ν-tempered representations of p-adic groups. I : l-adic case

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Abstract

The so-called tempered *complex* smooth representations of p-adic groups have been much studied and used, in connection with automorphic forms. Nevertheless, the smooth representations which are realized geometrically often have l-adic coefficients, so that archimedean estimates of their matrix coefficients hardly make sense. We investigate here a notion of tempered representation with coefficients in any normed field of characteristic $\neq p$. The theory turns out to be different according to the norm being Archimedean, non-Archimedean with $|p| \neq 1$ or non-Archimedean with $|p| = 1$.

In this paper, we concentrate on the last case. The main applications concern modular representation theory (*i.e.* on a positive characteristic field), and in particular the study of reducibility properties of the parabolic induction functors ; one of the main results is the generic irreducibility for induced families. Once a suitable theory of rational intertwining operators developed, this allows us to define Harish Chandra's μ-functions and show in some special cases how they track down the cuspidal constituents of parabolically induced representations. Besides, we discuss the admissibility of parabolic restriction functors and derive some lifting properties for supercuspidal modular representations.

Contents

1 Introduction

Let G be a p-adic group ; more precisely, $G = \mathfrak{G}(F)$ for some connected reductive algebraic group $\mathfrak{G}$ defined over a non-archimedean local field F, with residue characteristic p. As usual, for any ring R, a representation of G on a R-module is called smooth if the stabilizer of any vector is open. Since there exists a Haar measure with values in $\mathbb{Z}[\frac{1}{p}]$, we might think intuitively that the irreducible smooth K-representations of G, K being a field of characteristic $l \neq p$, are the K-valued points of

the "spectrum" of the non-commutative convolution ring $\mathcal{H}$ of $\mathbb{Z}[\frac{1}{p}]$-valued compactly supported smooth functions on G. When $K = \mathbb{C}$, some interesting "points", called tempered representations, are defined by $\mathbb{C}$-analytic criteria. These representations extend to continuous representations of a $\mathbb{C}$-analytic completion $\mathcal{S}$ of $\mathcal{H} \otimes \mathbb{C}$ which was defined by Harish-Chandra in a way analogous to the way one constructs the ring $C_0^\infty(X)$ of certain smooth functions on an algebraic complex affine variety X from its ring $\mathbb{C}[X]$ of algebraic functions. A natural question then is : what does happen if these constructions are carried over to the completion $\mathbb{C}_l$ of $\overline{\mathbb{Q}}_l$, for example ? As a motivation, recall that some techniques of commutative algebra to lift idempotents or solutions of polynomial equations from characteristic l to characteristic 0 involve l-adic completions. In our case, can we get informations on the modular representations from the study of l-adic analytic properties of *zero*-characteristic representations ?

More generally, if K is a normed field, can we find asymptotic conditions on the matrix coefficients of admissible K-representations, so that we obtain an interesting theory ?

The starting point of this note is the observation after Langlands, Casselman, Bernstein and Harish-Chandra's works that the theory of discrete series and tempered *complex* representations, while apparently analytic, can be described in purely algebraic and combinatorial terms, essentially due to the well known principle : *the asymptotic behavior of the coefficients may be derived from the central exponents of the Jacquet modules.* So let us fix a field K with characteristic different from p and endow it with a non-trivial valuation ν. We define in part 3 a notion of ν-tempered and ν-discrete series in a way parallel with the complex case. Exactly as in the classical case, we show that normalized parabolic induction preserves ν-temperedness whereas restriction *a priori* does not, we define a "weak" Jacquet module which preserves ν-temperedness, and the notion of "ν-discrete support". For our purposes, the main observation is that the famous "Langlands quotient theorem" which classifies irreducible $\mathbb{C}$-representations in terms of tempered $\mathbb{C}$-representations of Levi subgroups turns out to be still true in the case of K-representations, only replacing "tempered" by "ν-tempered". The adaptation of the classical proofs to our more general setting is performed in part 3.

After this observation, many questions arise, *e.g.* do ν-discrete series always exist ? Are there correspondences between discrete series associated to different valuations on the same field ? Are there interesting applications ? The following conjecture (partially a theorem) sums up what should be true in general, regarding the first two questions :

Conjecture 1.1 *i) Suppose ν is non-Archimedean with $\nu(p) = 0$, then an irreducible representation is a ν-discrete series if and only if it is cuspidal with ν-bounded central character.*

ii) Let $\pi \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(G)$ with finite order central character, then for any archimedean, resp. p-adic, valuation ν_∞, resp. ν_p, of $\overline{\mathbb{Q}}$, π is ν_p-discrete if and only if its image under Zelevinski's involution is ν_∞-discrete.

In this paper, we won't discuss point ii) of this conjecture ; we hope to explain elsewhere why it is true at least for classical groups. In the remaining of this introduction, we concentrate on the case $\nu(p) = 0$; statement i) actually contains two quite different sub-cases, namely that of mixed characteristic coefficients – meaning that the characteristic and the residual characteristic are different, or in other words that the restriction of ν to the prime field is non-trivial – and that of "equal characteristic". In the latter case, we will show in 4.5 that statement i) is true as soon as G admits discrete co-compact subgroups, which in turn is always true if the field F has characteristic zero. In the former case, we will explain in 5.7 how statement i) follows for classical groups from known facts on Plancherel measures (or reducibility points), due to Moeglin.

Our applications to modular theory proceed from these facts and the observation that whenever we formally assume that ν-discrete series are cuspidal, the Langlands quotient theorem may be sharpened as follows (see 3.14) : *the induced representations from Langlands ν-tempered data have to be irreducible.* This is particularly convenient to study (ir)reducibility properties of the parabolic induction functors. For example, using statement i) of 1.1 for *equal characteristic* coefficients we will show in part 5 (see 5.6) the following

Theorem 1.2 *Assume G admits discrete co-compact subgroups and let k be any algebraically closed field of characteristic $l \neq p$. Fix also a parabolic subgroup P with Levi component M and an irreducible smooth k-representation π of M.*

i) The set of those unramified k-characters ψ of M such that $i_{M,P}^G(\pi\psi)$ is irreducible is Zariski-dense in the algebraic k-torus of all unramified k-characters of M.

ii) The set of those unramified characters ψ of M such that $i_M^G(\pi\psi)$ has an elliptic *constituent consists in finitely many orbits under the group of unramified characters of G.*

iii) The set of those unramified characters ψ of M such that $i_M^G(\pi\psi)$ has a cuspidal *constituent consists in finitely many orbits under the group of unramified characters of G.*

Let us make some comments on this result. The first statement is known as "generic irreducibility" ; the proof here is new, even in the complex case and the result is new in positive characteristic. In the complex case, the only other proof known to the author relies on a unitary argument, which is unfortunately hidden in the redaction of [18]. More precisely, the argument of Waldspurger in *loc.cit.* is actually a reduction to the case where P is maximal and π is cuspidal. The latter case is not even proved in *loc.cit.*, essentially because it is considered trivial... and as a matter of fact it is so, *but* it relies on the classical "unitary trick" : any cuspidal, unitary and G-regular representation induces irreducibly (one can find the complete argument in [27, IV.2.2] for example). The reduction argument of Waldspurger certainly works also in the positive characteristic coefficients case, so that in principle it suffices to solve the "maximal-cuspidal" case. However we don't use this fact : in a sense, our arguments solve all cases simultaneously.

In point ii), a representation is called *elliptic* if its image in the Grothendieck group of finite length representations of G doesn't lie in the span of parabolically induced representations. For complex representations, this statement was proved in [3], with unitary arguments. The proof here is new and extends the result to positive characteristic.

Now, statement iii) needs some explanation for anyone acquainted with complex representation theory of p-adic groups but not with modular theory : as a matter of fact, when the characteristic of k divides the pro-order of a compact subgroup of G (so-called "non-banal" case), it may happen that cuspidal representations appear as subquotients of parabolically induced representations. We refer to [25] for terminology and basics on the differences between *cuspidal*, *supercuspidal*, and *projective* irreducible k-representations.

As for the assumption on G, we have already noticed that it is always fulfilled if the field F on which G is defined has characteristic 0. We mention here a work in progress by Bertrand Lemaire on techniques of lifting groups and Hecke algebras from local fields of characteristic p to p-adic fields, using "close fields" ; this work should imply the validity of the foregoing theorem when F has characteristic p.

As a major consequence of the generic irreducibility result, we can define an analog of Harish Chandra's j-function for an irreducible representation π as in the above theorem ; this is some rational function on the algebraic k-torus of unramified characters of M. The necessary extension of the classical theory of "meromorphic" intertwining operators to our setting is performed in part 7. In part 8, we apply this theory to modular representations. Proposition 8.2 connects the j-function of a $\overline{\mathbb{Q}}_l$-valued irreducible cuspidal and l-integral (*i.e.* admitting a stable $\overline{\mathbb{Z}}_l$-lattice) representation with that of an irreducible constituent of its reduction to $\overline{\mathbb{F}}_l$. Then, when l is non-banal, one may wonder if the latter j-function is of any use to track down those induced representations which have some cuspidal subquotient. Our results here are definitely partial ; one simple statement we obtain in this direction is the following :

Proposition 1.3 *Let G, P, M, π be as in the previous theorem. Assume further P to be maximal proper and π to be cuspidal. If $i_{M,P}^G(\pi)$ has a cuspidal subquotient then j_π has order of vanishing $\geqslant 2$ at the trivial unramified character.*

Recall that for complex representations, the zeros of the j-functions are known to be simple !

The next family of applications is the object of part 6. It uses the "mixed characteristic" case of statement i) of 1.1. The general subject is the study of lattices in l-adic representations.

For example, the first statement in the next proposition answers a natural question for people interested in reduction modulo l of l-adic representations : if a $\overline{\mathbb{Q}}_l$-representation admits stable $\overline{\mathbb{Z}}_l$-lattices, so its Jacquet modules do.

Proposition 1.4 *Assume that G is a classical group over a p-adic number field.*

i) If R is a noetherian Krull ring, then the parabolic restriction functors carry R-admissible R-torsion free representations to R-admissible R-torsion free representations (see 6.7).

ii) Any irreducible $\overline{\mathbb{F}}_l$-representation σ occurs as a subquotient of the mod-l reduction of a flat $\overline{\mathbb{Z}}_l$-admissible representation π_σ (see 6.8).

iii) In the former point, if σ is supercuspidal *and $\pi_\sigma \otimes \mathbb{Q}$ is irreducible then the latter must be supercuspidal (see 6.9).*

Now we briefly discuss the link with the following natural conjecture.

Conjecture 1.5 *For any compact open subgroup H, there is a compact-mod-center subset S_H in G which supports* all *bi-H-invariant cuspidal functions with values in* any *ring R.*

This second conjecture is addressed in another paper [10] – presently it is solved only for rank 1 groups ! However, some variants are already known : for example if "any ring R" is replaced by "any k-algebra R" for some field k, then the statement is known when the characteristic of k is *banal* (*i.e.* not dividing the order of a compact element) by [2] and [25, 2.15-2.17] (see the remark below 5.1 for a few more details).

We will prove in 4.10 that the latter conjecture implies statement i) of conjecture 1.1.

As another illustration of the ideas of this paper, we will prove in 5.7 the following

Proposition 1.6 *Assume $K = \overline{\mathbb{Q}}$ and π is an irreducible representation defined over $\overline{\mathbb{Z}}$ of the Levi component M of a maximal parabolic group P then conjecture 1.5 implies that $i_{M,P}^G(\pi\psi)$ is reducible only if $\psi(M \cap [G,G]) \subset \overline{\mathbb{Z}}[\frac{1}{p}]$.*

The second statement may be rephrased in terms of Plancherel measures : for any complex cuspidal representation with $\overline{\mathbb{Z}}$-valued central character of a Levi subgroup M, the poles of the associated μ-function are in $\overline{\mathbb{Z}}[\frac{1}{p}]$; this is far less explicit than Moeglin's results in the classical case [15] which assert that such poles actually are in $p^{\mathbb{Q}}$, not to speak of Shahidi's precise results in [19, Thm 8.1] when the inducing representation is generic ! The interesting feature of this statement is rather that it establises a somewhat surprising connection between supports of cuspidal functions and reducibility points.

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2 Notations and general facts

This section is mainly devoted to fixing notations and reminding known facts from the representation theory of G. Only our treatment of intertwining operators may be non-standard : we have to give a very general definition, even working when the coefficients are merely a ring. This definition is inspired by Waldspurger's proof of the rationality of classical intertwining operators in [27, IV].

2.1 *(Semi-)standard parabolic and Levi subgroups* : Let us fix some maximal split torus A_0 of G and some minimal parabolic subgroup P_0 containing A_0. Any parabolic subgroup containing A_0 will be called *semi-standard* ; if it contains also P_0 it will be called *standard*. A semi-standard (resp. standard) parabolic subgroup has a unique Levi factor containing A_0 ; such Levi subgroups will be called semi-standard (resp. standard). We abbreviate $M < G$ for "M is a standard Levi subgroup of G". Any $M < G$ determines a unique standard parabolic subgroup.

If M is a semi-standard Levi subgroup we note $W_M := \mathcal{N}_M(A_0)/\mathcal{Z}_M(A_0)$ its Weyl group relative to A_0, and we note A_M the (group of rational points of) the split component of the center of M.

For any locally pro-finite group H we note H^0 the subgroup generated by compact subgroups of H. An unramified character of H is a character which is trivial on H^0.

Eventually we fix a maximal open compact subgroup K such that $G = KP_0$.

2.2 *Some notations* : Hopefully the notations we introduce here coincide with those used by most authors in the domain. First of all, for any semi-standard M we put $a_M^* := X(A_M) \otimes \mathbb{R}$ where $X(A_M)$ is the lattice of rational characters of A_M. We also have $a_M^* = X(M) \otimes \mathbb{R}$ so that the canonical surjection $a_{M_0}^* \longrightarrow a_M^*$ induced by restriction $X(A_{M_0}) \longrightarrow X(A_M)$ has a canonical section which is induced by restriction $X(M) \longrightarrow X(M_0)$; we note $a_{M_0}^* = a_M^* \oplus a_{M_0}^{M*}$ the corresponding direct sum. It is convenient to endow $a_{M_0}^*$ with a W_G-invariant scalar product. The former decomposition is orthogonal w.r.t. this scalar product ; more precisely, if $\Sigma(M) \subset \Sigma(G)$ is the root system of M, then $a_M^* = \{\alpha \in a_{M_0}^*, \langle \alpha, \Sigma(M)\rangle = 0\}$ and $a_{M_0}^{M*}$ is the space generated by $\Sigma(M)$. If P is a parabolic subgroup containing M and $\Delta(P)$ denotes the set of simple roots of A_M in P, then we put

$$^+a_P^* := \sum_{\alpha \in \Delta(P)} \mathbb{R}_+^*.\alpha \subset a_M^* \quad \text{and} \quad (a_P^*)^+ := \{x \in a_M^*, \forall \alpha \in \Delta(P),\ \langle x, \alpha\rangle > 0\}$$

Notice that $^+a_P^* \subset a_M^{G*} := a_M^* \cap a_{M_0}^{G*}$. When P is a standard parabolic (hence $M < G$ and P is determined by M) we omit to specify P and just write $^+a_M^*$ and $(a_M^*)^+$.

Eventually if S is any subset of a_M^* we note $\overline{S}$ its closure for the real topology of a_M, and we note $-S$ its image by multiplication by -1. In the case $S = {}^+a_P^*$, we sometimes just write $-S = {}^-a_P^*$.

For any $M < G$, the set $\overline{(a_M^{G*})^+} := \overline{(a_M^*)^+} \cap a_M^{G*}$ is a closed convex cone which has a cellular decomposition into locally closed cones

$$\overline{(a_M^{G*})^+} = \bigsqcup_{M<N<G} (a_N^{G*})^+ \tag{2.3}$$

where $(a_G^{G*})^+$ is defined to be $\{0\}$. We will often use the following

Fact 2.4 *Let $N < G$ with associated standard parabolic subgroup $Q = NU_N$, and let $\mu \in (a_N^{G*})^+ \subset \overline{(a_{M_0}^{G*})^+}$. The set of those roots $\gamma \in \Sigma(G)$ such that $\langle \mu, \gamma\rangle > 0$ is the set $\Sigma(A_0, U_N)$ of roots of A_0 in Lie(U_N).*

Proof : By definition, $(a_N^{G*})^+$ is the set

$$\left\{x \in a_{M_0}^{G*},\ \ \alpha \in \Delta(P_0 \cap N) \Rightarrow \langle \alpha, x\rangle = 0 \quad \text{and} \quad \alpha \in \Delta(P_0) \setminus \Delta(P_0 \cap N) \Rightarrow \langle \alpha, x\rangle > 0\right\}.$$

Any $\gamma \in \Sigma(G)$ may be written as a sum $\gamma = \sum_{\alpha \in \Delta(P_0)} n_\alpha \alpha,\ \ n_\alpha \in \mathbb{Z}$ and we have the equivalence

$$\gamma \in \Sigma(A_0, U_N) \Leftrightarrow \exists \alpha \in \Delta(P_0) \setminus \Delta(P_0 \cap N) \text{ such that } n_\alpha > 0.$$

But since all n_α's must have the same sign, this is again equivalent to $\langle \mu, \gamma\rangle > 0$, by the description of $(a_N^{G*})^+$ above.

□

2.5 *Langlands' lemma* : the following assertion is known as "Langlands' combinatorial lemma". The present version is a translation into our notations of [6, IV.6.11] where a proof can be found.

Let $M < G$ and $\mu \in a_M^*$, then there is a unique $N < G$ such that $M < N$ and

$$\mu = {}^-\mu + \mu^+ + \mu_G \in \overline{{}^-a_M^{N*}} + (a_N^{G*})^+ + a_G$$

the decomposition being unique. The summands ${}^-\mu$, μ^+ and μ_G are mutually orthogonal and are defined as the orthogonal projections of μ on the respective convex closed sets $\overline{{}^-a_M^*}$, resp. $\overline{(a_M^*)^+}$ and a_G.

Here we write ${}^-a_M^{N*}$, (resp. $(a_M^{N*})^+$) for the projection of ${}^-a_M^*$ (resp. $(a_M^*)^+$) on a_M^{N*}. The latter a_M^{N*} has already been defined as the orthogonal space of a_N^* in a_M^*. Notice the following consequence of the uniqueness property : if we consider $\mu \in a_M^*$ as an element of $a_{M_0}^*$ and apply the lemma to get a $N_0 < G$ and a decomposition $\mu = {}^-\mu_0 + {\mu_0}^+ + \mu_G$, then we must have $N_0 = N$ and $({}^-\mu_0, {\mu_0}^+, \mu_G) = ({}^-\mu, \mu^+, \mu_G)$ with the notations of the lemma.

2.6 *General facts on the representation theory of G* : If R is a ring where p is invertible we note $Mod_R(G)$ the abelian category of all smooth representations of G with coefficients in R. Most of this paper deals with representations with coefficients in some field k of characteristic $\neq p$. Let us recall here some known facts in this context.

i) Assume k algebraically closed. If (π, V) is an irreducible smooth representation then it is admissible and has a central character [25, II.2.8].

ii) Let (π, V) be an admissible k-representation and χ a smooth character of A_G. Define

$$V_\chi := \{v \in V,\ \exists d \in \mathbb{N},\ \ \forall a \in A_G,\ \ (\pi(a) - \chi(a))^d v = 0\}$$

We call χ an exponent of A_G in π if $V_\chi \neq 0$ and we note $\mathcal{E}(A_G, \pi)$ the set of these exponents. When k is algebraically closed we have a G-equivariant decomposition

$$V = \bigoplus_{\chi \in \mathcal{E}(A_G, \pi)} V_\chi$$

which will be referred to as the exponent decomposition of (π, V).

iii) Assume that R contains a square root of p and choose such a root. Then one defines the *normalized* parabolic induction and restriction functors as in the complex coefficients case [25, p.97]. For a semi-standard parabolic subgroup P with Levi component M we use notations i_P^G, resp. r_P^M, for normalized induction, resp. restriction.

For $M < G$, we also write i_M^G and r_G^M, resp. $\overline{i_M^G}$ and $\overline{r_G^M}$, for the normalized induction and restriction along the standard parabolic subgroup associated to M, resp. along the opposite standard parabolic subgroup.

iv) Assume k algebraically closed. The parabolic functors send admissible, resp. finite length, representations to admissible, resp. finite length representations [25, II.5.13][1]

From [25, II.5.12], we can actually strengthen the length assertion in the following way. Let $M < G$ and $H_M \subset M$ an open compact subgroup of M. Then there is an open compact subgroup H of G such that for any admissible representation σ of M satisfying $\text{length}_{\mathcal{H}(M,H_M)}(\sigma^{H_M}) = \text{length}_M(\sigma)$, we have $\text{length}_{\mathcal{H}(G,H)}(i_M^G(\sigma)^H) = \text{length}_G(i_M^G(\sigma))$. Here the notations $\mathcal{H}(M, H_M)$ and $\mathcal{H}(G, H)$ denote the corresponding Hecke algebras with coefficients in k.

[1] Paragraph II.5 in [25] relies on the paper [16] of Moy and Prasad. According to one of the referees, the latter paper, as it is written, only applies to certain groups (after J.-K. Yu : quasi-split over a tamely ramified extension, with centralizer of a maximal split torus being an induced torus). In the meantime, J.-K. Yu has shown how to define Moy-Prasad's filtrations in the general case, so that graduate pieces on the Lie algebra and on the group correspond. Once these filtrations are defined, the arguments in [16] in principle work the same way. But we should warn the reader that it does not seem to have been yet written.

v) Besides the classical Frobenius reciprocity, there is the following adjointness property, which will be referred to as "Casselman reciprocity" or "second adjointness". For any *admissible* representations π of G and σ of M, we have [25, II.3.8]

$$Hom_G(i_P^G(\sigma),\pi) \simeq Hom_M(\sigma, r_G^{\overline{P}}(\pi))$$

where P is a semi-standard parabolic subgroup with Levi M and $\overline{P}$ is its opposite w.r.t M.

2.7 *Casselman's lemma* : Let (V,π) be an admissible representation of G over a field k, the contragredient of which is noted $(V^\vee,\pi^\vee)$, and $M < G$ with associated parabolic subgroup P. Casselman has constructed a natural M-equivariant non-degenerated pairing $\langle\ ,\ \rangle_M$ between $(V_P, r_G^M(\pi))$ and $(V_{\overline{P}}^\vee, \overline{r_G^M}(\pi^\vee))$ ([8, 4.2.3] for complex representations, [25, proof of II.3.8] in our more general setting). This pairing is responsible for what we called "Casselman's reciprocity" above. Another nice property of the pairing is the following one ([8, 4.3.3] or [27, I.4.3]). Let us define as usual a function

$$\begin{array}{rccl} H_0: & M_0 & \to & a_{M_0}^* \\ & m & \mapsto & H_0(m), \text{ such that } \forall x \in X(M),\ \langle H_0(m), x\rangle = \mathrm{val}_F(x(m)) \end{array}$$

Define a subset $M_0^+ := H_0^{-1}(\overline{(a_{M_0}^*)^+}) \subset M_0$. Notice that the set $A_{M_0} \cap M_0^+$ is the subset of those $a \in A_{M_0}$ such that $\mathrm{val}_F(\alpha(a)) \geqslant 0$ for each simple root α.

Fix $(v,v^\vee) \in V \times V^\vee$. There is $t > 0$ such that for any $m \in M_0^+$ we have

$$\langle \pi(m)v, v^\vee\rangle = \delta_{P_{m,t}}(m)^{1/2} \left\langle r_G^{M_{m,t}}(\pi)(m) v_{P_{m,t}}, v_{\overline{P_{m,t}}}^\vee \right\rangle_{M_{m,t}}$$

where $M_{m,t}$ is the standard Levi subgroup whose set of positive simple roots is $\Delta_{m,t} := \{\alpha \in \Delta_G,\ \langle \alpha, H_0(m)\rangle \leqslant t\}$, $P_{m,t}$ is the associated standard parabolic subgroup, and $(v_{P_{m,t}}, v_{\overline{P_{m,t}}}^\vee)$ stands for the image of $(v,v^\vee)$ in $V_{P_{m,t}} \times V_{\overline{P_{m,t}}}^\vee$.

