Non-abelian Lubin-Tate theory and elliptic representations

October 10th, 2007

What is non abelian Lubin-Tate theory ?

What is non abelian Lubin-Tate theory ?

Notation : K a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field k. Choose alg. closure $\overline{K}$ and let $W_K \subset \mathrm{Gal}(\overline{K}|K)$ be the Weil group.

What is non abelian Lubin-Tate theory ?

Notation : K a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field k. Choose alg. closure $\overline{K}$ and let $W_K \subset \mathrm{Gal}(\overline{K}|K)$ be the Weil group.

Lubin-Tate '65 : constructed explicitly abelian extensions of K via torsion points of one dimensional formal Lie groups.

What is non abelian Lubin-Tate theory ?

Notation : K a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field k. Choose alg. closure $\overline{K}$ and let $W_K \subset \mathrm{Gal}(\overline{K}|K)$ be the Weil group.

Lubin-Tate '65 : constructed explicitly abelian extensions of K via torsion points of one dimensional formal Lie groups.

Langlands '67 : local class field theory should be the case $d = 1$ of a family of correspondences between

- d-dimensional continuous representations of W_K
- smooth irreducible representations of $GL_d(K)$

What is non abelian Lubin-Tate theory ?

Notation : K a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field k. Choose alg. closure $\overline{K}$ and let $W_K \subset \mathrm{Gal}(\overline{K}|K)$ be the Weil group.

Lubin-Tate '65 : constructed explicitly abelian extensions of K via torsion points of one dimensional formal Lie groups.

Langlands '67 : local class field theory should be the case $d = 1$ of a family of correspondences between

- d-dimensional continuous representations of W_K
- smooth irreducible representations of $GL_d(K)$

NALT : aims at realizing Langlands' correspondences via étale cohomology of suitable moduli spaces of formal Lie groups.

An example : the Lubin-Tate tower

Lubin-Tate '66 : systematic study of 1-dimensional *formal $\mathcal{O}$-modules*.

An example : the Lubin-Tate tower

Lubin-Tate '66 : systematic study of 1-dimensional *formal $\mathcal{O}$-modules*.

1. Over $\overline{k}$, formal $\mathcal{O}$-modules are classified by their height. Say $d \in \mathbb{N} \mapsto \mathbb{H}_d$. The automorphism group of $\mathbb{H}_d$ is the group $\mathcal{O}_{D_d}^{\times}$ of invertible elements in the ring of integers of the division algebra with invariant $1/d$ over K.

An example : the Lubin-Tate tower

Lubin-Tate '66 : systematic study of 1-dimensional *formal $\mathcal{O}$-modules*.

1. Over $\overline{k}$, formal $\mathcal{O}$-modules are classified by their height. Say $d \in \mathbb{N} \mapsto \mathbb{H}_d$. The automorphism group of $\mathbb{H}_d$ is the group $\mathcal{O}_{D_d}^{\times}$ of invertible elements in the ring of integers of the division algebra with invariant $1/d$ over K.
2. The deformation space of $\mathbb{H}_d$ is a $W(\overline{k})$-analytic open disk of dimension $d-1$.

An example : the Lubin-Tate tower

Lubin-Tate '66 : systematic study of 1-dimensional *formal $\mathcal{O}$-modules*.

1. Over $\overline{k}$, formal $\mathcal{O}$-modules are classified by their height. Say $d \in \mathbb{N} \mapsto \mathbb{H}_d$. The automorphism group of $\mathbb{H}_d$ is the group $\mathcal{O}_{D_d}^{\times}$ of invertible elements in the ring of integers of the division algebra with invariant $1/d$ over K.
2. The deformation space of $\mathbb{H}_d$ is a $W(\overline{k})$-analytic open disk of dimension $d-1$.

Drinfeld '74 : the "space" $\mathcal{M}_{LT}$ of torsion sections of the universal deformation has a natural action of $GL_d(K)$.

