_σ-reductions separately. The condition on U_d implies that the modification ${}^\tau\phi \ldots {}^{\tau^{d-1}}\phi$ is an isomorphism along the graph of the (y_i). Hence in the stack $L_I^+\mathcal{G}_C\backslash \mathrm{Gr}_{I,\mathcal{G}_C}$ the two modifications $\mathcal{E} \to {}^\tau\mathcal{E}$ and $\mathcal{E} \to {}^{\tau^d}\mathcal{E}$ are isomorphic. Therefore it only remains to check, what happens near x. Since the map $\mathcal{E} \to {}^\tau\mathcal{E}$ is an isomorphism of $\mathcal{G}_\sigma$-torsors at x, we only need to check what happens for the B_σ-structures. But it follows that the Frobenius τ^d agrees with the Frobenius F of M. $\square$

The following lemma is a particular case of [GL17] 2.15, obtained by taking $r = 0$ and $n = 0$ in *loc. cit.*.

Lemma 4.5.11. *The map ϵ_d is formally smooth over U_d.*

4.5.4 The Deligne-Lusztig restriction of the cohomology

Recall that there is an action of $\mathcal{G}_\sigma(\mathbb{F}_x)$ on $\mathrm{Cht}_{\mathcal{G}_\sigma,I,N}$ and that the sheaf $\mathcal{F}_{\mathcal{G}_\sigma,I,N,W}$ is $\mathcal{G}_\sigma(\mathbb{F}_x)$-equivariant. Hence the sheaves $\mathcal{H}_{\mathcal{G}_\sigma,I,N,W}$ canonically lifts to $\mathrm{D}((X-N)^I, \Lambda[\mathcal{G}_\sigma(\mathbb{F}_x)])$.

We consider the variant of the map ϵ defined previously. Namely we have the map

$$\epsilon_I : \mathrm{Cht}_{\mathcal{G}_\sigma,N,I}/\Xi \to L_I^+\mathcal{G}_{\sigma,\mathrm{ad}}\backslash \mathrm{Gr}_{\mathcal{G}_\sigma,I}, \tag{4.47}$$

defined in section 4.4.2. We also have the map induced by the Lang map defined in section 4.2.1

$$q : M_\sigma/B_M \to M_\sigma(\mathbb{F}_x)\backslash M_\sigma/B_M = \frac{M_\sigma}{\mathrm{Ad}_{\mathrm{F}} B_M} \to \frac{U_M\backslash M_\sigma/U_M}{\mathrm{Ad}_{\mathrm{F}} T_M}. \tag{4.48}$$

It induces a map

$$\mathrm{Cht}_{\mathcal{G}_\sigma,N,I}/\Xi \times M_\sigma/B_M \to L_I^+\mathcal{G}_{\sigma,\mathrm{ad}}\backslash \mathrm{Gr}_{\mathcal{G}_\sigma,I} \times \frac{U_M\backslash M_\sigma/U_M}{\mathrm{Ad}_{\mathrm{F}} T_M}, \tag{4.49}$$

which descends to a map

$$\epsilon : \mathrm{Cht}_{\mathcal{G}_\sigma,N,I}/\Xi \times^{\mathcal{G}_\sigma(\mathbb{F}_x)} M_\sigma/B_M \to L^+_I\mathcal{G}_{\sigma,\mathrm{ad}}\backslash \mathrm{Gr}_{\mathcal{G}_\sigma,I} \times \frac{U_M\backslash M_\sigma/U_M}{\mathrm{Ad}_{\mathrm{F}}T_M}. \tag{4.50}$$

Let $w \in W_M$. Recall that we have the following diagram over $\overline{x}$

$$\begin{tikzcd} \frac{U_M\backslash B_M w B_M/U_M}{\mathrm{Ad}_{\mathrm{F}}T_M} \arrow[r, "j_w"] \arrow[d, equal] & \frac{U_M\backslash M_\sigma/U_M}{\mathrm{Ad}_{\mathrm{F}}T_M} \\ \mathrm{pt}/T^{w\mathrm{F}} \ltimes U_{M,w} \arrow[d, "k_w"] & \\ \mathrm{pt}/T^{w\mathrm{F}} & \end{tikzcd}$$

Lemma 4.5.12. *We have an isomorphism of sheaves on $(X-N)^I \times \overline{x}$, compatible with the $T^{w\mathrm{F}}$-action.*

$$^*\mathcal{R}^M_w((\mathcal{H}^{\leq\mu}_{I,N,\Xi,W})^{V_\sigma(\mathbb{F}_x)}) = \mathfrak{p}_!\epsilon^*(\mathrm{Sat}(W) \boxtimes^{\mathcal{G}_\sigma(\mathbb{F}_x)} j_{w,*}k^*_w\mathrm{Reg}_{T^{w\mathrm{F}}}), \tag{4.51}$$

where $(-)^{V_\sigma(\mathbb{F}_x)}$ denotes the invariants under $V_\sigma(\mathbb{F}_x)$, it is equipped with an action of $M_\sigma(\mathbb{F}_x)$ and we take the Deligne-Lusztig restriction with respect to this restriction, that is $\mathrm{RHom}_{M_\sigma(\mathbb{F}_x)}(\mathrm{R}\Gamma_c(Y_{M_\sigma}(\dot{w}),\Lambda),-)$.

Proof. By lemma 4.2.2, we have $\mathrm{R}\Gamma(M/B_M, q^*j_{w,*}k^*_w\mathrm{Reg}_{T^{w\mathrm{F}}}) = \mathrm{R}\Gamma(Y_{M_\sigma}(\dot{w}),\Lambda)$. The lemma then follows from the Kunneth formula. □

Remark 4.5.13. We have base changed everything to $\overline{x}$ to take into account the fact that $B_M w B_M$ is a priori only defined over $\overline{x}$ and not over x.

4.5.5 Compatibility of Harder-Narasimhan truncations

In 4.4.2, we fixed some Harder-Narasimhan truncations for both groups $\mathcal{G}_\sigma$ and $\mathcal{G}_C$. We now discuss the following compatibility of the morphism emb with these truncations.

Lemma 4.5.14. *Let $\mu' \in \Lambda_C$ and μ its projection onto Λ_σ, then we have $\mathrm{emb}^{-1}(\mathrm{Cht}^{\leq\mu'}_{\mathcal{G}_C,I\cup\{0\}}) \subset \mathrm{Cht}^{\leq\mu}_{\mathcal{G}_\sigma,I}$.*

Proof. We have a commutative diagram

$$\begin{tikzcd} \mathrm{Cht}_{\mathcal{G}_\sigma,I,N} \times^{M(\mathbb{F}_x)} M_\sigma/B_M \arrow[r] \arrow[dr] \arrow[ddr] & \mathrm{Cht}_{\mathcal{G}_C,I\sqcup\{0\},N^x} \arrow[d] \\ & \mathrm{Bun}_{\mathcal{G}_C} \arrow[d] \\ & \mathrm{Bun}_{\mathcal{G}_\sigma} \end{tikzcd}$$

where the map $\mathrm{Bun}_{\mathcal{G}_C} \to \mathrm{Bun}_{\mathcal{G}_\sigma}$ is induced by the map $\mathcal{G}_C \to \mathcal{G}_\sigma$. In terms of torsors, a $\mathcal{G}_C$-torsor is given by a $\mathcal{G}_\sigma$-torsor and a B_σ-reduction at x of this $\mathcal{G}_\sigma$-torsor, then this map simply forgets the B_σ-reduction. By our choice of representations of $\mathcal{G}_C$ and $\mathcal{G}_\sigma$, the image of $\mathrm{Bun}^{\leq\mu'}_{\mathcal{G}_C}$ in $\mathrm{Bun}_{\mathcal{G}_\sigma}$ is contained in $\mathrm{Bun}^{\leq\mu}_{\mathcal{G}_\sigma}$, this yields the desired inclusion. □

On cohomology sheaves, we then have two filtered cocartesian functors $\mathrm{Rep}^L G^I \to \mathrm{D}((X-N)^I, \Lambda)$ given by

(*i*). $(I, W, \mu) \mapsto \mathfrak{p}_!^{\leq\mu}(\epsilon_{\mathrm{ad}}^* \mathrm{Sat}(W) \boxtimes^{M(\mathbb{F}_x)} \nabla_w(\mathrm{Reg}_{T^{w\mathrm{F}}}) = \mathrm{R}_w^* \mathcal{H}_{I,\mathcal{G}_\sigma,N,W,\Xi}^{\leq\mu}$ and

(*ii*). $(I, W, \mu') \mapsto \mathfrak{p}_!^{\leq\mu'}(\mathrm{emb}_! \epsilon_{\mathrm{ad}}^* \mathrm{Sat}(W) \boxtimes^{M(\mathbb{F}_x)} \nabla_w(\mathrm{Reg}_{T^{w\mathrm{F}}}))$.

Using lemma 4.5.14, the adjunction maps $\mathrm{id} \to j_! j^*$ along all inclusions corresponding to $\mu' \mapsto \mu$ define a compatibility datum between these two cocartesian functors filtered with respect to partial Frobenii. In particular, on colimits, we get a canonical isomorphism

$$\varinjlim_{\mu} \mathrm{R}_w^* \mathcal{H}_{I,\mathcal{G}_\sigma,N,W,\Xi}^{\leq\mu} = \varinjlim_{\mu'} \mathfrak{p}_!^{\leq\mu'}(\mathrm{emb}_! \epsilon_{\mathrm{ad}}^* \mathrm{Sat}(W) \boxtimes^{M(\mathbb{F}_x)} \nabla_w(\mathrm{Reg}_{T^{w\mathrm{F}}})). \tag{4.52}$$

4.6 Iterated tame nearby cycles

Let S be a strict henselian trait over an algebraically closed field k of characteristic p and let Λ be a finite ring killed by a power of ℓ. We study iterated tame nearby cycles. This construction is an analog of a construction of Gaitsgory [Gai04], extended by Salmon [Sal23b] which relies on the unipotent nearby cycles of Beilinson . We follow their strategy and this construction essentially only requires to replace the unipotent nearby cycles by the tame ones.

We denote by s and η the closed and generic points of S, we denote by $\overline{\eta} \to \eta^t \to \eta$, an algebraic closure of η and the maximal tame extension respectively. We denote by S_t the normalization of S in η^t. Let $f : X \to S$ be a scheme of finite type over S, and denote by $X_t = X \times_S \eta^t$. Let us sum up the preceding data in the following diagrams

$$\begin{tikzcd} X_s \arrow[r, "i_t"] \arrow[d, equal] & X_t \arrow[d, "p_t"] & X_{\eta^t} \arrow[l, "j_t"'] \arrow[d, "p_t"] \\ X_s \arrow[r, "i"'] & X & X_\eta \arrow[l, "j"] \end{tikzcd}$$

lying over

$$\begin{tikzcd} s \arrow[r, "i_t"] \arrow[d, equal] & S_t \arrow[d] & \eta^t \arrow[l, "j_t"'] \arrow[d] \\ s \arrow[r, "i"'] & S & \eta \arrow[l, "j"] \end{tikzcd}$$

where $X_\eta = X \times_S \eta$ and similarly for s and η^t in place of η. The tame nearby cycle functors are given by, for $A \in \mathrm{D}_c^b(X_\eta, \Lambda)$,

$$\Psi^t(A) = i_t^* j_{t,*} p_t^* A.$$

This functor is equipped with a continuous action of $\mathrm{Gal}(\eta^t/\eta)$. We will write it Ψ_f^t if we want to put emphasis on the map f.