2.8 *The geometric lemma* : [1],[8]. Let P, Q be two semi-standard parabolic subgroups with Levi component M and N. Recall that the Bruhat ordering on the set $W_N \backslash W_G / W_M$ associated to the pair (Q,P) is defined by

$$\overline{w} \underset{QP}{\leqslant} \overline{w'} \text{ if and only if } Q\overline{w}P \subseteq \overline{Q\overline{w'}P}$$

where the closure is taken for the p-adic topology of G. Let (σ, V) be a representation of M with coefficients in a ring R containing a square root of p^{-1}, and let us fix the following model for induction :

$$i_P^G(V) := \left\{ \text{smooth functions } f: G \longrightarrow V,\ \forall p = mn \in P,\ f(gmn) = \delta_P^{\frac{1}{2}}(m^{-1})\sigma(m^{-1})f(g) \right\},$$

on which G acts by left translations. Then for any ordering $\preceq$ on $W_N \backslash W_G / W_M$ refining the Bruhat ordering $\leqslant$, with associate strict ordering $\prec$, define

$$\widetilde{F}_{QP}^{\prec\overline{w}}(V) := \left\{ f \in i_P^G(V),\ Supp(f) \cap \left(\bigcup_{\overline{w'} \prec \overline{w}} Q\overline{w'}P \right) = \emptyset \right\}$$

and similarly define $\widetilde{F}_{QP}^{\preceq\overline{w}}(V)$ by replacing $\prec$ by $\preceq$. Notice that for any $\overline{w}$,

$$\begin{array}{rcl} \widetilde{F}_{QP}^{\prec\overline{w}}(V) \subseteq \widetilde{F}_{QP}^{\preceq\overline{w}}(V) & = & \left\{ f \in i_P^G(V),\ Supp(f) \cap \overline{Q\overline{w}P} \subset Q\overline{w}P \right\} \\ & = & \left\{ f \in i_P^G(V),\ Supp(f) \cap Q\overline{w}P \text{ is compact-mod-}P \right\} \end{array}$$

Note $F^{\prec\overline{w}}_{QP}(V)$ the image of $\widetilde{F}^{\prec\overline{w}}_{QP}(V)$ in $r^N_Q \circ i^G_P(V)$. We obtain sub-functors of the functor $r^N_Q \circ i^G_P$ and for each $\overline{w}$ and each $w \in \overline{w}$ there are isomorphisms

$$F^{\prec\overline{w}}_{QP}/F^{\preceq\overline{w}}_{QP} \xrightarrow{\sim} F^{<\overline{w}}_{QP}/F^{\leqslant\overline{w}}_{QP} \xrightarrow{\sim} i^N_{N\cap w(P)} \circ w \circ r^{w^{-1}(N)\cap M}_{w^{-1}(Q)\cap M}.$$

In particular, if $\preceq$ is a total ordering, we get a filtration of the functor $r^N_Q \circ i^G_P$ with associate graduate the sum over $\overline{w}$ of the RHS above.

In applications, it will be sometimes convenient to fix a suitable choice of representatives w of double cosets $\overline{w} \in W_N\backslash W_G/W_M$. When M and N are standard, we will always choose the so-called "minimal length", or "Kostant" representatives, as in [1, 2.11] or [8]. This set of representatives will be generally noted ${}^QW^P$. For such a representative w, if P, resp. Q, is standard then $w(P)\cap N$, resp $w^{-1}(Q)\cap M$, is standard too.

2.9 *Regular representations and intertwining operators :* let Q and P be two semi-standard parabolic subgroups with the same Levi component M. In the previous paragraph, we have defined two natural sub-functors

$$F^{\leqslant\overline{1}}_{QP} \subset F^{<\overline{1}}_{QP} \subset r^M_Q \circ i^G_P : Mod_R(M) \longrightarrow Mod_R(M)$$

and have reminded that there is an isomorphism $F^{<1}_{QP}/F^{\leqslant 1}_{QP} \xrightarrow{\sim} \mathrm{Id}_{Mod_R(M)}$. We don't need yet the definition of this isomorphism, but the interested reader is sent to 7.11 where this definition is made explicit and used. The fundamental example is when $Q = \overline{P}$; then $F^{\leqslant 1}_{\overline{P}P} = 0$ and $F^{<1}_{\overline{P}P} = r^M_{\overline{P}} \circ \widetilde{F}^{<1}_{\overline{P}P}$ where $\widetilde{F}^{<1}_{\overline{P}P}$ is the set of functions supported on $\overline{P}P$ in $i^G_P(V)$.

Now let (σ, V) be any smooth R-representation of M. By restriction it gives a (smooth) representation of A_M which extends to a R-algebra morphism $R[A_M] \longrightarrow End_R(V)$. We will abuse notations and write $\sigma_{|R[A_M]}$ for this morphism. Let us define the following ideals of the ring $R[A_M]$

$$\begin{array}{ll} I_\sigma & := ker\ \sigma_{|R[A_M]} \\ I^{QP}_\sigma & := ker\,(r^M_Q \circ i^G_P/F^{<1}_{QP})(\sigma)_{|R[A_M]} \\ I^{Q,\overline{w}}_\sigma & := ker\ w(r^{M\cap w^{-1}(M)}_{M\cap w^{-1}(Q)}(\sigma))_{|R[A_M]} \text{ where } w \in \overline{w} \in W_M\backslash W_G/W_M \end{array}$$

In the following lemma and its proof we abbreviate $\leqslant := \underset{QP}{\leqslant}$.

Lemma 2.10 *The following properties are equivalent :*

i) $I_\sigma + I^{QP}_\sigma = R[A_M]$

ii) $I_\sigma + \bigcap_{\overline{w}<1} I^{Q,\overline{w}}_\sigma = R[A_M]$

and when R is an algebraically closed field and σ has finite length, they are also equivalent to

iii) $\forall \overline{w} < 1,\ \ \mathcal{E}(A_M,\sigma) \cap \mathcal{E}(A_M, w(r^{M\cap w^{-1}(M)}_{M\cap w^{-1}(Q)}(\sigma))) = \emptyset$

When these properties hold for, we say that σ is (P,Q)-regular.

Proof : Let us choose some total ordering $\preceq$ on the set $W_M\backslash W_G/W_M$ refining the Bruhat ordering $\leqslant$ and such that $\{\overline{w} \prec 1\} = \{\overline{w} < 1\}$. Applying the previous paragraph on the geometric lemma, we see that the representation $(r^M_Q \circ i^G_P)/F^{<1}_{QP}(\sigma)$ has a filtration with subquotients isomorphic to $i^M_{M\cap w(P)} \circ w \circ r^{M\cap w^{-1}(M)}_{M\cap w^{-1}(Q)}(\sigma)$, for $\overline{w} < 1$. Moreover, by faithfulness of parabolic induction, we have

$$ker\left(i^M_{M\cap w(P)} \circ w \circ r^{M\cap w^{-1}(M)}_{M\cap w^{-1}(Q)}(\sigma)\right)_{|R[A_M]} = ker\left(w \circ r^{M\cap w^{-1}(M)}_{M\cap w^{-1}(Q)}(\sigma)\right)_{|R[A_M]}$$

Hence, bounding the cardinal $|\{\overline{w} < 1\}|$ by $|W_G|$, we get

$$\left(\bigcap_{\overline{w}<1} I^{Q,\overline{w}}_\sigma\right)^{|W_G|} \subseteq I^{QP}_\sigma \subseteq \bigcap_{\overline{w}<1} I^{Q,\overline{w}}_\sigma.$$

The first inequality shows $ii) \Rightarrow i)$ and the second one shows $i) \Rightarrow ii)$. Assuming now R to be an algebraically closed field and σ to have finite length, we get from the definition of exponents the existence of some integer d such that

$$\left(\bigcap_{\chi \in \mathcal{E}(A_M,\sigma)} ker\ \chi \right)^d \subset I_\sigma \subset \bigcap_{\chi \in \mathcal{E}(A_M,\sigma)} ker\ \chi$$

and

$$\left(\bigcap_{\chi \in \bigcup_{w<1} \mathcal{E}(A_M, w\mathbf{r}^{M\cap w^{-1}M}_{M\cap w^{-1}Q}\sigma)} ker\ \chi \right)^d \subset I_\sigma^{QP} \subset \bigcap_{\chi \in \bigcup_{w<1} \mathcal{E}(A_M, w\mathbf{r}^{M\cap w^{-1}M}_{M\cap w^{-1}Q}\sigma)} ker\ \chi.$$

Both right inequalities imply that $i) \Rightarrow iii)$. Conversely by the Nullstellensatz, iii) implies

$$\bigcap_{\chi \in \mathcal{E}(A_M,\sigma)} ker\ \chi + \bigcap_{\chi \in \bigcup_{w<1} \mathcal{E}(A_M, w\mathbf{r}^{M\cap w^{-1}M}_{M\cap w^{-1}Q}\sigma)} ker\ \chi = R[A_M],$$

which by both left inequalities above implies i).

□

Now, consider the embedding

$$\sigma \simeq (F^{<1}_{QP}/F^{\leqslant 1}_{QP})(\sigma) \hookrightarrow (r^M_Q \circ i^G_P/F^{\leqslant 1}_{QP})(\sigma)$$

If the representation σ is (P,Q)-regular, then the representation theory of A_M shows that this embedding has a unique M-equivariant retraction r_σ. Namely, the retraction r_σ is given by the action of any element $i \in I_\sigma^{QP}$ such that $1_{R[A_M]} \in (i + I_\sigma)$.

In this case, we define the *intertwining operator*

$$J_{Q|P}(\sigma) : i^G_P(\sigma) \longrightarrow i^G_Q(\sigma) \tag{2.11}$$

to be the G-equivariant morphism associated by Frobenius reciprocity to the M-equivariant morphism $r^M_Q \circ i^G_P(\sigma) \longrightarrow (r^M_Q \circ i^G_P/F^{\leqslant 1}_{QP})(\sigma) \xrightarrow{r_\sigma} \sigma$.

In part 7, we will show how some known properties of classical complex intertwining operators still hold in this more general context. But in the beginning of this paper we won't need these properties and will only use the case when R is an algebraically closed field.

3 ν-tempered representations

In this section, we assume that $R = \mathcal{K}$ is a field of *characteristic different from p*, endowed with a norm $|.| : \mathcal{K} \longrightarrow \mathbb{R}_+$. We will call the function $\nu = -log|.| : \mathcal{K}^* \longrightarrow \mathbb{R}$ the associated *valuation*. We will be mainly concerned with non-Archimedean norms, in which case ν is a height 1 valuation in the usual sense, but in the beginning of our discussion we may allow any kind of norm. We will define and study ν-tempered representations, in analogy with complex tempered representations. Most of the statements and proofs are mere transcriptions of the classical ones, but sometimes we have to be careful with the characteristic. In some sense, the only "really new" statements are proposition 3.14 and lemma 3.16.

Let $M < G$ be a standard Levi subgroup of G, and recall M^0 is the subgroup of M generated by compact subgroups. We write $\Psi_\mathcal{K}(M) = Hom_\mathbb{Z}(M/M^0, \mathcal{K}^*)$ for the torus of unramified $\mathcal{K}$-valued characters and $\mathcal{C}_\mathcal{K}(A_M)$ for the group of smooth $\mathcal{K}$-characters of A_M. The valuation ν gives maps

$$\begin{array}{rccc} \nu : & \Psi_\mathcal{K}(M) & \to & a^*_M \\ & \psi & \mapsto & \nu \circ \psi \end{array} \quad \text{and} \quad \begin{array}{rccc} \nu : & \mathcal{C}_\mathcal{K}(A_M) & \to & a^*_M \\ & \chi & \mapsto & \nu \circ \chi \end{array}.$$

Here we use the standard isomorphism $X(A_M) \xrightarrow{\sim} Hom_{\mathbb{Z}}(A_M/{A_M}^0, \mathbb{Z})$ given by $\chi \mapsto -\mathrm{val}_F \circ \chi$. In contrast with the complex coefficients case, these maps ν need not be onto. Nevertheless, we have

Lemma 3.1 *Assume $\mathcal{K}$ algebraically closed. Then $(\mathrm{im}\ \nu)_M := \nu(\Psi_{\mathcal{K}}(M)) = \nu(\mathcal{C}_{\mathcal{K}}(A_M))$ and for $M < N$, the canonical map $a_M^* \longrightarrow a_N^*$, resp $a_N^* \longrightarrow a_M^*$, sends $(\mathrm{im}\ \nu)_M$ into $(\mathrm{im}\ \nu)_N$, resp. $(\mathrm{im}\ \nu)_N$ into $(\mathrm{im}\ \nu)_M$.*

Proof : Let us prove the first equality. If $\psi \in \Psi_{\mathcal{K}}(M)$ then by restriction it gives a smooth character of A_M and by definition we have $\nu(\psi) = \nu(\psi_{|A_M})$. Conversely, given a smooth character χ of A_M, we may first construct an unramified character χ^0 of A_M such that $\nu(\chi) = \nu(\chi^0)$. To do this, just choose a splitting $\iota :\ A_M/A_M^0 \hookrightarrow A_M$ of the canonical projection and put $\chi^0 := \chi \circ \iota$. Now, since $\mathcal{K}$ is algebraically closed, we may extend the character χ^0 of A_M/A_M^0 to a character ψ of M/M^0 and we get $\nu(\psi) = \nu(\chi)$.

Let us prove the second assertion. Let $\mu \in (im\ \nu)_M$ and choose $\chi \in \mathcal{C}_{\mathcal{K}}(A_M)$ such that $\mu = \nu(\chi)$. Since the projection $a_M^* \longrightarrow a_N^*$ is induced by the restriction map $X(A_M) \longrightarrow X(A_N)$, the image of μ coincides with $\nu(\chi_{|A_N})$. Now let $\mu \in (im\ \nu)_N$ and choose $\psi \in \Psi_{\mathcal{K}}(N)$ such that $\mu = \nu(\psi)$. Since the injection $a_N^* \hookrightarrow a_M^*$ is induced by the restriction map $X(N) \longrightarrow X(M)$, the image of μ in a_M^* coincides with $\nu(\psi_{|M})$.

□

We fix an algebraic closure $\mathcal{K}^a$ of $\mathcal{K}$. For any admissible smooth $\mathcal{K}$-representation π of G, define $\pi^a := \pi \otimes_{\mathcal{K}} \mathcal{K}^a$ the $\mathcal{K}^a$-representation of G obtained by scalars extension. It is still admissible. When $\mathcal{K}$ is assumed to be algebraically closed we drop the superscript a. In any case, $\mathcal{K}$ is assumed to contain a square root of p which is used to define normalized parabolic functors. These functors then commute with scalar extension.

Definition 3.2 *Assume $\mathcal{K}$ to be algebraically closed. An admissible representation of G over $\mathcal{K}$ is called*

- *"ν-tempered" if for any $M < G$ and any $\chi \in \mathcal{E}(A_M, r_G^M \pi)$ we have $-\nu(\chi) \in \overline{{}^+a_M^*}$.*
- *"a ν-discrete series" if for any $M < G$ and any $\chi \in \mathcal{E}(A_M, r_G^M \pi)$ we have $-\nu(\chi) \in {}^+a_M^*$.*

Let us rephrase this definition : recall $A_{M_0} \cap M_0^+ \subset A_{M_0}$ is the subset of those $a \in A_{M_0}$ such that $\mathrm{val}_F(\alpha(a)) \geqslant 0$ for each simple root α. Then a representation π is ν-tempered if and only if

$$\forall M < G,\ \forall a \in A_M \cap M_0^+,\ \forall \chi \in \mathcal{E}(A_M, r_G^M \pi),\ \text{ we have } |\chi(a)|_{\mathcal{K}} \leqslant 1.$$

Moreover it is a ν-discrete series if and only if

$$\forall M < G,\ \forall a \in A_M \cap M_0^+,\ \forall \chi \in \mathcal{E}(A_M, r_G^M \pi),\ \text{ we have } |\chi(a)|_{\mathcal{K}} < 1.$$

When $\mathcal{K} = \mathbb{C}$ and $|.|$ is the usual norm, this is just one of the usual equivalent definitions of temperedness and discreteness.

Examples : Simple applications of the foregoing definition show that :

- The trivial representation is non-tempered over $\mathbb{C}$. It is is ν-tempered non-discrete when $\mathcal{K} = \overline{\mathbb{Q}}_l$ with l-adic valuation, $l \neq p$, whereas it is ν-discrete when $\mathcal{K} = \overline{\mathbb{Q}}_p$ with p-adic valuation.
- The Steinberg representation is discrete over $\mathbb{C}$, tempered non-discrete over $\overline{\mathbb{Q}}_l$, and non-tempered over $\overline{\mathbb{Q}}_p$.
- A cuspidal representation is ν-discrete if and only if it is ν-tempered, if and only if its central character is "unitary" or "integrally valued" in the sense that it takes values in the subgroup $\nu^{-1}(0)$ of $\mathcal{K}^*$.

It will be convenient to have a definition over non-algebraically closed fields.

Lemma 3.3 *Let (π, V) be an admissible representation of G over $\mathcal{K}$. Then the following properties are equivalent :*

i) There is an extension ν^a of ν to $\mathcal{K}^a$ such that the representation (π^a, V^a) is ν^a-tempered.

ii) For any extension ν^a of ν to $\mathcal{K}^a$, the representation (π^a, V^a) is ν^a-tempered.

When these properties hold true, (π, V) is said to be ν-tempered. The same equivalence and the same definition apply with "discrete series" instead of "tempered".

Proof : Since extensions ν^a of ν correspond bijectively to embeddings $\mathcal{K}^a \hookrightarrow \widehat{\mathcal{K}}^a$ where $\widehat{\mathcal{K}}$ is the completion of $\mathcal{K}$ w.r.t. ν, they are transitively permuted by the action of the Galois group $Gal(\mathcal{K}^a/\mathcal{K})$ given by $(\gamma, \nu^a) \mapsto \nu^a \circ \gamma$. But for any $M < G$, the set of exponents $\mathcal{E}(A_M, \pi^a)$ is stable under the action of $Gal(\mathcal{K}^a/\mathcal{K})$ given by $(\gamma, \chi) \mapsto \gamma \circ \chi$. This yields the equivalence. □

As we are going to show in this section, many properties of complex tempered representations and discrete series hold true in our more general setting.

Lemma 3.4 *Assume $\mathcal{K}$ algebraically closed except in point i).*

i) Parabolic induction (normalized) preserves ν-temperedness.

ii) Parabolic restriction a priori *doesn't preserve ν-temperedness. But if π is a ν-tempered representation of G and $M < G$, then we can decompose $r_G^M(\pi) = r_G^M(\pi)^t \oplus r_G^M(\pi)^+$ with $r_G^M(\pi)^t$ tempered and $r_G^M(\pi)^+$ having no tempered subquotient.*

iii) If π is ν-tempered and if for any $M < G$ we have $r_G^M(\pi)^t = 0$, then π is a ν-discrete series.

Proof : Since ν-temperedness is checked on scalar extension to an algebraic closure, we may assume $\mathcal{K}$ algebraically closed to prove point i). For this proof, we send the reader to [27, III.2.3]. The argument there is based on a Jacquet module calculation using the geometric lemma which works the same way in our setting ; the only thing to do is to replace $Re(\chi)$ in *loc.cit.* by $-\nu(\chi)$.

In order to define the decomposition of $r_G^M(\pi)$ in point ii), we use the exponent decomposition and group the terms according to

$$r_G^M(\pi)^t := \bigoplus_{\chi \in \mathcal{E}(A_M, r_G^M(\pi)), \nu(\chi)=0} \left(r_G^M(\pi)\right)_\chi \quad \text{and} \quad r_G^M(\pi)^+ := \bigoplus_{\chi \in \mathcal{E}(A_M, r_G^M(\pi)), \nu(\chi)\neq 0} \left(r_G^M(\pi)\right)_\chi$$

By definition the second summand cannot be tempered. The proof that the first summand is tempered goes the same way as [27, III.3.1].

Eventually, point iii) is proved exactly as in [27, III.3.2]. □

Proposition 3.5 *(Discrete support) Assume $\mathcal{K}$ algebraically closed and let $\pi \in \mathrm{Irr}_{\mathcal{K}}(G)$ be ν-tempered. There exist $M < G$ and a ν-discrete representation $\sigma \in \mathrm{Irr}_{\mathcal{K}}(M)$ such that $\pi \hookrightarrow i_M^G(\sigma)$. Moreover the pair (M, σ) is unique up to G-conjugacy. Its conjugacy class is called the "discrete" support of π (by analogy with the notion of cuspidal support).*

Notice that in the complex case [27, III.4], the unitarity of tempered representations and of parabolic induction make it possible to strengthen the uniqueness statement by putting "π occurs as a constituent of $i_M^G(\sigma)$" instead of "$\pi \hookrightarrow i_M^G(\sigma)$". In our general context, we haven't unitarity hence no criterion for complete reducibility.

Proof : We follow [27, III.4.1]. Let $M < G$ be minimal for the condition $r_G^M(\pi)^t \neq 0$ and let σ be a quotient of the latter representation. Then from ii) in the previous lemma, σ is a ν-tempered representation of M, and from iii) in this lemma, it is also ν-discrete. By Frobenius reciprocity we get an embedding $\pi \hookrightarrow i_M^G(\sigma)$ whence the existence statement.

Suppose given a second pair (M', σ') with $M' < G$, σ' a ν-discrete representation of M' and an embedding $\pi \hookrightarrow i_{M'}^G(\sigma')$. By exactness of r_G^M and the $(t,+)$ decomposition in point ii) of the last lemma we get an injective map

$$r_G^M(\pi)^t \hookrightarrow r_G^M(i_{M'}^G(\sigma'))^t.$$

On the other hand, the geometric lemma shows that the RHS has a filtration whose successive quotients are of the form

$$i_{M\cap w(M')}^M(w(r_{M'}^{M'\cap w^{-1}(M)}(\sigma')^t)), \quad w \in {}^M W^{M'}.$$

But by ν-discreteness of σ' and point iii) in the last lemma, all these subquotients vanish unless $w(M') = M$. Since σ is an irreducible quotient of $r_G^M(\pi)^t$, it has to be isomorphic to some $w(\sigma')$. Hence the pair (M, σ) is associate to the pair (M', σ'). □

Definition 3.6 *In this paper, we will call* Langlands triple *any triple (M, σ, ψ) consisting of a standard Levi subgroup $M < G$, a ν-tempered representation σ and an unramified character $\psi \in \Psi_{\mathcal{K}}(M)$ of M such that $-\nu(\psi) \in (a_M^*)^+$.*

Lemma 3.7 *Let (M, σ, ψ) be a Langlands triple and P be the standard parabolic subgroup with Levi component M. Then the representation $\sigma\psi$ is $(P, \overline{P})$-regular in the sense of paragraph 2.10. In particular, there is a canonical intertwining operator*

$$J_{\overline{P}|P}(\sigma\chi) : i_M^G(\sigma\chi) \longrightarrow \overline{i_M^G}(\sigma\chi).$$

Proof : Let us use the geometric lemma to analyze the set of the various exponents of A_M in $\overline{r_G^M} \circ i_M^G(\sigma\psi)$. This set is

$$\left\{ w(\psi)_{|A_M} w(\chi)_{|A_M}, \ \ \chi \in \mathcal{E}\left(A_{M\cap w^{-1}(M)}, \overline{r_M^{M\cap w^{-1}(M)}}(\sigma)\right), \ \ w \in {}^P W^P \right\}.$$

Recall from 2.8 that ${}^P W^P$ is the set of minimal length representatives of double classes modulo W_M in W_G. By the definition of ν-temperedness, one shows as in [27, proof of III.2.3] (just replacing $Re(\chi)$ in *loc. cit.* by $-\nu(\chi)$) that for any w and any χ as above we have

$$-\nu(w(\chi)_{|A_M}) \in \overline{{}^+a_{\overline{P}}^*} = -(\overline{{}^+a_M^*}). \tag{3.8}$$

On the other hand, we claim that the negativity assumption on $\nu(\psi)$ implies that for any $w \neq 1$ as above we have

$$-\nu(w(\psi)_{|A_M}) \in -\nu(\psi) - (\overline{{}^+a_M^*} \setminus \{0\}). \tag{3.9}$$

To prove the claim, we first prove the following statement

$$\text{Let } \mu \in (a_{M_0}^*)^+. \ \text{Then} \quad \forall w \in W_G, \ \mu - w\mu \in \overline{{}^+a_{M_0}^*}. \tag{3.10}$$

We will prove this statement by induction on the length $l(w)$. Let us enumerate $\alpha_1, \cdots, \alpha_n$ the simple roots of A_{M_0} and note $\beta_1, \cdots, \beta_n$ the dual basis defined by $\langle \beta_i, \alpha_j \rangle = \delta_{ij}$. We need to check that for each $i = 1, \cdots, n$, we have $\langle \mu - w\mu, \beta_i \rangle \geqslant 0$. Let us write $\langle \mu - w\mu, \beta_i \rangle = \langle \mu, \beta_i - w^{-1}\beta_i \rangle$. If $l(w) = 1$, there is some $j \in \{1, \cdots, n\}$ such that w is the reflexion s_{α_j}. Then we have

$$\langle \mu, \beta_i - w^{-1}\beta_i \rangle = \delta_{ij} \langle \mu, \alpha_j \rangle \geqslant 0$$

whence the case $l(w) = 1$. For general w, write $w = s_{\alpha_j} w'$ with $l(w) = 1 + l(w')$. Then we have

$$\langle \mu, \beta_i - w^{-1}\beta_i \rangle = \langle \mu, \beta_i - {w'}^{-1}\beta_i \rangle + \langle \mu, {w'}^{-1}(\beta_i - s_{\alpha_j}\beta_i) \rangle$$

By induction we may assume the first summand to be nonnegative. Then the second one is 0 if $j \neq i$ and is $\langle \mu, w'^{-1}\alpha_i \rangle$ if $j = i$. But the latter is positive since the property $l(w'^{-1}s_{\alpha_j}) = 1 + l(w'^{-1})$ is equivalent to $w'^{-1}\alpha_i \in \overline{{}^{+}a^*_{M_0}}$. By induction, statement 3.10 follows. As a consequence, we get

$$\text{Let } \mu \in (a^*_M)^+. \text{ Then } \quad \forall w \in W_G,\ \mu - (w\mu)_{|A_M} \in \overline{{}^{+}a^*_M}.$$

Here the notation $_{|A_M}$ denotes the projection onto a^*_M. This statement follows from the fact that this projection sends $\overline{{}^{+}a^*_{M_0}}$ into $\overline{{}^{+}a^*_M}$.

