Continuous representation theory of p-adic Lie groups

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Contents of the lectures

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Basics on \( p \)-adic Lie groups.

**Definition:** a \( p \)-adic Lie group \( G \) is a group object in the category of locally \( \mathbb{Q}_p \)-analytic manifolds.

Locally \( \mathbb{Q}_p \)-analytic manifolds are defined in the same way as differentiable \( \mathbb{R} \)-manifolds. Smooth vector-valued functions on \( \mathbb{R}^n \) are replaced by **locally analytic** vector-valued functions on \( \mathbb{Z}_p^n \), i.e. functions that are locally given by convergent power series.

**Straightforward properties:**

- \( G \) is a totally disconnected locally compact topological group. In particular, the unit element has a basis of neighborhoods consisting of open compact subgroups.
- \( G \) has a Lie algebra \( \mathfrak{g} \) over \( \mathbb{Q}_p \).

**Rk:** one may replace \( \mathbb{Q}_p \) by a finite extension \( L \).
Nice open pro-$p$-subgroups

Let $G$ be a $d$-dimensional $p$-adic Lie group. Start with a local chart $\psi : \mathbb{Z}_p^d \hookrightarrow G$ with image an open neighborhood of $\psi(0) = e_G$. 

By definition there are:

- a power series $f(x, y) = \sum_{\alpha, \beta \in \mathbb{N}_d} a_{\alpha, \beta} x^\alpha y^\beta$ with $a_{\alpha, \beta} \in \mathbb{Q}_p^d$,
- an open neighborhood $p^h \mathbb{Z}_p^d$ of 0 in $\mathbb{Z}_p^d$, such that
- $f$ converges to an analytic function $p^h \mathbb{Z}_p^d \times p^h \mathbb{Z}_p^d \rightarrow \mathbb{Z}_p^d$ and $\psi(f(x, y)) = \psi(x)\psi(y)^{-1}$.

Note that $a_{0,0} = 0$ and that the linear part of $f$ is $x - y$.

For higher degree terms, convergence of $f(p^h, p^h)$ tells us that $(\alpha, \beta) \mapsto |a_{\alpha, \beta} p^{h(|\alpha|+|\beta|)}|$ cvges to 0. Since replacing $\psi$ by $\psi \circ p^{h'}$ multiplies $a_{\alpha, \beta}$ by $p^{h'(|\alpha|+|\beta|-1)}$, we thus may arrange for

- $a_{\alpha, \beta}$ to lie in $p\mathbb{Z}_p^d$ for all $\alpha, \beta$ with $|\alpha| + |\beta| > 1$, and
- $f$ to converge on the whole $\mathbb{Z}_p^d \times \mathbb{Z}_p^d$. (i.e. $h = 0$).
Nice open pro-$p$-subgroups (cont’d)

In this case, the image $H = \psi(\mathbb{Z}_p^d)$ of our chart is an open compact subgroup of $G$. More generally for any $n$, the subset $H_n := \psi(p^n\mathbb{Z}_p^d)$ is an open subgroup of $H$, and the collection of all $H_n$’s is a basis of neighborhoods of $e_G$.

Note that $H_n$ is normal in $H$ and $\psi$ induces a group isomorphism $(p^n\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p)^d \sim H_n/H_{n+1}$. In particular $H$ is pro-$p$.

The profinite group $H$ is an example of a uniform pro-$p$-group. The same group, equipped with the decreasing filtration $(H_n)_{n \in \mathbb{N}}$, is an example of a $p$-valued group, in the sense of Lazard.

**Definition**: a topologically finitely generated pro-$p$-group $G$ with lower $p$-series $(P_i(G))_{i \in \mathbb{N}}$ (inductively defined by $P_{i+1}(G) := P_i(G)^p[P_i(G),G]$) is called uniform if $[G,G] \subset G^p$ and the index $|P_i(G)/P_{i+1}(G)|$ is independent of $i$. 
A \( p \)-valuation on a group \( G \) is a map \( \omega : G \rightarrow [1/(p - 1), +\infty] \) such that \( \omega^{-1}(+\infty) = \{e_G\} \) and

1. \( \omega(gh^{-1}) \geq \min(\omega(g), \omega(h)) \)
2. \( \omega(ghg^{-1}h^{-1}) \geq \omega(g) + \omega(h) \)
3. \( \omega(g^p) = \omega(g) + 1 \)

It defines a non-increasing \( \mathbb{R} \)-filtration

\[
x \mapsto G_x := \{g \in G, \omega(g) \geq x\}
\]

whose graduate \( \text{gr}_{\omega}(G) \) is a graded torsion-free \( \mathbb{F}_p[\pi] \) Lie algebra, with bracket induced by commutators and \( \pi \)-action induced by the \( p^{th} \) power map in \( G \).