Let $n > 0$ be an integer prime to p and $\eta^t \to \eta_n \xrightarrow{p_n} \eta$ be the degree n-extension of η. Let n divide m and denote by $p_n^m : \eta_m \to \eta_n$ the corresponding map. It is clear that $\eta^t = \varprojlim_n \eta_n$. Denote by $\Lambda_n = p_{n,*}\Lambda$. The adjunction map $\mathrm{id} \to p^m_{n,*}p^{m,*}_n$ on η_n induces a morphism $i^m_n : \Lambda_n \to \Lambda_m$ on η. Consider the diagram

$$\begin{tikzcd}
X_s \arrow[r, "i_t"] \arrow[d, equal] & X_t \arrow[d, dashed, "p_n^t"] & X_{\eta^t} \arrow[l, "j_t"'] \arrow[d, "p_n^t"] \\
X_s \arrow[r, "i_n"] \arrow[d, equal] & X_n \arrow[d, dashed, "p_n"] & X_{\eta_n} \arrow[l, "j_n"'] \arrow[d, "p_n"] \\
X_s \arrow[r, "i"'] & X & X_\eta \arrow[l, "j"]
\end{tikzcd}$$

where all the squares are Cartesian. We have a natural map $i^*j_*(f^*\Lambda_n \otimes -) \to \Psi^t(-)$ of functors obtained as follows

$$\begin{aligned}
i^*j_*(f^*\Lambda_n \otimes -) &= i^*j_*p_{n,*}p_n^*(-) \\
&= i_n^*p_{n,*}j_{n,*}p_n^*(-) \\
&= i_t^*p_n^{t,*}j_n^*p_n^*(-) \\
&\to i_t^*j_{t,*}p_n^{t,*}p_n^*(-) \\
&= i_t^*j_{t,*}p_t^*(-) \\
&= \Psi^t(-)
\end{aligned}$$

where the only nontrivial map the base change map from the top right square. Note that this base map is compatible with the adjunction maps $\mathrm{id} \to p^m_{n,*}p^{m,*}_n$. We obtain an inductive diagram $(i^*j_*(f^*\Lambda_n \otimes (-))) \to \Psi^t(-)$, and therefore a map from the homotopy colimit of this diagram

$$\mathrm{can} : \varinjlim_n i^*j_*(f^*\Lambda_n \otimes (-)) \to \Psi^t(-).$$

Lemma 4.6.1. *The map* can *is an isomorphism.*

Proof. We only need to check that for all geometric points $x \to X_s$ the stalk at x is an isomorphism. We fix such a geometric point and $A \in \mathrm{D}^b_c(X, \Lambda)$, we first compute the stalk of the source of can. We have

$$\begin{aligned}
(\varinjlim_n i^*j_*(f^*\Lambda_n \otimes A))_x &= \varinjlim_n \mathrm{R}\Gamma((X_s)_{(x)}, i^*j_*(f^*\Lambda_n \otimes A)) \\
&= \varinjlim_n \mathrm{R}\Gamma(X_{(x)} \times_S \eta, (f^*\Lambda_n \otimes A)) \\
&= \varinjlim_n \mathrm{R}\Gamma(X_{(x)} \times_S \eta_n, A).
\end{aligned}$$

Now since $\eta^t = \varprojlim_n \eta_n$, we also have $\varprojlim_n X_{(x)} \times_S \eta_n = X_{(x)} \times_S \eta^t$. By [GV72] 8.7.7, we have

$$\varinjlim_n \mathrm{R}\Gamma(X_{(x)} \times_S \eta_n, A) = \mathrm{R}\Gamma(X_{(x)} \times_S \eta^t, A) = \Psi^t(A)_x.$$

$\square$

Consider now the following setup, S is a strict trait, $n > 0$ is a positive integer and $f : X \to S^n$ is a scheme of finite presentation. We define three functors.

(i). For a tuple $\underline{m} = (m_1, \dots, m_n)$, we denote by $\Lambda_{\underline{m}}$ the sheaf $\Lambda_{m_1} \boxtimes \cdots \boxtimes \Lambda_{m_n}$ on η^n. Following [Gai04] and [Sal23b] we define the functor $\Upsilon : \mathrm{D}^b_c(X_{\eta^n}, \Lambda) \to \mathrm{D}^b_c(X, \Lambda)$ by

$$\Upsilon(A) = \varinjlim_{\underline{n}} i^* j_*(f^* \Lambda_{\underline{n}} \otimes A) \tag{4.53}$$

where $j : X_{\eta^n} \to X$ and $i : X_{s^n} \to X$ are the inclusion.

(ii). Consider the Cartesian diagram

$$\begin{tikzcd} X_\Delta \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r, "\Delta"] & S^I \end{tikzcd}$$

where the map Δ is the diagonal. We then define Ψ_Δ the composition of the pullback to X_Δ and the nearby cycle functor with respect to the map $X_\Delta \to S$. Replacing nearby cycles by tame nearby cycles, we get a functor Ψ^t_I.

(iii). We define the functor $\Psi_1 \dots \Psi_n$ iteratively. Let A be a sheaf on X. Consider the projection onto the last factor $S^n \to S$, and the composition $X \to S^n \to S$, then apply the corresponding nearby cycles functor. The resulting special fiber, $X \times_S s$ now lives over S^{n-1}. We then iterate the construction. For an ordering $I = \{1, \dots, n\}$, we denote by $\Psi_I = \Psi_1 \dots \Psi_n$. Replace nearby cycles with tame nearby cycles, we get a functor Ψ^t_I.

Remark 4.6.2. In [Sal23b] and [Gai04], they work with unipotent nearby cycles while we work here with tame nearby cycles.

Lemma 4.6.3 (Tame variant of [Sal23b], 4.1)**.** (i). *There are natural transformations*

$$\Psi^t_1 \dots \Psi^t_n \leftarrow \Upsilon \to \Psi^t_\Delta(-_{|f^{-1}(\Delta)})[1-n]. \tag{4.54}$$

(ii). *Let $\pi : X \to X'$ be a morphism of finite type over S^n, then there is a natural base change map $\pi_! \Upsilon \to \Upsilon \pi_!$ making the following diagram commute*

$$\begin{tikzcd} \pi_! \Psi^t_I \arrow[d] & \pi_! \Upsilon \arrow[l] \arrow[r] \arrow[d] & \pi_! \Psi^t_\Delta[n-1] \arrow[d] \\ \Psi^t_I \pi_! & \Upsilon \pi_! \arrow[l] \arrow[r] & \Psi^t_\Delta[n-1]\pi_!. \end{tikzcd}$$

Moreover, all vertical maps are isomorphisms if π is proper.

(iii). *Let $\pi : X \to X'$ be a morphism of finite type over S^n, then there is a natural base change map $\pi^* \Upsilon \to \Upsilon \pi^*$ making the following diagram commute*

$$\begin{tikzcd} \pi^* \Psi^t_I \arrow[d] & \pi^* \Upsilon \arrow[l] \arrow[r] \arrow[d] & \pi^* \Psi^t_\Delta[n-1] \arrow[d] \\ \Psi^t_I \pi^* & \Upsilon \pi^* \arrow[l] \arrow[r] & \Psi^t_\Delta[n-1]\pi^*. \end{tikzcd}$$

Moreover, all vertical maps are isomorphisms if π is smooth.

(iv). Suppose that $n=2$ and $X = X_1 \times X_2$ and that the map $X \to S^2$ is the product of two maps $X_i \to S$, and let $A = A_1 \boxtimes A_2$ on X. Then the maps $\Upsilon(A) \to \Psi_1^t \Psi_2^t(A)$ and $\Upsilon(A) \to \Psi_\Delta^t$ are isomorphisms.

Proof. The proof of [Sal23b] 4.1 translates verbatim after replacing 'unipotent nearby cycles' by Ψ^t and $\mathcal{L}_m$ by Λ_m. We want to indicate where the map $\Lambda_{m_1} \otimes \Lambda_{m_2} \to \Lambda_{\mathrm{lcm}(m_1,m_2)}$ comes from. Consider the diagram

$$\begin{tikzcd} \eta_{\mathrm{lcm}(n,m)} \arrow[r, "\Delta_{n,m}"] \arrow[d, "p_{\mathrm{lcm}(n,m)}"'] & \eta_n \times \eta_m \arrow[d, "p_n \times p_m"] \\ \eta \arrow[r, "\Delta"'] & \eta \times \eta. \end{tikzcd}$$

The base change map yields an map $\Delta^*(p_n \times p_m)_*\Lambda \to p_{\mathrm{lcm}(n,m),*}\Lambda$ which is the desired map. □

4.7 Central functors and their monodromy

4.7.1 Monodromic sheaves and Verdier's lemma

We let Λ be a coefficient ring, in this section, we work over $\bar{\mathbb{F}}_q$. Let T be a torus over $\bar{\mathbb{F}}_q$, let $\pi_1^t(T)$ be the tame fundamental (geometric) group of T at the point 1. It is canonically isomorphic to

$$\pi_1^t(T) = X_*(T) \otimes \pi_1^t(\mathbb{G}_m) = X_*(T) \otimes \hat{\mathbb{Z}}^{(p)}(1). \tag{4.55}$$

Definition 4.7.1. We denote by $\mathrm{CH}_\Lambda(T)$

(i). If $\Lambda = \overline{\mathbb{Q}}_\ell$, then $\mathrm{CH}_\Lambda(T) = \mathrm{Hom}(\pi_1^t(T), \overline{\mathbb{Q}}_\ell^\times)^{\mathrm{tors}}$, where tors denotes the group of characters of finite order,

(ii). If $\Lambda = \overline{\mathbb{Z}}_\ell, \overline{\mathbb{F}}_\ell$ then $\mathrm{CH}_\Lambda(T) = \mathrm{Hom}(\pi_1^t(T)_{\ell'}, \Lambda^\times)^{\mathrm{tors}}$, where ℓ' denotes the prime to ℓ-part of the group.

For each $\chi \in \mathrm{CH}(T)$, there is a corresponding Kummer sheaf on T which we denote by $\mathcal{L}_\chi$.

Definition 4.7.2. Let X be a scheme with a T-action. A sheaf $A \in \mathrm{D}(X, \Lambda)$ is monodromic if its pullback along all T-orbits is a lisse and tame sheaf.

