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présentée par

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Contributions to the Langlands program

dirigée par Jean-François DAT et Laurent FARGUES

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Résumé

Cette thèse traite de deux problèmes dans le cadre du programme de Langlands. Pour le premier problème, dans la situation de GL_2 et un cocaractère non minuscule, nous fournissons un contre-exemple (sous certaines hypothèses naturelles) à la conjecture de Rapoport-Zink, communiquée par Laurent Fargues.

Le deuxième problème concerne un résultat dans le programme de Langlands p-adique. Soit A une algèbre $\mathbf{Q}_p$-affinoïde, au sens de Tate. Nous développons une théorie d'un espace localement convexe en A-modules parallèle au traitement dans le cas d'un corps par Schneider et Teitelbaum. Nous montrons qu'il existe une application d'intégration liant une catégorie de représentations localement analytiques en A -modules et des modules de distribution séparés *relatif*. Il existe une théorie de cohomologie localement analytique pour ces objets et une version du Lemme de Shapiro. Dans le cas d'un corps, ceci a été substantiellement développé par Kohlhaase. Comme une application, nous proposons une correspondance de Langlands p-adique en *families* : Pour un (φ, Γ)-module trianguline et *régulière* de dimension 2 sur l'anneau de Robba relatif $\mathscr{R}_A$ nous construisons une $\mathrm{GL}_2(\mathbf{Q}_p)$-représentation localement analytique en A-modules. Il s'agit d'un travail en commun avec Joaquin Rodrigues.

Abstract

This thesis deals with two problems within the Langlands program. For the first problem, in the situation of GL_2 and a non-minuscule cocharacter, we provide a counter-example (under some natural assumptions) to the Rapoport-Zink conjecture, communicated to us by Laurent Fargues.

The second problem deals with a result in the p-adic Langlands program. Let A be a $\mathbf{Q}_p$-affinoid algebra, in the sense of Tate. We develop a theory of locally convex A-modules parallel to the treatment in the case of a field by Schneider and Teitelbaum. We prove that there is an integration map linking a category of locally analytic representations in A-modules and separately continuous *relative* distribution modules. There is a suitable theory of locally analytic cohomology for these objects and a version of Shapiro's Lemma. In the case of a field this has been substantially developed by Kohlhaase. As an application we propose a p-adic Langlands correspondence in *families*: For a *regular* trianguline (φ, Γ)-module of dimension 2 over the relative Robba ring $\mathscr{R}_A$ we construct a locally analytic $\mathrm{GL}_2(\mathbf{Q}_p)$-representation in A-modules. This is joint work with Joaquin Rodrigues.

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Contents

1 Introduction

1.1 Un contre-exemple à la conjecture de Rapoport-Zink

Dans la première partie de cette thèse, nous fournissons un contre-exemple à la conjecture de Rapoport-Zink, cf. [49, §1.37] dans la situation suivante:

$$G = \mathrm{GL}_2, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}, \quad \mu = \begin{pmatrix} t^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Bien que la conjecture en loc.cit. est énoncée dans le cadre analytique rigide, nous adopterons le langage des espaces de Berkovich. Ici b est associé à l'isocristal $\mathbb{Q}_p \oplus \mathbb{Q}_p(2)$ et μ est un co-caractère non minuscule (défini sur $\breve{\mathbb{Q}}_p$) qui contrôle la filtration. Soit $\mathbb{A}^1_{\breve{\mathbb{Q}}_p}$ la ligne droite affine de Berkovich sur la completion de l'extension maximale non-ramifiée de $\mathbb{Q}_p$, $\breve{\mathbb{Q}}_p$. Pour chaque extension finie $L/\breve{\mathbb{Q}}_p$, $\mathbb{A}^1_{\breve{\mathbb{Q}}_p}(L)$ correspond à la classe d'équivalence des filtrations μ' défini sur L (dans la classe de conjugaison de $\{\mu\}$) telle que (b, μ') est admissible [1]. Dans ce cadre, la conjecture annonce grossiement (pour un énoncé précis, nous renvoyons le lecteur à la Conjecture 4.3).

Conjecture 1.1 (Rapoport-Zink)**.** *Il existe un $\mathbb{Q}_p$-système local $\mathcal{E}$ sur $\mathbb{A}^1_{\breve{\mathbb{Q}}_p}$ qui satisfait la propriété suivante: Pour tout point $\mu' \in \mathbb{A}^1_{\breve{\mathbb{Q}}_p}(L)$ la représentation galoisienne p-adique*

$$\mathrm{Gal}(\overline{L}/L) \xrightarrow{\pi_1^{\mathrm{dJ}}(\mu')} \pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p}) \to GL(\mathbb{Q}_p),$$

où le 2ème morphisme vient de $\mathcal{E}$, est isomorphe à la représentation cristalline déterminée par (b, μ') [2].

Donnons une esquisse pour savoir pourquoi la Conjecture 1.1 est fausse. On commence par définir un espace de modules (analogue à un espace de Rapoport-Zink) associé à $\mathcal{E}$ qui paramétrise les $\mathbb{Z}_p$-réseaux dans $\mathcal{E}$.

Definition 1.2. *Définir un espace de modules $\mathscr{M}(\mathcal{E}) \to \mathbb{A}^1_{\breve{\mathbb{Q}}_p}$, dont les valeurs à un $S \to \mathbb{A}^1_{\breve{\mathbb{Q}}_p}$ point étale sont donnés par*

$$\mathscr{M}(\mathcal{E})(S) := \{\mathcal{F}/S \mid \ \mathcal{F} \otimes \mathbb{Q}_p \cong \mathcal{E}|_S\},$$

où $\mathcal{F}$ est un $\mathbb{Z}_p$-système local (un système local de $\mathbb{Z}_p$-réseaux dans la terminologie utilisée par de Jong, cf. [18, §4]).

1. De manière équivalente par le travail de Colmez-Fontaine, cf. [12], la paire (b, μ') donne naissance à une représentation cristalline.

2. Ici $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p})$ est le (gros) groupe fondamental étale de recouvrements étales de $\mathbb{A}^1_{\breve{\mathbb{Q}}_p}$ comme défini dans [18, §2]. Dans loc.cit. il est noté par π_1.

Nous appelons le morphisme structurel

$$\pi_{\mathrm{dR}} : \mathscr{M}(\mathcal{E}) \to \mathbb{A}^1_{\breve{\mathbb{Q}}_p}$$

le morphisme de périodes de de-Rham. On démontre facilement que $\mathscr{M}(\mathcal{E})$ est représentable comme un $\breve{\mathbb{Q}}_p$-espace analytique et π_{dR} un recouvrement étale. Il est naturel de s'attendre à ce que $\mathscr{M}(\mathcal{E})$ a une structure de groupe, mais pour prouver cela, nous devons supposer que $\mathcal{E} \in \mathrm{Ext}^1(\mathbb{Q}_p, \mathbb{Q}_p(2))$ et aussi pour tout sous-groupe ouvert $\mathcal{U} \subset \mathbb{A}^1_{\breve{\mathbb{Q}}_p}$,

$$\mathrm{m}^* \eta|_{\mathcal{U}} = \mathrm{pr}_1^* \mathcal{E}|_{\mathcal{U}} + \mathrm{pr}_2^* \mathcal{E}|_{\mathcal{U}}, \tag{1}$$

où $\mathrm{m} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ est la loi de groupe de $\mathcal{U}$ et $\mathrm{pr}_{1,2} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ sont les première et seconde projections, respectivement [3].

Cela nous permet ensuite de travailler avec $\mathscr{M}(\mathcal{E})^{(0,0)}$, la composante connexe de $\mathscr{M}(\mathcal{E})$ autour de l'élément d'identité. Il se trouve que $\mathscr{M}(\mathcal{E})^{(0,0)}$ est un $\breve{\mathbb{Q}}_p$-groupe analytique de type p-divisible dans le sens de [26]. En utilisant un théorème de classification de $\mathbb{C}_p$-groupes analytiques de type p-divisible établi par Fargues, on peut montrer que $\mathcal{E}_{\mathbb{C}_p} \cong \mathcal{E}'_{\mathbb{C}_p}$, où $\mathcal{E}'$ est un certain $\mathbb{Q}_p$-système local sur $\mathbb{A}^1_{\breve{\mathbb{Q}}_p}$ qui est une extension de $\mathbb{Q}_p$ par $\mathbb{Q}_p(1)$. Le $\mathbb{Q}_p$-système local $\mathcal{E}'$ vient d'un espace de Rapoport-Zink honnête (l'espace de déformation d'une courbe elliptique ordinaire sur $\overline{\mathbb{F}}_p$ pour être precis).

En effet l'isomorphisme $\mathcal{E}_{\mathbb{C}_p} \cong \mathcal{E}'_{\mathbb{C}_p}$ semble étrange. Rappelons que $\mathcal{E}$ se trouve dans une suite exacte

$$0 \to \mathbb{Q}_p(2) \to \mathcal{E} \to \mathbb{Q}_p \to 0$$

et $\mathcal{E}'$ dans

$$0 \to \mathbb{Q}_p(1) \to \mathcal{E}' \to \mathbb{Q}_p \to 0.$$

Soient $\rho_1, \rho_2 \colon \pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p}) \to GL_2(\mathbb{Q}_p)$ les représentations correspondantes à $\mathcal{E}'$ and $\mathcal{E}$, respectivement. On munit $\mathrm{GL}_2(\mathbb{Q}_p)$ avec une action de $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p})$ via la conjugaison de ρ_2, la clé étant de considérer l'ensemble pointu

$$H^1(\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p}), GL_2(\mathbb{Q}_p))$$

et en particulier le 1-cocycle donné par

$$\begin{aligned} c \colon \pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p}) &\to GL_2(\mathbb{Q}_p) \\ \sigma &\mapsto \rho_1(\sigma)\rho_2(\sigma)^{-1}. \end{aligned}$$

Une analyse de c obtient une contradiction pour l'existence de $\mathcal{E}$.

Donnons un bref aperçu du contenu de la première partie de cette thèse.

3. L'addition dans (2) est considérée comme une somme de Baer dans le groupe abélien des extensions de $\mathbb{Q}_p$ par $\mathbb{Q}_p(2)$ sur $\mathcal{U} \times \mathcal{U}$.

Dans §4 nous rappelons la conjecture de Rapoport-Zink. Nous avons mis en place la situation qui mène à un contre-exemple et prouvons que $\mathscr{M}(\mathcal{E})^{(0,0)}$ est un objet en groupe dans la catégorie des $\breve{\mathbb{Q}}_p$-espaces analytiques. Par ailleurs π_{dR} est un morphisme de groupes.

Dans §5 nous montrons que $\mathscr{M}(\mathcal{E})^{(0,0)}$ est de type p-divisible.

Dans §6 nous rappelons l'espace de Rapoport-Zink qui déforme une courbe elliptique ordinaire sur $\overline{\mathbb{F}}_p$.

Dans §7 nous prouvons qu'il est impossible d'avoir deux $\mathbb{Q}_p$-représentations de $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p})$, dont l'une est une extension de $\mathbb{Q}_p$ par $\mathbb{Q}_p(1)$ et l'autre est une extension de $\mathbb{Q}_p$ par $\mathbb{Q}_p(2)$ qui sont isomorphes lorsqu'elles sont tirées vers $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\mathbb{C}_p})$.

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1.2 Une extension de la correspondance de Langlands p-adique

Dans la deuxième partie de cette thèse on étudie la correspondance de Langlands p-adique pour $\mathrm{GL}_2(\mathbf{Q}_p)$ en familles arithmétiques. Pour donner le contexte, rappelons les lignes générales de cette correspondance. Dans [9], [48] et [16], une bijection $V \mapsto \Pi(V)$ est établie entre les L-représentations [4] continues de dimension 2 absolument irreductibles du groupe de Galois absolu $\mathcal{G}_{\mathbf{Q}_p}$ de $\mathbf{Q}_p$ est les L-représentations Banach admissibles unitaires non-ordinaires de $\mathrm{GL}_2(\mathbf{Q}_p)$ qui sont topologiquement absolument irréductibles. Le foncteur inverse $\Pi \mapsto V(\Pi)$ est parfois appelé le foncteur Montréal, cf. [9, §IV].

La stratégie de base de la construction du foncteur $V \mapsto \Pi(V)$ est la suivante: par l'équivalence de Fontaine, la catégorie des représentations galoisienne en L-espaces vectoriels est équivalente à celui de (φ, Γ)-modules étales sur le corps de Fontaine $\mathscr{E}_L$ [5]. Cette dernière catégorie (-linéarisée) est considérée comme une mise à niveau parce que l'on peut souvent effectuer des calculs explicites. Un tel (φ, Γ)-module D peut être naturellement considéré comme un faisceau P^+-equivariant [6] sur $\mathbf{Z}_p$, où $P^+ = \left(\begin{smallmatrix} \mathbf{Z}_p - \{0\} & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ est un sous-semi-groupe du sous-groupe mirabolique $(= \left(\begin{smallmatrix} \mathbf{Q}_p^\times & \mathbf{Q}_p \\ 0 & 1 \end{smallmatrix}\right))$ de $\mathrm{GL}_2(\mathbf{Q}_p)$. Si U est un sous-ensemble ouvert compact de $\mathbf{Z}_p$, on note par $D \boxtimes U$ les sections locales sur U de ce faisceau. Dans [9], une involution magique w_D agissant sur $D \boxtimes \mathbf{Z}_p^\times$ est définie ce qui permet (on note que $\mathbf{P}^1(\mathbf{Q}_p)$ est construit en collant deux copies de $\mathbf{Z}_p$ au long de

4. À partir de maintenant, L sera le corps de coefficient, une extension fini de $\mathbf{Q}_p$.

5. Le corps $\mathscr{E}_L$ est défini comme des séries de Laurent $\sum_{n \in \mathbf{Z}} a_n T^n$ telles que $a_n \in L$ sont bornés et $\lim_{n \to -\infty} a_n = 0$. $\mathscr{E}$ est muni d'actions continues de $\Gamma = \mathbf{Z}_p^\times$ (on note σ_a, $a \in \mathbf{Z}_p^\times$, ses éléments) et un opérateur φ défini par les formules $\sigma_a(T) = (1+T)^a - 1$ et $\varphi(T) = (1+T)^p - 1$. Rappelons qu'un (φ, Γ)-module est un $\mathscr{E}$-module libre muni d'actions continus semi-linéaires de Γ est φ.

6. La matrice $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ codifie l'action de φ, $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ l'action de $\sigma_a \in \Gamma$ et $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ la multiplication par $(1+T)^b$.

$\mathbf{Z}_p^\times$) d'étendre D à un faisceau $\mathrm{GL}_2(\mathbf{Q}_p)$-équivariant sur [7] $\mathbf{P}^1$, qui est noté par $D \boxtimes_\omega \mathbf{P}^1$, où $\omega = (\det D)\chi^{-1}$ [8]. On récupère la représentation de Banach désirée $\Pi(V)$ (et sa duale) des constituants de $D \boxtimes_\omega \mathbf{P}^1$. Plus précisément, nous avons une suite exacte courte de $\mathrm{GL}_2(\mathbf{Q}_p)$-modules topologiques

$$0 \to \Pi(V)^* \otimes \omega \to D \boxtimes_\omega \mathbf{P}^1 \to \Pi(V) \to 0.$$

Soit $\mathscr{R}_L$ l'anneau de Robba [9] avec des coefficients dans L. Par une combinaison de résultats de Cherbonnier-Colmez ([5]) et Kedlaya ([38]), les catégories de (φ, Γ)-modules étale sur $\mathscr{E}_L$ et $\mathscr{R}_L$ sont équivalentes. Notons $D \mapsto D_{\mathrm{rig}}$ cette correspondance. On a des constructions analogues comme ci-dessus pour D_{rig} et, en particulier, on a un faisceau $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant $U \mapsto D_{\mathrm{rig}} \boxtimes U$ sur $\mathbf{P}^1$. Si on note $\Pi(V)^{\mathrm{an}}$ les vecteurs localement analytiques de $\Pi(V)$, on obtient une suite exacte

$$0 \mapsto (\Pi(V)^{\mathrm{an}})^* \otimes \omega \to D_{\mathrm{rig}} \boxtimes_\omega \mathbf{P}^1 \to \Pi(V)^{\mathrm{an}} \to 0.$$

Néanmoins, la construction de $D_{\mathrm{rig}} \boxtimes_\omega \mathbf{P}^1$ n'est pas une conséquence directe de $D \boxtimes_\omega \mathbf{P}^1$. C'est principalement parce que la formule définissant l'involution ne converge pas pour un (φ, Γ)-module sur $\mathscr{R}_L$ [10].

Inspiré par les calculs da la correspondance p-adique locale pour les (φ, Γ)-modules étale trianguline [11], Colmez ([11]) a récemment donné une construction directe, pour un (pas nécessairement étale) (φ, Γ)-module Δ (de rang 2) sur $\mathscr{R}_L$, d'une L-representation localement analytique $\Pi(\Delta)$ de $\mathrm{GL}_2(\mathbf{Q}_p)$. Plus précisément, on a le théorème suivant:

Théorème 1.3 ([11], Théorème 0.1)**.** *Il existe une extension unique de Δ à un faisceau $\mathrm{GL}_2(\mathbf{Q}_p)$-équivariant de type $\mathbf{Q}_p$-analytique [12] $\Delta \boxtimes_\omega \mathbf{P}^1$ sur $\mathbf{P}^1$ muni d'un caractère central ω. De plus, il existe une L-représentation unique admissible localement analytique $\Pi(\Delta)$, munie d'un caractère central ω, de $\mathrm{GL}_2(\mathbf{Q}_p)$, telle que*

$$0 \to \Pi(\Delta)^* \otimes \omega \to \Delta \boxtimes_\omega \mathbf{P}^1 \to \Pi(\Delta) \to 0.$$

7. À partir de maintenant, $\mathbf{P}^1$ signifiera $\mathbf{P}^1(\mathbf{Q}_p)$.
8. Le caractère $\det D$ est le caractère de $\mathbf{Q}_p^\times$ défini par les actions de φ et Γ sur $\wedge^2 D$. Si D est étale, il peut également être vu comme un caractère du groupe de Galois via la théorie de corps de classe locale. Le caractère $\chi\colon x \mapsto x|x|$ dénote le caractère cyclotomic. On voit les deux caractères commes des caractères de $\mathrm{GL}_2(\mathbf{Q}_p)$ en composant avec le déterminant.
9. Il est défini comme l'anneau des séries de Laurent $\sum_n a_n T^n$, $a_n \in L$, qui convergent sur une couronne $0 < v_p(T) \leq r$ pour un $r > 0$.
10. Pour construire l'involution sur D_{rig} dans le cas étale, on montre que w_D stabilise $D^\dagger \boxtimes \mathbf{Z}_p^\times$, où $D^\dagger$ est le (φ, Γ)-module sur les éléments surconvergents $\mathscr{E}_L^\dagger$ de $\mathscr{E}_L$ qui correspondent à D par la correspondance de Cherbonnier-Colmez , et qu'elle définit par continuité une involution sur $D_{\mathrm{rig}} \boxtimes \mathbf{Z}_p^\times$.
11. Un (φ, Γ)-module de rang 2 est trianguline si c'est une extension de (φ, Γ)-modules de rang 1.
12. Un faisceau $U \mapsto M \boxtimes U$ est de type $\mathbf{Q}_p$-analytique si, pour tout ouvert compact $U \subseteq \mathbf{P}^1$ et pour tout compact $K \subseteq \mathrm{GL}_2(\mathbf{Q}_p)$ qui stabilise U, l'espace $M \boxtimes U$ est de type-LF et un $\mathcal{D}(K)$-module continu, où $\mathcal{D}(K)$ est l'algèbre de distribution sur K.

Le but de ce travail est d'étudier cette correspondance dans le contexte des familles arithmétiques de (φ, Γ)-modules. Des résultats dans cette direction sur le côté ℓ-adique (i.e. la correspondance de Langlands locale *classique*, cf. [31]) ont été atteints par Emerton-Helm [24]. Les arguments dans [11] reposent fortement sur la théorie de cohomologie des représentations localement analytiques [43], et plus précisément sur le lemme de Shapiro. Puisque les auteurs ne sont pas au courant d'une référence pour ces résultats dans le contexte relatif, nous développons, dans un appendice (cf. §A), les définitions et propriétés nécessaires de $\mathrm{GL}_2(\mathbf{Q}_p)$-représentations localement analytiques en A-modules. Vu que ce point pourrait porter un intérêt particulier, nous le décrivons plus en détail dans §1.4 ci-dessous. Nous travaillerons exclusivement avec des espaces affinoïdes au sens de Tate, plutôt que de Berkovich ou Huber. Soit A une algèbre $\mathbf{Q}_p$-affinoïde et soit $\mathscr{R}_A$ l'anneau relatif de Robba sur A. Notre résultat principal peut être énoncé comme suit:

Théorème 1.4. *Soit A une algèbre $\mathbf{Q}_p$-affinoïde et soit Δ un (φ, Γ)-module trianguline sur $\mathscr{R}_A$ de rang 2 qui est une extension de $\mathscr{R}_A(\delta_2)$ par $\mathscr{R}_A(\delta_1)$, où $\delta_1, \delta_2 : \mathbf{Q}_p^\times \to A^\times$ sont des caractères localement analytiques satisfaisant certaines hypothèses de régularité*[13]*. Alors, il existe une extension de Δ à un faisceau $\mathrm{GL}_2(\mathbf{Q}_p)$-équivariant de type $\mathbf{Q}_p$-analytique $\Delta \boxtimes_\omega \mathbf{P}^1$ sur $\mathbf{P}^1$ muni d'un caractère central $\omega = \delta_1\delta_2\chi^{-1}$ et une $\mathrm{GL}_2(\mathbf{Q}_p)$-représentation (pas nécessairement unique) localement analytique*[14] *$\Pi(\Delta)$ en A-modules munie d'un caractère central ω, vivant dans une suite exacte*

$$0 \to \Pi(\Delta)^* \otimes \omega \to \Delta \boxtimes_\omega \mathbf{P}^1 \to \Pi(\Delta) \to 0.$$

On s'attend à ce que ce résultat ait des applications à l'étude de *variétés de Hecke*, cependant dans cette thèse nous ne faisons aucune tentative dans cette direction.

1.3 La construction de la correspondance

La construction de la correspondance suit les lignes générales de [11], mais plusieurs difficultés techniques apparaissent en chemin. Décrivons brièvement comment construire la correspondance $\Delta \mapsto \Pi(\Delta)$ et les problèmes supplémentaires qui se posent dans le cadre relatif (affinoïde).

À partir du calcul des vecteurs localement analytiques des séries principales unitaires ([8, Théorème 0.7]), on sait que, si D est un (φ, Γ)-module trianguline étale sur $\mathscr{E}_L$ de rang 2, alors $(\Pi(D))^{\mathrm{an}}$ est une extension des séries principales. L'idée de [11] est d'inverser intelligemment ce *dévissage* de $D_{\mathrm{rig}} \boxtimes_\omega \mathbf{P}^1$ afin de le construire à partir de ces pièces.

Pour le reste de cette introduction, soit $G = \mathrm{GL}_2(\mathbf{Q}_p)$ et $\overline{B}$ son sous-groupe inférieur de Borel et soient δ_1, δ_2 et ω comme dans le Théorème 1.4. En utilisant une version relative du dictionnaire p-adique analyse fonctionnelle, nous

13. Précisément, on suppose que $\delta_1\delta_2^{-1}$ est point par point jamais de la forme χx^i ou x^{-i} pour un $i \geq 0$.

14. Voir la Définition A.24 pour la définition d'une G-représentation localement analytique en A-modules.

construisons, pour $? \in \{+,-,\emptyset\}$, des faisceaux G-équivariants $\mathscr{R}_A^?(\delta_1) \boxtimes_\omega \mathbf{P}^1$ (munis d'un caracère central ω) de type $\mathbf{Q}_p$-analytique vivant dans une suite exacte

$$0 \to \mathscr{R}_A^+(\delta_1) \boxtimes_\omega \mathbf{P}^1 \to \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1 \to \mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1 \to 0.$$

De plus, on peut obtenir des identifications $B_A(\delta_2,\delta_1)^* \otimes \omega \cong \mathscr{R}_A^+(\delta_1) \boxtimes_\omega \mathbf{P}^1$ et $B_A(\delta_1,\delta_2) \cong \mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1$, où $B_A(\delta_1,\delta_2) = \mathrm{Ind}_B^G(\delta_1\chi^{-1} \otimes \delta_2)$ désigne la série principale localement analytique. Ces identifications nous permettent de considérer les séries principales localement analytiques (et ses duaux) comme (les sections globales) des faisceaux G-équivariants sur $\mathbf{P}^1$ d'intérêt.

Nous construisons ensuite le faisceau G-équivariant $\Delta \boxtimes_\omega \mathbf{P}^1$ sur $\mathbf{P}^1$ comme une extension de $\mathscr{R}_A(\delta_2) \boxtimes_\omega \mathbf{P}^1$ par $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$. Cela se fait, comme dans [11], en montrant que les extensions de $\mathscr{R}_A(\delta_2)$ par $\mathscr{R}_A(\delta_1)$ correspondent aux extensions de $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1$ par $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$. On montre alors qu'une extension de $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1$ par $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$ s'étend à une extension unique de $\mathscr{R}_A(\delta_2) \boxtimes_\omega \mathbf{P}^1$ par $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$. Dès que le faisceau $\Delta \boxtimes_\omega \mathbf{P}^1$ est construit, on montre que l'extension intermédiaire de $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1$ par $\mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1$ est scindée. Ainsi, on peut séparer les espaces qui sont Fréchets de ceux qui sont une limite inductive des espaces de Banach de manière à découper la représentation souhaitée $\Pi(\Delta)$.

Le fait que, pour $? \in \{+,-,\emptyset\}$, le P^+-module $\mathscr{R}_A^?(\delta_1)$ peut être vu comme des sections sur $\mathbf{Z}_p$ d'un faisceau G-équivariant sur $\mathbf{P}^1$, et que le semi-groupe $\overline{P}^+ = \left(\begin{smallmatrix} \mathbf{Z}_p - \{0\} & 0 \\ p\mathbf{Z}_p & 1 \end{smallmatrix}\right)$ stabilise $\mathbf{Z}_p$, montre que $\mathscr{R}_A^?(\delta_1) = \mathscr{R}_A^?(\delta_1) \boxtimes_\omega \mathbf{Z}_p$ est automatiquement muni d'une action supplémentaire de la matrice $\left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right)$. On note

$$\mathscr{R}_A^?(\delta_1,\delta_2) := (\mathscr{R}_A^?(\delta_1) \boxtimes_\omega \mathbf{Z}_p) \otimes \delta_2^{-1}$$

le $\overline{P}^+$-module ainsi défini. Le cœur technique pour prouver le Théorème 1.4 est un résultat de comparaison entre la cohomologie des semi-groupes $A^+ = \left(\begin{smallmatrix} \mathbf{Z}_p - \{0\} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ et $\overline{P}^+$ aux valeurs dans $\mathscr{R}_A(\delta_1\delta_2^{-1})$ et $\mathscr{R}_A(\delta_1,\delta_2)$, respectivement.

Théorème 1.5. *Le morphisme de restriction de $\overline{P}^+$ à A^+ induit un isomorphisme*

$$H^1(\overline{P}^+, \mathscr{R}_A(\delta_1,\delta_2)) \to H^1(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1})).$$

Le semi-groupe A^+ devrait être considéré comme codant l'action de φ et Γ. La difficulté, bien sûr, est de capturer l'action de l'involution et c'est l'idée sous-jacente pour considérer le semi-groupe $\overline{P}^+$. En effet $\overline{P}^+$ est plus proche de capturer l'action de l'involution par rapport à A^+. Le Théorème 1.5 est (essentiellement) en train de dire qu'un (φ,Γ)-module trianguline comme dans le Théorème 1.4 admet une extension à un faisceau G-équivariant sur $\mathbf{P}^1$.

Décrivons brièvement la démonstration du Théorème 1.5. L'idée principale est de réduire cette bijection au cas d'un point (i.e au cas où $A = L$ est une extension finie de $\mathbf{Q}_p$). La première étape consiste à construire un complexe de *Koszul* qui calcule la cohomologie de $\overline{P}^+$.

Proposition 1.6. *Soit M un $A[\overline{P}^+]$-module tel que l'action de $\overline{P}^+$ s'étend à une action de l'algèbre d'Iwasawa $\mathbf{Z}_p[[\overline{P}^+]]$. Alors le complexe*

$$\mathscr{C}_{\tau,\varphi,\gamma}(M) : 0 \to M \xrightarrow{X} M \oplus M \oplus M \xrightarrow{Y} M \oplus M \oplus M \xrightarrow{Z} M \to 0$$

où[15]

$$\begin{aligned} X(x) &= ((1-\tau)x, (1-\varphi)x, (\gamma-1)x) \\ Y(x,y,z) &= ((1-\varphi\delta_p)x + (\tau-1)y, (\gamma\delta_a - 1)x + (\tau-1)z, (\gamma-1)y + (\varphi-1)z) \\ Z(x,y,z) &= (\gamma\delta_a - 1)x + (\varphi\delta_p - 1)y + (1-\tau)z \end{aligned}$$

calcule la cohomologie de $\overline{P}^+$. C'est-à-dire $H^i(\mathscr{C}_{\tau,\varphi,\gamma}(M)) = H^i(\overline{P}^+, M)$.

La nature asymétrique de $\mathscr{C}_{\tau,\varphi,\gamma}(M)$ est due à la non-commutativité de $\overline{P}^+$. Une estimation brute des morphisms X, Y et Z conduit au corollaire suivant:

Corollaire 1.7. *Le complexe $\mathscr{C}_{\tau,\varphi,\gamma}(\mathscr{R}_A(\delta_1,\delta_2))$ est un complexe pseudo-cohérent concentré en degrés $[0,3]$. Dans la terminologie du corps du papier, $\mathscr{C}_{\tau,\varphi,\gamma}(\mathscr{R}_A(\delta_1,\delta_2)) \in \mathcal{D}^{[0,3]}_{\mathrm{pc}}(A)$*[16]*. En particulier, les groupes de cohomologie $H^i(\overline{P}^+, \mathscr{R}_A(\delta_1,\delta_2))$ sont les A-modules finis.*

Plus précisément, la preuve du Corollaire 1.7 est réduite à prouver la finitude d'un (φ,Γ)-cohomologie *twisté* de $\mathscr{R}_A(\delta_1,\delta_2)$, cf. Lemma 12.11.

Le problème avec $\mathscr{C}_{\tau,\varphi,\gamma}(M)$ est que les opérateurs δ_x sont difficiles à comprendre, rendant le complexe presque impraticable pour des calculs explicites. On peut cependant *linéariser* la situation et passer à l'algèbre de Lie, où les calculs sont souvent réalisables.

Proposition 1.8. *Pour $M \in \left\{\mathscr{R}_L^+(\delta_1,\delta_2), \mathscr{R}_L^-(\delta_1,\delta_2), \mathscr{R}_L(\delta_1,\delta_2)\right\}$, le complexe*

$$\mathscr{C}_{u^-,\varphi,a^+}(M) : 0 \to M \xrightarrow{X'} M \oplus M \oplus M \xrightarrow{Y'} M \oplus M \oplus M \xrightarrow{Z'} M \to 0,$$

où[17]

$$\begin{aligned} X'(x) &= ((\varphi-1)x, a^+x, u^-x) \\ Y'(x,y,z) &= (a^+x - (\varphi-1)y, u^-y - (a^+ + 1)z, (p\varphi - 1)z - u^-x) \\ Z'(x,y,z) &= u^-x + (p\varphi-1)y + (a^+ + 1)z \end{aligned}$$

calcule la cohomologie d'algèbre de Lie de $\overline{P}^+$. En particulier, $H^0(\tilde{P}, H^i(\mathscr{C}_{u^-,\varphi,a^+}(M))) = H^i(\overline{P}^+, M)$[18]*.*

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15. Ici $\tau = \left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right)$ et $\delta_x = \frac{\tau^x - 1}{\tau - 1}$ pour tout $x \in \mathbf{Z}_p^\times$.

16. Nous renvoyons le lecteur à §12.1 Pour la notion d'un complexe pseudo-cohérent et la définition de $\mathcal{D}^-_{\mathrm{pc}}(A)$.

17. Ici $a^+ = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ et $u^- = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ sont les éléments habituels de l'algèbre de Lie $\mathfrak{gl}_2$ de GL_2.

18. Ici $\tilde{P} = \left(\begin{smallmatrix} \mathbf{Z}_p^\times & 0 \\ p\mathbf{Z}_p & 1 \end{smallmatrix}\right)$, est le sous-groupe *non-discret* de $\overline{P}^+$.

Un calcul long et fastidieux mais direct conduit alors au corollaire suivant.

Corollaire 1.9. *Le L-espace vectoriel $H^2(\overline{P}^+, \mathscr{R}_L(\delta_1, \delta_2))$ est de dimension 1.*

Les Corollaires 1.7 et 1.9 permettent d'analyser une suite spectrale et de prouver le Théorème 1.5 dans le cas où A est réduit. On conclut alors par un argument d'induction sur l'indice de nilpotence du nilradical de A. Via le complexe $\mathscr{C}_{u^-,\varphi,a^+}(M)$ nous obtenons également une preuve alternative de [11, Proposition 5.18] dans le cas d'un (φ, Γ)-module cyclotomic. En chemin, nous montrons également un isomorphisme de comparaison entre la cohomologie continue et la cohomologie analytique pour certains (φ, Γ)-modules (cf. Proposition 11.4 pour un énoncé précis).

Cependant, armé avec le Théorème 1.5, le lecteur peut noter à ce stade qu'il y a une absence de théorie nécessaire pour conclure (ou même donner un sens) le Théorème 1.4. Les questions suivantes sont donc inévitables:

Q1 Qu'est-ce qu'un A-module localement convexe?

Q2 Qu'est-ce qu'une G-représentation localement analytique en A-modules?

Q3 Quelle est la relation entre les G-représentations localement analytique en A-modules et les modules munis d'une action (séparément) continue de l'algèbre de distribution relative $\mathscr{D}(G, A)$?

Nous fournissons un ensemble de réponses à ces questions (**A1-A3**) et prouvons certaines propriétés fondamentales concernant la théorie de la cohomologie localement analytique de $\mathscr{D}(G, A)$-modules, que nous décrivons dans la section suivante.

1.4 Familles analytiques de représentations localement analytiques

Rappelons que pour un $\mathbf{Q}_p$-groupe localement analytique H, une théorie des représentations localement analytiques du groupe H en L-espaces vectoriels a été développée par Schneider et Teitelbaum (cf. [53], [52], [54]). Afin de construire le A-module $\Pi(\Delta)$ du Théorème 1.4, muni d'une action de G localement analytique, on est forcé de développer un cadre raisonnable pour donner un sens à un tel objet. Il s'avère que, avec un certain soin, une grande partie de la théorie existante peut être étendue sans difficultés sérieuses au contexte relatif.

Definition 1.10 (A1). *Un A-module localement convexe est un A-module topologique dont la topologie sous-jacente est un localement convexe $\mathbf{Q}_p$-espace vectoriel. On note LCS_A la catégorie des A-modules localement convexes. Ses morphismes sont tous des morphismes A-linaires continus.*

Il existe une notion de dualité forte dans la catégorie LCS_A, cependant en dehors de nos applications, elle est mal-comportée (dans le sens où il y a peu d'objets réflexifs qui ne sont pas des A-modules libres). Soit H un $\mathbf{Q}_p$-groupe localement analytique.

Definition 1.11 (**A2**). *Nous définissons la catégorie* $\mathrm{Rep}^{\mathrm{la}}_A(H)$ *dont les objets sont tonnelés, Hausdorff, localement convexes* A*-modules* M *munis d'une action* A*-linéaire topologique de* H *tel que, pour tout* $m \in M$*, le morphisme d'orbite* $h \mapsto h \cdot m$ *est une fonction localement analytique de* H *à valeur dans* M*.*

On note $\mathrm{LA}(H, A)$ l'espace des fonctions localement analytiques de H à valeur dans A et $\mathscr{D}(H, A) = \mathrm{Hom}_{A,\mathrm{cont}}(\mathrm{LA}(H, A), A)$ (muni d'une topologie forte) son A-dual fort, l'espace des distributions de H à valeur dans A. Tous les deux $LA(H, A)$ et $\mathscr{D}(H, A)$ sont des A-modules localement convexes. Afin d'algebriser la situation, on procède comme dans [53] et montre qu'une représentation localement analytique de H est naturellement un module sur l'algèbre de distribution relative. Plus précisément, soit $\mathrm{Rep}^{\mathrm{la},LB}_A(H) \subseteq \mathrm{Rep}^{\mathrm{la}}_A(H)$ la sous-catégorie pleine se compose d'espaces qui sont de type A-LB et complets (i.e une limite inductive des espaces de Banach dont les morphismes de transition sont A-linéaires). Notre résultat principal dans §A peut être énoncé comme suit:

Théorème 1.12 (**A3**). *Chaque représentation localement analytique de* H *est munie d'une structure séparément continue* A*-linéaire d'un* $\mathscr{D}(H, A)$*-module*[19]*. De plus, la catégorie* $\mathrm{Rep}^{\mathrm{la},LB}_A(H)$ *est équivalente à la catégorie des modules* A*-modules localement convexes Hausdorff qui sont de type* A*-LB munis d'une action* $\mathscr{D}(H, A)$ *séparément continue avec des morphismes continu* $\mathscr{D}(H, A)$*-linéaires.*

L'idée de prouver le Théorème 1.12 est bien sûr de se réduire au résultat bien connu de Schneider-Teitelbaum, cf. [53, Theorem 2.2]. Pour cela, le résultat intermédiaire principal requis est la proposition suivante.

Proposition 1.13. *Il existe un isomorphisme de* A*-modules localement convexes*

$$\mathscr{D}(H, A) = \mathscr{D}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota} A.$$

Dans le cas où H est compact nous montrons que $\mathrm{LA}(H, A)$ satisfait à une propriété *bornée* que nous appelons A-régulière (Nous renvoyons le lecteur à la Definition A.20 et Lemma A.22 pour les énoncés precis). Cela suffit pour prouver la Proposition 1.13.

Remarque 1.14. La Proposition 1.13 suivrait immédiatement si $\mathrm{LA}(H, A)$ est complet (pour H compact). Au mieux de nos connaissances, il semble que cette question soit ouverte si la dimension de $H \geq 2$. Si $H \cong \mathbf{Z}_p$, on peut identifier $\mathrm{LA}(\mathbf{Z}_p, A)$ avec les puissances négatives (de T) dans $\mathscr{R}_A$ et conclure le résultat, cf. Lemma A.14. En particulier $\mathrm{LA}(\mathbf{Z}_p, A)$ est un exemple d'objet A-reflexif, qui n'est pas libre.

Finalement, avec l'équivalence de Théorème 1.12 en tête, nous passons notre attention aux questions cohomologiques concernant la catégorie $\mathrm{Rep}^{\mathrm{la}}_A(H)$.

Definition 1.15. *Soit* $\mathscr{G}_{H,A}$ *la catégorie des* A*-modules localement convexes complet Hausorff munis d'une structure d'un* $\mathscr{D}(H, A)$*-module* A*-linéaire séparément continu, prenant comme morphismes tous les morphismes continus*

19. Plus précisément, un morphisme A -bilineaire séparément continu $\mathscr{D}(H, A) \times M \to M$.

$\mathscr{D}(H,A)$-linéaires. Plus précisément, nous demandons que le morphisme

$$\mathscr{D}(H,A)\times M\to M$$

soit A-bilinéaire est séparément continu.

En suivant Kohlhaase ([43], [63]), on peut développer une théorie de cohomologie localement analytique pour la catégorie $\mathscr{G}_{H,A}$. On peut définir les groupes $H^i_{\mathrm{an}}(H,M)$ et $\mathrm{Ext}^i_{\mathscr{G}_{H,A}}(M,N)$ pour $i\geq 0$ et les objets M et N dans $\mathscr{G}_{H,A}$. Si H_2 est un sous-$\mathbf{Q}_p$-groupe fermé localement analytique de H_1, nous avons aussi un foncteur d'induction [20] $\mathrm{ind}_{H_2}^{H_1}\colon \mathscr{G}_{H_2,A}\to\mathscr{G}_{H_1,A}$. Notre but principal, en considérant une telle théorie, est de montrer la version relative suivante du lemme de Shapiro, qui est crucialement utilisé dans la construction de la correspondance $\Delta\mapsto\Pi(\Delta)$ du Théorème 1.4:

Proposition 1.16 (Relative Shapiro's Lemma)**.** *Soit H_1 un $\mathbf{Q}_p$-groupe localement analytique et soit H_2 un sous-$\mathbf{Q}_p$-groupe fermé localement analytique de H_1. Si M et N sont des objets de $\mathscr{G}_{H_2,A}$ et $\mathscr{G}_{H_1,A}$, respectivement , alors il y a des bijections A-linéaires*

$$\mathrm{Ext}^q_{\mathscr{G}_{H_1,A}}(\mathrm{ind}_{H_2}^{H_1}(M),N)\to\mathrm{Ext}^q_{\mathscr{G}_{H_2,A}}(M,N)$$

pour tout $q\geq 0$.

Donnons un bref aperçu du contenu de la deuxième partie de cette thèse. Dans §10, nous étendons le dictionnaire de l'analyse fonctionnelle p-adique au cadre relatif. Une question clé est d'établir que le faisceau $\mathscr{R}_A(\delta_1)\boxtimes_\omega\mathbf{P}^1$ est G-équivariant sur $\mathbf{P}^1$ et de type $\mathbf{Q}_p$-analytique.

Dans §11, on utilse la (φ,Γ)-cohomologie pour recalculer certains résultats de [4] (dans loc.cit. la (ψ,Γ)-cohomologie a été utilisée). Un résultat clé pour le chapitre suivant est la nullité de $H^2(A^+,\mathscr{R}_A(\delta_1\delta_2^{-1})$ si et seulement si $\delta_1\delta_2^{-1}$ est (point par point) jamais de la forme χx^i ou x^{-i} pour un $i\geq 0$ (i.e. $\delta_1\delta_2^{-1}$ est régulière).

Dans §12 et 13, le cœur technique du papier est réalisé. Nous commençons par prouver la finitude de la $\overline{P}^+$-cohomologie pour $\mathscr{R}_A(\delta_1,\delta_2)$. En utilisant le complexe d'algèbre de Lie nous fournissons une preuve alternative de [11, Proposition 5.18] (dans le cadre cyclotomic). Nous montrons que la dimension du groupe de cohomologie supérieure $H^2(\overline{P}^+,\mathscr{R}_L(\delta_1,\delta_2))$ est constant (de dimension 1) quand $\delta_1\delta_2^{-1}$ est régulière.

Dans §14, le Théorème 1.5 peut alors être établi.

Dans §15, le mécanisme général développé dans [11, §6] est utilisé pour construire $\Pi(\Delta)$ d'un (φ,Γ)-module régulière trianguline de rang 2 Δ, sur $\mathscr{R}_A$.

Dans l'appendice (§A) Nous établissons un cadre formel pour le résultat principal. Nous introduisons la catégorie de G-représentations localement analytiques en A-modules. Nous démontrons qu'il existe une relation entre cette

20. C'est le dual du foncteur d'Induction *standard*, typiquement noté $\mathrm{Ind}_{H_2}^{H_1}$, cf. Lemma A.55.

catégorie et une catégorie de modules sur l'algèbre de distribution relative dans le même esprit de [53]. Il existe une théorie de cohomologie localement analytique qui étend [43] et nous établissons une version relative du lemme de Shapiro. Ces résultats sont utilisés dans §15.

La dette que ce problème doit à Pierre Colmez sera évidente pour le lecteur. Les auteurs lui sont reconnaissants pour avoir suggéré ce problème et le remercient pour de nombreuses discussions sur ces différents aspects. Le premier auteur tient à remercier Jean-François Dat pour son encouragement continu au cours de cette étude. Ensuite, nous voulons remercier Kiran Kedlaya pour avoir passé d'innombrables heures à répondre à nos questions sur les anneaux de Robba et à suggérer un argument d'induction crucial. Nous tenons également à remercier Jean-François Dat, Jan Kohlhaase et Peter Schneider pour plusieurs discussions utiles sur ce que devrait être la catégorie des G-représentations localement analytiques en A-modules. Merci encore à Gabriel Dospinescu et Arthur-César Le Bras pour des conversations fructueuses sur divers sujets.

2 Part I: A counter-example to the Rapoport-Zink conjecture

3 Introduction

In this paper we provide a counter-example to the Rapoport-Zink conjecture [21], cf. [49, §1.37] in the situation

$$G = \mathrm{GL}_2, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}, \quad \mu = \begin{pmatrix} t^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Although the conjecture in loc.cit. is stated in the rigid analytic setting, we will adopt the language of Berkovich spaces. Here b is associated to the isocrystal $\mathbb{Q}_p \oplus \mathbb{Q}_p(2)$ and μ is a non-minuscule cocharacter (defined over $\breve{\mathbb{Q}}_p$) controlling the filtration. Consider the Berkovich affine line $\mathbb{A}^1_{\breve{\mathbb{Q}}_p}$ over the completion of the maximal unramified extension of $\mathbb{Q}_p$, $\breve{\mathbb{Q}}_p$. For every finite extension $L/\breve{\mathbb{Q}}_p$, $\mathbb{A}^1_{\breve{\mathbb{Q}}_p}(L)$ corresponds to the equivalence class of filtrations μ' defined over L (in the conjugacy class of $\{\mu\}$) such that (b, μ') is admissible [22]. In this setup the conjecture roughly states the following (for a precise statement we refer the reader to Conjecture 4.3).

Conjecture 3.1 (Rapoport-Zink)**.** *There exists a $\mathbb{Q}_p$-local system $\mathcal{E}$ over $\mathbb{A}^1_{\breve{\mathbb{Q}}_p}$ satisfying the following property: For any point $\mu' \in \mathbb{A}^1_{\breve{\mathbb{Q}}_p}(L)$ the p-adic Galois representation*

$$\mathrm{Gal}(\overline{L}/L) \xrightarrow{\pi_1^{\mathrm{dJ}}(\mu')} \pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p}) \to GL(\mathbb{Q}_p),$$

where the 2nd map is coming from $\mathcal{E}$, is isomorphic to the crystalline representation determined by (b, μ') [23]*.*

Let us provide a sketch for why Conjecture 3.1 is false. One begins by defining a moduli space (akin to a Rapoport-Zink space) associated to $\mathcal{E}$ which parametrizes the $\mathbb{Z}_p$-lattices in $\mathcal{E}$.

Definition 3.2. *Define a moduli-space $\mathscr{M}(\mathcal{E}) \to \mathbb{A}^1_{\breve{\mathbb{Q}}_p}$, whose values at an $S \to \mathbb{A}^1_{\breve{\mathbb{Q}}_p}$ étale point are given by*

$$\mathscr{M}(\mathcal{E})(S) := \{\mathcal{F}/S \mid \ \mathcal{F} \otimes \mathbb{Q}_p \cong \mathcal{E}|_S\},$$

where $\mathcal{F}$ is a $\mathbb{Z}_p$-local system (a local system of $\mathbb{Z}_p$-lattices in the terminology used by de Jong, cf. [18, §4]).

We call the structural map

$$\pi_{\mathrm{dR}} : \mathscr{M}(\mathcal{E}) \to \mathbb{A}^1_{\breve{\mathbb{Q}}_p}$$

21. Kedlaya announced a proof of a *modified* Rapoport-Zink conjecture in [41].
22. Equivalently by the work of Colmez-Fontaine, cf. [12], the pair (b, μ') gives rise to a crystalline representation.
23. Here $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p})$ is the étale fundamental group of étale covering spaces of $\mathbb{A}^1_{\breve{\mathbb{Q}}_p}$ as defined in [18, §2]. In loc.cit. it is denoted by π_1.

the de-Rham period map. It is not difficult to prove that $\mathscr{M}(\mathcal{E})$ is representable as a $\breve{\mathbb{Q}}_p$-analytic space and π_{dR} is an étale covering map. It is natural to expect that $\mathscr{M}(\mathcal{E})$ has a group structure, but to prove this we need to assume that $\mathcal{E} \in \mathrm{Ext}^1(\mathbb{Q}_p, \mathbb{Q}_p(2))$ and also for any open subgroup $\mathcal{U} \subset \mathbb{A}^1_{\breve{\mathbb{Q}}_p}$,

$$\mathrm{m}^*\,\eta|_{\mathcal{U}} = \mathrm{pr}_1^*\,\mathcal{E}|_{\mathcal{U}} + \mathrm{pr}_2^*\,\mathcal{E}|_{\mathcal{U}}, \tag{2}$$

where $\mathrm{m} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ is the group law of $\mathcal{U}$ and $\mathrm{pr}_{1,2} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ are the first and second projections, respectively [24].

We are then able to work with $\mathscr{M}(\mathcal{E})^{(0,0)}$, the connected component of $\mathscr{M}(\mathcal{E})$ around the identity element. It turns out that $\mathscr{M}(\mathcal{E})^{(0,0)}$ is an analytic $\breve{\mathbb{Q}}_p$-group of p-divisible type in the sense of [26]. Using a classification theorem of analytic $\mathbb{C}_p$-groups of p-divisible type established by Fargues, one is able to show that $\mathcal{E}_{\mathbb{C}_p} \cong \mathcal{E}'_{\mathbb{C}_p}$, where $\mathcal{E}'$ is a certain $\mathbb{Q}_p$-local system over $\mathbb{A}^1_{\breve{\mathbb{Q}}_p}$ which is an extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$. The $\mathbb{Q}_p$-local system $\mathcal{E}'$ is coming from an honest Rapoport-Zink space (the deformation space of an ordinary elliptic curve over $\overline{\mathbb{F}}_p$ to be precise).

Indeed the isomorphism $\mathcal{E}_{\mathbb{C}_p} \cong \mathcal{E}'_{\mathbb{C}_p}$ looks strange. Recall that $\mathcal{E}$ sits in an exact sequence

$$0 \to \mathbb{Q}_p(2) \to \mathcal{E} \to \mathbb{Q}_p \to 0$$

and $\mathcal{E}'$ in

$$0 \to \mathbb{Q}_p(1) \to \mathcal{E}' \to \mathbb{Q}_p \to 0.$$

Let $\rho_1, \rho_2 \colon \pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p}) \to GL_2(\mathbb{Q}_p)$ be the representations corresponding to $\mathcal{E}'$ and $\mathcal{E}$, respectively. Equipping $\mathrm{GL}_2(\mathbb{Q}_p)$ with an action of $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p})$ via conjugation of ρ_2, the key is to consider the pointed set

$$H^1(\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p}), GL_2(\mathbb{Q}_p))$$

and in particular the 1-cocycle given by

$$\begin{aligned} c\colon \pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p}) &\to GL_2(\mathbb{Q}_p) \\ \sigma &\mapsto \rho_1(\sigma)\rho_2(\sigma)^{-1}. \end{aligned}$$

An analysis of c obtains a contradiction for the existence of $\mathcal{E}$.

Finally we should note that in literature, there is enough to show that the Rapoport-Zink conjecture is false in our situation (without even assuming the assumptions we make in this paper). Indeed the existence of such a local system implies by [45, Theorem 1.5(iii)] that the associated filtered module with integrable connection must satisfy Griffith's transversality. But by repeating the proof as in the complex case [19, Proposition 1.1.14], this contradicts that μ is not minuscule, cf. [55, pg. 3]. The interest in the present case is the method of proof.

24. The addition in (2) is viewed as a Baer sum in the abelian group of extensions of $\mathbb{Q}_p$ by $\mathbb{Q}_p(2)$ over $\mathcal{U} \times \mathcal{U}$.

Let us give a brief overview of the structure of the paper.

In §2 we recall the Rapoport-Zink conjecture. We set up the situation which leads to a counter-example and prove that $\mathscr{M}(\mathcal{E})^{(0,0)}$ is a group object in the category of $\breve{\mathbb{Q}}_p$-analytic spaces. Furthermore π_{dR} is a group homomorphism.

In §3 we show that $\mathscr{M}(\mathcal{E})^{(0,0)}$ is of p-divisible type.

In §4 we recall the Rapoport-Zink space deforming an ordinary elliptic curve over $\overline{\mathbb{F}}_p$.

In §5 we prove that it is impossible to have two $\mathbb{Q}_p$-representations of $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p})$, one of which is an extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$ and the other an extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(2)$ which are isomorphic when pulled back to $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\mathbb{C}_p})$.

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4 The Rapoport-Zink conjecture

We provide a counter-example to the Rapoport-Zink conjecture when the cocharacter μ is not minuscule. Let us recall the conjecture, as established for example in their book, cf. [49, §1.37]. We follow closely the recount given by Urs Hartl, cf. [32, Chapter 2].

Let us first recall the definition of filtered isocystals. We denote by $K_0 := W(\overline{\mathbb{F}}_p)[\frac{1}{p}]$, the fraction field of the ring of Witt vectors over $\overline{\mathbb{F}}_p$. Let $\varphi = W(\mathrm{Frob}_p)$ be the Frobenius lift on K_0.

Definition 4.1. *An F-isocrystal over $\overline{\mathbb{F}}_p$ is a finite dimensional K_0-vector space D equipped with a φ-linear automorphism φ_D. If L is a finite field extension of K_0 and $\mathrm{Fil}^\bullet D_L$ is an exhaustive separated decreasing filtration of $D_L := D \otimes_{K_0} L$ by L-subspaces we say that $\underline{D} := (D, \varphi_D, \mathrm{Fil}^\bullet D_L)$ is a filtered isocrystal over L. We let $t_N(\underline{D})$ be the p-adic valuation of $\det \varphi_D$ and we let*

$$t_H(\underline{D}) = \sum_{i \in \mathbb{Z}} i \cdot \dim_L \mathrm{gr}^i_{\mathrm{Fil}^\bullet}(D_L)$$

We now need the notion of weak admissibility, which we just call admissible, after the work of Colmez-Fontaine showed that weakly admissible implies admissible, cf. [12].

Definition 4.2. *The filtered isocrystal $\underline{D}$ is called admissible if*

$$t_H(\underline{D}) = t_N(\underline{D}) \quad \textit{and} \quad t_H(\underline{D}') \leq t_N(\underline{D}')$$

for any subobject $\underline{D}' = (D', \varphi_D|_{D'}, \mathrm{Fil}^\bullet D'_L)$ of $\underline{D}$, where D' is any φ_D-stable K_0-subspace of D which is equipped with the induced filtration $\mathrm{Fil}^i D'_L = D'_L \cap \mathrm{Fil}^i D_L$ on $D'_L := D' \otimes_{K_0} L$.

To construct the first ingredient in the Rapoport-Zink conjecture, namely period spaces, let G be a reductive linear algebraic group over $\mathbb{Q}_p$. Fix a conjugacy class $\{\mu\}$ of cocharacters

$$\mu : \mathbb{G}_m \to G$$

defined over subfields of $\mathbb{C}_p$. Let E be the field of definition of the conjugacy class. Then E is a finite extension of $\mathbb{Q}_p$. Two cocharacters in this conjugacy class are called equivalent if they induce the same weight filtration on the category $\mathrm{Rep}_{\mathbb{Q}_p} G$ of finite dimensional $\mathbb{Q}_p$-rational representations of G. There is a projective variety $\mathcal{F}$ over E whose $\mathbb{C}_p$-valued points are in bijection with the equivalence classes of cocharacters (from the fixed conjugacy class of $\{\mu\}$). Namely for $V \in \mathrm{Rep}_{\mathbb{Q}_p} G$ and a cocharacter μ defined over L, one associates a filtration

$$\mathrm{Fil}^i_\mu V_L := \bigoplus_{j \geq i} V_{L,j}$$

of $V_L := V \otimes_{\mathbb{Q}_p} L$ given by the weight spaces

$$V_{L,j} := \left\{ v \in V_L | \ \mu(z) \cdot v = z^j v \text{ for all } z \in \mathbb{G}_m(L) \right\}.$$

This defined a closed embedding of $\mathcal{F}$ into a partial flag variety of V

$$\mathcal{F} \hookrightarrow \mathrm{Flag}(V) \otimes_{\mathbb{Q}_p} E,$$

where the points of $\mathrm{Flag}(V)$ evaluated at a $\mathbb{Q}_p$-algebra R are the filtrations F^i of $V \otimes_{\mathbb{Q}_p} R$ by R-submodules which are direct summands such that $\mathrm{rk}_R \, \mathrm{gr}^r_{F^\bullet}$ is the multiplicity of the weight i of the conjugacy class $\{\mu\}$ on V.

A pair (b, μ) with an element $b \in G(K_0)$ and a cocharacter $\mu : \mathbb{G}_m \to G$ defined over L/K_0 is called admissible if for some faithful representation $\rho : G \hookrightarrow \mathrm{GL}(V)$ in $\mathrm{Rep}_{\mathbb{Q}_p} G$ the filtered isocrystal

$$D_{b,\mu}(V) := (V \otimes_{\mathbb{Q}_p} L, \rho(b) \cdot \varphi, \mathrm{Fil}^\bullet_\mu V_L)$$

is admissible (we denote by $D_b(V)$ the isocrystal $(V \otimes_{\mathbb{Q}_p} L, \rho(b) \cdot \varphi)$). In fact this holds for any $V \in \mathrm{Rep}_{\mathbb{Q}_p} G$, cf. [49, Definition 1.18]. By the work of Colmez-Fontaine, the filtered isocrystal $D_{b,\mu}(V)$ is admissible iff it arises from a crystalline Galois representation $\mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}(U)$ via Fontaine's covariant functor

$$D_{b,\mu}(V) \cong D_{\mathrm{cris}}(U) := (U \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\mathrm{Gal}(\overline{L}/L)}.$$

This is an equivalence of categories from crystalline representations of $\mathrm{Gal}(\overline{L}/L)$ to admissible filtered isocrystals over L. We denote $V_{\mathrm{cris}}(D_{b,\mu}(V)) := U$ for Fontaine's covariant inverse functor. The assignment

$$\mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Rep}_{\mathbb{Q}_p}(\overline{L}/L), \quad V \mapsto V_{\mathrm{cris}}(D_{b,\mu}(V))$$

defines a tensor functor from $\mathrm{Rep}_{\mathbb{Q}_p} G$ to the category of continuous $\mathrm{Gal}(\overline{L}/L)$-representations in finite dimensional $\mathbb{Q}_p$-vector spaces.

Let $\breve{E} = EK_0$ be the completion of the maximal unramified extension of E. In what follows we consider cocharacters μ defined over complete extensions L of $\breve{E}$. Let $\breve{\mathcal{F}}^{\mathrm{an}}$ be the $\breve{E}$-analytic space associated with the variety $\breve{\mathcal{F}} := \mathcal{F} \otimes_E \breve{E}$. Rapoport and Zink define the p-adic period space associated with $(G, b, \{\mu\})$ as

$$\breve{\mathcal{F}}_b^{\mathrm{wa}} := \{\mu \in \breve{\mathcal{F}}^{\mathrm{an}} | \ (b, \mu) \text{ is admissible}\}$$

We are now ready to state the Rapoport-Zink conjecture in the form that we are interested in.

Conjecture 4.3. *There exists a unique largest arcwise connected dense open $\breve{E}$-analytic subspace $\breve{\mathcal{F}}_b^a \subset \breve{\mathcal{F}}_b^{\mathrm{wa}}$ invariant under $J(\mathbb{Q}_p)$ with $\breve{\mathcal{F}}_b^a(L) = \breve{\mathcal{F}}_b^{\mathrm{wa}}(L)$ for all finite extensions $L/\breve{E}$ and a tensor functor*

$$\underline{V} : \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathbb{Q}_p - \mathrm{Loc}_{\breve{\mathcal{F}}_b^a}$$

(where $\mathbb{Q}_p - \mathrm{Loc}_{\breve{\mathcal{F}}_b^a}$ is the category of local systems of $\mathbb{Q}_p$-vector spaces on $\breve{\mathcal{F}}_b^a$) with the following property: For any point $\mu \in \breve{\mathcal{F}}_b^a(L)$ with $L/\breve{E}$ finite, the tensor functor

$$\mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Rep}_{\mathbb{Q}_p}(\overline{L}/L), \quad V \mapsto V_{\mathrm{cris}}(D_{b,\mu}(V))$$

is isomorphic to the tensor functor

$$V \mapsto \underline{V}(V)_{\overline{\mu}}$$

which associates to a representation $V \in \mathrm{Rep}_{\mathbb{Q}_p} G$ the geometric fiber at μ of the corresponding local system $\underline{V}(V)$.

There has been some evidence to suggest that the conjecture is false when μ is not minuscule. Indeed for what follows we fix

$$G = \mathrm{GL}_2, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}, \quad \mu = \begin{pmatrix} t^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now $\mathrm{Rep}_{\mathbb{Q}_p} \mathrm{GL}_2$ is a tannakian category with tensor generator the standard representation. This means that any tensor functor $\underline{V} : \mathrm{Rep}_{\mathbb{Q}_p} \mathrm{GL}_2 \to \mathbb{Q}_p - \mathrm{Loc}_{\breve{\mathcal{F}}_b^{\mathrm{a}}}$ is determined by its value on the standard representation.

We want to show first that $\breve{\mathcal{F}}_b^{\mathrm{wa}} = \mathbb{A}^{1,\mathrm{an}}$. We take as the model filtration:

$$(0) = \mathrm{Fil}^3_\mu \subset \mathrm{Fil}^2_\mu = \mathrm{Fil}^1_\mu \subset \mathrm{Fil}^0_\mu = \breve{\mathbb{Q}}_p^2$$

and the isocrystal corresponding to b (associated to the standard representation) is $\mathbb{Q}_p \oplus \mathbb{Q}_p(2)$. We denote the canonical basis for $\mathbb{Q}_p^2$ by e_1, e_2 and let

$$V_- =< e_1 >, \quad V_+ =< e_2 >,$$

so that $b = p^2 \cdot \mathrm{id}_{V_-} \oplus \mathrm{id}_{V_+}$. Then the slope decomposition of the isocrystal, $D_b(\mathbb{Q}_p^2)$, associated to the standard representation has the form $D_b(\mathbb{Q}_p^2) = N_0 \oplus N_1$ with

$$N_0 = V_+ \otimes K_0, \quad N_1 = V_- \otimes K_0.$$

We consider the space $\mathscr{F} := \mathrm{Grass}_1(\mathbb{Q}_p^2)$ of subspaces F of dimension 1 (keeping track of $\mathrm{Fil}^2_\mu = \mathrm{Fil}^1_\mu$). It is the easy to see that F (considered as an L-point of $\mathscr{F}$) is admissible iff

$$F \cap (V_+ \otimes L) = (0).$$

That is it may be identified with an affine space of dimension 1, i.e. $\breve{\mathcal{F}}_b^{\mathrm{wa}} = \mathbb{A}^{1,\mathrm{an}}$, as promised.