On a uniform group \( G \), the integrally valued function

\[
g \mapsto \omega(g) := \sup\{i \in \mathbb{N}, g \in P_i(G)\}
\]

is a \( p \)-valuation. For our home-made open s/g \( H \), the Lie algebra \( \text{gr}_{\omega}(H) \) is commutative (all brackets vanish).
Coordinates on \( p \)-valued groups

Let \((G, \omega)\) be a complete \( p \)-valued group. Assume that \( \text{gr}^{\omega}(G) \) has finite rank over \( \mathbb{F}_p[\pi] \) and let \((g_i)_{i=1,\ldots,d}\) be representatives in \( G \) of a basis of \( \text{gr}^{\omega}(G) \) over \( \mathbb{F}_p[\pi] \) (in the graded sense).

**Theorem (Lazard) :** The map

\[
\Psi : \mathbb{Z}_p^d \rightarrow G \\
(x_1, \cdots, x_d) \mapsto g_1^{x_1} \cdot g_2^{x_2} \cdots g_d^{x_d}
\]

is a global analytic chart for \( G \).

Moreover we have

\[
\omega(g_1^{x_1} \cdots g_d^{x_d}) = \inf_{i=1,\ldots,d} (1 + \text{val}_p(x_i))
\]

An abstract group is called \( p \)-valuable if it admits a \( p \)-valuation for which it is complete with graduate of finite rank over \( \mathbb{F}_p[\pi] \).
Characterization of compact $p$-adic Lie groups

**Theorem (Lazard)**

A t.d. locally compact group admits a locally $\mathbb{Q}_p$-analytic structure if and only if it admits a $p$-valuable open pro-$p$-subgroup. Moreover this analytic structure is unique.

Consequences:

- Products of analytic groups are analytic.
- Any abstract group morphism between analytic groups is analytic.
- Closed subgroups of analytic groups are analytic.

Indeed all these statements can be checked most easily on $p$-valuable groups.
Let $G$ be a profinite group. The Iwasawa algebra of $G$ is the completed group ring

$$\mathbb{Z}_p[[G]] := \lim_{\leftarrow H} \mathbb{Z}_p[G/H]$$

where $H$ runs over open subgroups of $G$. It is a compact $\mathbb{Z}_p$-algebra.

**Example:** When $G = \mathbb{Z}_p$, this algebra is isomorphic to $\mathbb{Z}_p[[T]]$ and was studied in detail by Iwasawa.

**Elementary properties:**

- The canonical map $\mathbb{Z}_p[G] \rightarrow \mathbb{Z}_p[[G]]$ is injective and has dense image. Its restriction to $G$ is a homeo onto its image.
- If $G$ is pro-$p$ then $\mathbb{Z}_p[[G]]$ is local with radical $R := (p) + I$ ($I$ augm. ideal).
- The $R$-adic tgy is finer than the inverse limit tgy with equality if $G$ has finite type.
The following result is a major tool in the study of Iwasawa algebras and more general distributions algebras for $p$-adic Lie groups. Let $(G, \omega)$ be a $p$-valued complete group of finite rank and define for $x \in \mathbb{R}_+$:

$$\mathbb{Z}_p[G]_x := \text{submod. gen by elements } p^h(1 - g_1) \cdots (1 - g_k)$$

with $h + \omega(g_1) + \cdots \omega(g_k) \geq x$

This is a non-increasing $\mathbb{R}_+$-filtration of $\mathbb{Z}_p[G]$, in the sense that $\mathbb{Z}_p[G]_0 = \mathbb{Z}_p[G]$ and $\mathbb{Z}_p[G]_x \mathbb{Z}_p[G]_y \subset \mathbb{Z}_p[G]_{x+y}$ for all $x, y$.