To finish the proof of 3.9, it remains to prove that for any $\mu \in (a^*_M)^+$, if $w \in {}^PW^P$ is such that $w\mu = \mu$, then $w = 1$. First of all, recall from [1, 2.11] that elements in ${}^PW^P$ satisfy $w^{-1}(M \cap P_0) \subset P_0$. In other words, for any $\alpha \in \Delta(M \cap P_0)$ (the latter is the set of simple roots of A_0 in $\mathrm{Lie}(P_0 \cap M)$), we have $w^{-1}(\alpha) \in \overline{{}^{+}a^*_{M_0}}$. Now pick up $\alpha \in \Delta(P_0) \setminus \Delta(M \cap P_0)$. We have $\langle \mu, w^{-1}\alpha \rangle = \langle \mu, \alpha \rangle > 0$ which implies by fact 2.4 that $w^{-1}\alpha \in \overline{{}^{+}a^*_{M_0}}$. Finally we get $w^{-1}(\Delta(P_0)) \subset \overline{{}^{+}a^*_{M_0}}$ which means that w^{-1} has length 0 hence $w = 1$ (recall that $l(w) = |\{\alpha \in \Sigma(A_{M_0}, P_0), w(\alpha) \in \overline{{}^{-}a^*_{M_0}}\}|$).

Eventually, 3.8 and 3.9 show that $\sigma\psi$ is the only irreducible subquotient of $\overline{r^M_G} \circ i^G_M(\sigma\psi)$ whose central exponent is sent on $\nu(\psi)$. In particular the representation $\sigma\psi$ is $(P, \overline{P})$-regular in the sense of 2.10.

□

Notice that in the complex case, there is an alternative construction of intertwining operators via converging integrals (see [27, IV.1.1] for example) ; this construction also works in our more general context, if we assume further $\mathcal{K}$ to be complete. Anyway we won't use this fact here.

The relevance of the complex tempered representations in the representation theory is enlightened by the Langlands quotient theorem which classifies all irreducible complex representations in terms of tempered data. Due to the combinatorial nature of Langlands' original arguments, we have a similar result in our context.

Theorem 3.11 *(Langlands). Assume $\mathcal{K}$ algebraically closed.*

i) Let (M, σ, ψ) be a Langlands triple as in 3.6. Then the representation $i^G_M(\sigma\psi)$ has a unique irreducible quotient, noted $J(M, \sigma, \psi)$.

ii) If $J(M_1, \sigma_1, \psi_1) \simeq J(M_2, \sigma_2, \psi_2)$ then $M_1 = M_2$, $\nu(\psi_2\psi_1^{-1}) = 0$ and $\sigma_1\psi_1 \simeq \sigma_2\psi_2$

iii) Let $\pi \in \mathrm{Irr}_{\mathcal{K}}(G)$. There exists a triple (M, σ, ψ) as in i) such that $\pi \simeq J(P, \sigma, \psi)$.

There are two proofs of this fact in the complex case, one in [22] and one in [6, ch.XI]. The proof below is a mix of these two ones.

Proof : i) let (M, σ, ψ) be a Langlands triple. In the course of the proof of 3.7 we have shown that

$$dim_{\mathcal{K}}\left(Hom_G(i^G_M\sigma\psi, \overline{i^G_M}(\sigma\psi))\right) = 1$$

and have exhibited a distinguished generator $J_{\overline{P}|P}(\sigma\psi)$. Let π be any quotient of $i^G_M\sigma\psi$; we have an embedding $\sigma\psi \hookrightarrow \overline{r^M_G}\pi$ obtained by Casselman's reciprocity, see 2.6 v). Since $\sigma\psi$ is $(P, \overline{P})$-regular we get a retraction for this embedding. Hence, assuming π irreducible, Frobenius reciprocity provides us with an embedding $\pi \hookrightarrow \overline{i^G_M}(\sigma\psi)$. Since the composite $i^G_M(\sigma\psi) \longrightarrow \pi \longrightarrow \overline{i^G_M}(\sigma\psi)$ is non-zero, the above dimension equality implies that π is the only possible irreducible quotient of $i^G_M\sigma\psi$. Moreover, as an irreducible *subquotient* of $i^G_M\sigma\psi$, it is characterized by the property

$$\nu(\psi) \in \nu(\mathcal{E}(A_M, \overline{r^M_G}\pi)).$$

Proof of iii) : Let us fix some $\pi \in Irr_{\mathcal{K}}(G)$. Consider the set

$$\overline{\mathcal{E}}_\nu(\pi) := \bigcup_{M<G} \nu\left(\mathcal{E}(A_M, \overline{r^M_G}\pi)\right)$$

as a subset of $a^*_{M_0}$ and apply Langlands' lemma 2.5 to get for each $\mu \in -\overline{\mathcal{E}}_\nu(\pi)$ a decomposition $\mu = {}^-\mu + \mu^+ + \mu_G$, where $\mu^+ \in (a^{G*}_{N_\mu})^+$ for some $N_\mu < G$. Fix $\mu \in -\overline{\mathcal{E}}_\nu(\pi)$ such that μ^+ has maximal norm. Since $(\mu^+ + \mu_G) = \mu_{|A_{N_\mu}} \in -\nu(\mathcal{E}(A_{N_\mu}, \overline{r^{N_\mu}_G}\pi))$, we can find an irreducible subquotient ρ of $\overline{r^{N_\mu}_G}(\pi)$ whose central character satisfies $-\nu(\omega_\rho) = \mu^+ + \mu_G$. Using the exponent decomposition 2.6 ii), we may even assume ρ to be a subrepresentation of $\overline{r^{N_\mu}_G}(\pi)$.

Let ψ be any element of $\Psi_{\mathcal{K}}(N_\mu)$ such that $-\nu(\psi) = \mu^+ + \mu_G$ (use lemma 3.1) and put $\sigma := \rho\psi^{-1}$. We will show that σ is ν-tempered. Let us fix $M < N_\mu$ and pick some $\lambda \in -\nu(\mathcal{E}(A_M, \overline{r^M_{N_\mu}}\sigma))$. We have $\lambda \in a^{N_\mu *}_M$, hence λ and μ^+ are orthogonal and we have equalities

$$|\lambda|^2 + |\mu^+|^2 = |\lambda + \mu^+|^2 = |(\lambda + \mu^+)^+|^2 + |^-(\lambda + \mu^+)|^2.$$

Since $(\lambda + \mu^+ + \mu_G) \in -\nu(\mathcal{E}(A_M, \overline{r^M_G}\pi))$, our choice of μ implies that $|\mu^+| \geqslant |(\lambda + \mu^+)^+|$, hence the above equality implies that $|^-(\lambda + \mu^+)| \geqslant |\lambda|$. On another hand, since $-\mu^+ + (a^*_{M_0})^+ \supset (a^*_{M_0})^+$, we have the inequality

$$\mathrm{dist}(\lambda + \mu^+, \overline{(a^*_{M_0})^+}) = |^-(\lambda + \mu^+)| \leqslant |^-\lambda| = \mathrm{dist}(\lambda, \overline{(a^*_{M_0})^+})$$

Both inequalities imply that $|\lambda| = |^-\lambda|$, hence $\lambda = {}^-\lambda$, that is $\lambda \in \overline{{}^-a^{N_\mu *}_M}$. Letting λ vary, we get

$$-\nu(\mathcal{E}(A_M, \overline{r^M_{N_\mu}}\sigma)) \subset \overline{{}^-a^{N_\mu *}_M}, \quad \text{or equivalently} \quad -\nu(\mathcal{E}(A_{\overline{M}}, \overline{r^{\overline{M}}_{N_\mu}}\sigma)) \subset \overline{{}^+a^{N_\mu *}_{\overline{M}}}$$

putting $\overline{M} := w_M w_{N_\mu}(M) < N_\mu$ where $w_?$ is the longest element in $W_?$. Letting M vary, we get the ν-temperedness of σ. By our choice of σ and Casselman's reciprocity, we have a non-zero, hence surjective, morphism $i^G_{N_\mu}(\sigma\psi) \longrightarrow \pi$.

Proof of ii) : Let (M_i, σ_i, ψ_i) for $i = 1, 2$ be two Langlands triples and assume that the irreducible quotients $J(M_i, \sigma_i, \psi_i)$ given by point i) are isomorphic. From the proof of point i), this is equivalent to assuming that there is a non-zero morphism

$$i^G_{M_1}(\sigma_1\psi_1) \longrightarrow \overline{i^G_{M_2}}(\sigma_2\psi_2).$$

By Frobenius reciprocity, together with the geometric lemma, there exists a $w \in W_G$ having minimal length in its (W_{M_2}, W_{M_1}) double coset (so that $M_{2,w} := w(M_1) \cap M_2 < M_2$ and $M_{1,w} := w^{-1}(M_2) \cap M_1 < M_1$) and a morphism $i^{M_2}_{M_{2,w}} \circ w \circ \overline{r^{M_{1,w}}_{M_1}}(\sigma_1\psi_1) \longrightarrow \sigma_2\psi_2$. Applying then Casselman's reciprocity we get a non-zero morphism

$$w\left(\overline{r^{M_{1,w}}_{M_1}}(\sigma_1\psi_1)\right) \longrightarrow \overline{r^{M_{2,w}}_{M_2}}(\sigma_2\psi_2) \tag{3.12}$$

Hence, putting $\mu_i := -\nu(\psi_i) \in (a^*_{M_i})^+$, there are $\lambda_i \in -\nu(\mathcal{E}(A_{M_{i,w}}, \overline{r^{M_{i,w}}_{M_i}}(\sigma_i\psi_i)))$, $i = 1, 2$ such that $w(\lambda_1 + \mu_1) = \lambda_2 + \mu_2 =: \mu$. By ν-temperedness of the representations σ_i, we have $\lambda_i \in \overline{{}^-a^{M_i *}_{M_{i,w}}}$. Hence the decomposition of μ by Langlands' lemma 2.5 is given by

$${}^-\mu = \lambda_2 \in \overline{{}^-a^{M_2 *}_{M_{2,w}}}, \quad \mu^+ = (\mu_2)^+ \in (a^{G*}_{M_2})^+, \quad \text{and} \quad \mu_G = (\mu_2)_G \in a^*_G.$$

On the other hand, since w has minimal length in $W_{M_2} w W_{M_1} \subset W_G$ we have by [1, 2.11] $w(\overline{{}^+a^{M_1 *}_{M_0}}) \subset \overline{{}^+a^*_{M_0}}$. Since $w(a^*_{M_{1,w}}) \subset a^*_{M_{2,w}}$, it follows that $w(\overline{{}^-a^{M_1 *}_{M_{1,w}}}) \subset \overline{{}^-a^*_{M_{2,w}}}$, hence $w\lambda_1 \in \overline{{}^-a^*_{M_{2,w}}}$. Now let us look at the equation

$$\big(w\lambda_1 + {}^-(w\mu_1)\big) + (w\mu_1)^+ + (w\mu_1)_G = \lambda_2 + {\mu_2}^+ + (\mu_2)_G = \mu.$$

Of course we can subtract $(\mu_2)_G = (w\mu_1)_G = \mu_G$ from each expression. Putting $\mu^G := \mu - \mu_G$, we sum up

$$\begin{cases} (w\lambda_1 + {}^-(w\mu_1)) + (w\mu_1)^+ = \lambda_2 + {\mu_2}^+ = \mu^G \\ (\mu^G)^+ = {\mu_2}^+, \ \ {}^-(\mu^G) = \lambda_2 \\ (w\lambda_1 + {}^-(w\mu_1)) \in \overline{{}^-a^*_{M_{2,w}}}, \ \ (w\mu_1)^+ \in \overline{(a^{G*}_{M_{2,w}})^+} \end{cases}$$

Hence we have

$$|{\mu_2}^+| = \ \mathrm{dist}(\mu^G, \overline{{}^-a^*_{M_{2,w}}}) \leqslant |\mu^G - \big(w\lambda_1 + {}^-(w\mu_1)\big)\,| = |(w\mu_1)^+|. \tag{3.13}$$

Therefore we get

$$|\mu_2| = |{\mu_2}^+| + |(\mu_2)_G| \leqslant |(w\mu_1)^+| + |(w\mu_1)_G| \leqslant |w\mu_1| = |\mu_1|.$$

But since the (M_i, σ_i, ψ_i)'s play symmetric roles, the same argument shows that the converse inequality holds and thus we have $|\mu_1| = |\mu_2|$. Putting this in the above inequality, we get $|{\mu_2}^+| = |(w\mu_1)^+|$. Then 3.13 shows that $(w\lambda_1 + {}^-(w\mu_1))$ is the projection of μ^G onto $\overline{{}^-a^*_{M_{2,w}}}$ hence equals λ_2. But ${}^-(w\mu_1) = 0$ because $|w\mu_1| = |(w\mu_1)^+| + |(w\mu_1)_G|$, and eventually $w\lambda_1 = \lambda_2$. It follows that $w\mu_1 = \mu_2$.

Now we show that this implies $w = 1$, by an argument already used in the proof of 3.7. First of all, by the minimal length property of w we have $w^{-1}(M_2 \cap P_0) \subset P_0$, hence for any $\alpha \in \Delta(M_2 \cap P_0)$ we have $w^{-1}(\alpha) \in \overline{{}^+a^*_{M_0}}$. Let $\alpha \in \Delta(P_0) \setminus \Delta(M_2 \cap P_0)$. We have $\langle \mu_1, w^{-1}\alpha\rangle = \langle \mu_2, \alpha\rangle > 0$, and by 2.4 this implies $w^{-1}\alpha \in \overline{{}^+a^*_{M_0}}$. Eventually, we get $w^{-1}(\Delta(P_0)) \subset \overline{{}^+a^*_{M_0}}$ which means that w^{-1} has length 0 hence $w = 1$.

Thus we have shown that $\mu_1 = \mu_2$ and therefore $M_1 = M_2$. Moreover the non-zero morphism 3.12 is an isomorphism $\sigma_1\psi_1 \overset{\sim}{\longrightarrow} \sigma_2\psi_2$.

□

The following proposition is the main ingredient in our applications of these objects.

Proposition 3.14 *Assume $\mathcal{K}$ algebraically closed. The following properties are equivalent :*

i) For any $M < G$ and any $\pi \in \mathrm{Irr}_{\mathcal{K}}(M)$, if π is ν-discrete then it is cuspidal.

ii) For any $M < G$, the parabolic restriction r_G^M preserves ν-temperedness.

iii) For any Langlands' data (M, σ, χ) as in 3.11 i), the induced representation $\mathrm{i}_M^G(\sigma\chi)$ is irreducible.

Proof : i) implies ii). Fix a ν-tempered representation $\pi \in Irr_{\mathcal{K}}(G)$ and let (M, σ) be in the discrete support of π, as in proposition 3.5. Then under hypothesis i), σ has to be cuspidal. Now for any $N < G$, we have an embedding $\mathrm{r}_G^N(\pi) \hookrightarrow \mathrm{r}_G^N \circ \mathrm{i}_M^G\sigma$. The geometric lemma and the cuspidality of σ show that each subquotient of $\mathrm{r}_G^N\pi$ is a subquotient of some $\mathrm{i}^N_{w(M)}(w(\sigma))$ for some $w \in W_G$. But the latter is certainly ν-tempered, by 3.4 i), hence so is $\mathrm{r}_G^N\pi$.

ii) implies iii). We fix some Langlands' triple (M, σ, ψ) and pick an irreducible subrepresentation π of $\mathrm{i}_M^G(\sigma\psi)$. By 3.11 iii) $\pi \simeq J(N, \rho, \chi)$ for another Langlands' triple (N, ρ, χ). This yields a non-zero morphism $\mathrm{i}_N^G(\rho\chi) \longrightarrow \mathrm{i}_M^G(\sigma\psi)$. By Frobenius reciprocity, we have a non-zero morphism

$$\mathrm{r}_G^M\,\mathrm{i}_N^G(\rho\chi) \longrightarrow \sigma\psi$$

Hence by the geometric lemma there is some $w \in W_G$ such that w has minimal length in its double class $W_M w W_N$ (which implies that $M_w := M \cap w(N) < M$ and $N_w := N \cap w^{-1}(M) < N$ by [1, 2.12]) and a non-zero morphism

$$\mathrm{i}^M_{M_w} \circ w \circ \mathrm{r}_N^{N_w}(\rho\chi) \longrightarrow \sigma\psi.$$

Furthermore we can pick an irreducible constituent τ of $w(\mathrm{r}_N^{N_w}(\rho))$ and get a non-zero morphism $\mathrm{i}^M_{M_w}(\tau\chi^w\psi^{-1}) \longrightarrow \sigma$ or equivalently, by Casselman's reciprocity, a non zero morphism $\tau\chi^w\psi^{-1} \longrightarrow$

$\overline{r_M^{M_w}}(\sigma)$. Now, our hypothesis ii) implies that τ and $\overline{r_M^{M_w}}(\sigma)$ are ν-tempered. Hence we must have $\nu(\psi) = \nu(\chi)^w$. By an argument already used twice we show that this implies $w = 1$: first of all, by the minimal length property of w we have $w^{-1}(M \cap P_0) \subset P_0$, hence for any $\alpha \in \Delta(M \cap P_0)$ we have $w^{-1}\alpha \in \overline{{}^+a^*_{M_0}}$. Let $\alpha \in \Delta(P_0) \setminus \Delta(M \cap P_0)$. We have $\langle -\nu(\chi), w^{-1}\alpha\rangle = \langle -\nu(\psi), \alpha\rangle > 0$, and by 2.4 this implies $w^{-1}\alpha \in \overline{{}^+a^*_{M_0}}$. Eventually, we get $w^{-1}(\Delta(P_0)) \subset \overline{{}^+a^*_{M_0}}$ which means that w^{-1} has length 0 hence $w = 1$.

Hence $\nu(\chi) = \nu(\psi)$ and $M = N$. Now by the proof of 3.11 i), the set $\nu(\mathcal{E}(A_M, \overline{r_G^M} i_M^G(\sigma\psi)))$ contains $\nu(\psi)$ with multiplicity 1 and $J(M, \sigma, \psi)$ is the only constituent τ of $i_M^G(\sigma\psi)$ such that $\nu(\psi) \in \nu(\mathcal{E}(A_M, \overline{r_G^M}\tau))$. Since $\nu(\mathcal{E}(A_M, \overline{r_G^M}\pi))$, for the same reason, has to contain $\nu(\chi) = \nu(\psi)$, we deduce that both constituents have to coincide. Hence we have shown that $i_M^G(\sigma\psi)$ is irreducible.

iii) implies i). Let $\pi \in Irr_{\mathcal{K}}(G)$ be a ν-discrete representation. Assume that it is not cuspidal and choose a $N < G$ maximal and some non-zero quotient ρ of $r_G^N\pi$. By hypothesis iii) and 3.11 there is a Langlands' triple (relative to N) (M, σ, ψ) such that $\rho \simeq i_M^N(\sigma\psi)$. Let α be the simple root outside $\Delta(P_0 \cap N)$. There are two possibilities : either $-\nu(\psi) \in (a^*_M)^+$ or $\langle \alpha, -\nu(\psi)\rangle \leqslant 0$. The first case is impossible since it implies that $\pi \simeq i_M^G(\sigma\psi)$ and contradicts the uniqueness statement iii) of 3.11. So assume we are in the second case. Note P the *standard* parabolic subgroup of N containing M and $\overline{P}$ its opposite. Consider the semi-standard parabolic subgroup Q of G with Levi M such that $Q \cap N = \overline{P}$ and containing the standard unipotent radical associated with N (hence $\Delta(Q) = \{\alpha\} \cup \{-\beta\}_{\beta \in \Delta(P)}$). Since $\sigma\psi$ is a constituent of $\overline{r_N^M}(\rho)$, it is also a constituent of $r_Q^M\pi$. Hence $\nu(\psi) \in \nu(\mathcal{E}(A_M, r_Q^M\pi))$. But now for any $\gamma \in \Delta(Q)$ we have $\langle \gamma, -\nu(\psi)\rangle \leqslant 0$. In particular, $-\nu(\psi) \notin {}^+a^*_Q$, which contradicts the definition of being ν-discrete.

□

Remark 3.15 *If the equivalent properties of the proposition hold, an admissible $\mathcal{K}$-representation π of G is ν-tempered if and only if for any $M < G$ and any $\chi \in \mathcal{E}(A_M, r_G^M\pi)$ we have $\nu(\chi) = 0$.*

The next result will be useful for our applications to modular representation theory. It deals with parabolically induced modules from "almost" Langlands triples which by definition are data (M, σ, ψ) with σ a ν-tempered representation of $M < G$ and $\psi \in \Psi_{\mathcal{K}}(M)$ such that $-\nu(\psi) \in \overline{(a_M^{G*})^+}$ (closure).

Lemma 3.16 *Assume $\mathcal{K}$ to be algebraically closed and that the three equivalent points of proposition 3.14 hold. Let (M, σ, ψ) be an "almost" Langlands triple as above and $M < N < G$ the unique Levi subgroup such that $-\nu(\psi) \in (a_N^{G*})^+$ (recall the cellular decomposition of 2.3) . Then the length of $i_M^G(\sigma\psi)$ equals that of $i_M^N(\sigma\psi)$.*

Proof : Let $\phi : N \longrightarrow \mathcal{K}^*$ be an unramified character of N lifting the unramified character $\psi_{|A_N}$ of A_N. Then $\nu(\psi(\phi_{|M})^{-1}) \in a^*_M$ is the projection of $\nu(\psi)$ on a_M^{N*} along a^*_N ; it is zero since $\nu(\psi) \in a^*_N$. Hence the representation $i_M^N(\sigma\psi(\phi_{|M})^{-1}))$ is a ν-tempered representation by 3.4 i). Besides, $\nu(\phi)$ is the projection of $\nu(\psi)$ on a^*_N along a_M^{N*} hence it lies in $-(a_N^{G*})^+$. For any simple subquotient ρ of $i_M^N(\sigma\psi(\phi_{|M})^{-1}))$, the triple (N, ρ, ϕ) is therefore a Langland's datum, so that $i_N^G(\rho\phi)$ is irreducible. Since

$$i_M^G(\sigma\psi) \simeq i_N^G(\phi i_M^N(\sigma\psi(\phi_{|M})^{-1})))$$

the lemma follows. □

3.17 *ν-temperedness and coefficients* : Now recall that complex tempered representations may also be defined by some growth conditions on the matrix coefficients. To this end Harish-Chandra has defined a function Ξ on G by

$$\Xi(g) := \int_K \delta_{P_0}(kg)^{1/2} dk$$

where δ_{P_0} is the module character of P_0 extended by right K-invariance to $G = P_0K$. Here we will actually use another function ; starting from the Iwasawa decomposition $G = KM_0^+K$, we may define

$$\widetilde{\Xi}(g) := \delta_{P_0}(m)^{1/2}, \quad g \in KmK,\ m \in M_0^+$$

Notice that Ξ and $\widetilde{\Xi}$ actually are $\mathbb{Z}[\frac{1}{\sqrt{p}}]$-valued hence may be considered as $\mathcal{K}$-valued functions, thanks to our choice of a root of p. From the estimations in [27, II.1.1] the function $\widetilde{\Xi}$ could be used in place of Ξ for expressing the growth conditions on coefficients of *complex* tempered representations. We won't recall this complex case and refer the reader to [27, III.2]. Instead, we focus from now on to the non-Archimedean case, *i.e.* we assume $|.|_{\mathcal{K}}$ to be non-Archimedean.