Since the isocrystal corresponding to b is an extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(2)$, it is natural to assume the same for $\mathcal{E}$, that is there is an exact sequence:

Additional assumption 1.

$$\eta: \ 0 \to \mathbb{Q}_p(2) \to \mathcal{E} \to \mathbb{Q}_p \to 0,$$

where $\eta \in \mathrm{Ext}^1(\mathbb{Q}_p, \mathbb{Q}_p(2))$.

Next we show that we don't lose generality if we suppose that $\breve{\mathcal{F}}_b^a = \mathbb{A}^{1,\mathrm{an}}$, given the following additional assumption on $\mathcal{E} := \underline{V}$(standard representation):

Additional assumption 2. *For any open subgroup* $\mathcal{U} \subset \breve{\mathcal{F}}_b^a$,

$$\mathrm{m}^* \, \eta|_{\mathcal{U}} = \mathrm{pr}_1^* \, \eta|_{\mathcal{U}} + \mathrm{pr}_2^* \, \eta|_{\mathcal{U}},$$

where $\mathrm{m} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ *is the group law of* $\mathcal{U}$ *and* $\mathrm{pr}_{1,2} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ *are the first and second projections respectively. The addition is viewed as a Baer sum in the abelian group of extensions of* $\mathbb{Q}_p$ *by* $\mathbb{Q}_p(2)$

Remark 4.4. The above identity in $\mathrm{Ext}^1(\mathbb{Q}_p, \mathbb{Q}_p(2))$, where $\mathbb{Q}_p$ and $\mathbb{Q}_p(2)$ are viewed as $\mathbb{Q}_p$-local systems over $\mathcal{U} \times \mathcal{U}$, implies that there is an isomorphism

$$\mathrm{m}^* \, \mathcal{E}|_{\mathcal{U}} \cong \mathrm{pr}_1^* \, \mathcal{E}|_{\mathcal{U}} \oplus \mathrm{pr}_2^* \, \mathcal{E}|_{\mathcal{U}}$$

and the fact that $\mathrm{Hom}(\mathbb{Q}_p, \mathbb{Q}_p(2)) = 0$, implies that this isomorphism is unique.

Indeed given the above assumption on $\mathcal{E}$, we can take in particular $\mathcal{U} = B(0, \epsilon)$ for some $\epsilon > 0$ small enough, where $B(0, \epsilon)$ is the open ball around 0 of radius ϵ. We consider the multiplication by p map:

$$p : B(0, p\epsilon) \to B(0, \epsilon).$$

We see that if $\mathcal{E}$ is defined on $B(0, \epsilon)$ satisfying Conjecture 4.3 and Additional assumption 2 then so does $p^*\mathcal{E}$ defined over $B(0, p\epsilon)$. Repeating gives a $\mathbb{Q}_p$-local system, $\varprojlim_n p^{n*}\mathcal{E}$ defined over $\varprojlim_n B(0, p^n\epsilon) = \mathbb{A}^{1,\mathrm{an}}$. Thus we may suppose that $\breve{\mathcal{F}}_b^a = \mathbb{A}^{1,\mathrm{an}}$.

We now define a moduli-space which parametrizes the $\mathbb{Z}_p$-lattices in $\mathcal{E}$.

Definition 4.5. *Define a moduli-space* $\mathscr{M} \to \mathbb{A}^{1,\mathrm{an}}$, *whose values at an* $S \to \mathbb{A}^{1,\mathrm{an}}$ *étale point are given by*

$$\mathscr{M}(S) := \{\mathcal{F}/S \mid \ \mathcal{F} \otimes \mathbb{Q}_p \cong \mathcal{E}|_S\},$$

where $\mathcal{F}$ is a $\mathbb{Z}_p$-local system (a local system of $\mathbb{Z}_p$-lattices in the terminology used by de Jong, cf. [18, §4]). We call the structural map

$$\pi_{\mathrm{dR}} : \mathscr{M} \to \mathbb{A}^{1,\mathrm{an}}$$

the de-Rham period map, in analogy with the classical Gross-Hopkins period map, $\pi_{\mathrm{GH}} : \mathscr{M}_{\mathrm{LT}} \to \mathbb{P}^{1,\mathrm{an}}$.

The following proposition shows that $\mathscr{M}$ is representable by an analytic space.

Proposition 4.6. *The space $\mathscr{M}$ (which we denote by $\mathscr{M}(\mathcal{E})$ if necessary) as defined above is representable by an étale covering space over $\mathbb{A}^{1,\mathrm{an}}$. In particular the period morphism $\pi_{\mathrm{dR}} : \mathscr{M} \to \mathbb{A}^{1,\mathrm{an}}$ is étale and surjective.*

Proof. We note that $\mathbb{Z}_p - \underline{\mathrm{Loc}}$ is a stack over the category of k-analytic spaces with the étale topology (here we can take $k = \breve{\mathbb{Q}}_p$). This shows that $\mathscr{M}$ is at least a sheaf. Also $\mathcal{E}$ is given by the triple

$$\mathcal{E} = \left(\left\{U_i \to \mathbb{A}^{1,\mathrm{an}}\right\}, \mathcal{F}_i, \phi_{ij}\right)$$

where

1. $\left\{U_i \to \mathbb{A}^{1,\mathrm{an}}\right\}$ is an étale covering of $\mathbb{A}^{1,\mathrm{an}}$,
2. for each i there is given an object $\mathcal{F}_i \in \mathbb{Z}_p - \underline{\mathrm{Loc}}_{U_i}$,
3. for each pair i, j there is given an isomorphism $\phi_{ij} : \mathcal{F}_i|_{U_i \times_{\mathbb{A}^{1,\mathrm{an}}} U_j} \to \mathcal{F}_j|_{U_i \times_{\mathbb{A}^{1,\mathrm{an}}} U_j}$ in the fibre category of $\mathbb{Z}_p - \underline{\mathrm{Loc}} \otimes \mathbb{Q}_p$ over $U_i \times_{\mathbb{A}^{1,\mathrm{an}}} U_j$.

These data are subject to the cocycle condition $\mathrm{pr}_{ij}^*(\phi_{ij}) \circ \mathrm{pr}_{jk}^*(\phi_{jk}) = \mathrm{pr}_{ik}^*(\phi_{ik})$ on the triple product $U_i \times_{\mathbb{A}^{1,\mathrm{an}}} U_j \times_{\mathbb{A}^{1,\mathrm{an}}} U_k$. It follows immediately that $\mathscr{M}|_{U_i}$ is representable by a disjoint union of spaces finite étale over U_i. Hence [18, Lemma 2.3], implies that $\mathscr{M}$ is representable by an étale covering space (which we also denote by $\mathscr{M}$) over $\mathbb{A}^{1,\mathrm{an}}$.

□

For what follows we surpress 'an' from the notation (for example $\mathbb{A}^1$ will mean $\mathbb{A}^{1,\mathrm{an}}$). We note that there is a morphism

$$\mathscr{M}(\mathcal{E}) \to \coprod_{\mathbb{Z}^2} \mathrm{Sp}(\breve{\mathbb{Q}}_p).$$

The construction of this morphism is simple. Denote by u the morphism $\mathcal{E} \to \mathbb{Q}_p$ coming from the extension defining $\mathcal{E}$. If $S \to \mathbb{A}^1$ and $\mathcal{F} \in \mathscr{M}(\mathcal{E})(S)$, then $\mathcal{F} \cap \mathbb{Q}_p(2)$ is a lattice in $\mathbb{Q}_p(2)$ and $u(\mathcal{F})$ is a lattice in $\mathbb{Q}_p$. Denote the index of $\mathcal{F} \cap \mathbb{Q}_p(2)$ relative to $\mathbb{Z}_p(2)$ by a and the index of $u(\mathcal{F})$ relative to $\mathbb{Z}_p$ by b. The pair (a, b) then define the morphism. We can therefore rewrite

$$\mathscr{M}(\mathcal{E}) = \coprod_{(a,b) \in \mathbb{Z}^2} \mathscr{M}(\mathcal{E})^{(a,b)},$$

where $\mathscr{M}(\mathcal{E})^{(a,b)}$ is defined as the fiber of (a, b) under the above morphism.

Proposition 4.7. *$\mathscr{M}(\mathcal{E})^{(0,0)}$ is connected and is a group in the category of $\mathrm{Sp}(\breve{\mathbb{Q}}_p)$-Berkovich spaces. Furthermore the period morphism*

$$\pi_{\mathrm{dR}}\big|_{\mathscr{M}(\mathcal{E})^{(0,0)}} : \mathscr{M}(\mathcal{E})^{(0,0)} \to \mathbb{G}_a,$$

where $\mathbb{A}^1$ is identified with group structure $\mathbb{G}_a$, is a morphism of groups.

Proof. We denote by $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1)$ the étale fundamental group due to de Jong (pointed above $0 \in \mathbb{A}^1$), cf. [18] (there he simply denotes it by π_1). Recall that $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1)$ calculates all étale covers of the base space $\mathbb{A}^1$. The structure morphism $\mathbb{A}^1 \to \mathrm{Sp}(\breve{\mathbb{Q}}_p)$, defines a morphism

$$\pi_1^{\mathrm{dJ}}(\mathbb{A}^1) \to \mathrm{Gal}(\overline{\mathbb{Q}}_p/\breve{\mathbb{Q}}_p),$$

which when composed with the cyclotomic character gives a character χ of $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1)$. The character χ corresponds to the $\mathbb{Q}_p$-local system $\mathbb{Q}_p(1)$ over $\mathbb{A}^1$. Thus $\mathcal{E}$ is given by a morphism

$$\rho : \pi_1^{\mathrm{dJ}}(\mathbb{A}^1) \to \begin{pmatrix} \chi^2 & \alpha \\ 0 & 1 \end{pmatrix},$$

where $\alpha \in Z^1(\pi_1^{\mathrm{dJ}}(\mathbb{A}^1), \mathbb{Q}_p(2))$ is a continuous 1-cocycle (to see this, the fact that ρ is a homomorphism implies α is a 1-cocycle with values in $\mathbb{Q}_p(2)$ and together with ρ being continuous forces α to be a continuous 1-cocycle). We therefore have

$$\pi_0(\mathscr{M}(\mathcal{E})) = \mathrm{Im}(\rho)\backslash \mathrm{GL}_2(\mathbb{Q}_p)/\mathrm{GL}_2(\mathbb{Z}_p).$$

This follows from the fact that the number of orbits of the action of $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1)$ on $\mathscr{M}(\mathcal{E})_{\overline{0}}$ is equal to the number of connected components of $\mathscr{M}(\mathcal{E})$, cf. [18, Theorem 2.10]. We first prove that $\mathscr{M}(\mathcal{E})^{(0,0)}$ is connected. It suffices to show that $\mathrm{Im}(\alpha) = \mathbb{Q}_p$. Recall that for each finite extension $K/\breve{\mathbb{Q}}_p$, the fiber of $\mathcal{E}$ at $x \in \mathbb{A}^1(K)$ is given by the crystalline galois representation fabricated by the Colmez-Fontaine functor. For example at $x = 0$, this is just $\mathbb{Q}_p \oplus \mathbb{Q}_p(2)$. The point $x \in \mathbb{A}^1(K)$ gives a section

$$s_x : \mathrm{Gal}(\overline{K}/K) \to \pi_1^{\mathrm{dJ}}(\mathbb{A}^1).$$

For $x \neq 0$, we claim that the image of $\alpha \circ s_x$ is an open subgroup of $\mathbb{Q}_p$. Indeed $\rho \circ s_x$ is a group homomorphism from a compact group (namely $\mathrm{Gal}(\overline{K}/K)$) to the standard Borel subgroup $B(\mathbb{Q}_p)$. The image is thus a closed subgroup of $B(\mathbb{Q}_p)$ and hence a Lie subgroup. The unipotent part corresponds to $\alpha \circ s_x$ (which is non-trivial as $x \neq 0$) and thus has image, an open subgroup of $\mathbb{Q}_p$. To finish the proof of connectedness, we make use of the morphism

$$\times p : \mathbb{A}^1 \to \mathbb{A}^1.$$

This corresponds to an action of multiplying the image of $\alpha \circ s_x$ by p. This is enough to conclude that $\mathrm{Im}(\alpha) = \mathbb{Q}_p$.

We will now show that $\mathscr{M}(\mathcal{E})^{(0,0)}$ is a group object and the period morphism π_{dR} is a morphism of groups with $\mathbb{A}^1$ identified with $\mathbb{G}_a$. It is important to note that we are working in the category of $\mathrm{Sp}(\breve{\mathbb{Q}}_p)$-Berkovich spaces (in particular our morphisms to $\mathbb{A}^1$ are not fixed). We first describe the group law. Fix two morphisms $u : S \to \mathbb{A}^1$ and $v : S \to \mathbb{A}^1$. Given two points

$$\mathcal{F}_1 \in \mathscr{M}(\mathcal{E})^{(0,0)}(u : S \to \mathbb{A}^1) \text{ and } \mathcal{F}_2 \in \mathscr{M}(\mathcal{E})^{(0,0)}(v : S \to \mathbb{A}^1)$$

we note that both $\mathcal{F}_1$ and $\mathcal{F}_2$ are extensions of $\mathbb{Z}_p$ by $\mathbb{Z}_p(2)$. We can thus produce their Baer sum $\mathcal{F} := \mathcal{F}_1 + \mathcal{F}_2$. Explicitly, if we have

$$0 \to \mathbb{Z}_p(2) \xrightarrow{h} \mathcal{F}_1 \xrightarrow{g} \mathbb{Z}_p \to 0$$
$$0 \to \mathbb{Z}_p(2) \xrightarrow{h'} \mathcal{F}_2 \xrightarrow{g'} \mathbb{Z}_p \to 0,$$

then $\mathcal{F}$ is given by

$$\mathcal{F} := \{(f, f') \in \mathcal{F} \oplus \mathcal{F}' |\ g(f) = g'(f')\} \,/\, \{(h(b), 0) - (0, h'(b))|\ b \in \mathbb{Z}_p(2)\}\,.$$

Now we have

$$\mathcal{F}_1 \left[\frac{1}{p}\right] \cong u^*\mathcal{E} \text{ and } \mathcal{F}_2 \left[\frac{1}{p}\right] \cong v^*\mathcal{E}$$

Denote by $m : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ the addition law on $\mathbb{G}_a$ and $\mathrm{pr}_{1,2} : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ the two projections. We see that

$$\begin{aligned}\mathcal{F}\left[\frac{1}{p}\right] &= \mathcal{F}_1\left[\frac{1}{p}\right] + \mathcal{F}_2\left[\frac{1}{p}\right] \\ &= u^*\mathcal{E} + v^*\mathcal{E} \\ &= (\mathrm{pr}_1 \circ u \times v)^*\mathcal{E} + (\mathrm{pr}_2 \circ u \times v)^*\mathcal{E} \\ &= (u \times v)^*(\mathrm{pr}_1^*\mathcal{E} + \mathrm{pr}_2^*\mathcal{E}) \\ &= (u \times v)^* m^*\mathcal{E} \\ &= (u+v)^*\mathcal{E},\end{aligned}$$

where $u+v$ is the addition law under $m : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$. The above calculation shows that

$$\mathcal{F} \in \mathscr{M}(\mathcal{E})^{(0,0)}(u + v : S \to \mathbb{A}^1). \tag{3}$$

This gives us the required group law. The existence of an inverse follows from $\mathrm{Ext}^1(\mathbb{Z}_p, \mathbb{Z}_p(2))$ being a group. The canonical section $0 : \mathrm{Sp}(\breve{\mathbb{Q}}_p) \to \mathbb{A}^1$ forces the crystalline galois representation of $\mathcal{E}_0$ to be precisely $\mathbb{Q}_p \oplus \mathbb{Q}_p(2)$. This has a canonical lattice $\mathbb{Z}_p \oplus \mathbb{Z}_p(2)$ providing the identity section for $\mathscr{M}(\mathcal{E})^{(0,0)}$. Finally π_{dR} is a morphism of groups follows from (3). □

5 Analytic groups of p-divisible type

In this section we recall Fargue's notion of analytic groups of p-divisible type and prove that $\mathscr{M}(\mathcal{E})^{(0,0)}$ is such an object. For brevity we abuse notation slightly and write π_{dR} for $\pi_{\mathrm{dR}}\big|_{\mathscr{M}(\mathcal{E})^{(0,0)}}$. For an extensive study, we refer the reader to [26]. Fix a field K of complete valuation of rank 1, an extension of $\mathbb{Q}_p$ (e.g. $\breve{\mathbb{Q}}_p$).

Definition 5.1. *An analytic K-group of p-divisible type is an analytic K-group commutative G such that*

1. *The multiplication by p, $G \xrightarrow{\times p} G$, is a surjective and finite morphism.*
2. *If $|G|$ denotes the underlying topological space, then for all $x \in |G|$,*

$$\lim_{n\to+\infty} p^n x = 0.$$

Remark 5.2. Fargues's original definition is in the rigid-analytic setting. Here we work directly in the Berkovich setting. In our particular case, this creates no problems. We also have an equivalence of categoies:

$$\begin{aligned}&\{\text{paracompact hausdorff strictly } K-\text{analytic Berkovich spaces}\}\\ &\qquad\cong \{\text{quasi-paracompact qs rigid-analytic varieties}/K\}.\end{aligned}$$

such that if X is a quasi-paracompact qs rigid-analytic variety/K and X^{an} is its associated Berkovich space, then the underlying set of X is the set of points $x \in X^{\mathrm{an}}$ such that $[\mathscr{H}(x) : K] < \infty$.

We show that $\mathscr{M}(\mathcal{E})^{(0,0)}$ is an analytic $\breve{\mathbb{Q}}_p$-group of p-divisible type. The last section shows that is an analytic $\breve{\mathbb{Q}}_p$-group. Moreover by construction of the group law it is clearly also commutative.

Remark 5.3. The multiplication by p map, $p : \mathscr{M}(\mathcal{E})^{(0,0)}(\overline{\breve{\mathbb{Q}}}_p) \to \mathscr{M}(\mathcal{E})^{(0,0)}(\overline{\breve{\mathbb{Q}}}_p)$ is easily seen have closed image on the (classical) Tate points. That is those points whose residue field is a finite extension of $\breve{\mathbb{Q}}_p$. Indeed suppose that we have a sequence $px_i \xrightarrow{i\to\infty} x$ where x_i and x are such points. It suffices to show that $x = pz$ for some z. Now since π_{dR} is an étale covering map, it follows that it is an isomorphism around the identity element. That is there exists $V \subset \mathscr{M}(\mathcal{E})^{(0,0)}$ an analytic subgroup containing the identity element such that $\pi_{\mathrm{dR}}|_V : V \to (\mathbb{B}^\circ(0,\epsilon),+)$. Thus for $i,j > N$ for some $N > 0$, we have that $p(x_i - x_j) \in V^{\mathrm{rig}}$ and hence $x_i - x_j \in V^{\mathrm{rig}}$. It follows that $x_i \xrightarrow{i\to\infty} z$ for some z and hence $x = pz$.

We begin by showing that $\ker\pi_{\mathrm{dR}}$ is a p-divisible group. Note first that $\ker\pi_{\mathrm{dR}}$ is a $\breve{\mathbb{Q}}_p$-analytic group étale over $\breve{\mathbb{Q}}_p$. Thus it is of dimension 0 and is determined by its classical points. Now $\ker\pi_{\mathrm{dR}}$ is a p-divisible group iff it is p^∞-torsion (i.e. $\ker\pi_{\mathrm{dR}} = \ker\pi_{\mathrm{dR}}[p^\infty]$), $\ker\pi_{\mathrm{dR}}[p]$ is finite (cf. Lemma 5.6) and the multiplication by p on $\ker\pi_{\mathrm{dR}}$ is surjective (cf. Lemma 5.9). Note however for K a finite extension of $\breve{\mathbb{Q}}_p$

$$\ker\pi_{\mathrm{dR}}(K) = \ker(H^1(\mathrm{Gal}(\overline{K}/K), \mathbb{Z}_p(2)) \to H^1(\mathrm{Gal}(\overline{K}/K), \mathbb{Q}_p(2))),$$

where the map on the RHS is the obvious one, sending a representation V to $V \otimes \mathbb{Q}_p$. Finally since

$$H^1(\mathrm{Gal}(\overline{K}/K), \mathbb{Z}_p(2)) \left[\frac{1}{p}\right] = H^1(\mathrm{Gal}(\overline{K}/K), \mathbb{Q}_p(2)),$$

it follows that $\ker \pi_{\mathrm{dR}}(K) = \ker \pi_{\mathrm{dR}}[p^\infty](K)$ for every finite extension $K/\breve{\mathbb{Q}}_p$ and so $\ker \pi_{\mathrm{dR}} = \ker \pi_{\mathrm{dR}}[p^\infty]$, as desired. Together with Lemmas 5.6 and 5.9 we have proved the following theorem:

Theorem 5.4. *We have the following exact sequence:*

$$O \longrightarrow \Gamma \longrightarrow \mathscr{M}(\mathcal{E})^{(0,0)} \xrightarrow{\pi_{\mathrm{dR}}} \mathbb{A}^1_{\breve{\mathbb{Q}}_p} \longrightarrow 0,$$

where Γ is a p-divisible group over $\breve{\mathbb{Q}}_p$, π_{dR} is an étale surjection such that $\ker \pi_{\mathrm{dR}}$ *identifies with Γ.*

Proof. See the above paragraph ☐

Corollary 5.5. *$\mathscr{M}(\mathcal{E})^{(0,0)}$ is an analytic $\breve{\mathbb{Q}}_p$-group of p-divisible type.*

Proof. This follows from Theorem 5.4 and [26, Proposition 18]. ☐

It remains to prove $\ker \pi_{\mathrm{dR}}[p^\infty]$ is a p-divisible group. We have the following commutative diagram,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \ker \pi_{\mathrm{dR}}[p] \arrow[r] \arrow[d] & \mathscr{M}(\mathcal{E})^{(0,0)}[p] \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \ker \pi_{\mathrm{dR}} \arrow[r] \arrow[d, "\times p"] & \mathscr{M}(\mathcal{E})^{(0,0)} \arrow[r, "\pi_{\mathrm{dR}}"] \arrow[d, "\times p"] & \mathbb{A}^1 \arrow[d, "\times p"] \\
0 \arrow[r] & \ker \pi_{\mathrm{dR}} \arrow[r] & \mathscr{M}(\mathcal{E})^{(0,0)} \arrow[r, "\pi_{\mathrm{dR}}"] & \mathbb{A}^1
\end{tikzcd}$$

where all columns are left exact and the two bottom rows are left exact. Thus by a version of the nine-lemma, the top row is also left exact giving:

$$\ker \pi_{\mathrm{dR}}[p] \cong \mathscr{M}(\mathcal{E})^{(0,0)}[p].$$

Now for $[K : \breve{\mathbb{Q}}_p] < \infty$, it is clear that $\ker \pi_{\mathrm{dR}}(K)$ denotes the group of $\mathrm{Gal}(\overline{K}/K)$-crystalline representations which are extensions of $\mathbb{Z}_p$ by $\mathbb{Z}_p(2)$, whose associated $\mathbb{Q}_p$-representation is $\mathbb{Q}_p \oplus \mathbb{Q}_p(2)$ (the group law is given by the Baer sum on $\mathrm{Ext}^1(\mathbb{Z}_p, \mathbb{Z}_p(2))$, induced by the group law on $\mathscr{M}(\mathcal{E})^{(0,0)}(K))$. Thus we have an isomorphism of groups

$$\ker \pi_{\mathrm{dR}}[p](K) \cong H^1(\mathrm{Gal}(\overline{K}/K), \mathbb{Z}_p(2))[p].$$

We now give two proofs of the following lemma. The first is an indirect calculation on (φ, Γ)-modules. The second is more direct.

Lemma 5.6. *Given K as above and a p-adic $\mathrm{Gal}(\overline{K}/K)$-representation V, in $\mathbb{Z}_p$-modules, the p-torsion of $H^1(\mathrm{Gal}(\overline{K}/K), V)$ is finite and bounded as $[K : \breve{\mathbb{Q}}_p]$ increases. Thus in particular*

$$\ker \pi_{\mathrm{dR}}[p](\overline{\breve{\mathbb{Q}}}_p) = \varinjlim_K \ker \pi_{\mathrm{dR}}[p](K)$$

is finite.

Proof. Let $D := D(V)$ be the (φ, Γ)-module associated to V. By the thesis of Laurent Herr, cf. [34], we can commute the Galois cohomology of V directly in terms of D using the Herr complex ($C_{\psi,\gamma_K}(K, V)$ concentrated in degrees $[-1, 3]$):

$$0 \longrightarrow D \xrightarrow{(\psi-1,\gamma_K-1)} D \oplus D \xrightarrow{(\gamma_K-1)\,\mathrm{pr}_1-(\psi-1)\,\mathrm{pr}_2} D \longrightarrow 0,$$

where γ_K is a generator of $\Gamma_K := \mathrm{Gal}(K(\mu_{p^\infty})/K)$.

Theorem 5.7. $H^i(C_{\psi,\gamma_K}(K, V)) \cong H^i(\mathrm{Gal}(\overline{K}/K), V)$ *for all* $i \in \mathbb{N}$

Proof. This is proven in [34, Théorème 2.1 and Proposition 4.1]. ☐

We get the induced exact sequence (cf. [25, Lemme I.5.5]):

$$0 \longrightarrow \frac{D^{\psi=1}}{\gamma_K - 1} \longrightarrow H^1(\mathrm{Gal}(\overline{K}/K), V) \longrightarrow \left(\frac{D}{\psi - 1}\right)^{\Gamma_K} \longrightarrow 0.$$

The interest in this exact sequence is that the modules $D^{\psi=1}$ and $\frac{D}{\psi-1}$ have a natural interpretation in Iwasawa theory.

Proposition 5.8. $\frac{D}{\psi-1}$ *is a $\mathbb{Z}_p$-module of finte rank and no p-torsion.*

Proof. This is proven in [25, Proposition I.5.6 and Lemme I.7.1]. ☐

Thus regarding the question of p-torsion, we are reduced to the study of $\frac{D^{\psi=1}}{\gamma_K-1}$. We have the exact sequence:

$$0 \longrightarrow D^{\varphi=1} \longrightarrow D^{\psi=1} \xrightarrow{1-\varphi} \mathscr{C} \longrightarrow 0,$$

where the module $\mathscr{C} := (1-\varphi)D^{\psi=1}$ is free of rank $2[K : \breve{\mathbb{Q}}_p]$ over $\mathbb{Z}_p[[\Gamma_K]]$, cf. [6, Proposition 6.3.2]. This induces an exact sequence

$$0 \longrightarrow \frac{D^{\varphi=1}}{\gamma_K - 1} \longrightarrow \frac{D^{\psi=1}}{\gamma_K - 1} \longrightarrow \frac{\mathscr{C}}{\gamma_K - 1} \longrightarrow 0,$$

coming from taking the long exact sequence of Γ_K-cohomology and noting that Γ_K is pro-cyclic. Now $\mathbb{Z}_p[[\Gamma_K]] = \mathbb{Z}_p[\Delta] \otimes \mathbb{Z}_p[[T]]$, where Δ is the torsion submodule of Γ_K. Note also that the action of γ_K on $\mathscr{C}$ corresponds to multiplication by $T+1$. Thus the module $\frac{\mathscr{C}}{\gamma_K - 1}$ has no p-torsion. Finally, using freely the notation of Fontaine (cf. [25, §I]), we compute: $D^{\varphi=1} = (\mathbf{A} \otimes_{\mathbb{Z}_p} V)^{H_K, \varphi=1} = (\mathbf{A}^{\varphi=1} \otimes_{\mathbb{Z}_p} V)^{H_K} = V^{H_K}$. Thus all the p-torsion in $H^1(\mathrm{Gal}(\overline{K}/K), V)$ comes from $\frac{V^{H_K}}{\Gamma_K - 1}$ which is a finite $\mathbb{Z}_p$-module. Further as K increases, we will have at most

$$\frac{V}{\gamma_K - 1}[p]$$

p-torsion.

□

We now give a more direct proof of Lemma 5.6 in the case $V = \mathbb{Z}_p(2)$ and even calculate the order of $\ker \pi_{\mathrm{dR}}[p^n]$.

Lemma 5.9. *The group* $\ker \pi_{\mathrm{dR}}[p^n]$ *is of order* p^n*.*

Proof. We follow the notation of Lemma 5.6. We have an exact sequence of $\mathrm{Gal}(\overline{K}/K)$-modules

$$0 \to \mathbb{Z}_p(2) \xrightarrow{\times p^n} \mathbb{Z}_p(2) \to (\mathbb{Z}/p^n\mathbb{Z})(2) \to 0.$$

Taking the long exact sequence of cohomology we obtain that $H^1(\mathrm{Gal}(\overline{K}/K), \mathbb{Z}_p(2))[p^n]$ identifies with

$$\mathrm{coker}\left(H^0(\mathrm{Gal}(\overline{K}/K), \mathbb{Z}_p(2)) \to H^0(\mathrm{Gal}(\overline{K}/K), (\mathbb{Z}/p^n\mathbb{Z})(2))\right).$$

As $[K : \breve{\mathbb{Q}}_p]$ increases, we see that

$$H^0(\mathrm{Gal}(\overline{K}/K), \mathbb{Z}_p(2)) = 0 \text{ and } H^0(\mathrm{Gal}(\overline{K}/K), (\mathbb{Z}/p^n\mathbb{Z})(2)) = (\mathbb{Z}/p^n\mathbb{Z})(2).$$

This completes the proof. □

6 Rapoport-Zink spaces and $\mathbb{Q}_p$-local systems

This section is an application of [18, §6], applied to an ordinary elliptic curve over $\overline{\mathbb{F}}_p$. The deformation space of an ordinary abelian variety was studied by Serre-Tate and a summary is given in [36, §2]. Let us recall the definition of a Rapoport-Zink space (without EL or PEL-structure). Let H be a p-divisible group over a perfect field k of characteristic p, of height h and dimension d. Consider the category $\mathrm{Art}_{W(k)}$ of Artinian local algebras over $W(k)$ with residue field k. An object of $\mathrm{Art}_{W(k)}$ is a pair (R, j), where R is an Artinian local algebra with residue field k and $j : W(k) \to R$ is a local homomorphism of local rings.

Definition 6.1. *Let $R \in \mathrm{Art}_{W(k)}$. A deformation of H to R is a pair (G, ρ), where G is a p-divisible group over R and*

$$\rho : H \to G \otimes_R k$$

is an isomorphism locally for the fppf topology. Let Def_H be the associated functor on $\mathrm{Art}_{W(k)}$, taking R to the set of isomorphism classes of deformations (G, ρ) of H to R.

Recall the following theorem of Rapoport-Zink, cf. [49, Theorem 2.16].

Theorem 6.2. *The functor Def_H is representable by a formal scheme $\mathcal{M}$ over $\mathrm{Spf}\, W(k)$, which locally admits a finitely generated ideal of definition. Every irreducible component of the reduced subscheme is proper and $\mathcal{M}$ is non-canonically isomorphic to the functor represented by the formal spectrum*

$$\mathrm{Spf}\, W(k)[[T_1, \dots, T_{d(h-d)}]].$$

Now let H be the p-divisible group over $\overline{\mathbb{F}}_p$, associated to an ordinary elliptic curve over $\overline{\mathbb{F}}_p$,

$$H = \mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p.$$

To describe its isocrystal, let e_1, e_2 be the canonical basis for $\mathbb{Q}_p^2$ and let

$$V_- =< e_1 >, \;\; V_+ =< e_2 > .$$

As the element $b \in \mathrm{GL}_2(K_0)$ we take $b = p \cdot \mathrm{id}_{V_-} \oplus \mathrm{id}_{V_+}$. Then the isocrystal associated to H is $\mathbb{Q}_p \oplus \mathbb{Q}_p(1)$. The slope decomposition has the form $N = N_0 \oplus N_1$ with

$$N_0 = V_+ \otimes K_0, N_1 = V_- \otimes K_0.$$

We consider the space $\mathscr{F} := \mathrm{Grass}_1(\mathbb{Q}_p^2)$ of subspaces F of dimension 1. It is the easy to see that F (considered as an L-point of $\mathscr{F}$) is admissible iff

$$F \cap (V_+ \otimes L) = (0).$$

That is the admissible locus may be identified with an affine space of dimension 1.

We now return to study the functor Def_H, with H as above.

Definition 6.3. *Let S be a scheme such that p is locally nilpotent in $\mathcal{O}_S$. A p-divisible group $X \to S$ is said to be ordinary if X sits in a short exact sequence*

$$0 \to T \to X \to E \to 0$$

where T is a multiplicative p-divisible group (i.e. locally $T[p^n]$ is diagonalizable) and E is an étale p-divisible group. Such an exact sequence is unique up to unique isomorphism.

Remark 6.4. Suppose that X is an ordinary p-divisible group over $S = \mathrm{Spec}(K)$, where $K \supset \mathbb{F}_p$ is a perfect field. Then there exists a unique splitting of the short exact sequence $0 \to T \to X \to E \to 0$ over K.

Proposition 6.5. *Suppose that S is a scheme over $W(k)$ and p is locally nilpotent in $\mathcal{O}_S$. Let $S_0 = S \times \mathcal{O}_S/p\mathcal{O}_S$, the closed subscheme defined by the ideal $p\mathcal{O}_S$ of the structure sheaf $\mathcal{O}_S$. If $X \to S$ is a p-divisible group such that $X \times_S S_0$ is ordinary, then $X \to S$ is ordinary.*

Proof. This is a consequence of the rigidity of finite étale group schemes and commutative finite group schemes of multiplicative type, cf. [20, Exposé X]. □

Now H is an ordinary p-divisible group over $\overline{\mathbb{F}}_p$. It sits in a canonical split short exact sequence

$$0 \to \mu_{p^\infty} \to H \to \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

Let $T_i \to \operatorname{Spec}(W(\overline{\mathbb{F}}_p)/p^iW(\overline{\mathbb{F}}_p))$ and $E_i \to \operatorname{Spec}(W(\overline{\mathbb{F}}_p)/p^iW(\overline{\mathbb{F}}_p))$ be the multiplicative, respectively étale p-divisible group over $\operatorname{Spec}(W(\overline{\mathbb{F}}_p)/p^iW(\overline{\mathbb{F}}_p))$ which lifts μ_{p^∞}, respectively $\mathbb{Q}_p/\mathbb{Z}_p$, for each $i \geq 1$. Both T_i and E_i are unique up to unique isomorphism. Taking the limit of $T_i[p^n]$, respectively $E_i[p^n]$ as $i \to \infty$, we get a multiplicative, respectively étale p-divisible group $T \to \operatorname{Spec}(W(\overline{\mathbb{F}}_p))$, respectively $E \to \operatorname{Spec}(W(\overline{\mathbb{F}}_p))$. Let $\widehat{\mathbb{G}}_m$ be the completion of $\mathbb{G}_m$ along its unit section. The theorem below shows that the deformation space of H has a natural structure as a group object isomorphic to $\widehat{\mathbb{G}}_m$, of dimension 1.

Theorem 6.6.
1. *Every deformation $X \to \operatorname{Spec}(R)$ of H over an Artinian local $W(\overline{\mathbb{F}}_p)$-algebra R is an ordinary p-divisible group over R. Therefore X sits in an exact sequence*

$$0 \to T \times_{W(\overline{\mathbb{F}}_p)} R \to X \to E \times_{W(\overline{\mathbb{F}}_p)} R \to 0.$$

2. *The deformation functor Def_H has a natural structure via the Baer sum construction, as a functor from $\operatorname{Art}_{W(\overline{\mathbb{F}}_p)}$ to the category of Abelian groups. In particular the unit element in $\operatorname{Def}_H(R)$ corresponds to the split p-divisible group*

$$(T \times_{W(\overline{\mathbb{F}}_p)} E) \times_{W(\overline{\mathbb{F}}_p)} R.$$

3. *There is a natural isomorphism of functors*

$$\widehat{\mathbb{G}}_m \xrightarrow{\sim} \operatorname{Def}_H.$$

Proof. Statements 1 and 2 follow from Proposition 6.5. It remains to prove statement 3. By 1, over any Artinian local $W(\overline{\mathbb{F}}_p)$-algebra R, we see that $\operatorname{Def}_H(R)$ is the set of isomorphism classes of extensions of $E \times_{W(\overline{\mathbb{F}}_p)} R$ by $T \times_{W(\overline{\mathbb{F}}_p)} R$. We have also seen that $\operatorname{Def}_H(R)$ is naturally isomorphic to the inverse limit

$$\varprojlim_n \operatorname{Ext}^1_{\mathbb{Z}/p^n\mathbb{Z}}((\mathbb{Z}/p^n\mathbb{Z})_R, \mu_{p^n,R}),$$

where the Ext group is computed in the category of fppf-sheaves of $\mathbb{Z}/p^n\mathbb{Z}$-modules over $\operatorname{Spec}(R)$. This makes sense because the category of finite flat

R-group schemes in the category of fppf-sheaves is closed under extensions. Now in the fppf-topology, the Kummer sequence

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{z \mapsto z^n} \mathbb{G}_m \to 1$$

is always exact even if n is not invertible in the underlying scheme. So we can apply Kummer theory directly to compute $H^1_{\mathrm{fppf}}(\mathrm{Spec}(R), \mu_n)$ for any n, cf. [47, §4, pg. 125], for an explicit computation. By Kummer theory, we have

$$\mathrm{Ext}^1_{\mathbb{Z}/p^n\mathbb{Z}}((\mathbb{Z}/p^n\mathbb{Z})_R, \mu_{p^n,R}) = R^*/(R^*)^{p^n}.$$

Since $\mathfrak{m}_R$ is nilpotent and $R/\mathfrak{m}_R = \overline{\mathbb{F}}_p$ is perfect, it is easy to compute

$$\varprojlim_n R^*/(R^*)^{p^n} = 1 + \mathfrak{m}_R,$$

where the transition morphisms are given by restriction and are compatible with the transition morphisms in $\varprojlim_n \mathrm{Ext}^1_{\mathbb{Z}/p^n\mathbb{Z}}((\mathbb{Z}/p^n\mathbb{Z})_R, \mu_{p^n,R})$. Finally note that $\widehat{\mathbb{G}}_m(R) = 1 + \mathfrak{m}_R$. □

Remark 6.7. The above theorem holds (with the same proof) for the category $\mathcal{C}$ of complete local noetherian $W(\overline{\mathbb{F}}_p)$-algebras with residue field $\overline{\mathbb{F}}_p$. That is we can define Def_H as a functor on $\mathcal{C}$ and we would have $\widehat{\mathbb{G}}_m \xrightarrow{\sim} \mathrm{Def}_H$ as functors on $\mathcal{C}$. Hence there is a universal p-divisible group H^{univ} over Def_H.

Let R be a ring and $\mathrm{Nilp}_R^{\mathrm{op}}$ denote the category opposite of the category of R-algebras on which p is nilpotent. We now define the Tate module of a p-divisible group G over R. We follow closely [57, §3.3].

Definition 6.8. *The Tate module of G is defined as a sheaf*

$$T(G)(S) := \varprojlim_n G[p^n](S)$$

on $\mathrm{Nilp}_R^{\mathrm{op}}$. *Clearly $T(G)$ is a locally constant sheaf of $\mathbb{Z}_p$-modules.*

The next result says that the Tate-module commutes with analytification.

Proposition 6.9. *We have that*

$$T(G)^{\mathrm{an}} \cong \varprojlim_n G[p^n]^{\mathrm{an}}$$

as analytic sheaves on $\mathrm{Spec}(R)^{\mathrm{an}}$.

Proof. This follows from the proof of [57, Proposition 3.3.2]. □

Definition 6.10. *We define the rational Tate module of G as*

$$V(G)^{\mathrm{an}} := \varinjlim_p T(G)^{\mathrm{an}}$$

where the transition morphisms are multiplication by p

Remark 6.11. [57, Proposition 3.3.1] says that $T(G)$ is representable as a scheme over $\mathrm{Spec}(W(\overline{\mathbb{F}}_p))$ and since $p : T(G)^{\mathrm{an}} \to T(G)^{\mathrm{an}}$ is an open immersion, it follows that $V(G)^{\mathrm{an}}$ is representable by an analytic space over $\mathrm{Spec}(R)^{\mathrm{an}}$.

Fix a scheme X such that X^{an} is reduced and BT_X the stack of p-divisible groups over X equipped with the fppf topology. Consider the functors between stacks

$$\begin{aligned} T_p : \mathrm{BT}_X &\to \mathbb{Z}_p - \underline{\mathrm{Loc}}_{X^{\mathrm{an}}} \\ G &\mapsto T(G)^{\mathrm{an}} \end{aligned}$$

and

$$\begin{aligned} V_p : \mathrm{BT}_X &\to \mathbb{Q}_p - \underline{\mathrm{Loc}}_{X^{\mathrm{an}}} \\ G &\mapsto V(G)^{\mathrm{an}}. \end{aligned}$$

Proposition 6.12. *The functors T_p and V_p are fully faithful and exact.*

Proof. The fully faithful part is [18, Proposition 6.2(ii)]. Exactness is obvious. □

Take the p-adic period morphism associated to Def_H, $\pi : \mathrm{Def}_H^{\mathrm{an}} \to \mathbb{A}^{1,\mathrm{an}}$ and let $\mathcal{E}'$ denote $V(H^{\mathrm{univ}})^{\mathrm{an}}$, a $\mathbb{Q}_p$-local system on $\mathrm{Def}_H^{\mathrm{an}}$.

Proposition 6.13. *The rational Tate-module $\mathcal{E}'$ descends to a local system, which we denote again by $\mathcal{E}'$, of $\mathbb{Q}_p$-vector spaces on $\mathbb{A}^{1,\mathrm{an}}$.*

Proof. In order for $\mathcal{E}'$ to define a local system on $\mathbb{A}^{1,\mathrm{an}}$ it suffices to show that

- $\pi : \mathrm{Def}_H^{\mathrm{an}} \to \mathbb{A}^{1,\mathrm{an}}$ is a covering for the étale topology.
- There is descent datum $\tilde{\psi} : \mathrm{pr}_1^* \mathcal{E}' \xrightarrow{\sim} \mathrm{pr}_2^* \mathcal{E}'$ over $\mathrm{Def}_H^{\mathrm{an}} \times_{\mathbb{A}^{1,\mathrm{an}}} \mathrm{Def}_H^{\mathrm{an}}$ where $\mathrm{pr}_i : \mathrm{Def}_H^{\mathrm{an}} \times_{\mathbb{A}^{1,\mathrm{an}}} \mathrm{Def}_H^{\mathrm{an}} \to \mathrm{Def}_H^{\mathrm{an}}$ is projection onto the ith factor, such that $\tilde{\psi}$ satisfies the cocycle condition on $\mathrm{Def}_H^{\mathrm{an}} \times_{\mathbb{A}^{1,\mathrm{an}}} \mathrm{Def}_H^{\mathrm{an}} \times_{\mathbb{A}^{1,\mathrm{an}}} \mathrm{Def}_H^{\mathrm{an}}$.

The first point is well known and goes back to the work of Dwork. For the second point, let $\mathcal{D}$ be the constant filtered isocrystal over $\mathbb{A}^{1,\mathrm{an}}$ given by $(\mathbb{D}(H), \omega_{H^*} \hookrightarrow \mathbb{D}(H))$. We have canonical descent datum

$$\mathrm{id} : \mathrm{pr}_1^* \pi^* \mathcal{D} = \mathrm{pr}_2^* \pi^* \mathcal{D}$$

over $\mathrm{Def}_H^{\mathrm{an}} \times_{\mathbb{A}^{1,\mathrm{an}}} \mathrm{Def}_H^{\mathrm{an}}$. We have the object $\mathrm{pr}_i^* H^{\mathrm{univ,an}}$ as an object of the stack $\mathrm{BT}_{\mathrm{Def}_H^{\mathrm{an}} \times_{\mathbb{A}^{1,\mathrm{an}}} \mathrm{Def}_H^{\mathrm{an}}} \otimes \mathbb{Q}_p$. Note that $\mathrm{BT}_{\mathrm{Def}_H^{\mathrm{an}} \times_{\mathbb{A}^{1,\mathrm{an}}} \mathrm{Def}_H^{\mathrm{an}}}$ is stack on the quasi-étale topology of formal models of $\mathrm{Def}_H^{\mathrm{an}} \times_{\mathbb{A}^{1,\mathrm{an}}} \mathrm{Def}_H^{\mathrm{an}}$, whose values on a given formal model $\mathfrak{X}$ are p-divisible groups over $\mathfrak{X}$. Finally $\mathrm{BT}_{\mathrm{Def}_H^{\mathrm{an}} \times_{\mathbb{A}^{1,\mathrm{an}}} \mathrm{Def}_H^{\mathrm{an}}} \otimes \mathbb{Q}_p$ is the stack associated to $\mathrm{BT}_{\mathrm{Def}_H^{\mathrm{an}} \times_{\mathbb{A}^{1,\mathrm{an}}} \mathrm{Def}_H^{\mathrm{an}}}$ whose pre-stack were morphisms up to quasi-isogeny. The functor from this stack to filtered isocrystals is fully faithful by [18, Proposition 6.6] and sends $\mathrm{pr}_i^* H^{\mathrm{univ,an}}$ to $\mathrm{pr}_i^* \pi^* \mathcal{D}$. Thus one obtains descent datum

$$\psi : \mathrm{pr}_1^* H^{\mathrm{univ,an}} \xrightarrow{\sim} \mathrm{pr}_2^* H^{\mathrm{univ,an}}$$

in $\mathrm{BT}_{\mathrm{Def}_H^{\mathrm{an}} \times_{\mathbb{A}^{1,\mathrm{an}}} \mathrm{Def}_H^{\mathrm{an}}} \otimes \mathbb{Q}_p$. One applies the rational Tate module functor V_p together with Proposition 6.9 and takes

$$\tilde{\psi} := V_p(\psi) : \mathrm{pr}_1^* \mathcal{E}' \xrightarrow{\sim} \mathrm{pr}_2^* \mathcal{E}',$$

as required. This yields the local system $\mathcal{E}'$ on $\mathbb{A}^{1,\mathrm{an}}$. □

Note that $\mathcal{E}'$ sits in an exact sequence $0 \to \mathbb{Q}_p(1) \to \mathcal{E}' \to \mathbb{Q}_p \to 0$. The next proposition summarizes the work done in this section.

Proposition 6.14. *There is an isomorphism of $\breve{\mathbb{Q}}_p$-analytic groups*

$$\widehat{\mathbb{G}}_m \cong \mathscr{M}(\mathcal{E}')^{(0,0)}.$$

Proof. This follows from Theorem 6.6 and Proposition 6.13. □

7 Final Calculations

In this section we begin by showing that $\mathscr{M}(\mathcal{E})^{(0,0)}_{\mathbb{C}_p}$ is the analytification of an actual p-divisible group. First of all note that the property of being an analytic p-divisible group is stable by base change and so $\mathscr{M}(\mathcal{E})^{(0,0)}_{\mathbb{C}_p}$ is a $\mathbb{C}_p$-analytic group of p-divisible type. We have the following lemma giving the height and dimension of $\mathscr{M}(\mathcal{E})^{(0,0)}_{\mathbb{C}_p}$.

Lemma 7.1. $\mathscr{M}(\mathcal{E})^{(0,0)}_{\mathbb{C}_p}$ *has dimension 1 and height 1.*

Proof. First note that dimension and height are invariant by base change. Thus it suffices to show that $\mathscr{M}(\mathcal{E})^{(0,0)}$ has the specified dimension and height. We have that $\pi_{\mathrm{dR}} : \mathscr{M}(\mathcal{E})^{(0,0)} \to \mathbb{A}^1$ is an étale surjection. Since $\mathbb{A}^1$ has dimension 1, it follows that $\mathscr{M}(\mathcal{E})^{(0,0)}$ has dimension 1. By definition the height of $\mathscr{M}(\mathcal{E})^{(0,0)}$ is the height of its associated p-divisible group, $\mathscr{M}(\mathcal{E})^{(0,0)}[p^\infty]$. As remarked earlier, for K a finite extension of $\breve{\mathbb{Q}}_p$, we have

$$\begin{aligned}\mathscr{M}(\mathcal{E})^{(0,0)}[p^\infty](K) &= H^1(\mathrm{Gal}(\overline{K}/K), \mathbb{Z}_p(2))[p^\infty] \\ &= \ker(H^1(\mathrm{Gal}(\overline{K}/K), \mathbb{Z}_p(2)) \to H^1(\mathrm{Gal}(\overline{K}/K), \mathbb{Q}_p(2))).\end{aligned}$$

Following the proof of Lemma 5.6, we see that p-torsion in $H^1(\mathrm{Gal}(\overline{K}/K), \mathbb{Z}_p(2))$ comes from $\frac{\mathbb{Z}_p(2)^{H_K}}{\Gamma_K - 1}$. The latter is of the form $\mathbb{Z}_p/p^n\mathbb{Z}_p$ for $n > 0$ (since the action of $\gamma \in \Gamma_K$ is non-trivial in $\mathbb{Z}_p(2)^{H_K}$), which has exactly p, p-torsion. Hence the p-divisible group $\mathscr{M}(\mathcal{E})^{(0,0)}[p^\infty]$ must have height 1. □

The previous lemma together with a classification result of Fargues in [26], allows us to deduce the precise form of $\mathscr{M}(\mathcal{E})^{(0,0)}_{\mathbb{C}_p}$.

Proposition 7.2. *As $\mathbb{C}_p$-analytic p-divisible groups, we have $\mathscr{M}(\mathcal{E})^{(0,0)}_{\mathbb{C}_p} \cong \widehat{\mathbb{G}}^{\mathrm{an}}_{m,\mathbb{C}_p}$, where $\widehat{\mathbb{G}}_{m,\breve{\mathbb{Q}}_p}$ is the completion of $\mathbb{G}_{m,\breve{\mathbb{Q}}_p}$ along its unit section. This implies that the analytification, $\widehat{\mathbb{G}}^{\mathrm{an}}_{m,\breve{\mathbb{Q}}_p}$, is the open unit ball, with center 1, equipped with its natural multiplicative structure.*

Proof. By [26, Théorème 3.2] and Lemma 7.1, it follows that either $\mathscr{M}(\mathcal{E})^{(0,0)}_{\mathbb{C}_p} \cong \widehat{\mathbb{G}}^{\mathrm{an}}_{m,\mathbb{C}_p}$ or $\mathscr{M}(\mathcal{E})^{(0,0)}_{\mathbb{C}_p} \cong \mathbb{Q}_p/\mathbb{Z}_p \times \mathbb{G}^{\mathrm{an}}_a$. Since $\mathscr{M}(\mathcal{E})^{(0,0)}$ is connected, by [26, Proposition 6] $\mathscr{M}(\mathcal{E})^{(0,0)}_{\mathbb{C}_p}$ has finitely many connected components. Thus $\mathscr{M}(\mathcal{E})^{(0,0)}_{\mathbb{C}_p}$ is connected and this rules out $\mathscr{M}(\mathcal{E})^{(0,0)}_{\mathbb{C}_p} \cong \mathbb{Q}_p/\mathbb{Z}_p \times \mathbb{G}^{\mathrm{an}}_a$. Hence $\mathscr{M}(\mathcal{E})^{(0,0)}_{\mathbb{C}_p} \cong \widehat{\mathbb{G}}^{\mathrm{an}}_{m,\mathbb{C}_p}$, as claimed. □

We claim that we have $\mathcal{E}_{\mathbb{C}_p} \cong \mathcal{E}'_{\mathbb{C}_p}$. Indeed by Proposition 6.14 we have that $\mathscr{M}(\mathcal{E})^{(0,0)}_{\mathbb{C}_p} \cong \mathscr{M}(\mathcal{E}')^{(0,0)}_{\mathbb{C}_p} \cong \widehat{\mathbb{G}}^{\mathrm{an}}_{m,\mathbb{C}_p}$ as analytic groups. Now the logarithm $\widehat{\mathbb{G}}^{\mathrm{an}}_{m,\mathbb{C}_p} \to \mathbb{A}^1_{\mathbb{C}_p}$ is a $\mathbb{Q}_p/\mathbb{Z}_p$-torsor (it is an étale covering with kernel $\mathbb{Q}_p/\mathbb{Z}_p$). Let $\mathscr{H} := \varprojlim_{\times p} \widehat{\mathbb{G}}^{\mathrm{an}}_{m,\mathbb{C}_p}$ (viewed as a projective system, or a perfectoid space for the associated adic space). Then $\mathscr{H} \to \widehat{\mathbb{G}}^{\mathrm{an}}_{m,\mathbb{C}_p}$ is a $\mathbb{Z}_p$-torsor and $\mathscr{H} \to \mathbb{A}^1_{\mathbb{C}_p}$ is a $\mathbb{Q}_p$-torsor which is pro-étale. Now as $\widehat{\mathbb{G}}^{\mathrm{an}}_{m,\mathbb{C}_p}$ is an analytic group of p-divisible type, it is naturally equipped with an action of $\mathbb{Z}_p$. This induces an action of $\mathbb{Q}_p \rtimes \mathbb{Z}_p^*$ on $\mathscr{H}$ (the first component acts via the torsor structure $\mathscr{H} \to \mathbb{A}^1_{\mathbb{C}_p}$ and the second component is induced from $\mathbb{Z}_p^* \to \mathrm{Aut}(\widehat{\mathbb{G}}^{\mathrm{an}}_{m,\mathbb{C}_p})$. Now let

$$B^1 := \begin{pmatrix} \mathbb{Z}_p^* & \mathbb{Q}_p \\ & \mathbb{Z}_p^* \end{pmatrix}$$

and note that B^1 acts on $\mathscr{H}$ via $\begin{pmatrix} \lambda_1 & \mu \\ 0 & \lambda_2 \end{pmatrix} \mapsto \mu \rtimes \lambda_1\lambda_2^{-1} \in \mathbb{Q}_p \rtimes \mathbb{Z}_p^*$. Now consider

$$\mathrm{Ind}_{B^1}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathscr{H} := \mathscr{H} \times_{B^1} \mathrm{GL}_2(\mathbb{Q}_p).$$

We claim that $\mathrm{Ind}_{B^1}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathscr{H}$ corresponds to the infinite level Rapoport-Zink space associated to $\widehat{\mathbb{G}}^{\mathrm{an}}_{m,\mathbb{C}_p}$ (recall that we are deforming H, the ordinary p-divisible group). Indeed this follows from results of [29]. Taking $b' = 1$, $b = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, $G = \mathrm{GL}_2$, μ minuscule in Proposition 4.21 in loc.cit. gives (after taking connected components and noting $\tilde{J}_b^U = \mathrm{Spd}\ \mathbb{F}_p[[t^{1/p^\infty}]]$, cf. [56, Remark 18.2.5(3)]) $\mathscr{H}^\diamond \cong \mathcal{P}^\mu_{b,b',\mathbb{C}_p^\flat}$. The claim now follows from [29, Corollary 4.14].

Repeating the construction for $\mathscr{M}(\mathcal{E})^{(0,0)}_{\mathbb{C}_p}$ and $\mathscr{M}(\mathcal{E}')^{(0,0)}_{\mathbb{C}_p}$, we see that

$$\mathrm{Ind}_{B^1}^{\mathrm{GL}_2(\mathbb{Q}_p)} \varprojlim_{\times p} \mathscr{M}(\mathcal{E})^{(0,0)}_{\mathbb{C}_p} \cong \mathrm{Ind}_{B^1}^{\mathrm{GL}_2(\mathbb{Q}_p)} \varprojlim_{\times p} \mathscr{M}(\mathcal{E}')^{(0,0)}_{\mathbb{C}_p} \tag{4}$$

Now

$$\mathrm{Ind}_{B^1}^{\mathrm{GL}_2(\mathbb{Q}_p)} \varprojlim_{\times p} \mathscr{M}(\mathcal{E})^{(0,0)}_{\mathbb{C}_p} \times_{\mathrm{GL}_2(\mathbb{Q}_p)} \mathbb{Q}_p^2 \cong \mathcal{E}_{\mathbb{C}_p}$$

and similarly

$$\mathrm{Ind}_{B^1}^{\mathrm{GL}_2(\mathbb{Q}_p)} \varprojlim_{\times p} \mathscr{M}(\mathcal{E}')^{(0,0)}_{\mathbb{C}_p} \times_{\mathrm{GL}_2(\mathbb{Q}_p)} \mathbb{Q}_p^2 \cong \mathcal{E}'_{\mathbb{C}_p}.$$

By (4), this implies $\mathcal{E}_{\mathbb{C}_p} \cong \mathcal{E}'_{\mathbb{C}_p}$, as promised.

We now rephrase this condition in terms of fundamental groups. First we need a lemma relating the geometric étale fundamental group $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\mathbb{C}_p})$ and the étale fundamental group $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p})$.

Lemma 7.3. *There is an exact sequence of prodiscrete groups*

$$1 \to H \to \pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p}) \to \mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p) \to 1,$$

where H is the closure of the image of $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\mathbb{C}_p}) \to \pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p})$.

Proof. Exactness on the right follows from [18, Proposition 2.12]. Exactness on the left is by definition of H. Let's prove exactness in the middle. It is clear that the composition is trivial. This is because for every finite étale cover $X \to \mathrm{Sp}(\breve{\mathbb{Q}}_p)$, the base change $X \times_{\mathrm{Sp}(\breve{\mathbb{Q}}_p)} \mathbb{A}^1_{\mathbb{C}_p} \to \mathbb{A}^1_{\mathbb{C}_p}$ is a trivial cover. Let $U \subset \pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p})$ be an open subgroup and let $X \to \mathbb{A}^1_{\breve{\mathbb{Q}}_p}$ be the connected étale cover corresponding to U. Then U contains the image ($:= I$) of $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\mathbb{C}_p}) \to \pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p})$ iff $X_{\mathbb{C}_p} \cong \mathbb{A}^1_{\mathbb{C}_p}$, cf. [60, Proposition 5.5.5]. Note also that the intersection of all open subgroups containing I is the closure H of I in $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p})$, cf. Lemma 5.5.7(1) in loc.cit[25]. Finally [18, Lemma 2.14] gives that kernel of $\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p}) \to \mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p)$ coincides with H. This completes the proof.

□

Remark 7.4. Armed with Lemma 7.3 it is natural to proceed as follows. The inflation-restriction exact sequence applied to

$$1 \to H \to \pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p}) \to \mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p) \to 1,$$

gives a diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p), \mathbb{Q}_p(1)) \arrow[r] & H^1(\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p}), \mathbb{Q}_p(1)) \arrow[dr, "\gamma_1"] & \\
 & & & H^1(H, \mathbb{Q}_p) \\
0 \arrow[r] & H^1(\mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p), \mathbb{Q}_p(2)) \arrow[r] & H^1(\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p}), \mathbb{Q}_p(2)) \arrow[ur, "\gamma_2"] &
\end{tikzcd}$$

where we have identified $H^1(H, \mathbb{Q}_p(1))$ and $H^1(H, \mathbb{Q}_p(2))$ via a choice of basis for $\mathbb{Q}_p(1)$ as a $\mathbb{Q}_p$-vector space. Now the condition $\mathcal{E}_{\mathbb{C}_p} \cong \mathcal{E}'_{\mathbb{C}_p}$ means

25. Technically Lemma 5.5.7(1) in loc.cit is for profinite groups but the same proof works in our situation.

$\gamma_1([f_1]) = \gamma_2([f_2])$ (here $[-]$ denotes the class of the relevant 1-cocycle in the cohomology group, f_1 is a 1-cocycle corresponding to $\mathcal{E}'$ and f_2 is a 1-cocycle corresponding to $\mathcal{E}$). To obtain a contradiction we need to compare the images of γ_1 and γ_2. Unfortunately this is difficult to do (in this setup) due to a lack of morphism between $H^1(\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p}), \mathbb{Q}_p(1))$ and $H^1(\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p}), \mathbb{Q}_p(2))$ which is compatible with γ_1 and γ_2. To overcome this difficulty, the key is to keep track of $\mathcal{E}$ and $\mathcal{E}'$ via one cohomology group, albeit we will need to consider non-abelian cohomology of groups, so this will be a pointed set.

7.1 Non-abelian cohomology of groups

In this section we provide the set-up of non-abelian cohomology that is required. Consider the following situation: Let G be any profinite group and H a normal subgroup (we will apply what follows to our situation with $G = \pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p})$ and H as in Lemma 7.3). Suppose

$$\rho_1, \rho_2 \colon G \to \mathrm{GL}(V)$$

are two $\mathbb{Q}_p$-representations of G, of the same dimension (later we will set ρ_1 and ρ_2 to be the stalk at a geometric point of $\mathcal{E}'$ and $\mathcal{E}$, respectively). Define the following action of G on $\mathrm{GL}(V)$

$$\forall \sigma \in G,\ \forall g \in \mathrm{GL}(V),\ {}^{\sigma}g = \rho_2(\sigma) g \rho_2(\sigma)^{-1}.$$

The key to *comparing* ρ_1 and ρ_2 is to consider the following function

$$\begin{aligned} c \colon G &\to GL(V) \\ \sigma &\mapsto c_\sigma = \rho_1(\sigma)\rho_2(\sigma)^{-1}. \end{aligned}$$

Lemma 7.5. *The function c as defined above is a 1-cocycle, that is $c \in Z^1(G, GL(V))$. Moreover $c \sim 1$ iff $\rho_1 \cong \rho_2$.*

Proof. For any α, $\beta \in G$,

$$\begin{aligned} c_{\alpha\beta} &= \rho_1(\alpha\beta)\rho_2(\alpha\beta)^{-1} \\ &= \rho_1(\alpha)\rho_2(\alpha)^{-1}[\rho_2(\alpha)\rho_1(\beta)\rho_2(\beta)^{-1}\rho_2(\alpha)^{-1}] \\ &= c_\alpha \cdot^{\alpha} c_\beta. \end{aligned}$$

This proves c is indeed a 1-cocycle. For the last part note that $c \sim 1$ iff there exists $b \in \mathrm{GL}(V)$ such that $c_\sigma = b^{-1} \cdot^{\sigma} b = b^{-1}\rho_2(\sigma) b \rho_2(\sigma)^{-1}$. Plugging $c_\sigma = \rho_1(\sigma)\rho_2(\sigma)^{-1}$ gives precisely $\rho_1 \cong \rho_2$. This completes the proof. $\square$

Suppose now that $\rho_1|_H \cong \rho_2|_H$ (this will be the condition that $\mathcal{E}_{\mathbb{C}_p} \cong \mathcal{E}'_{\mathbb{C}_p}$) and so by Lemma 7.5, $c|_H \sim 1$. In this case it is natural to start analysing the inflation-restriction exact sequence (of pointed sets)

$$1 \to H^1(G/H, \mathrm{GL}(V)^H) \to H^1(G, \mathrm{GL}(V)) \to H^1(H, \mathrm{GL}(V)).$$

We now specialize to our situation. By Lemma 7.3, $G/H = \mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p)$. We will also replace $\mathrm{GL}(V)$ by the upper Borel subgroup B whose lower-right entries are 1. We begin by computing B^H.

