Finiteness properties of Hecke algebras of p-adic groups

June 8th, 2007

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Proposition *Assume $|G| \in R^\times$ and R noetherian. Then $R[H\backslash G/H]$ is a finitely generated module over its center which is a finitely generated R-algebra.*

Second version of main results. Assume R noetherian, $p \in R^\times$ and let $Mod_R(G)$ be the category of all smooth RG-modules.

Theorem *Assume G classical. Then any finitely generated $V \in Mod_R(G)$ is noetherian.*

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Strategy of proof : inductive arguments using parabolic induction and restriction functors. To initialize the induction let $Cusp_R(G)$ be the subcategory of cuspidal objects and Z_G be the center of G.

Lemma *Any f.g. $V \in Cusp_R(G)$ is $R[Z_G]$-admissible.*

Bernstein's strategy revisited

If $P = M.U \subset G$ is a parabolic subgroup, note i_P and r_P the parabolic functors. We have ring morphisms

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Then α is an isomorphism and the statement of Proposition follows easily.

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Frobenius reciprocity gives such a filtration : refine the partial order on standard parabolic subgroups in a total order $P_1, \cdots, P_g$. Then put $\mathcal{F}^0 := \mathrm{Id}$ and $\mathcal{F}^i := ker(\mathcal{F}^{i-1} \xrightarrow{Adj} i_{P_i} \circ r_{P_i} \circ \mathcal{F}^{i-1})$.

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Unfortunately, both the available proofs use noetherian properties !

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We now have an increasing filtration $\mathcal{F}_\bullet$ of Id such that for any object V, the graded piece $gr_i^{\mathcal{F}}(V)$ is a quotient of i_{P_i} of the maximal cuspidal quotient of $r_{\overline{P_i}}V$.

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Using the filtration and the induction hypothesis, we are left to show : *for any parabolic P and any $W \in Cusp_R(M)$ finitely generated, all cuspidal subquotients of $i_P W$ are finitely generated.*

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The main ingredient here is :

Theorem *Let K be a field with a valuation ν s.t. $\nu(p) = 0$, and $\sigma \in Irr_K(M)$ such that $\nu \circ \omega_{\sigma|Z_M \cap G^0}$ is not uniformly zero. Then $i_P \sigma$ has no cuspidal subquotient.*

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Fix a parabolic subgroup $P = MU$. Quite formal considerations (already in Bernstein) show that :

- i_P has a right adjoint i_P^* defined by $i_P^*(V) := \delta_P.\mathit{Hom}_G(\mathcal{C}_R^{\infty,c}(G/U), V)^\infty$.

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Need more notation:

$B(G)$ = extended building of G.
G_x = fixator of $x \in B(G)$.
G_x^+ = pro-p-radical of G_x.
For any $H \subset G$, put $H_x := H \cap G_x$ and $H_x^+ := H \cap G_x^+$.

Even more notation:

$RG =$ algebra of compactly supported R-valued distributions.
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$$e_{U_x^+} e_{\overline{U}_x} \varepsilon \in RG_x e_{U_x} e_{\overline{U}_x} \varepsilon.$$

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Say that a set $\mathcal{E}$ of idempotents of RM is *generating* if $\mathcal{C}_R^{\infty,c}(G) = \sum_{\varepsilon \in \mathcal{E}} \varepsilon.\mathcal{C}_R^{\infty,c}(G)$.

Corollary *If one can find a generating set of P-good and $\overline{P}$-good idempotents in RM, then the pair $(i_P, r_{\overline{P}})$ is adjoint.*

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Suppose $G = \mathcal{G}(F)$ and let $\underline{\mathcal{G}}$ be a smooth "connected" model of $\mathcal{G}$ on $\mathcal{O}_F$. Put $\underline{G} := \underline{\mathcal{G}}(\mathcal{O}_F)$ and let $\underline{G}^\dagger$ be the pro-p-radical of $\underline{G}$.

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Then, the respective schematic closures $\underline{\mathcal{P}}$, $\underline{\mathcal{M}}$, $\underline{\mathcal{U}}$ are smooth.

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Examples : $e_{\underline{G}^\dagger}$, and 1 if there's no nontrivial admissible pair.

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Theorem *Let* $(\mathcal{P}, \overline{\mathcal{P}})$ *be a* $\underline{\mathcal{G}}$*-admissible pair and* ε *an idempotent of* $R\underline{M}$ *which is essentially of depth zero. Then*

$$e_{\underline{U}^\dagger} e_{\overline{\underline{U}}} \varepsilon \in R\underline{G} e_{\underline{U}} e_{\overline{\underline{U}}} R\varepsilon.$$

Direct application of last theorem : fix $(\mathcal{P}, \overline{\mathcal{P}})$ and $x \in B(M)$, and let $\underline{\mathcal{G}}_x$ be the Bruhat-Tits model.

- *Minimal case*. If $\mathcal{M}$ is minimal, there's no p-pair in $\mathcal{M}_x$, so the unit 1 is trivially a $\mathcal{P}$-good idempotent.

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- *Depth zero case*. Since $e_{M_x^+}$ is P-good, the last theorem gives second adjointness on restriction to "depth 0" subcategories.

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