Definition 3.18 *We will say that a smooth function $f : G \longrightarrow \mathcal{K}$* essentially tends to 0 at infinity *if the sets $(|f|_{\mathcal{K}})^{-1}(\mathbb{R}_{\geqslant a})$ are compact-modulo-A_G for any $a \in \mathbb{R}_+^*$.*

For an admissible representation (π, V), we note $V^\vee$ its contragredient and for any $v, v^\vee \in V \times V^\vee$, the matrix coefficient $g \mapsto \langle gv, v^\vee\rangle$ is noted $c_{v,v^\vee}$.

Proposition 3.19 *An admissible representation (π, V) is ν-tempered, resp. ν-discrete, if and only if for any pair $(v_0, v_0^\vee) \in V \times V^\vee$, the function $\widetilde{\Xi}^{-1}c_{v_0,v_0^\vee}$ is $|.|_{\mathcal{K}}$-bounded, resp. essentially tends to 0 at infinity.*

Notice that in the archimedean case, there is an additional "polynomial" factor. The proof below explains why it disappears in the non-archimedean case.

Proof : We first treat the case when $\mathcal{K}$ is algebraically closed.

Estimating coefficients : we start recalling a general reduction argument in order to estimate coefficients, based on Casselman's lemma 2.7. This part of the argument is literally the same as in the complex case and the exposition is largely inspired from [27, III]. Nevertheless, the final estimate is different from the classical case, due to our hypothesis that $|.|_{\mathcal{K}}$ is non-archimedean. The notations being as in the statement of the proposition and as in 2.7, we are going to show there is a positive constant C such that

$$\forall g \in G,\ \ |\widetilde{\Xi}(g)^{-1}c_{v_0,v_0^\vee}(g)|_{\mathcal{K}} \leqslant C. \sup_{\substack{M<G \\ \chi\in\mathcal{E}(A_M, r_G^M\pi)}} \exp\left(\langle \nu(\chi), H_0(m_g)\rangle\right) \tag{3.20}$$

where m_g is any element in M_0^+ such that $g \in Km_gK$. Since $H_0(M_0^+) \subset \overline{(a_{M_0}^*)^+}$, this estimate shows that the function $\widetilde{\Xi}^{-1}c_{v,v^\vee}$ is bounded whenever π is ν-tempered. Moreover, since for any $x \in {}^+a_{M_0}^*$ and any $c \geqslant 0$, the set of elements in M_0^+ such that $\langle H_0(m), x\rangle \leqslant c$ is compact-modulo-A_G, the estimate also shows that the function $\widetilde{\Xi}^{-1}c_{v,v^\vee}$ essentially tends to 0 at infinity in the sense of 3.18 whenever π is ν-discrete.

Let us prove 3.20. Observe that by smoothness, the subsets $\pi(K).v_0 \subset V$ and $\pi(K).v_0^\vee \subset V^\vee$ are finite. Since by definition $|\widetilde{\Xi}(g)|_{\mathcal{K}} = |\delta_{P_0}(m_g)^{\frac{1}{2}}|_{\mathcal{K}}$, we have the following inequality

$$\forall g \in G,\ \ |\widetilde{\Xi}(g)^{-1}c_{v_0,v_0^\vee}(g)|_{\mathcal{K}} \leqslant \sup_{(v,v^\vee)\in\pi(K)v_0\times\pi(K)v_0^\vee} |\delta_{P_0}(m_g)^{-\frac{1}{2}}c_{v,v^\vee}(m_g)|_{\mathcal{K}}. \tag{3.21}$$

But by Casselman's lemma 2.7, there is some $t > 0$ such that for any $m \in M_0^+$ and any $(v, v^\vee) \in \pi(K)v_0 \times \pi(K)v_0^\vee$

$$c_{v,v^\vee}(m) = \delta_{P_{m,t}}(m)^{1/2}\left\langle r_G^{M_{m,t}}(\pi)(m)v_{P_{m,t}}, v^\vee_{\overline{P_{m,t}}}\right\rangle_{M_{m,t}}$$

Moreover, by definition of $M_{m,t}$ there are positive constants $C_{1,t}$ and $C_{2,t}$ such that

$$\forall m \in M_0^+,\ \ C_{1,t} \leqslant |\delta_{P_{m,t}}(m)^{1/2}\delta_{P_0}(m)^{-\frac{1}{2}}| \leqslant C_{2,t}$$

Now for all $M < G$, the set $\{m \in M_0^+,\ M_{m,t} = M\}$ is compact modulo the semi-group $A_M \cap M_0^+$. Let $X_*(A_M)$ be the free abelian group of co-characters of A_M and denote by $\lambda \in X_*(A_M) \mapsto \varpi_F^\lambda := \lambda(\varpi_F) \in A_M$ the evaluation map at some uniformizer ϖ_F of F. Then there is some compact subset $S_M \in M_0^+$ such that

$$\{m \in M_0^+,\ M_{m,t} = M\} \subset S_M \varpi_F^{X_*(A_M)}.$$

Writing P for the standard parabolic subgroup associated to M, let U_M be the set of finitely many vectors of the form $r_G^M(\pi)(s)v_P$ for $s \in S_M$ and let $C_M = \sup_{s\in S_M} |\delta_P(s)^{1/2}|_{\mathcal{K}}$. We have

$$|\delta_{P_0}(m_g)^{-\frac{1}{2}} c_{v,v^\vee}(m_g)|_{\mathcal{K}} \leqslant C_{2,t} C_{M_{m_g,t}} \sup_{u\in U_{m_g,t}} |\left\langle r_G^{M_{m_g,t}}(\pi)(\varpi_F^{\lambda_g})u, v^\vee_{P_{m_g,t}}\right\rangle_{M_{m_g,t}}|_{\mathcal{K}} \tag{3.22}$$

where λ_g is any element in $X_*(A_{M_{m_g,t}})$ such that $m_g \in S_{M_{m_g,t}}.\varpi_F^{\lambda_g}$. Notice that with this definition, there is a positive constant C_H such that

$$\forall x \in a^*_{M_0}, \forall g \in G,\ |\langle H_0(m_g), x\rangle - \langle \lambda_g, x\rangle| < C_H||x||. \tag{3.23}$$

Hence we are left to estimate finitely many functions of the form

$$f:\ \ \lambda \in X_*(A_M) \mapsto \left\langle r_G^M(\pi)(\varpi_F^\lambda)u, v_P^\vee\right\rangle_M$$

for some $M < G$. Let ρ be the (right) regular representation of $X_*(A_M)$ on the space $\mathcal{K}^{X_*(A_M)}$ of all $\mathcal{K}$-valued functions on $X_*(A_M)$. By definition of the exponents, there are integers d_χ, $\chi \in \mathcal{E}(A_M, r_G^M\pi)$ such that a function f as above is annihilated by all operators

$$\prod_{\chi\in\mathcal{E}(A_M, r_G^M\pi)} \left(\rho(\lambda) - \chi(\varpi_F^\lambda)\right)^{d_\chi},\ \ \lambda \in X_*(A_M).$$

But such a function can be written as in [27, I.2]

$$f(\lambda) = \sum_{\chi\in\mathcal{E}(A_M, r_G^M\pi)} \chi(\varpi_F^\lambda) Q_\chi(\lambda)$$

where $Q_\chi \in Sym(X(A_M) \otimes_{\mathbb{Z}} \mathcal{K})$ (the symmetric algebra). In other words, $Q_\chi(\lambda)$ is a polynomial expression with coefficients in $\mathcal{K}$ in the $\langle\lambda, \alpha\rangle \in \mathbb{Z}$, α exhausting some basis of $X(A_M)$. It follows that the function $\lambda \mapsto Q_\chi(\lambda)$ is $|.|_{\mathcal{K}}$-bounded on $X_*(A_M)$ *because we assumed* $|.|_{\mathcal{K}}$ *to be a non-archimedean norm*. Hence

$$\forall \lambda \in X_*(A_M),\ \ |f(\lambda)| \leqslant \sup_{\chi\in\mathcal{E}(A_M, r_G^M\pi)} |\chi(\varpi_F\lambda)|_{\mathcal{K}}. \tag{3.24}$$

Remembering that $|\chi(\varpi_F\lambda)|_{\mathcal{K}} = \exp(\langle\nu(\chi), \lambda\rangle)$, we get 3.20 from 3.21, 3.22, 3.23 and 3.24.

Estimating exponents : We use the following statement from [27, preuve de III.2.2], the proof of which is the same in our context as in the classical context and therefore is omitted. For any $\pi \in Irr_{\mathcal{K}}(G)$, any $M < G$ and any $\chi \in \mathcal{E}(A_M, r_G^M\pi)$, there exist a pair $(v, v^\vee)$ and $t > 0$ such that

$$\forall a \in A_M,\ \ (\forall \alpha \in \Delta(P),\ \mathrm{val}_F(\alpha(a)) \geqslant t) \Rightarrow \chi(a) = \delta_P(a)^{-1/2} \langle \pi(a)v, v^\vee\rangle \tag{3.25}$$

This shows that if the function $\widetilde{\Xi}^{-1} c_{v,v^\vee}$ is bounded, resp. tends to 0 at infinity, then the function χ on $A_M \cap M_0^+$ is bounded, resp. tends to 0 at infinity. Varying the exponent χ, we get the ν-temperedness, resp. the ν-discreteness, of π.

It remains to treat the non-algebraically closed case. Start with a ν-tempered, resp. ν-discrete, representation π and a pair $(v, v^\vee)$ as in the proposition. Choose a valuation ν^a of $\mathcal{K}^a$ extending ν. Then π^a is ν^a-tempered, resp ν^a-discrete, and by the foregoing discussion, the function $|\widetilde{\Xi}^{-1} c_{v,v^\vee}|_{\mathcal{K}^a}$ is bounded, resp. tends to 0 at infinity. But we have $|\widetilde{\Xi}^{-1} c_{v,v^\vee}|_{\mathcal{K}^a} = |\widetilde{\Xi}^{-1} c_{v,v^\vee}|_{\mathcal{K}}$.

Conversely, start with a representation (π, V) such that for any pair $(v, v^\vee)$, the function $|\widetilde{\Xi}^{-1} c_{v,v^\vee}|_{\mathcal{K}}$ is bounded, resp. tends to 0 at infinity. Since any coefficient $c_{v^a,v^{a\vee}}$ of π^a is a $\mathcal{K}^a$-linear combination of coefficients of π, functions of the type $|\widetilde{\Xi}^{-1} c_{v^a,v^{a\vee}}|_{\mathcal{K}^a}$ also are bounded, resp. tend to 0 at infinity. By the first part of the proof, the representation π^a is therefore ν^a-tempered, resp. ν^a-discrete. Varying ν^a, we find that π is ν-tempered, resp. ν-discrete.

□

4 Features of the case $\nu(p) = 0$

From now on, we assume $\nu(p) = 0$. In particular, for all $g \in G$, we have $|\widetilde{\Xi}(g)|_{\mathcal{K}} = 1$, and the asymptotic properties of coefficients in 3.19 simplify accordingly. The characteristic of $\mathcal{K}$ still may be positive, but we assume further that $\mathcal{K}$ is complete w.r.t its valuation ν.

4.1 *ν-completion of the Hecke algebra :* For a function $f : G \longrightarrow \mathcal{K}$, recall that we have defined in 3.18 a notion of "essential" convergence to 0 at infinity. We will also simply say that it "converges to zero at infinity" if for any $r \in \mathbb{R}_+^*$, the set $|f|_{\mathcal{K}}^{-1}(\mathbb{R}_{\geqslant r})$ is compact. For any compact open subgroup H of G, we put

$$\mathcal{S}_{\mathcal{K}}(H\backslash G/H) := \{f : \ H\backslash G/H \longrightarrow \mathcal{K}, \quad \text{which converge to 0 at infinity}\}$$

and endow it with the non-archimedean norm $||f|| := \sup_G |f(g)|_{\mathcal{K}}$. Besides, this space is nothing but the completion of the space $\mathcal{C}_{\mathcal{K}}^c(H\backslash G/H)$ of compactly supported bi-H-invariant $\mathcal{K}$-valued functions on G w.r.t. the norm $||.||$. Moreover we put

$$\mathcal{S}_{\mathcal{K}}(G) := \varinjlim_H \mathcal{S}_{\mathcal{K}}(H\backslash G/H)$$

which is a normed $\mathcal{K}$-space containing $\mathcal{C}_{\mathcal{K}}^c(G)$ as a dense subspace, but not complete for $||.||$.

Example : If $\mathcal{K}$ is a finite extension of $\mathbb{Q}_l$, then $\mathcal{S}_{\mathcal{K}}(H\backslash G/H)$ is just the l-adic completion of $\mathcal{C}_{\mathcal{K}}^c(H\backslash G/H)$. More precisely,

$$\mathcal{S}_{\mathcal{K}}(H\backslash G/H) \simeq \left(\varprojlim \mathcal{C}_{\mathbb{Z}/l^n\mathbb{Z}}^c(H\backslash G/H)\right) \otimes_{\mathbb{Z}_l} \mathcal{K}.$$

In particular, $\mathcal{S}_{\mathcal{K}}(H\backslash G/H)$ is endowed with a jointly continuous product which extends the convolution product of $\mathcal{C}_{\mathcal{K}}^c(H\backslash G/H)$.

We need also a "central-character" version of this construction. If $\omega : A_G \longrightarrow \mathcal{K}^*$ is a smooth character such that $\omega_{|H}$ is trivial, we put $\mathcal{C}_{\mathcal{K}}^c(H\backslash G/H, \omega)$ the set of bi-H-invariant functions with compact-modulo-A_G support and such that for all $a \in A_G$, we have $f(ag) = \omega(a)f(g)$. When $\nu(\omega) = 0$, we put

$$\mathcal{S}_{\mathcal{K}}(H\backslash G/H, \omega) := \{f \in \mathcal{C}_{\mathcal{K}}^c(H\backslash G/H, \omega) \ \text{ which essentially converge to 0 at infinity}\}$$

and $\mathcal{S}_{\mathcal{K}}(G, \omega) := \varinjlim_H \mathcal{S}_{\mathcal{K}}(H\backslash G/H, \omega)$.

Now, let dg be a Haar measure on G taking values in $\mathbb{Z}[\frac{1}{p}]$. We still write dg for the associated $\mathcal{K}$-valued Haar measure.

Lemma 4.2 *i) The functional $f \longrightarrow \int_G f(g)dg$ on $\mathcal{C}_{\mathcal{K}}^c(G)$ extends continuously to $\mathcal{S}_{\mathcal{K}}(G)$.*

ii) The convolution product of $\mathcal{C}_{\mathcal{K}}^c(G)$, resp. of $\mathcal{C}_{\mathcal{K}}^c(G, \omega)$, extends to a separately continuous product on $\mathcal{S}_{\mathcal{K}}(G)$, resp. on $\mathcal{S}_{\mathcal{K}}^c(G, \omega)$.

iii) If (π, V) is an admissible ν-tempered representation, then the structural morphism $\mathcal{C}^c_{\mathcal{K}}(G) \xrightarrow{\pi} End_{\mathcal{K}}(V)$ extends to $\mathcal{S}_{\mathcal{K}}(G)$. If π has a central character ω_π, then the structural morphism $\mathcal{C}^c_{\mathcal{K}}(G, \omega_\pi^{-1}) \xrightarrow{\pi} End_{\mathcal{K}}(V)$ extends to $\mathcal{S}_{\mathcal{K}}(G, \omega_\pi^{-1})$.

Proof : If $f \in \mathcal{S}_{\mathcal{K}}(G)$ is H-invariant, the series $\sum_{x \in G/H} f(x)\mathrm{vol}(H, dg)$ is convergent by completeness of $\mathcal{K}$ and the sum is independent of the choice of H by the packet summation theorem. This implies point i).

For f, g in $\mathcal{S}_{\mathcal{K}}(G)$, resp. in $\mathcal{S}_{\mathcal{K}}(G, \omega)$, the functions $x \mapsto f(yx)g(x^{-1})$, $x \in G$ are in $\mathcal{S}_{\mathcal{K}}(G)$, resp. in $\mathcal{S}_{\mathcal{K}}(G/A_G)$, and therefore we may put $f * g(y) := \int_G f(yx)g(x^{-1})dx$, resp. $f * g(y) := \int_{G/A_G} f(yx)g(x^{-1})dx$ by point i). Then one checks that

$$|f * g|_{\mathcal{K}}^{-1}(\mathbb{R}_{\geqslant r}) \subset \left(|f|_{\mathcal{K}}^{-1}(\mathbb{R}_{\geqslant r||g||^{-1}})\right) . \left(|g|_{\mathcal{K}}^{-1}(\mathbb{R}_{\geqslant r||f||^{-1}})\right)$$

which implies in turn that the product $f * g$ lies in $\mathcal{S}_{\mathcal{K}}(G)$, resp. in $\mathcal{S}_{\mathcal{K}}(G, \omega)$. Moreover by definition, we have $||f * g|| \leqslant ||f||.||g||$, whence point ii).

Now, let (V, π) be an admissible ν-tempered representation, $v \in V$ and $f \in \mathcal{S}_{\mathcal{K}}(G)$. For any $v^\vee \in V^\vee$, the function $g \mapsto f(g)\langle \pi(g)v, v^\vee \rangle$ lies in $\mathcal{S}_{\mathcal{K}}(G)$. Using point i), we may then define $\pi(f)v$ as the unique element of V satisfying :

$$\forall v^\vee \in V^\vee, \ \ \langle \pi(f)v, v^\vee \rangle = \int_G f(g)\langle \pi(g)v, v^\vee \rangle dg.$$

As a matter of fact, the RHS defines a linear form on $V^\vee$ which is fixed by any compact open subgroup H such that $f \in \mathcal{S}_{\mathcal{K}}(H \backslash G/H)$, hence which lies in $(V^\vee)^\vee \simeq V$ by admissibility.

When π has a central character ω_π, observe first that $\nu(\omega_\pi) = 0$. Then for $f \in \mathcal{S}_{\mathcal{K}}(G, \omega_\pi^{-1})$, the function $g \mapsto f(g)\langle \pi(g)v, v^\vee \rangle$ lies in $\mathcal{S}_{\mathcal{K}}(G/A_G)$ and we define the action by the same integral but over G/A_G.

□

Until the end of this section we assume that $\mathcal{K}$ is algebraically closed. Let (V, π) be an irreducible ν-discrete series with central character ω_π. Picking $v_0^\vee \in V^\vee$ and $v_0 \in V$, we get G-equivariant maps

$$\begin{array}{rccc} c_{v_0^\vee} : & V & \to & \mathcal{S}_{\mathcal{K}}(G, \omega_\pi) \\ & v & \mapsto & c_{v, v_0^\vee} \end{array} \quad \text{and} \quad \begin{array}{rccc} \pi_{v_0} : & \mathcal{S}_{\mathcal{K}}(G, \omega_\pi) & \to & V \\ & f & \mapsto & \pi(f^\vee)v_0 \end{array}, \tag{4.3}$$

where $f^\vee$ is defined by $f^\vee(x) := f(x^{-1})$. The composition of these two maps is an endomorphism of V given by the multiplication by some $d(v_0, v_0^\vee) \in \mathcal{K}$. By definition for all $(v_0, v_0^\vee, v, v^\vee)$ we have $d(v_0, v_0^\vee)\langle v, v^\vee \rangle = \int_{G/A_G} \langle g^{-1}v, v_0^\vee \rangle \langle gv_0, v^\vee \rangle dg$ hence by symmetry, there is some $d_\pi \in \mathcal{K}^*$ such that $d(v_0, v_0^\vee) = d_\pi \langle v_0, v_0^\vee \rangle$. The natural question in this context is whether d_π is zero or not ; in other words, does π have a formal degree ? Since we don't work with complex coefficients hence have no unitary trick, such questions generally are delicate.

The following result will be the central tool for our study of ν-discrete series and applications. It requires the group G to have discrete co-compact subgroups. By [5], this is true at least if G is defined over a characteristic zero field or if it is a linear group over a function field.

Lemma 4.4 *We assume here that the p-adic group G has discrete co-compact subgroups. If (V, π) is a ν-discrete series, then there is such a subgroup Γ such that some unramified twist $\psi\pi$ of π embeds G-equivariantly in the space $\mathcal{C}_{\mathcal{K}}(G/\Gamma)$ upon which G acts by left translation.*

Proof : First of all, we will fix, as we may, a twist $\psi\pi$ of π such that $\psi\pi$ has finite order central character ω. Then we may restrict our attention to subgroups Γ of the form $\Gamma = \Gamma'.Z$ where Γ' is a discrete co-compact subgroup of the derived group G' of G and Z is some discrete co-compact subgroup of A_G in the kernel of ω. For any such discrete co-compact $\Gamma = \Gamma'.Z \subset G$, we have a G-equivariant map

$$\begin{array}{rccc} t_\Gamma : & \mathcal{S}_{\mathcal{K}}(G, \omega) & \to & \mathcal{C}_{\mathcal{K}}(G/\Gamma) \\ & f & \mapsto & g \mapsto \sum_{\gamma \in \Gamma'} f(g\gamma) \end{array}$$

As a matter of fact, the sum is convergent by the discreteness of Γ' and the "essentially" zero limit at infinity of f. Moreover, given any non-zero function $f \in \mathcal{S}_{\mathcal{K}}(G,\omega)$, there is some Γ such that $t_\Gamma(f) \neq 0$. Indeed, if $g \in G$ is chosen such that $|f(g)|_{\mathcal{K}} = ||f||_{\mathcal{K}}$, it is sufficient to choose Γ such that $\Gamma \cap g^{-1}|f|_{\mathcal{K}}^{-1}(\mathbb{R}_{\geqslant ||f||_{\mathcal{K}}}) = \emptyset$ (this is possible since any Γ has a descending chain of finite index subgroups with trivial intersection).

Now the lemma follows from the fact that one can embed $\pi\psi$ into $\mathcal{S}_{\mathcal{K}}(G,\omega)$ as in 4.3. □

Proposition 4.5 *Same hypothesis on G as in 4.4. Assume $\mathcal{K}$ contains a field k such that $\nu_{|k} = 0$. Then any ν-discrete representation is cuspidal.*

Proof : First of all, the largest field $k \subset \mathcal{K}$ with $\nu_{|k} = 0$ has to be algebraically closed, since $\mathcal{K}$ is supposed to be so. Let (π, V) be a ν-tempered irreducible $\mathcal{K}$-representation and let us embed it in the space of smooth functions $\mathcal{C}_{\mathcal{K}}(G/\Gamma)$ for some discrete co-compact subgroup Γ as in lemma 4.4 (after maybe replacing π by an unramified twist). If H is an open pro-p-subgroup such that $\pi^H \neq 0$, we thus get an embedding of π^H into the finite dimensional $\mathcal{K}$-space $\mathcal{C}_{\mathcal{K}}(H\backslash G/\Gamma)$.

Now let $\mathcal{H}_k(G,H)$ be the (G,H) Hecke algebra with coefficients in k. The finite dimensional k-space $\mathcal{C}_k(H\backslash G/\Gamma)$ has a filtration $0 = V_0(k) \subset V_1(k) \subset \cdots \subset V_n(k) = \mathcal{C}_k(H\backslash G/\Gamma)$ by $\mathcal{H}_k(G,H)$-submodules such that each $V_n(k)/V_{n-1}(k)$ is simple over $\mathcal{H}_k(G,H)$. By base change, this leads to a corresponding filtration $0 = V_0(\mathcal{K}) \subset V_1(\mathcal{K}) \subset \cdots \subset V_n(\mathcal{K}) = \mathcal{C}_{\mathcal{K}}(H\backslash G/\Gamma)$ of $\mathcal{C}_{\mathcal{K}}(H\backslash G/\Gamma)$ by $\mathcal{H}_{\mathcal{K}}(G,H)$-submodules. Since k is algebraically closed, the $V_n(\mathcal{K})/V_{n+1}(\mathcal{K})$ remain simple over $\mathcal{H}_{\mathcal{K}}(G,H)$.

By uniqueness of Jordan-Hölder factors, this implies that the simple $\mathcal{H}_{\mathcal{K}}(G,H)$-module π^H is a base change of some simple $\mathcal{H}_k(G,H)$-module. In other words, (π, V) has a non-zero coefficient c_π taking values in k. But since $\nu_{|k} = 0$, the only possibility for c to essentially tend to zero at infinity is to have a compact-modulo-A_G support. Hence (π, V) is cuspidal.

