Non-abelian Lubin-Tate theory and elliptic representations

October 10th, 2007
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NALT: aims at realizing Langlands’ correspondences via étale cohomology of suitable moduli spaces of formal Lie groups.
An example: the Lubin-Tate tower

Lubin-Tate ’66: systematic study of 1-dimensional formal $\mathcal{O}$-modules.

1. Over $k$, formal $\mathcal{O}$-modules are classified by their height. Say $d \in \mathbb{N} \mapsto H^d$. The automorphism group of $H^d$ is the group $\mathcal{O} \times D_d$ of invertible elements in the ring of integers of the division algebra with invariant $1/d$ over $K$.

2. The deformation space of $H^d$ is a $W(k)$-analytic open disk of dimension $d-1$.

Drinfeld ’74: the “space” $M_{LT}$ of torsion sections of the universal deformation has a natural action of $GL_d(K)$.

Hence $H^\ast_c(M_{LT} \hat{\otimes} K, \mathbb{Q}_l)$ are endowed with an action of the triple product $GL_d(K) \times D \times D_d \times W_K$. 
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Harris-Taylor theorem '99

For an irreducible supercuspidal representation $\pi$ of $GL_d(K)$, we have

$$\text{Hom}_{GL_d(K)}(H_c^{d-1}(\mathcal{M}_{LT}), \pi) \xrightarrow{\sim} \mathcal{LJ}(\pi) \otimes \mathcal{L}(\pi)(?)$$
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- $\mathcal{L} \mathcal{J} : \mathcal{R}(G) \longrightarrow \mathcal{R}(D^{\times})$ is the map between Grothendieck groups which is dual to the transfer map from conjugacy classes in $D^{\times}$ to conjugacy classes in $G$.
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Supercuspidal representations have very special features
- they are the only projective/injective irreducible objects in the smooth category.
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- they are the only projective/injective irreducible objects in the smooth category.
- they correspond to irreducible representations of $W_K$.
- all representations of $GL_d(K)$ are obtained as subquotients of parabolically induced representations from supercuspidal ones.
Recent developments

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Unfortunately, $\mathcal{L}\mathcal{J}(\pi)$ may be 0.
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Interlude on representations of \( GL_d(K) \)

Let \( B \) be the Borel subgroup of upper triangular matrices in \( GL_d(K) \), and let \( S \) be the set of simple roots of the diagonal torus in \( B \).

Subsets of \( S \) are in natural bijection \( I \mapsto P_I \) with parabolic subgroups containing \( B \).
Interlude on representations of $GL_d(K)$ (continued)

Define $\pi_I := \mathcal{C}^\infty(G/P_I)/(\sum_{J \supset I} \mathcal{C}^\infty(G/P_J))$. 

It is known that $I \mapsto \pi_I$ is a bijection between the set of irreducible constituents of $\mathcal{C}^\infty(G/B)$ and the set of subsets of $S$. 

For convenience, identify $S \cong \{1, \ldots, d-1\}$ by numbering the first upper diagonal from left to right. 

Schneider-Stuhler computation for $i = 0, \ldots, d-1$, $H_{d-1+i}c(\Omega_{d-1}^\hat{\otimes}^\hat{\otimes} K, \mathbb{Q}_l) \cong \pi\{1, \ldots, i\}(-i)$. 

The $GL_d(K)$ action is asymmetric and the Galois action is not interesting. But uniformization theory suggests that more information should be encoded in a suitable cohomology complex. 

Want a complex $R\Gamma_c(\Omega_{d-1}^\hat{\otimes} K, \mathbb{Q}_l)$ in the derived category $D^b(G)$ of smooth representations. 

Observation: because weights are distinct, $R\Gamma_c$ is split in $D^b(G)$. 

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Theorem

(Holds for any split semi-simple $G$) For all $I, J \subseteq S$, put
\[ \delta(I, J) := |I \cup J| - |I \cap J|. \]

Let $I, J$ be two subsets of $S$, then:

\[ \text{Ext}_G^*(\pi_I, \pi_J) = \begin{cases} \overline{\mathbb{Q}}_l & \text{if } * = \delta(I, J) \\ 0 & \text{if } * \neq \delta(I, J) \end{cases}. \]
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- Let $I, J, K$ be three subsets of $S$ such that

$$\delta(I, J) + \delta(J, K) = \delta(I, K),$$

then the cup-product

$$\cup : \Ext^\delta(I, J)_G(\pi_I, \pi_J) \otimes_{\overline{\mathbb{Q}}_l} \Ext^\delta(J, K)_G(\pi_J, \pi_K) \longrightarrow \Ext^\delta(I, K)_G(\pi_I, \pi_K)$$

is an isomorphism.
Corollary

The algebra $\text{End}_{D^b(G)}(R\Gamma_c(\Omega^{d-1}, Q_l))$ is isomorphic to the algebra of $d \times d$ upper triangular matrices.
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We may choose an isomorphism so that the action of a fixed Frobenius lifting $\phi$ in $W_F$ be diagonal and that of the inertia group $I_K$ be given by the formula $i \mapsto \exp(t_l(i)N)$ with $N$ a nilpotent matrix in Jordan’s form.
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Proposition

The nilpotent matrix is the regular one. Equivalently we have $N^{d-1} \neq 0$. 
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Follows from the formula

$$\overline{Q}_l \otimes_{\overline{Q}_l[\Gamma]} R\Gamma_c(\Omega^{d-1} \hat{\otimes} \hat{K}, \overline{Q}_l) \simeq R\Gamma(\Omega^{d-1}/\Gamma \otimes \hat{K}, \overline{Q}_l).$$
A simple but somehow miraculous computation gives:

**Corollary**

*For each* $I \subseteq S$, we have

$$\text{RHom}_{D^b(G)}(R\Gamma_c(\Omega^{d-1}, \overline{\mathbb{Q}}_l), \pi_I)[1 - d] \sim \bigoplus_{k=0}^{\lvert I \rvert} \text{Sp}_{d_k}(i_k)[-\lvert I \rvert + 2k]$$
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Forgetting the graded structure, this gives

$$\mathcal{H}^*(\text{RHom}_{D^b(G)}(R\Gamma_c(\Omega^{d-1}, \mathbb{Q}_l), \pi_I)) \sim \mathcal{L}(\pi_I)(\frac{d - 1}{2})$$

where $\mathcal{L}$ denotes local Langlands correspondence.
Ongoing problems

- Find spaces to achieve geometric realization of Langlands’ correspondence for all representations.
- What happens for $\overline{\mathbb{F}}_l$ coefficients? Link with Broué’s conjecture for Deligne-Lusztig varieties.
- What about other period domains, other RZ spaces?