An example : the Lubin-Tate tower

Lubin-Tate '66 : systematic study of 1-dimensional *formal $\mathcal{O}$-modules*.

1. Over $\overline{k}$, formal $\mathcal{O}$-modules are classified by their height. Say $d \in \mathbb{N} \mapsto \mathbb{H}_d$. The automorphism group of $\mathbb{H}_d$ is the group $\mathcal{O}_{D_d}^{\times}$ of invertible elements in the ring of integers of the division algebra with invariant $1/d$ over K.
2. The deformation space of $\mathbb{H}_d$ is a $W(\overline{k})$-analytic open disk of dimension $d-1$.

Drinfeld '74 : the "space" $\mathcal{M}_{LT}$ of torsion sections of the universal deformation has a natural action of $GL_d(K)$.

Hence $H_c^*(\mathcal{M}_{LT}\widehat{\otimes}_K\overline{K}, \mathbb{Q}_l)$ are endowed with an action of the triple product $GL_d(K) \times D_d^{\times} \times W_K$.

Harris-Taylor theorem '99

For an irreducible *supercuspidal* representation π of $GL_d(K)$, we have

$$\mathit{Hom}_{GL_d(K)}(H_c^{d-1}(\mathcal{M}_{LT}), \pi) \underset{D^\times \times W_K}{\simeq} \mathcal{LJ}(\pi) \otimes \mathcal{L}(\pi)(?)$$

Harris-Taylor theorem '99

For an irreducible *supercuspidal* representation π of $GL_d(K)$, we have

$$\mathit{Hom}_{GL_d(K)}(H_c^{d-1}(\mathcal{M}_{LT}), \pi) \underset{D^\times \times W_K}{\simeq} \mathcal{LJ}(\pi) \otimes \mathcal{L}(\pi)(?)$$

- $\mathcal{LJ} : \mathcal{R}(G) \longrightarrow \mathcal{R}(D^\times)$ is the map between Grothendieck groups which is dual to the transfer map from conjugacy classes in $D^\times$ to conjugacy classes in G.
- $\mathcal{L}$ denotes Langlands correspondence.

Harris-Taylor theorem '99

For an irreducible *supercuspidal* representation π of $GL_d(K)$, we have

$$\mathit{Hom}_{GL_d(K)}(H_c^{d-1}(\mathcal{M}_{LT}), \pi) \underset{D^\times \times W_K}{\simeq} \mathcal{LJ}(\pi) \otimes \mathcal{L}(\pi)(?)$$

- $\mathcal{LJ} : \mathcal{R}(G) \longrightarrow \mathcal{R}(D^\times)$ is the map between Grothendieck groups which is dual to the transfer map from conjugacy classes in $D^\times$ to conjugacy classes in G.
- $\mathcal{L}$ denotes Langlands correspondence.

Supercuspidal representations have very special features

- they are the only projective/injective irreducible objects in the smooth category.
- they correspond to *irreducible* representations of W_K.

Harris-Taylor theorem '99

For an irreducible *supercuspidal* representation π of $GL_d(K)$, we have

$$Hom_{GL_d(K)}(H_c^{d-1}(\mathcal{M}_{LT}),\pi) \underset{D^\times \times W_K}{\simeq} \mathcal{LJ}(\pi)\otimes\mathcal{L}(\pi)(?)$$

- $\mathcal{LJ} : \mathcal{R}(G) \longrightarrow \mathcal{R}(D^\times)$ is the map between Grothendieck groups which is dual to the transfer map from conjugacy classes in $D^\times$ to conjugacy classes in G.
- $\mathcal{L}$ denotes Langlands correspondence.

Supercuspidal representations have very special features

- they are the only projective/injective irreducible objects in the smooth category.
- they correspond to *irreducible* representations of W_K.
- all representations of $GL_d(K)$ are obtained as subquotients of parabolically induced representations from supercuspidal ones.

Recent developments

Boyer '05 : computed *all* cohomology spaces.