The induced filtration on $\mathbb{Z}_p$ is the natural one, so that $\text{gr}^\omega(\mathbb{Z}_p[G])$ is a $\mathbb{F}_p[\pi] = \text{gr}^\omega(\mathbb{Z}_p)$-algebra.

**Remark:** One checks that the map

$$f \in \mathbb{Z}_p[G] \mapsto ||f||_{1/p} := p^{-\inf\{x,f \in \mathbb{Z}[G]_x\}}$$

is a multiplicative norm on $\mathbb{Z}_p[G]$. 
Theorem (Lazard)

1. The completion of $\mathbb{Z}_p[G]$ for the above filtration (or for the norm $||.||_\omega$) identifies with $\mathbb{Z}_p[[G]]$.
2. There is a canonical isomorphism of graded $\mathbb{F}_p[\pi]$-algebras

$$U(\text{gr}_\omega(G)) \xrightarrow{\sim} \text{gr}_\omega(\mathbb{Z}_p[G])$$

Here $U$ denotes the enveloping algebra.

In the case of our nice open cpct sgp $H$ defined above, the graded Lie algebra $\text{gr}_\bullet(H)$ is commutative, so we get

$$\text{Sym}_{\mathbb{F}_p[\pi]}(\text{gr}_\bullet H) \sim \mathbb{F}_p[\pi][T_1, \cdots, T_d] \xrightarrow{\sim} \text{gr}_\bullet(\mathbb{Z}_p[[H]]).$$

This result is fundamental in the study of Iwasawa algebras since it allows one to use techniques from filtered ring theory.
Filtered rings techniques (1/2)

Let $R$ be a ring and let $F_* R$ be a decreasing exhaustive and separated $\mathbb{Z}$-filtration of $R$. Assume that $R$ is complete for the filtration $F_*$ in the sense that $R = \varprojlim_n R/F_n R$. The first useful properties are:

- If $\text{gr}_* R$ is a (left or right) noetherian ring, so is $R$.
- If $\text{gr}_* R$ has no zero divisor, then $R$ has the same property.

Applied to the Iwasawa algebra of a compact $p$-adic Lie group, this gives

- $\mathbb{Z}_p[[G]]$ is left and right noetherian.
- If $G$ is $p$-valuable, $\mathbb{Z}_p[[G]]$ has no zero divisors.

Assume further on that $R$ is noetherian. The filtration $F_*$ is then an example of a **Zariskian filtration**. For such filtrations, many homological properties can be lifted from the associate graded ring. The following will be used in Teitelbaum’s second lecture:

- If $R \longrightarrow S$ is a morphism of Zariskian filtered rings such that $\text{gr}_* S$ is flat over $\text{gr}_* R$, then $S$ is flat over $R$. 
Filtered rings techniques (2/2)

If $M$ is a module on a ring $R$, one defines the \textit{grade} of $M$ by

$$j_R(M) = \inf\{k \in \mathbb{N}, \text{Ext}^k_R(M, R) \neq 0\}.$$

The Auslander condition on $M$ is:

$$\forall k \in \mathbb{N}, \forall N \subset \text{Ext}^k_R(M, R), j_R(N) \geq k.$$

The ring $R$ is called \textit{Auslander regular} if it is noetherian, has finite global dimension and any f.g. $R$-module $M$ satisfies the Auslander condition. When $R$ is commutative, Auslander regular = regular and, in this case $j_R(M)$ is the codimension of the support of $M$.

Let $R$ be Zariskian filtered. Then

- If $\text{gr}\_\cdot R$ is Auslander regular, so is $R$, and $\text{gld}(R) \leq \text{gld}(\text{gr}\_\cdot (R))$.
- For any module $M$ on $R$ and any “good” filtration on $M$ we have $j_R(M) = j_{\text{gr}\_\cdot (R)}(\text{gr}\_\cdot (M))$.

Application: If $G$ is $p$-valuable, $\mathbb{Z}_p[[G]]$ is Auslander regular of global dimension $\text{rank}(G) + 1$ (Venjakob, Brumer).
Application to the structure of modules

Motivated by classical Iwasawa theory \((G = \mathbb{Z}_p)\), there have been attempts to understand the structure of *finitely generated torsion modules* on an Iwasawa algebra, up to *pseudo-isomorphism*.