□

When $\mathcal{K}$ has mixed characteristic, we unfortunately cannot adapt the above argument to get the cuspidality of ν-discrete series. However, we have the weaker result 4.7 below, which will be enough to solve the case of classical groups. We need a preliminary lemma which has independent interest. Let us introduce the valuation ring $\mathcal{O} := \nu^{-1}(\mathbb{R}_{\geqslant 0})$ of $\mathcal{K}$.

Lemma 4.6 *Let $\mathcal{I}$ be any proper ideal of $\mathcal{O}$ and let (π, V) be a ν-discrete series. Then for any $\mathcal{O}$-valued coefficient c_π of π, the smooth $\mathcal{O}/\mathcal{I}$-valued function c_π mod $\mathcal{I}$ is cuspidal.*

Proof : We first check that π admits $\mathcal{O}$-valued coefficients. Observe that its central character ω_π is $\mathcal{O}$-valued since $\nu(\omega_\pi) = 0$ by definition of ν-discreteness. We will note $\overline{\omega_\pi}$ its reduction modulo $\mathcal{I}$. Since our hypothesis $\nu(p) = 0$ implies $|\widetilde{\Xi}|_{\mathcal{K}} = 1$, the matrix coefficients of π are $|.|_{\mathcal{K}}$-bounded by 3.19 hence become $\mathcal{O}$-valued after multiplication by some constant.

Now if c_π is as in the lemma. 3.19 tells us that it essentially tends to 0 at infinity, hence in particular the function

$$\begin{array}{ccc} G & \to & \mathcal{O}/\mathcal{I} \\ g & \mapsto & c_\pi(g) \mathrm{mod} \mathcal{I} \end{array}$$

has compact-modulo-A_G support. Moreover the G-submodule of $\mathcal{C}^c_{\mathcal{O}/\mathcal{I}}(G, \overline{\omega_\pi})$ generated by left translations of this function has to be $\mathcal{O}/\mathcal{I}$-admissible, because it is the image of that generated by left translations on c_π through the mod $\mathcal{I}$ map and because for any open pro-p-subgroup H, the H-invariant functor is exact on $\mathcal{O}$-valued smooth G-representations, since p is invertible in $\mathcal{O}$. Hence, this function is a cuspidal function. □

Proposition 4.7 *Same hypothesis on G as in 4.4. We assume here that $\mathcal{K}$ has characteristic 0. If (V, π) is a ν-discrete series with central character ω_π, then*

i) (V, π) has a formal degree in the sense of the discussion below 4.3, hence is a projective $\mathcal{S}_{\mathcal{K}}(G, \omega_\pi)$-module.

ii) For any embedding $\mathcal{K} \longrightarrow \mathbb{C}$, the complex representation $(V \otimes \mathbb{C}, \pi)$ is a (ordinary) discrete series.

Proof : First of all, we may change π by some unramified twist so that it has a finite order central character ω. Let $c_{v_0^\vee} : \pi \longrightarrow \mathcal{S}_{\mathcal{K}}(G, \omega)$ be the map defined in 4.3. With the notation introduced in the proof of 4.4, we fix a discrete co-compact subgroup $\Gamma = \Gamma'.Z$ such that $t_\Gamma \circ c_{v_0^\vee} \neq 0$. We also assume as we may that $\Gamma' \cap Z = \{1\}$.

Here, the most interesting property of $\mathcal{C}_{\mathcal{K}}(G/\Gamma)$ is that is is a semi-simple representation. A lazy way to see this is to observe that $\mathcal{K}$ must contain the algebraic closure of either one of the fields $\mathbb{Q}_l$, l prime or $\mathbb{Q}((X))$. The latter have the same cardinality as that of $\mathbb{C}$, hence their algebraic closure are abstractly isomorphic to $\mathbb{C}$. Thus there exists an embedding of fields $\mathbb{C} \hookrightarrow \mathcal{K}$! Then $\mathcal{C}_{\mathcal{K}}(G/\Gamma)$ is just a base change of $\mathcal{C}_{\mathbb{C}}(G/\Gamma)$ hence is semi-simple.

As a consequence there is a retraction $p : \mathcal{C}_{\mathcal{K}}(G/\Gamma) \longrightarrow V$ such that $p \circ t_\Gamma \circ c_{v_0^\vee} = Id$. We thus get the second assertion of point i). Now, the morphism $p \circ t_\Gamma : \mathcal{S}_{\mathcal{K}}(G) \longrightarrow V$ is given by a projective system $(v_H \in V^H)_{H \subset G}$ indexed by compact open subgroups in G such that for any $H \subset H'$, $\pi(e_{H'})v_H = v_{H'}$ with $e_{H'}$ the idempotent associated with H'. We then have

$$\forall H,\ \forall f \in \mathcal{S}_{\mathcal{K}}(G/H),\ \ p \circ t_\Gamma(f) = \pi(f)v_H.$$

Of course there is some H such that $v_H \neq 0$. Putting $v_0 := v_H$, we thus get $d(v_0, v_0^\vee) \neq 0$ (notations as in the discussion below 4.3), whence $d_\pi \neq 0$ and π has a formal degree.

Now put $f_\pi := d(v_0, v_0^\vee)^{-1} c_{v_0, v_0^\vee} \in \mathcal{S}_{\mathcal{K}}(G/Z)$ (recall $Z = A_G \cap \Gamma$ is co-compact in the center). We may formulate the Schur orthogonality relations in this context :

(4.8) $\quad Tr(\pi(f_\pi), V) = 1$ and for any ν-tempered (σ, W) with $\omega_{\sigma|Z} = 1$, $Tr(\sigma(f_\pi), W) = 0$

Now, let $m(\pi, \Gamma) := dim_{\mathcal{K}}(Hom_G(\pi, \mathcal{C}_{\mathcal{K}}(G/\Gamma)))$ be the multiplicity of π in $\mathcal{C}_{\mathcal{K}}(G/\Gamma)$. We will show that for "small" enough Γ, this multiplicity is proportional to the volume $vol(G/\Gamma)$. Then, given an embedding $\mathcal{K} \hookrightarrow \mathbb{C}$ as in point ii) (Notice that such embeddings exist if and only if $\mathcal{K}$ has the same absolute transcendence cardinal as $\mathbb{C}$) we see that the base change $(V \otimes \mathbb{C}, \pi)$ also has multiplicity proportional to $vol(G/\Gamma)$ in $\mathcal{C}_{\mathbb{C}}(G/\Gamma)$. By proposition [17, 1.3] on limit multiplicities, this implies that such a base change is a square-integrable representation, whence our point ii).

To prove this proportionality, we first remark that the admissible representation $\mathcal{C}_{\mathcal{K}}(G/\Gamma)$ is ν-tempered. Indeed the formula

$$\langle \phi, \phi' \rangle := \int_{G/\Gamma} \phi(x)\phi'(x)dx, \quad \phi, \phi' \in \mathcal{C}_{\mathcal{K}}(G/\Gamma)$$

identifies $\mathcal{C}_{\mathcal{K}}(G/\Gamma)$ with its contragredient, and the estimation

$$|c_{\phi,\phi'}(g)|_{\mathcal{K}} = |\int_{G/\Gamma} \phi(gx)\phi'(x)dx|_{\mathcal{K}} \leqslant \sup_{x \in G/\Gamma} |\phi(x)|_{\mathcal{K}} \sup_{x \in G/\Gamma} |\phi'(x)|_{\mathcal{K}}$$

shows that the matrix coefficients of $\mathcal{C}_{\mathcal{K}}(G/\Gamma)$ are $|.|_{\mathcal{K}}$-bounded. Since Z acts trivially, the action of G extends to an action of the algebra $\mathcal{S}_{\mathcal{K}}(G/Z)$ on $\mathcal{C}_{\mathcal{K}}(G/\Gamma)$.

It follows from 4.8 that $m(\pi, \Gamma) = Tr(f_\pi, \mathcal{C}_{\mathcal{K}}(G/\Gamma))$. But we have :

$$\begin{aligned}
Tr(f_\pi, \mathcal{C}_{\mathcal{K}}(G/\Gamma)) &= \int_{G/\Gamma} \left(\sum_{\gamma \in \Gamma'} f_\pi(g\gamma g^{-1}) \right) dg \\
&= \int_{G/\Gamma} \left(\sum_{\gamma \in c(\Gamma')} \sum_{\delta \in \Gamma/\Gamma_\gamma} f_\pi(g\delta\gamma\delta^{-1}g^{-1}) \right) dg \\
&= \sum_{\gamma \in c(\Gamma')} \left(\int_{G/\Gamma} \left(\sum_{\delta \in \Gamma/\Gamma_\gamma} f_\pi(g\delta\gamma\delta^{-1}g^{-1}) \right) dg \right) \\
&= \sum_{\gamma \in c(\Gamma')} \left(\int_{G/\Gamma_\gamma} f_\pi(g\gamma g^{-1}) dg \right)
\end{aligned}$$

The first line is due to the classical integral expression of f_π acting on $\mathcal{C}_{\mathcal{K}}(G/\Gamma)$. In the second line $c(\Gamma')$ is a set of representatives of conjugacy classes in Γ' and Γ_γ is the centralizer of γ in Γ. All equalities are just consequences of the packet summation property on non-archimedean complete fields.

Now in the last sum, the contribution of the unit conjugacy class is $f_\pi(1)\, vol(G/\Gamma)$. For $\gamma \neq 1 \in c(\Gamma)$, we will show that the contribution is zero, provided Γ is torsion-free. As a matter of fact γ then is a non-compact element thus determines a unique proper parabolic subgroup $P = MU$ of G such that $\gamma \in P$ contracts "strictly" U, see [11]. It follows that Γ_γ normalizes U and we may write :

$$\int_{G/\Gamma_\gamma} f_\pi(g\gamma g^{-1})dg = \int_{G/\Gamma_\gamma U} \left(\int_U f_\pi(xu\gamma u^{-1}x^{-1})du \right) dx$$

with appropriate choices of a Haar measure for functions on U and of a G-invariant measure dx for $\delta^{-1}_{\Gamma_\gamma U}$-equivariant functions on G (Once again these integrals are just countable convergent sums on a non-archimedean complete field and the equality of both sums is just the packet summation property). But by the same calculation as in [13, lemma 22] we have

$$\int_U f_\pi(gu\gamma u^{-1}g^{-1})du = D(\gamma)^{-1} \int_U f_\pi(g\gamma u g^{-1})du$$

where $D(\gamma)$ is the determinant of $ad(\gamma) - 1$ acting on the Lie algebra of U, which is non-zero by the strictly contracting action of γ on U. It remains to show that the last integral is zero.

Since f_π essentially tends to 0 at infinity, we may multiply it by a constant such that it becomes $\mathcal{O}$-valued. Then by lemma 4.6, the reductions modulo any proper ideal in $\mathcal{O}$ are cuspidal functions so in particular the corresponding integrals vanish. Thus our integral is an element in $\mathcal{O}$ which lies in any proper ideal : it has to be zero. □

Corollary 4.9 *If G is either of symplectic, orthogonal, unitary type or is an inner form of a linear group, then any ν-discrete representation is cuspidal.*

Proof : The proof relies on the following fact :

If G is as in the statement, then for any Levi subgroup $M < G$, any cuspidal $\pi \in \mathrm{Irr}_{\mathbb{C}}(M)$ such that $i_M^G(\pi)$ has a square-integrable constituent, we have $\omega_\pi(Z_M \cap [G,G]) \subset p^{\mathbb{Q}}$ (here Z_M is the center of M and ω_π the central character of π).

As a matter of fact, for linear groups we have the Bernstein-Zelevinski classification which tells us much more than what needed here ; for their inner forms, Tadic in [23] has solved the case of inner forms of general linear groups ; and for the other listed groups Moeglin in [15] has shown that the reducibility points from cuspidal inducing data have a very special form. Namely let $M < G$ be maximal, $\pi \in Irr_{\mathbb{C}}(M)$ be cuspidal such that $i_M^G(\pi)$ is reducible, then $\omega_\pi(Z_M \cap [G,G]) \subset p^{\mathbb{Q}}$. In particular this gives qualitative features of the poles of the Plancherel measure attached with any cuspidal pair (M,π) in this co-rank 1 situation. To go from the co-rank 1 to general Levi subgroups, we use the result of Harish-Chandra and Silberger [20, Thm 5.4.5.7]. It says that when a discrete series occurs in $i_M^G(\pi)$, the reduced root system of A_M in P contains $(Rk\,G - Rk\,M)$ linearly independent roots α, such that the induced representation $i_{P_\alpha}^{M_\alpha}(\pi)$ has also a (essentially) square-integrable constituent. Here M_α is the centralizer of $ker\ \alpha \subset A_M$ in G (it is a semi-standard subgroup containing M as a maximal Levi) and P_α is some (maximal) parabolic subgroup of M_α with Levi component M. This implies the italicized statement.

Now let (V,π) be a ν-discrete series of G. From point ii) in proposition 4.7, the above discussion, and the fact that $p^{\mathbb{Q}} \subset \mathcal{O}^*$, it follows that the cuspidal support of π is ν-integral hence ν-tempered. But by proposition 3.5 this is possible only if this cuspidal support is (V,π) itself.

□

Application : Under the hypothesis of the last corollary, we get from 3.14 ii) and the example below definition 3.2 the following classification of ν-tempered representations : *an irreducible*

representation of G is ν-tempered if and only if its cuspidal support consists of cuspidal representations with integrally valued central characters. In particular an unramified principal series is ν-tempered if and only if its Satake parameter takes values in $\nu^{-1}(0)$. The assumption on the group are likely to be superfluous, but the assumption $\nu(p) = 0$ is necessary.

To conclude this section, we mention a general criterion for proving cuspidality of ν-tempered representations. The proof is by direct application of lemma 4.6. We strongly believe that the hypothesis in this statement holds true in great generality. We have stated it as conjecture 1.5 in the introduction

Proposition 4.10 *Let us fix some finitely generated ideal $\mathcal{I}$ of $\mathcal{O}$, so that $\inf_{x\in\mathcal{I}} \nu(x) > 0$. Assume that for each compact open subgroup H of G there is a compact-mod-center subset $\Omega_{H,\mathcal{K}}$ of G supporting all the H-bi-invariant cuspidal functions in $\mathcal{C}^c_{\mathcal{O}/\mathcal{I}^n}(G)$ for any $n \in \mathbb{N}$. Then any ν-discrete representation is cuspidal.*

The combination of this proposition with 3.14 makes a striking link between cuspidal functions and irreducibility properties of parabolic induction : assuming the existence of uniform support for the former, we get irreducibility of induced modules from Langland's data. An application is given in 5.7.

5 Reducibility of parabolic induction

The following property has been known only for complex representations, the classical argument involving a unitary trick (see the introduction). The proof we present works on any coefficient field k of characteristic different from p, but requires the group to have discrete co-compact subgroups, since it relies on 4.5. Let us recall that this is in particular true for groups G defined over an extension of $\mathbb{Q}_p$ by [5].

Theorem 5.1 *(Generic irreducibility) Assume G has discrete co-compact subgroups. Let k be an algebraically closed field of characteristic $\neq p$. Let $M < G$ and $\sigma \in \mathrm{Irr}_k(M)$ be irreducible, then the induced representation $i_M^G(\sigma \otimes k(M/M^0))$ (M acting on both sides of the tensor product) is absolutely irreducible as a $k(M/M^0)$-representation. Equivalently, the representation $i_M^G(\sigma\psi)$ is irreducible for ψ in a Zariski-dense subset of $\Psi_k(M)$.*

Proof : We note $\mathcal{K} := k(M/M^0)$. Let β be a rational character of A_M such that $\mathrm{val}_F \circ \beta \in (a_M^*)^+$ and choose a valuation on $\mathcal{K}$ such that

$$\nu_{|k} = 0 \text{ and } \nu(m) = -\mathrm{val}_F \circ \beta(m) \text{ if } m \in A_M$$

(the restriction of ν to $k(A_M/A_M^0)$ is uniquely defined). Then define $\psi_{\mathcal{K}} : M/M^0 \longrightarrow \mathcal{K}$ to be the tautological character. By definition of ν, we have $-\nu(\psi_{\mathcal{K}}) = \mathrm{val}_F \circ \beta \in (a_M^*)^+$. Now consider the representation $\sigma_{\mathcal{K}} := \sigma \otimes_k \mathcal{K}$ obtained by scalar extension. It is certainly ν-tempered since its coefficients are linear combination of coefficients of σ which are bounded because $\nu_{|k} = 0$. It is also absolutely irreducible. Hence the triple $(M, \sigma_{\mathcal{K}}, \psi_{\mathcal{K}})$ is a Langlands' triple. Since $\nu_{|k} = 0$, by 4.5 there are no non-cuspidal ν-discrete series on the completion of $\mathcal{K}$, whence neither on $\mathcal{K}$ itself. Then we can apply proposition 3.14 iii) which tells us that $i_M^G(\sigma_{\mathcal{K}^a}\psi_{\mathcal{K}^a})$ is irreducible hence $i_M^G(\sigma_{\mathcal{K}}\psi_{\mathcal{K}})$ is absolutely irreducible.

Let us show the equivalence with the last statement. Using 2.6 iv), there is a compact open subgroup H such that $\mathrm{length}_{\mathcal{H}(G,H)}(i_M^G(\sigma\psi)^H) = \mathrm{length}_G(i_M^G(\sigma\psi))$ for any ψ. Therefore we recover a finite dimensional situation and our statement is classical : we have a family of finite dimensional modules over the algebra $\mathcal{H}(G,H)$ with parameters in an algebraic variety, and the Burnside theorem implies that the absolute irreducibility of these modules is a Zariski-open condition on this variety ; but an open set is dense if and only if it contains the generic points.

□

Remark : For general G, we can adapt the foregoing proof to show generic irreducibility on any field satisfying the following hypothesis

(H) : *For any compact open subgroup H and any $M < G$, the space of $H \cap M$-bi-invariant cuspidal functions in $\mathcal{C}_k^c(M)$ is finite dimensional.*

The proof goes the same way but to be able to use 3.14 iii), we use the criterion 4.10 to insure cuspidality of ν-discrete series. Let us briefly explain why the hypothesis (H) is valid on any "banal" characteristic field, "banal" meaning "not dividing the pro-order of a compact subgroup" ; this will also make more precise what is lacking in general. In banal characteristic, we know by [26] that cuspidal representations are injective objects in the category of representations with fixed central character. It follows that any bi-H-invariant cuspidal function is a linear combination of bi-H-invariant coefficients of *irreducible* cuspidal representations (just decompose the subrepresentation of the regular representation generated by the cuspidal function). Then by an argument due to Bernstein-Kazhdan in the complex case and written in [25, II.2.16] in positive characteristic, we get the desired uniform support.

5.2 *Subquotients of induced representations* : We go on with the context of 5.1. We would like to know a little more about what's going on outside the open set of $\Psi_k(M)$ where irreducibility occurs. To this aim we need criteria that behave in a nice algebraic way, as irreducibility does. We are interested here in the set of those unramified characters ψ such that $\text{length}(i_M^G(\sigma\psi)) > \text{length}(i_M^N(\sigma\psi))$ for every proper $M < N < G$. We will call these characters "discrete w.r.t σ" (this is a slightly different, in fact weaker, condition than that of being discrete in the sense of [3]). The set of these discrete characters is obviously stable under $\Psi_k(G)$ and is algebraically constructible according to the following lemma.

Lemma 5.3 *The length function $\psi \in \Psi_k(M) \mapsto l(\psi) := \text{length}(i_M^G(\sigma\psi)) \in \mathbb{N}$ is constructible. More precisely, the absolute-length function $\mathcal{Q} \in \text{Spec}(k[M/M^0]) \mapsto l(\mathcal{Q}) := \text{abslength}_G(i_M^G(\sigma \otimes k(\mathcal{Q})))$ where $k(\mathcal{Q})$ is the fraction field of $k[M/M^0]/\mathcal{Q}$ is constructible.*

Here the absolute length of a representation is the length of its scalar extension to an algebraic closure.

Proof : By [12, 0.9.3], it is enough to prove that for any subvariety $S \subset \Psi_k(M)$, there is a non-empty open subset U of S such that l is constant on U. But this follows from [3], step 2 of the proof of lemma 5.1., since we can recover a finite dimensional module situation by taking H-invariants, by 2.6 iv).

In fact, this even shows that l is upper semi-continuous. □

In the complex coefficients case the authors of [3] have shown, using a hermitian argument, that the set of discrete characters (in their sense) is a finite union of $\Psi_k(G)$-orbits in $\Psi_k(M)$. Here we give the analogous statement for discrete characters in our sense (which implies their statement) by an alternative argument which works for arbitrary characteristic of k, at least when the group has discrete co-compact subgroups or when the hypothesis (H) is fulfilled (see the remark below 5.1).

Since a constructible set contains the generic points of its closure, it will be sufficient to show that the residue field at some generic point of the closure of the set of discrete characters is necessarily that of a single $\Psi_k(G)$-orbit. Notice that it is the generic point of some $\Psi_k(G)$-stable irreducible subvariety Y of $\Psi_k(M)$. Such a variety is a single orbit if and only if the composite morphism $Y \hookrightarrow \Psi_k(M) \longrightarrow \Psi_k(A_G)$ is finite. Hence it suffices to show that if $\mathcal{Q}$ is the prime ideal in $k[M/M^0]$ corresponding to our generic point, then $k[M/M^0]/\mathcal{Q}$ is finite over $k[A_G/A_G^0]$.

For such an ideal $\mathcal{Q}$, we note $k(\mathcal{Q})$ the quotient field of $k[M/M^0]/\mathcal{Q}$ and $\overline{k(\mathcal{Q})}$ an algebraic closure of the latter.

Lemma 5.4 *Let $\mathcal{Q}$ be a prime ideal of $k[M/M^0]$ such that $k[M/M^0]/\mathcal{Q}$ is not finite over $k[A_G/A_G^0]$ and $\psi : M \longrightarrow \overline{k(\mathcal{Q})}$ the tautological character. There is a valuation $\nu : \overline{k(\mathcal{Q})} \longrightarrow \mathbb{R}$ such that $-\nu(\psi) \in \overline{(a_M^G)^+} \setminus \{0\}$ and $\nu_{|k} = 0$.*

Proof : Let $\{\alpha_1, \cdots, \alpha_n\} \subset a_M^*$ be the set of simple roots $\Delta(P)$ as in 2.2. Let β_i be the simple co-root in $X_*(A_M)\otimes\mathbb{Q}$ attached to α_i, so that the set $\{\beta_1, \cdots, \beta_n\}$ maps to a base of $X_*(A_M/A_G)\otimes\mathbb{Q}$. Since $X_*(A_M/A_G) \simeq A_M/(A_M^0 A_G)$ we can pick elements $\{a_1, \cdots, a_n\}$ in A_M such that a_i maps to $N\beta_i$ for some integer N. Thus for any valuation ν on $k(\mathcal{Q})$ we have $< \nu(\psi), \alpha_i >= \frac{1}{N}\nu(a_i)$; all we have to do is to find a valuation such that $\nu(a_i) \leqslant 0$ and $\nu(a_i) < 0$ for at least one i.

Let k_G be the quotient field of the image of $k[A_G/A_G^0]$ in $k(\mathcal{Q})$ and $k_G[a_i^{-1}]_{\mathcal{Q}}$ the sub-ring of $k(\mathcal{Q})$ generated by k_G and the (images of) a_i^{-1}'s. By Noether's normalization lemma, there are elements $x_1, \cdots, x_r$ in $k(\mathcal{Q})$, algebraically independent over k_G such that $k_G[a_i^{-1}]_{\mathcal{Q}}$ is finite over $k_G[x_1, \cdots x_r]$. Define a valuation on this polynomial ring by $\nu(x_i) = 1$ and $\nu(k_G) = 0$. We can extend it to a valuation on $k_G[a_i^{-1}]_{\mathcal{Q}}$, still noted ν. Necessarily we have $\nu(a_i^{-1}) \geqslant 0$ for each i and $\nu(a_i^{-1}) > 0$ for at least one (and even for r) a_i. Now, ν can be further extended to the quotient field of $k_G[a_i^{-1}]_{\mathcal{Q}}$, then to its finite extension $k(\mathcal{Q})$ and eventually to $\overline{k(\mathcal{Q})}$. □

Corollary 5.5 *Assume G has discrete co-compact subgroups or k satisfies (H) (remark below 5.1). The set of discrete characters w.r.t σ is a finite union of $\Psi_k(G)$-orbits in $\Psi_k(M)$.*

Proof : Assume $\mathcal{Q}$ is a prime ideal of $k[M/M^0]$ corresponding to a generic point of the constructible set of discrete characters w.r.t σ. By definition we have length$(i_M^G(\sigma \otimes \overline{k(\mathcal{Q})})) >$ length$(i_M^N(\sigma \otimes \overline{k(\mathcal{Q})}))$ (M acting on both sides of the tensor product) for any proper $M < N < G$.