Theorem 4.7.3 ([Ver83])**.** *Let X be a scheme with a T-action, the full subcategory of monodromic sheaves $\mathrm{D}(X)_{\mathrm{mon}}$ is stable and stable under the 6-operations. If A is a monodromic sheaf, then there is a canonical action of $\pi_1^t(T)$ on A. It is given by a morphism*

$$\phi_A : \Lambda[\pi_1(T)] \to \mathrm{End}(A), \tag{4.56}$$

which is called the canonical monodromy of A. Moreover this action commutes with all morphism of sheaves.

Definition 4.7.4. Let $\chi \in \mathrm{CH}_\Lambda(T)$ and let X be a scheme with a T-action. A monodromic sheaf $A \in \mathrm{D}(X, \Lambda)_{\mathrm{mon}}$ is χ-monodromic, if the canonical monodromy ϕ factors through the completion along the kernel of the morphism $\Lambda[\pi_1^t(T)] \to \Lambda$ determined by χ. We denote by $\mathrm{D}(X, \Lambda)_{\chi-\mathrm{mon}}$ the full subcategory of χ-monodromic sheaves on X.

Lemma 4.7.5. *Let X be a scheme with a T-action. We have a canonical decomposition*

$$\mathrm{D}(X,\Lambda) = \bigoplus_{\chi} \mathrm{D}(X,\Lambda)_{\chi}. \tag{4.57}$$

Proof. This is immediate since for $\chi \neq \chi'$, we have $\mathrm{Hom}_T(\chi,\chi') = 0$. $\square$

Consider the following situation. Let X be a scheme with an action of $\mathbb{G}_m$ and let $f : X \to \mathbb{G}_m$ be a map that is equivariant for the natural dilatation action on $\mathbb{A}^1$. Denote by X_0 and X° the inverse images of 0 and $\mathbb{G}_m$.

$$\begin{tikzcd} X_0 \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "f"'] & X^\circ \arrow[l, "j"'] \arrow[d, "f"] \\ 0 \arrow[r, "i"'] & \mathbb{A}^1 & \mathbb{G}_m \arrow[l, "j"] \end{tikzcd}$$

Theorem 4.7.6 ([Ver83])**.** *Consider (X, f) as above. Let A be a monodromic sheaf on X°, then*

(i). the nearby cycles $\Psi_f(A)$ is a monodromic sheaf on X_0 and $\Psi_f(A) = \Psi^t(A)$.

(ii). The canonical monodromy of $\pi_1(\mathbb{G}_m)$ on $\Psi_f(A)$ is the opposite of the monodromy on the nearby cycles.

4.7.2 Twisted equivariant sheaves

Let $\chi \in \mathrm{CH}_\Lambda(T)$. The sheaf $\mathcal{L}_\chi \in \mathrm{D}(T,\Lambda)$ is a multiplicative sheaf, that is, it is equipped with an isomorphism

$$m^*\mathcal{L}_\chi = \mathcal{L}_\chi \boxtimes \mathcal{L}_\chi, \tag{4.58}$$

where $m : T \times T \to T$ is the multiplication map. Since χ is a character of finite order, there exists $n > 0$, which is prime to ℓ if $\Lambda = \overline{\mathbb{F}}_\ell$ such that $\mathcal{L}_\chi^{\otimes n} = \Lambda_T$. The sheaf $\mathcal{L}_\chi$ is a direct summand of $p_{n,!}\Lambda$ where $p_n : T \times T$ is the map $t \mapsto t^n$.

Let X be scheme with a T-action and denote by $a : T \times X \to X$ the action map. We denote by $a_n : T \times X \to X$ the action dilated by n, that is, the action given by $a_n(t,x) = t^n.x$. We consider the quotient stack $X/_nT$ which is the quotient stack for this action. Since the group $T[n] = \ker p_n$ is of order invertible in Λ and acts trivially on X, the category $\mathrm{D}(X/_nT,\Lambda)$ splits canonically as

$$\mathrm{D}(X/_nT,\Lambda) = \oplus_\chi \mathrm{D}(X/(T,\mathcal{L}_\chi),\Lambda), \tag{4.59}$$

where χ ranges through the characters of $T[n]$. We refer to [LY20] Section 2 for a more detailed construction. We call the category $\mathrm{D}(X/(T,\mathcal{L}_\chi),\Lambda)$, the category of $(T,\mathcal{L}_\chi)$-equivariant sheaves. This terminology slightly deviates from the literature, as this category is called the category of χ-monodromic sheaves in *loc. cit.*

Remark 4.7.7. Note that if $\chi = 1$, then a $(T,\mathcal{L}_\chi)$-equivariant sheaf is simply a T-equivariant sheaf.

4.7.3 The central functor construction

Recall that we have fixed $T \subset \mathrm{Iw} \subset LG$ an Iwahori subgroup (defined over $\mathbb{F}_q$) and that Iw^0 is its unipotent radical. We consider the stack $\frac{\mathrm{Iw}^0\backslash LG/\mathrm{Iw}^0}{\mathrm{Ad}(T)}$ which we define as the quotient stack

$$\frac{\mathrm{Iw}^0\backslash LG/\mathrm{Iw}^0}{\mathrm{Ad}(T)} = (LG/\mathrm{Iw}^0)/\mathrm{Ad}(\mathrm{Iw}). \tag{4.60}$$

We denote by

(i). $\mathrm{D}_{\mathrm{Ad},\chi} = \mathrm{D}(\frac{\mathrm{Iw}^0\backslash LG/(\mathrm{Iw},\mathcal{L}_\chi)}{\mathrm{Ad}(T)}, \Lambda)$ as the category of sheaves that $(T, \mathcal{L}_\chi)$-equivariant sheaves for the action of T given by right translations

(ii). $\mathrm{D}_{\mathrm{Ad}_\mathrm{F}} = \mathrm{D}(\frac{\mathrm{Iw}^0\backslash LG/\mathrm{Iw}^0}{\mathrm{Ad}_\mathrm{F}(T)}, \Lambda)$.

(iii). $\mathrm{D}_{\mathrm{Ad}_\mathrm{F},\chi} = \mathrm{D}(\frac{\mathrm{Iw}^0\backslash LG/(\mathrm{Iw},\mathcal{L}_\chi)}{\mathrm{Ad}_\mathrm{F}(T)}, \Lambda)$ as in (i).

(iv). We denote by $\mathrm{Perv}_{\mathrm{Ad},\chi}$ and $\mathrm{Perv}_{\mathrm{Ad}_\mathrm{F}}$ the associated categories of perverse sheaves.

Remark 4.7.8. The category $\mathrm{D}_{\mathrm{Ad},\chi}$ is also equivalent to the category of sheaves on $\mathrm{Iw}^0\backslash LG/\mathrm{Iw}^0$ that are equivariant under $(T \times T, \mathcal{L}_\chi \boxtimes \mathcal{L}_\chi)$ where the action is given by $(t, t').x = txt'$.

We refer to [LY20] Section 4 and [Li22] for a precise construction of the monoidal structure on $\mathrm{D}_{\mathrm{Ad},\chi}$. Consider the following convolution diagram

$$\begin{tikzcd}
 & \mathrm{Iw}^0\backslash LG \times^{\mathrm{Iw}^0} LG/\mathrm{Iw}^0 \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[r, "m"] & \mathrm{Iw}^0\backslash LG/\mathrm{Iw}^0 \\
\mathrm{Iw}^0\backslash LG/\mathrm{Iw}^0 & & \mathrm{Iw}^0\backslash LG/\mathrm{Iw}^0
\end{tikzcd}$$

where pr_i are induced by the projections and m by the multiplication. Consider the actions of $T \times T$ on $\mathrm{Iw}^0\backslash LG \times^{\mathrm{Iw}^0} LG/\mathrm{Iw}^0$ defined as

Ad $(t, t').(x, y) = (txt'^{-1}, t'yt^{-1})$,

Ad_F $t.(x, y) = (txt'^{-1}, t'y\mathrm{F}(t^{-1}))$.

Let $A, B \in \mathrm{D}_{\mathrm{Ad},\chi}$ then the sheaf $\mathrm{pr}_1^* A \otimes \mathrm{pr}_2^* B$ descends is equivariant for the action of $T \times T$ hence descends to a sheaf

$$A \tilde{\boxtimes} B \in \mathrm{D}(\frac{\mathrm{Iw}^0\backslash LG \times^{\mathrm{Iw}} LG/\mathrm{Iw}^0}{\mathrm{Ad}(T)}, \Lambda). \tag{4.61}$$

The convolution product is defined as

$$A * B = m_! A \tilde{\boxtimes} B, \tag{4.62}$$

where $m : \frac{\mathrm{Iw}^0\backslash LG \times^{\mathrm{Iw}} LG/\mathrm{Iw}^0}{\mathrm{Ad}(T)} \to \mathrm{D}_{\mathrm{Ad},\chi}$.

Lemma 4.7.9 ([LY20], [Li22]). *The convolution product defines a monoidal structure on* $\mathrm{D}_{\mathrm{Ad},\chi}$.

Replacing $B \in \mathrm{D}_{\mathrm{Ad},\chi}$ by $B \in \mathrm{D}_{\mathrm{Ad_F},\chi}$ in the construction above first yields a sheaf $A\tilde{\boxtimes}B \in \mathrm{D}(\frac{\mathrm{Iw}^0\backslash LG\times^{\mathrm{Iw}}LG/\mathrm{Iw}^0}{\mathrm{Ad_F}(T)},\Lambda)$ and then a sheaf $A * B \in \mathrm{D}_{\mathrm{Ad_F},\chi}$. The same argument as in [LY20] and [Li22].

Lemma 4.7.10. *The bifunctor*

$$\begin{aligned}\mathrm{D}_{\mathrm{Ad},\chi} \times \mathrm{D}_{\mathrm{Ad_F},\chi} &\to \mathrm{D}_{\mathrm{Ad_F},\chi},\\ (A,B) &\mapsto A * B,\end{aligned}$$

defines a $\mathrm{D}_{\mathrm{Ad},\chi}$*-module structure on* $\mathrm{D}_{\mathrm{Ad_F},\chi}$.