Recent developments

Boyer '05 : computed *all* cohomology spaces. It turns out that

- the set of representations of $GL_d(K)$ is somehow "asymetric".
- Langlands correspondence is not realized for non supercuspidals.

Recent developments

Boyer '05 : computed *all* cohomology spaces. It turns out that

- the set of representations of $GL_d(K)$ is somehow "asymetric".
- Langlands correspondence is not realized for non supercuspidals.

Idea : more information might be encoded in a suitable cohomology complex.

Recent developments

Boyer '05 : computed *all* cohomology spaces. It turns out that

- the set of representations of $GL_d(K)$ is somehow "asymetric".
- Langlands correspondence is not realized for non supercuspidals.

Idea : more information might be encoded in a suitable cohomology complex.

Dat '06 : constructed a complex $R\Gamma_c(\mathcal{M}_{LT}) \in D^b(GL_d(K))$ with action of $D_d^\times \times W_K$, with the right cohomology, and showed

Recent developments

Boyer '05 : computed *all* cohomology spaces. It turns out that

- the set of representations of $GL_d(K)$ is somehow "asymetric".
- Langlands correspondence is not realized for non supercuspidals.

Idea : more information might be encoded in a suitable cohomology complex.

Dat '06 : constructed a complex $R\Gamma_c(\mathcal{M}_{LT}) \in D^b(GL_d(K))$ with action of $D_d^\times \times W_K$, with the right cohomology, and showed

$$RHom_{D^b(GL_d)}(R\Gamma_c(\mathcal{M}_{LT}), \pi) \underset{D^\times \times W_K}{\simeq} \mathcal{LJ}(\pi) \otimes \mathcal{L}(\pi)|-|^{\frac{d-1}{2}}$$

for *any* smooth irreducible representation π of $GL_d(K)$.

Recent developments

Boyer '05 : computed *all* cohomology spaces. It turns out that

- the set of representations of $GL_d(K)$ is somehow "asymetric".
- Langlands correspondence is not realized for non supercuspidals.

Idea : more information might be encoded in a suitable cohomology complex.

Dat '06 : constructed a complex $R\Gamma_c(\mathcal{M}_{LT}) \in D^b(GL_d(K))$ with action of $D_d^\times \times W_K$, with the right cohomology, and showed

$$RHom_{D^b(GL_d)}(R\Gamma_c(\mathcal{M}_{LT}), \pi) \underset{D^\times \times W_K}{\simeq} \mathcal{LJ}(\pi) \otimes \mathcal{L}(\pi)|-|^{\frac{d-1}{2}}$$

for *any* smooth irreducible representation π of $GL_d(K)$.

Unfortunately, $\mathcal{LJ}(\pi)$ may be 0.

A toy example : Drinfeld symmetric spaces

A toy example : Drinfeld symmetric spaces

Ω^{d-1} is the complement of all K-rational hyperplanes in $\mathbb{P}^{d-1}$. It is an open K-analytic subspace of $\mathbb{P}^{d-1}$.

A toy example : Drinfeld symmetric spaces

Ω^{d-1} is the complement of all K-rational hyperplanes in $\mathbb{P}^{d-1}$. It is an open K-analytic subspace of $\mathbb{P}^{d-1}$.

Schneider-Stuhler '91 : computed the l-adic cohomology spaces of Ω^{d-1}, as $GL_d(K) \times W_K$-modules.

A toy example : Drinfeld symmetric spaces

Ω^{d-1} is the complement of all K-rational hyperplanes in $\mathbb{P}^{d-1}$. It is an open K-analytic subspace of $\mathbb{P}^{d-1}$.

Schneider-Stuhler '91 : computed the l-adic cohomology spaces of Ω^{d-1}, as $GL_d(K) \times W_K$-modules.

Interlude on representations of $GL_d(K)$

Let B be the Borel subgroup of upper triangular matrices in $GL_d(K)$, and let S be the set of simple roots of the diagonal torus in B.