From the discussion above the former lemma, it is enough to show that $k[M/M^0]/\mathcal{Q}$ is finite over $k[A_G/A_G^0]$. Assuming the contrary, let $(\overline{k(\mathcal{Q})}, \nu)$ be as in the former lemma. By 4.5 or 4.10, our hypothesis imply that ν-discrete series are cuspidal. Hence we may apply lemma 3.16 and get a proper N such that length$(i_M^G(\sigma \otimes \overline{k(\mathcal{Q})}))$ = length$(i_M^N(\sigma \otimes \overline{k(\mathcal{Q})}))$ thus contradicting the discreteness property. □

5.6 *Proof of 1.2 ii) and iii)* : Both assertions are direct consequences of the former corollary, since neither elliptic representations nor cuspidal ones can be induced from a proper Levi subgroup.

5.7 *Proof of proposition 1.6* : Let π, M be as in the statement of this proposition and assume M, P are standard. We shall start proving that reducibility points occur only at *algebraic*-valued non-ramified characters. This is unconditional and does not use our assumption that conjecture 1.5 holds. Let us fix a $\mathbb{C}$-valued non ramified character ψ and assume that $\psi(M \cap G^0)$ is not contained in $\overline{\mathbb{Q}}$. If m is the generator of the free abelian group $(M \cap G^0)/M^0$ which contracts the unipotent radical of P, then $\psi(m)$ has to be transcendent and we may choose a valuation ν of the field $\overline{\mathbb{Q}}(\psi(m))$ trivial on $\overline{\mathbb{Q}}$ and with $\nu(\psi(m)) < 0$. Then $\pi \otimes \overline{\mathbb{Q}}(\psi(m))$ is a ν-tempered representation and $-\nu(\psi) \in (a_M^G)^+$. By 4.10 together with the remark below 5.1, the ν-discrete series are cuspidal, hence by 3.14, we find that $i_M^G(\pi\psi)$ must be irreducible.

Now let us assume $\psi(m)$ is not in $\overline{\mathbb{Z}}[\frac{1}{p}]^*$. Then one can find a non-p-adic non-Archimedean valuation ν of $\overline{\mathbb{Q}}$ such that either $\nu(\psi) \in (a_M^N)^+$ or $\nu(\psi) \in -(a_M^N)^+$. Note also that π is ν-tempered since it is defined over $\overline{\mathbb{Z}}$. Assume now $\nu(\psi) \in -(a_M^N)^+$: the statement in 1.5, together with propositions 4.10 and 3.14 imply that the induced representation $i_M^N(\rho\psi)$ is irreducible. Moreover it is isomorphic to $\overline{i_M^N}(\rho\psi)$ (via the intertwining operator of 3.7 for example). If alternatively $\nu(\psi) \in (a_M^N)^+$, then we may argue in the same way replacing parabolic by opposed parabolic.

6 Admissibility of parabolic restriction

Let R be a ring such that $p \in R^*$. As in the case of fields, call a smooth R-valued representation R-*admissible* if for all compact open subgroups H, the R-module of H-invariants is finitely generated over R. It is generally not known whether the parabolic restriction functors preserve the R-admissibility property. The usual proof of Jacquet-Casselman works on fields and more generally artinian rings. In this section we use our theory to generalize this property and proceed with applications to the problem of lifting representations.

We first connect our notion of ν-temperedness with a more intuitive notion of ν-integrality. This holds only under the assumption $\nu(p) = 0$. Moreover, we will consider only characteristic 0 valued fields $\mathcal{K}$. We still write $\mathcal{O} \subset \mathcal{K}$ for the valuation ring of ν.

Definition 6.1 *An admissible smooth $\mathcal{K}$-representation (π, V) of G is said to be ν-integral if there exists a G-stable $\mathcal{O}$-admissible $\mathcal{O}$-submodule in V which generates V over $\mathcal{K}$.*

For example, a $\mathcal{K}$-valued character ψ of G is ν-integral if and only if $\nu(\psi) = 0$ if and only if it is $\mathcal{O}$-valued. Because of the definition of integrality by a "finite type" property, it is easier to assume, as we will, that $\mathcal{O}$ is noetherian. Since it is a valuation ring, this is equivalent to assume that it is principal, which is equivalent to the valuation being discrete. This restriction is harmless regarding our applications.

Lemma 6.2 *Assume ν is a discrete valuation. An admissible smooth $\mathcal{K}$-representation (π, V) of G is ν-integral if and only if for all open pro-p-subgroups H of G, the H-invariant space V^H has a finitely generated $\mathcal{O}$-submodule which is stable under the Hecke algebra $\mathcal{H}_{\mathcal{O}}(G, H)$.*

Proof : Let us write $\mathcal{H}_{\mathcal{O}}(G)$ the Hecke algebra of all locally constant compactly supported $\mathcal{O}$-valued functions on G with convolution product defined w.r.t some $\mathcal{O}$-valued Haar measure. For an open pro-p-subgroup, we note e_H the idempotent of $\mathcal{H}_{\mathcal{O}}(G)$. We may and will identify the Hecke algebras $\mathcal{H}_{\mathcal{O}}(G, H)$ with algebras $e_H * \mathcal{H}_{\mathcal{O}}(G) * e_H$.

Fix some H. Observe that by noetherianity of $\mathcal{O}$, the assumption "V^H has a $\mathcal{O}$-finitely generated $e_H\mathcal{H}_{\mathcal{O}}(G)e_H$-stable submodule which generates V^H over K" is equivalent to "any $e_H\mathcal{H}_{\mathcal{O}}(G)e_H$-finitely generated $e_H\mathcal{H}_{\mathcal{O}}(G)e_H$-submodule of V^H is $\mathcal{O}$-finitely generated".

As a consequence, any finitely generated $\mathcal{H}_{\mathcal{O}}(G)$-submodule of V is $\mathcal{O}$-admissible.

Now let $(H_n)_{n\in\mathbb{N}}$ be a sequence of open pro-p-subgroup with trivial intersection. Define $V_1 := \sum \mathcal{H}_{\mathcal{O}}(G)e_{1i}$ where $(e_{1i})_i$ is a $\mathcal{K}$-basis of V^{H_1} and inductively define $V_n := \sum \mathcal{H}_{\mathcal{O}}(G)e_{ni}$ where $(e_{ni})_i$ is a basis of V^{H_n} such that $e_{H_{n-1}}e_{ni} \in V_{n-1}$. Then the $\mathcal{O}G$-submodule $\sum_n V_n$ is $\mathcal{O}$-admissible and generates V over $\mathcal{K}$.

□

Proposition 6.3 *Assume ν is a discrete valuation and let (π, V) be an admissible representation of G. Then it is ν-tempered if and only if it is ν-integral.*

Proof : Recall that our hypothesis $\nu(p) = 0$ together with proposition 3.19 imply that a $\mathcal{K}$-representation is ν-tempered if and only if its coefficients are $|.|_{\mathcal{K}}$-bounded. Assume first that (V, π) is ν-integral ; by [25, I.9.7] its contragredient $(V^\vee, \pi^\vee)$ is also ν-integral ; namely if $V_{\mathcal{O}}$ is a $\mathcal{O}$-model of V, $V_{\mathcal{O}}^\vee := \{v^\vee \in V^\vee, \langle V_{\mathcal{O}}, v^\vee\rangle \subset \mathcal{O}\}$ turns out to be a model for $V^\vee$ (the hypothesis $\nu(p)$ is crucial here). It is then clear that the coefficients of (V, π) are $|.|_{\mathcal{K}}$-bounded.

Conversely, assume now that (V, π) is tempered. By the proof of the former lemma, an admissible representation is ν-integral if and only if all its finitely generated subrepresentations are. Hence we may assume that (V, π) is finitely generated. Then let us fix some generators $v_i^\vee \in V^\vee, i = 1, \cdots, n$ of its contragredient. Recall that $\mathcal{C}_{\mathcal{K}}(G)$ stands for the space of $\mathcal{K}$-valued smooth functions on G. We endow it with the right regular representation. Consider the G-equivariant embedding

$$\begin{array}{rcl} V & \to & \mathcal{C}_{\mathcal{K}}(G)^n \\ v & \mapsto & (c_{v,v_i\vee})_{i=1,\cdots,n} \end{array}$$

Then by the first remark of the proof, the image has to be contained in the nth-power of the space $\mathcal{C}_{\mathcal{K}}^b(G)$ of $|.|_{\mathcal{K}}$-bounded smooth $\mathcal{K}$-functions on G. Let us identify V with its image and define $V_{\mathcal{O}} := V \cap \mathcal{C}_{\mathcal{O}}(G)^n$: clearly it is G-stable and generates V over K. To see that it is $\mathcal{O}$-admissible, fix an open compact subgroup H and a finite set $\{g_1, \cdots, g_m\}$ of elements of G such that the map

$$\begin{array}{rcl} V^H & \to & \mathcal{K}^{nm} \\ f & \mapsto & (f_1(g_1), \cdots, f_n(g_m)) \end{array}$$

be an embedding of $\mathcal{K}$-vector spaces. We obtain an embedding of $V_{\mathcal{O}}^H$ into $\mathcal{O}^{nm}$ which, by noetherianity of $\mathcal{O}$ shows that $V_{\mathcal{O}}^H$ is finitely generated. $\square$

Remark 6.4 *It follows from this proposition that an admissible representation is ν-integral if and only if its irreducible subquotients are.*

Now we address the problem of studying how admissibility over a ring goes through Jacquet functors. Let us first recall the following fact due to Jacquet-Casselman (see [8]) :

Let (π, V) be an admissible representation of G and $M < G$ with associated standard parabolic subgroup P. Let also H be an open pro-p-subgroup having Iwahori factorization with respect to the pair $(P, \overline{P})$ and put $H_M := H \cap M$. The natural projection $V \xrightarrow{j} r_G^M(V)$ takes V^H into $r_G^M(V)^{H_M}$. If $a \in A_M$ is strictly contracting on the unipotent radical of P (more explicitly, if $\mathrm{val}_F(\alpha(a)) > 0$ for all $\alpha \in \Delta(P)$), then

$$r_G^M(V)^{H_M} = \bigcup_{n \in \mathbb{N}} r_G^M(\pi)(a^{-n}) j(V^H) \tag{6.5}$$

This result holds on any ring R where p is invertible (see [25, preuve de II.3.2.ii)]). When R is a field it exhibits the LHS vector space as a *union* of vector spaces of dimension $\leqslant dim(V^H)$ and this implies the admissibility of $r_G^M(\pi)$ together with the surjectivity of $j : V^H \longrightarrow r_G^M(V)^{H_M}$. The latter argument works also if R is artinian but fails in general.

We will use some results of the previous parts to complete the argument for certain rings R. The first case is that of an algebra over a field.

Proposition 6.6 *Assume G has discrete co-compact subgroups, or k fulfills hypothesis (H) below 5.1. Let R be a commutative finite type k-algebra, $V \in Mod_R(G)$ be R-admissible and R-torsion free and $M < G$, then $r_G^M(V)$ is R-admissible.*

By torsion-free we mean that the maps "multiplication by r" on V are injective for all r. In particular, it implies that R is an integral domain.

Proof : First of all we use Noether's normalization lemma to reduce the problem to the case when R is a polynomial algebra. Indeed Noether's lemma gives an embedding $k[X_1, \cdots, X_n] =: R' \hookrightarrow R$ such that R is finite over R'. Viewed as a R'-representation, V is again admissible and torsion-free. Assuming we can prove the proposition for R', we get that $r_G^M(V)$ is R'-admissible hence *a fortiori* R-admissible.

From now on we assume $R = k[X_1, \cdots, X_n]$. The assumption on torsion means that, putting $\mathcal{K} := \mathrm{Frac}R$, V is a R-model of $V_{\mathcal{K}} := V \otimes_R \mathcal{K}$. Moreover, since it is defined by the injectivity of the "multiplication by r" maps, the exactness of r_G^M implies that $r_G^M(V)$ also is R-torsion-free.

Let $\nu : \mathcal{K} \longrightarrow \mathbb{R}$ be any discrete valuation of $\mathcal{K}$ such that $\nu(R) \subset \mathbb{R}_+$, then $V_{\mathcal{K}}$ is ν-tempered by 6.3. Using either 4.5 or 4.10, we know that ν-discrete series are cuspidal. Thereby we may apply 3.14 ii) : $r_G^M(V_{\mathcal{K}})$ is ν-tempered. More precisely, note π the action of G on $V_{\mathcal{K}}$ and π^a that on $V_{\mathcal{K}^a} := V \otimes_R \mathcal{K}^a$ where $\mathcal{K}^a$ is an algebraic closure of $\mathcal{K}$. By the remark 3.15, for any extension ν^a of ν to $\mathcal{K}^a$ and any exponent $\chi \in \mathcal{E}(A_M, r_G^M \pi^a)$, we have $\nu^a(\chi) = 0$. This means that χ takes values in $\mathcal{O}_{\nu^a}$, the valuation ring of ν^a. By [7, 1.4 Corollaire], R and all its finite normalizations (i.e its normalizations in any finite extension of $\mathcal{K}$) are Krull rings, so that we have, writing R^a for the integral closure of R in $\mathcal{K}^a$, $R^a = \bigcap_{\nu^a} \mathcal{O}_{\nu^a}$ where ν^a runs over all extensions of all discrete valuations of R. Hence the exponents in $\mathcal{E}(A_M, r_G^M \pi^a)$ actually take values in R^a.

Now let H be an open pro-p-subgroup having Iwahori factorization with respect to $(P, \overline{P})$ and put $H_M := H \cap M$. Fix some $a \in A_M$ strictly contracting as in the statement of Jacquet-Casselman's result 6.5 above. Writing more accurately $\mathcal{E}(A_M, r_G^M \pi)^H$ for the finite set of exponents occurring in $r_G^M(V_{\mathcal{K}^a})^{H_M}$, we know that there exists some positive integer d such that the operator $\prod_{\chi \in \mathcal{E}(A_M, r_G^M \pi)^H} (r_G^M(\pi)(a) - \chi(a))^d$ is zero on $r_G^M(V)^{H_M}$. Since the χ's are R^a-valued, this means that $r_G^M(\pi)(a^{-1})$ satisfies an integral polynomial equation over R, hence the ring $R[r_G^M(\pi)(a^{-1})]$ is a finite R-module. It follows by 6.5 that $r_G^M V^{H_M}$ is finite over R. Whence the proposition.

□

More interesting for applications is the case of rings as $\mathbb{Z}[\frac{1}{p}]$, $\mathbb{Z}_l$, or $\overline{\mathbb{Z}}_l$. Unfortunately we need more hypothesis on G.

Proposition 6.7 *Let G be a classical group (as in 4.9) over a p-adic field, R a $\mathbb{Z}[\frac{1}{p}]$-algebra, and let $V \in Mod_R(G)$ be R-admissible. Assume either R to be a Krull ring and V to be R-torsion free, or R to be a Dedekind ring. Then for any $M < G$, the parabolic restriction $r_G^M(V)$ is R-admissible.*

Proof : Let us note $\mathcal{K}$ the quotient field of R. In the Dedekind ring case, we may treat the torsion separately : by left-exactness of H-invariants, V_{tors} is still R-admissible. In particular, V_{tors}^H has finite length as an R-module. But then, assuming H has Iwahori factorization with respect to some pair of opposed parabolic subgroups $(P, \overline{P})$, Jacquet-Casselman's formula 6.5 shows that the length of any finite submodule of $r_G^M(V_{tors})^{H\cap M}$ is bounded by that of $j(V_{tors}^H)$ (notations of the former proof), whence $j(V_{tors}^H) = r_G^M(V_{tors})^{H\cap M}$ (compare with the original proof of Jacquet for fields [8, 3.3.4]).

From now on we assume V torsion-free and note $\mathcal{K}$ the quotient field of R. In particular, for any valuation $\nu : \mathcal{K} \longrightarrow \mathbb{R}$ non-negative on R, $V \otimes \mathcal{K}$ is ν-integral (see 6.3). Moreover we know by 4.9 that the ν-discrete series are cuspidal. By 3.14 ii) this implies that $r_G^M V \otimes \mathcal{K}$ is ν-integral hence ν-tempered. Then we may repeat the argument of the former proof, using our assumption that R is a Krull ring. □

In order to give an application of this, we first make the following remark

Lemma 6.8 *Assume G has discrete co-compact subgroups and let k be a perfect field of positive characteristic $\neq p$, $W(k)$ its Witt vectors, and K the quotient field of the latter. For any absolutely irreducible representation $\pi \in \mathrm{Irr}_k(G)$ we have*

i) π occurs as a subquotient of the reduction of a $W(k)$-admissible, torsion-free and generically irreducible $W(k)$-representation Π modulo the maximal ideal of $W(k)$.

ii) If π further is a projective or injective object in the category of admissible k-representations with central character ω_π, then π actually lifts to $W(k)$. Moreover π is projective and injective in the category of all representations of G with central character ω_π.

Proof : i) Let us first assume π to be cuspidal. Since any k-valued unramified character of G can be lifted to $W(k)$, using Teichmuller representatives, we may and will assume that the central character of ω_π has finite order. Then, arguing as in 4.4, we may embed π into the space of automorphic functions $\mathcal{C}_k(G/\Gamma)$ for some discrete co-compact subgroup Γ. This space is the reduction modulo the maximal ideal of $\mathcal{C}_{W(k)}(G/\Gamma)$, which is $W(k)$-admissible. Then there must be an irreducible direct summand of $\mathcal{C}_K(G/\Gamma)$ (since K has characteristic 0) whose intersection Π with $\mathcal{C}_{W(k)}(G/\Gamma)$ reduces to a (finite length) k representation with constituent π.

Now, if π is a general representation, it is contained in an induced representation $i_M^G(\sigma)$ with σ cuspidal. Let Σ be constructed from σ as above, we may take for Π the intersection of $i_M^G(\Sigma)$ with a suitable irreducible subrepresentation of $i_M^G(\Sigma \otimes K)$.

ii) Let us assume that the center is trivial for simplicity and that π is injective among admissible representations (in the projective case, just apply the following argument to $\pi^\vee$). We first remark that our injectivity hypothesis implies the cuspidality of π. As a matter of fact if π is not cuspidal, let $M < G$ with $r_G^M(\pi^\vee) \neq 0$: we claim that there is $\sigma \in Mod_k(M)$ such that $Ext_M^1(r_G^M \pi^\vee, \sigma) \neq 0$. Assuming this claim, we get $Ext_G^1(i_M^G(\sigma)^\vee, \pi) = Ext_G^1(\pi^\vee, i_M^G(\sigma)) \neq 0$ by Shapiro's lemma, which prevents π from being injective. To prove the claim, let us note I the maximal ideal of the ring $k[M/M^0]$ corresponding to the trivial character $M/M^0 \longrightarrow k^*$. Then consider the representation $r_G^M(\pi^\vee) \otimes (k[M/M^0]/I^2)$ where we let M act diagonally on the tensor product. We get an extension of $r_G^M(\pi^\vee)$ by itself which is not split since it is not on A_M.

Hence we may fix an open pro-p-subgroup H such that $\pi^H \neq 0$ and a discrete co-compact Γ such that π embeds into $\mathcal{C}_k(G/\Gamma)$. More precisely, if V is the space of π and $v_0^\vee \neq 0 \in V^{\vee H}$, Γ is

chosen such that the G-maps

$$c_{v_0^\vee}: \begin{array}{ccc} V & \to & \mathcal{C}_k^c(G/H) \\ v & \mapsto & c_{v,v_0^\vee} \end{array} \quad \text{and} \quad t_\Gamma: \begin{array}{ccc} \mathcal{C}_k^c(G/H) & \to & \mathcal{C}_k(G/\Gamma) \\ f & \mapsto & g \mapsto \sum_\Gamma f(g\gamma) \end{array}$$

have non-zero composition. But the injectivity of π among the admissible representations implies the existence of a retraction r such that $r \circ t_\Gamma \circ c_{v_0^\vee} = 1_V$; in particular $c_{v_0^\vee}$ itself has a retraction, so that π is also a projective object in the category of all k-representations.

Let us note $\mathcal{H}_k(G,H) = \mathcal{C}_k^c(H\backslash G/H)$ the relative (G,H)-Hecke algebra with coefficients in k and e_H its unit. The G-endomorphism $c_{v_0^\vee} \circ r \circ t_\Gamma$ of $\mathcal{C}_k^c(G/H)$ is given by right multiplication by the element $e := c_{v_0^\vee} \circ r \circ t_\Gamma(e_H)$. Since $(c_{v_0^\vee} \circ r \circ t_\Gamma)^2 = c_{v_0^\vee} \circ r \circ t_\Gamma$, this element e is an idempotent and the map $c_{v_0^\vee}$ induces an isomorphism $\pi \xrightarrow{\sim} \mathcal{C}_k^c(G/H)e = \mathcal{C}_k^c(G)e$.

Now, let λ be a uniformizer for $W(k)$ and $\nu : K \longrightarrow \mathbb{Z}$ be the corresponding valuation. The theorem [9, (6.7)] on lifting idempotents tells us that we can lift e to a primitive idempotent E in the λ-adic completion $\mathcal{S}_{W(k)}(G,H)$ of $\mathcal{H}_{W(k)}(G,H)$ (which is nothing but the integral part of the ν-Schwartz relative algebra on K). Then we can form the $W(k)G$ representation

$$\Pi := \mathcal{S}_{W(k)}(G)E$$

where $\mathcal{S}_{W(k)}(G)$ is the inductive limit of Banach algebras $\mathcal{S}_{W(k)}(G,J)$ for J varying among open pro-p-subgroups. By definition we have $\Pi \otimes_{W(k)} k \simeq \pi$. So it remains to check that Π is $W(k)$-admissible and torsion-free. The torsion-freeness is clear since E is idempotent and $\mathcal{S}_{W(k)}(G)$ is torsion-free over $W(k)$. But then for any open pro-p-subgroup J, $\Pi^J = \mathcal{S}_{W(k)}(J\backslash G/H)E$ is a complete torsion-free $W(k)$-module containing no K-line and with finite dimensional reduction π^J modulo λ. A classical argument of lifting generators of the special fiber shows that Π^J then has to be finitely generated over $W(k)$: if (e_i) is a finite set of elements in Π^J projecting to a k-basis of π^J, then any element in Π^J belongs to $\lambda\Pi^J + \sum_i W(k)e_i$ and, by induction to $\lambda^n\Pi^J + \sum_i W(k)e_i$, hence eventually to $\sum_i W(k)e_i$ since the latter is closed in Π^J for the λ-adic topology.

□

Corollary 6.9 *Going on with the notations of the former lemma, assume π is supercuspidal irreducible and G is a classical group defined on a p-adic number field. Then $\Pi \otimes K$ must be (super)cuspidal.*

Proof : Let $M < G$ and assume $r_G^M \Pi \otimes K$ is non zero. By 6.7, $\Sigma := r_G^M(\Pi)$ is $W(k)$-admissible hence reduces to a non-zero k-admissible representation σ. But since $\Pi \otimes K$ is a constituent of $i_M^G(\Sigma \otimes K)$, π has to be a constituent of $i_M^G(\sigma)$ which contradicts the supercuspidality assumption. □

Corollary 6.10 *Let k have* banal *characteristic l and assume G is a classical group over a p-adic number field. Each cuspidal irreducible k-representation lifts to $W(k)$, and conversely each cuspidal irreducible K-representation with $W(k)$-valued central character has* irreducible *reduction modulo l.*

Proof : In banal characteristic, all cuspidal irreducible representations are projective in the category of all k-representations with suitable central character, by [26]. Hence the statement is a combination of the latter corollary and of point ii) of the former lemma. Observe that the idempotent E which is constructed in $\mathcal{S}_{W(k)}(G,H)$ in the proof of point ii) of the former lemma actually lies in $\mathcal{H}_{W(k)}(G,H)$, due to 4.9. □

Remark : Although it was never published, it is generally accepted that Bernstein's decomposition of the category of all $\mathbb{C}G$-representations is valid (and follows the same pattern) for any algebraically closed field of *banal* characteristic. The last corollary shows that the reduction map induces a bijection between blocks for say $\overline{\mathbb{Q}_l}$-representations and those for $\overline{\mathbb{F}}_l$-representations (for banal l !), which surprisingly enough was not previously known. The following questions come naturally in this context :

i) Do all irreducible $\overline{\mathbb{F}}_l$-representations lift to characteristic zero ? Do all *elliptic* l-integral $\overline{\mathbb{Q}}_l$-representations have irreducible reduction ?

ii) Do all central idempotents of $End_G(\mathcal{C}^c_{\overline{\mathbb{Q}}_l}(G))$ provided by Bernstein's decomposition theorem actually live in $End_G(\mathcal{C}^c_{\overline{\mathbb{Z}}_l}(G))$? (The answer is yes for *cuspidal* central idempotents by the last corollary).