Roughly speaking, a pseudo-isomorphism is a morphism which becomes invertible in the quotient category of \(Mod_{fg} (\mathbb{Z}_p[[G]])\) by the subcategory of objects \(M\) of grade \(j_{\mathbb{Z}_p[[G]]}(M) > 1\) (*pseudo-null modules* as defined by Venjakob).

The main result in this direction is

**Theorem (Coates, Schneider and Sujatha)**

Let \(G\) be \(p\)-valuable and let \(M\) be a f.g. torsion left module over \(\mathbb{Z}_p[[G]]\). Then there are non-zero left ideals \(L_1, \ldots, L_m\) and an injection

\[
\varphi : \bigoplus_i \mathbb{Z}_p[[G]]/L_i \longrightarrow M/M_0
\]

with \(M_0\) and \(\text{coker}\varphi\) *pseudo-null*. 
Topological modules over Iwasawa algebras

So far we have only considered abstract module theory over Iwasawa algebras. We now turn to topological module theory.

**Definition:** A *topological* $\mathbb{Z}_p[[G]]$-*module* is a linearly topologized $\mathbb{Z}_p$-module $M$ endowed with a continuous action map $\mathbb{Z}_p[[G]] \times M \longrightarrow M$. Morphisms between such objects are $G$-equivariant continuous $\mathbb{Z}_p$-linear maps.

Nothing interesting can be said without imposing restriction on the topology. We will be interested here in the following subcategories.

- The subctgy $\text{Mod}_{co}(\mathbb{Z}_p[[G]])$ of compact topological modules.
- The subctgy $\text{Mod}_{ad}(\mathbb{Z}_p[[G]])$ of adic modules, i.e. such that $M \xrightarrow{\sim} \lim_{\leftarrow n} M/p^n$. 
The subcategory \( \text{Mod}_{co}(\mathbb{Z}_p[[G]]) \)

Let \( M \) be a locally compact linearly topologized \( \mathbb{Z}_p \)-module, and let us endow \( \text{End}_{\mathbb{Z}_p}^{\text{cont}}(M) \) with the linear topology of compact convergence. The following data on \( M \) are equivalent:

- A continuous action map \( \mathbb{Z}_p[[G]] \times M \rightarrow M \).
- A continuous map of \( \mathbb{Z}_p \)-algebras \( \mathbb{Z}_p[[G]] \rightarrow \text{End}_{\mathbb{Z}_p}^{\text{cont}}(M)_{cc} \).
- A continuous map of groups \( G \rightarrow \text{End}_{\mathbb{Z}_p}^{\text{cont}}(M)_{cc} \).
- A continuous action map \( G \times M \rightarrow M \).

The category \( \text{Mod}_{co}(\mathbb{Z}_p[[G]]) \) is abelian, and kernels and cokernels commute with the functor “forget the tgy”. We denote by

\[
\text{Mod}^{tf}_{co}(\mathbb{Z}_p[[G]])
\]

the subcategory of \( p \)-torsion free objects in \( \text{Mod}_{co}(\mathbb{Z}_p[[G]]) \). It is not abelian anymore but it is quasi-abelian. This means in particular that kernels and cokernels exist. Here the cokernel of a morphism is the usual cokernel divided by the closure of the torsion submodule.
The subcategory $\text{Mod}_{ad}(\mathbb{Z}_p[[G]])$

Let $M$ be a locally adic linearly topologized $\mathbb{Z}_p$-module, and let us endow $\text{End}_{\mathbb{Z}_p}^{\text{cont}}(M)$ with the linear topology of pointwise convergence. The following data on $M$ are equivalent:

- A continuous action map $\mathbb{Z}_p[[G]] \times M \rightarrow M$.
- A continuous map of $\mathbb{Z}_p$-algebras $\mathbb{Z}_p[[G]] \rightarrow \text{End}_{\mathbb{Z}_p}^{\text{cont}}(M)_{pc}$.
- A continuous map of groups $G \rightarrow \text{End}_{\mathbb{Z}_p}^{\text{cont}}(M)^{\times}$.
- A continuous action map $G \times M \rightarrow M$.

In particular the action of $G$ is continuous if and only if it is separately continuous.