Subsets of S are in natural bijection $I \mapsto P_I$ with parabolic subgroups containing B.

Interlude on representations of $GL_d(K)$ (continued)

Define $\pi_I := \mathcal{C}^\infty(G/P_I)/(\sum_{J \supset I} \mathcal{C}^\infty(G/P_J))$.

Interlude on representations of $GL_d(K)$ (continued)

Define $\pi_I := \mathcal{C}^\infty(G/P_I)/(\sum_{J \supset I} \mathcal{C}^\infty(G/P_J))$.

It is known that $I \mapsto \pi_I$ is a bijection between the set of irreducible constituents of $\mathcal{C}^\infty(G/B)$ and the set of subsets of S.

Interlude on representations of $GL_d(K)$ (continued)

Define $\pi_I := \mathcal{C}^\infty(G/P_I)/(\sum_{J \supset I} \mathcal{C}^\infty(G/P_J))$.

It is known that $I \mapsto \pi_I$ is a bijection between the set of irreducible constituents of $\mathcal{C}^\infty(G/B)$ and the set of subsets of S.

For convenience, identify $S \simeq \{1, \cdots, d-1\}$ by numbering the first upper diagonal from left to right.

Interlude on representations of $GL_d(K)$ (continued)

Define $\pi_I := \mathcal{C}^\infty(G/P_I)/(\sum_{J\supset I}\mathcal{C}^\infty(G/P_J))$.

It is known that $I \mapsto \pi_I$ is a bijection between the set of irreducible constituents of $\mathcal{C}^\infty(G/B)$ and the set of subsets of S.

For convenience, identify $S \simeq \{1,\cdots,d-1\}$ by numbering the first upper diagonal from left to right.

Schneider-Stuhler computation

for $i=0,\cdots,d-1$, $H_c^{d-1+i}(\Omega^{d-1}\widehat{\otimes}\widehat{\overline{K}},\overline{\mathbb{Q}}_l) \simeq \pi_{\{1,\cdots,i\}}(-i)$.

Interlude on representations of $GL_d(K)$ (continued)

Define $\pi_I := \mathcal{C}^\infty(G/P_I)/(\sum_{J\supset I}\mathcal{C}^\infty(G/P_J))$.

It is known that $I \mapsto \pi_I$ is a bijection between the set of irreducible constituents of $\mathcal{C}^\infty(G/B)$ and the set of subsets of S.

For convenience, identify $S \simeq \{1,\cdots,d-1\}$ by numbering the first upper diagonal from left to right.

Schneider-Stuhler computation

for $i = 0,\cdots,d-1$, $H_c^{d-1+i}(\Omega^{d-1}\widehat{\otimes}\widehat{\overline{K}},\overline{\mathbb{Q}}_l) \simeq \pi_{\{1,\cdots,i\}}(-i)$.

The $GL_d(K)$ action is asymetric and the Galois action is not interesting.

Interlude on representations of $GL_d(K)$ (continued)

Define $\pi_I := \mathcal{C}^\infty(G/P_I)/(\sum_{J\supset I}\mathcal{C}^\infty(G/P_J))$.

It is known that $I \mapsto \pi_I$ is a bijection between the set of irreducible constituents of $\mathcal{C}^\infty(G/B)$ and the set of subsets of S.

For convenience, identify $S \simeq \{1, \cdots, d-1\}$ by numbering the first upper diagonal from left to right.

Schneider-Stuhler computation

for $i = 0, \cdots, d-1$, $H_c^{d-1+i}(\Omega^{d-1}\widehat{\otimes}\widehat{\overline{K}}, \overline{\mathbb{Q}}_l) \simeq \pi_{\{1,\cdots,i\}}(-i)$.

The $GL_d(K)$ action is asymetric and the Galois action is not interesting.

But *uniformization theory* suggests that more information should be encoded in a suitable cohomology complex.