Lemma 7.6. *As a* $\mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p)$*-module,* $B^H \cong \mathbb{Q}_p(2)$.

Proof. Suppose $g = \begin{pmatrix} a & b \\ & 1 \end{pmatrix} \in B^H$. This means that for all $\sigma \in H$

$$g = \rho_2(\sigma) g \rho_2(\sigma)^{-1} \tag{5}$$

Computing the RHS of (5) gives

$$\begin{aligned} \rho_2(\sigma) g \rho_2(\sigma)^{-1} &= \begin{pmatrix} 1 & \alpha_2(\sigma) \\ & 1 \end{pmatrix} \begin{pmatrix} a & b \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha_2(\sigma) \\ & 1 \end{pmatrix} \\ &= \begin{pmatrix} a & b + (1-a)\alpha_2(\sigma) \\ & 1 \end{pmatrix} \end{aligned}$$

where $\alpha_2 \colon G \to \mathbb{Q}_p$ is determined by the class of $\mathcal{E}$ in $H^1(\pi_1^{\mathrm{dJ}}(\mathbb{A}^1_{\breve{\mathbb{Q}}_p}), \mathbb{Q}_p(2))$. Since $\alpha_2|_H$ is certainly non-trivial, we must have that $a = 1$. Finally to determine the action of $\sigma \in G$ on $\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \in B^H$, we compute

$$\begin{aligned} {}^{\sigma}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} &= \rho_2(\sigma) \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \rho_2(\sigma)^{-1} \\ &= \begin{pmatrix} \chi^2(\sigma) & \alpha_2(\sigma) \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \chi^{-2}(\sigma) & -\chi^{-2}(\sigma)\alpha_2(\sigma) \\ & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & b\chi^2(\sigma) \\ & 1 \end{pmatrix}. \end{aligned}$$

Thus it follows that the action of G on B^H is given by χ^2. This completes the proof. $\square$

We following lemma says that $H^1(\mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p), \mathbb{Q}_p(2))$ is enormous and it is therefore hopeless to study the cocycle c in $H^1(\mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p), \mathbb{Q}_p(2))$.

Lemma 7.7. *We have that* $H^1(\mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p), \mathbb{Q}_p(2))$ *is of dimension 1 over* $\breve{\mathbb{Q}}_p$.

Proof. Probably the quickest way to see this is via the exponential map of Bloch-Kato. The point is that if V is a semi-stable representation of $\mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p)$, whose Hodge-Tate weights are ≥ 2, then the Bloch-Kato exponential map

$$\exp_V \colon \mathbf{D}_{\mathrm{dR}}(V) \to H^1(\mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p), V)$$

is an isomorphism, cf. [1, Théorème 6.8]. Applying this to $V = \mathbb{Q}_p(2)$ gives the result. $\square$

To obtain a contradiction for the existence of $\mathcal{E}$ consider the following composition of maps

$$H^1(\mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p), \mathbb{Q}_p(2)) \xrightarrow{\alpha} H^1(G, B) \xrightarrow{\beta} H^1(\mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p), B), \tag{6}$$

where the first map is inflation and the second morphism is coming from the section $\mathrm{Sp}(\breve{\mathbb{Q}}_p) \to \mathbb{A}^1_{\breve{\mathbb{Q}}_p}$ defined by the origin point. Explicitly this composition sends a cocycle (class)

$$d\colon \mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p) \to \mathbb{Q}_p(2)$$

to the cocycle

$$\mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p) \to G \twoheadrightarrow \mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p) \xrightarrow{d} \mathbb{Q}_p(2) \hookrightarrow B, \tag{7}$$

where the first map is induced by the section $\mathrm{Sp}(\breve{\mathbb{Q}}_p) \to \mathbb{A}^1_{\breve{\mathbb{Q}}_p}$. Since the composition of the first two maps in (7) is the identity, the composition of α and β is given by the map on cohomology induced by the morphism

$$\mathbb{Q}_p(2) \hookrightarrow B.$$

We now compute $\beta([c])$ in two ways. First of all, we know that there exists $c'\colon \mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p) \to \mathbb{Q}_p(2)$ a 1-cocycle such that $\alpha([c']) = [c]$. On the other hand at the origin both representations ρ_1 and ρ_2 are split and so the cocycle c gets sent to a 1-cocycle (via the second map β) of the form

$$\begin{aligned} c''\colon \mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p) &\to B \\ \sigma &\mapsto \begin{pmatrix} \chi^{-1}(\sigma) & \\ & 1 \end{pmatrix}. \end{aligned}$$

But $\beta \circ \alpha([c'])$ is a 1-cocycle of the form (which we continue to denote by c')

$$\begin{aligned} c'\colon \mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p) &\to B \\ \sigma &\mapsto \begin{pmatrix} 1 & c'(\sigma) \\ & 1 \end{pmatrix}. \end{aligned}$$

By construction the two 1-cocycles are $c' \sim c''$ in $H^1(\mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p), B)$. This means that there exists $b \in B$ such that

$$b \begin{pmatrix} 1 & c'(\sigma) \\ & 1 \end{pmatrix} = \begin{pmatrix} \chi^{-1}(\sigma) & \\ & 1 \end{pmatrix} \rho_2(\sigma) b \rho_2(\sigma)^{-1}. \tag{8}$$

Rearranging (8) gives [26]

$$b \begin{pmatrix} \chi^2(\sigma) & \alpha_2(\sigma) + c'(\sigma) \\ & 1 \end{pmatrix} b^{-1} = \begin{pmatrix} \chi(\sigma) & \chi^{-1}(\sigma)\alpha_2(\sigma) \\ & 1 \end{pmatrix}. \tag{9}$$

26. Recall that $\rho_2(\sigma) = \begin{pmatrix} \chi^2(\sigma) & \alpha_2(\sigma) \\ & 1 \end{pmatrix}$.

Comparing the determinants of both sides of (9) gives a contradiction. This means that the $\mathbb{Q}_p$-local system $\mathcal{E}$ does not exist, giving a counter-example to Conjecture 4.3.

As a corollary of the previous sections, we conjecture the following result:

Conjecture 7.8. *For the datum*

$$G = \mathrm{GL}_2, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}, \quad \mu = \begin{pmatrix} t^2 & 0 \\ 0 & 1 \end{pmatrix}$$

there does not exist an adic space X over $\mathrm{Spa}(\breve{\mathbb{Q}}_p)$ such that

$$X^{\diamond} \cong \mathrm{Sht}_{(G,b,\mu)} .$$

Remark 7.9. Conjecture 7.8 is interesting because it advocates the importance of diamonds which are not coming from *classical* adic spaces. One could attempt to prove Conjecture 7.8 as follows (we will freely employ the notation of [56]): There is a period map $\pi_{\mathrm{GM}}\colon \mathrm{Sht}_{(G,b,\mu)} \to B^+_{\mathrm{dR}}/\mathrm{Fil}^2 \times_{\mathrm{Spd}(\mathbb{Q}_p)} \mathrm{Spd}(\breve{\mathbb{Q}}_p)$ and there is a natural morphism $\nu\colon B^+_{\mathrm{dR}}/\mathrm{Fil}^2 \times_{\mathrm{Spd}(\mathbb{Q}_p)} \mathrm{Spd}(\breve{\mathbb{Q}}_p) \to \mathbb{A}^1_{\breve{\mathbb{Q}}_p}$. One can show that there is a $\mathbb{Q}_p$-local system $\mathcal{F}$ over $B^+_{\mathrm{dR}}/\mathrm{Fil}^2 \times_{\mathrm{Spd}(\mathbb{Q}_p)} \mathrm{Spd}(\breve{\mathbb{Q}}_p)$ such that $\mathrm{Sht}_{(G,b,\mu)} \cong \mathscr{M}(\mathcal{F})$. The fact that there is a counterexample to the Rapoport-Zink conjecture means that $\mathcal{F}$ does not descend to a $\mathbb{Q}_p$-local system over $\mathbb{A}^1_{\breve{\mathbb{Q}}_p}$ via ν. One would expect that this implies $\mathscr{M}(\mathcal{F})$ does not come from an adic space.

8 Part II: Arithmetic families of (φ, Γ)-modules and locally analytic representations of $\mathrm{GL}_2(\mathbf{Q}_p)$

9 Introduction

9.1 An extension of the p-adic Langlands correspondence

The aim of this article is to study the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ in arithmetic families. To put things into context, let us recall the general lines of this correspondence. In [9], [48] and [16], a bijection $V \mapsto \Pi(V)$ between absolutely irreducible 2-dimensional continuous L-representations [27] of the absolute Galois group $\mathcal{G}_{\mathbf{Q}_p}$ of $\mathbf{Q}_p$ and admissible unitary non-ordinary Banach L-representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ which are topologically absolutely irreducible is established. The inverse functor $\Pi \mapsto V(\Pi)$ is sometimes referred to as the Montréal functor, cf. [9, §IV].

The basic strategy of the construction of the functor $V \mapsto \Pi(V)$ is the following: by Fontaine's equivalence, the category of local Galois representations in L-vector spaces is equivalent to that of étale (φ, Γ)-modules over Fontaine's field $\mathscr{E}_L$ [28]. The latter (linearized-) category is considered to be an upgrade as one can often perform explicit computations. Any such (φ, Γ)-module D can be naturally seen as a P^+-equivariant sheaf [29] over $\mathbf{Z}_p$, where $P^+ = \left(\begin{smallmatrix} \mathbf{Z}_p - \{0\} & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ is a sub-semi-group of the mirabolic subgroup $(= \left(\begin{smallmatrix} \mathbf{Q}_p^\times & \mathbf{Q}_p \\ 0 & 1 \end{smallmatrix}\right))$ of $\mathrm{GL}_2(\mathbf{Q}_p)$. If U is a compact open subset of $\mathbf{Z}_p$, we denote by $D \boxtimes U$ the sections over U of this sheaf. In [9], a magical involution w_D acting on $D \boxtimes \mathbf{Z}_p^\times$ is defined, allowing one (noting tht $\mathbf{P}^1(\mathbf{Q}_p)$ is built by glueing two copies of $\mathbf{Z}_p$ along $\mathbf{Z}_p^\times$) to extend D to a $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant sheaf over [30] $\mathbf{P}^1$, which is denoted $D \boxtimes_\omega \mathbf{P}^1$, where $\omega = (\det D)\chi^{-1}$ [31]. One then cuts out the desired Banach representation $\Pi(V)$ (and its dual) from the constituents of $D \boxtimes_\omega \mathbf{P}^1$. More precisely, we have a short exact sequence of topological $\mathrm{GL}_2(\mathbf{Q}_p)$-modules

$$0 \to \Pi(V)^* \otimes \omega \to D \boxtimes_\omega \mathbf{P}^1 \to \Pi(V) \to 0.$$

Let $\mathscr{R}_L$ denote the Robba ring [32] with coefficients in L. By a combination of results of Cherbonnier-Colmez ([5]) and Kedlaya ([38]), the categories of étale (φ, Γ)-modules over $\mathscr{E}_L$ and $\mathscr{R}_L$ are equivalent. Call $D \mapsto D_{\mathrm{rig}}$ this correspondence. We have analogous constructions as above for D_{rig} and, in particular,

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27. During all this text, L will denote the coefficient field, which is a finite extension of $\mathbf{Q}_p$.

28. The field $\mathscr{E}_L$ is defined as the Laurent series $\sum_{n \in \mathbf{Z}} a_n T^n$ such that $a_n \in L$ are bounded and $\lim_{n \to -\infty} a_n = 0$. $\mathscr{E}$ is equipped with continuous actions of $\Gamma = \mathbf{Z}_p^\times$ (we note σ_a, $a \in \mathbf{Z}_p^\times$, its elements) and an operator φ defined by the formulas $\sigma_a(T) = (1+T)^a - 1$ and $\varphi(T) = (1+T)^p - 1$. Recall that a (φ, Γ)-module is a free $\mathscr{E}$-module equipped with semi-linear continuous actions of Γ and φ.

29. The matrix $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ codifies the action of φ, $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ the action of $\sigma_a \in \Gamma$ and $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ the multiplication by $(1+T)^b$.

30. From now on, $\mathbf{P}^1$ will always mean $\mathbf{P}^1(\mathbf{Q}_p)$.

31. The character $\det D$ is the character of $\mathbf{Q}_p^\times$ defined by the actions of φ and Γ on $\wedge^2 D$. If D is étale, it can also be seen as a Galois character via local class field theory. The character $\chi \colon x \mapsto x|x|$ denotes the cyclotomic character. We see both characters as characters of $\mathrm{GL}_2(\mathbf{Q}_p)$ by composing with the determinant.

32. It is defined as the ring of Laurent series $\sum_n a_n T^n$, $a_n \in L$, converging on some annulus $0 < v_p(T) \le r$ for some $r > 0$.

we have a $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant sheaf $U \mapsto D_{\mathrm{rig}} \boxtimes U$ over $\mathbf{P}^1$. If we note $\Pi(V)^{\mathrm{an}}$ the locally analytic vectors of $\Pi(V)$, we get an exact sequence

$$0 \mapsto (\Pi(V)^{\mathrm{an}})^* \otimes \omega \to D_{\mathrm{rig}} \boxtimes_\omega \mathbf{P}^1 \to \Pi(V)^{\mathrm{an}} \to 0.$$

The construction however of $D_{\mathrm{rig}} \boxtimes_\omega \mathbf{P}^1$ is not a straightforward consequence of $D \boxtimes_\omega \mathbf{P}^1$. This is mainly because the formula defining the involution does not converge for a (φ, Γ)-module over $\mathscr{R}_L$ [33].

Inspired by the calculations of the p-adic local correspondence for trianguline [34] étale (φ, Γ)-modules, Colmez ([11]) has recently given a direct construction, for a (not necessarily étale) (φ, Γ)-module Δ (of rank 2) over $\mathscr{R}_L$, of a locally analytic L-representation $\Pi(\Delta)$ of $\mathrm{GL}_2(\mathbf{Q}_p)$. More precisely, we have the following theorem:

Theorem 9.1 ([11], Théorème 0.1)**.** *There exists a unique extension of Δ to a $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant sheaf of $\mathbf{Q}_p$-analytic type* [35] *$\Delta \boxtimes_\omega \mathbf{P}^1$ over $\mathbf{P}^1$ with central character ω. Moreover, there exists a unique admissible locally analytic L-representation $\Pi(\Delta)$, with central character ω, of $\mathrm{GL}_2(\mathbf{Q}_p)$, such that*

$$0 \to \Pi(\Delta)^* \otimes \omega \to \Delta \boxtimes_\omega \mathbf{P}^1 \to \Pi(\Delta) \to 0.$$

The purpose of the present work is to study this correspondence in the context of arithmetic families of (φ, Γ)-modules [36]. The arguments in [11] strongly rely on the cohomology theory of locally analytic representations developed in [43], and specifically on Shapiro's lemma. Since the authors are not aware of any reference for these results in the relative setting, we develop, in an appendix (cf. §A), the necessary definitions and properties of locally analytic $\mathrm{GL}_2(\mathbf{Q}_p)$-representations in A-modules. Since this point might carry some interest on its own, we describe it in more detail in §9.3 below. We will exclusively work with affinoid spaces in the sense of Tate, rather than Berkovich or Huber. Let A be a $\mathbf{Q}_p$-affinoid algebra and let $\mathscr{R}_A$ be the relative Robba ring over A. Our main result can be stated as follows:

Theorem 9.2. *Let A be a $\mathbf{Q}_p$-affinoid algebra and let Δ be a trianguline (φ, Γ)-module over $\mathscr{R}_A$ of rank 2 which is an extension of $\mathscr{R}_A(\delta_2)$ by $\mathscr{R}_A(\delta_1)$, where $\delta_1, \delta_2 : \mathbf{Q}_p^\times \to A^\times$ are locally analytic characters satisfying some regularity assumptions* [37]*. Then there exists a unique extension of Δ to a $\mathrm{GL}_2(\mathbf{Q}_p)$-*

33. To construct the involution on D_{rig} in the étale case, one shows that w_D stabilises $D^\dagger \boxtimes \mathbf{Z}_p^\times$, where $D^\dagger$ is the (φ, Γ)-module over the overconvergent elements $\mathscr{E}_L^\dagger$ of $\mathscr{E}_L$ corresponding to D by the Cherbonnier-Colmez correspondence, and that it defines by continuity an involution on $D_{\mathrm{rig}} \boxtimes \mathbf{Z}_p^\times$.

34. A rank 2 (φ, Γ)-module is trianguline if it is an extension of rank 1 (φ, Γ)-modules.

35. A sheaf $U \mapsto M \boxtimes U$ is of $\mathbf{Q}_p$-analytic type if, for every open compact $U \subseteq \mathbf{P}^1$ and every compact $K \subseteq \mathrm{GL}_2(\mathbf{Q}_p)$ stabilizing U, the space $M \boxtimes U$ is of LF-type and a continuous $\mathcal{D}(K)$-module, where $\mathcal{D}(K)$ is the distribution algebra over K.

36. Results concerning representations in famillies on the ℓ-adic side (i.e. the *classical* local Langlands correspondence, cf. [31]) have been achieved by Emerton-Helm in [24].

37. Precisely, we suppose that $\delta_1 \delta_2^{-1}$ is pointwise never of the form χx^i or x^{-i} for some $i \geq 0$.

equivariant sheaf of $\mathbf{Q}_p$-analytic type $\Delta \boxtimes_\omega \mathbf{P}^1$ over $\mathbf{P}^1$ with central character $\omega = \delta_1\delta_2\chi^{-1}$ and a (not necessarily unique) locally analytic $\mathrm{GL}_2(\mathbf{Q}_p)$-*representation*[38] $\Pi(\Delta)$ *in A-modules with central character ω, living in an exact sequence*

$$0 \to \Pi(\Delta)^* \otimes \omega \to \Delta \boxtimes_\omega \mathbf{P}^1 \to \Pi(\Delta) \to 0.$$

This result is expected to have applications to the study of *eigenvarieties*, however in this paper we make no attempt to say anything in this direction.

9.2 The construction of the correspondence

The construction of the correspondence follows the general lines of [11], but several technical difficulties appear along the way. Let's briefly describe how to construct the correspondence $\Delta \mapsto \Pi(\Delta)$ and the additional problems that arise in the relative (affinoid) setting.

From the calculation of the locally analytic vectors of the unitary principal series ([8, Théorème 0.7]), one knows that, if D is an étale trianguline (φ, Γ)-module over $\mathscr{E}_L$ of dimension 2, then $(\Pi(D))^{\mathrm{an}}$ is an extension of locally analytic principal series. The idea of [11] is to intelligently reverse this *dévissage* of $D_{\mathrm{rig}} \boxtimes_\omega \mathbf{P}^1$ in order to actually construct it from these pieces.

For the rest of this introduction let $G = \mathrm{GL}_2(\mathbf{Q}_p)$ and $\overline{B}$ be its lower Borel subgroup and let δ_1, δ_2 and ω be as in Theorem 9.2. Using a relative version of the dictionary of p-adic functional analysis, we construct, for $? \in \{+, -, \emptyset\}$, G-equivariant sheaves $\mathscr{R}_A^?(\delta_1) \boxtimes_\omega \mathbf{P}^1$ (with central character ω) of $\mathbf{Q}_p$-analytic type living in an exact sequence

$$0 \to \mathscr{R}_A^+(\delta_1) \boxtimes_\omega \mathbf{P}^1 \to \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1 \to \mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1 \to 0.$$

Moreover, one can get identifications $B_A(\delta_2, \delta_1)^* \otimes \omega \cong \mathscr{R}_A^+(\delta_1) \boxtimes_\omega \mathbf{P}^1$ and $B_A(\delta_1, \delta_2) \cong \mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1$, where $B_A(\delta_1, \delta_2) = \mathrm{Ind}_{\overline{B}}^G(\delta_1\chi^{-1} \otimes \delta_2)$ denotes the locally analytic principal series. These identifications allow us to consider the locally analytic principal series (and their duals) as (the global sections of) G-equivariant sheaves over $\mathbf{P}^1$ of interest.

We then construct the G-equivariant sheaf $\Delta \boxtimes_\omega \mathbf{P}^1$ over $\mathbf{P}^1$ as an extension of $\mathscr{R}_A(\delta_2) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$. This is done, as in [11], by showing that extensions of $\mathscr{R}_A(\delta_2)$ by $\mathscr{R}_A(\delta_1)$ correspond to extensions of $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$. One then shows that an extension of $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$ uniquely extends to an extension of $\mathscr{R}_A(\delta_2) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$. Once the sheaf $\Delta \boxtimes_\omega \mathbf{P}^1$ is constructed, one shows that the intermediate extension of $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1$ splits and thus one can separate the spaces that are Fréchets from those that are an inductive limit of Banach spaces so as to cut out the desired representation $\Pi(\Delta)$.

The fact that, for $? \in \{+, -, \emptyset\}$, the P^+-module $\mathscr{R}_A^?(\delta_1)$ can be seen as sections over $\mathbf{Z}_p$ of a G-equivariant sheaf over $\mathbf{P}^1$, and that the semi-group $\overline{P}^+ =$

38. See Definition A.24 for the definition of a locally analytic G-representation in A-modules.

$\left(\begin{smallmatrix} \mathbf{Z}_p - \{0\} & 0 \\ p\mathbf{Z}_p & 1 \end{smallmatrix}\right)$ stabilizes $\mathbf{Z}_p$, show that $\mathscr{R}_A^?(\delta_1) = \mathscr{R}_A^?(\delta_1) \boxtimes_\omega \mathbf{Z}_p$ is automatically equipped with an extra action of the matrix $\left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right)$. We call

$$\mathscr{R}_A^?(\delta_1, \delta_2) := (\mathscr{R}_A^?(\delta_1) \boxtimes_\omega \mathbf{Z}_p) \otimes \delta_2^{-1}$$

the $\overline{P}^+$-module thus defined. The technical heart for proving Theorem 9.2 is a comparison result between the cohomology of the semi-groups $A^+ = \left(\begin{smallmatrix} \mathbf{Z}_p - \{0\} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\overline{P}^+$ with values in $\mathscr{R}_A(\delta_1\delta_2^{-1})$ and $\mathscr{R}_A(\delta_1, \delta_2)$, respectively.

Theorem 9.3. *The restriction morphism from $\overline{P}^+$ to A^+ induces an isomorphism*

$$H^1(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2)) \to H^1(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1})).$$

The semi-group A^+ should be thought of as encoding the action of φ and Γ. The difficulty of course is to codify the action of the involution and this is the underlying idea for considering the semi-group $\overline{P}^+$. Indeed $\overline{P}^+$ should be thought of as getting closer to tracking the involution. Theorem 9.3 is (essentially) saying that a trianguline (φ, Γ)-module as in Theorem 9.2 admits an extension to a G-equivariant sheaf over $\mathbf{P}^1$.

Let us briefly describe the proof of Theorem 9.3. The main idea is to reduce this bijection to the case of a point (i.e to the case where $A = L$ is a finite extension of $\mathbf{Q}_p$). The first step is to build a *Koszul* complex which calculates $\overline{P}^+$-cohomology.

Proposition 9.4. *Let M be an $A[\overline{P}^+]$-module such that the action of $\overline{P}^+$ extends to an action of the Iwasawa algebra $\mathbf{Z}_p[[\overline{P}^+]]$. Then the complex*

$$\mathscr{C}_{\tau,\varphi,\gamma}(M) : 0 \to M \xrightarrow{X} M \oplus M \oplus M \xrightarrow{Y} M \oplus M \oplus M \xrightarrow{Z} M \to 0$$

where[39]

$$\begin{aligned} X(x) &= ((1-\tau)x, (1-\varphi)x, (\gamma-1)x) \\ Y(x,y,z) &= ((1-\varphi\delta_p)x + (\tau-1)y, (\gamma\delta_a - 1)x + (\tau-1)z, (\gamma-1)y + (\varphi-1)z) \\ Z(x,y,z) &= (\gamma\delta_a - 1)x + (\varphi\delta_p - 1)y + (1-\tau)z \end{aligned}$$

calculates $\overline{P}^+$-cohomology. That is $H^i(\mathscr{C}_{\tau,\varphi,\gamma}(M)) = H^i(\overline{P}^+, M)$.

The asymmetric nature of $\mathscr{C}_{\tau,\varphi,\gamma}(M)$ is due to the non-commutativity of $\overline{P}^+$. A crude estimation of the maps X, Y and Z leads to the following corollary.

Corollary 9.5. *The complex $\mathscr{C}_{\tau,\varphi,\gamma}(\mathscr{R}_A(\delta_1, \delta_2))$ is a pseudo-coherent complex concentrated in degrees $[0,3]$. In the terminology of the body of the paper, $\mathscr{C}_{\tau,\varphi,\gamma}(\mathscr{R}_A(\delta_1, \delta_2)) \in \mathcal{D}_{\mathrm{pc}}^{[0,3]}(A)$*[40]*. In particular the cohomology groups $H^i(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2))$ are finite A-modules.*

39. Here $\tau = \left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right)$ and $\delta_x = \frac{\tau^x - 1}{\tau - 1}$ for all $x \in \mathbf{Z}_p^\times$.

40. We refer the reader to §12.1 for the notion of a pseudo-coherent complex and the definition of $\mathcal{D}_{\mathrm{pc}}^-(A)$.

More precisely the proof of Corollary 9.5 is reduced to proving finiteness of a *twisted* (φ,Γ)-cohomology of $\mathscr{R}_A(\delta_1,\delta_2)$, cf. Lemma 12.11.

The issue with $\mathscr{C}_{\tau,\varphi,\gamma}(M)$ is that the operators δ_x are difficult to comprehend, rendering the complex almost impractical for explicit computations. One can however *linearize* the situation and pass to the Lie algebra, where calculations are often feasible.

Proposition 9.6. *If* $M \in \left\{\mathscr{R}_L^+(\delta_1,\delta_2), \mathscr{R}_L^-(\delta_1,\delta_2), \mathscr{R}_L(\delta_1,\delta_2)\right\}$ *then the complex*

$$\mathscr{C}_{u^-,\varphi,a^+}(M): 0 \to M \xrightarrow{X'} M \oplus M \oplus M \xrightarrow{Y'} M \oplus M \oplus M \xrightarrow{Z'} M \to 0,$$

where [41]

$$\begin{aligned} X'(x) &= ((\varphi-1)x, a^+x, u^-x) \\ Y'(x,y,z) &= (a^+x-(\varphi-1)y, u^-y-(a^++1)z, (p\varphi-1)z-u^-x) \\ Z'(x,y,z) &= u^-x+(p\varphi-1)y+(a^++1)z \end{aligned}$$

calculates the Lie-algebra cohomology of $\overline{P}^+$. *In particular,* $H^0(\tilde{P}, H^i(\mathscr{C}_{u^-,\varphi,a^+}(M))) = H^i(\overline{P}^+, M)$ [42].

A long, tedious but direct calculation then leads to the following corollary.

Corollary 9.7. *The L-vector space* $H^2(\overline{P}^+, \mathscr{R}_L(\delta_1,\delta_2))$ *is of dimension 1.*

Corollaries 9.5 and 9.7 allow for an analysis of a spectral sequence to take place and prove Theorem 9.3 in the case where A is reduced. One then concludes via an induction argument on the index of nilpotence of the nilradical of A. Via the complex $\mathscr{C}_{u^-,\varphi,a^+}(M)$ we also obtain an alternative proof of [11, Proposition 5.18] in the case of a cyclotomic (φ,Γ)-module. Along the way we show a comparison isomorphism relating continuous cohomology and analytic cohomology for certain (φ,Γ)-modules (cf. Proposition 11.4 for a precise statemtent).

Armed with Theorem 9.3, the reader may notice at this point however, that there is an absence of theory required to conclude (or even make sense of) Theorem 9.2. The following questions are therefore unavoidable:

Q1 What is a locally convex A-module?

Q2 What is a locally analytic G-representation in A-modules?

Q3 What is the relation between locally analytic G-representations in A-modules and modules equipped with a (separately) continuous action of the relative distribution algebra $\mathscr{D}(G,A)$?

We provide a set of answers to these questions (**A1-A3**) and prove some fundamental properties regarding the locally analytic cohomology theory of $\mathscr{D}(G,A)$-modules, which we describe in the following section.

41. Here $a^+ = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $u^- = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$ are the usual elements of the Lie algebra $\mathfrak{gl}_2$ of GL_2.

42. Here $\tilde{P} = \left(\begin{smallmatrix}\mathbf{Z}_p^\times&0\\p\mathbf{Z}_p&1\end{smallmatrix}\right)$, is the *non-discrete* subgroup of $\overline{P}^+$.

9.3 Analytic families of locally analytic representations

Recall that for a locally $\mathbf{Q}_p$-analytic group H, a theory of locally analytic representations of the group H in L-vector spaces has been developed by Schneider and Teitelbaum (cf. [53], [52], [54]). In order to construct the A-module $\Pi(\Delta)$ of Theorem 9.2, with a locally analytic action of G, we need to develop a reasonable framework to make sense of such an object. It turns out that, with some care, much of the existent theory can be extended without serious difficulties to the relative context.

Definition 9.8 (**A1**). *A locally convex A-module is a topological A-module whose underlying topology is a locally convex $\mathbf{Q}_p$-vector space. We let* LCS_A *be the category of locally convex A-modules. Its morphisms are all continuous A-linear maps.*

There is a notion of a strong dual in the category LCS_A, however outside of our applications, it is ill-behaved (in the sense that there are few reflexive objects which are not free A-modules). Let H be a locally $\mathbf{Q}_p$-analytic group.

Definition 9.9 (**A2**). *We define the category* $\mathrm{Rep}_A^{\mathrm{la}}(H)$ *whose objects are barrelled, Hausdorff, locally convex A-modules M equipped with a topological A-linear action of H such that, for every $m \in M$, the orbit map $h \mapsto h \cdot m$ is a locally analytic function on H with values in M.*

Denote $\mathrm{LA}(H, A)$ the space of locally analytic functions on H with values in A and $\mathscr{D}(H, A) = \mathrm{Hom}_{A,\mathrm{cont}}(\mathrm{LA}(H, A), A)$ (equipped with the strong topology) its strong A-dual, the space of A-valued distributions on H. Both $LA(H, A)$ and $\mathscr{D}(H, A)$ are locally convex A-modules. In order to algebrize the situation, one proceeds as in [53] and shows that a locally analytic representation of H is naturally a module over the relative distribution algebra. More precisely let $\mathrm{Rep}_A^{\mathrm{la},LB}(H) \subseteq \mathrm{Rep}_A^{\mathrm{la}}(H)$ denote the full subcategory consisting of spaces which are of A-LB-type (i.e inductive limit of Banach spaces whose transition morphism are A-linear) and complete. Then our main result in §A can be stated as follows:

Theorem 9.10 (**A3**). *Every locally analytic representation of H carries a separably continuous A-linear structure of $\mathscr{D}(H, A)$-module*[43]*. Moreover, the category* $\mathrm{Rep}_A^{\mathrm{la},LB}(H)$ *is isomorphic to the category of complete, Hausdorff locally convex A-modules which are of A-LB-type equipped with a separately continuous $\mathscr{D}(H, A)$-action with morphisms all continuous $\mathscr{D}(H, A)$-linear maps.*

The idea to prove Theorem 9.10 is of course to reduce to the well known result of Schneider-Teitelbaum, cf. [53, Theorem 2.2]. To achieve this, the main intermediary result required is the following proposition.

Proposition 9.11. *There is an isomorphism of locally convex A-modules*

$$\mathscr{D}(H, A) = \mathscr{D}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota} A.$$

43. More precisely, a separately continuous A-bilinear map $\mathscr{D}(H, A) \times M \to M$.

In the case where H is compact we show that $\mathrm{LA}(H, A)$ satisfies a boundedness property which we call A-regular (we refer the reader to Definition A.20 and Lemma A.22 for the precise statements). This is enough to prove Proposition 9.11.

Remark 9.12. Proposition 9.11 would also follow immediately if $\mathrm{LA}(H, A)$ is complete (for H compact). To the best of our knowledge this seems to be an open question if the dimension of $H \geq 2$. If $H \cong \mathbf{Z}_p$, one can identity $\mathrm{LA}(\mathbf{Z}_p, A)$ with the negative powers of $\mathscr{R}_A$ and conclude the result, cf. Lemma A.14. In particular $\mathrm{LA}(\mathbf{Z}_p, A)$ is an example of an A-reflexive object, which is not free as an A-module.

Finally with the equivalence of Theorem 9.10 in mind, we switch our attention to cohomological questions concerning the category $\mathrm{Rep}_A^{\mathrm{la}}(H)$.

Definition 9.13. *Let $\mathscr{G}_{H,A}$ denote the category of complete Hausdorff locally convex A-modules with the structure of a separately continuous A-linear $\mathscr{D}(H, A)$-module, taking as morphisms all continuous $\mathscr{D}(H, A)$-linear maps. More precisely we demand that the module structure morphism*

$$\mathscr{D}(H, A) \times M \to M$$

is A-bilinear and separately continuous.

Following Kohlhaase ([43], [63]), one can develop a locally analytic cohomology theory for the category $\mathscr{G}_{H,A}$. One can define groups $H^i_{\mathrm{an}}(H, M)$ and $\mathrm{Ext}^i_{\mathscr{G}_{H,A}}(M, N)$ for $i \geq 0$ and objects M and N in $\mathscr{G}_{H,A}$ [44]. If H_2 is a closed locally $\mathbf{Q}_p$-analytic subgroup of H_1, we also have an induction functor [45] $\mathrm{ind}_{H_2}^{H_1} : \mathscr{G}_{H_2,A} \to \mathscr{G}_{H_1,A}$. Our main purpose in considering such a theory is to show the following relative version of Shapiro's lemma, which is crucially used in the construction of the correspondence $\Delta \mapsto \Pi(\Delta)$ of Theorem 9.2:

Proposition 9.14 (Relative Shapiro's Lemma)**.** *Let H_1 be a locally $\mathbf{Q}_p$-analytic group and let H_2 be a closed locally $\mathbf{Q}_p$-analytic subgroup. If M and N are objects of $\mathscr{G}_{H_2,A}$ and $\mathscr{G}_{H_1,A}$, respectively, then there are A-linear bijections*

$$\mathrm{Ext}^q_{\mathscr{G}_{H_1,A}}(\mathrm{ind}_{H_2}^{H_1}(M), N) \to \mathrm{Ext}^q_{\mathscr{G}_{H_2,A}}(M, N)$$

for all $q \geq 0$.

Structure of the paper. In §10, we extend the dictionary of p-adic funcational analysis to the relative setting. A key issue is to establish that the sheaf $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$ is G-equivariant over $\mathbf{P}^1$ and is $\mathbf{Q}_p$-analytic.

In §11, we use (φ, Γ)-cohomology to recalculate some results from [4] (in loc.cit. (ψ, Γ)-cohomology was used). A key result for the subsequent chapter is

44. For example $\mathrm{Ext}^1_{\mathscr{G}_{H,A}}(M, N)$ classifies extensions of topological A-modules which are split.

45. This is the dual of the *standard* Induction functor, typically denoted $\mathrm{Ind}_{H_2}^{H_1}$, cf. Lemma A.55.

the nullity of $H^2(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1})$ iff $\delta_1\delta_2^{-1}$ is (pointwise) never of the form χx^i or x^{-i} for some $i \geq 0$ (i.e. $\delta_1\delta_2^{-1}$ is regular).

In §12 and 13, the technical heart of the paper is carried out. We begin by proving the finiteness of $\overline{P}^+$-cohomology for $\mathscr{R}_A(\delta_1, \delta_2)$. Using the Lie-algebra complex we provide a different proof of [11, Proposition 5.18] (in the cyclotomic setting). We show that the dimension of the higher cohomology group $H^2(\overline{P}^+, \mathscr{R}_L(\delta_1, \delta_2))$ is constant (of dimension 1) when $\delta_1\delta_2^{-1}$ is regular.

In §14, Theorem 9.3 is then established.

In §15, the general machinary developed in [11, §6] is used to construct $\Pi(\Delta)$ from a regular trianguline (φ, Γ)-module of rank 2 Δ, over $\mathscr{R}_A$.

In the appendix (§A) we establish a formal framework for the main result. We introduce the category of locally analytic G-representations in A-modules. We prove that there is a relationship between this category and a category of modules over the relative distribution algebra in the same spirit of [53]. There is a locally analytic cohomology theory extending that of [43] and we establish a relative version of Shapiro's Lemma. These results are used in §15.

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9.4 Notations

Let A be a $\mathbf{Q}_p$-affinoid algebra equipped with its Gauss-norm topology (making it a Banach space with norm $|\cdot|_A$ and $v_A = -\log_p |\cdot|_A$ a fixed valuation). We will denote

$$G = \mathrm{GL}_2(\mathbf{Q}_p), \quad A^+ = \begin{pmatrix} \mathbf{Z}_p \backslash \{0\} & 0 \\ 0 & 1 \end{pmatrix}, \quad P^+ = \begin{pmatrix} \mathbf{Z}_p \backslash \{0\} & \mathbf{Z}_p \\ 0 & 1 \end{pmatrix}, \quad \overline{P}^+ = \begin{pmatrix} \mathbf{Z}_p \backslash \{0\} & 0 \\ p\mathbf{Z}_p & 1 \end{pmatrix}.$$

As usual we note $\Gamma = \mathbf{Z}_p^\times$, $\mathbf{P}^1 = \mathbf{P}^1(\mathbf{Q}_p)$ and we assume $p > 2$ throughout. For $n \geq 1$ we set $r_n := \frac{1}{(p-1)p^{n-1}}$ and denote the element $\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$ by τ. For two continuous characters $\delta_1, \delta_2 \colon \mathbf{Q}_p^\times \to A^\times$ we will denote $\delta = \delta_1\delta_2^{-1}\chi^{-1}$ and $\omega = \delta_1\delta_2\chi^{-1}$ where $\chi(x) = x|x|$ corresponds to the cyclotomic character via local class field theory. We denote by $\kappa(\delta_1) := \delta_1'(1)$, the weight of δ_1.

10 Preliminaries

We start by recalling, in the relative case, some well-known constructions that will play a key role in our construction.

10.1 Dictionary of relative functional analysis

Let us first set up some notation and definitions.

10.1.1 Relative Laurent series rings

The theory of relative Robba rings has been expounded by Kedlaya-Liu in [40]. For $0 < r < s \leq \infty$ (with r and s rational, except possibly $s = \infty$), the relative Robba ring $\mathscr{R}_A$ is defined by setting,

$$\mathscr{R}_A^{[r,s]} = \mathscr{R}_{\mathbf{Q}_p}^{[r,s]} \widehat{\otimes}_{\mathbf{Q}_p} A; \quad \mathscr{R}_A^{]0,s]} = \varprojlim_{0<r<s} \mathscr{R}_A^{[r,s]}; \quad \mathscr{R}_A = \varinjlim_{s>0} \mathscr{R}_A^{]0,s]},$$

where $\mathscr{R}_{\mathbf{Q}_p}^{[r,s]}$ is the usual Banach ring of analytic functions on the rigid analytic annulus in the variable T with radii $r \leq v_p(T) \leq s$ with coefficients in $\mathbf{Q}_p$. The Banach ring $\mathscr{R}_A^{[r,s]}$ is equipped with valuation $v^{[r,s]}$ defined by:

$$v^{[r,s]} = \inf_{r \leq v_A(z) \leq s} v_A(f(z)) = \min\left(\inf_{k\in\mathbf{Z}}(v_A(a_k) + rk), \inf_{k\in\mathbf{Z}}(v_A(a_k) + sk)\right)$$

for $f = \sum_{k\in\mathbf{Z}} a_k T^k \in \mathscr{R}_A^{[r,s]}$.

This definition admits an interpretation in terms of rings of analytic functions over a rigid space: if $I = [r,s]$ and if $\mathbb{A}^{1,\mathrm{rig}}_{\mathbf{Q}_p} = \mathrm{Sp}(\mathbf{Q}_p\langle T\rangle)$ denotes the affine rigid line with parameter T, then noting B_I the admissible open subset of $\mathbb{A}^{1,\mathrm{rig}}_{\mathbf{Q}_p}$ defined by $v_p(T) \in I$, we have a natural isomorphism

$$\mathscr{R}_A^I \cong \mathcal{O}(\mathrm{Sp}(A) \times B_I).$$

We can also interpret these rings in terms of Laurent series (power series $\infty \in I$) with coefficients in A in the usual way. For $s < r_1$ we have an A-linear ring endomorphism $\varphi : \mathscr{R}_A^{[r,s]} \to \mathscr{R}_A^{[r/p,s/p]}$, sending T to $(1+T)^p - 1$, inducing an action of the operator φ over $\mathscr{R}_A$ and we also have a continuous action of the group Γ, commuting with that of φ, whose action is given by the formula $\sigma_a(T) = (1+T)^a - 1$, $a \in \mathbf{Z}_p^\times$, over all rings defined above.

Lemma 10.1 (Lemme 1.3 (v) [4]). *For every interval $I \subseteq]0,\infty]$, the ring $\mathscr{R}_A^I$ is a flat A-module. In particular, $\mathscr{R}_A$ is flat over A.*

10.1.2 Locally analytic functions and distributions

The Robba ring $\mathscr{R}_A$ is well interpreted in terms of distributions and locally analytic functions. Define $\mathscr{R}_A^+ := \mathscr{R}_A^{]0,\infty]}$ which is stable under the action of

φ and Γ (equipped with the subspace topology), and note $\mathscr{R}_A^- := \mathscr{R}_A/\mathscr{R}_A^+$ (with the induced action of φ and Γ equipped with the quoitent topology). The algebra of distributions with values in A is defined as[46]

$$\mathscr{D}(\mathbf{Z}_p, A) := \mathscr{D}(\mathbf{Z}_p, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p} A, \tag{10}$$

where the tensor product in (10) is independent of the choice of injective or projective tensor product (as $\mathscr{D}(\mathbf{Z}_p, \mathbf{Q}_p)$ is Fréchet and A is Banach). Let $\mathrm{LA}(\mathbf{Z}_p, A)$ be the space of locally analytic functions on $\mathbf{Z}_p$ with values in A. If $\mu \in \mathscr{D}(\mathbf{Z}_p, A)$, it's Amice transform is defined as

$$\mathscr{A}_\mu = \sum_{n\in\mathbf{N}} \int_{\mathbf{Z}_p} \binom{x}{n} T^n \mu(x) \in \mathscr{R}_A^+.$$

Finally, for $f \in \mathscr{R}_A$, we define its Colmez transform as (for all $x \in \mathbf{Z}_p$)

$$\phi_f(x) = \mathrm{res}_0((1+T)^{-x} f(T)\frac{dT}{1+T}) = \mathrm{res}_0((1+T)^{-x} f dt),$$

where for $f = \sum_{n\in\mathbf{Z}} a_n T^n$, we put $\mathrm{res}_0(f dT) = a_{-1}$ (as usual we set $t := \log(1+T)$). We then have the following result due to Chenevier, cf. [4, Proposition 2.8] (cf. also [37, Lemma 2.1.19]), generalizing those of Colmez, cf. [11, Théorème 2.3] (cf. also [51]):

Proposition 10.2.

- *The Amice transform $\mu \mapsto \mathscr{A}_\mu$ induces a topological isomorphism $\mathscr{D}(\mathbf{Z}_p, A) \cong \mathscr{R}_A^+$.*
- *The Colmez transform $f \mapsto \phi_f(x)$ induces a topological isomorphism $\mathscr{R}_A^- \cong \mathrm{LA}(\mathbf{Z}_p, A) \otimes \chi^{-1}$.*
- *If $\mu \in \mathscr{D}(\mathbf{Z}_p, A)$ and $f \in \mathscr{R}_A$, then $\int_{\mathbf{Z}_p} \phi_f \cdot \mu = \mathrm{res}_0(\sigma_{-1}(\mathscr{A}_\mu) f \frac{dT}{1+T})$.*
- *We have a (φ, Γ)-equivariant short exact sequence*

$$0 \to \mathscr{D}(\mathbf{Z}_p, A) \to \mathscr{R}_A \to \mathrm{LA}(\mathbf{Z}_p, A) \otimes \chi^{-1} \to 0.$$

The Robba ring $\mathscr{R}_A$ is equipped with a left inverse of φ constructed as follows: For $s < r_1$ the map $\oplus_{i=0}^{p-1} \mathscr{R}_A^{[r,s]} \to \mathscr{R}_A^{[r/p,s/p]}$ given by $(f_i)_{i=0,\dots,p-1} \mapsto \sum_{i=0}^{p-1} (1+T)^i \varphi(f_i)$ is a topological isomorphism and allows us to define $\psi : \mathscr{R}_A^{[r/p,s/p]} \to \mathscr{R}_A^{[r,s]}$ by $\varphi \circ \psi = p^{-1} \mathrm{Tr}_{\mathscr{R}_A^{[r/p,s/p]}/\varphi(\mathscr{R}_A^{[r,s]})}$. We also note $\psi : \mathscr{R}_A \to \mathscr{R}_A$ the induced operator, which is continuous, surjective and is a left inverse of φ.

10.1.3 Multiplication by a function

If $\mu \in \mathscr{D}(\mathbf{Z}_p, A)$ and $\alpha \in \mathrm{LA}(\mathbf{Z}_p, A)$, we define the distribution $\alpha\mu$ by the formula

$$\int_{\mathbf{Z}_p} \phi \cdot \alpha\mu := \int_{\mathbf{Z}_p} \alpha\phi \cdot \mu.$$

46. A priori this is different to DefinitionA.29 where the relative distribution algebra for a general locally $\mathbf{Q}_p$-analytic group is defined. Lemma A.31 says that these definitions are equivalent.

If $a \in \mathbf{Z}_p, n \in \mathbf{N}$ and if we take $\alpha = \mathbf{1}_{a+p^n\mathbf{Z}_p}$ the characteristic function of the compact open $a + p^n\mathbf{Z}_p \subseteq \mathbf{Z}_p$, then we note $\mathrm{Res}_{a+p^n\mathbf{Z}_p}$ the multiplication by α. Via the Amice transform this translates to

$$\mathscr{A}_{\mathrm{Res}_{a+p^n\mathbf{Z}_p}(\mu)} = \mathrm{Res}_{a+p^n\mathbf{Z}_p}\mathscr{A}_\mu,$$

where the restriction map on the RHS translates to $\mathrm{Res}_{a+p^n\mathbf{Z}_p} = (1+T)^a\varphi^n \circ \psi^n(1+T)^{-a}$, cf. [11, §2.1.1].

Since ψ is surjective, we have a Γ-equivariant exact sequence

$$0 \to \mathscr{D}(\mathbf{Z}_p^\times, A) \to \mathscr{R}_A^{\psi=0} \to \mathrm{LA}(\mathbf{Z}_p^\times, A) \otimes \chi^{-1} \to 0.$$

Lemma 10.3. *If $\mu \in \mathscr{D}(\mathbf{Z}_p, A)$ and $f \in \mathscr{R}_A$ then we have*

$$\psi(\mathscr{A}_\mu) = \mathscr{A}_{\psi(\mu)} \text{ and } \phi_{\psi(f)} = \psi(\phi_f),$$

where $\psi(\mu)$ is given by $\int_{\mathbf{Z}_p} \phi \cdot \psi(\mu) := \int_{p\mathbf{Z}_p} \phi(x/p) \cdot \mu$, and for any $\phi \in \mathrm{LA}(\mathbf{Z}_p, A)$, we set $\psi(\phi)(x) := \phi(px)$

Proof. In the case where A is a finite extension of $\mathbf{Q}_p$, this is [11, Proposition 2.2]. In our setup the same proof carries over. For the first equality we have that $(\varphi \circ \psi)(\mu) = \mathrm{Res}_{p\mathbf{Z}_p}\mu$ by construction and so $\varphi(\mathscr{A}_{\psi(\mu)}) = \mathrm{Res}_{p\mathbf{Z}_p}\mathscr{A}_\mu = \varphi \circ \psi\mathscr{A}_\mu$, from which we deduce the result.

For the second equality note that

$$\begin{aligned}
\phi_{\psi(f)}(x) &= \mathrm{res}_0((1+T)^{-x}\psi(f)dt) \\
&\overset{(i)}{=} \mathrm{res}_0(\psi((1+T)^{-px}f)dt) \\
&\overset{(ii)}{=} \mathrm{res}_0((1+T)^{-px}f dt) \\
&= \phi_f(px) \\
&= \psi(\phi_f)(x)
\end{aligned}$$

where (i) follows from the identity $\psi((1+T)^{-px}) = (1+T)^{-x}$ and (ii) follows from the identity $\mathrm{res}_0 \circ \psi = \mathrm{res}_0$.

□

The following corollary is now immediate.

Corollary 10.4. *If $\mu \in \mathscr{D}(\mathbf{Z}_p, A)$ (resp. $\phi \in \mathrm{LA}(\mathbf{Z}_p, A)$), the condition $\psi(\mu) = 0$ (resp. $\psi(\phi) = 0$) translates into μ (resp. ϕ) being supported on $\mathbf{Z}_p^\times$.*

10.1.4 The differential operator ∂

We define an A-linear differential operator $\partial : \mathscr{R}_A \to \mathscr{R}_A$ by the formula

$$\partial f := (1+T)\frac{df(T)}{dT}.$$

This operator plays an important role in the subsequent constructions that we will consider.

Lemma 10.5.
- *If $f \in \mathscr{R}_A$ then $\phi_{\partial f}(x) = x\phi_f(x)$.*
- *If $\mu \in \mathscr{D}(\mathbf{Z}_p, A)$, then $\partial \mathscr{A}_\mu = \mathscr{A}_{x\mu}$.*
- *∂ is bijective on $\mathscr{R}_A^{\psi=0}$.*

Proof. In the case when A is a finite extension of $\mathbf{Q}_p$, this is [11, Proposition 2.6 and Lemme 2.7]. In our setup the same proof carries over. For the first point, we have

$$\phi_{\partial f}(x) = \mathrm{res}_0((1+T)^{-x}\partial f\frac{dT}{1+T}) = \mathrm{res}_0((\partial((1+T)^{-x}f)+x(1+T)^{-x}f)\frac{dT}{1+T}) = x\phi_f(x),$$

where for the last equality, we have used the fact that $\mathrm{res}_0(\partial\mathscr{R}_A\frac{dT}{1+T}) = 0$. For the second point, we have

$$\mathscr{A}_{x\mu} = \int_{\mathbf{Z}_p} x(1+T)^x \cdot \mu = \int_{\mathbf{Z}_p} \partial(1+T)^x \cdot \mu = \partial\mathscr{A}_\mu.$$

Finally, the last point follows from the fact that ∂ is bijective on $\mathscr{D}(\mathbf{Z}_p^\times, A)$ and $\mathrm{LA}(\mathbf{Z}_p^\times, A)$ (the first two points show that it is just multiplication by x) and the short exact sequence $0 \to \mathscr{D}(\mathbf{Z}_p^\times, A) \to \mathscr{R}_A^{\psi=0} \to \mathrm{LA}(\mathbf{Z}_p^\times, A) \otimes \chi^{-1} \to 0$. □

10.1.5 $\mathbf{Q}_p$-Analytic sheaves and relative (φ, Γ)-modules

A crucial notion developed by Colmez is that of an analytic sheaf. This plays a greater role in the study of (φ, Γ)-modules for Lubin-Tate extensions (in the sense of [42]) associated to a finite extension $F \neq \mathbf{Q}_p$. In analogy with[11, Defintion 1.6], we define:

Definition 10.6. *Let H be a locally $\mathbf{Q}_p$-analytic semi-group and X an H-space (totally disconnected, compact space on which H acts by continuous endomorphisms). An H-sheaf $\mathscr{M}$ over X is the datum:*

1. *For every compact open $U \subset X$, a topological A-module $\mathscr{M} \boxtimes U$, with $\mathscr{M} \boxtimes \emptyset = 0$*
2. *For each $U \subset V$ of compact opens, there are continuous A-linear restriction maps:*
$$\mathrm{Res}_U^V \colon \mathscr{M} \boxtimes V \to \mathscr{M} \boxtimes U,$$
such that if $U = \cup_{i=1}^n U_i$ and $s_i \in \mathscr{M} \boxtimes U_i$ for $1 \leq i \leq n$, such that
$$\mathrm{Res}_{U_i \cap U_j}^{U_i} s_i = \mathrm{Res}_{U_i \cap U_j}^{U_j} s_j,$$
then there exists a unique $s \in \mathscr{M} \boxtimes U$, such that $\mathrm{Res}_{U_i}^U s = s_i$ for all i.
3. *There are continuous A-linear isomorphisms:*
$$g_U \colon \mathscr{M} \boxtimes U \cong \mathscr{M} \boxtimes gU$$
for every $g \in H$ and U compact open, satisfying the cocycle condition, $g_{hU} \circ h_U = (gh)_U$ for every $g, h \in H$ and U compact open. Moreover for every compact open U, the morphism $g \mapsto g_U$ is a continuous morphism of the stabiliser H_U (of U) to $\mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{M} \boxtimes U)$.

Since we will be primarily interested in attaching H-sheaves to (φ, Γ)-modules, we have the following definition, cf. [37, Definition 2.2.12].

Definition 10.7. *Let $r \in (0, r_1)$. A φ-module over $\mathscr{R}_A^{]0,r]}$ is a finite projective $\mathscr{R}_A^{]0,r]}$-module $M^{]0,r]}$ equipped with an isomorphism*

$$M^{]0,r]} \otimes_{\mathscr{R}_A^{]0,r]},\varphi} \mathscr{R}_A^{]0,r/p]} \cong M^{]0,r]} \otimes_{\mathscr{R}_A^{]0,r]}} \mathscr{R}_A^{]0,r/p]}$$

A (φ, Γ)-module over $\mathscr{R}_A^{]0,r]}$ is φ-module $M^{]0,r]}$ over $\mathscr{R}_A^{]0,r]}$ equipped with a commuting semilinear continuous action of Γ. A (φ, Γ)-module over $\mathscr{R}_A$ is the base change to $\mathscr{R}_A$ of a (φ, Γ)-module over $\mathscr{R}_A^{]0,r]}$ for some r. Let $\Phi\Gamma(\mathscr{R}_A)$ denote the category of (φ, Γ)-modules over $\mathscr{R}_A$. Morphisms are $\mathscr{R}_A$-linear morphisms commuting with the actions of φ and Γ.

In order to equip an H-sheaf with an action of a Lie algebra (so that one can perform explicit calculations), the following definition beckons.

Definition 10.8. *For $(H, X) \in \{(P^+, \mathbf{Z}_p), (G, \mathbf{P}^1)\}$, we say that an H-sheaf $\mathscr{M}$ over X is $\mathbf{Q}_p$-analytic if for all open compact $U \subset X$, $\mathscr{M} \boxtimes U$ is a locally convex A-module of A-LF-type (cf. Definition A.13) and a continuous $\mathscr{D}(K, A)$-module for all open compact subgroups $K \subset H$, stabilizing U.*

The point of Definition 10.8 is that a (φ, Γ)-module Δ over $\mathscr{R}_A$ naturally provides a $\mathbf{Q}_p$-analytic P^+-sheaf over $\mathbf{Z}_p$, which codifies its (φ, Γ)-structure, cf. [11, §1.3.3]. For $z \in \Delta$ one sets

$$\begin{pmatrix} p^k a & b \\ 0 & 1 \end{pmatrix} \cdot z := (1+T)^b \varphi^k \circ \sigma_a(z) \text{ if } k \in \mathbf{N}, a \in \mathbf{Z}_p^\times, b \in \mathbf{Z}_p.$$

If U is an open compact of $\mathbf{Z}_p$, we can write U as a finite disjoint union

$$\coprod_{i \in I} i + p^n \mathbf{Z}_p$$

and we define Res_U by the formula

$$\mathrm{Res}_U = \sum_{i \in I} \mathrm{Res}_{i + p^n \mathbf{Z}_p}$$

where we set

$$\mathrm{Res}_{i+p^n\mathbf{Z}_p} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \circ \varphi^n \circ \psi^n \circ \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}.$$

This turns out to be independent of choice as

$$z = \sum_{i \bmod p} \mathrm{Res}_{i+p\mathbf{Z}_p} z.$$

One then sets $\Delta \boxtimes U$ to be the image of Res_U. The aim is to show that for a trianguline (φ, Γ)-module over $\mathscr{R}_A$, we can extend the corresponding P^+-sheaf over $\mathbf{Z}_p$ to a G-sheaf over $\mathbf{P}^1$. Moreover the global sections of the latter will cut out the locally analytic G-representation in A-modules that we are attempting to attach to Δ.

We have the associated definition for (φ, Γ)-module over $\mathscr{R}_A$.

Definition 10.9. *A (φ, Γ)-module over $\mathscr{R}_A$ is said to be analytic if its associated P^+-sheaf over $\mathbb{Z}_p$ is analytic.*

Remark 10.10. Every (φ, Γ)-module over $\mathscr{R}_A$ is $\mathbf{Q}_p$-analytic (as the base field is $\mathbf{Q}_p$), cf. [2, Lemme 4.1] or [37, Lemma 2.2.14(3)], for why the action of Γ is locally analytic.

10.1.6 Multiplication by a character on $\mathscr{R}_A$

Let L be a finite extension of $\mathbf{Q}_p$. Let $N \geq 0$. For $f \in \mathscr{R}_L \boxtimes \mathbf{Z}_p^\times = \mathscr{R}_L^{\psi=0}$ and $i \in (\mathbf{Z}/p^N\mathbf{Z})^\times$, we write $f_i = \psi^N(1+T)^{-i} f$ so that

$$f = \sum_{i \in (\mathbf{Z}/p^N\mathbf{Z})^\times} (1+T)^i \varphi^N(f_i).$$

If $k \geq 0$, by the Leibnitz rule we can write

$$\partial^k f = \partial^k \left(\sum_{i \in (\mathbf{Z}/p^N\mathbf{Z})^\times} (1+T)^i \varphi^N(f_i) \right) = \sum_{i \in (\mathbf{Z}/p^N\mathbf{Z})^\times} \sum_{j=0}^{k} \binom{k}{j} i^{k-j} (1+T)^i p^{Nj} \varphi^N(\partial^j f_i).$$

This formula suggests the following proposition:

Proposition 10.11 (Proposition 2.9, [11])**.**

1. *If $\delta : \mathbf{Z}_p^\times \to L^\times$ is a locally analytic character and $f \in \mathscr{R}_L^{\psi=0}$, the expression*

$$\sum_{i \in \mathbf{Z}_p^\times (mod\ p^N)} \sum_{j=0}^{+\infty} \binom{\kappa(\delta)}{j} \delta(i) i^{-j} (1+T)^i p^{Nj} \varphi^N(\partial^j f_i),$$

   where $\kappa(\delta) = \delta'(1)$ is the weight of δ, converges in $\mathscr{R}^{\psi=0}$ for $N \geq N(\kappa)$ to an element $m_\delta(f)$ that does not depend on $N \geq N(\kappa)$ or on the choice of representatives of $\mathbf{Z}_p^\times (mod\ p^N)$. Here $N(\kappa)$ is an integer depending on κ.
2. *The map $m_\delta : \mathscr{R}_L \boxtimes \mathbf{Z}_p^\times \to \mathscr{R}_L \boxtimes \mathbf{Z}_p^\times$ thus defined, is continuous and stabilizes $\mathscr{R}_L^+ \boxtimes \mathbf{Z}_p^\times$. Furthermore it induces multiplication by δ on $\mathscr{D}(\mathbf{Z}_p^\times, L)$ and $\mathrm{LA}(\mathbf{Z}_p^\times, L)$.*

Let $N \geq 0$. For $f \in \mathscr{R}_A \boxtimes \mathbf{Z}_p^\times = \mathscr{R}_A^{\psi=0}$ and $i \in \mathbf{Z}_p^\times$, we write $f_i = \psi^n(1+T)^{-i} f$ so that

$$f = \sum_{i \in (\mathbf{Z}/p^N\mathbf{Z})^\times} (1+T)^i \varphi^n f_i.$$

We have the following relative version of Proposition 10.11:

Proposition 10.12. *If $\delta : \mathbf{Z}_p^\times \to A^\times$ is a locally analytic character and $f \in \mathscr{R}_A^{\psi=0}$, the expression*

$$\sum_{i \in \mathbf{Z}_p^\times (mod\ p^N)} \sum_{j=0}^{+\infty} \binom{\kappa(\delta)}{j} \delta(i) i^{-j} (1+T)^i p^{Nj} \varphi^N(\partial^j f_i),$$

where $\kappa(\delta) = \delta'(1)$ is the weight of δ, converges in $\mathscr{R}_A^{\psi=0}$ for N big enough (depending only on δ) to an element $m_\delta(f)$ that does not depend on N or on the choice of representatives of $\mathbf{Z}_p^\times(mod\ p^N)$.

Moreover, the map $m_\delta : \mathscr{R}_A \boxtimes \mathbf{Z}_p^\times \to \mathscr{R}_A \boxtimes \mathbf{Z}_p^\times$ thus defined, is continuous, stabilizes $\mathscr{R}_A^+ \boxtimes \mathbf{Z}_p^\times$ and induces the multiplication by δ on $\mathscr{D}(\mathbf{Z}_p^\times, A)$ and $\mathrm{LA}(\mathbf{Z}_p^\times, A)$.

Proof. For $A = L$ a finite extension of $\mathbf{Q}_p$, this is [11, Proposition 2.9] (cf. also [28, §3.2]). Since a similar argument works for the general case, we only provide a sketch here. We start by recalling some easy and standard estimations. For $0 < r < s < r_2$ and $g \in \mathscr{R}_A^{[r,s]}$ we have

- $v^{[r/p,s/p]}(\varphi(g)) = v^{[r,s]}(g)$.
- $v^{[pr,ps]}(\psi(g)) \geq v^{[r,s]}(g) - 1$.
- $v^{[r,s]}(\sigma_a(g)) = v^{[r,s]}(g)$ for all $a \in \mathbf{Z}_p^\times$.
- $v^{[r,s]}(\partial^k g) \geq v^{[r,s]}(g) - ks$.

By definition of the topology of the LF-space $\mathscr{R}_A$, we need to show that there exists an $s > 0$ such that, for all $0 < r < s$, the general term

$$f_j(\delta) = \binom{\kappa(\delta)}{j} \delta(i) i^{-j} (1+T)^i p^{Nj} \varphi^N(\partial^j f_i)$$

of the series defining $m_\delta(f)$ goes to zero (in $\mathscr{R}_A^{[r,s]}$) as j goes to $+\infty$.