We now recall the construction of the central functors and their twisted versions. We choose a uniformizer of π_x at x. This gives an identification of the completion at x of X with $\bar{\mathbb{F}}_q((t))$. We then identify the completion of X at x with the completion of $\mathbb{A}^1$ at 0 over $\mathbb{F}_x$. We consider the group scheme $\mathcal{G}_{C^0}^{\mathbb{A}^1}$ over $\mathbb{A}^1$, defined in the same way as $\mathcal{G}_{C^0}$. We consider $L\mathcal{G}_{C^0}^{\mathbb{A}^1}$, note that there is an action of $\mathbb{G}_m$ on the fiber at 0 of $(L\mathcal{G}_{C^0}^{\mathbb{A}^1})_0 = LG$. This one dimensional torus is called the rotation torus, on the $\bar{\mathbb{F}}_q$-points $LG(\bar{\mathbb{F}}_q) = G(\bar{\mathbb{F}}_q((t)))$ it acts by rescaling t. More specifically it is given by

$$z.g(t) = g(z^{-1}t), \tag{4.63}$$

where $z \in \mathbb{G}_m$ and $g(t) \in LG(\bar{\mathbb{F}}_q)$. There is also an action of $\mathbb{G}_m$ on $\mathbb{A}^1$ by dilatation, given by $(z,x) \mapsto zx$.

Theorem 4.7.11 ([BR22a] 4.3)**.** *There are actions of* $\mathbb{G}_m$ *on* $L\mathcal{G}_{C^0}$ *and* $L^+\mathcal{G}_{C^0}$ *such that the maps* $L\mathcal{G}_{C^0} \to \mathbb{A}^1$ *and* $L^+\mathcal{G}_{C^0}$ *are equivariant for the action of* $\mathbb{G}_m$ *and induce the rotation action on the fiber at* 0.

We consider the affine Grassmannian of $\mathcal{G}_{C^0}$ with two legs, that is $\mathrm{Gr}_{\mathcal{G}_{C^0},\{1,2\}}$, this is an (ind)-scheme over $\mathbb{A}^2$ and we restrict to to $\mathbb{A}^1 \times \{0\}$. The fiber at 0 is isomorphic to LG/Iw^0. Over $\mathbb{G}_m$, it is isomorphic to $\mathrm{Gr}_{G,\{1\}} \times LG/\mathrm{Iw}^0$. We consider the the following nearby cycles diagram

$$\begin{tikzcd} LG/\mathrm{Iw}^0 \arrow[r] \arrow[d] & (\mathrm{Gr}_{\mathcal{G}_{C^0}^{\mathbb{A}^1},\{1,2\}})_{\mathbb{A}^1\times 0} \arrow[d] & \mathrm{Gr}_{G,\{1\}} \times LG/\mathrm{Iw}^0 \arrow[l] \arrow[d] \\ 0 \arrow[r] & \mathbb{A}^1 & \mathbb{G}_m \arrow[l] \end{tikzcd}$$

Let $\chi \in \mathrm{CH}_\Lambda$, the χ-twisted central functor is defined as

$$\begin{aligned}\mathcal{Z}_\chi : \mathrm{Rep}_\Lambda \hat{G} &\to \mathrm{Perv}_{\mathrm{Ad},\chi}\\ W &\mapsto \Psi(\mathrm{Sat}(W) \boxtimes \mathcal{L}_\chi[\dim T])\end{aligned}$$

where $\mathcal{L}_\chi[\dim T]$ is the perverse shift of $\mathcal{L}_\chi$ on $T \subset LG/\mathrm{Iw}^0$. The perversity of $\mathcal{Z}_\chi$ follows from the fact that the nearby cycle functor is perverse t-exact.

Theorem 4.7.12 ([Gai01], [Gai04], [BFO09])**.** *(i). The functor* $\mathcal{Z}_\chi$ *well defined and is monoidal.*

(ii). The monodromy on the nearby cycles is tame.

(iii). The action of the monodromy on the nearby cycles is monoidal.

We want to point out a few key facts about the proof of this theorem. Firstly, the tameness of these nearby cycles follows from Verdier's theorem 4.7.6, see also [AB09] Section 5.2. The monoidality in [Gai01] follows from the properness of the convolution maps and the monoidality of the action of the monodromy follows in [Gai04] from the fact that both maps

$$\Psi_1\Psi_2 \leftarrow \Upsilon \rightarrow \Psi_\Delta, \tag{4.64}$$

are isomorphism when the considered nearby cycles are unipotent nearby cycles. In our setting, the same argument provides an isomorphism for tame nearby cycles.

4.7.4 Wakimoto sheaves

We assume that $\Lambda = \overline{\mathbb{Q}}_\ell$ and we fix $\chi \in \mathrm{CH}(T)$. We recall the construction of [AB09] and [BFO09] about Wakimoto sheaves. Let $\lambda \in X_*(T)$ we get a canonical point $\lambda(t) \in LG$ obtained as the image of t under $L\mathbb{G}_m \to LG$. We also fix a set of lifts $\dot{w}$ of the elements of the Weyl group of W and we assume that $\dot{ww'} = \dot{w}\dot{w}'$ if $\ell(ww') = \ell(w) + \ell(w')$. For an element $w_{aff} = w\lambda \in \tilde{W}$, we then get a point $\dot{w}_{aff} = \dot{w}\lambda(t) \in LG$.

Definition 4.7.13 (Standard and costandard)**.** Recall that $\tilde{W}$ denotes the extended affine Weyl group. For $w \in \tilde{W}$, using the choice of $\dot{w}$, the stratum $i_w : \mathrm{Iw}w\mathrm{Iw}^0/\mathrm{Iw}^0 \subset LG/\mathrm{Iw}^0$ is T-equivariantly isomorphic to $T \times \mathbb{A}^{\ell(w)}$ where $\ell(w)$ is the lenght of w. We define

- (i). The standard objects $\Delta_{w,\chi} = i_{w,!}\mathcal{L}_\chi \boxtimes \overline{\mathbb{Q}}_\ell[\dim T + \ell(w)]$,
- (ii). The costandard objects $\nabla_{w,\chi} = i_{w,*}\mathcal{L}_\chi \boxtimes \overline{\mathbb{Q}}_\ell[\dim T + \ell(w)]$.

Since the inclusion i_w is affine, all the sheaves $\Delta_{w,\chi}$ and $\nabla_{w,\chi}$ are perverse sheaves.

Theorem 4.7.14 ([AB09], [BFO09] for the $\overline{\mathbb{Q}}_\ell$ case, see Appendix 4.A for the $\overline{\mathbb{F}}_\ell$ case)**.** *(i). There is a fully faithful tensor functor $J_\chi : \mathrm{Rep}_\Lambda \hat{T} \to \mathrm{Perv}_{\mathrm{Ad},\chi}$ such that $J(\lambda) = \nabla_{\lambda,\chi}$ if $\lambda \in X_*^+$ is dominant and $J(\lambda) = \Delta_{w,\chi}$ is $\lambda \in X_*$ is antidominant. The objects in the essential image of J_χ are called χ-monodromic Wakimoto sheaves.*

(ii). Denote by $\mathrm{Perv}_{\mathrm{Ad},\chi}^{J-\mathrm{fil}}$ the category of sheaves equipped with a filtration whose graded pieces are Wakimoto sheaves. Then the central functor $\mathcal{Z}_\chi$ lifts to a monoidal functor

$$\mathcal{Z}_\chi : \mathrm{Rep}_\Lambda G \to \mathrm{Perv}_{\mathrm{Ad},\chi}^{J-\mathrm{fil}}. \tag{4.65}$$

(iii). The functor $\mathrm{gr} : \mathrm{Perv}_{\mathrm{Ad},\chi}^{J-\mathrm{fil}} \to J(\mathrm{Rep}_\Lambda \hat{T})$ that takes the graded pieces of the filtration is monoidal. The composition

$$\mathrm{Rep}_\Lambda \hat{G} \xrightarrow{\mathcal{Z}_\chi} \mathrm{Perv}_{\mathrm{Ad},\chi}^{J-\mathrm{fil}} \to J(\mathrm{Rep}_\Lambda \hat{T}) = \mathrm{Rep}_\Lambda \hat{T}, \tag{4.66}$$

is monoidal and isomorphic to the restriction from $\hat{G}$ to $\hat{T}$.

(iv). The canonical monodromy action on $\mathrm{gr}\mathcal{Z}_\chi$ is given by the action of $\chi \in \hat{T}$.

Remark 4.7.15. In [BFO09], the authors obtain the opposite monodromy χ^{-1}, this follows from the fact that we have normalized the rotation torus action using the normalization of [BR22a] which is the opposite normalization to the one of [BFO09].

4.8 Main theorems

4.8.1 Statement of the results

We now go back to our setup of section 4.4.2. The following theorem is an extension of theorem 5.2 of [Sal23b].

Theorem 4.8.1. *Consider the sheaf* $\mathrm{emb}_! \epsilon^*(\mathrm{Sat}_W \boxtimes \nabla_w(\mathrm{Reg}_{T^{w\mathrm{F}}}))$ *as before. This is a sheaf on* $\mathrm{Cht}_{\mathcal{G}_C, I\sqcup\{0\}, N^x}$*. The canonical map*

$$\varinjlim_{\mu} \mathfrak{p}_!^{\leq\mu} \Psi_1 \ldots \Psi_n \mathrm{emb}_! \epsilon_{\mathrm{ad}}^{\leq\mu,*}(\mathrm{Sat}_W \boxtimes \nabla_w(\mathrm{Reg}_{T^{w\mathrm{F}}})) \to \Psi_1 \ldots \Psi_n \mathfrak{p}_!^{\leq\mu} \mathrm{emb}_! \epsilon_{\mathrm{ad}}^{\leq\mu,*}(\mathrm{Sat}_W \boxtimes \nabla_w(\mathrm{Reg}_{T^{w\mathrm{F}}})), \tag{4.67}$$

is an isomorphism, where $I = \{i_1, \ldots, i_n\}$.

Note that this map *a priori* depends on the order of the coordinates.

Theorem 4.8.2. *Let* I, W *be as before, let* $s \in (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}}$ *and let* $j \in \mathbb{Z}$*, then*

(*i*). *The* $(\mathrm{Weil}_{F_x})^I$*-module* $e_s((H^j_{\mathcal{G}_\sigma, I, W, N})^{V_\sigma(\mathbb{F}_x)})$ *is tamely ramified, i.e. the action of* $\mathrm{Weil}^I_{F_x}$ *factors through* $(\mathrm{Weil}^t_{F_x})^I$.

(*ii*). *As an* $\mathrm{Exc}(F_x, \hat{G})$*-module,* $e_s((H^j_{\mathcal{G}_\sigma, I, W, N})^{V_\sigma(\mathbb{F}_x)})$ *is supported on* $\mathrm{ev}^{-1}_{\tau_{F_x}}(s)$.

Corollary 4.8.3. *Let* M^j_N *be the quasicoherent sheaf on* $\underline{Z}^{1,t,\circ}(F_x, \hat{G})/\hat{G}$ *corresponding to the functor* $(I, W) \mapsto (H^j_{\mathcal{G}_\sigma, I, W, N})^{V_\sigma(\mathbb{F}_x)}$*. Using the same notations as in theorem 4.8.2, the quasi-coherent sheaf* $e_s(M^j_N)$ *is supported on* $\mathrm{ev}^{-1}_{\tau_{F_x}}(s)$.