The category $\text{Mod}_{ad}(\mathbb{Z}_p[[G]])$ is only quasi-abelian, and cokernels don’t commute with the functor “forget the tgy”. We denote by

$$\text{Mod}^{tf}_{ad}(\mathbb{Z}_p[[G]])$$

the subcategory of torsion free objects in $\text{Mod}_{ad}(\mathbb{Z}_p[[G]])$. Again it is quasi-abelian.
Duality between adic and compact torsion free modules

For a topological torsion-free \( \mathbb{Z}_p \)-module \( M \), let’s put

\[ M^d := \text{Hom}_{\mathbb{Z}_p}^{\text{cont}}(M, \mathbb{Z}_p). \]

If \( M \) is compact, we endow \( M^d \) with the \( p \)-adic tplgy. It coincides with the compact convergence tpylgy, hence is complete and \( M^d \) is an adic \( \mathbb{Z}_p \)-module.

If \( M \) is adic, we endow \( M^d \) with the tpylgy of pointwise convergence. We then get a compact \( \mathbb{Z}_p \)-module.

Lemma (Shikhof)

These two constructions induce quasi-inverse anti-equivalences of categories between \( \text{Mod}_{\text{co}}^{tf}(\mathbb{Z}_p) \) and \( \text{Mod}_{\text{ad}}^{tf}(\mathbb{Z}_p) \).

Concretely any object in \( \text{Mod}_{\text{co}}^{tf}(\mathbb{Z}_p) \) is isomorphic to \( \mathbb{Z}_p^X \) for some set \( X \) and any object in \( \text{Mod}_{\text{ad}}^{tf}(\mathbb{Z}_p) \) is isomorphic to the \( p \)-adic completion \( \mathbb{Z}_p^{\langle X \rangle} \) of \( \mathbb{Z}_p(X) \). Duality carries \( \mathbb{Z}_p^X \) to \( \mathbb{Z}_p^{\langle X \rangle} \).
Equivariant duality

Let $M, N$ be two compact $\mathbb{Z}_p$-modules. Transposition gives a $\mathbb{Z}_p$-linear isomorphism $\text{Hom}_{\mathbb{Z}_p}^\text{cont}(M, N) \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}_p}^\text{cont}(N^d, M^d)$.

Lemma (ST)

The above isomorphism is a homeomorphism if one endows the LHS with the topology of compact convergence and the RHS with the topology of pointwise convergence. Hence we have

$$\text{Hom}_{\mathbb{Z}_p}^\text{cont}(M, N)_{cc} \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}_p}^\text{cont}(N^d, M^d)_{pc}.$$  

Corollary (ST)

The functor $M \mapsto M^d$ induce quasi-inverse anti-equivalences of categories between $\text{Mod}_{\text{co}}^{tf}(\mathbb{Z}_p[[G]])$ and $\text{Mod}_{\text{ad}}^{tf}(\mathbb{Z}_p[[G]])$. 
Finitely generated modules

A general result on compact rings says:

**Theorem**

The forgetful functor from the ctgry of f.g. topological $\mathbb{Z}_p[[G]]$-modules to the ctgry of f.g. abstract $\mathbb{Z}_p[[G]]$-modules is an equivalence of ctgries.

In particular any f.g. module over $\mathbb{Z}_p[[G]]$ carries a canonical (compact) topology making the action continuous. If $\mathbb{Z}_p[[G]]$ moreover is noetherian, e.g. if $G$ is a $p$-adic Lie group, then any (left or right) ideal in $\mathbb{Z}_p[[G]]$ is closed.

**Remark:** duality exchanges $\mathbb{Z}_p[[G]]$ and the space $C(G, \mathbb{Z}_p)$ of continuous function with the sup-norm topology. So $\mathbb{Z}_p[[G]]$ is the algebra of $\mathbb{Z}_p$-valued distributions on $G$. 
Variant

Let $K$ be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{O}$ be its ring of integers. Then we can form the topological $\mathcal{O}$-algebra $\mathcal{O}[[G]]$. All the foregoing results on the properties of the topological ring $\mathbb{Z}_p[[G]]$ and its topological modules apply to $\mathcal{O}[[G]]$. 
Banach representations

Let $K$ be a complete non-Archimedean field. Recall that a Banach space over $K$ is a vector space endowed with an ultrametric norm and complete w.r.t. this norm. Such spaces, with continuous $K$-linear maps as morphisms, form a $K$-linear category $\text{Ban}(K)$, which is quasi-abelian.