Observe that $f_j(\delta) \in \mathscr{R}_A^{]0,s]}$ whenever $s < r_N$ and $f \in \mathscr{R}_A^{]0,s]}$. We continue estimating the valuation of the terms appearing in the expression for $f_j(\delta)$. For any $0 < r < s < r_N$ we have

- $v^{[r,s]}((1+T)^i) = v^{[r,s]}(i^{-j}) = v^{[r,s]}(\delta(i)) = 0$.
- $v^{[r,s]}(\binom{\kappa(\delta)}{j}) = v_A(\binom{\kappa(\delta)}{j}) \geq j(\min(v_A(\kappa(\delta)), 0) - \frac{1}{p-1})$. Note by $C_\delta = \min(v_A(\kappa(\delta)), 0) - \frac{1}{p-1}$.
- $v^{[r,s]}(\varphi^N(\partial^j f_i)) \geq v^{[r,s]}(f) - N - jp^N s$ (as is shown by an immediate calculation using all the estimations made in the last paragraph).

Putting all this together, we get

$$v^{[r,s]}(f_j(\delta)) \geq v^{[r,s]}(f) + j(C_\delta + N - N/j - p^N s).$$

Observe that this estimation does not depend on r. So for any $N > 0$ large enough (and $s < r_N$) such that $C_\delta + N - N/j - p^N s > 0$ [47] and any $0 < r < s$, the general term $f_j(\delta)$ tends to zero in $\mathscr{R}_A^{[r,s]}$ as $j \to +\infty$ and thus the series converges. This completes the proof of the existence and continuity of $m_\delta(f)$.

47. Take, for instance, any $N > -C_\delta + 1$.

A small calculation shows that the value C_δ can be bounded only in terms of the valuation of $\delta(1+2p)$ and that, for f fixed, the formula defines a rigid analytic function on the A-points of the rigid analytic space $\mathfrak{X}$ whose A points parametrize continuous characters $\mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p^\times, A^\times)$ [48] with values in $\mathscr{R}_A^{[r,s]}$. Using the Zariski density of the points $x \mapsto x^k$ in $\mathfrak{X}$ [49], one shows, on the one hand the independence of $m_\delta(f)$ on the choice of a system of representatives of $\mathbf{Z}_p^\times (\mathrm{mod}\ p^N)$, and on the other hand, using the fact that, if $\delta(x) = x^k$, then $m_\delta(f) = \partial^k f$ and that ∂ extends to $\mathscr{R}_A$ the multiplication by x on $\mathscr{D}(\mathbf{Z}_p, A)$ and on $\mathrm{LA}(\mathbf{Z}_p, A)$, that $m_\delta(f)$ extends also the multiplication by δ for any locally analytic character δ. This completes the proof. $\square$

10.2 Duality

Let $\Phi\Gamma(\mathscr{R}_A)$ denote the category of (φ, Γ)-modules over $\mathscr{R}_A$, cf. Definition 10.7. In the process of constructing $\Pi(\Delta)$ we will need the notion of duality. If $\Delta \in \Phi\Gamma(\mathscr{R}_A)$, we set $\check{\Delta} = \mathrm{Hom}_{\mathscr{R}_A}(\Delta, \mathscr{R}_A(\chi))$ and denote by

$$\langle\ ,\ \rangle \colon \check{\Delta} \times \Delta \to \mathscr{R}_A(\chi)$$

the induced pairing. We impose a (φ, Γ)-structure on $\check{\Delta}$ by setting

$$\langle g \cdot \check{z}, g \cdot z \rangle := g \cdot \langle \check{z}, z \rangle$$

for all $\check{z} \in \check{\Delta}$, $z \in \Delta$ and $g \in \{\sigma_a, \varphi\}$. Note that $\check{\Delta} \in \Phi\Gamma(\mathscr{R}_A)$.

The pairing $\langle\ ,\ \rangle$ defines a new pairing

$$\begin{aligned} \{\ ,\ \} \colon \check{\Delta} \times \Delta &\to A \\ (\check{z}, z) &\mapsto \mathrm{res}_0(\langle \sigma_{-1}(\check{z}), z \rangle), \end{aligned}$$

where $\mathrm{res}_0\left(\sum_{k\in\mathbf{Z}} a_k T^k dT\right) = a_{-1}$. Assuming that Δ is free over $\mathscr{R}_A$, the point is that the pairing $\{\ ,\ \}$ identifies $\check{\Delta}$ and Δ as topological duals of Δ and $\check{\Delta}$ respectively, cf. [14, Proposition III.2.3].

10.3 Principal series

Let $\delta_1, \delta_2 \colon \mathbf{Q}_p^\times \to A^\times$ be two continuous characters. We define $B_A(\delta_1, \delta_2)$ to be the space of locally analytic functions $\phi \colon \mathbf{Q}_p \to A$, such that $\delta(x)\phi\left(\frac{1}{x}\right)$ extends to an analytic function on a neighbourhood of 0. We equip $B_A(\delta_1, \delta_2)$ with an action of G defined by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \phi\right)(x) = \delta_2(ad-bc)\delta(a-cx)\phi\left(\frac{dx-b}{a-cx}\right).$$

48. For the existence of $\mathfrak{X}$, cf. [37, Proposition 6.1.1].
49. i.e that any rigid analytic function on $\mathfrak{X}$ vanishing at those points vanish.

One can show that $B_A(\delta_1, \delta_2) = \mathrm{Ind}_{\overline{B}}^{G}(\delta_1\chi^{-1} \otimes \delta_2)$ (where $\overline{B}$ is the lower-half Borel subgroup of G). Here $\delta_1\chi^{-1} \otimes \delta_2$ is viewed as the character $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \mapsto \delta_1\chi^{-1}(a)\delta_2(d)$. For the definition of $\mathrm{Ind}_{\overline{B}}^{G}(\delta_1\chi^{-1} \otimes \delta_2)$, cf. Remark A.25. The topology of $B_A(\delta_1, \delta_2)$ is by definition the topology coming from $\mathrm{LA}(G/\overline{B}, A)$, cf. Definition A.15. This makes $B_A(\delta_1, \delta_2)$ into a Hausdorff, complete, locally convex A-module, cf. Definition A.4 and Lemma A.14.

The strong topological dual of $B_A(\delta_1, \delta_2)$, cf. Definition A.8, identifies with a space of distributions on $\mathbf{P}^1$ equipped with an action of G defined by

$$\int_{\mathbf{P}^1} \phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mu = \delta_1^{-1}\chi(ad - bc) \int_{\mathbf{P}^1} \delta(cx + d)\phi\left(\frac{ax + b}{cx + d}\right)\mu(x).$$

10.4 The G-module $\mathscr{R}_A(\delta) \boxtimes_\omega \mathbf{P}^1$

Suppose $\Delta \in \Phi\Gamma(\mathscr{R}_A)$ is of rank 2 and trianguline. In this section, we follow [11, §4.3], to construct the G-modules $\mathscr{R}_A(\delta) \boxtimes_\omega \mathbf{P}^1$ which will be the constituents of the G-module $\Delta \boxtimes_\omega \mathbf{P}^1$. Once $\Delta \boxtimes_\omega \mathbf{P}^1$ is constructed, we will see that one of its constituents is the representation $\Pi(\Delta)$, which we are searching for. Since we are only interested in constructing representations of $\mathrm{GL}_2(\mathbf{Q}_p)$, the constructions from [11] (where representations of $\mathrm{GL}_2(F)$, for $F/\mathbf{Q}_p$ a finite extension, are constructed) simplify considerably.

We start by recalling a structure result for arithmetic families of (φ, Γ)-modules.

Proposition 10.13 (Theorem 3.1.1,[37])**.** *Let A be a $\mathbf{Q}_p$-affinoid algebra and let $\Delta \in \Phi\Gamma(\mathscr{R}_A)$. There exists $r(\Delta)$ such that, for any $0 < r < r(\Delta)$, $\gamma - 1$ is invertible on $(\Delta^{]0,r]})^{\psi=0}$, and the $A[\Gamma, (\gamma-1)^{-1}]$-module structure on $(\Delta^{]0,r]})^{\psi=0}$ extends uniquely by continuity to a $\mathscr{R}_A^{]0,r]}(\Gamma)$-module structure for which $(\Delta^{]0,r]})^{\psi=0}$ is finite projective of rank $d = \mathrm{rank}_{\mathscr{R}_A} \Delta$. Moreover, if Δ is free over $\mathscr{R}_A$, then $(\Delta^{]0,r]})^{\psi=0}$ admits a set of d generators over $\mathscr{R}_A^{]0,r]}(\Gamma)$.*

Remark 10.14.
- The proof of the last statement of Proposition 10.13 can be found in the the proof [37, Theorem 3.1.1]. In general there exists a finite projective $\mathscr{R}_A$-module N such that $\Delta \oplus N$ is free of rank m over $\mathscr{R}_A$ and the proof in loc.cit. shows that $(\Delta^{]0,r_n]})^{\psi=0}$ admits a set of m generators over $\mathscr{R}_A(\Gamma)$.
- Taking direct limits we also get that $\Delta^{\psi=0}$ is a finite projective module over $\mathscr{R}(\Gamma)$ of rank d, admitting a set of m generators ($m = d$ if Δ is free).
- In the case when $\Delta = \mathscr{R}_A$, one can show that $\Delta^{\psi=0}$ is a free module of rank one over $\mathscr{R}_A(\Gamma)$ generated by $(1 + T)$, cf. [4, Proposition 2.14 and Remarque 2.15].

If $\Delta = \mathscr{R}_A$ we have a short exact sequence of Γ-modules

$$0 \to (\mathscr{R}_A^+)^{\psi=0} \to \mathscr{R}_A^{\psi=0} \to (\mathscr{R}_A^-)^{\psi=0} \to 0.$$

Recall that $(\mathscr{R}_A^+)^{\psi=0} = \mathscr{R}_A^+ \boxtimes \mathbf{Z}_p^\times \cong \mathscr{D}(\mathbf{Z}_p^\times, A)$ via the Amice transform, and that we have an involution w_* on it given by

$$\int_{\mathbf{Z}_p^\times} \phi(x) \cdot w_* \mu = \int_{\mathbf{Z}_p^\times} \phi(x^{-1}) \cdot \mu.$$

The involution is Γ-anti-linear in the sense that we have $w_* \circ \sigma_a = \sigma_a^{-1} \circ w_*$ for all $a \in \mathbf{Z}_p^\times$. We denote by $\iota : \mathscr{R}_A(\Gamma) \to \mathscr{R}_A(\Gamma)$ the involution defined by $\sigma_a \mapsto \sigma_a^{-1}$ on Γ.

Lemma 10.15. *There exists a unique $\mathscr{R}_A(\Gamma)$-anti-linear involution w_* with respect to ι* [50] *on $\mathscr{R}_A \boxtimes \mathbf{Z}_p^\times$ extending that on $\mathscr{R}_A^+ \boxtimes \mathbf{Z}_p^\times$. Moreover, w_* satisfies*

- $w_* = \partial w_* \partial$.
- $\nabla \circ w_* = -w_* \circ \nabla$.
- $w_* \circ \mathrm{Res}_{a+p^n\mathbf{Z}_p} = \mathrm{Res}_{a^{-1}+p^n\mathbf{Z}_p} \circ w_*$, *for all $a \in \mathbf{Z}_p^\times$, $n \geq 1$.*

Proof. Take a generator e of the free $\mathscr{R}_A(\Gamma)$-module $\mathscr{R}_A \boxtimes \mathbf{Z}_p^\times$ of rank one such that $e \in \mathscr{R}_A^+ \boxtimes \mathbf{Z}_p^\times$ (e.g $(1+T)$, cf. Remark 10.14). This forces

$$w_*(\lambda \cdot e) = \iota(\lambda) \cdot w_*(e)$$

for every $\lambda \in \mathscr{R}(\Gamma)$, where $w_*(e) \in \mathscr{R}_A^+ \boxtimes \mathbf{Z}_p^\times$ is well defined since $e \in \mathscr{R}_A^+ \boxtimes \mathbf{Z}_p^\times$.

For the rest of the properties we can use [11, Lemme 2.14] which shows that they hold for w_* acting on $\mathscr{R}_A^+ \boxtimes \mathbf{Z}_p^\times$ (the same proof carries over for any A). We only show the first one, the other two being immediate. Let $z = \lambda \cdot e \in \mathscr{R}_A \boxtimes \mathbf{Z}_p^\times$ for some $\lambda \in \mathscr{R}_A(\Gamma)$. We have

$$\partial \circ w_* \circ \partial(\lambda \cdot e) = \chi(\lambda)\ \partial \circ w_*(\lambda \cdot \partial e) = \chi(\lambda)\ \partial(\iota(\lambda) \cdot w_*(\partial e)) = \iota(\lambda) \cdot \partial \circ w_* \circ \partial(e) = \iota(\lambda) \cdot w_*(e) = w_*(z).$$

□

The following gives a relation between m_δ and w_*.

Lemma 10.16. *If $\delta \colon \mathbf{Z}_p^\times \to A^\times$ is a continuous character, then*

$$m_\delta \circ w_* = w_* \circ m_{\delta^{-1}}.$$

Proof. By Lemma 10.15, the identity is true for $\delta = x^k$ for all $k \in \mathbf{Z}$ (this is because $\partial^k = m_{x^k}$). Now the functions $\delta \mapsto m_\delta \circ w_*$ and $\delta \mapsto w_* \circ m_{\delta^{-1}}$ are rigid functions and coincide on x^k for all $k \in \mathbf{Z}$. Thus they coincide for all δ. □

If $\Delta \in \Phi\Gamma(\mathscr{R}_A)$, $\omega \colon \mathbf{Q}_p^\times \to A^\times$ (for applications ω will be $\delta_1\delta_2\chi^{-1}$ for any two continuous characters $\delta_1, \delta_2 \colon \mathbf{Q}_p^\times \to A^\times$) is a locally analytic character and ι is an involution on $\Delta \boxtimes \mathbf{Z}_p^\times$, we can define a module $\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1$ (cf. [11, §3.1.1] for details) equipped with an action of a group $\tilde{G}$ generated freely by a group $\tilde{Z}$ isomorphic to the torus $\{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right), a \in \mathbf{Q}_p^\times\}$ (acting on $\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1$ via multiplication by ω), a group $\tilde{A}^0 \cong \mathbf{Z}_p^\times$ (encoding the action of σ_a), a group $\tilde{U} \cong p\mathbf{Z}_p$ (encoding

50. i.e satisfying $w_* \circ \lambda = \iota(\lambda) \circ w_*$ for all $\lambda \in \mathscr{R}_A(\Gamma)$.

the multiplication by $(1+T)^b$, $b \in p\mathbf{Z}_p$) and the elements $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (encoding the action of φ) and $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Precisely, the $\tilde{G}$-module $\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1$ is defined as

$$\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1 = \{(z_1, z_2) \in \Delta \times \Delta \ : \ \mathrm{Res}_{\mathbf{Z}_p^\times}(z_1) = \iota(\mathrm{Res}_{\mathbf{Z}_p^\times}(z_2))\}$$

and the action of $\tilde{G}$ on an element $z = (z_1, z_2) \in \Delta \boxtimes_{\omega,\iota} \mathbf{P}^1$ is described by the following formulae:

- $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cdot z = (z_2, z_1)$.
- $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \cdot z = (\omega(a)z_1, \omega(a)z_2)$, $a \in \mathbf{Q}_p^\times$.
- $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot z = (\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)z_1, \omega(a)\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)z_2)$, $a \in \mathbf{Z}_p^\times$.
- If $z' = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)z$, then $\mathrm{Res}_{p\mathbf{Z}_p} z' = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)z_1$ and $\mathrm{Res}_{\mathbf{Z}_p} wz' = \omega(p)\psi(z_2)$.
- If $b \in p\mathbf{Z}_p$ and $z' = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \cdot z$ then $\mathrm{Res}_{\mathbf{Z}_p} z' = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \cdot z_1$ and $\mathrm{Res}_{p\mathbf{Z}_p} wz' = u_b(\mathrm{Res}_{p\mathbf{Z}_p}(z_2))$, where

$$u_b = \omega(1+b)\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) \circ \iota \circ \left(\begin{smallmatrix} (1+b)^{-2} & b(1+b)^{-1} \\ 0 & 1 \end{smallmatrix}\right) \circ \iota \circ \left(\begin{smallmatrix} 1 & (1+b)^{-1} \\ 0 & 1 \end{smallmatrix}\right) \text{ on } \Delta^+ \boxtimes p\mathbf{Z}_p.$$

Lemma 10.17. *The functor $M \mapsto M \boxtimes_{\omega,\iota} \mathbf{P}^1$ is an exact functor from P^+-modules living on $\mathbf{Z}_p$ to $\tilde{G}$-modules living on $\mathbf{P}^1(\mathbf{Q}_p)$.*

Proof. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of P^+-modules. We claim that we have an exact sequence $0 \to M' \boxtimes_{\omega,\iota} \mathbf{P}^1 \to M \boxtimes_{\omega,\iota} \mathbf{P}^1 \to M'' \boxtimes_{\omega,\iota} \mathbf{P}^1$. Let's show that the last arrow is surjective (for exactness in the middle and injectivity, the proof is similar). Let $(c,d) \in M'' \boxtimes_{\omega,\iota} \mathbf{P}^1$ and $(a,b) \in M \times M$ be any lifting. The element $\mathrm{Res}_{\mathbf{Z}_p^\times} a - \iota(\mathrm{Res}_{\mathbf{Z}_p^\times} b)$ maps to zero in M'' and so there exists an element $x \in M'$ such that $\mathrm{Res}_{\mathbf{Z}_p^\times} a - \iota(\mathrm{Res}_{\mathbf{Z}_p^\times} b) = x$. The element $(a - x, b) \in M \boxtimes_{\omega,\iota} \mathbf{P}^1$ maps then to (c,d). $\square$

For $\delta_1, \delta_2 : \mathbf{Q}_p^\times \to A^\times$ continuous characters, recall that we have set $\delta = \delta_1\delta_2^{-1}\chi^{-1}, \omega = \delta_1\delta_2\chi^{-1}$. We will soon be working with $\Delta \in \Phi\Gamma(\mathscr{R}_A)$ which is an extension of $\mathscr{R}_A(\delta_2)$ by $\mathscr{R}_A(\delta_1)$. Thus we need to *twist* appropriately the current involution w_*, cf. Lemma 10.15, on $\mathscr{R}_A \boxtimes \mathbf{Z}_p^\times$. We define an involution $\iota_{\delta_1,\delta_2}$ [51] acting on the module $\mathscr{R}_A(\delta_1) \boxtimes \mathbf{Z}_p^\times$ by the formula [52]

$$\iota_{\delta_1,\delta_2}(f \otimes \delta_1) = (\delta_1(-1) w_* \circ m_{\delta^{-1}}(z)) \otimes \delta_1.$$

We get in this way a module $\mathscr{R}_A(\delta_1) \boxtimes_{\omega,\iota_{\delta_1,\delta_2}} \mathbf{P}^1$, that we simply note $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$, equipped with an action of $\tilde{G}$. We show in what follows that this action of $\tilde{G}$ factorises through G.

Recall that, for a finite extension L of $\mathbf{Q}_p$ and $\Delta \in \Phi\Gamma(\mathscr{R})$ of rank 1 or 2 (if it is of rank 2, assume it is also trianguline), $\omega : \mathbf{Q}_p^\times \to L^\times$ locally analytic and ι an involution on $\Delta \boxtimes \mathbf{Z}_p^\times$, we have (cf. [11]) a G-module $\Delta \boxtimes_\omega \mathbf{P}^1$. the following lemma shows that, for $\Delta = \mathscr{R}_A(\delta_1)$, our construction specializes to that of Colmez.

51. The fact that i_{δ_1,δ_2} is an involution follows from Lemma 10.16.
52. By Proposition 10.12, this formula is well defined.

Lemma 10.18. *Let $\mathfrak{m} \subseteq A$ be a maximal ideal of A, $L = A/\mathfrak{m}$, $\Delta \in \Phi\Gamma(\mathscr{R}_A)$. Assume that Δ is either of rank 1 or 2 (if it is of rank 2, assume it is also trianguline) and ω, ι be as above. Then the $\tilde{G}$-module $(\Delta \boxtimes_\omega \mathbf{P}^1) \otimes_A L$ is canonically isomorphic to $(\Delta \otimes_A L) \boxtimes_{\omega \otimes L} \mathbf{P}^1$.*

Proof. This is immediate. The uniqueness of both involutions w_* defined in Lemma 10.15 above and in [11, Proposition 2.19] shows that they both coincide (since they do on $\mathscr{R}^+ \boxtimes \mathbf{P}^1$). □

The following result provides a link between the $- \boxtimes_\omega \mathbf{P}^1$ construction and principal series, cf. §10.3

Lemma 10.19. *We have*[53]

– $\mathscr{R}_A^+(\delta_1) \boxtimes_\omega \mathbf{P}^1 \cong B_A(\delta_2, \delta_1)^* \otimes \omega$

– $\mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1 \cong B_A(\delta_1, \delta_2)$.

Moreover $\mathscr{R}_A^+(\delta_1) \boxtimes_\omega \mathbf{P}^1$ and $\mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1$ are $\mathbf{Q}_p$-analytic sheaves.

Proof. This is essentially [11, Corollaire 4.11]. The same proof carries over with A in place of L (since one only checks that both actions of G coincide and the coefficient ring plays no role). The last part follows from Lemma A.55. □

For the rest of this paper we note $\mathscr{R}_A(\delta_1, \delta_2)$ to be the $\overline{P}^+$-module[54]

$$\mathscr{R}_A(\delta_1, \delta_2) := (\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{Z}_p) \otimes \delta_2^{-1}.$$

We set $\mathscr{R}_A^+(\delta_1, \delta_2)$ the sub-$\overline{P}^+$-module of $\mathscr{R}_A(\delta_1, \delta_2)$ corresponding to $\mathscr{R}_A^+$, and $\mathscr{R}_A^-(\delta_1, \delta_2)$ to be the quotient of $\mathscr{R}_A(\delta_1, \delta_2)$ by $\mathscr{R}_A^+(\delta_1, \delta_2)$.

Remark 10.20. As A^+-modules, $\mathscr{R}_A(\delta_1, \delta_2)$, $\mathscr{R}_A^+(\delta_1, \delta_2)$ and $\mathscr{R}_A^-(\delta_1, \delta_2)$ are respectively isomorphic to $\mathscr{R}_A(\delta_1\delta_2^{-1})$, $\mathscr{R}_A^+(\delta_1\delta_2^{-1})$ and $\mathrm{LA}(\mathbf{Z}_p, A) \otimes \delta$. The technical heart of this paper is to compare the $\overline{P}^+$-cohomology of $\mathscr{R}_A(\delta_1, \delta_2)$ and the A^+-cohomology of $\mathscr{R}_A(\delta_1\delta_2^{-1})$, cf. §14.

The following is the main result of this section, which is a relative version of [11, Proposition 4.12].

Proposition 10.21. *The action of $\tilde{G}$ on $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$ factorises through G and we have an exact sequence of G-modules*

$$0 \to B_A(\delta_2, \delta_1)^* \otimes \omega \to \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1 \to B_A(\delta_1, \delta_2) \to 0.$$

Moreover $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$ is a $\mathbf{Q}_p$-analytic sheaf.

53. By $B_A(\delta_1, \delta_2)^*$ we mean $\mathrm{Hom}_{A,\mathrm{cont}}(B_A(\delta_1, \delta_2), A)$ equipped with the strong dual topology, cf. Defintition A.8, where $(-)^*$ is denoted by $(-)'_b$ there.

54. We warn the reader that the module we call $\mathscr{R}_A(\delta_1, \delta_2)$ is not the one noted in the same way in [11, §4.3.2]. In our notation, $\mathscr{R}_A(\delta_1, \delta_2)$ corresponds to the module $\mathscr{R}_A(\delta_1, \delta_2, \eta)$ for $\eta = 1$ as defined in [11, §5.6].

Proof. We reduce the result to the case of a point, cf. [11, Proposition 4.12], using an inductive argument on the index $i \geq 0$ of nilpotence of A.

Suppose first that $i = 0$, i.e that A is reduced. Take $z \in \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$ and g in the kernel of $\tilde{G} \to G$. We need to show that $(g-1)z = (z_1, z_2) = 0$. Let $\mathfrak{m} \subseteq A$ be any maximal ideal of A and note $L = A/\mathfrak{m}$. Since $(\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1) \otimes_A L = \mathscr{R}_L(\delta_1) \boxtimes_\omega \mathbf{P}^1$, then we know by [11, Proposition 4.12] that $z_i = 0 \bmod \mathfrak{m}$. If we write $z_i = \sum_{n \in \mathbf{Z}} a_{n,i} T^n$, $i = 1, 2$, this means that $a_{n,i} = 0 \bmod \mathfrak{m}$ and hence, since this holds for every maximal ideal $\mathfrak{m}$ and since A is reduced, we deduce that $a_{n,i} = 0$ for every n and hence $z_i = 0$ as desired.

Suppose now the result is true for every affinoid algebra of index of nilpotence $\leq j$ and let A be an affinoid algebra whose nilradical N satisfies $N^{j+1} = 0$ and g be in the kernel of $\tilde{G} \to G$. We have the following short exact sequence

$$0 \to (\mathscr{R}_{A/N} \boxtimes_\omega \mathbf{P}^1) \otimes_{A/N} N^j \to \mathscr{R}_A \boxtimes_\omega \mathbf{P}^1 \to \mathscr{R}_{A/N^j} \boxtimes_\omega \mathbf{P}^1 \to 0.$$

By the base case of a reduced affinoid algebra and by the inductive hypothesis, the element $g - 1$ induces a linear endomorphism of the short exact sequence above which vanishes on $(\mathscr{R}_{A/N} \boxtimes_\omega \mathbf{P}^1) \otimes_{A/N} N^j$ and $\mathscr{R}_{A/N^j} \boxtimes_\omega \mathbf{P}^1$ respectively. Therefore it vanishes on $\mathscr{R}_A \boxtimes \mathbf{P}^1$, which shows the desired result.

For the second part, we first observe that, if we call $K_m = \left(\begin{smallmatrix} 1+p^m\mathbf{Z}_p & p^m\mathbf{Z}_p \\ p^m\mathbf{Z}_p & 1+p^m\mathbf{Z}_p \end{smallmatrix}\right)$, then the decomposition $K_m = \left(\begin{smallmatrix} 1 & 0 \\ p^m\mathbf{Z}_p & 1+p^m\mathbf{Z}_p \end{smallmatrix}\right)\left(\begin{smallmatrix} 1+p^m\mathbf{Z}_p & p^m\mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ [55] shows that it is enough to show the existence of an action of the distribution algebra of $\overline{U}^m = \left(\begin{smallmatrix} 1 & 0 \\ p^m\mathbf{Z}_p & 1 \end{smallmatrix}\right)$ for some $m \geq 0$.

We claim that, as a consequence of the identity $H^1_{\mathrm{an}}(\overline{U}^1, \mathscr{R}_A(\delta_1, \delta_2)) \cong H^1(\overline{U}^1, \mathscr{R}_A(\delta_1, \delta_2))$ of Proposition 11.4, the action of $\overline{U}^1$ on $\mathscr{R}_A(\delta)$ is locally analytic and hence extends to a separately continuous action of $\mathscr{D}(\overline{U}^1, A)$ (since $\mathscr{R}_A$ is barrelled it will also be jointly continuous, cf. [23, §0.3.11]). Indeed, calling $M = \mathscr{R}_A(\delta_1, \delta_2)$, which is an LF-space, we need to show that, for any $m \in M$, the orbit map $o_m \colon \overline{U}^1 \to M$ is locally analytic. By the definition of a locally analytic function, one reduces to the case where M is Banach.

Consider now any continuous 1-coycle $c \colon \overline{U}^1 \to M$ such that $c(\tau) = m$. Then c defines a function (which we continue to denote by c)

$$c \colon \mathbf{Z}_p \to M : \quad a \mapsto c(\tau^a) = \frac{\tau^a - 1}{\tau - 1} c(\tau)$$

which is, by Proposition 11.4, cohomologous to a locally analytic 1-cocycle. Since 1-coboundaries are trivially locally analytic, it follows that c is locally analytic. By expanding $\tau^a = \sum_{n \geq 0} \binom{a}{n} (\tau - 1)^n$ one gets that

$$c(a) = \sum_{n \geq 1} \binom{a}{n} (\tau - 1)^{n-1} m,$$

55. This decomposition follows by noting that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ ca^{-1} & d-bca^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$, for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_m$.

which shows that the Mahler coefficients of the function c are given by $a_n(c) = (\tau-1)^{n-1}m$. But the Mahler coefficients of the orbit map are nothing but $a_n(o_m) = (\tau-1)^n m$, thus showing that o_m is a locally analytic function, completing the proof. $\square$

11 Cohomology of (φ,Γ)-modules

In this section we recalculate some results of [4] using (φ,Γ)-cohomology. We calculate higher cohomology groups studied in [11, §5], in preparation to extend the results in loc.cit. to the affinoid setting. Let us begin by recalling the definition of analytic cohomology. Let H be a $\mathbf{Q}_p$-analytic semi-group (e.g. A^+, $\overline{P}^+$, G). Let M be a complete, Hausdorff locally convex A-module (cf. Definition A.4) with the structure of a seperately continuous A-linear $\mathscr{D}(H,A)$-module (i.e. M is an object of the category $\mathscr{G}_{H,A}$, cf. Definition A.39). We note $\mathrm{LA}^\bullet(H,M)$ to be the complex

$$0 \to \mathrm{LA}^0(H,M) \xrightarrow{d_1} \mathrm{LA}^1(H,M) \xrightarrow{d_2} \cdots,$$

where $\mathrm{LA}^n(H,M) := \mathrm{LA}(H^n,M)$ and d_{n+1} is the differential

$$d_{n+1}c(g_0,\dots,g_n) = g_0\cdot c(g_1,\dots,g_n)+\sum_{i=0}^{n-1}(-1)^{i+1}c(g_0,\dots,g_ig_{i+1},\dots g_n)+(-1)^{n+1}c(g_0,\dots,g_{n-1}).$$

Throughout $H^i_{\mathrm{an}}(H,M)$ will denote the ith cohomology group of this complex. For a detailed introduction to H^*_{an} (although in a slightly different setting) we refer the reader to the paper of Kohlhasse, cf. [43]. Finally $H^i(H,M)$ will denote continuous (semi-)group cohomology.

For $n \in \mathbf{N}$, denote by $U^n = \begin{pmatrix} 1 & p^n\mathbf{Z}_p \\ 0 & 1 \end{pmatrix}$

Lemma 11.1. *If $\Delta \in \Phi\Gamma(\mathscr{R}_A)$, then $H^i_{\mathrm{an}}(U^n,\Delta) = 0 \ \ \forall n \in \mathbf{N}$, if $i = 0,1$.*

Proof. For $i=0$, we note that $H^0_{\mathrm{an}}(U^n,\Delta) = \Delta^{(1+T)^{p^n}=1}$. For $i=1$, we have a map

$$H^1_{\mathrm{an}}(U^n,\Delta) \hookrightarrow H^1(U^n,\Delta),$$

since the continuous 1-coboundaries are in correspondence with the locally analytic 1-coboundaries. Finally U^n is procyclic and so

$$H^1(U^n,\Delta) = \Delta/\left((1+T)^{p^n}-1\right) = 0,$$

as $(1+T)^{p^n}-1$ is invertible in $\mathscr{R}_A$. $\square$

Denote by $A^0 = \begin{pmatrix} \mathbf{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix}$. and let Δ be a (φ,Γ)-module over $\mathscr{R}_A$. We construct a natural map

$$\Theta^\Delta : \mathrm{Ext}^1(\mathscr{R}_A,\Delta) \to H^1_{\mathrm{an}}(A^+,\Delta).$$

Let $\widetilde{\Delta}$ be an extension of $\mathscr{R}_A$ by Δ and let $e \in \widetilde{\Delta}$ be a lifting of $1 \in \mathscr{R}_A$. Then $g \mapsto (g-1)e$, $g \in A^+$, is an analytic 1-cocycle and induces an element of $H^1_{\mathrm{an}}(A^+, \Delta)$ independent of the choice of e. Thus we obtain the desired map.

Proposition 11.2. *For any (φ, Γ)-module Δ over $\mathscr{R}_A$, Θ^Δ is an isomorphism.*

Proof. For injectivity of Θ^Δ, let $\widetilde{\Delta}$ be an extension of $\mathscr{R}_A$ by Δ in the category of (φ, Γ)-modules whose image under Θ^Δ is zero. Let $e \in \widetilde{\Delta}$ be a lifting of $1 \in \mathscr{R}_A$. Then there exists $d \in \Delta$, such that $(g-1)e = (g-1)d$ for all $g \in A^+$. Then $g(e-d) = e-d$ for all $g \in A^+$ and thus $\widetilde{\Delta} = \Delta \oplus \mathscr{R}_A$ as a (φ, Γ)-module. For surjectivty of Θ^Δ, given a 1-cocycle $g \mapsto c(g) \in \Delta$, we can extend the (φ, Γ)-module structure on Δ to the $\mathscr{R}_A$-module $\widetilde{\Delta} = \Delta \oplus \mathscr{R}_A e$, such that $\varphi(e) = e + c(\varphi)$ and $\gamma(e) = e + c(\gamma)$ for $\gamma \in \Gamma$. □

Next we relate $H^i_{\mathrm{an}}(A^+, \Delta)$ to a Lie-algebra cohomology, where calculations can be made explicit. We denote by Φ^+ the semi-group $\begin{pmatrix} p^{\mathbf{N}} & 0 \\ 0 & 1 \end{pmatrix}$, where $\varphi = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$. We have that $A^+ = \Phi^+ \times A^0$ (this decomposition breaks up the φ-action and the Γ-action). For Δ a (φ, Γ)-module over $\mathscr{R}_A$, we denote by $H^i_{\mathrm{Lie}}(A^+, \Delta)$ to be the cohomology groups of the complex:

$$0 \longrightarrow \Delta \xrightarrow{x \mapsto (\nabla x, (\varphi - 1)x)} \Delta \oplus \Delta \xrightarrow{(a,b) \mapsto (\varphi-1)a - \nabla b} \Delta \longrightarrow 0.$$

We will be interested in the A^0-invariants, $H^i_{\mathrm{Lie}}(\Delta) := H^0(A^0, H^i_{\mathrm{Lie}}(A^+, \Delta))$. By a simple calculation we see that:

$$H^0_{\mathrm{an}}(A^+, \Delta) = H^0_{\mathrm{Lie}}(\Delta) = \Delta^{\varphi=1, \Gamma=1}.$$

Now let $\widetilde{\Delta}$ be an extension of $\mathscr{R}_A$ by Δ and let $e \in \widetilde{\Delta}$be a lifting of $1 \in \mathscr{R}_A$. Then $(\nabla_{\widetilde{\Delta}} e, (\varphi - 1)e)$ is a 1-cocycle in the above complex which is Γ-invariant, whose class does not depend on e. Thus we obtain a map:

$$\Theta^\Delta_{\mathrm{Lie}} : H^1_{\mathrm{an}}(A^+, \Delta) \to H^1_{\mathrm{Lie}}(\Delta).$$

Lemma 11.3. *For any (φ, Γ)-module over $\mathscr{R}_A$, $\Theta^\Delta_{\mathrm{Lie}}$ is an isomorphism.*

Proof. Copy the proof of [11, Lemme 5.6]. □

11.1 Continuous vs. analytic cohomology

In this section we show the following comparison between locally analytic cohomology defined by Lazard, cf. [44, Chapitre V, §2.3] and continuous group cohomology.

Proposition 11.4. *Let $\delta_1, \delta_2 \colon \mathbf{Q}_p^\times \to A^\times$ be continuous characters. If*

$$M \in \left\{ \mathscr{R}_A^+(\delta_1, \delta_2), \mathscr{R}_A^-(\delta_1, \delta_2), \mathscr{R}_A(\delta_1, \delta_2) \right\},$$

then the natural application

$$H^i_{\mathrm{an}}(\overline{P}^+, M) \to H^i(\overline{P}^+, M)$$

is an isomorphism for all $i \geq 0$ (here $H^i(\overline{P}^+, M)$ denotes continuous cohomology).

Proof. From the exact sequence of $\overline{P}^+$-modules

$$0 \to \mathscr{R}^+_A(\delta_1, \delta_2) \to \mathscr{R}_A(\delta_1, \delta_2) \to \mathscr{R}^-_A(\delta_1, \delta_2) \to 0,$$

it suffices to prove the result for $\mathscr{R}^+_A(\delta_1, \delta_2)$ and $\mathscr{R}^-_A(\delta_1, \delta_2)$. So suppose $M \in \mathscr{R}^+_A(\delta_1, \delta_2), \mathscr{R}^-_A(\delta_1, \delta_2)$. Indeed for $? \in \{\mathrm{an}, \emptyset\}$ we have spectral sequences

$$H^i_?(A^+, H^j_?(\overline{U}^1, M)) \implies H^{i+j}_?(\overline{P}^+, M) \tag{11}$$

and for $j \geq 0$ fixed

$$H^i_?(A^0, H^k_?(\Phi^+, H^j_?(\overline{U}^1, M))) \implies H^{i+k}_?(A^+, H^j_?(\overline{U}^1, M)), \tag{12}$$

where $\overline{U}^1 = \begin{pmatrix} 1 & 0 \\ p\mathbf{Z}_p & 1 \end{pmatrix}$. We claim that

$$H^j_{\mathrm{an}}(\overline{U}^1, M) \to H^j(\overline{U}^1, M). \tag{13}$$

is an isomorphism. Indeed this is true for $j = 0$ and for $j = 1$ it is enough to prove that a continuous 1-cocycle $c \colon \overline{U}^1 \to M$ satisfying

$$c(\tau^a) = \frac{\tau^a - 1}{\tau - 1} c(\tau)$$

for all $a \in \mathbf{Z}_p$ is locally analytic. Indeed this follows from the fact that $M \boxtimes_\omega \mathbf{P}^1$ is a $\mathbf{Q}_p$-analytic sheaf, cf. Lemma 10.19 (note that $c(\tau^a) = \sum_{n \geq 1} \binom{a}{n} (\tau - 1)^{n-1} c(\tau)$ and $(\tau - 1)^n c(\tau)$ are the Mahler coefficients of the locally analytic function $\overline{U}^1 \to M$ given by $g \mapsto g \cdot c(\tau)$, cf. [15, §IV.2]). Since Φ^+ is discrete we have an isomorphism

$$H^k_{\mathrm{an}}(\Phi^+, H^j_{\mathrm{an}}(\overline{U}^1, M)) \to H^k(\Phi^+, H^j(\overline{U}^1, M)).$$

The same argument (for proving (13) is an isomorphism) gives an isomorphism

$$H^i_{\mathrm{an}}(A^0, H^k_{\mathrm{an}}(\Phi^+, H^j_{\mathrm{an}}(\overline{U}^1, M))) \to H^i(A^0, H^k(\Phi^+, H^j(\overline{U}^1, M))).$$

Spectral sequences (11) and (12) now give the result. □

Remark 11.5. In the setting of Proposition 11.4, one cannot apply Lazard's classical result [44, Théorème 2.3.10] because M is not of finite type over A. Note also that a similar proof yields isomorphisms $H^i_{\mathrm{an}}(A^+, M) \xrightarrow{\sim} H^i(A^+, M)$ for all $i \geq 0$.

11.2 The cohomology of Φ^+

Let $\delta : \mathbf{Q}_p^\times \to A^\times$ be a continuous character. We next compute explicitly some Φ^+-cohomology. Note that as Φ^+ is discrete, analytic cohomology coincides with standard (continuous) cohomology. In particular we will be interested in the groups $H^i(\Phi^+, \mathscr{R}_A^- \otimes \delta)$ and $H^i(\Phi^+, \mathscr{R}_A^+ \otimes \delta)$ viewed as A^0-modules. Since Φ^+ is infinite cyclic, these cohomology groups vanish for $i \geq 2$.

11.2.1 The case of $\mathscr{R}_A^-$

We begin with some notation. If $N \geq 0$, we set $\mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A) \subset \mathrm{LA}(\mathbf{Z}_p^\times, A)$ to be the free sub A-module of rank $N+1$ consisting of polynomial functions of degree at most N with coefficients in A. Observing that $\mathrm{LA}(\mathbf{Z}_p^\times, A) \cap \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A) = \emptyset$ we set

$$T_N := \mathrm{LA}(\mathbf{Z}_p^\times, A) \oplus \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A).$$

Here is a lemma describing the kernel and cokernel of $1 - \alpha\varphi$ on the locally analytic functions:

Lemma 11.6. *Let $\alpha \in A^\times$. Then*

1. *$1 - \alpha\varphi \colon \mathscr{R}_A^- \to \mathscr{R}_A^-$ is injective.*
2. *If $N \geq 0$ is large enough, then $T_N + (1-\alpha\varphi)\mathscr{R}_A^- = \mathscr{R}_A^-$ and $T_N \cap (1 - \alpha\varphi)\mathscr{R}_A^- = (1-\alpha\varphi)\mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A)$.*

Proof. We first prove injectivity. Note that this is already proved in [4, Lemme 2.9(ii)]. We repeat the argument here. If $\phi \in \mathscr{R}_A^-$ is in the kernel of $1 - \alpha\varphi$, then $\phi = \alpha^n \varphi^n(\phi)\ \forall\, n \in \mathbf{N}$. Recall the action of φ on $\mathscr{R}_A^-$:

$$(\varphi \cdot \phi)(x) = \begin{cases} \phi\left(\frac{x}{p}\right) & \text{if } x \in p\mathbf{Z}_p \\ 0 & \text{if } x \notin p\mathbf{Z}_p \end{cases}$$

Thus ϕ is zero on $p^n\mathbf{Z}_p^\times$ and hence $\phi = 0$, as desired.

We next prove the second assertion. Let $N \geq 0$ be such that $|\alpha^{-1}p^{N+1}| < 1$, so that in particular $1 - \alpha p^{-j} = -\alpha p^{-j}(1 - \alpha^{-1}p^j) \in A^\times$ for all $j > N$. Let $\phi \in \mathrm{LA}(\mathbf{Z}_p, A)$ and n be such that ϕ is analytic on every ball $i + p^n\mathbf{Z}_p$ ($i \in \mathbf{Z}_p^\times$), then we can write $\mathbf{1}_{p^n\mathbf{Z}_p}\phi = \sum_{j=0}^{+\infty} a_j \mathbf{1}_{p^n\mathbf{Z}_p} x^j$ and so the function

$$\phi - (1-\alpha\varphi)\left(\sum_{j=N+1}^{+\infty} \frac{a_j}{1-\alpha p^{-j}} \mathbf{1}_{p^{n-1}\mathbf{Z}_p} x^j\right),$$

which is well defined since the elements $1 - \alpha p^{-j} \in A^\times$, can be expressed as the sum of a polynomial of degree N and a locally analytic function vanishing on $p^n\mathbf{Z}_p$. Hence every $\phi \in \mathscr{R}_A^-$ is of the form $\phi_1 + P_\phi + (1-\alpha\varphi)\phi_2$, with $P_\phi \in \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A)$ and $\phi_1, \phi_2 \in \mathrm{LA}(\mathbf{Z}_p, A)$ such that ϕ_1 vanishes in a neighbourhood of 0. In particular ϕ_1 is of the form $\sum_{i=0}^{n-1} \phi_{1,i}$, with $\phi_{1,i} \in \mathrm{LA}(p^i\mathbf{Z}_p^\times, A)$. Writing

$\varphi^i\psi^i(\phi_{1,i}) = (1-(1-\alpha\varphi))^i\psi^i(\alpha^{-i}\phi_{1,i})$ and upon expanding $(1-(1-\alpha\varphi))^i$ expresses $\phi_{1,i}$ as a sum of elements in $(1-\alpha\varphi)\mathscr{R}_A^-$ and $\psi^i(\alpha^{-i}\phi_{1,i}) \in \mathrm{LA}(\mathbf{Z}_p^\times, A)$.

We calculate next the intersection $T_N \cap (1-\alpha\varphi)\mathscr{R}_A^-$. If $(1-\alpha\varphi)\phi = \phi' + P$ for some $\phi \in \mathrm{LA}(\mathbf{Z}_p, A)$, $\phi' \in \mathrm{LA}(\mathbf{Z}_p^\times, A)$ and $P \in \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A)$, then we have $\psi((1-\alpha\varphi)\phi) = \psi(P)$ (as $\mathrm{LA}(\mathbf{Z}_p^\times, A) = \mathrm{LA}(\mathbf{Z}_p, A)^{\psi=0}$). Thus $(\psi-\alpha)\cdot\phi = \psi(P)$ and hence $\phi(x) = \alpha^{-1}(\phi(px) - P(px))$ for all x (recalling that $(\psi\cdot\phi)(x) = \phi(px)$). Repeating gives $\phi(x) = \alpha^{-n}\phi(p^n x) - \alpha^{-n}P(p^n x) - \alpha^{n-1}P(p^{n-1}x) - \ldots - \alpha^{-1}P(px)$, which shows that ϕ is analytic on $\mathbf{Z}_p$. Writing $\phi(x) = \sum_{i\geq 0} a_i x^i$, $P(x) = \sum_{i=0}^N b_n x^n$ on $\mathbf{Z}_p$ with $a_i, b_i \in A$, the equality $\phi(px) - \alpha\phi(x) = P(px)$ gives

$$\sum_{i=0}^{+\infty}(p^n - \alpha)a_i x^i = \sum_{i=0}^{N} b_n x^n$$

on $\mathbf{Z}_p$. This gives $(p^n-\alpha)a_i = 0$ for $i > N$ and thus $a_i = 0$ (since $(p^n - \alpha) = p^n(1-\alpha p^{-n}) \in A^\times$), which implies that $\phi \in \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A)$ and hence $T_N \cap (1-\alpha\varphi)\mathscr{R}_A^- \subseteq (1-\alpha\varphi)\mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A)$. To prove the reverse inclusion, we note that, if $P(x) = \sum_{i=0}^N a_i x^i \in \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A)$, then

$$\begin{aligned}(1-\alpha\varphi)P &= \mathbf{1}_{\mathbf{Z}_p^\times}\cdot P + \mathbf{1}_{p\mathbf{Z}_p}\cdot\sum_{i=0}^{N}(1-\alpha p^{-i})a_i x^i \\ &= \alpha\mathbf{1}_{\mathbf{Z}_p^\times}\cdot\sum_{i=0}^{N}p^{-i}a_i x^i + \sum_{i=0}^{N}(1-\alpha p^{-i})a_i x^i \in T_N\end{aligned}$$

□

The following three statements are now immediate from the above lemma, the identity $H^i_{\mathrm{an}}(\Phi^+, M) = H^i(\Phi^+, M)$ and the description of the continuous cohomology of a cyclic group.

Corollary 11.7. $H^0(\Phi^+, \mathscr{R}_A^- \otimes \delta) = 0$.

Corollary 11.8. *For N large enough, we have a short exact sequence (of A^0-modules)*

$$0 \to (1-\alpha\varphi)\mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A)\otimes\delta \to T_N\otimes\delta \to H^1(\Phi^+, \mathscr{R}_A^-\otimes\delta) \to 0.$$

11.2.2 The case of $\mathscr{R}_A^+$.

As in the preceding section, the calculation of the cohomology of the group Φ^+ acting on $\mathscr{R}_A^+$ will reduce to the following lemma

Lemma 11.9. *Let $\alpha \in A^\times$, consider $1-\alpha\varphi\colon \mathscr{R}_A^+ \to \mathscr{R}_A^+$ and let N be large enough. Then*

$$\ker(1-\alpha\varphi) = \ker(1-\alpha\varphi\colon \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A)^*),$$

$$\mathrm{coker}(1-\alpha\varphi) = \mathrm{coker}(1-\alpha\varphi\colon \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A)^*).$$

In particular:

1. $\ker(1-\alpha\varphi) = \bigoplus_{i=0}^{N} \mathrm{Ann}(1-\alpha p^i)t^i$ *and* $1-\alpha\varphi$ *is injective if* α *is such that* $(1-\alpha p^i)$ *is not a zero divisor for any* i.

2. $1-\alpha\varphi$ *is surjective if* $1-\alpha p^i \in A^\times$ *for all* $i \in \mathbf{N}$.

Proof. This is essentially [4, Lemme 2.9 (ii)], of which the idea of proof comes from [7, Lemme A.1]. We provide a sketch here. Choose $N \geq 0$ an integer large enough so that $|\alpha p^{N+1}| < 1$. Then $1-\alpha\varphi$ is invertible on $T^{N+1}\mathscr{R}_A^+$. To conclude, it suffices to remark that there is a φ-stable decomposition:

$$\mathscr{R}_A^+ = (\bigoplus_{i=0}^{N} At^i) \oplus T^{N+1}\mathscr{R}_A^+$$

and that $(1-\alpha\varphi)(t^j) = (1-\alpha p^j)t^j$. We get then the desired result for the kernel and cokernel of $1-\alpha\varphi$ observing that Amice transform identifies $\bigoplus_{0\leq i\leq N} At^i$ with the dual of $\mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A)$. □

Remark 11.10. Observe that if $\alpha = p^{-i}$ for some $i \in \mathbf{N}$, then the kernel of $1-\alpha\varphi : \mathscr{R}_A^+ \to \mathscr{R}_A^+$ contains $A \cdot t^i$, which is identified with the free A-module of rank one generated by the distribution sending a function f to $f^{(i)}(0)$ via the Amice transformation.

Corollary 11.11. *Let* $j \in \{0,1\}$ *and* N *be large enough. We have*

$$H^j(\Phi^+, \mathscr{R}_A^+ \otimes \delta) \cong H^j(\Phi^+, \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A)^* \otimes \delta).$$

In particular

- $H^0(\Phi^+, \mathscr{R}_A^+ \otimes \delta) \neq 0$ *if and only if* $1-\delta(p)p^i$ *divides zero for some* $i \in \mathbf{N}$.
- $H^1(\Phi^+, \mathscr{R}_A^+ \otimes \delta) \neq 0$ *if and only if there exists an* $i \in \mathbf{N}$ *such that* $1-\delta(p)p^i$ *vanishes at some point of* $\mathrm{Sp}(A)$.

11.3 The A^0-cohomology

We next compute some A^0-cohomology, for which we use the description of analytic cohomology in terms of the action of the Lie algebra. Fix a continuous character $\eta : \mathbf{Z}_p^\times \to A^\times$, viewing it naturally as a character of A^0.

11.3.1 The case of $\mathrm{LA}(\mathbf{Z}_p^\times, A) \otimes \eta$.

Lemma 11.12. $\nabla + \kappa(\eta)$ *is surjective on* $\mathrm{LA}(\mathbf{Z}_p^\times, A)$ *and its kernel is generated by the set of* $\phi\eta$, *with* ϕ *locally constant.*

Proof. We compute for $\phi \in \mathrm{LA}(\mathbf{Z}_p^\times, A)$,

$$\nabla\phi(x) = \lim_{a\to 1} \frac{\phi(x/a) - \phi(x)}{a-1} = -x\phi'(x)$$

and thus

$$(\nabla + \kappa(\eta))(\phi\eta) = -x(\phi'\eta + \kappa(\eta)x^{-1}\phi\eta) + \kappa(\eta)\phi\eta = \nabla(\phi)\cdot\eta,$$

where the first equality follows from the relation $\eta'(x) = \eta'(1)\eta(x)x^{-1}$. We see that to show (1), multiplying by η if necessary (recall that η takes values in $A^\times$ so multiplication by η is invertible), we can assume $\kappa(\eta) = 0$. The description of the kernel is clear and surjectivity follows from surjectivity of $\phi \mapsto \phi'$ (as we can easily see integrating a power series). $\square$

Proposition 11.13.

1. *$H^0_{\mathrm{an}}(A^0, \mathrm{LA}(\mathbf{Z}_p^\times, A) \otimes \eta)$ is a free A-module of rank 1 generated by $\mathbf{1}_{\mathbf{Z}_p^\times}\eta$.*
2. *$H^1_{\mathrm{an}}(A^0, \mathrm{LA}(\mathbf{Z}_p^\times, A) \otimes \eta) = 0$.*

Proof. Note that for $i \in \{0, 1\}$ and $M \in \Phi\Gamma(\mathscr{R}_A)$, $H^i_{\mathrm{an}}(A^0, M)$ is computed by the A^0-invariants of the cohomology of the complex

$$0 \to M \xrightarrow{\nabla} M \to 0.$$

To see this, it suffices to repeat Lemma 11.3, ignoring the action of Φ^+. As $\nabla(\phi \otimes \eta) = ((\nabla + \kappa(\eta))\phi) \otimes \eta$, the two assertions follow from lemma 11.12: the first cohomology group is trivial since $(\nabla + \kappa(\eta))$ is surjective, and if $\phi\eta \otimes \eta$ is killed by ∇ and fixed by A^0, then $\phi(ax)\eta(x) = \phi(x)\eta(x)$ for every $a \in A^0$ and so ϕ is constant on $\mathbf{Z}_p^\times$, hence the result. $\square$

11.4 The A^+-cohomology

In this section we denote $\delta\colon \mathbf{Q}_p^\times \to A^\times$ a continuous character, $\alpha = \delta(p)$ and $\beta = \delta(a)$. Let N be large enough and set $\mathrm{Pol}_{\le N} = \mathrm{Pol}_{\le N}(\mathbf{Z}_p, A)$. We next calculate the A^+-cohomology of $\mathscr{R}_A^- \otimes \delta$, $\mathscr{R}_A^+ \otimes \delta$ and hence that of $\mathscr{R}_A \otimes \delta$.

11.4.1 The case of $\mathscr{R}_A^- \otimes \delta$.

Lemma 11.14. *We have $H^0(A^+, \mathscr{R}_A^- \otimes \delta) = 0$.*

Proof. This follows from Corollary 11.7. $\square$

Lemma 11.15. *The groups $H^i(A^+, \mathscr{R}_A^- \otimes \delta)$, $i = 1, 2$, live in an exact sequence of A-modules*

$$\begin{aligned} 0 \quad \to \quad & ((1-\alpha\varphi)\mathrm{Pol}_{\le N} \otimes \delta)^\Gamma \xrightarrow{f} (T_N \otimes \delta)^\Gamma \to H^1(A^+, \mathscr{R}_A^- \otimes \delta) \\ \to \quad & H^1(A^0, (1-\alpha\varphi)\mathrm{Pol}_{\le N} \otimes \delta) \xrightarrow{g} H^1(A^0, T_N \otimes \delta) \to H^2(A^+, \mathscr{R}_A^- \otimes \delta) \to 0 \end{aligned}$$

Proof. Inflation-restriction and Corollary 11.7 give

$$H^1(A^+, \mathscr{R}_A^- \otimes \delta) = H^0(A^0, H^1(\Phi^+, \mathscr{R}_A^- \otimes \delta)).$$

The result follows then by taking the long exact sequence of A^0-cohomology associated to the short exact sequence of A^0-modules of Corollary 11.8. $\square$

In order to calculate $H^i(A^+, \mathscr{R}^- \otimes \delta)$, we need to examine the cokernel of f and the kernel of g. We first begin with a lemma stating some preliminary reductions.

Lemma 11.16.

1. $(T_N \otimes \delta)^\Gamma = A \cdot \langle \mathbf{1}_{\mathbf{Z}_p^\times} \delta \rangle \oplus \bigoplus_{i=0}^N \mathrm{Ann}(1 - \beta a^{-i}) \cdot x^i$.
2. $((1-\alpha\varphi)\mathrm{Pol}_{\leq N} \otimes \delta)^\Gamma = (1-\alpha\varphi) \bigoplus_{i=0}^N \mathrm{Ann}(1-\beta a^{-i}) \cdot x^i$
3. $H^1(A^0, T_N \otimes \delta) = H^1(A^0, \mathrm{Pol}_{\leq N} \otimes \delta)$.

Proof. Since $T_N = \mathrm{LA}(\mathbf{Z}_p^\times, A) \oplus \mathrm{Pol}_{\leq N}$ (as A^0-modules), the first point follows from Proposition 11.13(1) and the fact that $(1-\beta\gamma)(a_i x^i) = a_i(1-\beta a^{-i})x^i$.

The second point follows from the same calculation, the fact that γ commutes with φ and Lemma 11.6(1).

The third point is a consequence of Proposition 11.13(2). $\square$

Lemma 11.17. $H^2(A^+, \mathscr{R}_A^- \otimes \delta) = \oplus_{i=0}^N A/(1-\alpha p^{-i}, 1-\beta a^{-i})$.

Proof. As calculated in the proof of Lemma 11.6, if $Q(x) = a_0 + a_1 x + \ldots + a_N x^N \in \mathrm{Pol}_{\leq N}$, then $(1-\alpha\varphi)Q \in T_N = \mathrm{LA}(\mathbf{Z}_p^\times, A) \oplus \mathrm{Pol}_{\leq N}$ is given by

$$(1-\alpha\varphi)Q = \alpha \mathbf{1}_{\mathbf{Z}_p^\times} \cdot \sum_{i=0}^N a_i p^{-i} x^i \oplus \sum_{i=0}^N a_i (1-\alpha p^{-i}) x^i. \tag{14}$$

Using Lemma 11.16(3), we need to calculate the cokernel of the following composition:

$$H^1(A^0, (1-\alpha\varphi)\mathrm{Pol}_{\leq N} \otimes \delta) \to H^1(A^0, T_N \otimes \delta) \xrightarrow{\sim} H^1(A^0, \mathrm{Pol}_{\leq N} \otimes \delta)).$$

By equation (14) above, this map sends the class of $(1-\alpha\varphi)Q$ in $H^1(A^0, (1-\alpha\varphi)\mathrm{Pol}_{\leq N} \otimes \delta)$ to the class of $\sum_{i=0}^N a_i(1-\alpha p^{-i})x^i$ in $H^1(A^0, (1-\mathrm{Pol}_{\leq N} \otimes \delta))$. Since $\mathrm{Pol}_{\leq N} = \oplus_{i=0}^N A \cdot x^i$ and since the action of φ and γ commute, we easily see that this cokernel is given by $\oplus_{i=0}^N A/(1-\alpha p^{-i}, 1-\beta a^{-i})$ as claimed. $\square$

Corollary 11.18. *$H^2(A^+, \mathscr{R}_A^- \otimes \delta) = 0$ if and only if δ is pointwise never of the form x^i for any $i \geq 0$.*

Proof. This follows immediately from the last lemma, observing that, if δ is pointwise never of the form x^i if and only $(1-\alpha p^{-i}, 1-\beta a^{-i}) = A$ for all $i \in \mathbf{N}$. $\square$

An explicit description of the cokernel of f and the kernel of g seems a difficult task, but we can describe them in a particular case that will be of interest to us.

Proposition 11.19. *Let $\delta : \mathbf{Q}_p^\times \to A^\times$ be such that δ is pointwise never of the form x^i for any $i \geq 0$. Then $H^1(A^+, \mathscr{R}_A^- \otimes \delta)$ is a free A-module of rank 1.*

Proof. First note that as δ is pointwise never of the form x^i for any $i \geq 0$, $(1-\alpha p^{-i}, 1-\beta a^{-i}) = A$. The key observation, which appears in the proof of [4, Théorème 2.29], is that, if $a, b \in A$ are such that $(a,b) = A$, then multiplication by a is bijective on A/bA and on $\mathrm{Ann}(b)$. Indeed, if $u, v \in A$ are such that $au + bv = 1$, then multiplication by u provides an inverse for this map.

We first show that $\ker(g) = 0$. As in the proof of Lemma 11.17, we consider the following composition

$$H^1(A^0, (1-\alpha\varphi)\mathrm{Pol}_{\leq N} \otimes \delta) \to H^1(A^0, T_N \otimes \delta) \xrightarrow{\sim} H^1(A^0, \mathrm{Pol}_{\leq N} \otimes \delta))$$

and we show in this case that it is injective. It is easy to see, from the formulas describing the action of φ and Γ, that the problem reduces to showing that, for every $i = 0, \ldots, N$, if $b_i \in A$ is such that $a_i(1-\alpha p^{-i}) = (1-\beta a^{-i})b_i$, then $b_i \in (1-\alpha p^{-i})A$, which is a consequence of the fact that multiplication by $1-\beta a^{-i}$ is bijective on $A/(1-\alpha p^{-i})A$.

In a similar way, one can show that $\mathrm{coker}(f)$ is a free A-module of rank one using Lemma 11.16(1) and (2), and the fact that multiplication by $(1-\alpha p^{-i})$ is bijective on $\mathrm{Ann}(1-\beta a^{-i})$. This shows that $H^1(A^+, \mathscr{R}_A \otimes \delta)$ is a free A-module of rank one and completes the proof.

□

11.4.2 The case of $\mathscr{R}_A^+ \otimes \delta$.

Lemma 11.20. *Let N be large enough. Then*

1. $H^0(A^0, H^0(\Phi^+, \mathscr{R}_A^+ \otimes \delta)) = \bigoplus_{i=0}^N \mathrm{Ann}(1-\alpha p^i, 1-\beta a^i) \cdot t^i$.
2. $H^1(A^0, H^0(\Phi^+, \mathscr{R}_A^+ \otimes \delta)) = \bigoplus_{i=0}^N \mathrm{Ann}(1-\alpha p^i)/(1-\beta a^i) \cdot t^i$.
3. $H^0(A^0, H^1(\Phi^+, \mathscr{R}_A^+ \otimes \delta)) = \bigoplus_{i=0}^N \mathrm{Ann}(1-\beta a^i \colon A/(1-\alpha p^i)) \cdot t^i$.
4. $H^1(A^0, H^1(\Phi^+, \mathscr{R}_A^+ \otimes \delta)) = \bigoplus_{i=0}^N A/(1-\alpha p^i, 1-\beta a^i) \cdot t^i$.

Proof. This follows easily from Lemma 11.9 and Corollary 11.11, observing that $(1-\beta\sigma_a)(t^i) = (1-\beta a^i)t^i$. □

Proposition 11.21. *Let $\delta : \mathbf{Q}_p^\times \to A^\times$ be such that δ is pointwise never of the form x^{-i} for any $i \geq 0$. Then $H^i(A^+, \mathscr{R}_A^+ \otimes \delta) = 0$ for $i \in \{0, 1, 2\}$.*

Proof. First note that as δ is pointwise never of the form x^{-i} for any $i \geq 0$, $(1-\alpha p^i, 1-\beta a^i) = A$. This implies, as in the proof of Proposition 11.19, that multiplication by $1-\beta a^i$ is bijective on $\mathrm{Ann}(1-\alpha p^i)$ and on $A/(1-\alpha p^i)$.

The vanishing of $H^0(A^+, \mathscr{R}_A^+ \otimes \delta)$ follows from Lemma 11.20(1) and the fact that multiplication by $1-\beta a^i$ on $\mathrm{Ann}(1-\alpha p^i)$ is injective.

The vanishing of $H^2(A^+, \mathscr{R}_A^+ \otimes \delta)$ follows from Lemma 11.20(4) and the fact that multiplication by $1-\beta a^i$ on $A/(1-\alpha p^i)$ is surjective.