Proof. By lemma 4.4.35, corollary 4.8.3 follows from theorem 4.8.2. ☐

Corollary 4.8.4. *All parameters attached by [GL17] to depth* 0 *cuspidal representations are tame and the diagram*

$$\begin{tikzcd} \mathrm{Irr}^0 H(K) \arrow[r, "\mathrm{LLC}^{\mathrm{GL}}"] \arrow[d, "\mathrm{LS}"'] & (\underline{Z}^{1,t}(K, \hat{H}) /\!\!/ \hat{H})(\Lambda) \arrow[d, "\mathrm{ev}_{\tau_K}"] \\ (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}} \arrow[r, equal] & (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}} \end{tikzcd}$$

commutes.

Remark 4.8.5. The recent result of Li-Huerta [LH23] shows that the semisimple Langlands correspondence of Lafforgue and Genestier [GL17] and the one of Fargues and Scholze [FS21] agree hence our result is also valid for the Fargues-Scholze correspondence.

4.8.2 Commutation of nearby cycles and pushforward

For the proof of theorem 4.8.1, we will follow Salmon's proof [Sal23b] Theorem 5.2 with the necessary generalizations.

Lemma 4.8.6. *Let $I = \{i_1, \dots, i_n\}$, $K \in \mathrm{Perv}_{\mathrm{Ad_F}}$ and $V \in \mathrm{Rep}_\Lambda({}^LG)^I$ then the sheaf*

$$\varinjlim_\mu \mathfrak{p}_!^{\leq\mu} \epsilon^*_{d,\mathrm{ad}}(\mathrm{Sat}(V) \boxtimes K)$$

is constant on $\bar\eta \times_{\bar{\mathbb{F}}_q} \cdots \times_{\bar{\mathbb{F}}_q} \bar\eta$.

Corollary 4.8.7. *Let $\zeta : I \to J$ be a morphism of finite sets. Let $V \in \mathrm{Rep}_\Lambda({}^LG)^I$ and V_ζ is restriction along the morphism $({}^LG)^J \to ({}^LG)^I$ induced by ζ. Let $K \in \mathrm{Perv}_{\mathrm{Ad_F}}$. Then there is a canonical isomorphism*

$$\Psi_I \varinjlim_\mu \mathfrak{p}_!^{\leq\mu} \epsilon^*_{d,\mathrm{ad}}(\mathrm{Sat}(V) \boxtimes K) = \Psi_J \varinjlim_\mu \mathfrak{p}_!^{\leq\mu} \epsilon^*_{d,\mathrm{ad}}(\mathrm{Sat}(V_\zeta) \boxtimes K). \tag{4.68}$$

Lemma 4.8.8. *Let $\zeta : I \to I'$ be a morphism of finite sets, $K \in \mathrm{Perv}_{\mathrm{Ad_F}}$, J a finite set and let $W \in \mathrm{Rep}_\Lambda({}^LG)^J$ and $V \in \mathrm{Rep}_\Lambda({}^LG)^I$. There is a canonical isomorphism*

$$\Psi_I \mathrm{emb}_! \epsilon^*_{d,\mathrm{ad}}(\mathrm{Sat}(W \boxtimes V) \boxtimes K) = \Psi_{I'} \mathrm{emb}_! \epsilon^*_{d,\mathrm{ad}}(\mathrm{Sat}(W \boxtimes V_\zeta) \boxtimes K). \tag{4.69}$$

Proof of theorem 4.8.1. We can now apply the proof of [Sal23b], we reproduce the key argument here. Let $V \in \mathrm{Rep}_\Lambda({}^LG)^J$ and $W \in \mathrm{Rep}_\Lambda({}^LG)^I$. We show that the canonical map $\varinjlim_\mu \mathfrak{p}^{\leq\mu}_{I\cup J,!} \Psi_J \epsilon^*(\mathrm{Sat}(W \boxtimes V) \boxtimes \nabla(\mathrm{Reg}_{T^{w\mathrm{F}}})) \to \Psi_J \varinjlim_\mu \mathfrak{p}^{\leq\mu}_{I\cup J,!} \epsilon^*(\mathrm{Sat}(W \boxtimes V) \boxtimes \nabla(\mathrm{Reg}_{T^{w\mathrm{F}}}))$ is an isomorphism. First note that there is a canonical isomorphism over $(X-N)^{I\cup\{0\}}$

$$\mathrm{Cht}_{\mathcal{G}_C,N,I\cup\{0\},W\boxtimes 1} \simeq \mathrm{Cht}_{\mathcal{G}_C N,I,W} \times (X-N) \tag{4.70}$$

where $\mathrm{Cht}_{\mathcal{G}_C,N,I\cup\{0\},W\boxtimes 1}$ is the closure of the support of $\epsilon^*(\mathrm{Sat}(W) \boxtimes 1)$ and 1 denotes the trivial representation of $\hat G$.

Let J_1, J_2, J_3 be three disjoint copies of J. As in *loc. cit*, consider the following composition.

$$\begin{aligned}
&\Psi_J \varinjlim_\mu \mathfrak{p}^{\leq\mu}_{I\cup J,!} \epsilon^*(\mathrm{Sat}(W \boxtimes V) \boxtimes \nabla(\mathrm{Reg}_{T^{w\mathrm{F}}})) \\
&\simeq \Psi_{J_1} \varinjlim_\mu \mathfrak{p}^{\leq\mu}_{I\cup J_1\cup J_2,!} \Psi_{J_2} \epsilon^*(\mathrm{Sat}(W \boxtimes V \boxtimes 1) \boxtimes \nabla(\mathrm{Reg}_{T^{w\mathrm{F}}})) \\
&\to \Psi_{J_1} \varinjlim_\mu \mathfrak{p}^{\leq\mu}_{I\cup J_1\cup J_2,!} \Psi_{J_2} \epsilon^*(\mathrm{Sat}(W \boxtimes V \boxtimes (V^* \otimes V)) \boxtimes \nabla(\mathrm{Reg}_{T^{w\mathrm{F}}})) \\
&\simeq \Psi_{J_1} \varinjlim_\mu \mathfrak{p}^{\leq\mu}_{I\cup J_1\cup J_2\cup J_3,!} \Psi_{J_2} \Psi_{J_3} \epsilon^*(\mathrm{Sat}(W \boxtimes V \boxtimes V^* \boxtimes V) \boxtimes \nabla(\mathrm{Reg}_{T^{w\mathrm{F}}})) \\
&\to \Psi_{J_1} \Psi_{J_2} \varinjlim_\mu \mathfrak{p}^{\leq\mu}_{I\cup J_1\cup J_2\cup J_3,!} \Psi_{J_3} \epsilon^*(\mathrm{Sat}(W \boxtimes V \boxtimes V^* \boxtimes V) \boxtimes \nabla(\mathrm{Reg}_{T^{w\mathrm{F}}})) \\
&\simeq \Psi_{J_2} \varinjlim_\mu \mathfrak{p}^{\leq\mu}_{I\cup J_2\cup J_3,!} \Psi_{J_3} \epsilon^*(\mathrm{Sat}(W \boxtimes (V \otimes V^*) \boxtimes V) \boxtimes \nabla(\mathrm{Reg}_{T^{w\mathrm{F}}})) \\
&\to \Psi_{J_2} \varinjlim_\mu \mathfrak{p}^{\leq\mu}_{I\cup J_2\cup J_3,!} \Psi_{J_3} \epsilon^*(\mathrm{Sat}(W \boxtimes 1 \boxtimes V) \boxtimes \nabla(\mathrm{Reg}_{T^{w\mathrm{F}}})) \\
&\simeq \varinjlim_\mu \mathfrak{p}^{\leq\mu}_{I\cup J_3,!} \Psi_{J_3} \epsilon^*(\mathrm{Sat}(W \boxtimes V) \boxtimes \nabla(\mathrm{Reg}_{T^{w\mathrm{F}}}))
\end{aligned}$$

where the maps are

(i). the first one is 4.70,

(ii). the second one is obtained from the canonical map $1 \to V^* \otimes V$,

(iii). the third one is the cocartesianity to pass from $(V^* \otimes V)$ to $V^* \boxtimes V$ from lemma 4.8.8.

(iv). the fourth one is the base change map $\mathfrak{p}_! \Psi_{J_2} \to \Psi_{J_2} \mathfrak{p}_!$,

(v). the fifth one is from lemma 4.8.7 using fusion to pass from $(V \boxtimes V^*)$ to $(V \otimes V^*)$ for the nearby cycles $\Psi_{J_1} \Psi_{J_2}$,

(vi). the sixth one is by functoriality of $V \otimes V^* \to 1$,

(vii). the last one is by 4.70.

We refer to *loc. cit.* for the argument that the previous composition is indeed the inverse of the canonical map. □

Proof of lemma 4.8.6. If there were no K in the statement this would be the first part of the main theorem of [Xue20d]. We need to extend the argument. The argument of *loc. cit.* requires two properties of the functors

$$(I, W) \mapsto \mathfrak{p}_{I,!}^{\leq \mu} \epsilon_{d,\mathrm{ad}}^* (\mathrm{Sat}(W) \boxtimes K)$$

to hold.

(i). That this defines a cocartesian functor filtered with respect to partial Frobenius morphisms,

(ii). That the Eichler shimura relations of [Laf18] Proposition 7.1 hold for all $v \in X - N$.

We now show that these two properties hold. Note that if $\Lambda = \overline{\mathbb{Q}}_\ell$ then it is shown in [Sal23b] Section 3 that the first point implies the second, his proof however does not extend to the modular setting as it requires dividing by $n!$ for all n.

Partial Frobenius. The partial Frobenius morphisms are constructed as in [Xue20a] Section 7.1.