We end this section with a lemma which is needed in section 8, and is also of independent interest.

Lemma 6.11 *Let π be an irreducible l-integral $\overline{\mathbb{Q}}_l$ representation of G. Then there is some finitely generated $\overline{\mathbb{Z}}_l G$-lattice π^l in π such that $\pi^l \otimes \overline{\mathbb{F}}_l$ is semi-simple.*

Proof : Let V be the space of the representation π. We first recover a finite dimensional situation by choosing (as we may from [25, II.5]) an open pro-p-subgroup such that the length of the reduction of any $\mathcal{H}(G,H)$-stable $\overline{\mathbb{Z}}_l$-lattice in V^H is the same as that of $r_l(\pi)$. Then we want to show the existence of such a stable lattice $\omega^l \subset V^H$ such that $\omega^l \otimes \overline{\mathbb{F}}_l$ is a semi-simple $\mathcal{H}_{\overline{\mathbb{F}}_l}(G,H)$-module.

By finite generation of $\mathcal{H}(G,H)$ there is a finite extension E of $\mathbb{Q}_l$ and a model (π, V_E) of (π, V) on E (compare [25, 4.7]). Let O_E be the ring of integers of E and k_E its residue field. Fix a $\mathcal{H}(G,H)$-stable $\mathcal{O}_E$-lattice ω in V_E^H and choose a O_E-basis for this lattice. So we get a map $\mathcal{H}_{O_E}(G,H) \longrightarrow \mathcal{M}_N(O_E)$, the $N \times N$ matrix algebra with N the dimension of V^H.

Let us look at the map $\mathcal{H}_{k_E}(G,H) \longrightarrow \mathcal{M}_N(k_E)$ obtained by reduction of the former one. This map factors through some parabolic subalgebras of $\mathcal{M}_N(k_E)$ and we fix a minimal one $\mathcal{P}$: one can construct it by lifting a composition series of the underlying $\mathcal{H}_{k_E}(G,H)$-module $\omega \otimes \overline{\mathbb{F}}_l$ for example. After maybe changing the O_E basis we started with, we even can assume that $\mathcal{P}$ is a "standard" upper block-triangular parabolic algebra. Then the diagonal blocks correspond to $\mathcal{H}_{\overline{\mathbb{F}}_l}(G,H)$-simple subquotients of $\omega \otimes \overline{\mathbb{F}}_l$.

Therefore, with this choice of a O_E-basis, we see that the $\mathcal{H}_{O_E}(G,H)$-module ω is equivalent to a morphism $\mathcal{H}_{O_E}(G,H) \longrightarrow \mathcal{M}_N(O_E)$ factoring through a standard (upper) parahoric subalgebra of $\mathcal{M}_N(O_E)$. More precisely, there are integers $n_1, \cdots, n_L$ with L the length of $r_l(\pi)$, such that $n_1 + \cdots + n_L = N$ and the morphism in question factors through the subalgebra of all matrices congruent modulo λ_E to an upper block triangular matrix of type $(n_1, \cdots, n_L)$, where λ_E is a uniformizer for E. Now we enlarge E by taking a L-root of λ_E, say $\lambda_{E'}$. Then we extend V_E and ω to E' and $O_{E'}$ and make the change of basis associated to the diagonal matrix $D := (1, \cdots, 1; \lambda_{E'}^{-1}, \cdots, \lambda_{E'}^{-1}; \cdots; \lambda_{E'}^{-(L-1)}, \cdots \lambda_{E'}^{-(L-1)})$ where the entries are repeated n_1, then n_2, etc... and n_L times. Now a simple computation shows that we get a map $\mathcal{H}_{O_{E'}}(G,H) \longrightarrow \mathcal{M}_N(O_{E'})$ factoring through the algebra of all matrices congruent to a block-*diagonal* matrix of type $(n_1, n_2, \cdots, n_L)$ modulo $\lambda_{E'}$. In particular the reduction modulo $\lambda_{E'}$ of this morphism factorizes through a Levi subalgebra and actually is semi-simple. In other words, the reduction of the $\mathcal{H}(G,H)$-stable $O_{E'}$-lattice $\omega' := D(\omega \otimes O_{E'})$ is semi-simple as a $\mathcal{H}_{k_{E'}}(G,H)$-module.

Now we extend back the scalars to $\overline{\mathbb{Z}}_l$ and write π^l for the $\overline{\mathbb{Z}}_l G$-submodule of V generated by the $\omega' \subset V^H$ we have just constructed. We know that π^l is a finitely generated $\overline{\mathbb{Z}}_l G$-lattice and write $\overline{\pi}$ its reduction. By construction $\overline{\pi}^H$ is a semi-simple $\mathcal{H}_{\overline{\mathbb{F}}_l}(G,H)$-module. Let $\sigma \subset \overline{\pi}$ be an irreducible subrepresentation of $\overline{\pi}$. Our choice of H insures that σ^H is a non-zero (necessarily simple) $\mathcal{H}_{\overline{\mathbb{F}}_l}(G,H)$-submodule of $\overline{\pi}^H$. Let $\rho_H \subset \overline{\pi}^H$ be a stable complement of σ^H and write ρ for the representation generated by ρ inside $\overline{\pi}$. Since $\overline{\pi}^H$ generates $\overline{\pi}$ we clearly have $\overline{\pi} = \rho + \sigma$. Moreover since $\rho \cap \sigma$ must be different from σ, it is zero, hence ρ is a stable complement for σ and $\overline{\pi}$ is semi-simple.

$\square$

7 Some remarks on intertwining operators

The theory of intertwining operators for p-adic groups was mainly developed by Harish Chandra in a rather analytic way : the very definition of these operators was obtained by convergence of some integrals in a convenient situation and then extended by meromorphic continuation. Silberger first showed the rationality of these operators, but still with analytic arguments, and only with Bernstein's (unpublished) work it became a little more transparent that this theory could be developed in a purely algebraic way.

Given two parabolic subgroups P, Q with the same Levi component, we have proposed in 2.10 a general and purely algebraic condition of (P, Q)-regularity for a representation of M, and, when this condition is fulfilled, we have defined an intertwining operator $J_{Q|P}(\sigma)$.

In this section we build up a generalization of the classical theory from this definition, with three aims : first we want to prove a functoriality property for these intertwining operators : this will be lemma 7.2 and will be useful for the next section. Secondly, we want to define rational intertwining operators and Harish-Chandra's j-functions in the general situation of a coefficient field of characteristic $\neq p$ (the non-vanishing of the j-function we obtain needs the generic irreducibility result so we will assume at this point that the hypothesis on G in 5.1 are fulfilled). Eventually we want to show that our intertwining operators satisfy most of the known properties of classical complex intertwining operators : this should be clear for the experts but certainly need to be checked.

The exposition owes a lot to Waldspurger's manuscript on the Plancherel theorem [27], where many analytic arguments of Harish-Chandra's original proof were already "algebraized".

7.1 *Functoriality for intertwining operators* : The next lemma was inspired by [27, IV.1.1.(8)]. Let $M < G$ and P, Q be parabolic subgroups with Levi component M. Assume given

- an epimorphism of noetherian integral domains $R \xrightarrow{f} R'$ with kernel noted $\mathcal{P}$,
- a f, M-equivariant epimorphism $\pi_R \xrightarrow{\phi} \sigma_{R'}$ between torsion-free R-valued (resp. R'-valued) representations of M,

such that

i) putting $K = \mathrm{Frac}R$ and $k = \mathrm{Frac}R'$, the representation $\pi_k := \pi_R \otimes_R k$ is (P, Q)-regular in the sense of 2.10.

ii) the representation $r_Q^M \circ i_P^G(\pi_R)$ is R-admissible and its base change to K has finite length.

Lemma 7.2 *The representations $\sigma_k := \sigma_{R'} \otimes_{R'} k$, $\pi_K := \pi_R \otimes_R K$, and $\pi_{R_{\mathcal{P}}} := \pi_R \otimes_R R_{\mathcal{P}}$ where $R_{\mathcal{P}}$ is the localization of R at $\mathcal{P}$ are (P, Q)-regular, and the squares*

$$\begin{tikzcd} i_P^G(\pi_{R_{\mathcal{P}}}) \arrow[r, "J_{Q|P}(\pi_{R_{\mathcal{P}}})"] \arrow[d, "i_P^G(\phi)"] & i_Q^G(\pi_{R_{\mathcal{P}}}) \arrow[d, "i_Q^G(\phi)"] \\ i_P^G(\sigma_k) \arrow[r, "J_{Q|P}(\sigma_k)"] & i_Q^G(\sigma_k) \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} i_P^G(\pi_{R_{\mathcal{P}}}) \arrow[r, "J_{Q|P}(\pi_{R_{\mathcal{P}}})"] \arrow[d, hook] & i_Q^G(\pi_{R_{\mathcal{P}}}) \arrow[d, hook] \\ i_P^G(\pi_K) \arrow[r, "J_{Q|P}(\pi_K)"] & i_Q^G(\pi_K) \end{tikzcd}$$

are commutative.

Proof : For any R-algebra $\widetilde{R}$ we put

$$S_{\widetilde{R}} := im\left(\widetilde{R}[A_M] \longrightarrow End_{\widetilde{R}}(V_{\pi_R} \otimes_R \widetilde{R})\right)$$

and

$$T_{\widetilde{R}} := im\left(\widetilde{R}[A_M] \longrightarrow End_{\widetilde{R}}((r_Q^G \circ i_P^G/F_{QP}^{<1})(V_{\pi_R} \otimes_R \widetilde{R}))\right).$$

First step : the rings S_R and T_R are finitely generated R-modules. The canonical map $S_R \otimes_R \widetilde{R} \longrightarrow S_{\widetilde{R}}$ and $T_R \otimes_R \widetilde{R} \longrightarrow T_{\widetilde{R}}$ are always surjective and if $\widetilde{R}$ is flat over R, they are bijective.

Let us first consider the case of S_R. For any open pro-p-subgroup H of M, let us put $S_R^H := im\left(R[A_M] \longrightarrow End_R(V_{\pi_R}^H)\right)$. We have canonical surjective maps $S_R \longrightarrow S_R^H$, and the smoothness of π_R implies that the limit map $S_R \longrightarrow \varprojlim_H S_R^H$ is injective. Since by assumption ii), π_R is R-admissible, $V_{\pi_R}^H$ is finitely generated over R and so $End_R(V_{\pi_R}^H)$ and S_R^H are. By assumption ii) again, π_K has finite length hence in particular is generated by its H-fixed vectors for some open pro-p-subgroup H. This implies that the canonical morphism $S_K \xrightarrow{\sim} S_K^H$ is bijective. Since V_{π_R} is R-torsion free this shows by [9, (2.38)] that for any $H' \subset H$ the map $S_R^{H'} \longrightarrow S_R^H$ is injective, and therefore bijective. Hence the limit $\varprojlim S_R^{H'}$ stabilizes to the finitely generated R-module S_R^H and moreover the embedding $S_R \longrightarrow \varprojlim S_R^H$ is bijective.

Now the surjectivity of the canonical map $S_R \otimes_R \widetilde{R} \longrightarrow S_{\widetilde{R}}$ comes from the definition of $S_{\widetilde{R}}$. When $\widetilde{R}$ is flat over R, [9, (2.38)] shows that the canonical map $S_R^H \otimes_R \widetilde{R} \longrightarrow S_{\widetilde{R}}^H$ is bijective. Hence passing to the limit, the map $S_R \otimes_R \widetilde{R} \xrightarrow{\sim} S_{\widetilde{R}}$ turns out to be injective, and therefore bijective.

The same proofs work for T_R by our assumption ii).

Second step : the kernel of the canonical morphisms $S_R \otimes_R k \longrightarrow S_k$ and $T_R \otimes_R k \longrightarrow T_k$ are nilpotent ideals.

By the first step, we may replace R by any flat $\widetilde{R}$ inserted in some diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\phi"] & \widetilde{R} \arrow[dl, "\widetilde{\phi}"] \\ k & \end{tikzcd}$$

In particular we may assume R to be local and henselian with residue field k. Now the morphism $S_R \otimes_R k \longrightarrow S_k$ is a morphism between finite dimensional commutative algebras over k. Since such algebras are products of local k-algebras, this morphism has nilpotent kernel if and only if the image of any primitive idempotent is non-zero. Let e be a primitive idempotent in $S_R \otimes_R k$. Since R is henselian, there is an idempotent E in S_R projecting to e. This idempotent induces a non-trivial decomposition $V_{\pi_R} = im(E) \oplus ker(E)$. Hence the image of e in S_k induces in turn a *non-trivial* decomposition $V_{\pi_k} = (im(E) \otimes_R k) \oplus (ker(E) \otimes_R k)$. This image has therefore to be non-zero. The same proof works for T replacing S.

Third step : the representation $\pi_{R_{\mathcal{P}}}$ is (P,Q)-regular.

Recall the notations of 2.10. By definition for any R-algebra $\widetilde{R}$, we have $I_{\pi_{\widetilde{R}}} = ker(\widetilde{R}[A_M] \longrightarrow S_{\widetilde{R}})$ and $I_{\pi_{\widetilde{R}}}^{QP} = ker(\widetilde{R}[A_M] \longrightarrow T_{\widetilde{R}})$. It follows from the previous step that the image of $I_{\pi_{R_{\mathcal{P}}}}$, resp of $I_{\pi_{R_{\mathcal{P}}}}^{QP}$, in $k[A_M]$ contains a power of I_{π_k}, resp. $I_{\pi_k}^{QP}$. Since by assumption i) we have $I_{\pi_k} + I_{\pi_k}^{QP} = k[A_M]$, it follows that

$$\mathcal{P}R_{\mathcal{P}}[A_M] + I_{\pi_{R_{\mathcal{P}}}} + I_{\pi_{R_{\mathcal{P}}}}^{QP} = R_{\mathcal{P}}[A_M].$$

But by step one, $R_{\mathcal{P}}[A_M]/(I_{\pi_{R_{\mathcal{P}}}} + I_{\pi_{R_{\mathcal{P}}}}^{QP})$ is a finitely generated $R_{\mathcal{P}}$-module. Hence by our noetheriannity assumption on R and Nakayama's lemma, this module is 0, whence the desired (P,Q)-regularity.

Fourth step : the representation π_K is (P,Q)-regular and so σ_k is.

By flatness of K over $R_{\mathcal{P}}$, we have $I_{\pi_{R_{\mathcal{P}}}} \otimes_{R_{\mathcal{P}}} K \xrightarrow{\sim} I_{\pi_K}$ and similarly for I^{QP}, hence the regularity of π_K follows from that of $\pi_{R_{\mathcal{P}}}$.

Now, since $\pi_k \xrightarrow{f} \sigma_k$ is onto, we have $I_{\sigma_k} \supset I_{\pi_k}$ and $I_{\sigma_k}^{QP} \supset I_{\pi_k}^{QP}$. Hence the regularity of σ_k follows from our assumption i).

Last step : commutative diagrams

Let us fix $i \in I^{QP}_{\pi_{R_{\mathcal{P}}}}$ such that $i \in 1 + I_{\pi_{R_{\mathcal{P}}}}$. Consider the following diagram, where vertical maps are functorially induced by ϕ, the left horizontal maps are the canonical projections and the right ones are the canonical isomorphisms :

$$\begin{tikzcd}
r^M_Q \circ i^G_P(\pi_{R_{\mathcal{P}}}) \arrow[r] \arrow[d, "r^M_Q \circ i^G_P(\phi)"] & r^M_Q \circ i^G_P/F^{\leqslant 1}_{QP}(\pi_{R_{\mathcal{P}}}) \arrow[r, "i"] \arrow[d] & F^{<1}_{QP}/F^{\leqslant 1}_{QP}(\pi_{R_{\mathcal{P}}}) \arrow[r] \arrow[d] & \pi_{R_{\mathcal{P}}} \arrow[d, "\phi"] \\
r^M_Q \circ i^G_P(\sigma_k) \arrow[r] & r^M_Q \circ i^G_P/F^{\leqslant 1}_{QP}(\sigma_k) \arrow[r, "f(i)"] & F^{<1}_{QP}/F^{\leqslant 1}_{QP}(\sigma_k) \arrow[r] & \sigma_k
\end{tikzcd}$$

By definition it is commutative. Applying the functor i^G_Q to its exterior square, we get the right hand square of the following diagram

$$\begin{tikzcd}
i^G_P(\pi_{R_{\mathcal{P}}}) \arrow[r] \arrow[d, "i^G_P\phi"] & i^G_Q \circ r^M_Q \circ i^G_P(\pi_{R_{\mathcal{P}}}) \arrow[r] \arrow[d, "i^G_Q \circ r^M_Q \circ i^G_P(\phi)"] & i^G_Q(\pi_{R_{\mathcal{P}}}) \arrow[d, "i^G_Q\phi"] \\
i^G_P(\sigma_k) \arrow[r] & i^G_P \circ r^M_Q \circ i^G_P(\sigma_k) \arrow[r] & i^G_Q(\sigma_k)
\end{tikzcd}$$

where the left horizontal arrows are functorially given by the adjointness map $1_{Mod_R(G)} \longrightarrow i^G_Q \circ r^M_Q$. Now, by definition the composite horizontal arrows are respectively $J_{Q|P}(\pi_{R_{\mathcal{P}}})$ and $J_{Q|P}(\sigma_k)$ and the commutative exterior square is the first one of the lemma. The commutativity of the second one is proved in the same way. $\square$

7.3 *Rational intertwining operators* : Let k be a field of characteristic $\neq p$, M a standard Levi subgroup of G and σ an absolutely irreducible smooth representation of M over k.

Put $\mathcal{K} := k(M/M^0)$ and note ψ_{un} the tautological unramified character $M \longrightarrow \mathcal{K}^*$. Define $\sigma_{\mathcal{K}} := \sigma \otimes_k \mathcal{K}$ the base change of σ and $\sigma_{un} := \sigma_{\mathcal{K}} \otimes \psi_{un}$. We claim that for any semi-standard parabolic subgroup P with Levi component M, σ_{un} is $(P, \overline{P})$-regular in the sense of 2.10. Indeed, choose a $\nu \in -(a^*_P)^+$ and view it as an additive character $\nu : M/M^0 \longrightarrow \mathbb{R}$, thanks to the identification $a^*_M \simeq Hom_{\mathbb{Z}}(M/M^0, \mathbb{R})$ of 2.2. By setting $\nu(k) = 0$, ν extends uniquely to a valuation on $\mathcal{K}$ and the triple $(M, \sigma \otimes_k \mathcal{K}, \psi_{un})$ is a (semi-standard version of) Langland's triple as in 3.6. Hence the claim follows from 3.7.

As a consequence, for any pair (P, Q) of parabolic subgroups containing M there is an intertwining operator

$$J_{Q|P}(\sigma_{un}) : \ i^G_P(\sigma_{un}) \longrightarrow i^G_Q(\sigma_{un}).$$

This may be viewed as a rational family of intertwining operators on the algebraic torus $\Psi_k(M)$. More precisely, any $\psi \in \Psi_k(M)$ such that $\sigma\psi$ is (P, Q)-regular as in 2.10, is a regular point of the rational operator $J_{Q|P}(\sigma_{un})$ and we have

$$J_{Q|P}(\sigma_{un})(\psi) = J_{Q|P}(\sigma\psi). \tag{7.4}$$

To see this, just apply lemma 7.2 in the following situation : $R := k[M/M^0]$, $R' := k$, $f := \psi$, $\pi_R := \sigma \otimes k[M/M^0]$ and $\sigma_{R'} := \sigma\psi$ (compare with the proof of [27, IV.1.1.(8)]).

In the next section, we will need the following statement on pôles of $J_{\overline{P}|P}(\sigma_{un})$ in the maximal-cuspidal case. This is of course reminiscent of [27, IV.1.2]. If $a \in A_M$, we note $\overline{a}$ its image in M/M^0.

Lemma 7.5 *Assume P maximal and proper in G and σ cuspidal. If $\mathcal{N}_G(M) = M$, then $J_{\overline{P}|P}(\sigma_{un})$ is regular. If $s \in \mathcal{N}_G(M) \setminus M$, then $J_{\overline{P}|P}(\sigma_{un})$ is singular at σ only if $\omega_\sigma = \omega^s_\sigma$. In this case, for any $a \in (A_M \cap G^0)$, the operator $(\overline{a} - \overline{a}^{-1})J_{\overline{P}|P}(\sigma_{un})$ is regular. In particular the poles are simple.*

Proof : If $\mathcal{N}_G(M) = M$, then $r_{\overline{P}}^M \circ i_P^G(\sigma\psi) = \sigma\psi$ hence $\sigma\psi$ is $(P, \overline{P})$-regular and the whole $J_{\overline{P}|P}(\sigma_{un})$ is regular on $\Psi_k(M)$.

Assume $s \in \mathcal{N}_G(M) \setminus M$, then we have an exact sequence $0 \longrightarrow \sigma \longrightarrow r_{\overline{P}}^M \circ i_P^G(\sigma) \longrightarrow \sigma^s \longrightarrow 0$ hence σ is (P, Q)-regular (and therefore is a regular point of $J_{\overline{P}|P}(\sigma_{un})$ by the discussion above the lemma) unless $\omega_\sigma = \omega_\sigma^s$.

By the same exact sequence as above for σ_{un} instead of σ, we see (with the notations of 2.10) that $I_{\sigma_{un}}$ is the kernel of the map $k(M/M^0)[A_M] \longrightarrow k(M/M^0)$ induced by $a \in A_M \mapsto \omega_\sigma(a)\overline{a}$ whereas $I_{\sigma_{un}}^{QP}$ is the kernel of the map $k(M/M^0)[A_M] \longrightarrow k(M/M^0)$ induced by $a \in A_M \mapsto \omega_\sigma^s(a)\overline{a}^s$. Let us choose $a \in A_M$ such that $\overline{a} \neq \overline{a}^s$. Then we have

$$\frac{a - \omega_\sigma(a)^s\overline{a}^s}{\omega_\sigma(a)\overline{a} - \omega_\sigma(a)^s\overline{a}^s} = \frac{\omega_\sigma^{-1}(a).a - \overline{a}^s}{\overline{a} - \overline{a}^s} \in I_{\sigma_{un}}^{QP} \cap (1 + I_{\sigma_{un}})$$

Hence the multiplication by this element gives the retraction leading to $J_{\overline{P}|P}(\sigma_{un})$ as in 2.11. In particular, if $a \in A_M \cap G^0$ and $\overline{a} \neq 1$, then since s induces a non-trivial automorphism of the rank 1 free abelian group $(M \cap G^0)/M^0$, we have $\overline{a}^s = \overline{a}^{-1}$ which finishes the proof.

□

7.6 *Definition of j-functions* : Here we assume k to be algebraically closed and we will apply the generic irreducibility property of 5.1. In particular, we assume that G or k satisfies the assumptions required in 5.1.

Let us keep the notations of the former paragraph 7.3. The generic irreducibility theorem tells us that the $\mathcal{K}$-representations $i_P^G(\sigma_{un})$ and $i_{\overline{P}}^G(\sigma_{un})$ are absolutely irreducible. Since both intertwining operators $J_{\overline{P}|P}(\sigma_{un})$ and $J_{P|\overline{P}}(\sigma_{un})$ are non-zero, they must be invertible. Hence the composite $J_{\overline{P}|P}(\sigma_{un}) \circ J_{P|\overline{P}}(\sigma_{un})$ is an automorphism of $i_P^G(\sigma_{un})$ and therefore has to be the scalar multiplication by some $j_P(\sigma) \in \mathcal{K} = k(M/M^0)$.

Using 7.8 i) below and arguing as in [27, IV.3.(1)], we see that $j_P(\sigma)$ doesn't depend on P containing M so we may just write j_σ.

7.7 *Some properties of intertwining operators* : In the remaining of this section, we will prove proposition 7.8 below. The results of this proposition are well-known in the complex coefficients case (at least points i) and ii)), and a proof can be found in [27, IV] for example. Unfortunately, all the proofs known to the author make use of analytic arguments and it is not so clear how they apply to the general setting we have introduced in 2.10.

Although we don't use these results in the rest of the paper, it seemed quite natural to the author to state and prove them here.

Let us introduce some notations : if P, Q are two semi-standard parabolic subgroups with the same Levi component M, we set $d(P, Q) = |\Sigma_{red}(P) \cap \Sigma_{red}(\overline{Q})|$ where $\Sigma_{red}(?)$ stands for the set of reduced roots of A_M in ?.