Interlude on representations of $GL_d(K)$ (continued)

Define $\pi_I := \mathcal{C}^\infty(G/P_I)/(\sum_{J \supset I} \mathcal{C}^\infty(G/P_J))$.

It is known that $I \mapsto \pi_I$ is a bijection between the set of irreducible constituents of $\mathcal{C}^\infty(G/B)$ and the set of subsets of S.

For convenience, identify $S \simeq \{1, \cdots, d-1\}$ by numbering the first upper diagonal from left to right.

Schneider-Stuhler computation

for $i = 0, \cdots, d-1$, $H_c^{d-1+i}(\Omega^{d-1} \widehat{\otimes} \widehat{\overline{K}}, \overline{\mathbb{Q}}_l) \simeq \pi_{\{1,\cdots,i\}}(-i)$.

The $GL_d(K)$ action is asymetric and the Galois action is not interesting.

But *uniformization theory* suggests that more information should be encoded in a suitable cohomology complex. Want a complex $R\Gamma_c(\Omega^{d-1} \widehat{\otimes} \widehat{\overline{K}}, \overline{\mathbb{Q}}_l)$ in the derived category $D^b(G)$ of *smooth* representations.

Interlude on representations of $GL_d(K)$ (continued)

Define $\pi_I := \mathcal{C}^\infty(G/P_I)/(\sum_{J\supset I}\mathcal{C}^\infty(G/P_J))$.

It is known that $I \mapsto \pi_I$ is a bijection between the set of irreducible constituents of $\mathcal{C}^\infty(G/B)$ and the set of subsets of S.

For convenience, identify $S \simeq \{1,\cdots,d-1\}$ by numbering the first upper diagonal from left to right.

Schneider-Stuhler computation

for $i=0,\cdots,d-1$, $H_c^{d-1+i}(\Omega^{d-1}\widehat{\otimes}\widehat{\overline{K}},\overline{\mathbb{Q}}_l) \simeq \pi_{\{1,\cdots,i\}}(-i)$.

The $GL_d(K)$ action is asymetric and the Galois action is not interesting.

But *uniformization theory* suggests that more information should be encoded in a suitable cohomology complex. Want a complex $R\Gamma_c(\Omega^{d-1}\widehat{\otimes}\widehat{\overline{K}},\overline{\mathbb{Q}}_l)$ in the derived category $D^b(G)$ of *smooth* representations.

Observation : because weights are distinct, $R\Gamma_c$ is split in $D^b(G)$.

Theorem

(Holds for any split semi-simple G) For all $I, J \subseteq S$, put $\delta(I,J) := |I \cup J| - |I \cap J|$.

- *Let I, J be two subsets of S, then :*

$$\mathit{Ext}^*_G(\pi_I, \pi_J) \quad = \quad \begin{cases} \overline{\mathbb{Q}}_l & \textit{if} \quad * = \delta(I,J) \\ 0 & \textit{if} \quad * \neq \delta(I,J) \end{cases} \quad .$$

Theorem

(Holds for any split semi-simple G) For all $I, J \subseteq S$, put $\delta(I,J) := |I \cup J| - |I \cap J|$.

- *Let I, J be two subsets of S, then :*

$$\mathit{Ext}^*_G(\pi_I, \pi_J) \quad = \quad \begin{cases} \overline{\mathbb{Q}}_l & \text{if} \quad * = \delta(I,J) \\ 0 & \text{if} \quad * \neq \delta(I,J) \end{cases} \quad .$$

- *Let I, J, K be three subsets of S such that $\delta(I,J) + \delta(J,K) = \delta(I,K)$, then the cup-product*

$$\cup \ : \ \mathit{Ext}^{\delta(I,J)}_G(\pi_I, \pi_J) \otimes_{\overline{\mathbb{Q}}_l} \mathit{Ext}^{\delta(J,K)}_G(\pi_J, \pi_K) \longrightarrow \mathit{Ext}^{\delta(I,K)}_G(\pi_I, \pi_K)$$

is an isomorphism.