Eichler-Shimura relations. We adapt to our setting an argument of [XZ17]. We have a cocartesian functor filtered with respect to partial Frobenius endomorphism $(I, W) \mapsto \mathfrak{p}_!^{\leq \mu} \epsilon_{\mathrm{ad}}^* (\mathrm{Sat}(W) \boxtimes K)$. Let us denote it by $\mathcal{H}_{I,W,N,K}^{\leq \mu}$. Let $V \in \mathrm{Rep}_\Lambda {}^L G$, the Eichler-Shimura relations is the statement that the morphism

$$\sum_i (-1)^i \mathrm{F}_0^{\deg(v) i} S_{\Lambda^{\dim V - i} V, v} \tag{4.71}$$

vanishes in $\mathrm{Hom}((\mathcal{H}_{I \sqcup \{0\}, W \boxtimes V, N, K}^{j, \leq \mu})_{|(X-N)^I \times v}, (\mathcal{H}_{I \sqcup \{0\}, W \boxtimes V, N, K}^{j, \leq \mu + \kappa})_{|(X-N)^I \times v})$, where $v \in X - N$ is a place, F_0 is the partial Frobenius at the leg 0, κ is large enough and $S_{\Lambda^{\dim V - i} V, v}$ is the excursion operator defined in [Laf18] Section 6.1. Let $\bar{z} \to (X - N)^I$ and $\bar{v} \to v$ be geometric points and let $\tilde{v} \to \bar{z} \times \bar{v}$ be a geometric point of this product. Let us introduce the auxiliary cocartesian functor

$$\tilde{H}_{J, V'}^{j, \leq \mu} = (\mathcal{H}_{I \sqcup J, W \boxtimes V', N, K}^{j, \leq \mu})_{|\Delta(\tilde{v})}, \tag{4.72}$$

where $\Delta(\tilde{v})$ is the geometric point of $(X - N)^I \times v^J$ given by $\tilde{v} \to \bar{z} \times \bar{v} \to \bar{z} \times \Delta(v) \to (X - N)^I \times v^J$. On $\varinjlim_{\leq \mu} \tilde{H}_{J, V'}^{j, \leq \mu}$ there is an action of $\mathrm{Gal}(\bar{v}/v)^I$ coming from the partial Frobenius morphisms. Hence we have collection of functors $\tilde{H}_J^j : \mathrm{Rep}_\Lambda ({}^L G_{\mathrm{loc}, v})^J \to \mathrm{Rep}_\Lambda \mathrm{Gal}(\bar{v}/v)^I$ where ${}^L G_{\mathrm{loc}, v}$ is the local L-group at v. By theorem 4.4.33 and lemma 4.4.34, and since the moduli of unramified parameters is isomorphic to ${}^LG/\hat{G}$, there are coherent sheaves $\mathcal{M}^j_{\tilde{\mathcal{H}}}$ on $\hat{G}.\mathrm{F}/\hat{G} \subset {}^LG/\hat{G}$ and a spectral sequence $H^p(\hat{G}, \mathcal{M}^q_{\tilde{\mathcal{H}}} \otimes V') \Rightarrow \tilde{H}^{p+q}_{J,V'}$ for all $V' \in \mathrm{Rep}_\Lambda(({}^LG_{\mathrm{loc},v})^J)$. It is then enough to show that the endomorphism $\sum_i (-1)^i \mathrm{F}_0^{\deg(v)i} S_{\Lambda^{\dim V - i} V, v}$ of $\mathcal{M}^q_{\tilde{\mathcal{H}}} \otimes V$ is zero. Under the isomorphism $\mathcal{M}^q_{\tilde{\mathcal{H}}} \otimes_\Lambda V = \mathcal{M}^q_{\tilde{\mathcal{H}}} \otimes_{\mathcal{O}_{{}^LG/\hat{G}}} \mathcal{E}_V$ the endomorphism $\sum_i (-1)^i \mathrm{F}_0^{\deg(v)i} S_{\Lambda^{\dim V - i} V, v}$ corresponds to the endomorphism $\sum_i (-1)^i \mathrm{F}_0^{\deg(v)i} \mathrm{Tr}(\mathrm{F}_0^{\deg(v)}, \Lambda^{\dim(V)-i} V)$ which is zero by the Cayley-Hamilton theorem. □

Proof of corollary 4.8.7. We know by 4.8.6 that the sheaf $\varinjlim_\mu \mathfrak{p}_!^{\leq \mu} \epsilon^*_{d,\mathrm{ad}}(\mathrm{Sat}(V) \boxtimes K)$ is constant on $\bar{\eta} \times_{\bar{\mathbb{F}}_q} \cdots \times_{\bar{\mathbb{F}}_q} \bar{\eta}$. The nearby cycle functor Ψ_I is isomorphic to the functor $\mathrm{D}(\bar{\eta}^n, \Lambda) \to \mathrm{D}(\bar{x}, \Lambda)$ given by $M \mapsto M_{\bar{\eta}_I}$ where $\bar{\eta}_I$ is a generic geometric point of X^I. Similarly, $\Psi_{I'}$ is isomorphism to $M \mapsto M_{\bar{\eta}_{I'}}$, the choice of a specialization morphism $\bar{\eta}_I \to \bar{\eta}_{I'}$ induces the desired isomorphism. □

Proof of lemma 4.8.8. We have a diagram of stacks

$$\begin{tikzcd}
\mathrm{Cht}_{\mathcal{G}_\sigma, I, N} \times^{M_\sigma(\mathbb{F}_x)} M_\sigma/B_M \arrow[r] \arrow[d] & \mathrm{Cht}_{\mathcal{G}_C, I \sqcup \{0\}, N^x} \arrow[d] \\
(\mathrm{Cht}_{\mathcal{G}_\sigma, I, N} \times^{M_\sigma(\mathbb{F}_x)} M_\sigma/B_M)/\Xi \arrow[r] & \mathrm{Cht}_{\mathcal{G}_C, I \sqcup \{0\}, N^x}/\Xi
\end{tikzcd}$$

since the vertical map are étale and surjective it is enough to show the desired isomorphism before modding out by Ξ. Similarly, since the map ϵ is smooth, we only need to show the corresponding statement on affine Grassamannians which now follows from theorem 4.7.12. □

4.8.3 Control of the monodromy

In this section we show theorem 4.8.2. We start by showing the tameness assertion. There is a canonical decomposition

$$(H^j_{\mathcal{G}_\sigma, I, W, N})^{V_\sigma(\mathbb{F}_x)} = (H^j_{\mathcal{G}_\sigma, I, W, N})^{V_\sigma(\mathbb{F}_x), \mathrm{tame}} \oplus (H^j_{\mathcal{G}_\sigma, I, W, N})^{V_\sigma(\mathbb{F}_x), \mathrm{wild}} \tag{4.73}$$

where $(-)^{\mathrm{tame}}$ (resp. $(-)^{\mathrm{wild}}$) denotes the direct factor where $P^I_{F_x} \subset \mathrm{Weil}^I_{F_x}$, the product of the wild inertia subgroups, acts trivially (resp. non trivially on any irreducible subquotient). The tameness assertion is then equivalent the property

$$(H^j_{\mathcal{G}_\sigma, I, W, N})^{V_\sigma(\mathrm{F}_x), \mathrm{wild}} = 0. \tag{4.74}$$

Similarly, since $P^I_{F_x}$ is a pro-p-group, the complex $(H_{\mathcal{G}_\sigma, I, W, N})^{V_\sigma(\mathrm{F}_x)}$ splits as

$$(H_{\mathcal{G}_\sigma, I, W, N})^{V_\sigma(\mathbb{F}_x)} = (H_{\mathcal{G}_\sigma, I, W, N})^{V_\sigma(\mathbb{F}_x), \mathrm{tame}} \oplus (H_{\mathcal{G}_\sigma, I, W, N})^{V_\sigma(\mathbb{F}_x), \mathrm{wild}} \tag{4.75}$$

such that $H^j((H_{\mathcal{G}_\sigma, I, W, N})^{V_\sigma(\mathbb{F}_x), \mathrm{tame}}) = (H^j_{\mathcal{G}_\sigma, I, W, N})^{V_\sigma(\mathbb{F}_x), \mathrm{tame}}$ (resp. with $(-)^{\mathrm{wild}}$). It is then enough to show that $(H_{\mathcal{G}_\sigma, I, W, N})^{V_\sigma(\mathbb{F}_x), \mathrm{wild}} = 0$. Since the collection of functors $({}^*\mathcal{R}_w)_{w \in W_M}$ is conservative by theorem 4.2.1, it is enough to show that for all $w \in W_M$, we have

$${}^*\mathcal{R}_w((H_{\mathcal{G}_\sigma, I, W, N})^{V_\sigma(\mathbb{F}_x), \mathrm{wild}}) = 0. \tag{4.76}$$

Fix an ordering $I = \{1, \dots, n\}$. We have

$$\begin{aligned}
{}^*\mathcal{R}_w((H_{\mathcal{G}_\sigma,I,W,N})^{V_\sigma(\mathbb{F}_x)}) &= \Psi_1 \dots \Psi_n {}^*\mathcal{R}_w((\mathcal{H}_{\mathcal{G}_\sigma,I,W,N})^{V_\sigma(\mathbb{F}_x)}) \\
&= \Psi_1 \dots \Psi_n \varinjlim_{\mu} \mathfrak{p}_!^{\leq\mu} \epsilon^*(\mathrm{Sat}(W) \boxtimes \nabla_w(\mathrm{Reg}_{T^{w\mathrm{F}}})) \\
&= \varinjlim_{\mu} \mathfrak{p}_!^{\leq\mu} \epsilon^* \Psi_1 \dots \Psi_n(\mathrm{Sat}(W) \boxtimes \nabla_w(\mathrm{Reg}_{T^{w\mathrm{F}}})),
\end{aligned}$$

where the first line follows from the computation of nearby cycles on $(X-N)^I$, the second one from lemma 4.5.12 and the third one from theorem 4.8.1 and the smoothness of ϵ. As these isomorphisms are equivariant for the action of $\mathrm{Weil}_{F_x}^I$, it is enough to show that the action of $\mathrm{Weil}_{F_x}^I$ on $\Psi_1 \dots \Psi_n(\mathrm{Sat}(W) \boxtimes \nabla_w(\mathrm{Reg}_{T^{w\mathrm{F}}})$ is tame by theorem 4.7.12. This establishes the first point of theorem 4.8.2.

Let us now show the second point. Let $s \in (\hat{T} /\!\!/ W)^{\hat{\mathrm{F}}}$. We want to show that the $\mathrm{Exc}(F_x, \hat{G})$-module $e_s((H^j_{\mathcal{G}_\sigma,I,W,N})^{V_\sigma(\mathbb{F}_x)})$ is supported on $\mathrm{ev}_{\tau_{F_x}}^{-1} s$. Let $I_s \subset \mathcal{O}(\hat{G})^{\hat{G}}$ the ideal defining the point $s \in \hat{G} /\!\!/ \hat{G}$. It is enough to show that for all $f \in I_s$, the excursion operator $F_{f,\tau_{F_x}} \in \mathrm{Exc}$ is nilpotent on $e_s((H^j_{\mathcal{G}_\sigma,I,W,N})^{V_\sigma(\mathbb{F}_x)})$. It is then enough to show that for all $J, V \in \mathrm{Rep}_\Lambda({}^LG)^J, x \in V^{\hat{G}}$ and $\xi \in (V^*)^{\hat{G}}$ such that the function on $\hat{G}^J$ given by $f(g) = \langle g.x, \xi\rangle \in \mathcal{O}(\hat{G}^J)$ lies in $(I_s\mathcal{O}(\hat{G}))^{\otimes J}$, that the excursion operator $F_{J,V,x,\xi,(\tau_{F_x})_{j\in J}}$ is nilpotent on $e_s((H^j_{\mathcal{G}_\sigma,I,W,N})^{V_\sigma(\mathbb{F}_x)})$. This endomorphism is obtained as the composition

$$\begin{aligned}
e_s((H^j_{\mathcal{G}_\sigma,I,W,N})^{V_\sigma(\mathbb{F}_x)}) &= e_s((H^j_{\mathcal{G}_\sigma,I\sqcup\{0\},W\boxtimes 1,N})^{V_\sigma(\mathbb{F}_x)}) \\
&\xrightarrow{x} e_s((H^j_{\mathcal{G}_\sigma,I\sqcup J,W\boxtimes V,N})^{V_\sigma(\mathbb{F}_x)}) \\
&\xrightarrow{(\tau_{F_x})_{j\in J}} e_s((H^j_{\mathcal{G}_\sigma,I\sqcup J,W\boxtimes V,N})^{V_\sigma(\mathbb{F}_x)}) \\
&\xrightarrow{\xi} e_s((H^j_{\mathcal{G}_\sigma,I\sqcup\{0\},W\boxtimes 1,N})^{V_\sigma(\mathbb{F}_x)}) \\
&= e_s((H^j_{\mathcal{G}_\sigma,I,W,N})^{V_\sigma(\mathbb{F}_x)}).
\end{aligned}$$