In what concerns the vanishing of $H^1(A^+, \mathscr{R}_A^+ \otimes \delta)$, the inflation-restriction sequence gives

$$0 \to H^1(A^0, H^0(\Phi^+, \mathscr{R}_A^+ \otimes \delta)) \to H^1(A^+, \mathscr{R}_A^+ \otimes \delta) \to H^0(A^0, H^1(\Phi^+, \mathscr{R}_A^+ \otimes \delta)) \to 0.$$

The result is now an easy consequence of Lemma 11.20(2) and (3) and the fact that multiplication by $1-\beta a^i$ is surjective on $\mathrm{Ann}(1-\alpha p^i)$ and injective on $A/(1-\alpha p^i)$, respectively. □

11.4.3 The case of $\mathscr{R}_A \otimes \delta$

The calculation of $H^i(A^+, \mathscr{R}_A \otimes \delta)$ is now formal using the (long exact sequence of A^+-cohomology associated to the) short exact sequence of A^+-modules

$$0 \to \mathscr{R}_A^+ \otimes \delta \to \mathscr{R}_A \to \mathscr{R}_A^- \otimes \delta\chi^{-1} \to 0.$$

Moreover in the so-called regular case, we are able to compute it explicitly. [56].

Proposition 11.22. *Let $\delta : \mathbf{Q}_p^\times \to A^\times$ be such that δ is pointwise never of the form χx^i nor of the form x^{-i} for any $i \geq 0$. Then $H^0(A^+, \mathscr{R}_A \otimes \delta) = H^2(A^+, \mathscr{R}_A \otimes \delta) = 0$ and $H^1(A^+, \mathscr{R}_A \otimes \delta)$ is a free A-module of rank 1.*

Proof. The long exact sequence of A^+-cohomology associated to the short exact sequence of A^+-modules

$$0 \to \mathscr{R}_A^+ \otimes \delta \to \mathscr{R}_A \to \mathscr{R}_A^- \otimes \delta\chi^{-1} \to 0$$

and the fact that $H^0(A^+, \mathscr{R}_A^- \otimes \delta\chi^{-1}) = H^2(A^+, \mathscr{R}_A^+ \otimes \delta) = 0$, cf. Lemma 11.14 and Proposition 11.21 yields

$$0 \to H^1(A^+, \mathscr{R}_A^+ \otimes \delta) \to H^1(A^+, \mathscr{R}_A \otimes \delta) \to H^1(A^+, \mathscr{R}_A^- \otimes \delta\chi^{-1}) \to 0,$$

and so

$$H^2(A^+, \mathscr{R}_A \otimes \delta) \cong H^2(\mathscr{R}_A^- \otimes \delta\chi^{-1}).$$

The result follows then from Proposition 11.19 and Proposition 11.21. □

Remark 11.23.

- The calculations of this section show that, for $M \in \{\mathscr{R}_A^+, \mathscr{R}_A^-, \mathscr{R}_A\}$, the A-modules $H^i(A^+, M \otimes \delta)$ are finite (as also proved in [4]).
- $H^2(A^+, \mathscr{R}_A \otimes \delta) = 0$ if and only if δ is pointwise never of the form χx^i, $i \in \mathbf{N}$. Indeed, this is a necessary condition by Corollary 11.18. For the converse first note that if δ is never of the form χx^i nor x^{-i} for any $i \in \mathbf{N}$, then $H^2(A^+, \mathscr{R}_A \otimes \delta) = 0$ by Proposition 11.22. On the other hand, if δ reduces to x^{-i} for some $i \geq 0$ at some point of $\mathrm{Sp}(A)$, we use the following argument to reduce to the case of a point (A a finite extension of $\mathbf{Q}_p$). The finiteness of the A-module $H^2(A^+, \mathscr{R}_A \otimes \delta)$, the vanishing of $H^3(A^+, \mathscr{R}_A \otimes \delta)$, the fact that $\mathscr{R}_A$ is a flat A-module (cf. Lemma 10.1) and the Tor-spectral sequence

  $$\mathrm{Tor}_{-p}(H^q(A^+, \mathscr{R}_A \otimes \delta), A/\mathfrak{m}) \Rightarrow H^{p+q}(A^+, \mathscr{R}_{A/\mathfrak{m}} \otimes \delta)$$

  show that $H^2(A^+, \mathscr{R}_A \otimes \delta) \otimes A/\mathfrak{m} = H^2(A^+, \mathscr{R}_{A/\mathfrak{m}} \otimes \delta)$ for every maximal ideal $\mathfrak{m} \subseteq A$. Since $H^2(A^+, \mathscr{R}_{A/\mathfrak{m}} \otimes \delta) = 0$ (cf. [11, Théorème 5.16]), we can conclude that $H^2(A^+, \mathscr{R}_A \otimes \delta) = 0$ by Nakayama's lemma.

56. This result is already proved by similar methods in [4, Théorème 2.29].

12 Relative cohomology

Over the next few sections we prove an isomorphism between the $\overline{P}^{+}$-cohomology and the A^{+}-cohomology with coefficients in $\mathscr{R}_A(\delta_1, \delta_2)$ assuming $\delta_1\delta_2^{-1}$ is regular. This is a generalization of a result of Colmez, who proves it for the case where A is a finite extension of $\mathbf{Q}_p$ and we indeed reduce the general result to that case using some arguments on derived categories inspired by [37].

12.1 Formalism of derived categories

In this section we fix a noetherian ring A. Let $\mathcal{D}^{-}(A)$ denote the derived category of A-modules bounded above. We begin by recalling the notion of a pseudo-coherent complex. For a detailed explanation we refer the reader to [58, Tag 064N]

Definition 12.1.

1. *An object $K^{\bullet}$ of $\mathcal{D}^{-}(A)$ is pseudo-coherent if it is quasi-isomorphic to a bounded above complex of finite free A-modules. We denote by $\mathcal{D}^{-}_{\mathrm{pc}}(A) \subseteq \mathcal{D}^{-}(A)$ the full subcategory of pseudo-coherent objects of $\mathcal{D}^{-}(A)$.*
2. *An A-module M is called pseudo-coherent if $M[0] \in \mathcal{D}^{-}_{\mathrm{pc}}(A)$.*

We have the following simple Lemma detecting when a module is pseudo-coherent.

Lemma 12.2. *An A-module M is pseudo-coherent iff there exists an infinite resolution*

$$\cdots \to A^{\oplus n_1} \to A^{\oplus n_0} \to M \to 0$$

Proof. This is just rephrasing part (2) of Definition 12.1. □

Since A is noetherian, Lemma 12.2 can be further strenghened to the following.

Lemma 12.3. *An A-module M is pseudo-coherent iff it is finite.*

Proof. We first show that a finite A-module M is pseudo-coherent. Indeed since M is finite, one may choose a surjection $A^{\oplus n_0} \to M$. Then having constructed an exact complex of finite free A-modules of length t, we can extend by choosing a surjection

$$A^{\oplus n_{t+1}} \to \ker(A^{\oplus n_t} \to A^{\oplus n_{t-1}}).$$

Here we have implicitly used that a submodule of a finite A-module is finite. Conversely, a pseudo-coherent module is finite by Lemma 12.2. □

The following Lemma allows us to use induction-type arguments when trying to prove results concerning pseudo-coherent complexes.

Lemma 12.4. *Let $K^{\bullet} \in \mathcal{D}^{-}(A)$. The following are equivalent*

1. *$K^{\bullet} \in \mathcal{D}^{-}_{\mathrm{pc}}(A)$.*

2. *For every integer m, there exists a bounded complex $E^\bullet$ (depending on m) of finite free A-modules and a morphism $\alpha : E^\bullet \to K^\bullet$ such that $H^i(\alpha)$ is an isomorphism for $i > m$ and $H^m(\alpha)$ is surjective.*

Proof. Suppose (1) holds. Let $E^\bullet$ be a bounded above complex of finite free A-modules and let $E^\bullet \to K^\bullet$ be a quasi-isomorphism. Consider the naive truncation at place m

$$F_m^\bullet : \cdots \to 0 \to E^m \to E^{m+1} \to \cdots .$$

Then the induced maps $F_m^\bullet \to K^\bullet$ satisfy condition (2).

Suppose (2) holds. We are going to construct our bounded above complex $E^\bullet$ of finite free A-modules (which will be quasi-isomorphic to $K^\bullet$) by descending induction. Since $K^\bullet$ is bounded above, there is an integer a, such that $K^n = 0$, $\forall n \geq a$. By descending induction on $n \in \mathbf{Z}$, we are going to construct a complex

$$F_n^\bullet : \cdots \to 0 \to F^n \to F^{n+1} \to \cdots \to F^{a-1} \to 0 \to \cdots$$

and a morphism $\alpha_n : F_n^\bullet \to K^\bullet$, such that $H^i(\alpha_n)$ is an isomorphism for $i > n$ and a surjection for $i = n$. For the base case $n = a$, we can take $F^i = 0$ $\forall i$. Now consider the induction step. Let $C^\bullet = \mathrm{cone}(F_n^\bullet \xrightarrow{\alpha_n} K^\bullet)$. The long exact sequence of cohomology coming from the triangle

$$F_n^\bullet \to K^\bullet \to C^\bullet \to F_n^\bullet[1]$$

gives $H^i(C^\bullet) = 0$ for $i \geq n$. It is easy to see that condition (2) is stable by extensions and so in particular $C^\bullet$ satisfies condition (2). We claim that $H^{n-1}(C^\bullet)$ is a finite A-module. Indeed choose a bounded complex $D^\bullet$ of finite free A-modules and a morphism $\beta : D^\bullet \to C^\bullet$ inducing isomorphism on cohomology in degrees $\geq n$ and a surjection in degree $n-1$. It suffices to show $H^{n-1}(D^\bullet)$ is a finite A-module. Let t be the largest integer such that $E^t \neq 0$. If $t = n-1$, then the result is clear. If $t > n-1$, then $D^{t-1} \to D^t$ is surjective as $H^t(D^\bullet) = 0$. As D^t is free, we see that $D^{t-1} = D' \oplus D^t$. It suffices to prove the result for the complex $(D')^\bullet$, which is the same as $D^\bullet$ except has D' in degree $t-1$ and 0 in degree t. The result follows by induction. Hence $H^{n-1}(C^\bullet)$ is a finite A-module as claimed.

Choose a finite free A-module F^{n-1} and a map $p : F^{n-1} \to C^{n-1}$ such that the composition $F^{n-1} \to C^{n-1} \to C^n$ is zero and such that F^{n-1} surjects onto $H^{n-1}(C^\bullet)$. Since $C^{n-1} = K^{n-1} \oplus F^n$ (by definition of the cone), we can write $p = (\alpha^{n-1}, -d^{n-1})$. The vanishing of the composition $F^{n-1} \to C^{n-1} \to C^n$, implies these maps fit into a morphism of complexes:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & F^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "\alpha^{n-1}"] & F^n \arrow[r] \arrow[d, "\alpha_n^n"] & F^{n+1} \arrow[r] \arrow[d, "\alpha_n^{n+1}"] & \cdots \\
\cdots \arrow[r] & K^{n-2} \arrow[r] & K^{n-1} \arrow[r] & K^n \arrow[r] & K^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Moreover we obtain a morphism of triangles

$$\begin{tikzcd}
(F^n \to \cdots) \arrow[r] \arrow[d] & (F^{n-1} \to \cdots) \arrow[r] \arrow[d] & F^{n-1} \arrow[d, "p"] \\
(F^n \to \cdots) \arrow[r] & K^\bullet \arrow[r] & C^\bullet
\end{tikzcd}$$

By the octaeder axiom for triangulated categories, our choice of p implies that the map of complexes

$$(F^{n-1} \to \cdots) \to K^\bullet$$

induces an isomorphism in degrees $\geq n$ and a surjection in degree $n-1$. $\square$

Remark 12.5. The above proof also shows the following useful fact. If a complex in $\mathcal{D}^-_{\mathrm{pc}}(A)$ has trivial cohomology in degrees strictly greater than b, then it is quasi-isomorphic to a complex $P^\bullet \in \mathcal{D}^-_{\mathrm{pc}}(A)$ with $P^i = 0\ \forall i \geq b+1$ and each P^j is finite free $\forall j \in \mathbf{Z}$. The proof also shows that $\mathcal{D}^-_{\mathrm{pc}}(A)$ is stable by extensions.

Since A is noetherian, we have the following simple criterion for detecting whether an object in $\mathcal{D}^-(A)$ is pseudo-coherent.

Proposition 12.6. *An object $K^\bullet \in \mathcal{D}^-(A)$ is pseudo-coherent iff $H^i(K^\bullet)$ is a finite A-module for all i.*

Proof. If $K^\bullet \in \mathcal{D}^-(A)$ is pseudo-coherent then every cohomology $H^i(K^\bullet)$, is a finite A-module. For the converse suppose that $H^i(K^\bullet)$ is a finite A-module for all i. By Lemmas 12.3 and 12.4, $H^i(K^\bullet)[0]$ satisfies condition (2) of Lemma 12.4. Let n be the largest integer such that $H^n(K^\bullet)$ is non-zero. We will prove the Proposition by induction on n. We have the distinguished triangle

$$\tau_{\leq n-1}K^\bullet \to K^\bullet \to H^n(K^\bullet)[-n].$$

Fix an integer k. Now $H^n(K^\bullet)[-n]$ satisfies condition (2) of Lemma 12.4 for $m = k$. Since condition (2) for $m = k$ is stable under extensions, $K^\bullet$ satisfies condition (2) for $m = k$ if $\tau_{\leq n-1}K^\bullet$ satisfies condition (2) for $m = k$. The result follows by induction. $\square$

12.2 The Koszul complex

Let a be a generator of $\mathbf{Z}_p^\times$ and note

$$\gamma = \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad \varphi = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad \tau = \left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right)$$

which are generators of the group $\overline{P}^+$ satisfying the following relations:

$$\varphi\gamma = \gamma\varphi,$$

$$\gamma\tau = \tau^{a^{-1}}\gamma,$$

$$\varphi\tau^p = \tau\varphi,$$

giving a finite presentation of the group generated by those elements (proof: using those relations we can write any other relation as $\varphi^x\tau^y\gamma^z = 1$ which in turn implies $x = y = z = 0$) We thus have the following result. We need a lemma describing the nilpotent nature of $\tau - 1$

In this subsection we compute a complex that computes $\overline{P}^+$-cohomology, cf. Lemma 12.8. This is an analogue of the Koszul complex in a non-commutative setting. The reader can compare this with the complex constructed in [62, §1.5.1] which calculates Galois cohomology. In loc.cit. the construction is somewhat simpler because of the non-triviality of the center (of the group under consideration). Let M be a $\overline{P}^+$-module such that the action of $\overline{P}^+$ extends to an action of the Iwasawa algebra $\mathbf{Z}_p[[\overline{P}^+]]$ and define

$$\mathscr{C}_{\tau,\varphi,\gamma} : 0 \to M \xrightarrow{X} M \oplus M \oplus M \xrightarrow{Y} M \oplus M \oplus M \xrightarrow{Z} M \to 0 \tag{15}$$

where

$$\begin{aligned} X(x) &= ((1-\tau)x, (1-\varphi)x, (\gamma-1)x) \\ Y(x,y,z) &= ((1-\varphi\delta_p)x + (\tau-1)y, (\gamma\delta_a - 1)x + (\tau-1)z, (\gamma-1)y + (\varphi-1)z) \\ Z(x,y,z) &= (\gamma\delta_a - 1)x + (\varphi\delta_p - 1)y + (1-\tau)z \end{aligned}$$

where,

$$\delta_p = \frac{1-\tau^p}{1-\tau} = 1 + \tau + \ldots + \tau^{p-1},$$

for $a \in \mathbf{Z}_p^\times, b \in \mathbf{Z}_p$

$$\frac{\tau^{ba}-1}{\tau^a - 1} = \sum_{n\geq 1} \binom{ba}{n} (\tau^a - 1)^{n-1} \in \mathbf{Z}_p[[\tau - 1]],$$

$$\delta_a = \frac{\tau^a - 1}{\tau - 1}$$

which is a well defined element since, as $\tau^{p^n} \to 1$ as n tends to $+\infty$, $\tau - 1$ is topologically nilpotent in the Iwasawa algebra $\mathbf{Z}_p[[\tau-1]] = \mathbf{Z}_p[[\overline{U}]] \subseteq \mathbf{Z}_p[[\overline{P}^+]]$.

The construction of $\mathscr{C}_{\tau,\varphi,\gamma}$ is obtained from taking successive fibers of smaller complexes. Define

$$\mathscr{C}_\tau : 0 \to M \xrightarrow{D} M \to 0 \tag{16}$$

where

$$D(x) := (\tau - 1)x$$

and

$$\mathscr{C}_{\tau,\varphi} : 0 \to M \xrightarrow{E} M \oplus M \xrightarrow{F} M \to 0 \tag{17}$$

where

$$E(x) = ((\tau - 1)x, (\varphi - 1)x)$$
$$F(x, y) = (\varphi\delta_p - 1)x + (1 - \tau)y$$

We now define morphisms between the complexes. We note by $[\varphi - 1] : \mathscr{C}_\tau \to \mathscr{C}_\tau$ the morphism:

$$\begin{array}{ccccccc}
\mathscr{C}_\tau : 0 & \longrightarrow & M & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi-1} & & \downarrow{\scriptstyle \varphi\delta_p-1} & & \\
\mathscr{C}_\tau : 0 & \longrightarrow & M & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

and $[\gamma - 1] : \mathscr{C}_{\tau,\varphi} \to \mathscr{C}_{\tau,\varphi}$ the morphism:

$$\begin{array}{ccccccccc}
\mathscr{C}_{\tau,\varphi} : 0 & \longrightarrow & M & \longrightarrow & M \oplus M & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma-1} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \gamma\delta_a-1} & & \\
\mathscr{C}_{u^-,\varphi} : 0 & \longrightarrow & M & \longrightarrow & M \oplus M & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

where $s(x, y) = ((\gamma\delta_a - 1)x, (\gamma - 1)y)$

Lemma 12.7. *There are distinguised triangles*

$$\mathscr{C}_{\tau,\varphi} \to \mathscr{C}_\tau \xrightarrow{[\varphi-1]} \mathscr{C}_\tau$$

and

$$\mathscr{C}_{\tau,\varphi,\gamma} \to \mathscr{C}_{\tau,\varphi} \xrightarrow{[\gamma-1]} \mathscr{C}_{\tau,\varphi}$$

in $\mathcal{D}^-(A)$.

Proof. This is evident from the definition of the cone of a morphism in $\mathcal{D}^-(A)$ and the relations $\varphi\delta_p \cdot \gamma\delta_a = \gamma\delta_a \cdot \varphi\delta_p$, $(\gamma\delta_a - 1)(\tau - 1) = (\tau - 1)(\gamma - 1)$ and $(\varphi\delta_p - 1)(\tau - 1) = (\tau - 1)(\varphi - 1)$. □

We now show that $C_{\tau,\varphi,\gamma}$ computes $\overline{P}^+$-cohomology. Recall the complex

$$\mathscr{C}_{\varphi,\gamma} : 0 \to M \xrightarrow{E'} M \oplus M \xrightarrow{F'} M \to 0$$

where

$$E'(x) = ((1 - \varphi)x, (\gamma - 1)x)$$
$$F'(x, y) = (\gamma - 1)x + (\varphi - 1)y$$

calculates the A^+-cohomology of M. There is an obvious restriction morphism $\mathscr{C}_{\tau,\varphi,\gamma} \to \mathscr{C}_{\varphi,\gamma}$ whose kernel (as a morphism in the abelian category of chain complexes) is

$$\mathscr{C}^{\text{twist}}_{\varphi,\gamma}: 0 \to M \xrightarrow{E''} M \oplus M \xrightarrow{F''} M \to 0$$

where

$$E''(x) = ((1-\varphi\delta_p)x, (\gamma\delta_a - 1)x)$$
$$F''(x,y) = ((\gamma\delta_a - 1)x + (\varphi\delta_p - 1)y$$

Lemma 12.8. *The complex $\mathscr{C}_{\tau,\varphi,\gamma}$ calculates the $\overline{P}^+$-cohomology groups. That is $H^i(\mathscr{C}_{\tau,\varphi,\gamma}) = H^i(\overline{P}^+, M)$.*

Proof. This is just a reinterpretation of the Hochschild-Serre spectral sequence. We have a distinguished triangle

$$\mathscr{C}_{\tau,\varphi,\gamma} \to \mathscr{C}_{\varphi,\gamma} \xrightarrow{1-\tau} \mathscr{C}^{\text{twist}}_{\varphi,\gamma} \tag{18}$$

in the derived category $\mathcal{D}^-(A)$, where the morphism

$$1-\tau\colon \mathscr{C}_{\varphi,\gamma} \to \mathscr{C}^{\text{twist}}_{\varphi,\gamma}$$

is component-wise just $1-\tau$. Let $\overline{U} = \begin{pmatrix} 1 & 0 \\ p\mathbf{Z}_p & 1 \end{pmatrix}$ so that $\overline{P}^+ = \overline{U} \rtimes A^+$. For a semi-group G we denote by R^G denote the derived functor of $(-)^G$. For M a $\overline{P}^+$-module, we claim that [57]

$$R^{\overline{U}}(M) = (0 \to M \xrightarrow{1-\tau} M^* \to 0)$$

where M^* is isomorphic to M as $\overline{U}$-modules, but is equipped with a *twisted* (φ,γ)-action (which we denote by $(\widetilde{\varphi},\widetilde{\gamma})$):

$$\widetilde{\varphi}\cdot m := \varphi\delta_p \cdot m \text{ and } \widetilde{\gamma}\cdot m := \gamma\delta_a \cdot m.$$

First note that $1-\tau\colon M \to M^*$ is indeed a morphism of $\overline{P}^+$-modules (this follows from the relations $(\gamma\delta_a - 1)(\tau - 1) = (\tau-1)(\gamma-1)$ and $(\varphi\delta_p - 1)(\tau-1) = (\tau-1)(\varphi-1)$). Now $H^1(\overline{U}, M)$ is equipped with a natural (φ,γ)-action (which we denote by (φ',γ')):

$$\varphi'\cdot c_\tau := \varphi\cdot c_{\tau^p} \text{ and } \gamma'\cdot c_\tau := \gamma\cdot c_{\tau^a},$$

where c_τ is the value of the 1-cocycle c with $[c] \in H^1(\overline{U}, M)$, at τ. To prove the claim it suffices to show that $\varphi\cdot c_{\tau^p} = \varphi\delta_p\cdot c_\tau$ and $\gamma\cdot c_{\tau^a} = \gamma\delta_a\cdot c_\tau$. However these follow from the fact that c is a 1-cocycle. Thus by the Hochschild-Serre spectral sequence we have

$$R^{\overline{P}^+}(M) = R^{A^+}(0 \to M \xrightarrow{1-\tau} M^* \to 0).$$

57. Here $R^{\overline{U}}$ is viewed as a function from $\mathcal{D}^+(\overline{P}^+ - \text{Mod})$ to $\mathcal{D}^+(A^+ - \text{Mod})$.

Therefore applying R^{A^+} to the distinguished triangle

$$(0 \to M \xrightarrow{1-\tau} M^* \to 0) \to M \xrightarrow{1-\tau} M^*$$

gives the distinguished triangle

$$R^{\overline{P}^+}(M) \to R^{A^+}(M) \xrightarrow{1-\tau} R^{A^+}(M^*) \tag{19}$$

and it is easy to see that $R^{A^+}(M) = \mathscr{C}_{\varphi,\gamma}$ and $R^{A^+}(M^*) = \mathscr{C}_{\varphi,\gamma}^{\mathrm{twist}}$. The result now follows from comparing the triangles (18) and (19). $\square$

12.3 Finiteness of cohomology

In this subsection we show that the cohomology groups

$$H^i(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2))$$

are finite-type A-modules. The idea is to reduce the problem to finiteness of A^+-cohomology, cf. [4], [37] and finiteness of *twisted* A^+-cohomology.

The first thing to note is that the complexes $\mathscr{C}_{\tau,\varphi,\gamma}$ are well defined for $M \in \{\mathscr{R}_A^+(\delta), \mathscr{R}_A(\delta), \mathscr{R}_A^-(\delta)\}$, which is a consequence of the following lemma.

Lemma 12.9. *Let $M \in \{\mathscr{R}_A^-(\delta_1, \delta_2), \mathscr{R}_A(\delta_1, \delta_2), \mathscr{R}_A^+(\delta_1, \delta_2)\}$. The action of $\overline{P}^+$ extends by continuity to an action of the distribution algebra $\mathscr{D}(\overline{P}^+, A)$. In particular, M is equipped with an action of the Iwasawa algebra $\mathbf{Z}_p[[\overline{P}^+]]$.*

Proof. For the proof of this lemma, we use some facts of §10.4 (which is independent of the present section). For $M \in \{\mathscr{R}_A^+(\delta_1, \delta_2), \mathscr{R}_A^-(\delta_1, \delta_2)\}$, the result is a consequence of the isomorphisms $\mathscr{R}_A^+(\delta_1) \boxtimes_\omega \mathbf{P}^1 \cong B_A(\delta_2, \delta_1)^* \otimes \omega$ and $\mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1 \cong B_A(\delta_1, \delta_2)$ of Lemma 10.19, the fact that the locally analytic principal series are equipped with an action of the distribution algebra $\mathscr{D}(G, A)$ and the fact that, since $\overline{P}^+$ stabilizes $\mathbf{Z}_p$, then $\mathscr{R}_A^{(\pm)}(\delta_1, \delta_2) = (\mathscr{R}_A^{(\pm)}(\delta_1) \boxtimes_\omega \mathbf{Z}_p) \otimes \delta_2^{-1}$ inherits an action of the distribution algebra $\mathscr{D}(\overline{P}^+, A)$, and in particular an action of the Iwasawa algebra $\mathbf{Z}_p[[\overline{P}^+]]$. For $M = \mathscr{R}_A(\delta_1, \delta_2)$, the result follows by the same arguments noting that, since $\mathscr{R}(\delta_1) \boxtimes_\omega \mathbf{P}^1$ is an extension of $\mathscr{R}_A^-(\delta) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A^+(\delta) \boxtimes_\omega \mathbf{P}^1$ in the category of separately continuous $\mathscr{D}(G, A)$-modules, it is also equipped with an action of $\mathscr{D}(G, A)$. $\square$

The main theorem of this subsection is the following.

Theorem 12.10. *If $M = \mathscr{R}_A(\delta_1, \delta_2)$ then $\mathscr{C}_{\tau,\varphi,\gamma} \in \mathcal{D}^-_{\mathrm{pc}}(A)$. In particular, the A-modules $H^i(\overline{P}^+, \mathscr{R}(\delta_1, \delta_2))$ are finite.*

Proof. Recall that we have the distinguished triangle

$$\mathscr{C}_{\tau,\varphi,\gamma} \to \mathscr{C}_{\varphi,\gamma} \to \mathscr{C}_{\varphi,\gamma}^{\mathrm{twist}}$$

in the derived category $\mathcal{D}^-(A)$. By Theorem 4.4.2, [37], $\mathscr{C}_{\varphi,\gamma} \in \mathcal{D}^-_{\mathrm{pc}}(A)$. Thus by Lemma 12.8, to prove the result it is enough to show $\mathscr{C}_{\varphi,\gamma}^{\mathrm{twist}} \in \mathcal{D}^-_{\mathrm{pc}}(A)$. This now follows from Lemma 12.11 $\square$

Lemma 12.11. *For $M = \mathscr{R}(\delta_1, \delta_2)$, the A-modules $H^i(\mathscr{C}^{\mathrm{twist}}_{\varphi,\gamma})$ are finite.*

Proof. To prove this lemma, we still proceed by a dévissage argument. We define a complex

$$\mathscr{C}_{\varphi\delta_p} : 0 \longrightarrow M \xrightarrow{1-\varphi\delta_p} M \longrightarrow 0$$

and we observe that we have a distinguished triangle

$$\mathscr{C}^{\mathrm{twist}}_{\varphi,\gamma} \to \mathscr{C}_{\varphi\delta_p} \xrightarrow{1-\gamma\delta_a} \mathscr{C}_{\varphi\delta_p}.$$

Moreover, by taking long exact sequences associated to the short exact sequence $0 \to \mathscr{R}^+_A(\delta_1, \delta_2) \to \mathscr{R}_A(\delta_1, \delta_2) \to \mathscr{R}^-_A(\delta_1, \delta_2) \to 0$, it is enough to show finiteness for $\mathscr{R}^+_A(\delta_1, \delta_2)$ and $\mathscr{R}^-_A(\delta_1, \delta_2)$. The lemma follows from 12.12. □

Lemma 12.12. *For $M \in \{\mathscr{R}^+(\delta_1, \delta_2), \mathscr{R}^-(\delta_1, \delta_2)\}$, the A-modules $H^i(\mathscr{C}^{\mathrm{twist}}_{\varphi,\gamma})$ are finite.*

Proof. The case of $\mathscr{R}^+(\delta_1, \delta_2)$ follows directly from lemma 12.13 below, which shows that the cohomology of the complex $\mathscr{C}_{\varphi\delta_p}$ is already of finite type.

For $\mathscr{R}^-(\delta_1, \delta_2)$, the long exact sequence associated to the triangle $\mathscr{C}^{\mathrm{twist}}_{\varphi,\gamma} \to \mathscr{C}_{\varphi\delta_p} \xrightarrow{1-\gamma\delta_a} \mathscr{C}_{\varphi\delta_p}$ yields

$$0 \to H^0(\mathscr{C}^{\mathrm{twist}}_{\varphi,\gamma}) \to H^0(\mathscr{C}_{\varphi\delta_p}) \xrightarrow{1-\gamma\delta_a} H^0(\mathscr{C}_{\varphi\delta_p}) \to H^1(\mathscr{C}^{\mathrm{twist}}_{\varphi,\gamma}) \to H^1(\mathscr{C}_{\varphi\delta_p}) \xrightarrow{1-\gamma\delta_a} H^0(\mathscr{C}_{\varphi\delta_p})$$
$$\to H^2(\mathscr{C}^{\mathrm{twist}}_{\varphi,\gamma}) \to 0,$$

and the result follows then from lemmas 12.14 and 12.15 □

Lemma 12.13. *The operator $1 - \varphi\delta_p : \mathscr{R}^+_A(\delta_1, \delta_2) \to \mathscr{R}^+_A(\delta_1, \delta_2)$ has finite kernel and cokernel.*

Proof. The proof of this lemma is an adaptation of lemma 11.9. Let N be big enough such that $|\delta(p)p^N| < 1$. We show that $1 - \varphi\delta_p : T^N\mathscr{R}^+_A(\delta_1, \delta_2) \to T^N\mathscr{R}^+_A(\delta_1, \delta_2)$ is bijective. For that, we construct an inverse of this operator by proving that $\sum_{k\geq 0}(\varphi\delta_p)^k$ converges. Observe that $(\varphi\delta_p)^k = \varphi^k\delta_{p^k}$ and that the operator $\delta_{p^k} = 1 + \tau + \ldots + \tau^{p^k-1} = p^k + (\tau - 1) + \ldots + (\tau^{p^k-1} - 1)$ is bounded (independently of k) by a constant C.

By identifying $\mathscr{R}^+_A(\delta_1, \delta_2)$ with the space of analytic functions on the open unit ball equipped with the Fréchet topology given by the family of norms $(|\cdot|_{[0,r]})_{0<r<1}$ and the action of φ twisted by $\delta(p)$, we have (cf. lemme 2.9.(ii), [4]) $|\varphi^k(T^N)|_{[0,r]} \leq C_r p^{-Nk}$ for some constant $C_r > 0$ and hence, for $f \in T^N\mathscr{R}^+(\delta_1, \delta_2)$ and any $0 < r < 1$,

$$|(\varphi\delta_p)^k(f)|_{[0,r]} = |\varphi^k\delta_{p^k}(f)|_{[0,r]} \leq CC_r|f|_{[0,r]}\left(\frac{\lambda}{p^N}\right)^k,$$

which shows that the expression $\sum_{k\geq 0}(\varphi\delta_p)^k$ converges. We deduce that the kernel and cokernel are, respectively, a submodule and a quotient of $\mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A)$. This concludes the proof. □

Lemma 12.14.

– *The operator* $1-\varphi\delta_p : \mathscr{R}_A^-(\delta_1,\delta_2) \to \mathscr{R}_A^-(\delta_1,\delta_2)$ *is injective.*
– *If* $N \geq 0$ *is big enough, then* $\mathscr{R}_A^-(\delta_1,\delta_2) = (1-\varphi\delta_p)\mathscr{R}_A^-(\delta_1,\delta_2) + \mathrm{LA}(\mathbf{Z}_p^\times, A) + \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A)$.

Proof. For the first point, exactly as in the proof of lemma 11.6, if $(1-\varphi\delta_p)f = 0$ then $\varphi\delta_p f = f$ and, applying this and the identity $(\varphi\delta_p)^n = \varphi^n\delta_{p^n}$ successively [58], we get $\varphi^n\delta_{p^n} f = f$ and so f is supported on $p^n\mathbf{Z}_p$ for all $n \geq 0$ and hence vanishes everywhere.

We now prove the second assertion. By a direct calculation solving a differential equation locally, we can show that every $\phi \in \mathscr{R}_A^-(\delta_1,\delta_2)$ is of the form $\phi_1 + P_\phi + (1-\alpha\varphi)\phi_2$, with $P_\phi \in \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A)$, $\phi_2 \in \mathscr{R}_A^-$ and ϕ_1 who is zero in a neighbourhood of 0, and thus of the form $\sum_{i=0}^{n-1}\phi_{1,i}$, with $\phi_{1,i} \in \mathrm{LA}(p^i\mathbf{Z}_p^\times, A)$. Writing $\varphi^i\psi^i(\phi_{1,i}) = \varphi^i\delta_{p^i}\cdot\delta_{p^i}^{-1}\psi^i = (\varphi\delta_p)^i(\delta_p^{-1}\psi)^i = (1-(1-\varphi\delta_p))^i(\delta_p^{-1}\psi)^i(\alpha^{-i}\phi_{1,i})$ and upon expanding $(1-(1-\varphi\delta_p))^i$ expresses $\phi_{1,i}$ as a sum of elements in $(1-\varphi\delta_p)\mathscr{R}_A^-(\delta_1,\delta_2)$ and $\psi^i(\alpha^{-i}\phi_{1,i}) \in \mathrm{LA}(\mathbf{Z}_p^\times, A)$. □

Lemma 12.15. *The operator* $1-\gamma\delta_a : \mathrm{LA}(\mathbf{Z}_p^\times, A) \to \mathrm{LA}(\mathbf{Z}_p^\times, A)$ *has finite kernel and cokernel (as* A*-modules)*

Proof. For the sake of brevity write $M = \mathrm{LA}(\mathbf{Z}_p^\times, A)$. We have a morphism of complexes (in the abelian category of chain complexes)

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "1-\gamma"] \arrow[d, "1-\tau"] & M \arrow[r] \arrow[d, "1-\tau"] & 0 \\ 0 \arrow[r] & M \arrow[r, "1-\gamma\delta_a"] & M \arrow[r] & 0 \end{tikzcd}$$

Note that the cokernel of this morphism of complexes vanishes by Lemma 13.10 (the same proof carries over with L replaced by A). Thus by Proposition 11.13, it suffices to show that $M^{\tau=1}$ is a finite A-module. Take $f \in M^{\tau=1}$. Then by definition of the action of τ on $\mathscr{R}_A^-(\delta_1,\delta_2)$ we have

$$f(x) = \delta(1-px)f\left(\frac{x}{1-px}\right).$$

Repeating this procedure we see that the value of $f(x)$ determines the value of $f\left(\frac{x}{1-kpx}\right)$ for all $k \in \mathbf{Z}$. Now $1-p\mathbf{Z}$ is dense in $1-p\mathbf{Z}_p$ and so $(1-p\mathbf{Z})^{-1}$ is dense in $1-p\mathbf{Z}_p$. By continuity of f, this implies that the values $f(1)$, $f(2)$, …, $f(p-1)$ determine f completely. This proves the result.

□

58. We denote $\delta_{p^n} = \frac{1-\tau^{p^n}}{1-\tau}$.

13 The $\overline{P}^+$-cohomology

In this section we fix a finite extension L of $\mathbf{Q}_p$, two continuous characters $\delta_1, \delta_2 : \mathbf{Z}_p^\times \to L^\times$ and we consider the modules $\mathscr{R}_L^+(\delta_1, \delta_2), \mathscr{R}_L(\delta_1, \delta_2)$ and $\mathscr{R}_L^-(\delta_1, \delta_2)$ (for brevity we will omit the subscript L). We systematically calculate all $\overline{P}^+$-cohomology groups of these modules, which will be essential in comparing them to their A^+-cohomology. This section is inspired by combining two observations. The first is that if M is equipped with a continuous action of $\overline{P}^+$ such that this action induces an action of the Lie algebra of $\overline{P}^+$, then we can simplify cohomological calculations by passing to the Lie algebra. The second is that we have a good enough understanding of the infinitesimal action of the Lie algebra on a (φ, Γ)-module so as to be able to make explicit computations (cf. [2], [22]). For the commodity of the reader, the main results of this section can be summarized as follows:

Proposition 13.1.

- *Let* $M_+ = \mathscr{R}^+(\delta_1, \delta_2)$.
  1. *If* $\delta_1\delta_2^{-1} \notin \{x^{-i}, i \in \mathbf{N}\}$, *then* $H^j(\overline{P}^+, M_+) = 0$ *for all* j.
  2. *If* $\delta_1\delta_2^{-1} = \mathbf{1}_{\mathbf{Q}_p^\times}$, *then* $\dim_L H^j(\overline{P}^+, M_+) = 1, 1, 1, 0$ *for* $j = 0, 1, 2, 3$.
  3. *If* $\delta_1\delta_2^{-1} = x^{-i}$, $i \geq 1$, *then* $\dim_L H^j(\overline{P}^+, M_+) = 1, 3, 3, 1$ *for* $j = 0, 1, 2, 3$.
- *Let* $M_- = \mathscr{R}^-(\delta_1, \delta_2)$.
  1. *If* $\delta_1\delta_2^{-1} \notin \{\chi x^i, i \in \mathbf{N}\}$, *then* $\dim_L H^j(\overline{P}^+, M_-) = 0, 1, 1, 0$ *for* $j = 0, 1, 2, 3$.
  2. *If* $\delta_1\delta_2^{-1} = \chi x^i$, $i \in \mathbf{N}$, *then* $\dim_L H^j(\overline{P}^+, M_-) = 0, 2, 2, 1$ *for* $j = 0, 1, 2, 3$.
- *Let* $M = \mathscr{R}(\delta_1, \delta_2)$.
  1. *If* $\delta_1\delta_2^{-1} \notin \{x^{-i}, i \in \mathbf{N}\} \cup \{\chi x^i, i \in \mathbf{N}\}$, *then* $\dim_L H^j(\overline{P}^+, M) = 0, 1, 1, 0$, *for* $j = 0, 1, 2, 3$.
  2. *If* $\delta_1\delta_2^{-1} = \mathbf{1}_{\mathbf{Q}_p^\times}$, *then* $\dim_L H^j(\overline{P}^+, M) = 1, 2, 2, 0$, *for* $j = 0, 1, 2, 3$.
  3. *If* $\delta_1\delta_2^{-1} = x^{-i}, i \geq 1$, *then* $\dim_L H^j(\overline{P}^+, M) = 1, 3, 2, 0$, *for* $j = 0, 1, 2, 3$.
  4. *If* $\delta_1\delta_2^{-1} = \chi x^i, i \in \mathbf{N}$, *then* $\dim_L H^j(\overline{P}^+, M) = 0, 2, 2, 1$, *for* $j = 0, 1, 2, 3$.

Remark 13.2. Observe that the result about $H^1(\overline{P}^+, \mathscr{R}^+(\delta_1, \delta_2))$ when $\delta_1\delta_2^{-1} = x^{-i}$, $i \geq 1$, is in contradiction with [11, Lemme 5.21]. There seems to be a mistake in loc.cit., where the twisted action of A^+ on $H^1(\overline{U}, M)$ is not taken into account. This changes slightly the results of [11], getting unicity of the correspondence only for the non-pathological case $\delta_1\delta_2^{-1} \notin \{x^{-i}, i \geq 1\}$ (indeed, the restriction $H^1(\overline{P}^+, \mathscr{R}(\delta_1, \delta_2)) \to H^1(A^+, \mathscr{R}(\delta_1, \delta_2))$ turns out to be only surjective, but not injective). The authors plan to study this supplementary extensions in more detail in the near future.

13.1 The Lie algebra complex

We note

$$a^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ a^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ u^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ u^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

the usual generators of the Lie algebra $\mathfrak{gl}_2$ of GL_2. We note that $[a^+, u^-] = -u^-$ and $p\varphi u^- = u^-\varphi$.

Denote by $H^i_{\mathrm{Lie}}(\overline{P}^+, M)$ the cohomology groups of the complex

$$\mathscr{C}_{u^-,\varphi,a^+} : 0 \to M \xrightarrow{X'} M \oplus M \oplus M \xrightarrow{Y'} M \oplus M \oplus M \xrightarrow{Z'} M \to 0 \tag{20}$$

where

$$X'(x) = ((\varphi - 1)x, a^+x, u^-x)$$
$$Y'(x, y, z) = (a^+x - (\varphi - 1)y, u^-y - (a^+ + 1)z, (p\varphi - 1)z - u^-x)$$
$$Z'(x, y, z) = u^-x + (p\varphi - 1)y + (a^+ + 1)z$$

Let $\tilde{P} := \begin{pmatrix} \mathbf{Z}_p^\times & 0 \\ p\mathbf{Z}_p & 1 \end{pmatrix}$. Note that $\tilde{P}$ is a p-pro-subgroup of $\overline{P}^+$.

Lemma 13.3. *If $M \in \{\mathscr{R}^+(\delta_1, \delta_2), \mathscr{R}^-(\delta_1, \delta_2), \mathscr{R}(\delta_1, \delta_2)\}$, the natural application*

$$H^i(\overline{P}^+, M) \to H^0(\tilde{P}, H^i_{\mathrm{Lie}}(\overline{P}^+, M))$$

is an isomorphism.

Proof. The same proof as Lemma 12.8 shows that there is a spectral sequence

$$H^i_{\mathrm{Lie}}(A^+, H^j_{\mathrm{Lie}}(\overline{U}, M)) \Rightarrow H^{i+j}_{\mathrm{Lie}}(\overline{P}^+, M)$$

where $H^i_{\mathrm{Lie}}(\overline{U}, M)$ is defined to be the cohomology of the complex

$$0 \to M \xrightarrow{u^-} M \to 0.$$

For the definition of $H^j_{\mathrm{Lie}}(A^+, -)$, cf. [11, §5.2]. The result now follows from [61, Corollary 21] by taking $\tilde{P}$-invariants on both sides. □

Remark 13.4. For future calculations, we need to explicit the action of $\tilde{P}$ on the different Lie algebra cohomology groups. Recall that this group acts naturally on the module and by its adjoint action on the Lie algebra. Take for instance $(x, y, z) \in M^{\oplus 3}$ a 1-cocycle on the Lie algebra complex $\mathscr{C}_{u^-,\varphi,a^+}$ representing some cohomology class. An easy calculation shows that, if $\sigma_a \in A^0$, then, as cohomology classes

$$\sigma_a \cdot (x, y, z) = (\sigma_a x, \sigma_a y, a\sigma_a z).$$

If we want to calculate the action of τ, say, on 1-coycles, in the same way, we get

$$\tau(x,y,z) = (\tau x + \tau\varphi\frac{\tau^{1-p}-1}{\log(\tau)}z, \tau\, y - p\tau\, z, \tau z).$$

The formula for the first coordinate is obtained by using the fact that Lie algebra cohomology is calculated by 'differentiating locally analytic cocycles at the identity' (cf. [61]), and it can be taken as a formal formula (since there might be some convergence problems) but it will be enough for us (in general, one should replace τ by τ^n for some n big enough).

13.2 Deconstructing cohomology

In order to compute cohomology we build the complex $\mathscr{C}_{u^-,\varphi,a^+}$ from smaller complexes. Define

$$\mathscr{C}_{u^-} : 0 \to M \xrightarrow{D} M \to 0,$$

where

$$D(x) := u^- x$$

and

$$\mathscr{C}_{u^-,\varphi} : 0 \to M \xrightarrow{E} M \oplus M \xrightarrow{F} M \to 0,$$

where

$$E(x) = (u^- x, (\varphi - 1)x)$$
$$F(x,y) = (p\varphi - 1)x - u^- y.$$

We now define morphisms between the complexes. We note by $[\varphi - 1] : \mathscr{C}_{u^-} \to \mathscr{C}_{u^-}$ the morphism:

$$\begin{array}{ccccccc}
\mathscr{C}_{u^-} : 0 & \longrightarrow & M & \longrightarrow & M & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \varphi-1} & & \Big\downarrow{\scriptstyle p\varphi-1} & & \\
\mathscr{C}_{u^-} : 0 & \longrightarrow & M & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

and $[a^+] : \mathscr{C}_{u^-,\varphi} \to \mathscr{C}_{u^-,\varphi}$ the morphism:

$$\begin{array}{ccccccccc}
\mathscr{C}_{u^-,\varphi} : 0 & \longrightarrow & M & \longrightarrow & M \oplus M & \longrightarrow & M & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle a^+} & & \Big\downarrow{\scriptstyle s} & & \Big\downarrow{\scriptstyle a^+ + 1} & & \\
\mathscr{C}_{u^-,\varphi} : 0 & \longrightarrow & M & \longrightarrow & M \oplus M & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

where $s(x,y) = ((a^+ + 1)x, a^+ y)$

Lemma 13.5. *We have the following distinguished triangles in $\mathcal{D}^-(L)$:*

$$\mathscr{C}_{u^-,\varphi} \to \mathscr{C}_{u^-} \xrightarrow{[\varphi-1]} \mathscr{C}_{u^-},$$

$$\mathscr{C}_{u^-,\varphi,a^+} \to \mathscr{C}_{u^-,\varphi} \xrightarrow{[a^+]} \mathscr{C}_{u^-,\varphi}.$$

Proof. This is evident from the definition of the cone of a morphism in $\mathcal{D}^-(L)$. ☐

The following lemma will be the cornerstone of our cohomology calculations.

Lemma 13.6.

1. $H^0(\mathscr{C}_{u^-,\varphi,a^+}) = H^0([a^+]\colon H^0(\mathscr{C}_{u^-,\varphi}))$.
2. *We have the following exact sequences in cohomology:*

$$0 \to H^1([a^+]\colon H^0(\mathscr{C}_{u^-,\varphi})) \to H^1(\mathscr{C}_{u^-,\varphi,a^+}) \to H^0([a^+]\colon H^1(\mathscr{C}_{u^-,\varphi})) \to 0, \tag{21}$$

$$0 \to H^1([\varphi-1]\colon H^0(\mathscr{C}_{u^-})) \to H^1(\mathscr{C}_{u^-,\varphi}) \to H^0([\varphi-1]\colon H^1(\mathscr{C}_{u^-})) \to 0. \tag{22}$$

3. *We have the following exact sequences in cohomology:*

$$0 \to H^1([a^+] : H^1(\mathscr{C}_{u^-,\varphi})) \to H^2(\mathscr{C}_{u^-,\varphi,a^+}) \to H^0([a^+] : H^2(\mathscr{C}_{u^-,\varphi})) \to 0, \tag{23}$$

$$H^2(\mathscr{C}_{u^-,\varphi}) \cong H^1([\varphi-1] : H^1(\mathscr{C}_{u^-})). \tag{24}$$

4. $H^3(\mathscr{C}_{u^-,\varphi,a^+}) = H^1([a^+]\colon H^2(\mathscr{C}_{u^-,\varphi}))$.

Proof. This follows from taking long exact sequences in cohomology from the triangles in Lemma 13.5. ☐

The module from which we are taking cohomology should be clear from context, and whenever it is not specified, it means that the result holds for any such module. Notations should be clear, for instance, we have $H^0([a^+]\colon H^0(\mathscr{C}_{u^-,\varphi})) = M^{u^-=0,\varphi=1,a^+=0}$, $H^1([a^+]\colon H^0(\mathscr{C}_{u^-,\varphi})) = \operatorname{coker}([a^+]\colon H^0(\mathscr{C}_{u^-,\varphi})) = \operatorname{coker}(a^+\colon M^{u^-=0,\varphi=1} \to M^{u^-=0,\varphi=1})$, et cetera desunt.

Remark 13.7. From the definition of the action of the group on the different terms of the Hochschild-Serre spectral sequence, or on the Lie algebra cohomology, as the composition of the natural action on the module with inner automorphisms on the group or on the Lie algebra, we can calculate the explicit action of $\tilde{P}$ on each constituent component of the exact sequences appearing in Lemma 13.6 and Lemma 13.6. For instance, the action of A^0 on the third term of Equation (22), on the terms of Equation (24) and on those of Lemma 13.6(4) is twisted by χ (this comes from the identity $\sigma_a^{-1}u^-\sigma = au^-$), while it acts as usual on the other terms (since A^0 commutes with itself and with φ). We can check that τ acts as usual on each separate term (but the sequences do not split as a sequence of $\overline{U}$-modules).

13.3 The Lie algebra cohomology of $\mathscr{R}^-(\delta_1, \delta_2)$

The following conglomerate of technical lemmas on the action of the Lie algebra on $\mathscr{R}^-(\delta_1, \delta_2)$ will culminate in the main Proposition 13.24 of this section, calculating the $\overline{P}^+$-cohomology on this module, following the strategy suggested by Lemma 13.6.

Lemma 13.8. *Call $M = \mathscr{R}^-(\delta_1, \delta_2)$ and let $f \in M$. Under the identification (as modules) $M = \mathrm{LA}(\mathbf{Z}_p, L)$, the infinitesimal actions of a^+, u^- and φ are given by*[59]

$$(a^+ f)(x) = \kappa(\delta) f(x) - x f'(x),$$

$$(u^- f)(x) = \kappa(\delta) x f(x) - x^2 f'(x),$$

$$(\varphi f)(x) = \delta(p) f\left(\frac{x}{p}\right).$$

Proof. First note tat, for $\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \in \overline{P}^+$ and $f \in \mathscr{R}^-(\delta_1, \delta_2)$, the action of $\overline{P}^+$ on $\mathscr{R}^-(\delta_1, \delta_2)$ is given by

$$\left(\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \cdot f\right)(x) = \delta(a - bx) f\left(\frac{x}{a - bx}\right).$$

The action of φ is now evident and that of a^+ follows from a direct calculation.

Viewing $\mathscr{R}^-(\delta_1, \delta_2)$ as the module $\mathscr{R}^-$ equipped with action of $\overline{P}^+$, we have, by [22, Théorème 1.1],

$$u^- = -t^{-1} \nabla(\nabla + \kappa(\delta_2 \delta_1^{-1})),$$

where here $\nabla = t\frac{d}{dt}$. Recall that, by the dictionary of functional analysis, multiplication by t and $\frac{d}{dt}$ become, respectively, the operations of derivation and multiplication by x on $\mathrm{LA}(\mathbf{Z}_p, L)$. In particular we have $\nabla(f) = f + x f'$. The description of the action of u^- follows now from a direct computation. □

By an inoffensive abuse of language, we will talk in the sequel about the action of the elements a^+ and u^- on $\mathrm{LA}(\mathbf{Z}_p, L)$ (resp. $\mathrm{LA}(\mathbf{Z}_p^\times, L)$), by which we mean their action on $\mathscr{R}^-(\delta_1, \delta_2)$ under the identification $\mathscr{R}^-(\delta_1, \delta_2) = \mathrm{LA}(\mathbf{Z}_p, L)$ (resp. $\mathscr{R}^-(\delta_1, \delta_2) \boxtimes \mathbf{Z}_p^\times = \mathrm{LA}(\mathbf{Z}_p^\times, L)$).

13.3.1 Calculation of $H^0(\mathscr{C}_{u^-,\varphi,a^+})$:

Proposition 13.9. *Let $M = \mathscr{R}^-(\delta_1, \delta_2)$. We have $H^0(\mathscr{C}_{u^-,\varphi,a^+}) = 0$.*

Proof. This follows immediately from the injectivity of $1 - \delta(p)\varphi$ on $\mathrm{LA}(\mathbf{Z}_p, L)$. □

59. Observe that, in the formula for φ below, $f\left(\frac{x}{p}\right)$ is taken to be zero whenever $z \in \mathbf{Z}_p^\times$, so the precise formula should be $(\varphi f)(x) = \mathbf{1}_{p\mathbf{Z}_p}(x) \delta(p) f\left(\frac{x}{p}\right)$.

13.3.2 Calculation of $H^2(\mathscr{C}_{u^-,\varphi})$:

Lemma 13.10. *The operator u^- restricted to $\mathrm{LA}(\mathbf{Z}_p^\times, L)$ is surjective on $\mathrm{LA}(\mathbf{Z}_p^\times, L)$.*

Proof. This is an easy exercise on power series that we leave to the reader. □

Lemma 13.11. *If $M = \mathscr{R}^-(\delta_1, \delta_2)$ then:*

1. *If $\delta(p) \notin \{p^i |\ i \geq -1\}$, or if $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) \neq i$, then $H^2(\mathscr{C}_{u^-,\varphi}) = 0$.*
2. *Otherwise*[60] *$H^2(\mathscr{C}_{u^-,\varphi})$ is of dimension 1 naturally generated by $[x^{i+1}]$.*

Proof. Suppose first $\delta(p) \notin \{p^i |\ i \geq -1\}$. Note that, by [11, Lemme 5.9],

$$M = \mathrm{LA}(\mathbf{Z}_p^\times) \oplus (p\varphi - 1)M.$$

By Lemma 13.6,

$$H^2(\mathscr{C}_{u^-,\varphi}) = \mathrm{coker}([\varphi - 1] : M/u^- M).$$

Thus it suffices to show that the map $p\varphi - 1 : M/u^- M \to M/u^- M$ is surjective. This follows from the fact that

$$M \xrightarrow{p\varphi-1} M \to M/u^- M$$

is surjective (where the second map is the natural quotient map) since u^- is surjective on $\mathrm{LA}(\mathbf{Z}_p^\times, L)$ by Lemma 13.10.

Suppose now $\delta(p) \in \{p^i |\ i \geq -1\}$. In this case, by [11, Lemme 5.9], we have

$$M = \big(\mathrm{LA}(\mathbf{Z}_p^\times) + (p\varphi - 1)M\big) \oplus L \cdot x^{i+1}, \tag{25}$$

where $\mathrm{LA}(\mathbf{Z}_p^\times) \cap (p\varphi - 1)M = L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^{i+1}$.

Suppose first $i \geq 0$. If $\kappa(\delta) \neq i$ then $(\kappa(\delta) - i)^{-1} u^- x^i = x^{i+1}$. Thus in this case the map $M \xrightarrow{p\varphi-1} M \to M/u^- M$ is surjective and the result follows. On the other hand if $\kappa(\delta) = i$ then x^{i+1} is not in the image of u^- and in this case $H^2(\mathscr{C}_{u^-,\varphi}) = L \cdot x^{i+1}$.

Finally consider the case $i = -1$. In this case, as $\mathbf{1}_{\mathbf{Z}_p}$ is never in the image of u^- the result follows. □

13.3.3 Calculation of $H^3(\mathscr{C}_{u^-,\varphi,a^+})$:

At this stage, we can already deduce the following.

Proposition 13.12. *Let $M = \mathscr{R}^-(\delta_1, \delta_2)$.*

- *If $\delta(p) = p^i$, $i \geq -1$, and $\kappa(\delta) = i$, then $H^3(\mathscr{C}_{u^-,\varphi,a^+})$ is of dimension 1 naturally generated by $[x^{i+1}]$.*
- *Otherwise $H^3(\mathscr{C}_{u^-,\varphi,a^+}) = 0$.*

Proof. This follows from Lemma 13.6(4) and Lemma 13.11, by observing that the action of $[a^+]$ on $H^2(\mathscr{C}_{u^-,\varphi}) = M/(u^-, p\varphi - 1)$ is given by $a^+ + 1$, and using the formula $(a^+ + 1)[x^{i+1}] = (\kappa(\delta) - i)[x^{i+1}]$. □

60. i.e if $\delta(p) = p^{-1}$, or if $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) = i$.

13.3.4 Calculation of $H^1(\mathscr{C}_{u^-,\varphi})$:

The following lemma describes the kernel of u^- acting on $\mathscr{R}^-(\delta_1, \delta_2)$ in the appropriate way so as to calculate (cf. Corollary 13.14) the left term of Equation (22) of Lemma 13.6. Define [61]

$$X_{\kappa(\delta)} := \left\{ f \in \mathrm{LA}(\mathbf{Z}_\mathrm{p}^\times) |\ f(x) = \sum_{i \in (\mathbf{Z}/p^n\mathbf{Z})^\times} c_i \left(\frac{x}{i}\right)^{\kappa(\delta)} \mathbf{1}_{i+p^n\mathbf{Z}_p} \text{ for some } n > 0 \right\}.$$

Lemma 13.13. *If $M = \mathscr{R}^-(\delta_1, \delta_2) = \mathrm{LA}(\mathbf{Z}_p, L)$. Then*

1. *If $\delta(p) \notin \{p^i |\ i \geq 0\}$, or if $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) \neq i$, then*

$$M^{u^-=0} = X_{\kappa(\delta)} \oplus (1-\varphi)M^{u^-=0}.$$

2. *Otherwise*

$$M^{u^-=0} = \left(X_{\kappa(\delta)} + (1-\varphi)M^{u^-=0}\right) \oplus L \cdot x^i,$$

Furthermore $X_{\kappa(\delta)} \cap (1-\varphi)M^{u^-=0}$ is the line $L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i$.

Proof. (1): First suppose that $\delta(p) \neq p^{-1}$. Take $f \in M^{u^-=0}$. Since $\delta(p) \notin \{p^i |\ i \geq -1\}$, by [11, Lemme 5.9], we can uniquely write $f = f_1 + (1-\varphi)f_2$ where f_1 is supported on $\mathbf{Z}_p^\times$ and $f_2 \in M$. Thus $0 = u^- f = u^- f_1 + u^-(1-\varphi)f_2 = u^- f_1 + (1-p\varphi)u^- f_2$. We deduce, again using [11, Lemme 5.9], that $u^- f_1 = u^- f_2 = 0$. Solving the differential equation $u^- f_1 = 0$ gives precisely $f_1 \in X_{\kappa(\delta)}$.

Suppose now $\delta(p) = p^{-1}$. Repeating the same procedure as above, since $p\delta(p) = 1$, in this case [11, Lemme 5.9] gives $u^- f_1 = b\mathbf{1}_{\mathbf{Z}_p^\times}$ and $(1-p\varphi)u^- f_2 = -b\mathbf{1}_{\mathbf{Z}_p^\times}$ for some $b \in L$. This implies $u^- f_2 = -b\mathbf{1}_{\mathbf{Z}_p}$. This equation has no solution unless $b = 0$ (as can be easily seen upon expanding in power series around zero), in which case we obtain again $u^- f_1 = u^- f_2 = 0$.

Suppose finally that $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) \neq i$. By [11, Lemme 5.9], we can write $f = f_1 + (1-\varphi)f_2 + ax^i$, where f_1 is supported on $\mathbf{Z}_p^\times$, $f_2 \in M$ and $a \in L$. Thus $0 = u^- f = u^- f_1 + u^-(1-\varphi)f_2 + au^- x^i = u^- f_1 + (1-p\varphi)u^- f_2 + a(\kappa(\delta) - i)x^{i+1}$. The latter implies

$$0 = u^- f_1 + (1-p\varphi)u^- f_2 \tag{26}$$

$$0 = a(\kappa(\delta) - i)x^{i+1} \tag{27}$$

Again by [11, Lemme 5.9] (this time with $\alpha = p\delta(p) = p^{i+1}$), the first equation implies $u^- f_1 = b\mathbf{1}_{\mathbf{Z}_p^\times} x^{i+1}$ and $(1-p\varphi)u^- f_2 = -b\mathbf{1}_{\mathbf{Z}_p^\times} x^{i+1}$ for some $b \in L$. This implies $u^- f_2 = -bx^{i+1}$. But all the solutions to the equation $u^- f_2 = -bx^{i+1}$

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61. Observe that, upon developing $(x/i)^{\kappa(\delta)} = \sum_{k \geq 0} \binom{\kappa(\delta)}{k} (x/i - 1)^n$ and observing that $v_p(\binom{\kappa(\delta)}{k}) \geq k(\min(\kappa(\delta), 0) - \frac{1}{p-1})$, we see that $(x/i)^{\kappa(\delta)}$ is a well defined analytic function on $i + p^n\mathbf{Z}_p$ for n big enough.

are of the form $f_2 = -b\frac{x^i}{\kappa(\delta)-i} + f_2'$ where $f_2' \in M^{u^-=0}$ is any element. Observe finally that the set

$$\{b \cdot \big(f_1 - (1-\varphi)\frac{x^i}{\kappa(\delta)-i}\big) |\ b \in L,\ f_1 \in \mathrm{LA}(\mathbf{Z}_p^\times, L),\ u^- f_1 = \mathbf{1}_{\mathbf{Z}_p^\times} x^{i+1}\}$$

is exactly $X_{\kappa(\delta)}$. On the other hand, since $\kappa(\delta) \neq i$, equation (27) forces $a = 0$. This shows that $M^{u^-=0}$ is the sum of $X_{\kappa(\delta)}$ and $(1-\varphi)M^{u^-=0}$.

We now show that it is indeed a direct sum: if $\delta(p)$ is not equal to p^i for some $i \geq 0$ then this is immediate. Suppose then that $\delta(p) = p^i$, $i \geq 0$ and $\kappa(\delta) \neq i$ and suppose that $f_1 = (1-\varphi)g$ for some $g \in M^{u^-=0}$, $f_1 \in X_{\kappa(\delta)}$. Then by the same lemma [11, Lemme 5.9] we get

$$f_1 = b' \mathbf{1}_{\mathbf{Z}_p^\times} x^i = (1-\varphi)g \tag{28}$$

for some $b' \in L$. Then $0 = u^- f_1 = b'(\kappa(\delta)-1)\mathbf{1}_{\mathbf{Z}_p^\times} x^{i+1}$ so that $b' = 0$ and hence $g = 0$ and $f_1 = 0$ as well.