A Banach representation of a topological group $G$ is a Banach space with a continuous action map $G \times V \rightarrow V$. They form a category $\text{Ban}_G(K)$.

When $K$ is spherically complete (e.g. discretely valued), Banach spaces are brelled, hence satisfy the so-called Banach-Steinhaus theorem. Therefore, if $G$ is locally compact the following data are equivalent:

- A continuous action map $G \times V \rightarrow V$
- A separately continuous map $G \times V \rightarrow V$
- A continuous multiplicative map $G \rightarrow \mathcal{L}(V, V)_s$. 


However, the foregoing definition does not lead to a reasonable theory, even for $p$-adic Lie groups:

- There may exist non-trivial $G$-equivariant continuous maps between two non-isomorphic topologically irreducible Banach representations.
- For $G = \mathbb{Z}_p$, there are (many) infinite dimensional topologically irreducible Banach representations.

One reason for such phenomena is the lack of a $K$-valued Haar measure on a $p$-adic Lie group $G$, i.e. a left $G$-invariant continuous linear form on the space $C(G, K)$ of continuous maps on $G$.

In order to get a reasonable theory, Schneider and Teitelbaum relate Banach space representations to Iwasawa topological modules using the duality theory we have presented.

We assume from now on that $K$ is a finite extension of $\mathbb{Q}_p$ and will describe a convenient duality for Banach space representations.
Banach spaces over $K$ and torsion-free adic $\mathcal{O}$-modules

Let $\mathcal{O}$ be the ring of integers of $K$ and let $M$ be a torsion-free adic topological $\mathcal{O}$-module. Then the $K$-vector space $M_K := M \otimes K$ is complete for the norm $m \mapsto \sup\{|\lambda|, \lambda \in K, \lambda.m \in M\}$. This construction is functorial and we have

Lemma

The functor

$$\text{Mod}^{tf}_{ad}(\mathcal{O})_\mathbb{Q} \longrightarrow \text{Ban}(K)$$

thus obtained is an equivalence of categories.

Proof.

Let $\text{Ban}(K)^{\leq 1}$ denote the ctgry of all Banach spaces $V$ over $K$ such that with $\|V\| \subset |K|$, and with morphisms all norm decreasing $K$-linear maps. This is a $\mathcal{O}$-linear category and the obvious functor $\text{Ban}(K)^{\leq 1}_\mathbb{Q} \longrightarrow \text{Ban}(K)$ is an equivalence.

The functor $M \mapsto M_K$ factors through $\text{Ban}(K)^{\leq 1}$ and the functor $V \mapsto V^\circ$ (unit ball) is a quasi-inverse.
$G$-equivariant duality

Let $G$ be a profinite group. We have the following equivariant version of the last lemma:

**Lemma**

The functor $M \mapsto M_K$ induces an equivalence of categories

$$\text{Mod}^{tf}_{ad}(O[[G]])_\mathbb{Q} \xrightarrow{\sim} \text{Ban}_G(K)$$

The point is that, by compacity of $G$ one can change the norm to an equivalent $G$-equivariant one.

Composing with duality between adic/compact $O[[G]]$-modules:

**Theorem (ST)**

The functor $M \mapsto M_K^d$ induces an antiequivalence of categories

$$\text{Mod}^{tf}_{co}(O[[G]])_\mathbb{Q} \xrightarrow{\sim} \text{Ban}_G(K).$$

This result enables Schneider and Teitelbaum to propose a convenient notion of *admissibility*. 
Admissibility for Banach representations

Let $G$ be a profinite group.

**First definition:** A Banach representation of $G$ is called admissible if it is of the form $M^d_K$ for some finitely generated torsion-free $\mathcal{O}[[G]]$-module $M$.

In other words, letting $D^c(G, K) = K[[G]] := \mathcal{O}[[G]] \otimes K$ be the algebra of distributions on $G$, a Banach representation $V$ is admissible if its dual $V'$ is a finitely generated $K[[G]]$-module.