This endomorphism is obtained after applying $H^j(-)$ to corresponding morphism of complex $e_s((H_{\mathcal{G}_\sigma,I,W,N})^{V_\sigma(\mathbb{F}_x)}) \to e_s((H_{\mathcal{G}_\sigma,I,W,N})^{V_\sigma(\mathbb{F}_x)})$ hence it enough to show the corresponding property for the complex. Let (w,χ) be a pair $w \in W_M$ and χ a character of $T_M^{w\mathrm{F}}$ such that the pair (w,χ) corresponds to s. Since the Deligne-Lusztig restriction functors are conservative, it is enough to show the statement for the complex ${}^*\mathcal{R}_{w,\chi}((H_{\mathcal{G}_\sigma,I,W,N})^{V_\sigma(\mathbb{F}_x)})$. As before we have

$$ {}^*\mathcal{R}_{w,\chi}((H_{\mathcal{G}_\sigma,I,W,N})^{V_\sigma(\mathbb{F}_x)}) = \varinjlim_{\mu} \mathfrak{p}_!^{\leq\mu} \epsilon^* \Psi_1 \dots \Psi_n(\mathrm{Sat}(W) \boxtimes \nabla_w(e_\chi \mathrm{Reg}_{T^{w\mathrm{F}}})). \tag{4.77}$$

It is therefore enough to show this property for the object $\Psi_1 \dots \Psi_n(\mathrm{Sat}(W) \boxtimes \nabla_w(e_\chi \mathrm{Reg}_{T^{w\mathrm{F}}}))$. By theorem 4.7.12, this object is a perverse sheaf on $\frac{\mathrm{Iw}^0\backslash LG/\mathrm{Iw}^0}{\mathrm{Ad}_{\mathrm{F}}T}$. It has a filtration coming from a composition series of $e_\chi\mathrm{Reg}_{T^{w\mathrm{F}}}$, its graded pieces are the sheaves $\Psi_1 \dots \Psi_n(\mathrm{Sat}(W) \boxtimes \nabla_w(\chi))$. By theorem 4.8.1, we have

$$\Psi_1 \dots \Psi_n(\mathrm{Sat}(W) \boxtimes \nabla_w(\chi)) = \mathcal{Z}_\chi(W) * \nabla_w(\chi). \tag{4.78}$$

The excursion operator on central functors is then the composition

$$\begin{aligned}\mathcal{Z}_\chi(W) * \nabla_w(\chi) &= \mathcal{Z}_\chi(1) * \mathcal{Z}_\chi(W) * \nabla_w(\chi) \\ &\xrightarrow{x} \mathcal{Z}_\chi(V) * \mathcal{Z}_\chi(W) * \nabla_w(\chi) \\ &\xrightarrow{(\tau_{F_x})_{j\in J}} \mathcal{Z}_\chi(V) * \mathcal{Z}_\chi(W) * \nabla_w(\chi) \\ &\xrightarrow{\xi} \mathcal{Z}_\chi(W) * \nabla_w(\chi).\end{aligned}$$

By theorem 4.7.14, the functor $\mathcal{Z}_\chi$ is equipped with a monoidal functorial filtration. We consider this filtration for the sheaves $\mathcal{Z}_\chi(1)$ and $\mathcal{Z}_\chi(V)$ in the previous composition. Passing to graded pieces yields a composition

$$\begin{aligned}\mathcal{Z}_\chi(W) * \nabla_w(\chi) &= \mathrm{gr}\mathcal{Z}_\chi(1) * \mathcal{Z}_\chi(W) * \nabla_w(\chi) \\ &\xrightarrow{x} \mathrm{gr}(\mathcal{Z}_\chi(V)) * \mathcal{Z}_\chi(W) * \nabla_w(\chi) \\ &\xrightarrow{\mathrm{gr}(\tau_{F_x})_{j\in J}} \mathrm{gr}(\mathcal{Z}_\chi(V)) * \mathcal{Z}_\chi(W) * \nabla_w(\chi) \\ &\xrightarrow{\xi} \mathcal{Z}_\chi(W) * \nabla_w(\chi).\end{aligned}$$

By theorem 4.7.14, the action of the composition $\xi \circ \mathrm{gr}(\tau_{F_x}) \circ x$ is induced by the function $1 \to 1$ obtained by multiplying by $f(\chi)$. Since $f(\chi) = 0$, on the graded pieces this composition is 0, hence it is nilpotent before taking the graded pieces. This concludes the proof of theorem 4.8.2.

4.8.4 Consequences for the local Langlands correspondence

In this section, we want to show corollary 4.8.4. Let $\pi \in \mathrm{Irr}^0_\Lambda(G(F_x))$ be a depth 0 irreducible cuspidal representation of $G(F_x)$. By theorem 4.3.5, there exists a depth 0 type (σ, τ) for π. By [GL17], there exists a level $N = x + N^x$ such that $\mathrm{Hom}_{\mathbb{T}_\sigma}(\pi^{\mathcal{G}_\sigma(\mathcal{O}_{F_x})_{0+}}, H^{V_\sigma(\mathbb{F}_x)}_{\{0\},1,N}) \neq 0$, where $\mathbb{T}_\sigma$ denotes the local Hecke algebra of $G(F_x)$ with level $\mathcal{G}_\sigma(\mathcal{O}_{F_x})_{0+}$. In particular $\mathrm{Hom}_{M^{\mathbb{F}}_\sigma}(\tau, H^{V_\sigma(\mathbb{F}_x)}_{\{0\},1,N}) \neq 0$. Let $s = \mathrm{LS}(\pi)$, by theorem 4.8.2, $H^{V_\sigma(\mathbb{F}_x)}_{\{0\},1,N}$ is an $\mathrm{Exc}(F_x, \hat{G})$-module supported on $\mathrm{ev}^{-1}(s)$ hence by the main theorem of [GL17], for any π' such that $\pi'^{\mathcal{G}_\sigma(\mathcal{O}_{F_x})_{0+}} \neq 0$ that appears as a subquotient of $e_s H^{V_\sigma(\mathbb{F}_x)}_{\{0\},1,N}$, its Langlands parameter $\sigma_{\pi'}$ is tame and satisfies $\sigma_{\pi'}(\tau_{F_x})_{\mathrm{ss}} \sim s$.

4.A Filtration by Wakimoto sheaves

In this appendix we want to extend theorem 4.7.14 to the modular setting. The proof consists essentially in reproducing the argument of [AB09], [BFO09], [BR22a], [ARon] to the correct setting. All categories considered have coefficients in $\overline{\mathbb{F}}_\ell$. All the geometric objects are considered as schemes (or stacks) defined over $\bar{\mathbb{F}}_q$.

We will use the following notations (some of them were introduced in the core of this paper, we recall them).

(*i*). $\tilde{W} = W \ltimes X_*$ is the extended affine Weyl group.

(*ii*). $\mathrm{Fl} = LG/\mathrm{Iw}^0$.

(iii). $i_w : \mathrm{Fl}_w = \mathrm{Iw}w\mathrm{Iw}/\mathrm{Iw}^0 \to \mathrm{Fl}$ be the inclusion.

(iv). Given $\chi \in \mathrm{CH}_{\overline{\mathbb{F}}_\ell}(T)$, we have $\mathrm{D}_{\mathrm{Ad}}, \mathrm{D}_{\mathrm{Ad},\chi}$ the category of sheaves that are $\mathrm{Ad}(I)$-equivariant on Fl and χ-equivariant for the right action of T.

(v). Given $\chi \in \mathrm{CH}_{\overline{\mathbb{F}}_\ell}(T)$ and a T-equivariant isomorphism $\mathrm{Fl}_w = T \times \mathbb{A}^{\ell(w)}$,

(a) $\Delta_{w,\chi} = i_{w,!}(\mathcal{L}_\chi \boxtimes (\overline{\mathbb{F}}_\ell)_{\mathbb{A}^{\ell(w)}})[\dim T + \ell(w)]$,

(b) $\nabla_{w,\chi} = i_{w,*}(\mathcal{L}_\chi \boxtimes (\overline{\mathbb{F}}_\ell)_{\mathbb{A}^{\ell(w)}})[\dim T + \ell(w)]$

(c) $\mathrm{IC}_{w,\chi} = i_{w,!*}(\mathcal{L}_\chi \boxtimes (\overline{\mathbb{F}}_\ell)_{\mathbb{A}^{\ell(w)}})[\dim T + \ell(w)]$

the standard, costandard and IC sheaves respectively.

(vi). We define Perv_χ the category of perverse sheaves of D_χ. Since the embedding i_w are affine all sheaves $\Delta_{w,\chi}, \nabla_{w,\chi}$ and $\mathrm{IC}_{w,\chi}$.

(vii). We have the central functor

$$\mathcal{Z}_\chi : \mathrm{Rep}_{\overline{\mathbb{F}}_\ell} \hat{G} \to \mathrm{Perv}_\chi, \tag{4.79}$$

that was defined in section 4.7.

Remark 4.A.1. The group $\tilde{W}$ naturally acts on $\mathrm{CH}_{\overline{\mathbb{F}}_\ell}(T)$ through its quotient W.

Remark 4.A.2. The sheaves $\Delta_{w,\chi}, \nabla_{w,\chi}, \mathrm{IC}_{w,\chi}$ are perverse sheaves which are $(T, \mathcal{L}_{w\chi})$-equivariant for the left action of T on Fl.

Remark 4.A.3. We fix a collection of elements $\dot{w} \in \mathrm{N}_G(T)$ lifting the elements of W such that $\dot{w}\dot{w}' = \dot{\overline{ww'}}$ if $\ell(w) + \ell(w') = \ell(ww')$. For $\lambda \in X_*$, we have a canonical map $L^+T \to LG$, the image of t determines a point t^λ in Fl_λ. Given $w \in \tilde{W}$, we can write it uniquely as $w = \lambda w_f$, we set $\dot{w} = t^\lambda \dot{w}$.