(2): This follows from the same arguments as last two paragraphs, noting the following differences. On the one hand, if $\kappa(\delta) = i$, then the value a in equation (27) is free to take any value in L. On the other hand, if $\kappa(\delta) = i$, then there is no solution to the equation $u^- f_2 = -bx^{i+1}$ unless $b = 0$ (expand around zero). In that case equation (26) gives $u^- f_1 = u^- f_2 = 0$. Finally, note that $L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i \in (1-\varphi)M^{u^-=0}$, whence the result. ∎

Observing that $\mathrm{coker}([\varphi - 1]\colon H^0(\mathscr{C}_{u^-}) = \mathrm{coker}(\varphi - 1\colon M^{u^-=0})$, we get the following:

Corollary 13.14. *If $M = \mathscr{R}^-(\delta_1, \delta_2)$ then*

1. *If $\delta(p) \notin \{p^i |\ i \geq 0\}$, or if $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) \neq i$, then*

$$H^1([\varphi - 1]\colon H^0(\mathscr{C}_{u^-})) = X_{\kappa(\delta)}.$$

2. *Otherwise*

$$H^1([\varphi - 1]\colon H^0(\mathscr{C}_{u^-})) = \big(X_{\kappa(\delta)}/L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i\big) \oplus L \cdot [x^i].$$

Proof. This is an immediate consequence of Lemma 13.13. ∎

We now proceed to calculate the right side term of Equation (22) of Lemma 13.6. Note that the action of $[\varphi - 1]$ on $H^1(\mathscr{C}_{u^-}) = M/u^- M$ is given by $p\varphi - 1$.

Lemma 13.15. *If $M = \mathscr{R}^-(\delta_1, \delta_2)$ then*

1. *If $\delta(p) \notin \{p^i |\ i \geq -1\}$, or if $\delta(p) = p^i$, $i \geq 0$, and $\kappa(\delta) \neq i$, then $H^0([\varphi - 1]\colon H^1(\mathscr{C}_{u^-})) = 0$.*
2. *Otherwise $H^0([\varphi - 1]\colon H^1(\mathscr{C}_{u^-})) = L \cdot [x^{i+1}]$.*

Proof. (1): Suppose first that $\delta(p) \notin \{p^i \mid i \geq -1\}$. Take $f \in M$ and suppose that $(1-p\varphi)f = u^- h$ for some $h \in M$. We write $h = h_1 + (1-\varphi)h_2$ with h_1 supported on $\mathbf{Z}_p^\times$. Then $(1-p\varphi)f = u^- h = u^- h_1 + (1-p\varphi)u^- h_2$. By uniqueness and by injectivity of $(1-p\varphi)$ we obtain $f = u^- h_2$.

Now suppose $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) \neq i$. Take $f \in M$ and suppose that $(1-p\varphi)f = u^- h$ for some $h \in M$. We write $h = h_1 + (1-\varphi)h_2 + ax^i$ with h_1 supported on $\mathbf{Z}_p^\times$ and $a \in L$. Then $(1-p\varphi)f = u^- h = u^- h_1 + (1-p\varphi)u^- h_2 + a(\kappa(\delta) - i)x^{i+1}$. This forces $a = 0$ and $(1-p\varphi)(f - u^- h_2) = b\mathbf{1}_{\mathbf{Z}_p^\times} x^{i+1}$ for some $b \in L$. Hence $f - u^- h_2 = bx^{i+1}$ and thus $f = u^-\left(h_2 + \frac{b}{\kappa(\delta)-i}x^i\right)$.

(2): Suppose first $\delta(p) = p^{-1}$. Take $f \in M$ and suppose that $(1-p\varphi)f = u^- h$ for some $h \in M$. We write $h = h_1 + (1-\varphi)h_2$ with h_1 supported on $\mathbf{Z}_p^\times$. Then $(1-p\varphi)f = u^- h = u^- h_1 + (1-p\varphi)u^- h_2$. Then $u^- h_1 = b\mathbf{1}_{\mathbf{Z}_p^\times} = (1-p\varphi)(f - u^- h_2)$. The last equality implies that $f - u^- h_2 = b\mathbf{1}_{\mathbf{Z}_p}$ and hence $f = b\mathbf{1}_{\mathbf{Z}_p}$ modulo u^-. In this case, if $b \neq 0$, then $b\mathbf{1}_{\mathbf{Z}_p}$ is not in the image of u^-. Hence

$$\ker(p\varphi - 1 : M/u^- M) = L \cdot [\mathbf{1}_{\mathbf{Z}_p}].$$

Finally if $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) = i$, then the question is whether x^{i+1} is in the image of u^-. Indeed taking $f(x) \in \mathrm{LA}(\mathbf{Z}_p, L)$ and expanding $u^- f$ in a small ball around $x = 0$, we see that the coefficient of x^{i+1} is 0. Hence x^{i+1} is not in the image of u^-, as we have already pointed out. This completes the proof. □

We can at this stage easily deduce the following corollary, which gives a complete description of $H^1(\mathscr{C}_{u^-,\varphi})$:

Proposition 13.16. *Let $M = \mathscr{R}^-(\delta_1, \delta_2)$. Then*

1. *If $\delta(p) \notin \{p^i \mid i \geq -1\}$, or if $\delta(p) = p^i$, $i \geq 0$, and $\kappa(\delta) \neq i$, then $H^1(\mathscr{C}_{u^-,\varphi}) = X_{\kappa(\delta)}$.*
2. *If $\delta(p) = p^{-1}$, then $H^1(\mathscr{C}_{u^-,\varphi})$ lives in an exact sequence*

$$0 \to X_{\kappa(\delta)} \to H^1(\mathscr{C}_{u^-,\varphi}) \to L \cdot [\mathbf{1}_{\mathbf{Z}_p}] \to 0.$$

3. *If $\delta(p) = p^i$, $i \geq 0$, and $\kappa(\delta) = i$ then $H^1(\mathscr{C}_{u^-,\varphi})$ lives in an exact sequence*

$$0 \to \left(X_{\kappa(\delta)}/L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i\right) \oplus L \cdot [x^i] \to H^1(\mathscr{C}_{u^-,\varphi}) \to L \cdot [x^{i+1}] \to 0.$$

Proof. This is an immediate consequence of Lemmas 13.6, 13.15 and Corollary 13.14. □

13.3.5 Calculation of $H^1(\mathscr{C}_{u^-,\varphi,\gamma})$.

We have already explicitly calculated the second exact sequence (22) of Lemma 13.6 in Proposition 13.16. The left hand side term of Equation (21) is easy to deal with:

Lemma 13.17. *If $M = \mathscr{R}^-(\delta_1, \delta_2)$ then $H^0(\mathscr{C}_{u^-,\varphi}) = 0$.*

Proof. This follows immediately from the injectivity of $\delta(p)\varphi - 1$ on $\mathrm{LA}(\mathbf{Z}_p, L)$, cf. Lemma 11.6(1). $\square$

We calculate the kernel of $[a^+]$ on $\ker([\varphi - 1]\colon H^1(\mathscr{C}_{u^-}))$:

Lemma 13.18. *If $M = \mathscr{R}^-(\delta_1, \delta_2)$ then*

1. *If $\delta(p) \notin \{p^i |\ i \geq -1\}$, or if $\delta(p) = p^i$ for some $i \geq -1$ and $\kappa(\delta) \neq i$, then $[a^+]$ on $H^0([\varphi - 1]\colon H^1(\mathscr{C}_{u^-}))$ is injective.*
2. *Otherwise the kernel of $[a^+]$ on $H^0([\varphi - 1]\colon H^1(\mathscr{C}_{u^-}))$ is (naturally isomorphic to) $L \cdot [x^{i+1}]$.*

Proof. We use Lemma 13.15. If $\delta(p) \notin \{p^i |\ i \geq 0\}$ or if $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) \neq i$, then $\ker(p\varphi - 1\colon M/u^-M) = 0$ and the result is obvious. If $\kappa(\delta) = i$ then

$$\ker(p\varphi - 1\colon M/u^-M) = L \cdot [x^{i+1}].$$

In this case $(a^+ + 1)[x^{i+1}] = 0$.

Suppose now $\delta(p) = p^{-1}$. In this case

$$\ker(p\varphi - 1\colon M/u^-M) = L \cdot [\mathbf{1}_{\mathbf{Z}_p}]$$

and, since $(a^+ + 1)\mathbf{1}_{\mathbf{Z}_p} = (\kappa(\delta) + 1)\mathbf{1}_{\mathbf{Z}_p}$, the result now follows depending on whether $\kappa(\delta) = -1$ or not. $\square$

The last needed ingredient is the kernel of $[a^+]$ on $H^1([\varphi - 1]\colon H^0(\mathscr{C}_{u^-}))$.

Lemma 13.19. *If $M = \mathscr{R}^-(\delta_1, \delta_2)$ then*

1. *If $\delta(p) \notin \{p^i |\ i \geq 0\}$, or if $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) \neq i$, then*
$$\ker([a^+]\colon H^1([\varphi - 1]\colon H^0(\mathscr{C}_{u^-}))) = X_{\kappa(\delta)}.$$
2. *Otherwise*
$$\ker([a^+]\colon H^1([\varphi - 1]\colon H^0(\mathscr{C}_{u^-}))) = \left(X_{\kappa(\delta)}/L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i\right) \oplus L \cdot [x^i].$$

Proof. This is an immediate consequence of Lemma 13.14, noting that $a^+X_{\kappa(\delta)} = 0$ and that $a^+x^i = 0$ whenever $\kappa(\delta) = i$. $\square$

Now we can calculate the first Lie algebra cohomology group with values in $\mathscr{R}^-(\delta_1, \delta_2)$.

Proposition 13.20. *Let $M = \mathscr{R}^-(\delta_1, \delta_2)$. Then*

1. *If $\delta(p) \notin \{p^i |\ i \geq -1\}$, or if $\delta(p) = p^i$ for some $i \geq -1$ and $\kappa(\delta) \neq i$, then*
$$H^1(\mathscr{C}_{u^-,\varphi,a^+}) = X_{\kappa(\delta)}.$$
2. *If $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) = i$, then $H^1(\mathscr{C}_{u^-,\varphi,a^+})$ lives in an exact sequence*
$$0 \to X_{\kappa(\delta)}/L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i \oplus L \cdot [x^i] \to H^1(\mathscr{C}_{u^-,\varphi,a^+}) \to L \cdot [x^{i+1}] \to 0$$

3. *If $\delta(p)=p^{-1}$ and $\kappa(\delta)=-1$ then $H^1(\mathscr{C}_{u^-,\varphi,a^+})$ lives in an exact sequence*

$$0\to X_{\kappa(\delta)}\to H^1(\mathscr{C}_{u^-,\varphi,a^+})\to L\cdot[\mathbf{1}_{\mathbf{Z}_p}]\to 0.$$

Proof. By Lemma 13.17 and the short exact sequence (21) of Lemma 13.6 we have

$$H^1(\mathscr{C}_{u^-,\varphi,a^+})\cong \ker([a^+]\colon H^1(\mathscr{C}_{u^-,\varphi})).$$

Since the short exact sequence (22) splits as a sequence of a^+-modules, we have

$$0\to\ker(a^+\colon M^{u^-=0}/(\varphi-1))\to\ker([a^+]\colon H^1(\mathscr{C}_{u^-,\varphi}))\to\ker(a^++1\colon (M/u^-M)^{p\varphi=1})\to 0.$$

Now (1) (resp. (2), resp. (3)) follows from (1) (resp. (2), resp. (1)) of Lemma 13.19 and (1) (resp. (2), resp. (2)) of Lemma 13.18. □

13.3.6 Calculation of $H^2(\mathscr{C}_{u^-,\varphi,\gamma})$:

We start by calculating the left side term of equation (23) of Lemma 13.6:

Lemma 13.21. *If $M=\mathscr{R}^-(\delta_1,\delta_2)$ then*

1. *If $\delta(p)\notin\left\{p^i \mid i\geq -1\right\}$, or if $\delta(p)=p^i$ for some $i\geq -1$ and $\kappa(\delta)\neq i$, then*

$$\operatorname{coker}([a^+]:H^1(\mathscr{C}_{u^-,\varphi}))=X_{\kappa(\delta)}.$$

2. *If $\delta(p)=p^i$ for some $i\geq 0$ and $\kappa(\delta)=i$, we have a short exact sequence*

$$0\to X_{\kappa(\delta)}/L\cdot\mathbf{1}_{\mathbf{Z}_p^\times}x^i\oplus L\cdot[x^i]\to\operatorname{coker}([a^+]:H^1(\mathscr{C}_{u^-,\varphi}))\to L\cdot[x^{i+1}]\to 0.$$

3. *If $\delta(p)=p^{-1}$ and $\kappa(\delta)=-1$ then we have a short exact sequence*

$$0\to X_{\kappa(\delta)}\to\operatorname{coker}([a^+]:H^1(\mathscr{C}_{u^-,\varphi}))\to L\cdot[\mathbf{1}_{\mathbf{Z}_p}]\to 0.$$

Proof.

1. Suppose first that $\delta(p)\notin\left\{p^i \mid i\geq -1\right\}$, or that $\delta(p)=p^i$ for some $i\geq 0$ and $\kappa(\delta)\neq i$. By Proposition 13.16(1), we have $H^1(\mathscr{C}_{u^-,\varphi})=X_{\kappa(\delta)}$. The result follows then by noting that the action of $[a^+]$ on this space is given by a^+, and that $a^+X_{\kappa(\delta)}=0$ (since $xa^+f=u^-f$).
   We now deal with the case where $\delta(p)=p^{-1}$ and $\kappa(\delta)\neq -1$. In this case $H^1(\mathscr{C}_{u^-,\varphi})$ is described by (2) of Proposition 13.16 and the result follows from Lemma 13.22(1) below.
2. If $\delta(p)=p^i$ for some $i\geq 0$ and $\kappa(\delta)=i$, then $H^1(\mathscr{C}_{u^-,\varphi})$ is described by (3) of Proposition 13.16. Note that $[a^+]$ acts as a^+ on the left term of the exact sequence, and as a^++1 on the right hand side term. The result follows then by Lemma 13.22(2) and by noting that $a^+x^i=0$.
3. This case follows in the same way, using Proposition 13.16(2) and Lemma 13.22(2).

□

The next lemma describes the cokernel of $[a^+]$ on $H^0([\varphi-1]\colon H^1(\mathscr{C}_{u^-}))$.

Lemma 13.22. *If $M = \mathscr{R}^-(\delta_1,\delta_2)$ and $\delta(p) = p^i$ for some $i \geq -1$ then*

1. *If $\kappa(\delta) \neq i$ then* $\operatorname{coker}([a^+]\colon H^0([\varphi-1]\colon H^1(\mathscr{C}_{u^-}))) = 0$.
2. *If $\kappa(\delta) = i$ then* $\operatorname{coker}([a^+]\colon H^0([\varphi-1]\colon H^1(\mathscr{C}_{u^-})))$ *is (naturally isomorphic to) $L\cdot[x^{i+1}]$.*

Proof. We use Lemma 13.15. Suppose first $i \geq 0$. If $\kappa(\delta) \neq i$ then $\ker(p\varphi - 1 : M/u^-M) = 0$ and the result is obvious. If $\kappa(\delta) = i$ then

$$\ker(p\varphi - 1 : M/u^-M) = L\cdot[\mathbf{1}_{\mathbf{Z}_p}x^{i+1}].$$

In this case $(a^+ + 1)x^{i+1} = 0$.

Suppose now $i = -1$. In this case

$$\ker(p\varphi - 1 : M/u^-M) = L\cdot[\mathbf{1}_{\mathbf{Z}_p}]$$

and so $(a^+ + 1)\mathbf{1}_{\mathbf{Z}_p} = (\kappa(\delta)+1)\mathbf{1}_{\mathbf{Z}_p}$. The result now follows depending on whether $\kappa(\delta) = -1$ or not. □

We finally calculate the right hand side term $\ker([a^+]\colon H^2(\mathscr{C}_{u^-,\varphi}))$ of equation (23) of Proposition 13.6.

Lemma 13.23. *If $M = \mathscr{R}^-(\delta_1,\delta_2)$. Then*

1. *If $\delta(p) \notin \left\{p^i \mid i \geq -1\right\}$, or if $\delta(p) = p^i$ for some $i \geq -1$ and $\kappa(\delta) \neq i$, then* $\ker([a^+]\colon H^2(\mathscr{C}_{u^-,\varphi})) = 0$.
2. *Otherwise* $\ker([a^+]\colon H^2(\mathscr{C}_{u^-,\varphi})) = L\cdot[x^{i+1}]$.

Proof.

1. If $\delta(p) \neq p^{-1}$ the result follows since, by Lemma 13.11, we know that $H^2(\mathscr{C}_{u^-,\varphi}) = 0$. If $\delta(p) = p^{-1}$ (and hence $\kappa(\delta) \neq -1$) then $H^2(\mathscr{C}_{u^-,\varphi}) = L\cdot\mathbf{1}_{\mathbf{Z}_p}$ and, since $(a^+ + 1)\mathbf{1}_{\mathbf{Z}_p} = (\kappa(\delta)+1)\mathbf{1}_{\mathbf{Z}_p}$, it is injective.
2. Note first that $[a^+]$ acts on $H^2(\mathscr{C}_{u^-,\varphi})$ as $a^+ + 1$. By (2) of Lemma 13.11, $H^2(\mathscr{C}_{u^-,\varphi}) = L\cdot x^{i+1}$ and the result follows since $(a^+ + 1)x^{i+1} = 0$.

□

We are now ready to compute $H^2(\mathscr{C}_{u^-,\varphi,a^+})$:

Proposition 13.24. *If $M = \mathscr{R}^-(\delta_1,\delta_2)$ then*

1. *If $\delta(p) \notin \left\{p^i \mid i \geq -1\right\}$, or if $\delta(p) = p^i$ for some $i \geq -1$ and $\kappa(\delta) \neq i$, then*

$$H^2(\mathscr{C}_{u^-,\varphi,a^+}) = X_{\kappa(\delta)}.$$

2. *If $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) = i$, then $H^2(\mathscr{C}_{u^-,\varphi,a^+})$ lives in an exact sequence*

$$0 \to Y_i \to H^2(\mathscr{C}_{u^-,\varphi,a^+}) \to L \cdot [x^{i+1}] \to 0,$$

where

$$0 \to X_{\kappa(\delta)}/L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i \oplus L \cdot [x^i] \to Y_i \to L \cdot [x^{i+1}] \to 0.$$

3. *If $\delta(p) = p^{-1}$ and $\kappa(\delta) = -1$ then $H^2(\mathscr{C}_{u^-,\varphi,a^+})$ lives in an exact sequence*

$$0 \to Y_{-1} \to H^2(\mathscr{C}_{u^-,\varphi,a^+}) \to L \cdot [\mathbf{1}_{\mathbf{Z}_p}] \to 0,$$

where

$$0 \to X_{\kappa(\delta)} \to Y_{-1} \to L \cdot [\mathbf{1}_{\mathbf{Z}_p}] \to 0.$$

Proof. (1) (resp. (2), resp. (3)) follows from (1) (resp. (2), resp (3)) of Lemma 13.21 and (1) (resp. (2), resp. (2)) of Lemma 13.23. □

13.4 The $\overline{P}^+$-cohomology of $\mathscr{R}^-(\delta_1, \delta_2)$

We can now just calculate the $\tilde{P}$-invariants of the Lie algebra cohomology to calculate the $\overline{P}^+$-cohomology of $\mathscr{R}^-(\delta_1, \delta_2)$.

13.4.1 Calculation of $H^0(\overline{P}^+, \mathscr{R}^-(\delta_1, \delta_2))$:

Lemma 13.25. *Let $M = \mathscr{R}^-(\delta_1, \delta_2)$. Then $H^0(\overline{P}^+, M) = 0$.*

Proof. Obvious from Proposition 13.9. □

13.4.2 Calculation of $H^1(\overline{P}^+, \mathscr{R}^-(\delta_1, \delta_2))$:

Lemma 13.26. *If $M = \mathscr{R}^-(\delta_1, \delta_2)$ then*

1. *If $\delta(x) \neq x^i$ for any $i \geq 0$ then $H^1(\overline{P}^+, M)$ is of dimension 1 and generated by $\mathbf{1}_{\mathbf{Z}_p^\times} \delta \otimes \delta$.*
2. *If $\delta(x) = x^i$ for some $i \geq 0$ then $H^1(\overline{P}^+, M)$ is of dimension 2.*

Proof. Start observing that the action of $\tilde{P}$ on the Lie algebra cohomology is explicitly given in Remark 13.4 (cf. also Remark 13.7): the action of τ on each of the extremities of the exact sequences of Lemma 13.6 is the usual one, while the action of A^0 is given by the usual one, except for the term $\ker([\varphi - 1] \colon H^1(\mathscr{C}_{u^-}))$, on which its action is given by the usual action twisted by χ.

- Suppose we are under the hypothesis of Proposition 13.20(1). Then

$$H^1(\mathscr{C}_{u^-,\varphi,a^+}) = X_{\kappa(\delta)}.$$

Note $\gamma = \sigma_a \in A^0$ a topological generator. Let $f \in H^0(A^0, X_{\kappa(\delta)})$ and write

$$f(x) = \sum_{i \in (\mathbf{Z}/p^n\mathbf{Z})^\times} c_i \left(\frac{x}{i}\right)^{\kappa(\delta)} \mathbf{1}_{i+p^n\mathbf{Z}_p}, \quad n \geq 0.$$

Since $\gamma f = f$, we have

$$\begin{aligned} \sum_{i \in (\mathbf{Z}/p^n\mathbf{Z})^\times} c_i \left(\frac{x}{i}\right)^{\kappa(\delta)} \mathbf{1}_{i+p^n\mathbf{Z}_p} &= \delta(a) \sum_{i \in (\mathbf{Z}/p^n\mathbf{Z})^\times} c_i \left(\frac{x}{ia}\right)^{\kappa(\delta)} \mathbf{1}_{ia+p^n\mathbf{Z}_p} \\ &= \delta(a) \sum_{i \in (\mathbf{Z}/p^n\mathbf{Z})^\times} c_{ia^{-1}} \left(\frac{x}{i}\right)^{\kappa(\delta)} \mathbf{1}_{i+p^n\mathbf{Z}_p}. \end{aligned}$$

Thus $\delta(a)c_{ia^{-1}} = c_i$ which implies $c_a = c_1\delta(a)$, for any $a \in \mathbf{Z}_p^\times$. This implies

$$f(x) = c_1\delta(x)\mathbf{1}_{\mathbf{Z}_p^\times}.$$

Since $\delta\mathbf{1}_{\mathbf{Z}_p^\times}$ is fixed by τ, the result follows from Lemma 13.3.

- We now place ourselves under the hypothesis of Proposition 13.20(2). We have

$$0 \to X_{\kappa(\delta)}/L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i \oplus L \cdot [x^i] \to H^1(\mathscr{C}_{u^-,\varphi,a^+}) \to L \cdot [x^{i+1}] \to 0. \tag{29}$$

To calculate the A^0-invariants of $X := X_{\kappa(\delta)}/L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i$, we just consider the short exact sequence of Γ-modules

$$0 \to L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i \to X_{\kappa(\delta)} \to X \to 0$$

and take the associated long exact sequence. One easily sees that

– if $\delta(x) \neq x^i$, then $H^0(A^0, L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i) = H^1(A^0, L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i) = 0$ so that $H^0(A^0, X) = H^0(A^0, X_{\kappa(\delta)}) = L \cdot \mathbf{1}_{\mathbf{Z}_p^\times}\delta$.

– If $\delta(x) = x^i$, then A^0 fixes $\mathbf{1}_{\mathbf{Z}_p^\times} x^i$, so we get a long exact sequence

$$0 \to L\cdot\mathbf{1}_{\mathbf{Z}_p^\times}\delta \to L\cdot\mathbf{1}_{\mathbf{Z}_p^\times}\delta \to H^0(A^0, X) \to L\cdot\mathbf{1}_{\mathbf{Z}_p^\times}\delta \xrightarrow{\alpha} H^1(A^0, X_{\kappa(\delta)}) \to H^1(A^0, X) \to 0.$$

Now $H^1_{\mathrm{Lie}}(A^0, X_{\kappa(\delta)}) = X_{\kappa(\delta)}$ and so $H^1(A^0, X_{\kappa(\delta)}) = H^0(A^0, X_{\kappa(\delta)}) = L \cdot 1_{\mathbf{Z}_p^\times}\delta$. It follows that α is an isomorphism and so $H^0(A^0, X) = 0$. Note also that this implies $H^1(A^0, X) = 0$.

Thus, if $\delta(x) \neq x^i$, $H^1(\mathscr{C}_{u^-,\varphi,a^+})^{\tilde{P}} = L \cdot \mathbf{1}_{\mathbf{Z}_p^\times}\delta$. Suppose now $\delta(x) = x^i$. Since the action of A^0 on each of the terms of Equation (29) is locally constant, taking invariants is exact and we obtain

$$0 \to L \cdot [x^i] \to H^1(\mathscr{C}_{u^-,\varphi,a^+})^{A^0} \to L \cdot [x^{i+1}] \to 0.$$

taking $\overline{U}$-invariants of this exact sequence (note that $\tau([x^{i+1}]) = [\frac{x^{i+1}}{1-px}] = [x^{i+1}] + p[x^{i+2}] + p[x^{i+3}] + \ldots = [x^{i+1}] \mod u^-$ if $\delta(x) = x^i$), we obtain the desired result.

- Finally, suppose that hypothesis of Proposition 13.20(3) hold. We have

$$0 \to X_{\kappa(\delta)} \to H^1(\mathscr{C}_{u^-,\varphi,a^+}) \to L \cdot \mathbf{1}_{\mathbf{Z}_p} \to 0,$$

and the result follows similarly. □

13.4.3 Calculation of $H^2(\overline{P}^+, \mathscr{R}^-(\delta_1, \delta_2))$:

Lemma 13.27. *If $M = \mathscr{R}^-(\delta_1, \delta_2)$ then*

1. *If $\delta \neq x^i$ for any $i \geq 0$, then $H^2(\overline{P}^+, M)$ is of dimension 1 and generated by $\mathbf{1}_{\mathbf{Z}_p^\times}\delta \otimes \delta$.*
2. *If $i \geq 0$ and $\delta(x) = x^i$ then $H^2(\overline{P}^+, M)$ is of dimension 3.*

Proof. - Suppose first that we are in the case of Proposition 13.24(1). Then $H^2(\mathscr{C}_{u^-,\varphi,a^+}) = X_{\kappa(\delta)}$, and the result follows as in Lemma 13.26 above.

- We now deal with the case of Proposition 13.24(2). We have

$$0 \to Y_i \to H^2(\mathscr{C}_{u^-,\varphi,a^+}) \to L \cdot [x^{i+1}] \to 0, \tag{30}$$

$$0 \to X_{\kappa(\delta)}/L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i \oplus L \cdot x^i \to Y_i \to L \cdot [x^{i+1}] \to 0.$$

Again, the action of $\tilde{P}$ on each component is described in Remark 13.7. As before, since the action of $\tilde{P}$ is locally constant, taking $\tilde{P}$-invariants of the short exact sequence (30) gives

$$0 \to (Y_i)^{\tilde{P}} \to H^2(\mathscr{C}_{u^-,\varphi,a^+})^{\tilde{P}} \to (L \cdot [x^{i+1}])^{\tilde{P}} \to 0.$$

Now, $[x^{i+1}]$ is invariant under the action of A^0 and τ if and only if $\delta(x) = x^i$ for some $i \geq 0$, and the $\tilde{P}$-invariants of Y_i were calculated in Lemma 13.26. This allows us to conclude.

- The case of Proposition 13.24(3) is treated similarly. □

13.4.4 Calculation of $H^3(\overline{P}^+, \mathscr{R}^-(\delta_1, \delta_2))$:

Lemma 13.28. *Let $M = \mathscr{R}^-(\delta_1, \delta_2)$.*

- *If $\delta = x^i$, $i \geq 0$, then $H^3(\overline{P}^+, \mathscr{R}^-(\delta_1, \delta_2))$ is of dimension 1 naturally generated by $[x^{i+1}]$.*
- *Otherwise $H^3(\overline{P}^+, \mathscr{R}^-(\delta_1, \delta_2)) = 0$.*

Proof. This follows by taking $\tilde{P}$-invariants to the results of Proposition 13.12, by observing that the action of τ is the natural one, and that of A^0 is twisted by χ. □

13.5 The $\overline{P}^+$-cohomology of $\mathscr{R}^+(\delta_1, \delta_2)$: a first reduction

In this section we calculate all $\overline{P}^+$-cohomology groups of $\mathscr{R}^+(\delta_1, \delta_2)$ as described in Proposition 13.1.

We first start with a lemma that allows us to reduce, as we have already done before for the A^+-cohomology (cf. §11.2.2), the calculation of $H^i(\overline{P}^+, \mathscr{R}^+(\delta_1, \delta_2))$ to that of $H^i(\overline{P}^+, \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, L)^*(\delta_1, \delta_2))$ for $N \geq 0$ big enough, where $\mathrm{Pol}_{\leq N}(\mathbf{Z}_p, L)^*(\delta_1, \delta_2)$ denotes the sub-module of $\mathscr{R}^+(\delta_1, \delta_2)$ corresponding to $\mathrm{Pol}_{\leq N}(\mathbf{Z}_p, L)^*$ under the identification (as L-vector spaces) $\mathscr{R}^+(\delta_1, \delta_2) = \mathscr{R}^+$. We also recall that, under the Amice transform, we have an identification $\mathrm{Pol}_{\leq N}(\mathbf{Z}_p, L)^*(\delta_1, \delta_2) = \oplus_{i=0}^N L \cdot t^i$. This module is stable under the action of $\overline{P}^+$ by [11, Lemme 5.20] [62], and the action of $\overline{P}^+$ is explicitly given by

$$\sigma_a(t^j) = \delta_1 \delta_2^{-1}(a) a^j t^j, \quad \varphi(t^j) = \delta_1 \delta_2^{-1}(p) p^j t^j, \quad \tau(t^j) = \sum_{h=0}^{j} \binom{\kappa - h}{j - h} p^{j-h}\, t^h,$$

where we have set $\kappa = -\kappa(\delta_1 \delta_2^{-1}) - 1$. Observe that, if $\kappa \in \{0, 1, \dots, N-1\}$ and $j = \kappa + 1$, then $\tau t^j = t^j$.

Lemma 13.29. *We have, for every i and for N big enough,*

$$H^i(\overline{P}^+, \mathscr{R}^+(\delta_1, \delta_2)) = H^i(\overline{P}^+, \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, L)^*(\delta_1, \delta_2)).$$

Proof. This follows from the Hochschild-Serre spectral sequence and the same arguments of Lemma 11.9. Observe that it suffices to take N such that $|\delta(p)| < p^N$. □

13.6 The Lie algebra cohomology of $\mathrm{Pol}_{\leq N}(\mathbf{Z}_p, L)^*(\delta_1, \delta_2)$

From now on until the end of this section, call $M = \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, L)^*(\delta_1, \delta_2)$, which we identify with $\oplus_{i=0}^N L \cdot t^i$ equipped with the corresponding action of $\overline{P}^+$. Let us now calculate the Lie algebra action on the module M.

Lemma 13.30. *For $f \in M$, the infinitesimal actions of a^+ and u^- and the action of φ are given by*

$$(a^+ f)(t) = (\kappa(\delta_1) - \kappa(\delta_2)) f(t) + t f'(t),$$

$$(u^- f)(t) = (\kappa(\delta_2) - \kappa(\delta_1) - 1) f'(t) - t f''(t),$$

$$(\varphi f)(t) = \delta_1 \delta_2^{-1}(p) f(pt).$$

62. The first $\overline{P}^+$-cohomology group of this module is calculated in [11, Lemme 5.21] but, as we mentioned earlier, there are some small mistakes there, whence the incompatibility with our results.

Proof. First note that as a A^+-module, $\mathscr{R}^+(\delta_1,\delta_2)$ is identified with $\mathscr{R}^+(\delta_1\delta_2^{-1})$. The action of φ is now evident. For the action of a^+ note that $(\sigma_a f)(t) = \delta_1\delta_2^{-1}(a)f(at)$ for any $a \in \mathbf{Z}_p^\times$. The derivative of the function $g(a) := \delta_1\delta_2^{-1}(a)f(at)$ evaluated at $a = 1$ is precisely $(\kappa(\delta_1) - \kappa(\delta_2))f(t) + tf'(t)$.

Finally viewing $\mathscr{R}^-(\delta_1,\delta_2)$ as the module $\mathscr{R}^-$ equipped with action of $\overline{P}^+$, we have by, [22, Théorème 1.1],

$$u^- = -t^{-1}\nabla(\nabla - \kappa(\delta_2\delta_1^{-1})),$$

where here $\nabla = t\frac{d}{dt}$. Thus

$$\begin{aligned}(u^- f)(t) &= -\frac{d}{dt}(\nabla - \kappa(\delta_2\delta_1^{-1}))(f)(t)\\ &= -\frac{d}{dt}(tf'(t) - \kappa(\delta_2\delta_1^{-1})f(t))\\ &= (\kappa(\delta_2) - \kappa(\delta_1) - 1)f'(t) - tf''(t)\end{aligned}$$

□

13.6.1 Calculation of $H^0(\mathscr{C}_{u^-,\varphi,a^+})$:

Call $\kappa = \kappa(\delta_1\delta_2^{-1})$.

Lemma 13.31.

1. *If $\delta_1\delta_2^{-1}(p) = p^{-i}$, $i \geq 0$, and $\kappa = -i$, then $H^0(\mathscr{C}_{u^-,\varphi,a^+}) = L \cdot t^i$.*
2. *Otherwise $H^0(\mathscr{C}_{u^-,\varphi,a^+}) = 0$.*

Proof. The formula $u^- t^j = j(-\kappa - j)t^{j-1}$ shows that $M^{u^-=0} = L \cdot t^0$ if $\kappa \notin \{-i, N \geq i \geq 1\}$, and $M^{u^-=0} = L \cdot t^0 \oplus L \cdot t^i$ if $\kappa = -i, N \geq i \geq 1$.

Suppose $\kappa = -i, i \geq 1$. In this case, t^0 is not killed by a^+ and,since we have $a^+ t^j = (\kappa + j)t^j$ and $\varphi(t^j) = \delta_1\delta_2^{-1}(p)p^j t^j$, we see that the term t^i is in the kernel of a^+ and $\varphi - 1$ if and only if $\delta_1\delta_2^{-1}(p) = p^{-i}$.

On the other cases, the term t^0 is in the kernel of a^+ and $\varphi - 1$ if and only if $\kappa = 0$ and $\delta_1\delta_2^{-1}(p) = 1$. This completes the proof. □

13.6.2 Calculation of $H^1(\mathscr{C}_{u^-,\varphi,a^+})$:

In the following we note $\kappa = \kappa(\delta_1\delta_2^{-1})$. Observe also, for the following statements, that $\delta_1\delta_2^{-1}(p) = \delta(p)$ (since $\chi(p) = 1$).

Lemma 13.32.

1. *If $\delta(p) \notin \{p^{-i}, i \geq 0\}$, or if $\delta(p) = p^{-i}$ for some $N \geq i \geq 1$ and $\kappa \neq -i$, then $H^1(\mathscr{C}_{u^-,\varphi}) = 0$.*
2. *If $\delta(p) = 1$ then $H^1(\mathscr{C}_{u^-,\varphi}) = L \cdot [t^0]$.*
3. *If $\delta(p) = p^{-i}$ for some $i \geq 1$ and $\kappa = -i$ then $H^1(\mathscr{C}_{u^-,\varphi})$ lives in an exact sequence*

$$0 \to L \cdot [t^i] \to H^1(\mathscr{C}_{u^-,\varphi}) \to L \cdot [t^{i-1}] \to 0.$$

Proof. We use equation (21) of Lemma 13.6 to calculate this group. First observe that, if $\kappa \notin \{-i, N \geq i \geq 1\}$, then $H^1(\mathscr{C}_{u^-}) = L \cdot [t^N]$ and $H^1(\mathscr{C}_{u^-}) = L \cdot [t^{i-1}] \oplus L \cdot [t^N]$ if $\kappa = -i$, $N \geq i \geq 1$. The result now follows by considering the kernel of $[\varphi - 1]$ on $H^1(\mathscr{C}_{u^-})$ and the cokernel of $[\varphi - 1]$ on $H^0(\mathscr{C}_{u^-})$. Observe that N being chosen big enough, the term $[t^N]$ ends up always being killed. □

We now calculate the extremities of the short exact sequence of (21) of Lemma 13.6.

Lemma 13.33.

1. *If $\delta(p) \notin \{p^{-i}, i \geq 0\}$, or if $\delta(p) = p^{-i}$, $i \geq 0$, and $\kappa \neq -i$, then $H^1([a^+] \colon H^0(\mathscr{C}_{u^-,\varphi})) = 0$.*
2. *Otherwise $H^1([a^+] \colon H^0(\mathscr{C}_{u^-,\varphi})) = L \cdot [t^i]$.*

Proof. Observe first he formula

$$a^+ t^i = (\kappa + i) t^i.$$

1. In the first case we have $H^0(\mathscr{C}_{u^-,\varphi})$ so in particular $H^1([a^+] : H^0(\mathscr{C}_{u^-,\varphi})) = 0$.
2. If $\delta(p) = p^{-i}$ for some $i \geq 0$ and $\kappa = -i$, then $H^0(\mathscr{C}_{u^-,\varphi}) = L \cdot t^i$ and $a^+ t^i = 0$ giving the result.

□

Lemma 13.34.

1. *If $\delta(p) \notin \left\{p^{-i} \mid i \geq 0\right\}$, or if $\delta(p) = p^{-i}$, $i \geq 0$, and $\kappa \neq -i$, then $H^0([a^+] \colon H^1(\mathscr{C}_{u^-,\varphi})) = 0$.*
2. *If $\delta(p) = 1$ and $\kappa = 0$, then $H^0([a^+] \colon H^1(\mathscr{C}_{u^-,\varphi})) = L \cdot [t^0]$.*
3. *Otherwise we have an exact sequence*

$$0 \to L \cdot [t^i] \to H^0([a^+] \colon H^1(\mathscr{C}_{u^-,\varphi})) \to L \cdot [t^{i-1}] \to 0.$$

Proof.

1. If $\delta(p) \notin \left\{p^{-i} \mid i \geq 0\right\}$, or if $\delta(p) = p^{-i}$ for some $i \geq 1$ and $\kappa \neq -i$, the group $H^1(\mathscr{C}_{u^-,\varphi})$ is already zero by Lemma 13.32(i). If $\delta(p) = 1$ and $\kappa \neq 0$, then, by Lemma 13.32(ii), $H^1(\mathscr{C}_{u^-,\varphi}) = L \cdot [t^0]$ and a^+ acts via multiplication by κ, so it is injective.
2. This follows easily from Lemma 13.32(ii).
3. If $\delta(p) = p^{-i}$ for some $i \geq 1$ and $\kappa = -i$, then by Lemma 13.32(3) we have an exact sequence

$$0 \to L \cdot [t^i] \to H^1(\mathscr{C}_{u^-,\varphi}) \to L \cdot [t^{i-1}] \to 0,$$

   where $[a^+]$ acts as a^+ on the LHS term and as $a^+ + 1$ on the RHS term. We deduce the result since $a^+ t^i = (a^+ + 1) t^{i-1} = 0$.

$\square$

We can now put everything together to calculate the Lie algebra cohomology.

Proposition 13.35. *Let $M = \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, L)^*(\delta_1, \delta_2)$. Then*

1. *If $\delta(p) \notin \{p^{-i}, i \geq 0\}$, or if $\delta(p) = p^{-i}$, $i \geq 0$, and $\kappa \neq -i$, then $H^1(\mathscr{C}_{u^-,\varphi,a^+}) = 0$.*
2. *If $\delta(p) = 1$ and $\kappa = 0$, then $H^1(\mathscr{C}_{u^-,\varphi,a^+})$ lives in a short exact sequence*

$$0 \to L \cdot [t^0] \to H^1(\mathscr{C}_{u^-,\varphi,a^+}) \to L \cdot [t^0] \to 0.$$

3. *Otherwise we have short exact sequences*

$$0 \to L \cdot [t^i] \to H^1(\mathscr{C}_{u^-,\varphi,a^+}) \to Z_i \to 0,$$

$$0 \to L \cdot [t^i] \to Z_i \to L \cdot [t^{i-1}] \to 0.$$

Proof. This is an immediate consequence of Lemma 13.33 and Lemma 13.34. $\square$

13.6.3 Calculation of $H^2(\mathscr{C}_{u^-,\varphi,a^+})$:

Using the same methods based on Lemma 13.6, we calculate $H^2(\mathscr{C}_{u^-,\varphi,a^+})$ for the module $M = \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, L)^*(\delta_1, \delta_2)$. The following series of lemmas systematically calculate each term appearing on Lemma 13.6. Since the calculations are in the same spirit, we leave the easy proofs to the reader. Call $\kappa = \kappa(\delta_1\delta_2^{-1})$ as before.

The following two lemmas calculate $H^2(\mathscr{C}_{u^-,\varphi})$ and the kernel of $[a^+]$ acting on it.

Lemma 13.36.

1. *If $\kappa = -i$, $i \geq 1$, and $\delta(p) = p^{-i}$, then $H^2(\mathscr{C}_{u^+,\varphi}) = L \cdot [t^{i-1}]$.*
2. *Otherwise $H^2(\mathscr{C}_{u^+,\varphi}) = 0$.*

Lemma 13.37.

1. *If $\kappa = -i$, $i \geq 1$, and $\delta(p) = p^{-i}$, then $H^0([a^+]\colon H^2(\mathscr{C}_{u^-,\varphi})) = L \cdot [t^{i-1}]$.*
2. *Otherwise $H^0([a^+]\colon H^2(\mathscr{C}_{u^-,\varphi})) = 0$.*

We calculate now the cokernel of $[a^+]$ on $H^1(\mathscr{C}_{u^-,\varphi})$.

Lemma 13.38.

1. *If $\delta(p) \notin \left\{p^{-i}, i \geq 0\right\}$, or if $\delta(p) = p^{-i}$ for some $N \geq i \geq 0$ and $\kappa \neq -i$, then $H^1([a^+]\colon H^1(\mathscr{C}_{u^-,\varphi})) = 0$.*
2. *If $\delta(p) = 1$ and $\kappa = 0$, then $H^1([a^+]\colon H^1(\mathscr{C}_{u^-,\varphi})) = L \cdot [t^0]$.*
3. *If $\delta(p) = p^{-i}$ for some $i \geq 1$ and $\kappa = -i$ then $H^1([a^+]\colon H^1(\mathscr{C}_{u^-,\varphi}))$ lives in an exact sequence*

$$0 \to L \cdot [t^i] \to H^1([a^+]\colon H^1(\mathscr{C}_{u^-,\varphi})) \to L \cdot [t^{i-1}] \to 0.$$

We conclude

Proposition 13.39.

1. *If* $\delta(p) \notin \{p^{-i}, i \geq 0\}$, *or if* $\delta(p) = p^{-i}$ *for some* $N \geq i \geq 0$ *and* $\kappa \neq -i$, *then* $H^2(\mathscr{C}_{u^-,\varphi,a^+}) = 0$.
2. *If* $\delta(p) = 1$ *and* $\kappa = 0$, *then* $H^2(\mathscr{C}_{u^-,\varphi,a^+}) = L \cdot [t^0]$.
3. *If* $\delta(p) = p^{-i}$ *for some* $i \geq 1$ *and* $\kappa = -i$ *then* $H^2(\mathscr{C}_{u^-,\varphi,a^+})$ *lives in an exact sequence*

$$0 \to Z \to H^2(\mathscr{C}_{u^-,\varphi,a^+}) \to L \cdot [t^{i-1}],$$

$$0 \to L \cdot [t^i] \to Z \to L \cdot [t^{i-1}] \to 0.$$

13.6.4 Calculation of $H^3(\mathscr{C}_{u^-,\varphi,a^+})$:

Proposition 13.40.

1. *If* $\kappa = -i$, $i \geq 1$, *and* $\delta(p) = p^{-i}$, *then* $H^3(\mathscr{C}_{u^+,\varphi,a^+}) = L \cdot [t^{i-1}]$.
2. *Otherwise* $H^3(\mathscr{C}_{u^+,\varphi,a^+}) = 0$.

Proof. Immediate from Lemma 13.36. □

13.7 The $\overline{P}^+$-cohomology of $\mathscr{R}^+(\delta_1, \delta_2)$

We take $\tilde{P}$-invariants to calculate group cohomology from the Lie algebra cohomology.

13.7.1 Calculation of $H^0(\overline{P}^+, \mathscr{R}^+(\delta_1, \delta_2))$

Lemma 13.41.

1. *If* $\delta_1\delta_2^{-1} = x^{-i}$, $i \geq 0$, *then* $H^0(\mathscr{C}_{u^-,\varphi,a^+}) = L \cdot t^i$.
2. *Otherwise* $H^0(\mathscr{C}_{u^-,\varphi,a^+}) = 0$.

Proof. This follows by taking $\tilde{P}$-invariants on Lemma 13.31 and observing that τ fixes t^i. □

13.7.2 Calculation of $H^1(\overline{P}^+, \mathscr{R}^+(\delta_1, \delta_2))$

Lemma 13.42. *If* $M = \mathscr{R}^+(\delta_1, \delta_2)$ *then*

1. *If* $\delta_1\delta_2^{-1} \notin \{x^{-i} \mid i \geq 0\}$ *then* $H^1(\overline{P}^+, M) = 0$.
2. *If* $\delta_1\delta_2^{-1} = x^{-i}$ *for some* $i \geq 0$ *then* $H^1(\overline{P}^+, M)$ *is of dimension* 2.

Proof. To calculate the fixed points by $\tilde{P}$ on the Lie algebra cohomology after all the identifications we have made, recall that we have rendered these actions explicit in Remark 13.4.

1. Suppose first that we are in the situation of Proposition 13.35(1). In this case $H^1(\mathscr{C}_{u^-,\varphi,a^+}) = 0$ and the result is immediate.
   Now assume we are in the case of Proposition 13.35(2), then we have
   $$0 \to L \cdot [t^0] \to H^1(\mathscr{C}_{u^-,\varphi,a^+}) \to L \cdot [t^0] \to 0.$$
   Note also that, by Remark 13.4, A^0 acts on each term on the extremities via multiplication by $\delta_1\delta_2^{-1}$ (which is not trivial by hypothesis). We conclude by taking A^0 invariants of this sequence.
   Finally, suppose that the hypothesis of Proposition 13.35(3) are satisfied. Then we have exact sequences
   $$0 \to L \cdot [t^i] \to H^1(\mathscr{C}_{u^-,\varphi,a^+}) \to Z_i \to 0,$$
   $$0 \to L \cdot [t^i] \to Z_i \to L \cdot [t^{i-1}] \to 0.$$
   Again, by Remark 13.4, A^0 acts on the first term of the two SES's as multiplication by $\delta_1\delta_2^{-1}|_{\mathbf{Z}_p^\times} \neq \chi^{-i}$ and on the second term of the second SES via multiplication by $\chi\delta_1\delta_2^{-1}|_{\mathbf{Z}_p^\times} \neq \chi^{-i+1}$. The result also follows by taking A^0-invariants.
2. If $\delta_1\delta_2^{-1} = \mathbf{1}_{\mathbf{Q}_p^\times}$, then we are in the situation of Proposition 13.35(2), and the result follows since everything is $\tilde{P}$-invariant.
   If $\delta_1\delta_2^{-1} = x^{-i}$ for some $i \geq 1$, we are in the situation of Proposition 13.35(3). Note that, by Remark 13.4, τ acts on each of the extremities of the long exact sequences of 13.35(3) just as τ. In this case, the term t^{i-1} of the second short exact sequence is not fixed by the action of τ, so taking $\tilde{P}$-invariants in the short exact sequences gives the result.

□

13.7.3 Calculation of $H^2(\overline{P}^+, \mathscr{R}^+(\delta_1, \delta_2))$

Lemma 13.43.

1. *If $\delta_1\delta_2^{-1} \notin \{x^{-i}, i \geq 0\}$, then $H^2(\overline{P}^+, M) = 0$.*
2. *If $\delta_1\delta_2^{-1} = \mathbf{1}_{\mathbf{Q}_p^\times}$ then $H^2(\overline{P}^+, M)$ is of dimension one naturally isomorphic to $L \cdot [t^0]$.*
3. *If $\delta_1\delta_2^{-1} = x^{-i}$, $i \geq 1$, then $H^2(\overline{P}^+, M)$ is of dimension 3.*

Proof. This follows by taking $\overline{P}^+$-invariants in Lemma 13.39. □

13.7.4 Calculation of $H^3(\overline{P}^+, \mathscr{R}^+(\delta_1, \delta_2))$

Lemma 13.44.

1. *If $\delta_1\delta_2^{-1} = x^{-i}$, $i \geq 1$, then $H^3(\overline{P}^+, M)$ is of dimension 1 naturally generated by $[t^{i-1}]$.*
2. *Otherwise $H^3(\overline{P}^+, M) = 0$.*

Proof. Immediate from Lemma 13.40. □

13.8 The $\overline{P}^+$-cohomology of $\mathscr{R}(\delta_1, \delta_2)$

Call, for $* \in \{+, -, \emptyset\}$, M_* the module $\mathscr{R}^*(\delta_1, \delta_2)$. We use the short exact sequence

$$0 \to M_+ \to M \to M_- \to 0 \tag{31}$$

in order to calculate the $\overline{P}^+$-cohomology of M. We restate one of the propositions announced at the beginning of this section.

Proposition 13.45. *Let $M = \mathscr{R}(\delta_1, \delta_2)$. Then*

1. *If $\delta_1\delta_2^{-1} \notin \{x^{-i}, i \in \mathbf{N}\} \cup \{\chi x^i, i \in \mathbf{N}\}$, then $\dim_L H^j(\overline{P}^+, M) = 0, 1, 1, 0$, for $j = 0, 1, 2, 3$.*
2. *If $\delta_1\delta_2^{-1} = \mathbf{1}_{\mathbf{Q}_p^\times}$, then $\dim_L H^j(\overline{P}^+, M) = 1, 2, 2, 0$, for $j = 0, 1, 2, 3$.*
3. *If $\delta_1\delta_2^{-1} = x^{-i}, i \in \mathbf{N}$, then $\dim_L H^j(\overline{P}^+, M) = 1, 3, 2, 0$, for $j = 0, 1, 2, 3$.*
4. *If $\delta_1\delta_2^{-1} = \chi x^i, i \in \mathbf{N}$, then $\dim_L H^j(\overline{P}^+, M) = 0, 2, 2, 1$, for $j = 0, 1, 2, 3$.*

Proof. For a $\overline{P}^+$-module N we note, for simplicity $H^i(N) = H^i(\overline{P}^+, N)$. Consider the long exact sequence on cohomology associated to the sequence of Equation 31

$$\begin{aligned} 0 &\to H^0(M_+) \to H^0(M) \to H^0(M_-) \to H^1(M_+) \to H^1(M) \to H^1(M_-) \\ &\to H^2(M_+) \to H^2(M) \to H^2(M_-) \to H^3(M_+) \to H^3(M) \to H^3(M_-) \to 0. \end{aligned}$$

Recall that we have already calculated (cf. Lemmas 13.25, 13.26, 13.27, 13.28, 13.41, , 13.42, 13.43, 13.44)all of the $H^j(M_\pm)$.

If $\delta_1\delta_2^{-1} \notin \{x^{-i}, i \in \mathbf{N}\}$, then $H^j(M_+)$ for all j, which implies that $H^j(M) = H^j(M_-)$.

If $\delta_1\delta_2^{-1} = x^{-i}$, $i \in \mathbf{N}$, then $H^0(M_-) = H^3(M_-) = 0$ and the map $H^1(M) \to H^1(M_-)$ is the zero map (cf. [11, Corollaire 5.23(ii)], which is independent of [11, Lemme 5.21]). We hence have $H^1(M) = H^1(M_+)$, and an exact sequence

$$0 \to H^1(M_-) \to H^2(M_+) \to H^2(M) \to H^2(M_-) \to H^3(M_+) \to H^3(M) \to 0.$$

By the same arguments as the ones in the proof of [11, Théorème 5.16], we can show that in this case the map $H^3(M_+) \to H^3(M)$ is the zero map, hence we deduce $H^3(M) = 0$ and $dim_L H^2(M) = 2$, which completes the proof. $\square$

14 A relative cohomology isomorphism

Definition 14.1. *We call a character $\delta : \mathbf{Q}_p^\times \to A^\times$ regular if pointwise (meaning the reduction for each maximal ideal $\mathfrak{m} \subset A$) it is never of the form χx^i or x^{-i} for some $i \geq 0$.*

Remark 14.2. By [4, Corollaire 2.11], if δ is regular then $H^2(A^+, \mathscr{R}_A(\delta)) = 0$. Moreover in the setting of a point (A is a finite extension of $\mathbf{Q}_p$), $\delta_1\delta_2^{-1} : \mathbf{Q}_p^\times \to L^\times$ is regular implies the pair (δ_1, δ_2) is generic in the sense of [11].

The following is a relative version of [11, Proposition 5.18].

Proposition 14.3. *Suppose A is reduced. Let δ_1, $\delta_2 : \mathbf{Q}_p^\times \to A^\times$ such that $\delta_1\delta_2^{-1}$ is regular. Then the restriction morphism from $\overline{P}^+$ to A^+, induces a surjection:*

$$H^1(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2)) \to H^1(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1})).$$

Proof. We work at the derived level. For the sake of brevity let $C^\bullet_{\overline{P}^+}$ denote the Koszul complex $\mathscr{C}_{\tau,\varphi,\gamma}$ of §12.2. Similarly let $C^\bullet_{A^+}$ denote the complex $\mathscr{C}_{\varphi,\gamma}$. We have a canonical morphism:

$$C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2)) \to C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1}))$$

in $\mathcal{D}^-_{\mathrm{pc}}(A)$. Let

$$C^\bullet := \mathrm{Cone}(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2)) \to C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1})))$$

and note that $C^\bullet \in \mathcal{D}^-_{\mathrm{pc}}(A)$ by Remark 12.5. The distinguished triangle

$$C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2)) \to C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1})) \to C^\bullet$$

induces a long exact sequence in cohomology

$$\cdots \to H^1(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2))) \to H^1(C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1}))) \to H^1(C^\bullet) \to \cdots$$

Moreover since $\mathscr{R}_A(\delta_1, \delta_2)$ is a flat A-module and $\mathscr{R}_A(\delta_1, \delta_2) \otimes_A A/\mathfrak{m} \cong \mathscr{R}_{A/\mathfrak{m}}(\delta_1, \delta_2)$ for any maximal ideal $\mathfrak{m} \subset A$, it follows that

$$C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2)) \otimes^{\mathbf{L}} A/\mathfrak{m} \cong C^\bullet_{\overline{P}^+}(\mathscr{R}_{A/\mathfrak{m}}(\delta_1, \delta_2)).$$

Similarly we have

$$C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1})) \otimes^{\mathbf{L}} A/\mathfrak{m} \cong C^\bullet_{A^+}(\mathscr{R}_{A/\mathfrak{m}}(\delta_1\delta_2^{-1})).$$

Hence the morphism $A \to A/\mathfrak{m}$ induces a morphism of distinguished triangles which by the functoriality of the truncation operators gives a morphism of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2))) \arrow[r] \arrow[d] & H^1(C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1}))) \arrow[r, "\gamma"] \arrow[d] & H^1(C^\bullet) \arrow[r, "\gamma_1"] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^1(C^\bullet_{\overline{P}^+}(\mathscr{R}_{A/\mathfrak{m}}(\delta_1, \delta_2))) \arrow[r, "\alpha"] & H^1(C^\bullet_{A^+}(\mathscr{R}_{A/\mathfrak{m}}(\delta_1\delta_2^{-1}))) \arrow[r, "\beta"] & H^1(C^\bullet \otimes^{\mathbf{L}} A/\mathfrak{m}) \arrow[r, "\beta_1"] & \cdots
\end{tikzcd}$$

By [11, Proposition 5.18] (see also Proposition 13.1), α is an isomorphism and so β is the zero morphism. We claim that γ is the zero morphism as well. To do this, we take advantage of the spectral sequence:

$$\mathrm{Tor}_{-p}(H^q(C^\bullet), A/\mathfrak{m}) \implies H^{p+q}(C^\bullet \otimes^{\mathbf{L}} A/\mathfrak{m}),$$

whose 2nd page takes the form

$$
\begin{tikzcd}
0 \arrow[drr] & 0 & 0 \\
\mathrm{Tor}_2(H^2(C^\bullet), A/\mathfrak{m}) \arrow[drr] & \mathrm{Tor}_1(H^2(C^\bullet), A/\mathfrak{m}) & H^2(C^\bullet)\otimes_A A/\mathfrak{m} \\
\mathrm{Tor}_2(H^1(C^\bullet), A/\mathfrak{m}) \arrow[drr] & \mathrm{Tor}_1(H^1(C^\bullet), A/\mathfrak{m}) & H^1(C^\bullet)\otimes_A A/\mathfrak{m} \\
\mathrm{Tor}_2(H^0(C^\bullet), A/\mathfrak{m}) & \mathrm{Tor}_1(H^0(C^\bullet), A/\mathfrak{m}) & H^0(C^\bullet)\otimes_A A/\mathfrak{m}
\end{tikzcd}
$$

The long exact sequence in cohomology

$$\cdots \to H^3(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1,\delta_2))) \to H^3(C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1}))) = 0 \to H^3(C^\bullet) \to 0 \to \cdots$$

implies that $H^3(C^\bullet) = 0$ hence explaining the top row. Moreover since $\delta_1\delta_2^{-1}$ is regular, $H^2(C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1}))) = 0$ by [4, Corollaire 2.11] (see also Remark 11.23). By Proposition 13.1, $H^3(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1,\delta_2))) = 0$ and thus from the long exact sequence

$$\cdots \to 0 \to H^2(C^\bullet) \to H^3(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1,\delta_2))) = 0 \to 0 \to \cdots$$

we deduce that $H^2(C^\bullet) = 0$. Hence the spectral sequence degenerates at the 2nd page in degree 1 cohomology and so $H^1(C^\bullet \otimes^{\mathbf{L}} A/\mathfrak{m}) = H^1(C^\bullet)\otimes_A A/\mathfrak{m}$. Similarly the spectral sequence

$$\mathrm{Tor}_{-p}(H^q(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1\delta_2^{-1}))), A/\mathfrak{m}) \implies H^{p+q}(C^\bullet_{\overline{P}^+}(\mathscr{R}_{A/\mathfrak{m}}(\delta_1\delta_2^{-1})))$$

implies $H^2(C^\bullet_{\overline{P}^+}(\mathscr{R}_{A/\mathfrak{m}}(\delta_1\delta_2^{-1}))) = H^2(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1\delta_2^{-1})))\otimes_A A/\mathfrak{m}$. Thus in the diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & H^1(C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1}))) \arrow[r, "\gamma"] \arrow[d] & H^1(C^\bullet) \arrow[r, "\gamma_1"] \arrow[d] & H^2(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1,\delta_2))) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & H^1(C^\bullet_{A^+}(\mathscr{R}_{A/\mathfrak{m}}(\delta_1\delta_2^{-1}))) \arrow[r, "\beta"] & H^1(C^\bullet \otimes^{\mathbf{L}} A/\mathfrak{m}) \arrow[r, "\beta_1"] & H^2(C^\bullet_{\overline{P}^+}(\mathscr{R}_{A/\mathfrak{m}}(\delta_1,\delta_2))) \arrow[r] & 0
\end{tikzcd}
$$

we have $\beta_1 = \gamma_1 \otimes A/\mathfrak{m}$. By Proposition 13.1, β_1 is an isomorphism of dimension 1 vector spaces over $A/\mathfrak{m}$ for every $\mathfrak{m}\subset A$. Thus γ_1 is a surjective morphism of locally free A-modules, cf. [37, Lemma 2.1.8(1)], which are locally of dimension 1. Hence it is an isomorphism and so γ is the zero morphism, as desired. This completes the proof. □

Proposition 14.4. *Suppose A is reduced. Let δ_1, $\delta_2 : \mathbf{Q}_p^\times \to A^\times$ such that $\delta_1\delta_2^{-1}$ is regular. Then the restriction morphism from $\overline{P}^+$ to A^+, induces an injection:*

$$H^1(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2)) \to H^1(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1})).$$

Proof. Keeping the notation used in the proof of Proposition 14.3, since $H^1(C^\bullet)$ is locally free, $\mathrm{Tor}_1(H^1(C^\bullet), A/\mathfrak{m}) = 0$. But the spectral sequence

$$\mathrm{Tor}_{-p}(H^q(C^\bullet), A/\mathfrak{m}) \implies H^{p+q}(C^\bullet \otimes^{\mathbf{L}} A/\mathfrak{m})$$

abuts to 0 in degree 0 as α is an isomorphism. This implies that $H^0(C^\bullet) \otimes A/\mathfrak{m} = 0$ for every maximal ideal $m \subset A$. By Nakayama's Lemma it follows that $H^0(C^\bullet)$ is 0 and we deduce the result. □

Theorem 14.5. *Suppose A is reduced. Let δ_1, $\delta_2 : \mathbf{Q}_p^\times \to A^\times$ such that $\delta_1\delta_2^{-1}$ is regular. Then the restriction morphism from $\overline{P}^+$ to A^+, induces an isomorphism:*

$$H^1(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2)) \to H^1(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1})).$$

Proof. This is a consequence of Propositions 14.3 and 14.4. □

We are now ready to handle the case when A is non-reduced. We begin by proving a slightly enhanced version of Theorem 14.5.

Proposition 14.6. *Suppose A is reduced and M a finite A-module (equipped with trivial $\overline{P}^+$-action). Let δ_1, $\delta_2 : \mathbf{Q}_p^\times \to A^\times$ such that $\delta_1\delta_2^{-1}$ is regular. Then the restriction morphism from $\overline{P}^+$ to A^+, induces an isomorphism:*

$$H^1(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2) \otimes_A M) \to H^1(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1}) \otimes_A M).$$

Proof. To prove surjectivity, we follow the proof of Proposition 14.3. In fact the only thing that needs to be checked is that

$$H^1(\overline{P}^+, \mathscr{R}_{A/\mathfrak{m}}(\delta_1, \delta_2) \otimes_A M) \to H^1(A^+, \mathscr{R}_{A/\mathfrak{m}}(\delta_1\delta_2^{-1}) \otimes_A M)$$

is an isomorphism. Denote by $M' := M \otimes_A A/\mathfrak{m}$, so that the above is equivalent to showing

$$H^1(\overline{P}^+, \mathscr{R}_{A/\mathfrak{m}}(\delta_1, \delta_2) \otimes_{A/\mathfrak{m}} M') \to H^1(A^+, \mathscr{R}_{A/\mathfrak{m}}(\delta_1\delta_2^{-1}) \otimes_{A/\mathfrak{m}} M'), \tag{32}$$

is an isomorphism. Now M' is flat over $A/\mathfrak{m}$ and so by the Tor-spectral sequence

$$H^1(\overline{P}^+, \mathscr{R}_{A/\mathfrak{m}}(\delta_1, \delta_2) \otimes_{A/\mathfrak{m}} M') \cong H^1(\overline{P}^+, \mathscr{R}_{A/\mathfrak{m}}(\delta_1, \delta_2)) \otimes_{A/\mathfrak{m}} M'$$

and similarly

$$H^1(A^+, \mathscr{R}_{A/\mathfrak{m}}(\delta_1, \delta_2) \otimes_{A/\mathfrak{m}} M') \cong H^1(A^+, \mathscr{R}_{A/\mathfrak{m}}(\delta_1, \delta_2)) \otimes_{A/\mathfrak{m}} M'.$$

Thus the morphism 32 is an isomorphism by [11, Proposition 5.18]. The rest of the proof of Proposition 14.3 goes through with $C^\bullet$ replaced by $C^\bullet \otimes^{\mathbf{L}} M$. For injectivity the proof of Proposition 14.4 remains unchanged except with $C^\bullet$ replaced by $C^\bullet \otimes^{\mathbf{L}} M$. This completes the proof. □

We now need a lemma that guarantees the connection morphisms are 0 in a certain long exact sequence.