Moreover since $\text{Mod}_{fg}^{tf}(\mathcal{O}[[G]])_\mathbb{Q} \simeq \text{Mod}_{fg}(K[[G]])$, we have an anti-equivalence of categories

$$\text{Mod}_{fg}(K[[G]]) \xrightarrow{\sim} \text{Ban}_{G}^{adm}(K).$$

In particular, if $K[[G]]$ is noetherian (e.g. if $G$ is analytic) then $\text{Ban}_{G}^{adm}(K)$ is an abelian category. Moreover kernels and cokernels commute with all forgetful functors. Thus the theory of admissible Banach representations becomes purely algebraic.
The case of $p$-adic Lie groups

We give here a more intrinsic formulation of the admissibility condition under the assumption that $G$ is analytic. Let $k = \mathcal{O}/\varpi_K \mathcal{O}$ be the residue field of $K$.

**Proposition (ST, Breuil)**

A Banach representation $V$ of $G$ is admissible if and only if for any (resp. for one) open $p$-valuable subgroup $H$ of $G$ and any (resp. one) $H$-invariant bounded open $\mathcal{O}$-submodule $N \subset V$, the $k$-vector space $(N \otimes \mathcal{O} k)^H$ is finite dimensional.

**Proof** : Let $V, N$ as above and put $M := N^d \in \text{Mod}^{tf}_{co}(\mathcal{O}[[H]])$, so that $V = M^d_K$. By topological Nakayama lemma for compact modules, $M$ is f.g. over $\mathcal{O}[[H]]$ iff $\dim_k(M/R_H M) < \infty$. where $R_H$ is the radical of $\mathcal{O}[[H]]$. Therefore the claim follows from the bijectivity of the following injection

\[
(N \otimes \mathcal{O} k)^H \rightarrow (\text{Hom}^{cont}_\mathcal{O}(M, \mathcal{O}) \otimes \mathcal{O} k)^H
\]

\[
\rightarrow \text{Hom}^{cont}_\mathcal{O}(M, k)^H = \text{Hom}_\mathcal{O}(M/R_H M, k)
\]
Good properties of admissible Banach representations

As a consequence of the previous equivalence of categories, we get a bijection

\[
\begin{array}{c}
\text{Isom. classes of} \\
\text{topologically irreducible}
\end{array}
\quad \sim \quad
\begin{array}{c}
\text{Isom. classes} \\
\text{of simple} \\
K[[G]] - \text{modules}
\end{array}
\]

Moreover any non-zero \(G\)-equivariant between two admissible topologically irreducible Banach representations is an isomorphism, thus correcting a previously mentioned pathology.

For the group \(G = \mathbb{Z}_p\), since \(K[[G]] \simeq K \otimes \mathcal{O}[[T]]\), we see that all topologically irreducible admissible Banach representations are finite dimensional, which is in accordance with the analogy with compact commutative Lie groups.
Remarks on non-compact $p$-adic Lie groups

A Banach representation of an arbitrary $p$-adic Lie group is called *admissible* if it is admissible w.r.t any, or equivalently one, compact open subgroup $G_0$.

Note that the dual $V'$ of a Banach representation is a module over the distribution algebra $D_c(G, K) = \mathcal{C}(G, K)'$ and we have $D_c(G, K) = \bigoplus_{g \in G/G_0} g.K[[G_0]]$ as a $K[[G_0]]$ module.

**Definition:** A Banach representation $V$ of $G$ is called unitary if the topology of $V$ can be defined by a $G$-equivariant norm.

Equivalently $V$ is unitary if it contains a bounded open $G$-equivariant lattice $N$. 
Sources of Banach representations

1. **Spaces of continuous functions**: If $X$ is a compact set with a continuous action of $G$, then $C(X, K)$ with the sup norm is a Banach representation of $G$. If $X/G$ is finite then $C(X, K)$ is admissible. Example (or variant): principal series for reductive groups (see later).

2. **Completion of representations**: Start with a smooth or locally algebraic or locally analytic $K$-representation $V$ of $G$, and assume given a $G$-invariant $\mathcal{O}$-submodule $N$ which generates $V$ over $K$. Define a $G$-invariant seminorm on $V$ by $||v|| = \sup\{|\lambda|, \lambda \in K, \lambda x \in N\}$ and complete. We get (possibly zero) a unitary Banach representation $\hat{V}(N)$.

3. **$(\varphi, \Gamma)$-modules**: In their lectures, Berger and Colmez explain how to construct a Banach representation of $GL_2(\mathbb{Q}_p)$ from a $(\varphi, \Gamma)$-module.