Corollary

The algebra End$_{D^b(G)}(R\Gamma_c(\Omega^{d-1}, \overline{\mathbb{Q}}_l))$ *is isomorphic to the algebra of* $d \times d$ *upper triangular matrices.*

Corollary

The algebra $\mathit{End}_{D^b(G)}(R\Gamma_c(\Omega^{d-1}, \overline{\mathbb{Q}}_l))$ *is isomorphic to the algebra of* $d \times d$ *upper triangular matrices.*

We may choose an isomorphism so that the action of a fixed Frobenius lifting ϕ in W_F be diagonal and that of the inertia group I_K be given by the formula $i \mapsto \exp(t_l(i)N)$ with N a nilpotent matrix in Jordan's form.

Corollary

The algebra $\mathrm{End}_{D^b(G)}(R\Gamma_c(\Omega^{d-1}, \overline{\mathbb{Q}}_l))$ *is isomorphic to the algebra of* $d \times d$ *upper triangular matrices.*

We may choose an isomorphism so that the action of a fixed Frobenius lifting ϕ in W_F be diagonal and that of the inertia group I_K be given by the formula $i \mapsto \exp(t_l(i)N)$ with N a nilpotent matrix in Jordan's form.

Proposition

The nilpotent matrix is the regular one. Equivalently we have $N^{d-1} \neq 0$*.*

Corollary

The algebra $\mathrm{End}_{D^b(G)}(R\Gamma_c(\Omega^{d-1}, \overline{\mathbb{Q}}_l))$ *is isomorphic to the algebra of* $d \times d$ *upper triangular matrices.*

We may choose an isomorphism so that the action of a fixed Frobenius lifting ϕ in W_F be diagonal and that of the inertia group I_K be given by the formula $i \mapsto \exp(t_l(i)N)$ with N a nilpotent matrix in Jordan's form.

Proposition

The nilpotent matrix is the regular one. Equivalently we have $N^{d-1} \neq 0$.

Follows from the formula
$\overline{\mathbb{Q}}_l \otimes^{L}_{\overline{\mathbb{Q}}_l[\Gamma]} R\Gamma_c(\Omega^{d-1} \widehat{\otimes} \widehat{\overline{K}}, \overline{\mathbb{Q}}_l) \simeq R\Gamma(\Omega^{d-1}/\Gamma \otimes \widehat{\overline{K}}, \overline{\mathbb{Q}}_l)$.

A simple but somehow miraculous computation gives :

Corollary

For each $I \subseteq S$, we have

$$R\mathit{Hom}_{D^b(G)}(R\Gamma_c(\Omega^{d-1}, \overline{\mathbb{Q}}_l), \pi_I)[1-d] \xrightarrow{\sim} \bigoplus_{k=0}^{|I|} \mathsf{Sp}_{d_k}(i_k)[-|I|+2k]$$

A simple but somehow miraculous computation gives :

Corollary

For each $I \subseteq S$, we have

$$RHom_{D^b(G)}(R\Gamma_c(\Omega^{d-1}, \overline{\mathbb{Q}}_l), \pi_I)[1-d] \xrightarrow{\sim} \bigoplus_{k=0}^{|I|} Sp_{d_k}(i_k)[-|I|+2k]$$

Forgetting the graded structure, this gives

$$\mathcal{H}^*(RHom_{D^b(G)}(R\Gamma_c(\Omega^{d-1}, \overline{\mathbb{Q}}_l), \pi_I)) \xrightarrow{\sim} \mathcal{L}(\pi_I)(\frac{d-1}{2})$$

where $\mathcal{L}$ denotes local Langlands correspondence.

Ongoing problems

- Find spaces to achieve geometric realization of Langlands' correspondence for *all* representations.
- What happens for $\overline{\mathbb{F}}_l$ coefficients ? Link with Broué's conjecture for Deligne-Lusztig varieties.
- What about other period domains, other RZ spaces ?