Lemma 14.7. *Let A be an $\mathbf{Q}_p$-affinoid algebra and $I \subset A$ an ideal of A. Let $\delta_1,\ \delta_2 : \mathbf{Q}_p^\times \to A^\times$ such that $\delta_1\delta_2^{-1}$ is regular. The short exact sequence*

$$0 \to I \to A \to A/I \to 0$$

induces an injective morphism

$$H^2(\overline{P}^+, \mathscr{R}_A(\delta_1,\delta_2) \otimes_A I) \to H^2(\overline{P}^+, \mathscr{R}_A(\delta_1,\delta_2)).$$

Proof. By Lemma 13.3, it suffices to show that the morphism

$$H^2_{\mathrm{Lie}}(\overline{P}^+, \mathscr{R}_A(\delta_1,\delta_2) \otimes_A I) \to H^2_{\mathrm{Lie}}(\overline{P}^+, \mathscr{R}_A(\delta_1,\delta_2))$$

coming from the long exact sequence in Lie algebra cohomology of $\overline{P}^+$ is injective. Recall the Lie algebra complex from §13.1:

$$\mathscr{C}_{u^-,\varphi,a^+} : 0 \to M \xrightarrow{A} M \oplus M \oplus M \xrightarrow{B} M \oplus M \oplus M \xrightarrow{C} M \to 0$$

where

$$\begin{aligned} A(x) &= ((\varphi-1)x, a^+x, u^-x) \\ B(x,y,z) &= (a^+x - (\varphi-1)y, u^-y - (a^+ + 1)z, (p\varphi-1)z - u^-x) \\ C(x,y,z) &= u^-x + (p\varphi-1)y + (a^+ + 1)z. \end{aligned}$$

It suffices to show that if $(x,y,z) \in \mathscr{R}_A^{\oplus 3}$ such that $B(x,y,z) \in \mathscr{R}_A^{\oplus 3} \otimes_A I$, then $(x,y,z) \in \mathscr{R}_A^{\oplus 3} \otimes_A I$. Since

$$H^2_{\mathrm{Lie}}(A^+, \mathscr{R}_A(\delta_1,\delta_2) \otimes_A I) \to H^2_{\mathrm{Lie}}(A^+, \mathscr{R}_A(\delta_1,\delta_2))$$

is injective (this follows from the fact that $H^2_{\mathrm{Lie}}(A^+, \mathscr{R}_L(\delta_1,\delta_2)) = 0$ for every finite extension L of $\mathbf{Q}_p$), it follows that $(x,y) \in \mathscr{R}_A^{\oplus 2} \otimes_A I$. Thus the problem is the following: $f \in \mathscr{R}_A$ such that

$$(a^+ + 1)f \in \mathscr{R}_A \otimes_A I \text{ and } (p\varphi - 1)f \in \mathscr{R}_A \otimes_A I$$

and one must show that $f \in \mathscr{R}_A \otimes_A I$.

Call $F = p\delta(p)\varphi - 1$ and denote by $\mathscr{R}_I := \mathscr{R}_A \otimes_A I$. We will show that, if $f \in \mathscr{R}_A$ is such that $F(f) \in \mathscr{R}_I$, then $f \in \mathscr{R}_I$. We first observe that this statement is true for $\mathscr{R}_A^{\pm}$. Indeed:

- If $\phi \in \mathrm{LA}(\mathbf{Z}_p, A)$ is such that $F(\phi) \in \mathrm{LA}(\mathbf{Z}_p, I)$, then call $\overline{\phi} \in \mathrm{LA}(\mathbf{Z}_p, A/I)$ the reduction of ϕ modulo I. We have then $F(\overline{\phi}) = 0$, but F is injective on $\mathrm{LA}(\mathbf{Z}_p, A/I)$, so $\overline{\phi} = 0$, which translates into $\phi \in \mathrm{LA}(\mathbf{Z}_p, I)$.
- The claim that, for $\mu \in \mathscr{R}_A^+$, then $F(\mu) \in \mathscr{R}_I^+$ implies $\mu \in \mathscr{R}_I^+$ follows from a direct calculation by looking at the coefficients of the power series expression of μ.

Consider now the following commutative diagram given by the residue map $f = \sum_{n\in\mathbf{Z}} a_n T^n \mapsto [x \mapsto \phi_f(x) = \mathrm{res}_0((1+T)^{-x} f(T)\frac{dT}{1+T}) = \sum_{n\geq 0} a_{-n-1}\binom{-x-1}{n}]$:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{R}_A^+ \arrow[r] \arrow[d, "F"] & \mathscr{R}_A \arrow[r] \arrow[d, "F"] & \mathrm{LA}(\mathbf{Z}_p, A) \arrow[r] \arrow[d, "F"] & 0 \\ 0 \arrow[r] & \mathscr{R}_A^+ \arrow[r] & \mathscr{R}_A \arrow[r] & \mathrm{LA}(\mathbf{Z}_p, A) \arrow[r] & 0 \end{tikzcd}$$

Let $f = \sum_{n\in\mathbf{Z}} a_n T^n \in \mathscr{R}_A$ be such that $F(f) \in \mathscr{R}_I$. Then we have $F(\phi_f) \in \mathrm{LA}(\mathbf{Z}_p, I)$ and this implies by the first point of the last paragraph that $\phi_f \in \mathrm{LA}(\mathbf{Z}_p, I)$ and hence $a_n \in I$ for all $n < 0$. Hence, if we write $f = f^- + f^+$, where $f^- = \sum_{n<0} a_n T^n$ and $f^+ = \sum_{n\geq 0} a_n T^n$, we get that $F(f^-) \in \mathscr{R}_I$ (because $f^- \in \mathscr{R}_I$!) and hence $F(f^+) = F(f) - F(f^-) \in \mathscr{R}_I$, which implies that $f^+ \in \mathscr{R}_I$ and allows us to conclude that $f \in \mathscr{R}_I$. This completes the proof.

□

Remark 14.8. The statement of Lemma 14.7 is not surprising as $\overline{P}^+$ acts trivially on the coefficient algebra A in $\mathscr{R}_A(\delta_1, \delta_2)$.

Theorem 14.9. *Let δ_1, $\delta_2 : \mathbf{Q}_p^\times \to A^\times$ such that $\delta_1\delta_2^{-1}$ is regular. Then the restriction morphism from $\overline{P}^+$ to A^+, induces an isomorphism:*

$$H^1(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2)) \to H^1(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1})).$$

Proof. Since A is in particular noetherian its nilradical is nilpotent. Thus it is natural to proceed via induction on the index of nilpotence $i \geq 0$. The base case $i = 0$ (meaning A is reduced) is Theorem 14.5. Suppose by induction the result is true for index i and suppose now $N^{i+1} = 0$. For the sake of brevity denote $X_{N^i} := \mathscr{R}_A(\delta_1, \delta_2) \otimes_A N^i$, $X_A := \mathscr{R}_A(\delta_1, \delta_2)$ and $X_{A/N^i} := \mathscr{R}_A(\delta_1, \delta_2) \otimes_A A/N^i$. The short exact sequence

$$0 \to X_{N^i} \to X_A \to X_{A/N^i} \to 0$$

gives a commutative diagram

$$\begin{tikzcd} H^0(\overline{P}^+, X_{A/N^i}) \arrow[r, "\beta'"] \arrow[d] & H^1(\overline{P}^+, X_{N^i}) \arrow[r] \arrow[d, "\alpha_1"] & H^1(\overline{P}^+, X_A) \arrow[r] \arrow[d, "\alpha_2"] & H^1(\overline{P}^+, X_{A/N^i}) \arrow[r, "\rho"] \arrow[d, "\alpha_3"] & H^2(\overline{P}^+, X_{N^i}) \\ H^0(A^+, X_{A/N^i}) \arrow[r, "\beta"] & H^1(A^+, X_{N^i}) \arrow[r] & H^1(A^+, X_A) \arrow[r] & H^1(A^+, X_{A/N^i}) \arrow[r] & 0. \end{tikzcd}$$

The two rows come from long exact sequences in cohomology and commutativity comes from functoriality of the restriction morphism $H^i(\overline{P}^+, -) \to H^i(A^+, -)$. By Lemma 14.7, the connecting morphism ρ is the zero morphism. Identifying X_{A/N^i} with $\mathscr{R}_{A/N^i}(\delta_1, \delta_2)$ and X_{N^i} with $\mathscr{R}_{A/N}(\delta_1, \delta_2) \otimes_{A/N} N^i$; we see that

α_1 is an isomorphism by Proposition 14.6 and α_3 is an isomorphism by the inductive step. On the other hand by [4, Proposition 2.10] the morphism

$$H^0(A^+, X_A) \to H^0(A^+, X_{A/N^i})$$

is surjective. Thus β is the zero morphism. By the commutativity of the first square, this implies that β' is the zero morphism. Thus by the 5-Lemma, α_2 is an isomorphism and this proves the result. $\square$

15 Construction of the correspondence

In this section we construct, following [11, Chapter 6], the correspondence $\Delta \mapsto \Pi(\Delta)$ for a regular (φ, Γ)-module Δ over the relative Robba ring $\mathscr{R}_A$, interpolating the analogous construction of loc. cit. at the level of points. The construction is inspired from the calculation of the locally analytic vectors in the unitary principal series case (corresponding to the case when Δ is trianguline and étale), cf. [7], [8], [10] and involves a detailed study of the Jordan-Hölder components of the G-module $\Delta \boxtimes_\omega \mathbf{P}^1$. We will see that the sought-after representation $\Pi(\Delta)$ is cut out from these constituents.

There are a couple of differences between the approach taken in [11] and the one taken in the present paper that merits to be pointed out. On the one hand, in constrast to loc. cit. where the construction is carried out more generally for the case of a Lubin-Tate (φ, Γ)-module, we work in the usual cyclotomic context, which simplifies many of the proofs and steps. This is mainly due to the rich structure of the $\mathscr{R}_A(\Gamma)$-module $\Delta^{\psi=0}$ for $\Delta \in \Phi\Gamma(\mathscr{R}_A)$. Secondly, we are only able to carry out the construction for *regular* trianguline (φ, Γ)-modules (cf. Definition 15.1). Indeed it is only for those objects that our cohomology comparison theorem of the last chapter works. The last modification between our approach and the one taken in loc. cit. is found in the argument showing that the middle extension of $B_A(\delta_1, \delta_2)^* \otimes \omega$ by $B_A(\delta_1, \delta_2)$ in the proof of Theorem 15.2 splits, where we give a more direct method.

15.1 The main result

We begin with a definition.

Definition 15.1. *Let Δ be a trianguline (φ, Γ)-module over $\mathscr{R}_A$, which is an extension of $\mathscr{R}_A(\delta_2)$ by $\mathscr{R}_A(\delta_1)$. We say that Δ is regular if $\delta_1\delta_2^{-1} \colon \mathbf{Q}_p^\times \to A^\times$ is regular in the sense of Definition 14.1.*

The following theorem is a relative version of [11, Theorem 6.11] in the case when the pair (δ_1, δ_2) is generic and also pointwise not of the form x^{-i} for some $i \geq 0$, cf. Remark 14.2.

Theorem 15.2. *Suppose Δ is a regular (φ, Γ)-module over $\mathscr{R}_A$ such that*

$$0 \to \mathscr{R}_A(\delta_1) \to \Delta \to \mathscr{R}_A(\delta_2) \to 0.$$

Then there exists a locally analytic A-representation $\Pi(\Delta)$ [63] *of* $\mathrm{GL}_2(\mathbf{Q}_p)$, *with central character* ω, *such that we have an exact sequence*

$$0 \to \Pi(\Delta)^* \otimes \omega \to \Delta \boxtimes_\omega \mathbf{P}^1 \to \Pi(\Delta) \to 0.$$

Moreover $\Pi(\Delta)$ *is an extension of* $B_A(\delta_2, \delta_1)$ *by* $B_A(\delta_1, \delta_2)$. *Furthermore if* Δ *is a non-trivial extension of* $\mathscr{R}_A(\delta_2)$ *by* $\mathscr{R}_A(\delta_1)$ *then* $\Pi(\Delta)$ *is a non-trivial extension of* $B_A(\delta_2, \delta_1)$ *by* $B_A(\delta_1, \delta_2)$.

Remark 15.3. Contrary to [11, Theorem 6.11], unless A is a finite extension of $\mathbf{Q}_p$, then $\Pi(\Delta)$ will not be of compact type (this is because A is not of compact type as a locally convex $\mathbf{Q}_p$-vector space). Thus $\Pi(\Delta)$ is almost never an admissible G-representation in the sense of [54]. It is however of A-LB-type, cf. Definition A.13.

Before we begin to prove Theorem 15.2 we need to define and construct the G-module $\Delta \boxtimes_\omega \mathbf{P}^1$.

15.2 Notations

We let $\mathrm{Ext}^1(\mathscr{R}_A(\delta_2), \mathscr{R}_A(\delta_1))$ denote the group of extensions of $\mathscr{R}_A(\delta_2)$ by $\mathscr{R}_A(\delta_1)$ in the category of (φ, Γ)-modules over $\mathscr{R}_A$. Note that, since every (φ, Γ)-module over $\mathscr{R}_A$ is analytic (cf. Remark 10.10), this last group coincides with the extension group $\mathrm{Ext}^1_{\mathrm{an}}(\mathscr{R}_A(\delta_2), \mathscr{R}_A(\delta_1))$ in the category of analytic (φ, Γ)-modules over $\mathscr{R}_A$ [64].

Let H be a finite dimensional locally $\mathbf{Q}_p$-analytic group. We refer the reader to the appendix for the necessary definitions and properties of the theory of locally analytic H-representations in A-modules. We just recall that $\mathscr{G}_{H,A}$ (cf. Definition A.39) denotes the category of complete Hausdorff locally convex A-modules equipped with a separately continuous A-linear $\mathscr{D}(H, A)$-module structure and we let $\mathrm{Ext}^1_H(M, N)$ [65] denote the group of extensions of M by N in the category $\mathscr{G}_{H,A}$.

Example 15.4. From Lemma 10.19 and Proposition 10.21, it follows that, if $? \in \{+, -, \emptyset\}$, then the spaces $\mathscr{R}^?_A(\delta_i) \boxtimes_\omega \mathbf{P}^1$ are objects of the category $\mathscr{G}_{G,A}$.

If H_2 is a closed locally $\mathbf{Q}_p$-analytic subgroup of a locally $\mathbf{Q}_p$-analytic group H_1, we have an induction functor $\mathrm{ind}_{H_1}^{H_2} \colon \mathscr{G}_{H_2,A} \to \mathscr{G}_{H_1,A}$, cf. Lemma A.52. We cite the following fact from the appendix that will be of much use to us, cf. Proposition A.54.

63. It is probably unique but we are unable to show this. This comes down to knowing that the Ext^1 of certain principal series is a free A-module of rank 1.

64. This fact can also be seen by using the bijections $\mathrm{Ext}^1(\mathscr{R}_A(\delta_2), \mathscr{R}_A(\delta_1)) = H^1(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1})) = H^1_{\mathrm{an}}(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1})) = \mathrm{Ext}^1_{\mathrm{an}}(\mathscr{R}_A(\delta_2), \mathscr{R}_A(\delta_1))$ where the equalities follow from [4, Lemme 2.2], Proposition 11.4 and Proposition 11.2, respectively.

65. Note that this is called $\mathrm{Ext}^1_{\mathscr{G}_{H,A}}(M, N)$ in the appendix. We warn the reader that $\mathscr{G}_{H,A}$ is not an abelian category and so one needs to define precisely what the group of extensions means, cf. Definition A.47.

Proposition 15.5 (Relative Shapiro's Lemma). *Let H_1 be a locally $\mathbf{Q}_p$-analytic group and let H_2 be a closed locally $\mathbf{Q}_p$-analytic subgroup. If M and N are objects of $\mathscr{G}_{H_2,A}$ and $\mathscr{G}_{H_1,A}$, respectively, then there are A-linear bijections*

$$\mathrm{Ext}^q_{H_1}(\mathrm{ind}_{H_2}^{H_1}(M), N) \to \mathrm{Ext}^q_{H_2}(M, N)$$

for all $q \geq 0$.

15.3 Extensions of $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$

Denote by $\overline{P} = \begin{pmatrix} \mathbf{Q}_p^\times & 0 \\ \mathbf{Q}_p & 1 \end{pmatrix}$ the lower-half mirabolic subgroup of the lower-half Borel $\overline{B} = \begin{pmatrix} \mathbf{Q}_p^\times & 0 \\ \mathbf{Q}_p & \mathbf{Q}_p^\times \end{pmatrix}$ and $\overline{U}^1 = \begin{pmatrix} 1 & 0 \\ p\mathbf{Z}_p & 1 \end{pmatrix}$. We are now ready to state the first result toward a proof of Theorem 15.2, which is essentially a formal consequence of Theorem 14.9.

Theorem 15.6. *Let δ_1, $\delta_2 \colon \mathbf{Q}_p^\times \to A^\times$ such that $\delta_1\delta_2^{-1}$ is regular. Then there is a natural isomorphism*

$$\mathrm{Ext}^1_G(\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1, \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1) \cong \mathrm{Ext}^1(\mathscr{R}_A(\delta_2), \mathscr{R}_A(\delta_1)).$$

Proof. Denote by $\mathscr{R}_A(\delta_1, \delta_2) \boxtimes \mathbf{P}^1$ the $\overline{P}$-module[66] $(\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1) \otimes \delta_2^{-1}$, so that $\mathscr{R}_A(\delta_1, \delta_2)$ is identified with the sub-$\overline{P}^+$-module $(\mathscr{R}(\delta_1) \boxtimes_\omega \mathbf{Z}_p) \otimes \delta_2^{-1}$ of $\mathscr{R}_A(\delta_1, \delta_2) \boxtimes \mathbf{P}^1$. The proof is done in several steps and follows the proof of [11, Théorème 6.1] in the case where A is a finite extension of $\mathbf{Q}_p$.

(Step 1) We first descend from G to $\overline{P}$ using Shapiro's Lemma. Since $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1 \cong \mathrm{Ind}_{\overline{B}}^G(\delta_1\chi^{-1} \otimes \delta_2)^* \otimes \omega$, cf. Lemma 10.19, using Lemma A.55 we get that

$$\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1 \cong \mathrm{Ind}_{\overline{B}}^G(\delta_2^{-1} \otimes \delta_1^{-1}\chi)^* \cong \mathrm{ind}_{\overline{B}}^G(\delta_2 \otimes \chi^{-1}\delta_1).$$

So by Proposition 15.5 (for $q = 1$) we get

$$\mathrm{Ext}^1_G(\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1, \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1) \cong \mathrm{Ext}^1_{\overline{B}}(\delta_2 \otimes \chi^{-1}\delta_1, \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1).$$

Since we are only interested in locally analytic representations with a central character ω, we don't loose any information by passing from $\overline{B}$ to $\overline{P}$ (since both $\delta_2 \otimes \chi^{-1}\delta_1$ and $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$ have the same central character, namely ω) and thus we have

$$\mathrm{Ext}^1_{\overline{B}}(\delta_2 \otimes \chi^{-1}\delta_1, \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1) \cong \mathrm{Ext}^1_{\overline{P}}(\delta_2 \otimes \chi^{-1}\delta_1, \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1).$$

Then, since (as $\overline{P}$-modules) $\mathscr{R}_A(\delta_1, \delta_2) \boxtimes \mathbf{P}^1 \cong (\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1) \otimes (\delta_2^{-1} \otimes \chi\delta_1^{-1})$, the RHS in the above equality is also equal to

$$H^1_{\mathrm{an}}(\overline{P}, \mathscr{R}_A(\delta_1, \delta_2) \boxtimes \mathbf{P}^1).$$

66. Here δ_2 is seen as a character of $\overline{P}$, by setting $\delta_2 \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} = \delta_2(a)$.

(Step 2) We now descend from $\overline{P}$ to $\overline{P}^+$. That is the restriction of $\mathscr{R}_A(\delta_1,\delta_2)\boxtimes \mathbf{P}^1$ to a $\overline{P}^+$-module induces an isomorphism

$$H^1_{\mathrm{an}}(\overline{P},\mathscr{R}_A(\delta_1,\delta_2)\boxtimes \mathbf{P}^1) = H^1_{\mathrm{an}}(\overline{P}^+,\mathscr{R}_A(\delta_1,\delta_2)\boxtimes \mathbf{P}^1).$$

This is shown in the exact same way as in [11, Lemme 6.4].

(Step 3) Finally we descend from $\mathscr{R}_A(\delta_1,\delta_2)\boxtimes \mathbf{P}^1$ to $\mathscr{R}_A(\delta_1,\delta_2)$. More precisely we show that the inclusion $\mathscr{R}_A(\delta_1,\delta_2)\subset \mathscr{R}_A(\delta_1,\delta_2)\boxtimes \mathbf{P}^1$ (as $\overline{P}^+$-modules) induces an isomorphism

$$H^1_{\mathrm{an}}(\overline{P}^+,\mathscr{R}_A(\delta_1,\delta_2)) \cong H^1_{\mathrm{an}}(\overline{P}^+,\mathscr{R}_A(\delta_1,\delta_2)\boxtimes \mathbf{P}^1).$$

Indeed by the long exact sequence in cohomology associated to the short exact sequence

$$0\to \mathscr{R}_A(\delta_1,\delta_2)\to \mathscr{R}_A(\delta_1,\delta_2)\boxtimes \mathbf{P}^1 \to Q\to 0, \tag{33}$$

where we define $Q := (\mathscr{R}_A(\delta_1,\delta_2)\boxtimes \mathbf{P}^1)/\mathscr{R}_A(\delta_1,\delta_2)\boxtimes \mathbf{Z}_p)$ as a $\overline{P}^+$-module, it suffices to show that $H^0_{\mathrm{an}}(\overline{P}^+,Q) = H^1_{\mathrm{an}}(\overline{P}^+,Q)=0$.

First observe that $Q=\mathscr{R}_A(\delta_1,\delta_2)\boxtimes(\mathbf{P}^1-\mathbf{Z}_p)$ as $\overline{U}^1$-modules and that $H^1_{\mathrm{an}}(\overline{U}^1,Q)=0$. Indeed, since $\overline{U}^1 = \begin{pmatrix}0&1\\p&0\end{pmatrix}U^0\begin{pmatrix}0&p^{-1}\\1&0\end{pmatrix}$, it is enough to show that $H^1_{\mathrm{an}}(U^0,\mathscr{R}_A)=0$, which follows from Lemma 11.1. For the same reason $H^0_{\mathrm{an}}(\overline{U}^1,Q)=0$ and so $H^0_{\mathrm{an}}(\overline{P}^+,Q)=0$.

Finally, let $c\mapsto c_g$ be a locally analytic 1-cocycle over $\overline{P}^+$ with values in Q. By adding a coboundary we can assume that $c_g=0$ for every $g\in\overline{U}^1$. For $a\in \mathbf{Z}_p\backslash\{0\}$, let $\alpha(a)=\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)$. By the relation $\alpha(a)\left(\begin{smallmatrix}1&0\\ap&1\end{smallmatrix}\right)=\left(\begin{smallmatrix}1&0\\p&1\end{smallmatrix}\right)\alpha(a)$ we get $c_{\alpha(a)}=\left(\begin{smallmatrix}1&0\\p&1\end{smallmatrix}\right)c_{\alpha(a)}$ and thus $c_{\alpha(a)}=0$ for every $a\in\mathbf{Z}_p\backslash\{0\}$ since $\left(\begin{smallmatrix}1&0\\p&1\end{smallmatrix}\right)-1$ is injective on Q (since $w\left(\left(\begin{smallmatrix}1&0\\p&1\end{smallmatrix}\right)-1\right)w=\left(\begin{smallmatrix}1&p\\0&1\end{smallmatrix}\right)-1$ is injective on $\mathscr{R}(\delta_1,\delta_2)\boxtimes_\omega p\mathbf{Z}_p$). Thus $c_g=0$ for all $g\in\overline{P}^+$.

Steps 1-3 show that

$$\mathrm{Ext}^1_G(\mathscr{R}_A^+(\delta_2)\boxtimes_\omega \mathbf{P}^1,\mathscr{R}_A(\delta_1)\boxtimes_\omega \mathbf{P}^1)\cong H^1_{\mathrm{an}}(\overline{P}^+,\mathscr{R}_A(\delta_1,\delta_2)).$$

The result now follows from Theorem 14.9 and Proposition 11.4.

□

15.4 The G-module $\Delta\boxtimes_\omega \mathbf{P}^1$

In this section we show, following [11, §6.3], the existence of an unique extension $\Delta\boxtimes_\omega\mathbf{P}^1$ extending that of $\mathscr{R}_A^+(\delta_2)\boxtimes_\omega$ by $\mathscr{R}_A(\delta_1)\boxtimes_\omega\mathbf{P}^1$ associated to a trianguline (φ,Γ)-module $\Delta\in\mathrm{Ext}^1(\mathscr{R}_A(\delta_2),\mathscr{R}_A(\delta_1))$ over $\mathscr{R}_A$ by Theorem 15.6. As in the introduction, we observe that working in the context of cyclotomic (φ,Γ) simplifies considerably several proofs and constructions.

We begin by a lemma permitting to extend the involution.

Proposition 15.7. *Let $\Delta, \Delta_1 \in \Phi\Gamma(\mathscr{R}_A)$ be in an exact sequence $0 \to \Delta_1 \to \Delta \xrightarrow{\alpha} \mathscr{R}_A(\delta) \to 0$, for $\delta\colon \mathbf{Q}_p^\times \to A^\times$ locally analytic, and let $\Delta_+ = \alpha^{-1}(\mathscr{R}_A^+(\delta)) \subseteq \Delta$. Let $j_+\colon \mathscr{R}_A^+(\Gamma) \to \mathscr{R}_A^+(\Gamma)$ and $j\colon \mathscr{R}_A(\Gamma) \to \mathscr{R}_A(\Gamma)$ be the involutions defined by $\sigma_a \mapsto \delta(a)\sigma_a^{-1}$. The any $\mathscr{R}_A^+(\Gamma)$-anti-linear involution $\iota\colon \Delta_+ \boxtimes \mathbf{Z}_p^\times \to \Delta_+ \boxtimes \mathbf{Z}_p^\times$ with respect to j_+*[67] *stabilizing $\Delta_1 \boxtimes \mathbf{Z}_p^\times$ extends uniquely to an $\mathscr{R}_A(\Gamma)$-anti-linear involution with respect to j on $\Delta \boxtimes \mathbf{Z}_p^\times$.*

Proof. As $\mathscr{R}_A^+(\Gamma)$-modules we have

$$\Delta_+ \boxtimes \mathbf{Z}_p^\times \cong (\Delta_1 \boxtimes \mathbf{Z}_p^\times) \oplus \mathscr{R}_A^+(\Gamma) \cdot e_2,$$

for some $e_2 \in \Delta_+ \boxtimes \mathbf{Z}_p^\times$, with $\sigma_a(e_2) = \delta(a)e_2$, $a \in \mathbf{Z}_p^\times$; and similarly, as $\mathscr{R}_A(\Gamma)$-modules we have

$$\Delta \boxtimes \mathbf{Z}_p^\times \cong (\Delta_1 \boxtimes \mathbf{Z}_p^\times) \oplus \mathscr{R}_A(\Gamma) \cdot e_2,$$

so that, in particular, the module $\Delta_+ \boxtimes \mathbf{Z}_p^\times$ contains a basis of $\Delta \boxtimes \mathbf{Z}_p^\times$ as a $\mathscr{R}_A(\Gamma)$-module. Since the involution ι' we are looking for (on $\Delta \boxtimes \mathbf{Z}_p^\times$) extends ι and is $\mathscr{R}_A(\Gamma)$-anti-linear, we are forced to set, for any $z = z_1 + \lambda e_2$, $z_1 \in \Delta_1$, $\lambda \in \mathscr{R}_A(\Gamma)$,

$$\iota'(z) = \iota(z_1) + j(\lambda)\iota(e_2).$$

Since every element of Δ can be uniquely written in this way, we deduce the result. □

Denote by $G^+ = \begin{pmatrix} \mathbf{Z}_p & \mathbf{Z}_p \\ p\mathbf{Z}_p & \mathbf{Z}_p^\times \end{pmatrix} \cap G$, $\overline{B}^+ = \begin{pmatrix} \mathbf{Z}_p \backslash \{0\} & 0 \\ p\mathbf{Z}_p & \mathbf{Z}_p^\times \end{pmatrix}$ and note that $P^+ \subset G^+$, $\overline{B}^+ \subset G^+$ and that G^+ stabilizes $\mathbf{Z}_p$ so that, if M is a G-equivariant sheaf over $\mathbf{P}^1$, then $M \boxtimes \mathbf{Z}_p$ inherits an action of G^+. The next result explicitly describes the isomorphism of Theorem 15.6 and gives the construction of the G-module $\Delta \boxtimes_\omega \mathbf{P}^1$ for a regular (φ, Γ)-module Δ over $\mathscr{R}_A$.

Proposition 15.8. *Let M be a non-trivial extension of $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$. Then:*

1. *M constains a unique G^+-submodule Δ_+ which is an extension of $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{Z}_p$ by $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{Z}_p$.*
2. *There exists a unique $\Delta \in \Phi\Gamma(\mathscr{R}_A)$ which is an extension of $\mathscr{R}_A(\delta_2)$ by $\mathscr{R}_A(\delta_1)$ such that Δ_+ is identified with the inverse image of $\mathscr{R}_A^+(\delta_2)$ in Δ.*
3. *$\Delta_+ \boxtimes \mathbf{Z}_p^\times$ is stable under w and, if we denote by ι the involution of $\Delta_+ \boxtimes \mathbf{Z}_p^\times$ induced by w, then $M = \Delta_+ \boxtimes_{\omega,\iota} \mathbf{P}^1$.*
4. *The involution ι extends uniquely to a $\mathscr{R}_A(\Gamma)$-anti-linear involution (with respect to j defined above) on $\Delta \boxtimes \mathbf{Z}_p^\times$ and $\Delta \boxtimes_{w,\iota} \mathbf{P}^1$ is a G-module which is an extension of $\mathscr{R}_A(\delta_2) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$.*

Proof. We follow the proof of [11, Proposition 6.7,].

67. i.e. satisfying $\iota \circ \lambda = j_+(\lambda) \circ \iota$

1. Note that G^+ stablizes $\mathbf{Z}_p$ and so $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{Z}_p$ is a G^+-module. Step 1 of Theorem 15.6 shows that

$$\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{Z}_p \cong \mathrm{Ind}_{B^+}^{G^+}(\delta_2^{-1} \otimes \delta_1^{-1}\chi)^* = \mathrm{ind}_{B^+}^{G^+}(\delta_2 \otimes \chi^{-1}\delta_1).$$

We have an exact sequence of G^+-modules

$$0 \to \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{Z}_p \to \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1 \to Q' \to 0,$$

where we define $Q' := (\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1)/(\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{Z}_p)$ as a G^+-module. Proposition 15.5 gives

$$\begin{aligned}\mathrm{Ext}^1_{G^+}(\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{Z}_p, Q') &= \mathrm{Ext}^1_{\overline{B}^+}(\delta_2 \otimes \chi^{-1}\delta_1, Q') \\ &\overset{(i)}{=} \mathrm{Ext}^1_{\overline{P}^+}(\delta_2 \otimes \chi^{-1}\delta_1, Q') \\ &\overset{(ii)}{=} 0\end{aligned}$$

where (i) follows from the fact that $\delta_2 \otimes \chi^{-1}\delta_1$ and $\mathscr{R}_A(\delta_1) \boxtimes_\omega (\mathbf{P}^1 - \mathbf{Z}_p)$ have the same central character ω, and (ii) follows from Step 3 of the proof of Theorem 15.6. This proves the existence of a G^+-submodule $\Delta_+ \subset M$. We now prove uniqueness. If $X \subset M$ is a G^+-extension of $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{Z}_p$ by $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{Z}_p$ and $e \in X$ is a lift of $1 \otimes \delta_2 \in \mathscr{R}^+(\delta_2)$, then $\left(\left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right) - 1\right) \cdot e \in \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{Z}_p$. Writing $e = e' + z$ for some $e' \in \Delta_+$ and $z \in Q'$, we see that $\left(\left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right) - 1\right) \cdot z \in \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{Z}_p$. Thus $z = 0$ as $\left(\left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right) - 1\right)$ is injective on Q' (noting that $Q' = \mathscr{R}_A(\delta_1) \boxtimes_\omega (\mathbf{P}^1 - \mathbf{Z}_p)$ as $\overline{U}^1$-modules).

2. This follows from the fact that extensions (as A^+-modules) of $\mathscr{R}_A^+(\delta_2)$ by $\mathscr{R}_A(\delta_1)$ are in correspondence with extensions of $\mathscr{R}_A(\delta_2)$ by $\mathscr{R}_A(\delta_1)$.

3. Since Δ_+ is a G^+-module, by restricting to P^+, we can think of it as a P^+-module living on $\mathbf{Z}_p$. Thus the notation $\Delta_+ \boxtimes \mathbf{Z}_p^\times$ makes sense.
   For $i \in \mathbf{Z}_p^\times$ the identity

$$w \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & i^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -i^{-1} & 0 \\ p & i \end{pmatrix}$$

   and the fact that $\begin{pmatrix} -i^{-1} & 0 \\ p & i \end{pmatrix} \in G^+$ implies that

$$w(\Delta_+ \boxtimes (i + p\mathbf{Z}_p)) \subset \Delta_+ \boxtimes (i^{-1} + p\mathbf{Z}_p) \tag{34}$$

   The inclusion in (34) is in fact an equality since w is an involution. This shows the stability of $\Delta_+ \boxtimes \mathbf{Z}_p^\times$ by w.
   To show the equality of the statement, it suffices to show that the sequence

$$0 \to \Delta_+ \boxtimes \mathbf{Z}_p^\times \xrightarrow{x \mapsto (x, -wx)} \Delta_+ \oplus \Delta_+ \xrightarrow{(y,z) \mapsto y + w \cdot z} M \to 0 \tag{35}$$

is exact. The exactness on the middle is clear since any section supported on $\mathbf{Z}_p$ such that it's involution is also supported on $\mathbf{Z}_p$ is necessarily supported on $\mathbf{Z}_p^\times$. Finally, to prove surjectivity of $\Delta_+ \oplus \Delta_+ \to M$ in (35), it suffices to note that induced applications $\mathscr{R}_A(\delta_1) \oplus \mathscr{R}_A(\delta_1) \to \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$ and $\mathscr{R}_A^+(\delta_2) \oplus \mathscr{R}_A^+(\delta_2) \to \mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1$ are surjective.

4. The existence and uniqueness of ι follows from Proposition 15.7. For the last part it suffices to show that the action of $\tilde{G}$ on $\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1$ factorizes via G. First note that if A is a finite extension of $\mathbf{Q}_p$ the result follows from [11, Proposition 6.7(iv)].

   We now proceed by induction on the index $i \geq 0$ of nilpotence of A. Suppose first that A is reduced. Take $(z_1, z_2) \in \Delta \boxtimes_{\omega,\iota} \mathbf{P}^1$ and g in the kernel of $\tilde{G} \to G$. It suffices to show that $y = (g-1)z = 0$. Call $y = (y_1, y_2)$. Let $\mathfrak{m} \subset A$ be a maximal ideal. By Lemma 10.18 and by the result for the case of a point, $y_i = 0 \bmod \mathfrak{m}$. If we write $y_i = \sum_{n \in \mathbf{Z}} a_{n,i} T^n \oplus \sum_{n \in \mathbf{Z}} a_{n,i} T^n$ for $i = 1, 2$ we see that $a_{n,i} = 0 \bmod \mathfrak{m}$ and hence $y_i = 0$ so that $y = 0$, as desired.

   Suppose now the result is true for every affinoid algebra of index of nilpotence $\leq j$ and let A be an affinoid algebra whose nilradical N satisfies $N^{j+1} = 0$ and g be in the kernel of $\tilde{G} \to G$. We have the following short exact sequence (note that Δ is a flat A-module because it is an extension of flat A-modules and so is $\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1$ who is topologically isomorphic to two copies of Δ)

$$0 \to (\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1) \otimes_A N^j \to \Delta \boxtimes_{\omega,\iota} \mathbf{P}^1 \to (\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1) \otimes_A A/N^j \to 0.$$

   We can identify $(\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1) \otimes_A A/N^j$ with $(\Delta \otimes_A A/N^j) \boxtimes_{\omega,\iota} \mathbf{P}^1$ and $(\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1) \otimes_A N^j$ with $(\Delta \otimes_A A/N) \boxtimes_{\omega,\iota} \mathbf{P}^1 \otimes_{A/N} N^j$. The result now follows by the inductive hypothesis and the base case.

□

From now on we denote by $\Delta \boxtimes_\omega \mathbf{P}^1$ the module $\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1$ constructed in Proposition 15.8.

15.5 The representation $\Pi(\Delta)$

We are now almost ready to construct the representation $\Pi(\Delta)$ and prove Theorem 15.2. We will need some preparation results. We start by showing that $H^1(\overline{P}^+, \mathscr{R}_A^-(\delta_1, \delta_2))$ is a free A-module in the *quasi*-regular case [68].

Proposition 15.9. *Let $\delta_1, \delta_2 \colon \mathbf{Q}_p^\times \to A^\times$ be locally analytic characters such that $\delta_1 \delta_2^{-1}$ is quasi-regular. Then there is an isomorphism*

$$H^1(\overline{P}^+, \mathscr{R}_A^-(\delta_1, \delta_2)) \cong A.$$

68. Here quasi-regular means that pointwise $\delta_1 \delta_2^{-1}$ is never of the form χx^i for some $i \geq 0$. Clearly regular implies quasi-regular.

Proof. This will follow from Proposition 11.19 and Lemma 15.11 below. $\square$

Lemma 15.10. *$\mathscr{R}_A^-$ is a flat A-module.*

Proof. For $0 < r \le s \le \infty$ rings $\mathscr{R}_A^{[r,s]}$ are Banach A-algebras of countable type. Thus by [39, Lemma 1.3.8], we can identify $\mathscr{R}_A^{[r,s]}/\mathscr{R}_A^{[r,\infty]}$ with the completed direct sum $\widehat{\oplus}_{i\in I} Ae_i$ where $(e_i)_{i\in I}$ form a potentially orthonormal basis. We note in the following $(\mathscr{R}_A^{[r,s]})^-$ the module $\mathscr{R}_A^{[r,s]}/\mathscr{R}_A^{[r,\infty]}$. Under this identification, if $I \subseteq A'$ is a finitely generated ideal of A, then

$$I \otimes_A (\mathscr{R}_A^{[r,s]})^- \cong \widehat{\oplus}_{i\in I} Ie_i.$$

This implies that the morphism $I \otimes_A (\mathscr{R}_A^{[r,s]})^- \to (\mathscr{R}_A^{[r,s]})^-$ is injective. Thus $(\mathscr{R}_A^{[r,s]})^-$ is a flat A-module. Observe that, in fact, the quotient $\mathscr{R}_A^{[r,s]}/\mathscr{R}_A^{[r,\infty]}$ does not depend on r and coincides with $\mathscr{R}_A^{]0,s]}/\mathscr{R}_A^{]0,\infty]}$ so that this last module is flat. Finally, since filtered colimits are exact, this implies that

$$\varinjlim_{s>0} \mathscr{R}_A^{]0,s]}/\mathscr{R}_A^{]0,\infty]} = \mathscr{R}_A/\mathscr{R}_A^+ = \mathscr{R}_A^-$$

is a flat A-module.

$\square$

Lemma 15.11. *Let $\delta_1, \delta_2 \colon \mathbf{Q}_p^\times \to A^\times$ such that $\delta_1\delta_2^{-1}$ is quasi-regular. The restriction morphism*

$$H^1(\overline{P}^+, \mathscr{R}_A^-(\delta_1,\delta_2)) \to H^1(A^+, \mathscr{R}_A^-(\delta))$$

is an isomorphism.

Proof. This is precisely the same proof as Theorem 14.9 with $\mathscr{R}_A^-$ replacing $\mathscr{R}_A$. The key points are the following

- The morphism
$$C^\bullet_{\overline{P}^+}(\mathscr{R}_A^-(\delta_1,\delta_2)) \to C^\bullet_{A^+}(\mathscr{R}_A^-(\delta))$$
is in $\mathcal{D}^-_{\mathrm{pc}}(A)$ by Lemmas 11.14 and 12.12 and Proposition 11.19.
- $\mathscr{R}_A^-$ is flat A-module, cf. Lemma 15.10. This means that for any maximal ideal $\mathfrak{m} \subset A$ we have
$$C^\bullet_{\overline{P}^+}(\mathscr{R}_A^-(\delta_1,\delta_2)) \otimes^{\mathbf{L}} A/\mathfrak{m} \cong C^\bullet_{\overline{P}^+}(\mathscr{R}_{A/\mathfrak{m}}^-(\delta_1,\delta_2))$$
and
$$C^\bullet_{A^+}(\mathscr{R}_A^-(\delta)) \otimes^{\mathbf{L}} A/\mathfrak{m} \cong C^\bullet_{A^+}(\mathscr{R}_{A/\mathfrak{m}}^-(\delta)).$$
- Since $\delta_1\delta_2^{-1}$ is quasi-regular, $H^2(C^\bullet_{A^+}(\mathscr{R}_A^-(\delta))) = 0$ by Corollary 11.18.
- The result is true when A is a finite extension of $\mathbf{Q}_p$, cf. [11, Lemme 5.24].

$\square$

This completes the proof of Proposition 15.9. Finally we need a lemma which identifies $\check{\Delta} \boxtimes_{\omega^{-1}} \mathbf{P}^1$ as the topological dual of $\Delta \boxtimes_\omega \mathbf{P}^1$ (equipped with the strong topology).

Lemma 15.12. *If Δ is an extension of $\mathscr{R}_A(\delta_2)$ by $\mathscr{R}_A(\delta_1)$ and if the G-module $\Delta \boxtimes_\omega \mathbf{P}^1$ exists, then its dual is $\check{\Delta} \boxtimes_{\omega^{-1}} \mathbf{P}^1$.*

Proof. This is the same proof as [11, Proposition 3.2], so we just provide a sketch. It suffices to construct a perfect pairing which identifies $\Delta \boxtimes_\omega \mathbf{P}^1$ and $\check{\Delta} \boxtimes_{\omega^{-1}} \mathbf{P}^1$ as the topological duals of one another (as topological $A[G]$-modules). To construct this pairing first note that if (Δ, ω, ι) is compatible, then so is $(\check{\Delta}, \omega^{-1}, \iota^*)$, where

$$\iota^* \colon \check{\Delta} \boxtimes \mathbf{Z}_p^\times \to \check{\Delta} \boxtimes \mathbf{Z}_p^\times$$

is the involution of $\check{\Delta} \boxtimes \mathbf{Z}_p^\times$ adjoint to that of ι with respect to the pairing $\{\ ,\ \}$ (cf. §10.2 for the definition of $\{\ ,\ \}$). To see this one first defines a pairing

$$\{\ ,\ \}_{\mathbf{P}^1} \colon (\check{\Delta} \boxtimes_{\omega^{-1}, \iota^*} \mathbf{P}^1) \times (\Delta \boxtimes_{\omega, \iota} \mathbf{P}^1) \to A$$

by the formula

$$\{\check{z}, z\}_{\mathbf{P}^1} := \{\mathrm{Res}_{\mathbf{Z}_p} \check{z}, \mathrm{Res}_{\mathbf{Z}_p} z\} + \{\mathrm{Res}_{p\mathbf{Z}_p} w \cdot \check{z}, \mathrm{Res}_{p\mathbf{Z}_p} w \cdot z\}.$$

One then proceeds to check that $\{g \cdot \check{z}, g \cdot z\}_{\mathbf{P}^1} = \{\check{z}, z\}_{\mathbf{P}^1}$ for all $g \in \tilde{G}$. Finally one notes that since $\{\ ,\ \}$ is perfect, so is $\{\ ,\ \}_{\mathbf{P}^1}$. ☐

Proof of Theorem 15.2. By Proposition 10.21, Theorem 15.6 and Proposition 15.8 we have that [69]

$$\Delta \boxtimes_\omega \mathbf{P}^1 = [B_A(\delta_2, \delta_1)^* \otimes \omega - B_A(\delta_1, \delta_2) - B_A(\delta_1, \delta_2)^* \otimes \omega - B_A(\delta_2, \delta_1)].$$

We begin by showing that the middle extension $[B_A(\delta_1, \delta_2) - B_A(\delta_1, \delta_2)^* \otimes \omega]$ is split in the category $\mathscr{G}_{G,A}$. We compute

$$\begin{aligned} \mathrm{Ext}^1_G(B(\delta_1, \delta_2)^* \otimes \omega, \mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1) &\overset{(i)}{\cong} \mathrm{Ext}^1_{\overline{B}}(\delta_2 \otimes \chi^{-1}\delta_1, \mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1) \\ &\overset{(ii)}{\cong} \mathrm{Ext}^1_{\overline{P}}(\delta_2 \otimes \chi^{-1}\delta_1, \mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1) \end{aligned}$$

where (i) follows from Proposition 15.5 and (ii) follows from the fact that both $(\delta_2 \otimes \chi^{-1}\delta_1)$ and $\mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1$ have the same central character. Then as $\overline{P}$-modules

$$(\mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1) \otimes (\delta_2^{-1} \otimes \chi\delta_1^{-1}) \cong (\mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1) \otimes \delta_2^{-1}.$$

Let us denote by $\mathscr{R}_A^-(\delta_1, \delta_2) \boxtimes \mathbf{P}^1$ the $\overline{P}$-module $(\mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1) \otimes \delta_2^{-1}$, so that we get

$$\mathrm{Ext}^1_{\overline{P}}(\delta_2 \otimes \chi^{-1}\delta_1, \mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1) = H^1_{\mathrm{an}}(\overline{P}, \mathscr{R}_A^-(\delta_1, \delta_2) \boxtimes \mathbf{P}^1).$$

69. The notation $M = [M_1 - M_2 - \ldots - M_n]$ means that M admits an increasing filtration $0 \subseteq F_1 \subseteq \ldots \subseteq F_n = M$ by sub-objects such that $M_i = F_i/F_{i-1}$ for $i = 1, \ldots, n$.

Finally, by [11, Lemme 6.4]

$$H^1_{\mathrm{an}}(\overline{P}, \mathscr{R}^-_A(\delta_1,\delta_2)\boxtimes \mathbf{P}^1) = H^1_{\mathrm{an}}(\overline{P}^+, \mathscr{R}^-_A(\delta_1,\delta_2)\boxtimes \mathbf{P}^1).$$

Putting this calculations together, we conclude that

$$\mathrm{Ext}^1_G(B(\delta_1,\delta_2)^*\otimes\omega, \mathscr{R}^-_A(\delta_1)\boxtimes_\omega \mathbf{P}^1) = H^1_{\mathrm{an}}(\overline{P}^+, \mathscr{R}^-_A(\delta_1,\delta_2)\boxtimes \mathbf{P}^1).$$

Consider the commutative diagram:

$$\begin{tikzcd}
H^1_{\mathrm{an}}(\overline{P}^+, \mathscr{R}_A(\delta_1,\delta_2)) \arrow[r, "\sim"] \arrow[d] & H^1_{\mathrm{an}}(\overline{P}^+, \mathscr{R}_A(\delta_1,\delta_2)\boxtimes \mathbf{P}^1) \arrow[d] \\
H^1_{\mathrm{an}}(\overline{P}^+, \mathscr{R}^-_A(\delta_1,\delta_2)) \arrow[r] & H^1_{\mathrm{an}}(\overline{P}^+, \mathscr{R}^-_A(\delta_1,\delta_2)\boxtimes \mathbf{P}^1),
\end{tikzcd}$$

where the top horizontal arrow is an isomorphism by Theorems 14.9 and 15.6. By reinterpreting the extensions in terms of cohomology classes, we see that showing that the middle extension $[B_A(\delta_1,\delta_2) - B_A(\delta_1,\delta_2)^*\otimes\omega]$ splits is equivalent to showing that the right vertical arrow is the zero morphism. Now, by Proposition 15.9, $H^1_{\mathrm{an}}(\overline{P}^+, \mathscr{R}^-_A(\delta_1,\delta_2))$ is a free A-module of rank 1 and the same proof as the first point in [11, Remarque 5.26] now shows that the bottom horizontal arrow

$$H^1_{\mathrm{an}}(\overline{P}^+, \mathscr{R}^-_A(\delta_1,\delta_2)) \to H^1_{\mathrm{an}}(\overline{P}^+, \mathscr{R}^-_A(\delta_1,\delta_2)\boxtimes \mathbf{P}^1)$$

is the zero morphism and so is the right vertical arrow, proving the claim.

It follows that $\Delta\boxtimes_\omega \mathbf{P}^1$ is an extension of Π_1 by $\Pi_2^*\otimes\omega$, where Π_1 and Π_2 are extensions of $B_A(\delta_2,\delta_1)$ by $B_A(\delta_1,\delta_2)$. By Lemma 15.12, it follows that $\Pi_1=\Pi_2$. We now define $\Pi(\Delta) := \Pi_2$.

Furthermore if Δ is a non-trivial extension of $\mathscr{R}_A(\delta_2)$ by $\mathscr{R}_A(\delta_1)$, then so is the extension of $B_A(\delta_1,\delta_2)^*\otimes\omega$ by $\mathscr{R}_A(\delta_1)\boxtimes_\omega \mathbf{P}^1$ (recalling that

$$\mathrm{Ext}^1_G(\mathscr{R}^+_A(\delta_2)\boxtimes_\omega \mathbf{P}^1, \mathscr{R}_A(\delta_1)\boxtimes_\omega \mathbf{P}^1) \cong \mathrm{Ext}^1(\mathscr{R}_A(\delta_2), \mathscr{R}_A(\delta_1)),$$

cf. Theorem 15.6). This implies that $\Pi(\Delta)$ is a non-trivial extension of $B_A(\delta_2,\delta_2)$ by $B_A(\delta_1,\delta_2)$ and finishes the proof.

$\square$

A The category of locally analytic G-representations in A-modules

Let A be an affinoid $\mathbf{Q}_p$-algebra (in the sense of Tate). Unless otherwise stated H will be a locally $\mathbf{Q}_p$-analytic group (for applications H will be a closed locally $\mathbf{Q}_p$-analytic subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$). We attempt to generalise some definitions from [50] in the case where the base coefficient is an $\mathbf{Q}_p$-affinoid algebra.

There are some results in this direction in [35, §3] although our approach is different. In particular, our aim is to give a reasonable definition of the category of locally analytic H-representations in A-modules, $\mathrm{Rep}_A^{\mathrm{la}}(H)$, analogous to the definition in [53, §3] (in the case where A is a finite extension of $\mathbf{Q}_p$), and to study the (locally analytic) cohomology of such a representation.

A.1 Preliminaries and definitions

In what follows if V and W are two locally convex $\mathbf{Q}_p$-vector spaces and in the situation that the bijection

$$V \otimes_{\mathbf{Q}_p,\iota} W \to V \otimes_{\mathbf{Q}_p,\pi} W$$

is a topological isomorphism, we write simply $V \otimes_{\mathbf{Q}_p} W$ and $V \widehat{\otimes}_{\mathbf{Q}_p} W$ to denote the topological tensor product and its completion, respectively. Note that H is strictly paracompact (this means that every open covering of H admits a locally finite refinement of pairwise disjoint open subsets) and so it admits a covering of pairwise disjoint open compact subsets.

We need to define a notion of a locally convex A-module. First we recall the definition of a *topological* A-module.

Definition A.1. *A topological A-module is an A-module endowed with a topology such that module addition $+\colon M \times M \to M$ and scalar multiplication $\cdot\colon A \times M \to M$ are continuous functions (where the domains of these functions are endowed with product topologies).*

Definition A.2. *Let M be an A-module. A seminorm q on M is a function $q\colon M \to \mathbf{R}$ such that*

- *$q(am) = |a|q(m)$ for all $a \in A$ and $m \in M$, where $|\cdot|$ is some non-zero multiplicative seminorm on A.*
- *$q(m+n) \leq \max\{q(m), q(n)\}$ for any $m, n \in M$.*

Let $(q_i)_{i\in I}$ be a family of seminorms on an A-module M. We define a topology on M to be the coarsest topology on M such that

1. All $q_i\colon M \to \mathbf{R}$, for $i \in I$, are continuous.
2. All translation maps $m + -\colon M \to M$, for $m \in M$, are continuous.

Remark A.3. One would at a first glance be tempted to define a locally convex A-module as a topological A-module whose underlying topology is given by a family of seminorms in the above sense. The problem with this definition is twofold. The $\mathbf{Q}_p$-affinoid algebra A equipped with the topology defined by the Gauss norm say, will not necessarily be a locally convex A-module (unless A is reduced). This is essentially due to the fact that the Gauss norm is not necessarily multiplicative on A. On the other hand the topology on A defined by the seminorms coming from the Berkovich spectrum $\mathcal{M}(A)$ coincides with the topology induced by the spectral seminorm ($f \in A \mapsto \max_{x\in\mathcal{M}(A)}|f(x)|$). Under this topology A will indeed be a locally convex A-module but not necessarily Hausdorff.

Due to Remark A.3 we define a locally convex A-module in the following way.

Definition A.4. *A locally convex A-module is a topological A-module whose underlying topology is a locally convex $\mathbf{Q}_p$-vector space. We let LCS_A be the category of locally convex A-modules. Its morphisms are all continuous A-linear maps.*

Remark A.5. Let us show that this definition is coherent in the case when $A = L$ is a finite extension of $\mathbf{Q}_p$. That is, that a locally convex $\mathbf{Q}_p$-vector space equipped with a continuous multiplication by L is also an L-locally convex vector space.

We employ the notion of [50, §4]. Let L be a finite extension of $\mathbf{Q}_p$. It is clear that any locally convex L-vector space in the sense of loc.cit. satisfies the conditions of Definition A.4. On the other hand, let M be a locally convex $\mathbf{Q}_p$-vector space (whose topology is defined by a family of lattices $\mathscr{B}$) equipped with a continuous multiplication by L. We show that we can equip M with a system of lattices $\mathscr{B}'$ satisfying conditions (lc1) and (lc2) of [50, §4] with K there replaced by L defining the same topology as $\mathscr{B}$. Let x_i be a $\mathbf{Z}_p$-basis of $\mathcal{O}_L$. For $U \in \mathscr{B}$, set $U' = \sum_i x_i U$ and denote $\mathscr{B}' = \{U' : U \in \mathscr{B}\}$. It is easy to see that this family of $\mathcal{O}_L$-lattices satisfies conditions (lc1) and (lc2). We show that the topology defined by $\mathscr{B}'$ coincides with that defined by $\mathscr{B}$.

- Let $U \in \mathscr{B}$. Since multiplication by x_i is an homeomorphism (multiplication by x_i^{-1} being a continuous inverse) $x_i U$ for all i is open (in the topology defined by $\mathscr{B}$). Thus so is $\sum x_i U = U'$, which shows that the topology defined by $\mathscr{B}$ is finer than that defined by $\mathscr{B}'$.

- On the otherhand, let $U \in \mathscr{B}$. We show now that there exists $V' \in \mathscr{B}'$ such that $V' \subseteq U$. Let V be such that $V \subseteq x_i^{-1} U$ for all i (again we use the fact that multiplication by x_i^{-1} is open and (lc2)). Then $\sum x_i V \subseteq \sum_i x_i x_i^{-1} U \subseteq U$. This completes the proof of the claim.

Lemma A.6. *A equipped with its norm topology is a barrelled, complete Hausdorff locally convex A-module.*

Proof. Consider the induced Gauss norm on A, $|\cdot|$. By [3, §3.1, Proposition 5(i)], $|\cdot|$ is a $\mathbf{Q}_p$-algebra norm. Thus A equipped with its norm topology is a locally convex $\mathbf{Q}_p$-vector space. Moreover since $|\cdot|$ is sub-multiplicative ($|ab| \leq |a| \cdot |b|$), multiplication by A is continuous. Finally since the topology is defined by a norm, it is Hausdorff and A is a $\mathbf{Q}_p$-Banach algebra for this norm. Note that all Banach spaces are barrelled, cf. [50, page 40]. □

We now prove that there is a well defined Hausdorff completion for a locally convex A-module.

Lemma A.7. *For any locally convex A-module M there exists an up to unique topological isomorphism unique complete Hausdorff topological A-module $\widehat{M}$ together with a continuous A-linear map*

$$c_M \colon M \to \widehat{M}$$

such that the following universal property holds: For any continuous A-linear map $f\colon M \to N$ into a complete Hausdorff locally convex A-module N there is a unique continuous A-linear map $\widehat{f}\colon \widehat{M} \to N$ such that

$$f = \widehat{f} \circ c_M.$$

Proof. Uniqueness follows from the universal property. For the existence, replacing M by $M/\overline{\{0\}}$ if necessary, we may assume that M is Hausdorff (note that $\overline{\{0\}}$ is a locally convex A-module and thus so is $M/\overline{\{0\}}$). We consider M as a locally convex $\mathbf{Q}_p$-vector space and let $\widehat{M}$ be as in [50, Proposition 7.5]. We show that $\widehat{M}$ is a topological A-module. It is easy to see that $\widehat{A \times M} = A \times \widehat{M}$ (as locally convex $\mathbf{Q}_p$-vector spaces). By the universal property in loc.cit. the A-module structure $A \times M \to M$ extends to a continuous morphism $\alpha\colon A \times \widehat{M} \to \widehat{M}$. Since $M \to \widehat{M}$ is an injection and M is dense in $\widehat{M}$, it follows that α exhibits $\widehat{M}$ as a topological A-module. Finally since c_M is an injection it is A-linear. $\square$

From now on when A is considered as a locally convex A-module, we will assume it is equipped with its Gauss-norm topology, $(A, |\cdot|_A)$.

We need the notion of the *strong* dual of a locally convex A-module. This will be much less well behaved compared to the classical situation when A is a finite extension of $\mathbf{Q}_p$. For example if the dimension of H is greater than 1 then we do not even know if $\mathrm{LA}(H, A)$ is A-reflexive, cf. Conjecture A.18 and Remark A.32.

Definition A.8. *Let M be a locally convex A-module. As an abstract A-module, we define the*

$$M_b' := \mathrm{Hom}_{A,\mathrm{cont}}(M, A)$$

Now equip $\mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(M, A)$ with the strong $\mathbf{Q}_p$-locally convex topology and give

$$M_b' \subseteq \mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(M, A)$$

the induced subspace topology. We call M_b' equipped with this topology, the strong dual of M. We say M is A-reflexive if the canonical morphism $M \to (M_b')_b'$ is a topological isomorphism.

Remark A.9. Let M be a locally convex A-module. The strong dual M_b' can equivalently be defined with the topology obtained by taking the sets $\{f\colon M \to A | f(B) \subseteq U\}$ for $B \subseteq M$ a bounded set and $U \subseteq A$ open, as a system of neighbourhoods of 0.

Let $R \in \{\mathbf{Q}_p, A\}$. For M and N locally convex R-modules, we will sometimes write $\mathcal{L}_{R,b}(M, N) := \mathrm{Hom}_{R,\mathrm{cont}}(M, N)$ equipped with the strong topology. If M is a locally convex $\mathbf{Q}_p$-vector space, then we will denote the *classical* strong dual of M, cf. [50, Chapter 1, §9], by $M'_{\mathbf{Q}_p,b}$.

We now prove that M_b' as given by Definition A.8 is indeed a locally convex A-module.

Lemma A.10. *If M is a locally convex A-module, then so is its strong dual M_b'.*

Proof. By [50, §5], M'_b is a locally convex $\mathbf{Q}_p$-vector space. So it suffices to show that multiplication by A is continuous. This comes down to chasing definitions. Let $|\cdot|$ be the Gauss norm on A. For any bounded set B of M (viewing M as a locally convex $\mathbf{Q}_p$-vector space), we have the seminorm (on $\mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(M,A)$)

$$p_B(f) := \sup_{v\in B}|f(v)|.$$

The locally convex topology on $\mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(M,A)$ is then defined by the family of seminorms $\{p_B\}_{B\in\mathscr{B}}$ where $\mathscr{B}$ is the set of all bounded subsets of M. For any finitely many seminorms $p_{B_1}, p_{B_2}, \ldots, p_{B_r}$ in the given family and any real number $\epsilon > 0$ the open sets

$$\{f \in \mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(M,A)|\ p_{B_1}, p_{B_2}, \ldots, p_{B_r}(f) \leq \epsilon\}$$

form a basis around 0 of $\mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(M,A)$. For any $a \in A$ we have $p_B(af) \leq |a|p_B(f)$ and now it is easy to see that multiplication by A is continuous on $\mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(M,A)$. Thus it is also continuous on $\mathrm{Hom}_{A,\mathrm{cont}}(M,A)$. □

Example A.11. By [37, Lemma 2.1.19] (cf. also [17, 5.5 Proposition]), we have topological isomorphisms [70]

$$\mathcal{L}_{A,b}(\mathscr{R}_A^-, A) = \mathscr{R}_A^+, \quad \mathcal{L}_{A,b}(\mathscr{R}_A, A) = \mathscr{R}_A.$$

Note that, if we denote by $\mathscr{R}_A^\sim$ the sub-A-module of $\mathscr{R}_A$ given by Laurent series whose non-negative powers of T vanish, equipped with it's induced topology, then we have $(\mathscr{R}_A^+)^\perp = \mathscr{R}_A^+$ and $(\mathscr{R}_A^\sim)^\perp = \mathscr{R}_A^\sim$ (the orthogonal is taken with respect to the natural separately continuous pairing $\mathscr{R}_A \times \mathscr{R}_A \to A$, $(f,g) \mapsto \mathrm{rés}_0(f(T)g(T)dT)$). This shows that $\mathscr{R}_A^+$ and $\mathscr{R}_A^\sim$ are closed sub-A-modules of $\mathscr{R}_A$ and that we have a topological decomposition $\mathscr{R}_A = \mathscr{R}_A^+ \oplus \mathscr{R}_A^\sim$ (cf. [50, Proposition 8.8(ii)]). We can conclude, by using [50, Lemma 5.3(ii)], that we have a topological isomorphism of locally convex A-modules $\mathscr{R}_A^- \cong \mathscr{R}_A^\sim$ and $\mathscr{R}_A^+ \cong \mathscr{R}_A/\mathscr{R}_A^\sim$. On the other hand, the same argument of [37, Lemma 2.1.19] shows that

$$\mathcal{L}_{A,b}(\mathscr{R}_A^+, A) = \mathscr{R}_A^-.$$

Indeed, as in loc. cit., the inverse to the natural map $\mathscr{R}_A \to \mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{R}_A, A)$ induced by the pairing $\mathrm{rés}_0$ is given by associating, to any $\mu \in \mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{R}_A, A)$, the power series $\sum_{n\in\mathbf{Z}} \mu(T^{-1-n})T^n$. This series lies in $\mathscr{R}_A^\sim$ if and only if $\mu(T^{-1-n}) = 0$ for all $n \geq 0$, or equivalently $\mu(\mathscr{R}_A^\sim) = 0$. This shows in particular that the spaces $\mathscr{R}_A^+$ and $\mathscr{R}_A^-$ are A-reflexive.