4. **Completed cohomology**: In his lectures, Emerton presumably will study some kind of completion of the (étale) cohomology of the modular curve with infinite $p$-level structure.
Principal series of reductive groups

Let \( G \) be the group of rational points of some reductive group over \( \mathbb{Q}_p \) and let \( P \) be a parabolic subgroup and \( \chi \) be a continuous character of \( P \). We can form

\[
\text{Ind}^G_P(\chi) := \{ f \in C(G, K), \forall p \in P, f(gp) = \chi(p)^{-1}f(g) \}.
\]

Since \( G/P \) is compact, it is an admissible Banach representation of \( G \). More precisely, choose a compact open subgroup \( G_0 \) of \( G \) such that \( G = G_0P \). E.g. if \( G = \text{GL}_n(\mathbb{Q}_p) \) and \( P \) is the Borel subgroup of upper triangular matrices, take \( G_0 = \text{GL}_n(\mathbb{Z}_p) \).

Then we have

\[
\text{Ind}^G_P(\chi)'_{\mid G_0} = K[[G_0]] \otimes_{K[[P \cap G_0]]} \chi K.
\]

It is strongly believed that these “parabolically induced” Banach representation always have finite length.
An irreducibility conjecture

To simplify, consider the case of a split reductive group over $\mathbb{Q}_p$ and assume that $P = B$ is a Borel subgroup. Let $T$ be a maximal torus in $B$ and assume also that $\rho := \frac{1}{2} \sum_{\alpha \in \Phi(T, \text{Lie}(B))} \alpha \in X^*(T)$ (this is true when $G$ is simply connected).

**Conjecture (Schneider, ICM06)**

The principal series $\text{Ind}_B^G(\chi)$ is topologically irreducible unless there is a positive coroot $\alpha^\vee \in X^*(T)$ such that $(\chi \rho) \circ \alpha^\vee = (\cdot)^m$ for some positive integer $m$.

Here is what is known on this conjecture:

**Theorem (ST)**

The conjecture is true for $\text{GL}_2$.

**Theorem (many people)**

The principal series $\text{Ind}_B^G(\chi)$ is topologically irreducible unless there is a positive coroot $\alpha^\vee \in X^*(T)$ such that $d(\chi \rho \circ \alpha^\vee) = m$ for some positive integer $m$. 
Completion of locally algebraic representations:

Let $V$ be a smooth or locally algebraic $K$-representation of $G$, let $N$ be a $G$-invariant lattice in $V$ (not containing any $K$-line).

By definition the completion $\hat{V}(N)$ is admissible if and only if $N \otimes_{\mathcal{O}} k$ is admissible, as a smooth $k$-representation of $G$.

Note that $\hat{V}(N)$ indeed depends on the lattice $N$. More precisely, let $N' \subset N$ be another $G$-invariant generating $\mathcal{O}$-submodule, then the canonical $G$-equivariant continuous map $\hat{V}(N') \longrightarrow \hat{V}(N)$ is an isomorphism if and only if $N'$ and $N$ are commensurable which means that there is $\lambda \in K$ such that $N \subset \lambda N'$.

**Question**

*How to parametrize classes of commensurability of lattices?*

Note that there is a distinguished comm. class. Namely that of finitely $\mathcal{O}[G]$-generated lattices.
A nice example

Let $G := \text{GL}_2(\mathbb{Q}_p)$ and $V = V_k := \text{St} \otimes \text{Sym}^{k-2}(K^2) \otimes |\det|^{\frac{k-1}{2}}$, where St is the smooth Steinberg representation of $G$ and $k > 2$.

Breuil has noticed that the completion $B(k)$ of $V_k$ along a finitely generated lattice is not admissible.

**Theorem (Breuil, Colmez)**

There is a family of commensurability classes of lattices in $V_k$, parametrized by an element $\mathcal{L} \in K$, such that the corresponding completions $B(k, \mathcal{L})$ satisfy the following:

- They are admissible and topologically irreducible.
- They are pairwise non-isomorphic
- The “canonical” map $B(k) \longrightarrow B(k, \mathcal{L})$ is a quotient map.

Relevance of this construction: the $B(k, \mathcal{L})$ are the “$p$-adic langlands correspondent” of the semi-stable non-crystalline 2-dimensional $K$-representations of $\text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p)$, see Colmez lectures.