Remark 4.A.4. There is a well defined convolution

$$\mathrm{D}_\chi \times \mathrm{D}_{\chi'} \to \mathrm{D}_{\chi'}, \tag{4.80}$$

defined as in section 4.7.3. We refer to [LY20] for a careful discussion about this kind of convolution and to lemma 3.3 and 3.4 of *loc. cit.* for a proof of the following lemma.

Lemma 4.A.5. *With the choice of trivialization of 4.A.3, we have canonical isomorphisms*

(i). $\Delta_{w,w\chi'} * \Delta_{w',\chi'} = \Delta_{ww',\chi'}$ *if* $\ell(w) + \ell(w') = \ell(ww')$,

(ii). $\nabla_{w,w\chi'} * \nabla_{w',\chi'} = \nabla_{ww',\chi'}$ *if* $\ell(w) + \ell(w') = \ell(ww')$,

(iii). $\Delta_{w^{-1},w\chi} * \nabla_{w,\chi} = \Delta_{e,\chi} = \nabla_{w^{-1},w\chi} * \Delta_{w,\chi}$.

From now on we fix the choices of $\dot{w}$ of 4.A.3 and we fix a total order $\leq$ on $\tilde{W}$ refining the Bruhat order. From now on we stick to the strategy of [AB09] Section 3.6. We also refer to [ARon] Chapter 4. We now construct the Wakimoto sheaves.

Lemma 4.A.6 (Construction of the Wakimoto sheaves)**.** *We fix* $\chi \in \mathrm{CH}_{\overline{\mathbb{F}}_\ell}$. *There exists a fully faithful tensor functor*

$$J = \mathrm{Rep}_{\overline{\mathbb{F}}_\ell}\hat{T} \to \mathrm{Perv}_\chi, \tag{4.81}$$

such that

$$J(\lambda) = \nabla_{\chi,\lambda}, \tag{4.82}$$

if λ *is dominant.*

Proof. Arguing as in [AB09], the functor defined $X_*^+ \to \mathrm{Perv}_\chi, \lambda \mapsto \nabla_{\chi,\lambda}$ is monoidal. As all the objects $\nabla_{\chi,\lambda}$ are $\otimes$-invertible, it extends to a monoidal functor $X_* \to \mathrm{Perv}_\chi$. Taking direct sums of each objects yields the desired $\otimes$-functor $\mathrm{Rep}_{\overline{\mathbb{F}}_\ell}\hat{T} \to \mathrm{Perv}_\chi$. $\square$

Objects in the image of J are called Wakimoto sheaves. We define Perv_χ^J for the essential image of J. We define $\mathrm{Perv}_\chi^{J-\mathrm{fil}}$ for the category of objects of Perv_χ equipped with a decreasing X_*-filtration such that all the λ-graded pieces are isomorphic to $V_\lambda \otimes J(\lambda)$ where V_λ is a $\overline{\mathbb{F}}_\ell$-vector space.

Theorem 4.A.7 (Analog of [AB09] Theorems 4 and 6 and [BFO09] 2.5)**.** (i). *There is a unique monoidal lift of* $\mathcal{Z}_\chi$ *to* $\mathrm{Perv}_\chi^{J-\mathrm{fil}}$.

(ii). *The composition*

$$\mathrm{Rep}_{\overline{\mathbb{F}}_\ell}\hat{G} \to \mathrm{Perv}_\chi^{\mathrm{J-fil}} \xrightarrow{\mathrm{gr}} \mathrm{Perv}_\chi^J \simeq \mathrm{Rep}_{\overline{\mathbb{F}}_\ell}\hat{T}, \tag{4.83}$$

is monoidal and monoidally isomorphic to the restriction functor $\mathrm{Rep}_{\overline{\mathbb{F}}_\ell}\hat{G} \to \mathrm{Rep}_{\overline{\mathbb{F}}_\ell}\hat{T}$.

(iii). *The monodromy endomorphism (coming from the nearby cycles) acts by* $\chi \in \hat{T}$ *on this functor.*

Lemma 4.A.8 (Analog of [AB09] lemma 13)**.** (i). *We have* $\mathrm{Hom}_{\mathrm{Perv}_\chi}(J(\lambda), J(\mu)) \neq 0$ *only if* $\lambda \leq \mu$ *and* $\mathrm{Hom}_{\mathrm{Perv}_\chi}(J(\lambda), J(\lambda)) = \overline{\mathbb{F}}_\ell$.

(ii). *The forgetful functor* $\mathrm{Perv}_\chi^{J-\mathrm{fil}} \to \mathrm{Perv}_\chi$ *is faithful and an object in the essential image of the forgetful functor has a unique filtration compatible with the order* $\leq$.

Proof. The first point comes from the fact that $\mathrm{Fl}_\mu \subset \overline{\mathrm{Fl}_\lambda}$ only if $\mu \leq \lambda$ and that Fl_λ is open in the support of $J(\lambda)$.

The second point reduces down to the fact that an object in Perv_χ has at most one J-filtration. Let us show this fact. If A is such an object and $A_{\geq\lambda}, A'_{\geq\lambda}$ are two such filtrations on A we want to show that $A_{\geq\lambda} = A'_{\geq\lambda}$. We proceed by induction. If A has only one term in its filtration, the statement is clear. Let $\lambda \in X_*$ and assume by induction that for all $\mu > \lambda, A_{\geq\mu} = A'_{\geq\mu}$. Then we have $A_{>\lambda} = A'_{>\lambda}$, after replacing A by $A/A_{>\lambda}$, we can assume that $A_{\geq\lambda}$ is the first term of the filtration, but then the condition (i) forces $A_{\geq\lambda} = J(\lambda)^k$ for some k, the same applies to $A'_{\geq\lambda}$. $\square$

Definition 4.A.9. (i). An object $X \in \mathrm{Perv}_\chi$ is convolution exact if the functor $X * -$ is t-exact.

(ii). An object $X \in \mathrm{Perv}_\chi$ is weakly central if for all $L \in \mathrm{Perv}_\chi$, we have $L * X \simeq X * L$, (though we do not assume this isomorphism to be functorial).

Definition 4.A.10. Let $w = \lambda.w_f \in \tilde{W}$ and assume that $w_f.\chi = \chi$ so that $\nabla_{w_f,\chi} \in \mathrm{Perv}_\chi$. We define $J_w = J_\lambda * \nabla_{w_f,\chi} \in \mathrm{D}_\chi$ and we call those sheaves the extended Wakimoto sheaves.

Proposition 4.A.11 ([AB09] Theorem 5)**.** *The object J_w is perverse.*

Lemma 4.A.12 ([AB09] Proposition 5)**.** *Let $X \in \mathrm{Perv}_\chi$ be a convolution exact object then X has a filtration whose graded pieces are extended Wakimoto sheaves, if furthermore X is weakly central then then only J_w that appear are those such that $w \in X_*$.*

The proof of the above lemma and proposition in *loc. cit.* only uses some fact about the perverse t-structure as well as the geometry of the convolution map which all hold in the present setting.

Lemma 4.A.13 (Compare with [AB09] Lemma 9)**.** *For λ dominant,*

$$j_\lambda^* \mathcal{Z}_\chi(V_\lambda) \neq 0, \tag{4.84}$$

where V_λ denotes the Weyl module of highest weight λ.

Proof. We reduce to the characteristic 0 situation. By *loc. cit.*, Fl_λ is open in the support of $\mathcal{Z}_\chi(V_\lambda)$. Consider a lift of χ to $\overline{\mathbb{Z}}_\ell$, the construction of the central functor is then the reduction mod ℓ of the $\overline{\mathbb{Z}}_\ell$-version of the central functor. After inverting ℓ, we know that this stalk is non-zero and free of rank one by [BFO09]. By constructibility of the $\overline{\mathbb{Z}}_\ell$-version, the mod$\ell$-reduction is therefore nonzero. □

Proof. We now prove theorem 4.A.7. By theorem 4.7.12, the object $\mathcal{Z}_\chi(V)$ is weakly central since it comes equipped with a central structure and is convolution exact. By lemma 4.A.12 there is Wakimoto filtration on $\mathcal{Z}_\chi(V)$. By lemma 4.A.8, this filtration is unique. By the lemmas $16, 17$ and 18 of [AB09], which hold for arbitrary monoidal categories, this filtration is monoidal and the composition $\mathrm{gr}\mathcal{Z}_\chi$ is monoidal. Composing with the inverse on J, we get a monoidal functor $\mathrm{Rep}_{\overline{\mathbb{F}}_\ell} \hat{G} \to \mathrm{Rep}_{\overline{\mathbb{F}}_\ell} \hat{T}$, which therefore corresponds to a morphism of $\hat{T} \to \hat{G}$. It only remains to check that this is the inclusion of the maximal torus. By 4.A.13, λ is direct summand of $\mathrm{gr}\mathcal{Z}_\chi(V_\lambda)$, this implies that $\hat{T} \to \hat{G}$ is injective and is identified with the prescribed maximal torus. This yields the first two points of theorem 4.A.7. □

Lemma 4.A.14. *Consider the action of $(\mathbb{G}_m)_{\mathrm{rot}}$, and let $\lambda \in \mathring{X}_*$. The sheaves $\mathrm{IC}_{\lambda,\chi}$ and J_λ are $\lambda(\chi^{-1})$ monodromic where $\lambda : \hat{T} \to \mathbb{G}_m = (\mathbb{G}_m)^\vee_{\mathrm{rot}}$ is considered as a cocharacter of $\hat{T}$.*

Proof. By [BR22a] Section 4.4, the normalized loop rotation on Fl is given on Fl_λ by $z.t^\lambda = \lambda(z^{-1})t^\lambda$. Hence the rotation monodromy on $\mathrm{IC}_{\lambda,\chi}$ is given by $\lambda(\chi^{-1})$. For the case of J_λ, we proceed by induction on the length of λ and reduce to the case where λ is of minimal lenght and thus $J_\lambda = \mathrm{IC}_{\lambda,\chi}$. □

This last lemma yields the last point of theorem 4.A.7.

Lemma 4.A.15. *The monodromy action coming from the nearby cycles on $\mathcal{Z}_\chi(V)$ is given on $\mathrm{gr}(\mathcal{Z}_\chi(V))$ by the action of χ.*

Proof. Since by point (ii) of theorem 4.A.7, under the functor J, $\mathrm{gr}(\mathcal{Z}_\chi(V)$ corresponds to $\mathrm{Res}_{\hat{T}}^{\hat{G}} V$ and the graded piece corresponding to $\lambda \in X_*$ corresponds to the direct summand of of weight λ. The torus monodromy acts on this direct summand by the character $\lambda(\chi^{-1})$ by lemma 4.A.14. By theorem 4.7.6, the action of the monodromy on the nearby cycles is given by $\lambda(\chi)$. This also concludes the proof of theorem 4.A.7. □

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