Finiteness properties of Hecke algebras of $p$-adic groups

June 8th, 2007
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**Proposition** Assume $|G| \in R^\times$ and $R$ noetherian. Then $R[H\backslash G/H]$ is a finitely generated module over its center which is a finitely generated $R$-algebra.
Second version of main results. Assume $R$ noetherian, $p \in R^\times$ and let $\text{Mod}_R(G)$ be the category of all smooth $RG$-modules.

**Theorem**  Assume $G$ classical. Then any finitely generated $V \in \text{Mod}_R(G)$ is noetherian.
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Let $Z(Mod_R(G))$ be the center of the category $Mod_R(G)$.

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Strategy of proof : inductive arguments using parabolic induction and restriction functors. To initialize the induction let $Cusp_R(G)$ be the subcategory of cuspidal objects and $Z_G$ be the center of $G$.

**Lemma**  Any f.g. $V \in Cusp_R(G)$ is $R[Z_G]$-admissible.
Bernstein’s strategy revisited

If $P = M.U \subset G$ is a parabolic subgroup, note $i_P$ and $r_P$ the parabolic functors. We have ring morphisms

$$\mathcal{Z}(\text{Cusp}_R(M)) \longrightarrow \text{End}(i_P|\text{Cusp}_R(M)) \longleftarrow \mathcal{Z}(\text{Mod}_R(G)).$$
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Assume the following two properties hold :

- $\text{Mod}_R(G) = \bigoplus_{\{M\}} i_P(\text{Cusp}_R(M))^{ab}$.
- $r_P$ is faithful on $i_P(\text{Cusp}_R(M))^{ab}$. 
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Then $\alpha$ is an isomorphism and the statement of Proposition follows easily.
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Frobenius reciprocity gives such a filtration: refine the partial order on standard parabolic subgroups in a total order $P_1, \cdots, P_g$. Then put $\mathcal{F}^0 := \text{Id}$ and $\mathcal{F}^i := \ker(\mathcal{F}^{i-1} \xrightarrow{\text{Adj}} i_{P_i} \circ r_{P_i} \circ \mathcal{F}^{i-1})$. 

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Unfortunately, both the available proofs use noetherian properties!
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We now have an increasing filtration $\mathcal{F}_\bullet$ of $\text{Id}$ such that for any object $V$, the graded piece $\text{gr}^{\mathcal{F}_i}_i(V)$ is a quotient of $i_{P_i}$ of the maximal cuspidal quotient of $r_{P_i}V$. 
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Using the filtration and the induction hypothesis, we are left to show: 

for any parabolic $P$ and any $W \in \text{Cusp}_R(M)$ finitely generated, all cuspidal subquotients of $i_PW$ are finitely generated.
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The main ingredient here is:

**Theorem** Let $K$ be a field with a valuation $\nu$ s.t. $\nu(p) = 0$, and $\sigma \in \text{Irr}_K(M)$ such that $\nu \circ \omega_{\sigma}|_{Z_M \cap G^0}$ is not uniformly zero. Then $i_P\sigma$ has no cuspidal subquotient.
How to prove second adjointness?
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Fix a parabolic subgroup $P = MU$. Quite formal considerations (already in Bernstein) show that:

- $i_P$ has a right adjoint $i^*_P$ defined by $i^*_P(V) := \delta_P.Hom_G(C^\infty,c(R)(G/U),V)^\infty$. 

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- There is a natural transformation $i^*_P \xrightarrow{\varphi} r_P$ defined by evaluating $\alpha$ on any function of the form $1_{U_cH_M,\alpha U}$. 

The problem is to show the latter is an isomorphism.
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Need more notation:

$B(G) = \text{extended building of } G$.

$G_x = \text{fixator of } x \in B(G)$.

$G_x^+ = \text{pro-}p\text{-radical of } G_x$.

For any $H \subset G$, put $H_x := H \cap G_x$ and $H_x^+ := H \cap G_x^+$. 
Even more notation:

\[ RG = \text{algebra of compactly supported } R\text{-valued distributions.} \]
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**Definition** Let \( P = MU \) be a parabolic subgroup and \( x \in B(M) \). An idempotent \( \varepsilon \in RM_x \) is said to be \( P \)-good if for any embedding \( B(M) \hookrightarrow B(G) \) we have

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e_{U_x}e_{\overline{U}_x}\varepsilon \in RG_xe_{U_x}e_{\overline{U}_x}\varepsilon.
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**Proposition**  If \( \varepsilon \) is \( P \)-good and \( \overline{P} \)-good, then \( \varphi_V \) restricts to an isomorphism \( \varepsilon(i_P^* V) \xrightarrow{\sim} \varepsilon(r_{\overline{P}} V) \).
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Say that a set \( \mathcal{E} \) of idempotents of \( RM \) is generating if

\[
\mathcal{C}_R^\infty,^c(G) = \sum_{\varepsilon \in \mathcal{E}} \varepsilon \cdot \mathcal{C}_R^\infty,^c(G).
\]

**Corollary**  If one can find a generating set of \( P \)-good and \( \overline{P} \)-good idempotents in \( RM \), then the pair \((i_P, r_{\overline{P}})\) is adjoint.
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Suppose $G = G(F)$ and let $\overline{G}$ be a smooth “connected” model of $G$ on $\mathcal{O}_F$. Put $\underline{G} := \overline{G}(\mathcal{O}_F)$ and let $G^\dagger$ be the pro-$p$-radical of $G$. 
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**Definition** A parabolic pair $(\mathcal{P} = \mathcal{M}\mathcal{U}, \overline{\mathcal{P}} = \mathcal{M}\overline{\mathcal{U}})$ is called $\overline{G}$-admissible if $\mathcal{M}$ is the centralizer of a split torus of $\overline{G}$ whose schematic closure in $\overline{G}$ is a torus.

Then, the respective schematic closures $\overline{\mathcal{P}}, \overline{\mathcal{M}}, \overline{\mathcal{U}}$ are smooth.
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**Definition**  A central idempotent $\varepsilon \in RG$ is called essentially of depth zero if for any $\mathcal{G}$-admissible pair $(\mathcal{P}, \mathcal{P})$, we have

$$\varepsilon \in RG^\dagger e_{U^\dagger} e_{U^\dagger} RG^\dagger.$$

Examples: $e_{G^\dagger}$, and 1 if there’s no nontrivial admissible pair.
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**Theorem** Let $(\mathcal{P}, \overline{\mathcal{P}})$ be a $G$-admissible pair and $\varepsilon$ an idempotent of $RM$ which is essentially of depth zero. Then

$$e_U^\dagger e_U^\varepsilon \in RG e_U e_U R \varepsilon.$$
Direct application of last theorem: fix $(\mathcal{P}, \overline{\mathcal{P}})$ and $x \in B(M)$, and let $G_x$ be the Bruhat-Tits model.

- **Minimal case.** If $M$ is minimal, there’s no $p$-pair in $M_x$, so the unit 1 is trivially a $\mathcal{P}$-good idempotent.
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- **Depth zero case.** Since $e^+_{M_x}$ is $P$-good, the last theorem gives second adjointness on restriction to “depth 0” subcategories.
How to produce $P$-good idempotents? 2: types theory.

Connections with types theory: let $(J, \theta)$ consist

- either of a group of the form $J(\Lambda, \beta)$ and a semisimple character in Stevens’ sense (classical groups),

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It follows that $\theta|_{J \cap M}$ is $P$-good for any $P = MU$. Moreover any $(J, \theta)$ may be “extended” to $(J, \theta_M)$. So if such data form a generating set, get second adjointness.

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