We need to recall some classical definitions adapted to our context:

70. Note that, identifying $\mathscr{R}_A^- = \varinjlim_s \mathscr{R}_A^{]0,s]}/\mathscr{R}_A^{]0,+\infty]}$, one can see (cf. Lemma A.22 and Lemma A.30 below) that $\mathcal{L}_{A,b}(\mathscr{R}_A^-, A) = \varprojlim_s \mathcal{L}_{A,b}(\mathscr{R}_A^{]0,s]}/\mathscr{R}^{]0,+\infty]}, A)$ is Fréchet (observe that the space $\mathscr{R}_A^{]0,s]}/\mathscr{R}_A^{]0,+\infty]}$ with its topology defined by $v^{]0,s]}$, where $v^{]0,s]}$ is the valuation induced by $v^{[r,s]}$ for any $r < s$, is a Banach space).

Definition A.12. *A Cauchy net in a locally convex A-module M is a net $(m_i)_{i\in I}$ in M (a family of vectors m_i in M where the index set I is directed) such that for every $\epsilon > 0$ and every seminorm q_α (defining the topology on M), $\exists\kappa$ such that for all $\lambda, \mu > \kappa$, $q_\alpha(m_\lambda - m_\mu) < \epsilon$. M is complete if and only if every Cauchy net converges.*

Definition A.13. *Let $R \in \{\mathbf{Q}_p, A\}$ and let M be a locally convex R-module. We say M is a Fréchet space if it is metrizable and complete. We say M is R-LB-type if it can be written as a countable increasing union of R-Banach spaces with R-linear injective transition morphisms. We say M is R-LF-type if it can be written as a countable increasing union of locally convex R-modules which are Fréchet spaces with R-linear injective transition morphisms.*

A.2 Relative non-archimedean functional analysis

Here are our first (and main) examples of locally convex A-modules:

Lemma A.14. *$\mathscr{R}_A^+$, $\mathscr{R}_A^+ \boxtimes \mathbf{P}^1$, $\mathscr{R}_A^-$, $\mathscr{R}_A^- \boxtimes \mathbf{P}^1$, $\mathscr{R}_A$ and $\mathscr{R}_A \boxtimes \mathbf{P}^1$ are complete Hausdorff locally convex A-modules. Moreover $\mathscr{R}_A^+$ and $\mathscr{R}_A^+ \boxtimes \mathbf{P}^1$ are Fréchet spaces, $\mathscr{R}_A^-$ and $\mathscr{R}_A^- \boxtimes \mathbf{P}^1$ are of A-LB-type and $\mathscr{R}_A$ and $\mathscr{R}_A \boxtimes \mathbf{P}^1$ are of A-LF-type. For $? \in \{+, -, \emptyset\}$ we have that*

$$\mathscr{R}_A^? = \mathscr{R}_{\mathbf{Q}_p}^? \widehat{\otimes}_{\mathbf{Q}_p} A,$$

$$\mathscr{R}_A^? \boxtimes \mathbf{P}^1 = (\mathscr{R}_{\mathbf{Q}_p}^? \boxtimes \mathbf{P}^1) \widehat{\otimes}_{\mathbf{Q}_p} A.$$

Proof. We first prove the statement for $\mathscr{R}_A^?$, $? \in \{+, -, \emptyset\}$. By [4, lemme 1.3(i)] we have an isomorphism as Fréchet spaces (in the category of locally convex $\mathbf{Q}_p$-vector spaces) $\mathscr{R}_A^+ = \mathscr{R}_{\mathbf{Q}_p}^+ \widehat{\otimes}_{\mathbf{Q}_p} A$. Thus $\mathscr{R}_A^+$ is a locally convex $\mathbf{Q}_p$-vector space. Multiplication by A is clearly continuous on $\mathscr{R}_{\mathbf{Q}_p}^+ \otimes_{\mathbf{Q}_p} A$ (the latter is also a locally convex $\mathbf{Q}_p$-vector space) and so by Lemma A.7, the completion $\mathscr{R}_A^+$ is a locally convex A-module.

By example A.11, $\mathscr{R}_A^-$ is A-reflexive and so by Remark A.32 we have an isomorphism $\mathscr{R}_A^- = \mathscr{R}_{\mathbf{Q}_p}^- \widehat{\otimes}_{\mathbf{Q}_p} A$. Moreover $\mathscr{R}_A^- = \varinjlim_s \mathscr{R}_A^{]0,s]}/\mathscr{R}_A^{]0,+\infty]}$ is A-LB-type.

Finally by definition $\mathscr{R}_A$ is A-LF-type and since $\mathscr{R}_A = \mathscr{R}_A^+ \oplus \mathscr{R}_A^-$ (as topological A-modules), it is also Hausdorff and complete. We compute

$$\begin{aligned}
\mathscr{R}_A &= \mathscr{R}_A^+ \oplus \mathscr{R}_A^- \\
&= (\mathscr{R}_{\mathbf{Q}_p}^+ \widehat{\otimes}_{\mathbf{Q}_p} A) \oplus (\mathscr{R}_{\mathbf{Q}_p}^- \widehat{\otimes}_{\mathbf{Q}_p} A) \\
&= (\mathscr{R}_{\mathbf{Q}_p}^+ \oplus \mathscr{R}_{\mathbf{Q}_p}^-) \widehat{\otimes}_{\mathbf{Q}_p} A \\
&= \mathscr{R}_{\mathbf{Q}_p} \widehat{\otimes}_{\mathbf{Q}_p} A.
\end{aligned}$$

We finally observe that, if $M \in \{\mathscr{R}_A^+, \mathscr{R}_A^-, \mathscr{R}_A\}$, then $M \boxtimes \mathbf{P}^1$ (with its topology induced by the inclusion $M \boxtimes \mathbf{P}^1 \subseteq M \times M$) is topologically isomorphic to $M \times M$, the isomorphism given by $z \mapsto (\mathrm{Res}_{\mathbf{Z}_p}(z), \psi(\mathrm{Res}_{\mathbf{Z}_p}(wz)))$ with inverse $(z_1, z_2) \mapsto (z_1, \varphi(z_2) + w(\mathrm{Res}_{\mathbf{Z}_p^\times}(z_1)))$. $\square$

Let M be a Hausdorff locally convex A-module. We define a locally convex A-module $\mathrm{LA}(H,M)$ of all M-valued *locally analytic* functions on H.

Definition A.15. *An M-index $\mathcal{I}$ on H is a family of triples*

$$\{(H_i, \phi_i, M_i)\}_{i\in I}$$

where the H_i are pairwise disjoint open subsets of H which cover H, each $\phi_i\colon H_i \to \mathbf{Q}_p^d$ is chart[71] *for the manifold H whose image is an affinoid ball and $M_i \hookrightarrow M$ is a continuous linear injection from an A-Banach space M_i into M. Let $\mathcal{F}_{\phi_i}(M_i)$ be the set of all functions $f\colon H_i \to M_i$ such that $f\circ\phi_i^{-1}$ is an M_i-valued holomorphic function on the affinoid ball $\phi_i(H_i)$. Note that $\mathcal{F}_{\phi_i}(M_i)$ is an A-Banach space. We set*

$$\mathcal{F}_{\mathcal{I}}(M) := \prod_{i\in I}\mathcal{F}_{\phi_i}(M_i),$$

where $\mathcal{F}_{\mathcal{I}}(M)$ is equipped with the direct product topology (in particular it is a locally convex A-module). We then define[72]

$$\mathrm{LA}(H,M) := \varinjlim_{\mathcal{I}}\mathcal{F}_{\mathcal{I}}(M)$$

equipped with the $\mathbf{Q}_p$-locally convex inductive limit topology.

Remark A.16. In Definition A.15, in order to see that $\mathrm{LA}(H,M)$ is a locally convex A-module, one needs to check that multiplication by A on $\mathrm{LA}(H,M)$ is continuous. Indeed since, $\cdot\colon A\times M\to M$ is continuous, then so is $A\times\mathcal{F}_{\phi_i}(M_i)\to\mathcal{F}_{\phi_i}(M_i)$. This implies that multiplication $B_{\mathcal{I}}\colon A\times\mathcal{F}_{\mathcal{I}}(M)\to\mathcal{F}_{\mathcal{I}}(M)$ is continuous. Denote by $B\colon A\times\mathrm{LA}(H,M)\to\mathrm{LA}(H,M)$, the multiplication by A on $\mathrm{LA}(H,M)$. The continuity of B follows from the continuity of the $B_{\mathcal{I}}$ (cf. the 3rd paragraph of the proof of Lemma A.22 where a similar problem is proved).

Lemma A.17. *Let $(V,||\cdot||)$ be a $\mathbf{Q}_p$-Banach space and let $V_0'\subset V'_{\mathbf{Q}_p,b}$ be the unit ball. Given a constant C and a vector $v\in V$, if $|l(v)|_p\le C$ for all $l\in V_0'$ then $||v||\le C$.*

Proof. This is a direct consequence of the Hahn-Banach theorem. Applying [50, Proposition 9.2] with $U:=V$, $q:=||\cdot||$, $U_o:=\mathbf{Q}_p v$ and $|l_o(v)|_p\ge||v||$ for all $a\in\mathbf{Q}_p$ we obtain a linear form $l\in V_0'$ and $|l(v)|_p\ge||v||$. ☐

To kick-start our study of locally convex A-modules and their relationship to $\mathrm{Rep}^{\mathrm{la}}_A(H)$ (cf. Definition A.24) we need to know that $\mathrm{LA}(H,A)$ is well-behaved. Féaux states explicitly in his thesis (cf. [27]) that he does not know if $\mathrm{LA}(H,A)$ is complete. The completeness of $\mathrm{LA}(H,A)$ has since become somewhat of a folklore conjecture:

71. To be more precise one takes $\mathbb{H}_i$ an affinoid rigid analytic space over $\mathbf{Q}_p$ isomorphic to a closed ball, so that ϕ_i induces an isomorphism $\phi_i'\colon H_i\xrightarrow{\sim}\mathbb{H}_i(\mathbf{Q}_p)$.

72. This colimit is taking place in the category of $\mathbf{Q}_p$-locally convex vector spaces.

Conjecture A.18. *If H is a compact locally $\mathbf{Q}_p$-analytic group then* $\mathrm{LA}(H, A)$ *is complete.*

Remark A.19. If Conjecture A.18 is true then one can show that $\mathrm{LA}(H, A) \cong \mathrm{LA}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi} A \cong \mathrm{LA}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota} A$, cf. Remark A.32. If H is of dimension 1 then Conjecture A.18 is true, cf. lemma A.14.

Although we are unable to prove Conjecture A.18 we show that $\mathrm{LA}(H, A)$ is sufficiently well-behaved for applications.

Definition A.20. *Let $R \in \{\mathbf{Q}_p, A\}$. We call a Hausdorff R-LB-type $V = \varinjlim_n V_n$,* R-regular *if, for every bounded subset B of V, there exists an n such that V_n contains B and B is bounded in V_n.*

Remark A.21. Let $R \in \{\mathbf{Q}_p, A\}$. By [23, Proposition 1.1.10 and 1.1.11], a Hausdorff semi-complete R-LB-type is R-regular.

Before we state our next result we need some notation. Via Mahler expansions (cf. [44, III. Théorème 1.2.4]), the set of continuous functions from $\mathbf{Z}_p^d$ to $\mathbf{Q}_p$ can be viewed as the space of all series

$$\sum_{\alpha \in \mathbf{N}^d} c_\alpha \binom{x}{\alpha}$$

with $c_\alpha \in \mathbf{Q}_p$ such that $|c_\alpha| \to 0$ as $|\alpha| \to 0$. Here as usual

$$\binom{x}{\alpha} := \binom{x_1}{\alpha_1} \cdots \binom{x_d}{\alpha_d}$$

and

$$|\alpha| := \sum_{i=1}^{d} \alpha_i$$

for $x = (x_1, \ldots, x_d) \in \mathbf{Z}_p^d$ and multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{N}^d$. We are grateful for G. Dospinescu for supplying us with the idea for the following lemma.

Lemma A.22. *If H is a compact locally $\mathbf{Q}_p$-analytic group then* $\mathrm{LA}(H, A)$ *is A-regular.*

Proof. By choosing a covering of H by a finite number of open subsets isomorphic (as locally $\mathbf{Q}_p$-analytic manifolds) to $\mathbf{Z}_p^d$ for some $d \in \mathbf{N}$, we suppose that $H = \mathbf{Z}_p^d$, cf. [21, Corollary 8.34]. For each $h \in \mathbf{N}$ and $f \in \mathrm{LA}(H, A)$, write $f = \sum_{n \in \mathbf{N}^d} a_n(f)\binom{x}{\alpha}$, $a_n(f) \in A$, the Mahler expansion of the continuous function f. By Amice's theorem (cf. [44, III. Théorème 1.3.8] or [13, Théorème I.4.7]), we have

$$\mathrm{LA}_h(\mathbf{Z}_p^d, \mathbf{Q}_p) = \widehat{\bigoplus_{n \in \mathbf{N}^d}} \mathbf{Q}_p \cdot k_{n,h} \binom{x}{n}$$

where $k_{n,h} := \lfloor p^{-h}n_1 \rfloor! \dots \lfloor p^{-h}n_d \rfloor!$, and where $\mathrm{LA}_h(\mathbf{Z}_p^d, \mathbf{Q}_p)$ denotes the space of functions which are analytic on every ball of poly-radius $(h, \dots, h)$. One also obtains

$$\mathrm{LA}_h(\mathbf{Z}_p^d, A) = \widehat{\bigoplus_{n \in \mathbf{N}^d}} A \cdot k_{n,h} \binom{x}{n},$$

which shows in particular that each $\mathrm{LA}_h(\mathbf{Z}_p^d, A)$ is complete. By definition, we also have

$$\mathrm{LA}(\mathbf{Z}_p^d, A) = \varinjlim_{h \in \mathbf{N}} \mathrm{LA}_h(\mathbf{Z}_p^d, A).$$

We denote the norms on $\mathrm{LA}_h(H, \mathbf{Q}_p)$ and $\mathrm{LA}_h(H, A)$ by $|\cdot|_{h,\mathbf{Q}_p}$ and $|\cdot|_{h,A}$, respectively and $v_{h,\mathbf{Q}_p}$ and $v_{h,A}$ their respective valuations.

Consider the $\mathbf{Q}_p$-bilinear application

$$\begin{aligned} B\colon A'_{\mathbf{Q}_p,b} \times \mathrm{LA}(H, A) &\to \mathrm{LA}(H, \mathbf{Q}_p) \\ (l, f) &\mapsto l \circ f. \end{aligned}$$

Note that the Mahler coefficients of $l \circ f$ are then just given by $a_n(l \circ f) = l(a_n(f))$.

We show that the map B above is continuous. Note (as is easily seen by looking at Mahler expansions) that restriction gives, for every $h \in \mathbf{N}$, $\mathbf{Q}_p$-bilinear forms

$$\begin{aligned} B_h\colon A'_{\mathbf{Q}_p,b} \times \mathrm{LA}_h(H, A) &\to \mathrm{LA}_h(H, \mathbf{Q}_p) \\ (l, f) &\mapsto l \circ f. \end{aligned}$$

We claim that B is continuous if and only if B_h is continuous for every $h \in \mathbf{N}$. Indeed, by the definition of the locally convex final topology, the topology of $\mathrm{LA}(H, \mathbf{Q}_p)$ is defined by the family of all lattices $L \subseteq \mathrm{LA}(H, \mathbf{Q}_p)$ such that $L \cap \mathrm{LA}_h(H, \mathbf{Q}_p)$ is open (in $\mathrm{LA}_h(H, \mathbf{Q}_p)$) for all $h \in \mathbf{N}$. So let L be such a lattice. We want to show that $L' := B^{-1}(L)$ is open (in $A'_{\mathbf{Q}_p,b} \times \mathrm{LA}(H, A)$). Note first that L' is a lattice: if $a \in \mathbf{Z}_p$ and if $(x, y) \in L'$ then $B(a(x, y)) = a^2 B(x, y) \in L$ so that $a(x, y) \in L'$ and, if $(x, y) \in A'_{\mathbf{Q}_p,b} \times \mathrm{LA}(H, A)$, then there exists an $a \in \mathbf{Q}_p^\times$ such that $aB(x, y) \in L$ and, writing $a = a'/p^k$, $a' \in \mathbf{Z}_p$, we also have that $(a')^2 B(x, y) \in L$, whence $a'(x, y) \in L'$, which proves that L' is a lattice. So (again by definition of the locally convex final topology), L' is open if and only if $L' \cap (A'_{\mathbf{Q}_p,b} \times \mathrm{LA}_h(H, A))$ is open for every $h \in \mathbf{N}$. Noting $B_{h,n}\colon A'_{\mathbf{Q}_p,b} \times \mathrm{LA}_h(H, A) \to \mathrm{LA}_n(H, \mathbf{Q}_p)$ for $n \geq h$ the composition of B_h with the natural continuous inclusion $\mathrm{LA}_h(H, \mathbf{Q}_p) \to \mathrm{LA}_n(H, \mathbf{Q}_p)$, we have that

$$\begin{aligned} L' \cap (A'_{\mathbf{Q}_p,b} \times \mathrm{LA}_h(H, A)) &= B^{-1}(L) \cap (A'_{\mathbf{Q}_p,b} \times \mathrm{LA}_h(H, A)) \\ &= B^{-1}(L \cap \cup_{n \geq h} \mathrm{LA}_n(H, \mathbf{Q}_p)) \cap (A'_{\mathbf{Q}_p,b} \times \mathrm{LA}_h(H, A)) \\ &= \cup_{n \geq h} [B^{-1}(L \cap \mathrm{LA}_n(H, \mathbf{Q}_p)) \cap (A'_{\mathbf{Q}_p,b} \times \mathrm{LA}_h(H, A))] \\ &= \cup_{n \geq h} B_{h,n}^{-1}(L \cap \mathrm{LA}_n(H, \mathbf{Q}_p)), \end{aligned}$$

which is open if each B_h (and hence $B_{h,n}$) is continuous. This proves the claim. We finally prove that each map B_h is continuous. Indeed $|B_h(l,f)|_{h,\mathbf{Q}_p} = |l\circ f|_{h,\mathbf{Q}_p} \leq ||l|| |f|_{h,A}$ (where $||\cdot||$ is the norm on $A'_{\mathbf{Q}_p,b}$) and so B is continuous.

Let $T \subset \mathrm{LA}(H,A)$ be a bounded subset and consider

$$S := A'_0 \times T \subset A'_{\mathbf{Q}_p,b} \times \mathrm{LA}(H,A)$$

where $A'_0 \subset A'_{\mathbf{Q}_p,b}$ is the unit ball. Then S is bounded and so is $B(S)$, since B is continuous. By [23, Proposition 1.1.11], $\mathrm{LA}(H,\mathbf{Q}_p)$ is $\mathbf{Q}_p$-regular and so for some $h \geq 1$, $B(S)$ is contained in $\mathrm{LA}_h(H,\mathbf{Q}_p)$ and there exists a constant C such that

$$v_{h,\mathbf{Q}_p}(l\circ f) = \inf_{n\in\mathbf{N}^d} v_p(l(a_n(f))) - v_p(k_{n,h}) \geq C$$

for all $l \in A'_0$. Lemma A.17 now implies that $v_A(a_n(f)) - v_p(k_{n,h}) \geq C$ for all $n \in \mathbf{N}^d$. Now

$$\begin{aligned} v_A(a_n(f)) - v_p(k_{n,h+1}) &= v_A(a_n(f)) - v_p(k_{n,h}) + v_p(k_{n,h}) - v_p(k_{n,h+1}) \\ &\geq C + v_p(k_{n,h}) - v_p(k_{n,h+1}) \xrightarrow{|n|\to+\infty} +\infty. \end{aligned}$$

This implies that $f \in \mathrm{LA}_{h+1}(\mathbf{Z}_p^d, A)$ for all $f \in T$. We now compute

$$\begin{aligned} v_{h+1,A}(f) &= \inf_{n\in\mathbf{N}^d} v_A(a_n(f)) - v_p(k_{n,h+1}) \\ &= \inf_{n\in\mathbf{N}^d} v_A(a_n(f)) - v_p(k_{n,f}) + v_p(k_{n,h}) - v_p(k_{n,h+1}) \\ &\geq C \end{aligned}$$

since $v_p(k_{n,h}) - v_p(k_{n,h+1}) \geq 0$. This shows that T is contained and bounded in $\mathrm{LA}_{h+1}(\mathbf{Z}_p^d, A)$. Thus $\mathrm{LA}(H,A)$ is A-regular. $\square$

Remark A.23. For an alternative proof of the fact that the $\mathbf{Q}_p$-bilinear map B in the proof of Lemma A.22 is continuous, note that B is the composition of the morphisms

$$\begin{aligned} B\colon A'_{\mathbf{Q}_p,b} \times \mathrm{LA}(H,A) &\xrightarrow{\mathrm{id}\times\alpha} A'_{\mathbf{Q}_p,b} \times (A\widehat{\otimes}_{\mathbf{Q}_p,\pi}\mathrm{LA}(H,\mathbf{Q}_p)) \\ &\to A'_{\mathbf{Q}_p,b}\widehat{\otimes}_{\mathbf{Q}_p,\pi}(A\widehat{\otimes}_{\mathbf{Q}_p,\pi}\mathrm{LA}(H,\mathbf{Q}_p)) \\ &= (A'_{\mathbf{Q}_p,b}\widehat{\otimes}_{\mathbf{Q}_p,\pi}A)\widehat{\otimes}_{\mathbf{Q}_p,\pi}\mathrm{LA}(H,\mathbf{Q}_p) \\ &\xrightarrow{\beta} \mathrm{LA}(H,\mathbf{Q}_p), \end{aligned}$$

where α is the morphism of [23, Proposition 2.2.10] (cf. the discussion immediately after loc.cit.) which is a continuous bijection. The last morphism β is induced from the pairing of A and $A'_{\mathbf{Q}_p,b}$.

A.3 Relative locally analytic representations

In this section we define the category $\mathrm{Rep}_A^{\mathrm{la}}(H)$ and we study the structure of a locally analytic representation over the relative distribution algebras, generalizing some fundamental work of Schneider and Teitelbaum to our relative setting.

The following definition is similar to [33, Définition 3.2].

Definition A.24. *An object M in $\mathrm{Rep}_A^{\mathrm{la}}(H)$ is a barrelled, Hausdorff, locally convex A-module equipped with a topological*[73] *A-linear action of H, such that, for each $m \in M$, the orbit map $h \mapsto h \cdot m$ is an element in $\mathrm{LA}(H, M)$. Morphisms are given by continuous A-linear H-maps.*

Remark A.25 (locally analytic induced representation)*.* Let G be a locally $\mathbf{Q}_p$-analytic group, H a closed locally $\mathbf{Q}_p$-analytic subgroup and suppose that G/H is compact. Let M be an object of $\mathrm{Rep}_A^{\mathrm{la}}(H)$, which is Banach. Then

$$\mathrm{Ind}_H^G(M) := \left\{ f \in \mathrm{LA}(G, M) \mid \forall h_i \in H_i : \; f(h_1 h_2) = h_2^{-1} \cdot f(h_1) \right\}$$

identifies (as topological A-modules) with $\mathrm{LA}(G/H, M)$, cf. [27, Satz 4.3.1]. Moreover $\mathrm{Ind}_H^G(M)$ (equipped with the natural action of G: $(g \cdot f)(x) := f(g^{-1}x)$) is an object of $\mathrm{Rep}_A^{\mathrm{la}}(G)$, cf. Satz 4.1.5 in loc.cit.

To track the action of $\mathscr{D}(G)$[74] on $\mathrm{LA}(G/H, M)$ induced by the above isomorphism, we need to explicit this isomorphism. Any choice of a section $G/H \to G$ gives an isomorphism of locally $\mathbf{Q}_p$ analytic manifolds $G \cong G/H \times H$. This gives an isomorphism $\mathrm{LA}(G, M) \cong \mathrm{LA}(G/H \times H, M)$ and the space $\mathrm{Ind}_H^G(M) \subseteq \mathrm{LA}(G, M)$ is identified with the sub-module $\{f \; : \; f(\overline{g}, h) = h^{-1} \cdot f(\overline{g}, 1), \; \overline{g} \in G/H, h \in H\}$ of $\mathrm{LA}(G/H \times H, M)$. On the other hand, the composition[75]

$$\mathrm{LA}(G/H, M) \to \mathrm{LA}(H, \mathrm{End}(M)) \times \mathrm{LA}(G/H, M) \to \mathrm{LA}(G/H \times H, M)$$

$$\tilde{f} \mapsto (\rho^{-1}, \tilde{f}) \mapsto [(\overline{g}, h) \mapsto \rho(h)^{-1} \cdot \tilde{f}(\overline{g})],$$

where we have noted $\rho^{-1} : H \to \mathrm{GL}(M)$ the representation of H on M, induces an isomorphism between $\mathrm{LA}(G/H, M)$ and the image of $\mathrm{Ind}_H^G(M)$ in $\mathrm{LA}(G, M)$.

It will turn out that every complete object of $\mathrm{Rep}_A^{\mathrm{la}}(H)$ carries a structure of a $\mathscr{D}(H, A)$[76]-module. The following lemma is essentially [63, Proposition 1.3].

Lemma A.26. *Let M be a locally convex $\mathbf{Q}_p$-module and let N be a locally convex A-module. Then $\tilde{f}(a \otimes x) = af(x)$ defines a topological A-linear isomorphism*

$$\mathcal{L}_{\mathbf{Q}_p, b}(M, N) \xrightarrow{\sim} \mathcal{L}_{A, b}(M \otimes_{\mathbf{Q}_p, \pi} A, N)$$

73. We say that the H-action on M is *topological* if H induces continuous endomorphisms of M.

74. This is the distribution algebra with coefficients in $\mathbf{Q}_p$, cf. Definition A.29.

75. we refer the reader to the proof of [27, Satz 4.3.1].

76. cf. Definition A.29.

Proof. For $f \in \mathcal{L}_{\mathbf{Q}_p,b}(M,N)$, the map $\tilde{f}$ is given by the composition of the continuous map $M \otimes_{\mathbf{Q}_p,\pi} A \to N \otimes_{\mathbf{Q}_p,\pi} A$ induced by $a \otimes x \mapsto a \otimes f(x)$ and the (continuous) map $N \otimes_{\mathbf{Q}_p,\pi} A \to N$ induced by the A-module structure on N, so it is well defined. The inverse of $f \mapsto \tilde{f}$ is given by $g \mapsto g_0$, where $g_0(x) = g(x \otimes 1)$. This shows that the map of the statement induces an A-linear bijection.

The fact that it's a topological isomorphism follows from [50, Corollary 17.5(iii)]: the map $\alpha\colon M \to M \otimes_{\mathbf{Q}_p,\pi} A$ defined by $m \mapsto m \otimes 1$ induces an homeomorphism between M and it's image, and so the image of any bounded subset of M is bounded in $M \otimes_{\mathbf{Q}_p,\pi} A$ and, conversely, the inverse image (i.e its intersection with $M \subseteq M \otimes_{\mathbf{Q}_p,\pi} A$) of any bounded subset of $M \otimes_{\mathbf{Q}_p,\pi} A$ is bounded in M. To show that $f \mapsto \tilde{f}$ is continuous, it is enough to show that, if $B \subseteq M \otimes_{\mathbf{Q}_p,\pi} A$ is bounded and if $U \subseteq N$ is an open set [77], then the inverse image of the set $\{f\colon M \otimes_{\mathbf{Q}_p,\pi} A \to N \mid f(B) \subseteq U\}$ is open in $\mathcal{L}_{\mathbf{Q}_p,b}(M,N)$, but this inverse image is nothing but [78] $\{f\colon M \to N \mid f(\alpha^{-1}(B)) \subseteq U)\}$, which is open since $\alpha^{-1}(B)$ is bounded in M. Conversely, if $B \subseteq M$ is bounded and $U \subseteq N$ is open, then the inverse image of $\{f\colon M \to N \mid f(B) \subseteq U\}$ by the map $g \mapsto g_0$ is $\{g\colon M \otimes_{\mathbf{Q}_p,\pi} A \to N \mid g(\alpha(B)) \subseteq U\}$, which is open since $\alpha(B)$ is bounded in $M \otimes_{\mathbf{Q}_p,\pi} A$, and shows that the inverse map is continuous. This completes the proof. ☐

Corollary A.27. *In the setting of Lemma A.26, if in addition N is Hausdorff and complete, then*

$$\mathcal{L}_{\mathbf{Q}_p,b}(M,N) \xrightarrow{\sim} \mathcal{L}_{A,b}(M \widehat{\otimes}_{\mathbf{Q}_p,\pi} A, N)$$

Proof. This is an immediate consequence of Lemma A.26 above and the universal property of Hausdorff completion, cf. [50, Corollary 7.7]. ☐

Remark A.28. Since [50, Corollary 17.5(iii)] holds for projective as well as inductive tensor product, both Lemma A.26 and Corollary A.27 hold with inductive tensor product replacing the projective one.

Definition A.29. *We define the space of distributions on H with values in A as the strong dual of $\mathrm{LA}(H,A)'_b$ (cf. Definition A.8)*

$$\mathscr{D}(H,A) := \mathrm{LA}(H,A)'_b.$$

Lemma A.30. *Let $R \in \{\mathbf{Q}_p, A\}$. If $V = \varinjlim_n V_n$ is R-regular (cf. Definition A.20) then for any locally convex R-module, the natural map $\mathcal{L}_{R,b}(V,W) \to \varprojlim_n \mathcal{L}_{R,b}(V_n,W)$ is a topological isomorphism which is R-linear.*

Proof. This is the same proof as [23, Proposition 1.1.22]. The crucial point in loc.cit. is that a Hausdorff semi-complete R-LB-type is R-regular. ☐

77. One can assume U is an A^0-module where A^0 is the subset of A whose norm is less than 1. Indeed by choosing a presentation of $A = \mathbf{Q}_p\langle T_1, T_2, \ldots, T_n\rangle/I$, one can consider $U' = \sum \overline{T}_i \cdot U$. One then repeats the argument of Remark A.5.

78. It is enough to suppose $B = B_M \otimes_{\mathbf{Q}_p} A^0$ where B_M is a bounded lattice in M.

Lemma A.31. *Let H be a compact locally $\mathbf{Q}_p$-analytic group. We have an isomorphism of locally convex A-modules*

$$\mathscr{D}(H, A) = \mathscr{D}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi} A.$$

In particular, $\mathscr{D}(H, A)$ is an A-Fréchet space.

Proof. By [50, Proposition 20.9] we have

$$\mathscr{D}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi} A = \mathcal{L}_{\mathbf{Q}_p,b}(\mathscr{D}(H, \mathbf{Q}_p)'_{\mathbf{Q}_p,b}, A).$$

We conclude by observing that (we use the notation from the proof of Lemma A.22)

$$\begin{aligned}
\mathcal{L}_{\mathbf{Q}_p,b}(D(H, \mathbf{Q}_p)'_{\mathbf{Q}_p,b}, A) &\overset{(i)}{=} \mathcal{L}_{\mathbf{Q}_p,b}(\mathrm{LA}(H, \mathbf{Q}_p), A) \\
&\overset{(ii)}{=} \varprojlim_n \mathcal{L}_{\mathbf{Q}_p,b}(\mathrm{LA}_n(H, \mathbf{Q}_p), A) \\
&\overset{(iii)}{=} \varprojlim_n \mathcal{L}_{A,b}(\mathrm{LA}_n(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi} A, A) \\
&\overset{(iv)}{=} \varprojlim_n \mathcal{L}_{A,b}(\mathrm{LA}_n(H, A), A) \\
&\overset{(v)}{=} \mathcal{L}_{A,b}(\mathrm{LA}(H, A), A) \\
&\overset{(vi)}{=} \mathscr{D}(H, A),
\end{aligned}$$

where (i) follows by reflexivity of $\mathrm{LA}(H, \mathbf{Q}_p)$ (cf. [53, Lemma 2.1 and Theorem 1.1]), (ii) follows from Lemma A.30, (iii) follows from Corollary A.27, (iv) is an immediate consequence of the definition of $\mathrm{LA}_n(H, A)$, (v) is a consequence of Lemmas A.22 and A.30 and (vi) is by definition. The last assertion follows since $\mathscr{D}(H, \mathbf{Q}_p)$ is Fréchet and A is Banach so their completed projective tensor product is Fréchet. This completes the proof. □

Remark A.32. For H a compact locally $\mathbf{Q}_p$-analytic group, the natural morphism

$$\alpha\colon \mathrm{LA}(H, A) \to \mathrm{LA}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi} A$$

is (cf. the discussion immediately after [23, Proposition 2.2.10]) a continuous bijection. We do not know whether it is actually a topological isomorphism, cf. Conjecture A.18. By [46, Theorem 2], this is the case if $\mathrm{LA}(H, A)$ is complete (note that $\mathrm{LA}_h(H, A)$ has a Schauder basis, by Amice's theorem, so it has the approximation property). We claim that α is a topological isomorphism if and only if $\mathrm{LA}(H, A)$ is a reflexive A-module. Indeed, this follows from the following equalities:

$$\begin{aligned}
\mathcal{L}_{A,b}(\mathscr{D}(H, A), A) &= \mathcal{L}_{A,b}(\mathscr{D}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi} A, A) \\
&= \mathcal{L}_{\mathbf{Q}_p,b}(\mathscr{D}(H, \mathbf{Q}_p), A) \\
&= \mathcal{L}_{\mathbf{Q}_p,b}(\mathscr{D}(H, \mathbf{Q}_p), \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi} A \\
&= \mathrm{LA}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi} A.
\end{aligned}$$

The aim is now to obtain a version of Lemma A.31 without the assumption that H is compact.

Lemma A.33. *Let $\{H_i\}_{i\in I}$ be pairwise disjoint compact open subsets which cover H. Then there is a (A-linear) topological isomorphism*

$$\mathscr{D}(H,A)=\bigoplus_i \mathscr{D}(H_i,A).$$

Moreover $\mathscr{D}(H,A)$ is complete and Hausdorff.

Proof. We have a topological isomorphism

$$\mathrm{LA}(H,A)=\prod_i \mathrm{LA}(H_i,A).$$

The claim now follows from the fact that there is a topological isomorphism

$$(\prod_i \mathrm{LA}(H_i,A))'_b=\bigoplus_i \mathrm{LA}(H_i,A)'_b$$

To see this last fact, one repeats the same proof for [50, Proposition 9.11]. Finally $\mathscr{D}(H,A)$ is complete and Hausdorff follows from [50, Corollary 5.4 and Lemma 7.8]. ☐

Lemma A.34. *Let H be a locally $\mathbf{Q}_p$-analytic group. We have an isomorphism of locally convex A-modules*

$$\mathscr{D}(H,A)=\mathscr{D}(H,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota}A.$$

Proof. This is an immediate consequence of Lemmas A.31 and A.33. Let $\{H_i\}_{i\in I}$ be pairwise disjoint compact open subsets which cover H.

$$\begin{array}{ccc}
\bigoplus_i \left[\mathscr{D}(H_i,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p,\iota} A\right] & \xrightarrow{\sim} & \mathscr{D}(H,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p,\iota} A \\
\downarrow & & \downarrow \\
\bigoplus_i \left[\mathscr{D}(H_i,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota} A\right] & \longrightarrow & \mathscr{D}(H,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota} A.
\end{array}$$

Now [23, Lemma 1.1.30] implies that the top horizontal arrow is a topological isomorphism. By definition the right vertical arrow is a topological embedding (since $\mathscr{D}(H,\mathbf{Q}_p)$ is Hausdorff, so is $\mathscr{D}(H,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p} A$, cf. [50, Corollary 17.5(i)]) that identifies its target with the completion of its source. We will show that the same is true for the left vertical arrow which will imply that the bottom horizontal arrow is a topological isomorphism, as required. Since the composite of the top horizontal arrow and the right vertical arrow is a topological embedding, the same is true for the left vertical arrow. It clearly has dense image and the target ($=\mathscr{D}(H,A)$) is complete. This completes the proof. ☐

Remark A.35. In the setting of Lemma A.34, [30, I.1.3 Proposition 6] [79] shows that

$$\mathscr{D}(H,A) = \mathscr{D}(H,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi}A$$

is a topological isomorphism.

The following is a relative version of the integration map constructed in [53, Theorem 2.2].

Lemma A.36. *Let H be a locally $\mathbf{Q}_p$-analytic group and let M be a complete Hausdorff locally convex A-module. There is a unique A-linear map*

$$I : \mathrm{LA}(H,M) \to \mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{D}(H,A),M),$$

satisfying $I(\phi)(\delta_h \otimes 1) = \phi(h)$ for all $\phi \in \mathrm{LA}(H,M)$ and all $h \in H$. Here $\delta_h \in \mathscr{D}(H,\mathbf{Q}_p)$ such that $\delta_h(f) := f(h)$ for all $f \in \mathrm{LA}(H,\mathbf{Q}_p)$.

Moreover, if M is A-LB-type (cf. Definition A.13) then this map is a bijection.

Proof. By [53, Theorem 2.2] (cf. also the comment immediately after its proof), one has a unique map

$$I_{\mathbf{Q}_p} : \mathrm{LA}(H,M) \to \mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(\mathscr{D}(H,\mathbf{Q}_p),M),$$

satisfying $I_{\mathbf{Q}_p}(\phi)(\delta_h \otimes 1) = \phi(h)$, $h \in H$, and which is bijective if M is of $\mathbf{Q}_p$-LB-type. Note that this map is clearly A-linear.

By Lemma A.33, one reduces to show the result for H compact. So assume H is compact. By Corollary A.27 (where we forget the topologies), since M is Hausdorff and complete, there is an A-linear bijection

$$r\colon \mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(\mathscr{D}(H,\mathbf{Q}_p),M) \xrightarrow{\sim} \mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{D}(H,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi}A,M).$$

Moreover, Lemma A.31 gives an isomorphism

$$s\colon \mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{D}(H,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi}A,M) \xrightarrow{\sim} \mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{D}(H,A),M).$$

The composition of all these maps ($s \circ r \circ I_{\mathbf{Q}_p}$) gives the desired map

$$I : \mathrm{LA}(H,M) \to \mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{D}(H,A),M).$$

The result follows. ∎

Before we state the main result of this section we need to equip $\mathscr{D}(H,A)$ with a ring structure (in particular a convolution product). We show that the convolution product on $(\mathscr{D}(H,\mathbf{Q}_p),*)$, cf. [53, §2], extends to $\mathscr{D}(H,A)$. Indeed by Lemma A.34 we have an isomorphism of locally convex A-modules

$$\mathscr{D}(H,A) = \mathscr{D}(H,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota}A.$$

79. The proposition essentially states that direct sums commute with (completion) of projective tensor product.

We define for $h_1, h_2 \in H$ an A-bilinear, separately continuous map

$$\begin{aligned} *_A \colon (\mathscr{D}(H, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p,\iota} A) \times (\mathscr{D}(H, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p,\iota} A) &\to (\mathscr{D}(H, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p,\iota} A) \\ (\delta_{h_1} \otimes 1) \times (\delta_{h_2} \otimes 1) &\mapsto (\delta_{h_1} * \delta_{h_2} \otimes 1). \end{aligned}$$

Since the Dirac distributions δ_h for $h \in H$ are dense in $\mathscr{D}(H, \mathbf{Q}_p)$, cf. [53, Lemma 3.1], $*_A$ is well defined. Note that $*_A$ is separately continuous since $*$ is separately continuous, cf. [53, Proposition 2.3]. It is clear that $*_A$ extends uniquely to an A-bilinear, separately continuous map (which we denote by $*$, abusing notation)

$$*\colon \mathscr{D}(H, A) \times \mathscr{D}(H, A) \to \mathscr{D}(H, A).$$

The following lemma summarizes the above discussion.

Lemma A.37. $(\mathscr{D}(H, A), *)$ *is an associative A-algebra with $\delta_1 \otimes 1$ ($1 \in H$ is the unit element) as the unit element. Futhermore the convolution $*$ is separately continuous and A-bilinear.*

Let $\mathrm{Rep}_A^{\mathrm{la},LB}(H) \subseteq \mathrm{Rep}_A^{\mathrm{la}}(H)$ be the full subcategory consisting of spaces which are A-LB-type and complete. As a result we obtain the following corollary.

Corollary A.38. *The category of $\mathrm{Rep}_A^{\mathrm{la},LB}(H)$ is isomorphic to the category of complete, Hausdorff locally convex A-modules which are of A-LB-type equipped with a separately continuous $\mathscr{D}(H, A)$-action (more precisely the module structure morphism $\mathscr{D}(H, A) \times M \to M$ is A-bilinear and separately continuous) with morphisms all continuous $\mathscr{D}(H, A)$-linear maps.*

Proof. This is an immediate consequence of Lemma A.36. □

A.4 Locally analytic cohomology and Shapiro's lemma

In this section we prove Shapiro's Lemma for a relative version of the cohomology theory developed by Kohlhaase in [43]. We should warn the reader that Lazard's definition of locally analytic cohomology of a locally $\mathbf{Q}_p$-analytic group via analytic cochains, cf. [44, Chapitre V, §2.3] (or [59] for a modern treatment), does not always coincide with the cohomology groups defined in [43]. Futhermore the cohomology groups defined by Kohlhasse are finer than that of Lazard, in the sense that they themeselves carry a locally convex topology. In what follows however, we will ignore this extra structure. Let us first explain the setup. Let H be a locally $\mathbf{Q}_p$-analytic group (for applications H will be a closed locally $\mathbf{Q}_p$-analytic subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$). We will follow closely the treatment in [43], albeit in a relative setting. In particular we are able to reduce many of the arguments to the case considered in loc.cit. The key is lemma A.34.

Definition A.39. *Let $\mathscr{G}_{H,A}$ denote the category of complete Hausdorff locally convex A-modules with the structure of a separately continuous A-linear $\mathscr{D}(H,A)$-module, taking as morphisms all continuous $\mathscr{D}(H,A)$-linear maps. More precisely we demand that the module structure morphism*

$$\mathscr{D}(H,A)\times M\to M$$

is A-bilinear and separately continuous.

Remark A.40. Alternatively, one sees that $\mathscr{G}_{H,A}$ can be also defined as the category of complete Hausdorff locally convex $\mathbf{Q}_p$-modules with the structure of a separately continuous $\mathscr{D}(H,A)$-module, taking as morphisms all continuous $\mathscr{D}(H,A)$-linear maps.

As a consequence of Lemma A.37 and the fact that $\mathscr{D}(H,A)$ is complete and Hausdorff (cf. Lemma A.33) the convolution product $(\mathscr{D}(H,A),*)$ induces a unique continuous A-linear map [80]

$$\mathscr{D}(H,A)\widehat{\otimes}_{A,\iota}\mathscr{D}(H,A)\to\mathscr{D}(H,A). \tag{36}$$

We now endow $\mathscr{G}_{H,A}$ (and LCS_A) with the structure of an exact category. A sequence in $\mathscr{G}_{H,A}$ (or LCS_A)

$$\cdots\to M_{i-1}\xrightarrow{\alpha_{i-1}}M_i\xrightarrow{\alpha_i}M_{i+1}\xrightarrow{\alpha_{i+1}}\cdots$$

is called *s-exact* if $M_i=K_i\oplus L_i$ (as topological A-modules) where $K_i:=\ker(\alpha_i)$ and α_i induces an isomorphism (as topological A-modules) between L_i and K_{i+1}.

Remark A.41. A sequence in $\mathscr{G}_{H,A}$

$$0\to M\to N\to P\to 0$$

is s-exact iff it is split in the category of topological A-modules.

Definition A.42. *An object P of $\mathscr{G}_{H,A}$ is called* s-projective *if the functor* $\mathrm{Hom}_{\mathscr{G}_{H,A}}(P,\cdot)$ *transforms all short s-exact sequences*

$$0\to M_1\to M_2\to M_3\to 0$$

in $\mathscr{G}_{H,A}$ into exact sequences of A-modules.

Lemma A.43. *If M is any complete Hausdorff locally convex A-module, then*

$$\mathscr{D}(H,A)\widehat{\otimes}_{A,\iota}M$$

is an object of $\mathscr{G}_{H,A}$.

80. The tensor product $\mathscr{D}(H,A)\widehat{\otimes}_{A,\iota}\mathscr{D}(H,A)$ deserves some explanation. First one forms the abstract tensor product $\mathscr{D}(H,A)\otimes_A\mathscr{D}(H,A)$ and equips it with the *injective* tensor product topology. This means that the topology is universal for separately continuous A-bilinear maps $\beta\colon V\times W\to U$ where V,W and U are locally convex A-modules. Then one takes the Hausdorff completion.

Proof. Indeed $\mathscr{D}(H,A)\widehat{\otimes}_{A,\iota}M$, being Hausdorff and complete by definition, it suffices to remark that by tensoring the identity map on M with (36) we obtain a continuous A-linear map

$$\mathscr{D}(H,A)\widehat{\otimes}_{A,\iota}(\mathscr{D}(H,A)\widehat{\otimes}_{A,\iota}M) \cong (\mathscr{D}(H,A)\widehat{\otimes}_{A,\iota}\mathscr{D}(H,A))\widehat{\otimes}_{A,\iota}M \to \mathscr{D}(H,A)\widehat{\otimes}_{A,\iota}M.$$

□

We'll call an object of the form $\mathscr{D}(H,A)\widehat{\otimes}_{A,\iota}M$ (for M any complete Hausdorff locally convex A-module) in $\mathscr{G}_{H,A}$ *s-free*. Notice that s-free does not imply it is free as an A-module. As one expects, s-projective modules can be viewed as direct summands of an s-free module.

Lemma A.44. *An object P of $\mathscr{G}_{H,A}$ is s-projective if and only if it is a direct summand (in $\mathscr{G}_{H,A}$) of an s-free module.*

Proof. First note that for any complete Hausdorff locally convex A-module M and any object N of $\mathscr{G}_{H,A}$ there is a natural continuous A-linear bijection

$$\mathrm{Hom}_{\mathscr{G}_{H,A}}(\mathscr{D}(H,A)\widehat{\otimes}_{A,\iota}M, N) \to \mathrm{Hom}_{A,\mathrm{cont}}(M,N).$$

This is the same proof as the first paragraph of the proof of Lemma A.26 (with A replaced by $\mathscr{D}(H,A)$). The result now follows from [63, Proposition 1.4]. □

We will be interested in considering the cohomology of objects in $\mathscr{G}_{H,A}$ and so we need the notion of a resolution.

Definition A.45. *If M is an object of $\mathscr{G}_{H,A}$ then by an* s-projective s-resolution *of M we mean an s-exact sequence*

$$\cdots \to X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} M$$

in $\mathscr{G}_{H,A}$ in which all objects X_i are s-projective.

For an object M of $\mathscr{G}_{H,A}$ let $B_{-1}(H,M) := M$ and for $q \geq 0$ let

$$B_q(H,M) := \mathscr{D}(H,A)\widehat{\otimes}_{A,\iota}B_{q-1}(H,M)$$

with its structure of an s-free module. For $q \geq 0$ define

$$d_q(\delta_0\otimes\cdots\otimes\delta_q\otimes m) := \sum_{i=0}^{q-1}(-1)^i\delta_0\otimes\ldots\delta_i\delta_{i+1}\otimes\cdots\otimes\delta_q\otimes m+(-1)^q\delta_0\otimes\cdots\otimes\delta_{q-1}\otimes\delta_q m.$$

Lemma A.46. *For any object M of $\mathscr{G}_{H,A}$ the sequence $(B_q(H,M),d_q)_{q\geq 0}$ is an s-projective s-resolution of M in $\mathscr{G}_{H,A}$.*

Proof. This is an immediate consequence of [43, Proposition 2.4]. The critical point is that by Lemma A.34

$$B_q(H,M) = \mathscr{D}(H,A)\widehat{\otimes}_{A,\iota}B_{q-1}(H,M) = \mathscr{D}(H,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota}B_{q-1}(H,M)$$

and so the $B_q(H,M)$ defined above coincide with the ones defined in [43]. One proves that the splitting is as A-modules and not just as $\mathbf{Q}_p$-vector spaces by exhibiting a contracting homotopy consisting of continuous A-linear maps, cf. [63, §2]. □

Definition A.47. *If M and N are objects of $\mathscr{G}_{H,A}$ we define* $\operatorname{Ext}^q_{\mathscr{G}_{H,A}}(M,N)$ *to be the qth cohomology group of the complex* $\operatorname{Hom}_{\mathscr{G}_{H,A}}(B_\bullet(H,M),N)$ *for any* $q \geq 0$.

Remark A.48. Definition A.47 is independent of the s-projective s-resolution one takes for M. To see this it suffices to show that if M is s-projective and N any object of $\mathscr{G}_{H,A}$ then $\operatorname{Ext}^q_{\mathscr{G}_{H,A}}(M,N)$ is trivial for all $q > 0$. The proof is the same as [63, Proposition 2.2(b)].

Remark A.49. For any two objects M and N of $\mathscr{G}_{H,A}$, as usual $\operatorname{Ext}^1_{\mathscr{G}_{H,A}}(M,N)$ is the set of equivalence classes of s-exact sequences

$$0 \to N \to E \to M \to 0$$

with objects E of $\mathscr{G}_{H,A}$. The point is that for P an object of $\mathscr{G}_{H,A}$, there are natural maps $\delta^*\colon \operatorname{Ext}^q_{\mathscr{G}_{H,A}}(P,M) \to \operatorname{Ext}^{q+1}_{\mathscr{G}_{H,A}}(P,N)$ such that

$$0 \to \operatorname{Hom}_{\mathscr{G}_{H,A}}(P,N) \to \operatorname{Hom}_{\mathscr{G}_{H,A}}(P,E) \to \operatorname{Hom}_{\mathscr{G}_{H,A}}(P,M) \xrightarrow{\delta^*} \operatorname{Ext}^1_{\mathscr{G}_{H,A}}(P,N) \to \cdots$$

is exact. To construct a bijection between $\operatorname{Ext}^1_{\mathscr{G}_{H,A}}(M,N)$ and the set of equivalence classes of s-exact sequences $S\colon 0 \to N \to E \to M \to 0$, one takes $P = M$ and sends S to $\delta^*(\mathrm{id}_M)$. One then checks that this gives a bijection.

As in the setting of [43], one can identify the categories of separately continuous left and right $\mathscr{D}(H,A)$-modules. If M and N are objects of $\mathscr{G}_{H,A}$, M a right module, we define $M \widetilde{\otimes}_{\mathscr{D}(H,A),\iota} N$ to be the quotient of $M \widehat{\otimes}_{A,\iota} N$ by the image of the natural map

$$\begin{aligned} M \widehat{\otimes}_{A,\iota} \mathscr{D}(H,A) \widehat{\otimes}_{A,\iota} N &\to M \widehat{\otimes}_{A,\iota} N \\ m \otimes \delta \otimes n &\mapsto m\delta \otimes n - m \otimes \delta n, \end{aligned}$$

where $m \in M$, $n \in N$ and $\delta \in \mathscr{D}(H,A)$. The induced topology is the quotient topology.

Lemma A.50. *For any complete Hausdorff locally convex A-module M and any object N of $\mathscr{G}_{H,A}$ there is a natural A-linear topological isomorphism*

$$(M \widehat{\otimes}_{A,\iota} \mathscr{D}(H,A)) \widetilde{\otimes}_{\mathscr{D}(H,A),\iota} N \cong M \widehat{\otimes}_{A,\iota} N.$$

If the object P of $\mathscr{G}_{H,A}$ is s-projective then the functor $P \widetilde{\otimes}_{\mathscr{D}(H,A),\iota} (\cdot)$ takes s-exact sequences in $\mathscr{G}_{H,A}$ to exact sequences of A-modules. If P is s-free this functor takes s-exact sequences in $\mathscr{G}_{H,A}$ to s-exact sequences in LCS_A.

Proof. The first part is [63, Proposition 1.5]. The second part follows from that fact that $(-)\widehat{\otimes}_{\mathbf{Q}_p,\iota} M$ preserves the s-exactness of sequences of locally convex A-modules and Lemma A.44. ◻

Let H_1 be a locally $\mathbf{Q}_p$-analytic group and let H_2 be a closed locally $\mathbf{Q}_p$-analytic subgroup. For an object M of $\mathscr{G}_{H_2,A}$ we set

$$\mathrm{ind}_{H_2}^{H_1}(M) := \mathscr{D}(H_1, A)\widetilde{\otimes}_{\mathscr{D}(H_2,A),\iota} M. \tag{37}$$

For (37) to be a functor [81], we need the following lemma.

Lemma A.51. *The (right) $\mathscr{D}(H_2, A)$-module $\mathscr{D}(H_1, A)$ is s-free. In particular there is an A-linear topological isomorphism*

$$\mathscr{D}(H_1, A) \cong \mathscr{D}(H_1/H_2, A)\widehat{\otimes}_{A,\iota}\mathscr{D}(H_2, A).$$

Proof. The proof of [43, Lemma 5.2] gives that

$$\mathscr{D}(H_1, \mathbf{Q}_p) \cong \mathscr{D}(H_1/H_2, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota}\mathscr{D}(H_2, \mathbf{Q}_p) \tag{38}$$

We now compute

$$\begin{aligned}
\mathscr{D}(H_1, A) &\overset{(i)}{=} \mathscr{D}(H_1, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota} A \\
&\overset{(ii)}{=} \mathscr{D}(H_1/H_2, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota}\mathscr{D}(H_2, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota} A \\
&\overset{(iii)}{=} \mathscr{D}(H_1/H_2, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota}\mathscr{D}(H_2, A) \\
&= \mathscr{D}(H_1/H_2, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota} A\widehat{\otimes}_{A,\iota}\mathscr{D}(H_2, A) \\
&\overset{(iv)}{=} \mathscr{D}(H_1/H_2, A)\widehat{\otimes}_{A,\iota}\mathscr{D}(H_2, A)
\end{aligned}$$

where (i), (iii) and (iv) follows from Lemma A.34, and (ii) follows from (38). This completes the proof. ☐

We are now ready to prove the following lemma. The proof is similar to the proof of [43, Proposition 5.1].

Lemma A.52. *The functor*

$$\begin{aligned}
\mathrm{ind}_{H_2}^{H_1} \colon \mathscr{G}_{H_2,A} &\to \mathscr{G}_{H_1,A} \\
M &\mapsto \mathrm{ind}_{H_2}^{H_1}(M)
\end{aligned}$$

takes s-exact sequences to s-exact sequences and s-projective objects to s-projective objects.

Proof. Lemmas A.50 and A.51 imply that there is a natural A-linear topological isomorphism

$$\mathrm{ind}_{H_2}^{H_1}(M) = \mathscr{D}(H_1/H_2, A)\widehat{\otimes}_{A,\iota} M. \tag{39}$$

Thus $\mathrm{ind}_{H_2}^{H_1}(M)$ is Hausdorff and complete. Its structure of a separately continuous $\mathscr{D}(H_1, A)$-module is the one induced from the s-free module $\mathscr{D}(H_1, A)\widehat{\otimes}_{A,\iota} M$. The final assertion follows from Lemmas A.50 and A.51, and the fact that $\mathrm{ind}_{H_2}^{H_1}(M)$ respects direct sums. ☐

81. It is a priori not clear that $\mathrm{ind}_{H_2}^{H_1}(M)$ is an object of $\mathscr{G}_{H_1,A}$.

Lemma A.53 (Relative Frobenius reciprocity)**.** *If M and N are objects of $\mathscr{G}_{H_2,A}$ and $\mathscr{G}_{H_1,A}$, respectively, then there is an A-linear bijection*

$$\mathrm{Hom}_{\mathscr{G}_{H_1,A}}(\mathrm{ind}_{H_2}^{H_1}(M), N) \to \mathrm{Hom}_{\mathscr{G}_{H_2,A}}(M, N)$$

Proof. From the proof of Lemma A.44 we have an A-linear bijection

$$\begin{aligned}\alpha\colon \mathrm{Hom}_{\mathscr{G}_{H_1,A}}(\mathscr{D}(H_1,A)\widehat{\otimes}_{A,\iota}M, N) &\to \mathrm{Hom}_{A,\mathrm{cont}}(M,N)\\ g &\mapsto \alpha(g)\end{aligned}$$

where $\alpha(g)(m) := g(1\otimes m)$. It follows directly from the definitions that a continuous $\mathscr{D}(H_1,A)$-linear map g from the left factors through $\mathrm{ind}_{H_2}^{H_1}(M)$ $(= \mathscr{D}(H_1/H_2,A)\widehat{\otimes}_{A,\iota}M)$ if and only if $\alpha(g)$ is $\mathscr{D}(H_2,A)$-linear. □

Proposition A.54 (Relative Shapiro's Lemma)**.** *Let H_1 be a locally $\mathbf{Q}_p$-analytic group and let H_2 be a closed locally $\mathbf{Q}_p$-analytic subgroup. If M and N are objects of $\mathscr{G}_{H_2,A}$ and $\mathscr{G}_{H_1,A}$, respectively, then there are A-linear bijections*

$$\mathrm{Ext}^q_{\mathscr{G}_{H_1,A}}(\mathrm{ind}_{H_2}^{H_1}(M), N) \to \mathrm{Ext}^q_{\mathscr{G}_{H_2,A}}(M, N)$$

for all $q \geq 0$.

Proof. Choose an s-projective s-resolution $X_\bullet \to M$ in $\mathscr{G}_{H_2,A}$ (e.g. $(B_q(H,M),d_q)_{q\geq 0}$). By Lemma A.52, the complex $\mathrm{ind}_{H_2}^{H_1}(X_\bullet) \to \mathrm{ind}_{H_2}^{H_1}(M)$ is an s-projective s-resolution of $\mathrm{ind}_{H_2}^{H_1}(M)$. By Lemma A.53 there is an A-linear isomorphism of complexes

$$\mathrm{Hom}_{\mathscr{G}_{H_1,A}}(\mathrm{ind}_{H_2}^{H_1}(X_\bullet), N) \to \mathrm{Hom}_{\mathscr{G}_{X_{H_2},A}}(X_\bullet, N).$$

The result now follows. □

The next result relates locally analytic induction $\mathrm{Ind}_{H_2}^{H_1}$, cf. Remark A.25 and the functor $\mathrm{ind}_{H_2}^{H_1}$.

Lemma A.55. *Let $\delta\colon H_2 \to A^\times$ be a locally analytic character and suppose H_1/H_2 is compact and of dimension 1. Then $\mathrm{Ind}_{H_2}^{H_1}\delta$ and $\left(\mathrm{Ind}_{H_2}^{H_1}\delta\right)'_b$ are objects of $\mathscr{G}_{H_1,A}$ (where $\left(\mathrm{Ind}_{H_2}^{H_1}\delta\right)'_b$ is equipped with H_1-action: $(h_1\cdot F)(f) := F(h_1^{-1}\cdot f)$ for $F \in \left(\mathrm{Ind}_{H_2}^{H_1}\delta\right)'_b$, $f \in \mathrm{Ind}_{H_2}^{H_1}\delta$ and $h_1 \in H_1$) , and we have an isomorphism*

$$\left(\mathrm{Ind}_{H_2}^{H_1}\delta\right)'_b \cong \mathrm{ind}_{H_2}^{H_1}\delta^{-1}$$

in the category $\mathscr{G}_{H_1,A}$.

Proof. Indeed by Remark A.25, $\mathrm{Ind}_{H_2}^{H_1}\delta \cong \mathrm{LA}(H_1/H_2, A)$ as locally convex A-modules. By Remark A.19 [82] the latter is a complete locally convex A-module

82. It is important that H_1/H_2 is of dimension 1 here.

of A-LB-type. Thus $\mathrm{Ind}_{H_2}^{H_1}\delta$ is an object of $\mathrm{Rep}_A^{\mathrm{la},LB}(H_1)$. By Corollary A.38, it follows that $\mathrm{Ind}_{H_2}^{H_1}\delta$ is an object of $\mathscr{G}_{H_1,A}$ as claimed.

Now as locally convex A-modules

$$\begin{aligned}\left(\mathrm{Ind}_{H_2}^{H_1}\delta\right)'_b &\cong \mathrm{LA}(H_1/H_2,A)'_b\\ &\overset{(i)}{\cong} \mathscr{D}(H_1/H_2,A)\\ &\overset{(ii)}{\cong} \mathrm{ind}_{H_2}^{H_1}\delta^{-1}\end{aligned}$$

where (i) is by definition and (ii) follows from (39). By [27, §4.3] we get a continuous H_1-equivariant A-linear composition [83]

$$\mathrm{ind}_{H_2}^{H_1}\delta^{-1} \hookrightarrow \mathscr{D}(H_1,A) \twoheadrightarrow \left(\mathrm{Ind}_{H_2}^{H_1}\delta\right)'_b, \tag{40}$$

which is an isomorphism.

Thus we get a continuous H_1-equivariant A-linear topological isomorphism

$$\alpha\colon \mathrm{ind}_{H_2}^{H_1}\delta^{-1} \xrightarrow{\sim} \left(\mathrm{Ind}_{H_2}^{H_1}\delta\right)'_b.$$

Thus $\left(\mathrm{Ind}_{H_2}^{H_1}\delta\right)'_b$ is a locally analytic H_1-representation in A-modules. By [53, Proposition 3.2], it follows that $\left(\mathrm{Ind}_{H_2}^{H_1}\delta\right)'_b$ is an object of $\mathscr{G}_{H_1,A}$. Since (40) is H_1-invariant, by continuity, α is $\mathscr{D}(H_1,A)$-linear. This completes the proof. □

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