Proposition 7.8 *Let M be a standard Levi subgroup of G and (σ, V) be a R-representation of M.*

i) (Multiplicativity) Let (O, P, Q) be three parabolic subgroups with Levi component M such that $d(O, Q) = d(O, P) + d(P, Q)$ and $d(O, P) = 1$. If σ is (O, Q)-regular then it is (P, Q)-regular as well as (O, P)-regular and we have $J_{Q|O}(\sigma) = J_{Q|P}(\sigma) \circ J_{P|O}(\sigma)$.

ii) (Compatibility with induction 1) Let P and Q be two parabolic subgroups with Levi component M which are contained in some parabolic subgroup O with Levi component N. If σ is (P, Q)-regular, then it is $(P \cap N, Q \cap N)$-regular and we have, with natural identifications and notations, $J_{Q|P}^G(\sigma) = i_O^G(J_{Q\cap N|P\cap N}^N(\sigma))$.

iii) (Compatibility with induction 2) Let P and Q be two parabolic subgroups with Levi component M, and N a Levi subgroup of G containing M such that $P \cap N = Q \cap N$ and such that

PN and QN be parabolic subgroups (with Levi component N). If $i^N_{P\cap N}(\sigma)$ is (PN,QN)-regular, then σ is (P,Q)-regular and we have, with natural identifications and notations, $J_{Q|P}(\sigma) = J_{QN|PN}(i^N_{P\cap N}(\sigma))$.

7.9 *Proof of the regularity statements of proposition 7.8 :*
Point i) : Recall the notations I_σ and $I_\sigma^{Q\overline{w}}$, $\overline{w} \in W_M\backslash W_G/W_M$ of 2.10. Our hypothesis is that

$$I_\sigma + \bigcap_{\overline{w} \underset{QO}{<} 1} I_\sigma^{Q,\overline{w}} = R[A_M]$$

and we want to prove that

$$I_\sigma + \bigcap_{\overline{w} \underset{QP}{<} 1} I_\sigma^{Q,\overline{w}} = R[A_M] \quad \text{and} \quad I_\sigma + \bigcap_{\overline{w} \underset{PO}{<} 1} I_\sigma^{P,\overline{w}} = R[A_M]. \tag{7.10}$$

We claim that $\overline{w} \underset{QP}{<} 1 \Rightarrow \overline{w} \underset{QO}{<} 1$. Indeed, $\overline{w} \underset{QP}{<} 1$ by definition means $Q\overline{w}P \subset \overline{QP}$, hence

$$Q\overline{w}PO \subset \overline{QP}O \subseteq \overline{QPO} = \overline{QO},$$

the last equality being a consequence of $d(O,Q) = d(O,P) + d(P,Q)$. This is enough to get the first statement of 7.10.

Of course we also have $\overline{w} \underset{PO}{<} 1 \Rightarrow \overline{w} \underset{QO}{<} 1$ for the same reason as above. Then we have to compare $I_\sigma^{Q,\overline{w}}$ and $I_\sigma^{P,\overline{w}}$, and for that we need to compare representations $r^{M\cap w^{-1}M}_{M\cap w^{-1}Q}\sigma$ and $r^{M\cap w^{-1}M}_{M\cap w^{-1}P}\sigma$, for an element $w \in \overline{w}$, with $\overline{w} \underset{PO}{<} 1$.

We assume now that $d(O,P) = 1$. In this case there is a parabolic subgroup R containing O and P, with Levi component N containing M as a maximal Levi subgroup . Then we have $Q \cap N = P \cap N = \overline{O \cap N}$. On another hand, we claim that $\overline{w} \underset{PO}{<} 1 \Rightarrow \overline{w} \in W_M\backslash W_N/W_M$. Indeed, writing U_R for the radical of R, we have

$$\overline{PO} = \overline{(P\cap N)(O\cap N)U_R} = \overline{\overline{(P\cap N)(O\cap N)}U_R} = \overline{N.U_R} = R.$$

It follows that for $w \in \overline{w} \underset{P0}{<} 1$

$$M \cap w^{-1}(Q) = M \cap w^{-1}(Q\cap N) = M \cap w^{-1}(P \cap N) = M \cap w^{-1}P.$$

But then $I_\sigma^{P,\overline{w}} = I_\sigma^{Q,\overline{w}}$ and the second part of 7.10 follows, under our assumption $d(O,P) = 1$.

Point ii) : Here our hypothesis is $I_\sigma + \bigcap_{\overline{w} \underset{QP}{<} 1} I_\sigma^{Q,\overline{w}} = R[A_M]$ and we want to prove $I_\sigma + \bigcap_{\overline{w} \underset{Q_NP_N}{<} 1} I_\sigma^{Q_N,\overline{w}} = R[A_M]$ where we have noted $P_N := P \cap N$ and $Q_N := Q \cap N$ to have notations shorter.

We will consider the set $W_M\backslash W_N/W_M$ as a subset of the set $W_M\backslash W_G/W_M$. For $\overline{w} \in W_M\backslash W_N/W_M$ and any $w \in \overline{w}$, notice that $M \cap w^{-1}(Q) = M \cap w^{-1}(Q_N)$ so that $I_\sigma^{Q,\overline{w}} = I_\sigma^{Q_N,\overline{w}}$. On the other hand, we claim that for $\overline{w} \in W_M\backslash W_N/W_M$, we have $\overline{w} \underset{Q_NP_N}{<} 1 \Rightarrow \overline{w} \underset{QP}{<} 1$. Indeed, assuming $Q_N\overline{w}P_N \subset \overline{Q_NP_N}$, we get

$$Q\overline{w}P = Q(Q_N\overline{w}P_N)P \subset Q\overline{Q_NP_N}P \subset \overline{QP}.$$

Point ii) follows from these two facts.

Point iii) : Our hypothesis here is

$$I_{i_{P\cap N}\sigma} + \bigcap_{\overline{w} \underset{QNPN}{<} 1} I^{QN,\overline{w}}_{i_{P\cap N}\sigma} = R[A_N]$$

and we want to show

$$I_\sigma + \bigcap_{\overline{w} \underset{QP}{<} 1} I_\sigma^{Q,\overline{w}} = R[A_M].$$

Notice first that if $w \in W_G$, we have $w \underset{QP}{<} 1 \Rightarrow w \underset{QNPN}{<} 1$ by the same argument as in the proof of point ii). Now let $\overline{w} \in W_N \backslash W_G / W_N$, we claim that

$$I^{QN,\overline{w}}_{i_{P\cap N}(\sigma)} \subset \bigcap_{\overline{w'} \mapsto \overline{w}} (I_\sigma^{Q,\overline{w'}} \cap R[A_N])$$

where $\overline{w'}$ runs through the fiber of the projection map $W_M \backslash W_G / W_M \longrightarrow W_N \backslash W_G / W_N$ at $\overline{w}$. Admitting this claim, the hypothesis implies

$$(I_\sigma \cap R[A_N]) + \bigcap_{\overline{w} \underset{PQ}{<} 1} (I_\sigma^{Q,\overline{w}} \cap R[A_N]) = R[A_N]$$

which *a fortiori* yields what we wanted.

To prove the claim, we fix some $w \in \overline{w}$ and apply the geometric lemma to the representation $w(r_{N\cap w^{-1}(QN)} \circ i_{P\cap N}\sigma)$. It turns out that the latter has a filtration whose subquotients have the form $w(i_{N\cap w^{-1}N\cap v(P)} \circ v \circ r_{M\cap (wv)^{-1}(QN)}\sigma)$, v running in the set $W_{N\cap w^{-1}N} \backslash W_N / W_M$. The annihilator in $R[A_N]$ of such a representation is the same as that of $wv(r_{M\cap (wv)^{-1}(QN)}\sigma)$ (because induction is a faithful functor), which in turn is contained in that of $wv(r_{M\cap (wv)^{-1}(Q)}\sigma)$, the latter being $I_\sigma^{Q,\overline{wv}} \cap R[A_N]$. This gives the claim.

7.11 *Proof of the statements on intertwining operators in proposition 7.8 :* First we have to make explicit the isomorphism $F^{<1}_{QP}/F^{\leqslant 1}_{QP} \xrightarrow{\sim} \mathrm{Id}_{Mod_R(M)}$ which is used in 2.9 to define intertwining operators. In fact it is simply induced by the map

$$\begin{array}{rcl} \widetilde{F}^{<1}_{QP}(V) & \to & V \\ f & \mapsto & \int_{U_Q/(U_P\cap U_Q)} f(u)du = \int_{U_Q\cap U_{\overline{P}}} f(u)du \end{array}.$$

Notice that the first integral is well defined since by definition of $\widetilde{F}^{<1}_{QP}$, $(Supp\ f) \cap QP$ is compact-mod-P. The equality with the second integral comes from the decomposition $U_Q = (U_Q \cap U_{\overline{P}})(U_Q \cap U_P)$. For the remaining assertions of proposition 7.8, we will use the following

Lemma 7.12 *Let (σ, V) be a (P,Q)-regular R-representation of M. Then the operator $J_{Q|P}(\sigma)$ is the unique G-equivariant map $i_P^G(\sigma) \longrightarrow i_Q^G(\sigma)$ such that*

$$\forall f \in \widetilde{F}^{<1}_{QP}(V), \ \ J_{Q|P}(\sigma)(f)(1_G) = \int_{U_{\overline{P}}\cap U_Q} f(u)du.$$

Proof : Let J be any G-equivariant map $i_P^G(\sigma) \longrightarrow i_Q^G(\sigma)$ and let $K : \ r_Q^M \circ i_P^G(\sigma) \longrightarrow \sigma$ the M-equivariant map associated to J by Frobenius reciprocity. We have the equality

$$\forall f \in i_P^G(V), \ \ J(f)(1_G) = K(f_Q)$$

where f_Q is the image of f in $r_Q^M \circ i_P^G(V)$. Let us note $K_{Q|P}(\sigma)$ the K associated with $J_{Q|P}(\sigma)$. The following statement is equivalent to that of the lemma : $K_{Q|P}(\sigma)$ *is the unique M-equivariant map such that*

$$\forall f \in \widetilde{F}^{<1}_{QP}(V), \ \ K_{Q|P}(\sigma)(f_U) = \int_{(U_P\cap U_Q)\backslash U_Q} f(u)du.$$

But this is exactly how we have defined $J_{Q|P}(\sigma)$ in 2.11. □

Point i) : Let us first restate the hypothesis $d(O,Q) = d(O,P) + d(P,Q)$ in the following way

$$U_{\overline{O}} \cap U_Q = (U_{\overline{O}} \cap U_P)(U_{\overline{P}} \cap U_Q).$$

Notice also the characterization of $\widetilde{F}^{<1}_{QO}(V)$

$$\widetilde{F}^{<1}_{QO}(V) = \{f \in i^G_O V,\ \ Supp\ f \cap (U_{\overline{O}} \cap U_Q) \text{ is compact}\}, \tag{7.13}$$

which is equivalent to our original definition since the natural map $(U_{\overline{O}} \cap U_Q) \longrightarrow QO/O$ is an homeomorphism. Hence if $f \in \widetilde{F}^{<1}_{QO}(V)$, we have in particular

i) $\forall v \in U_Q \cap U_{\overline{P}}$, the function $i^G_O(\sigma)(v)f$ has compact support on $U_{\overline{O}} \cap U_P$ (*i.e.* lies in $\widetilde{F}^{<1}_{PO}(V)$).

ii) $\forall v \in U_Q \cap U_{\overline{P}}$, we have

$$J_{P|O}(\sigma)(f)(v) = J_{P|O}(\sigma)(i^G_O(\sigma)(v^{-1})f)(1) = \int_{U_{\overline{O}} \cap U_P} f(vu)du$$

hence in particular, the function $J_{P|O}(\sigma)(f) \in i^G_P V$ has compact support on $U_{\overline{P}} \cap U_Q$ (*i.e.* lies in $\widetilde{F}^{<1}_{QP}(V)$).

It follows from both points that

$$J_{Q|P}(\sigma) \circ J_{P|O}(\sigma)(f)(1) = \int_{U_{\overline{P}} \cap U_Q} \int_{U_{\overline{O}} \cap U_P} f(vu)dudv = \int_{U_{\overline{O}} \cap U_Q} f(x)dx. = J_{Q|O}(\sigma)(f)(1)$$

But by lemma 7.12, the above equality is actually valid for any f, whence point i).

Point ii) : First of all we fix the following isomorphism

$$\begin{array}{rcl} i^G_P(V) & \rightarrow & i^G_O \circ i^N_{P_N}(V) \\ f & \mapsto & \Big(g \mapsto (f_g :\ n \in N \mapsto \delta^{\frac{1}{2}}_O(n) f(gn))\Big) \end{array}$$

and similarly for Q instead of P. Hence, if $f \in i^G_P(V)$, we have

$$\forall g \in G,\ \ \big(i^G_O(J_{Q_N|P_N}(\sigma))(f)\big)_g = J_{Q_N|P_N}(\sigma)(f_g).$$

In particular $i^G_O(J_{Q_N|P_N}(\sigma))(f)(1) = J_{Q_N|P_N}(\sigma)(f_1)(1)$. Now let $f \in \widetilde{F}^{<1}_{QP}(V)$. The set $(Supp\ f) \cap (U_Q \cap U_{\overline{P}})$ is compact, hence so is the set $(Supp\ f_1) \cap (U_{Q_N} \cap U_{\overline{P_N}})$ since $U_Q \cap U_{\overline{P}} = U_{Q_N} \cap U_{\overline{P_N}}$. Thus by 7.13 we have $f_1 \in \widetilde{F}^{<1}_{Q_N P_N}(V)$ and we can compute

$$i^G_O(J_{Q_N|P_N}(\sigma))(f)(1) = \int_{U_Q \cap U_{\overline{P}}} f(u)du = J_{Q|P}(\sigma)(f)(1).$$

By lemma 7.12 this is enough to prove point ii).

Point iii) : we have the decompositions of unipotent radicals $U_Q = U_{Q\cap N}.U_{QN}$ and $U_P = U_{P\cap N}.U_{PN}$. They imply that the natural inclusion $U_{QN} \cap U_{\overline{PN}} \hookrightarrow U_Q \cap U_{\overline{P}}$ is a homeomorphism. We identify $i^G_P(V)$ and $i^G_{PN}(i^N_{P\cap N}(V))$ as in the previous proof. If $\widetilde{f} \in i^G_{PN}(i^N_{P\cap N}(V))$, the corresponding function $f \in i^G_P(V)$ satisfies $\delta^{\frac{1}{2}}_{PN}(n)f(gn) = \widetilde{f}(g)(n)$ for all $g \in G, n \in N$. In particular, for any $n \in N$, we have $Supp\ i^G_P(\sigma)(n)f \subset Supp\ \widetilde{f}$ so that, using 7.13, if $\widetilde{f} \in \widetilde{F}^{<1}_{QN,PN}(i_{P\cap N}(V))$

then for any $n \in N$, the function $i_P^G(\sigma)(n)f$ is in $\widetilde{F}_{QP}^{<1}(V)$. Hence we compute

$$\begin{aligned}
\widetilde{J_{Q|P}(\sigma)}(f)(1)(n) &= \delta_{QN}^{\frac{1}{2}}(n) J_{Q|P}(\sigma)(f)(n) = \delta_{QN}^{\frac{1}{2}}(n) J_{Q|P}(\sigma)(i_P^G(\sigma)(n^{-1})f)(1) \\
&= \delta_{QN}^{\frac{1}{2}}(n) \int_{U_Q \cap U_{\overline{P}}} f(nu)du = \delta_{QN}^{\frac{1}{2}}(n) \int_{U_{Q_N} \cap U_{\overline{P_N}}} f(nu)du \\
&= \delta_{QN}^{\frac{1}{2}}(n) \int_{U_{Q_N} \cap U_{\overline{P_N}}} \delta_{QN}^{\frac{-1}{2}} \delta_{PN}^{\frac{1}{2}}(n) f(un)du \\
&= \int_{U_{Q_N} \cap U_{\overline{P_N}}} \widetilde{f}(u)(n)du = J_{QN|PN}(i_{P\cap N}(\sigma))(\widetilde{f})(1)(n)
\end{aligned}$$

Hence point iii) follows from lemma 7.12.

Corollary 7.14 *Let $M < G$ and σ be an irreducible representation of M over an algebraically closed field (of characteristic $\neq p$). For any semi-standard Levi subgroup N, note j_σ^N the j-function relative to N attached to σ. Then we have*

$$j_\sigma^G = \prod_{\alpha \in \Sigma_{red}(P)} j_\sigma^{M_\alpha}$$

Proof : We can label the set $\Sigma_{red}(P) = \{\alpha_1, \cdots, \alpha_r\}$ in such a way that there are two sequences of semi-standard parabolic subgroups $(P_0 = P, P_1, \cdots, P_r = \overline{P})$ and $(Q_0, \cdots, Q_r)$ such that each P_i has semi-standard Levi component M and each Q_i has semi-standard Levi component M_i (that Levi subgroup containing M and the root subgroup attached to α_i). Then we have $d(P, \overline{P}) = \sum_i d(P_i, P_{i+1})$ and for each i, $d(P_i, P_{i+1}) = 1$. Hence we get from 7.8 i) and ii) the following decomposition

$$J_{\overline{P}|P}^G(\sigma) = \prod_{i=r}^{1} i_{Q_i}^G (J_{P_i \cap M_i | P_{i-1} \cap M_i}^{M_i}(\sigma)).$$

Similarly we have

$$J_{P|\overline{P}}^G(\sigma) = \prod_{i=1}^{r} i_{Q_i}^G (J_{P_{i-1} \cap M_i | P_i \cap M_i}^{M_i}(\sigma))$$

hence by multiplying both and noticing $J_{P_{i-1} \cap M_i | P_i \cap M_i}^{M_i}(\sigma) J_{P_i \cap M_i | P_{i-1} \cap M_i}^{M_i}(\sigma) = j_\sigma^{M_i}$ we get the corollary.

□

The product is in $k(M/M^0)$, but one can be more precise : each $j_\sigma^{M_\alpha}$ is invariant by $\Psi_k(M_\alpha)$, as a function on $\Psi_k(M)$, so that we have $j_\sigma^{M_\alpha} \in k(M \cap M_\alpha^0/M^0) \subset k(M/M^0)$. Let m_α be the unique generator of the rank 1 abelian group $M \cap M_\alpha^0/M^0$, such that there is a positive power $m_\alpha^n \in A_M$ with $|\alpha(m_\alpha^n)|_F \leqslant 1$, then one can write :

$$\forall \psi \in \Psi_k(M), \quad j_\sigma^{M_\alpha}(\psi) = c. \prod_{i \in I_\alpha} (\psi(m_\alpha) - x_{\sigma,i}^\alpha)^{n_{\sigma,i}^\alpha}, \;\; x_{\sigma,i}^\alpha \in k, \; n_{\sigma,i}^\alpha \in \mathbb{Z}^*, \; c \in k^*$$

Corollary 7.15 *Let $M < N < G$, $\sigma \in \mathrm{Irr}_k(M)$ and $\pi \in \mathrm{Irr}_k(N)$ a quotient of $i_{P\cap N}^N(\sigma)$ (where P is the standard parabolic subgroup with Levi M). Then for any $\alpha \in \Sigma_{red}(A_N, PN)$ we have*

$$j_\pi^{N_\alpha} = \prod_{\beta \in \Sigma_{red}(A_M, P), \beta_{|A_N} \in \mathbb{N}^*.\alpha} (j_\sigma^{M_\beta})_{|\Psi_k(N)}$$

Proof : Since $\{\beta \in \Sigma_{red}(A_M, P), \beta_{|A_N} \in \mathbb{N}^*.\alpha\} = \Sigma_{red}(A_M, P \cap N_\alpha) \setminus \Sigma_{red}(A_M, P \cap N)$, we may simplify notations by assuming $N_\alpha = G$ and proving

$$j_\pi^G = \prod_{\beta \in \Sigma_{red}(A_M, P) \setminus \Sigma_{red}(A_M, P \cap N)} (j_\sigma^{M_\beta})_{|\Psi_k(N)}.$$

Let us first identify the RHS. We note Q the parabolic subgroup with Levi M containing $P \cap N$ and the unipotent radical of $\overline{PN}$. By 7.8 iii), we have $J_{\overline{PN}|PN}(i_{P\cap N}(\sigma_{un})) = J_{Q|P}(\sigma_{un})$, hence the product

$$J_{PN|\overline{PN}}(i_{P\cap N}(\sigma_{un}))J_{\overline{PN}|PN}(i_{P\cap N}(\sigma_{un}))$$

is multiplication by some $j_\sigma^{PQ} \in k(M/M^0)$ which as in the former corollary can be computed

$$j_\sigma^{PQ} = \prod_{\beta\in\Sigma_{red}(A_M,P)\cap\Sigma_{red}(A_M,\overline{Q})} (j_\sigma^{M_\beta}) = \prod_{\beta\in\Sigma_{red}(A_M,P)\setminus\Sigma_{red}(A_M,P\cap N)} (j_\sigma^{M_\beta}).$$

Hence we have to show $j_\pi = (j_\sigma^{PQ})_{|\Psi_k(N)}$. Notice that the inclusion $\Psi_k(N) \hookrightarrow \Psi_k(M)$ corresponds to the ring morphism $f : k[M/M^0] \longrightarrow k[N/N^0]$ induced by the inclusion $M \subset N$. We will apply lemma 7.2 to the following situation :

- The parabolic subgroups (P, Q) of this lemma will be here $(PN, \overline{PN})$.
- $R = k[M/M^0]$ and $R' = k[N/N^0]$ with f given above.
- $\pi_R := i_{P\cap N}(\sigma \otimes k[M/M^0])$ where as usual M acts diagonally on the tensor product.
- $\sigma_{R'} := \pi \otimes k[N/N^0]$ and the morphism $\pi_R \longrightarrow \sigma_{R'}$ is induced by some surjective morphism $i_{P\cap N}(\sigma) \longrightarrow \pi$.

The two required conditions above 7.2 are fulfilled, hence we may apply the lemma to the pair $(PN, \overline{PN})$ and then once again to the pair $(\overline{PN}, PN)$. Putting $\mathcal{P} := ker\ f$, the lemma shows that the multiplication by j_σ^{PQ} sends $i_P^G(\sigma\otimes R_{\mathcal{P}})$ into itself. But $End_G(i_P^G(\sigma\otimes R_{\mathcal{P}}))$ is a finitely generated $R_{\mathcal{P}}$-module (take H-invariants for a suitable H) hence j_σ^{PQ} is integral on $R_{\mathcal{P}}$ and therefore lies in $R_{\mathcal{P}}$ since the latter is integrally closed in Frac$R$. The commutative diagrams of 7.2 then show that $f(j_\sigma^{PQ}) = j_\pi$ (still writing f for its extension $R_{\mathcal{P}} \longrightarrow k$) or in other words that $j_\pi = (j_\sigma^{PQ})_{|\Psi_k(N)}$. □

We end this section with the following remark, to be used in the next section : let P be a semi-standard parabolic subgroup with Levi component M. If $v \in \mathcal{N}_G(A_0)$, the map $f \mapsto (\rho(v)f : \ g \mapsto f(gv))$ provides an isomorphism of functors $\lambda_P(v) : \ i_P^G \xrightarrow{\sim} i_{v(P)}^G \circ v$. If (σ, V) is a (P, Q)-regular representation of M, then $v(\sigma)$ is $(v(P), v(Q))$-regular and we have

$$\lambda_Q(v)(\sigma) \circ J_{Q|P}(\sigma) = J_{v(Q)|v(P)}(v(\sigma)) \circ \lambda_P(v)(\sigma). \tag{7.16}$$

8 Modular Plancherel measures

In this section, we first study the link between the j-function of an l-integral $\overline{\mathbb{Q}}_l$-representation and the j-function of a Jordan-Holder factor in its reduction modulo l. Then we specialize to the co-rank 1 case (meaning we induce from a maximal proper Levi subgroup) where Silberger in the complex case has described the j-function. We use it to track down cuspidal subquotients of the induced representation. Our results here are certainly incomplete.

8.1 *Compatibility of j with reduction modulo l :* Let us start with a l-integral $\overline{\mathbb{Q}}_l$-representation π of a Levi subgroup $M < G$ with associated $j_\pi \in \overline{\mathbb{Q}}_l(M/M^0)^*$ as in 7.6. If the special fiber divisor $\Psi_{\overline{\mathbb{F}}_l}(M)$ of $\Psi_{\overline{\mathbb{Z}}_l}(M)$ is not contained in the singular locus of j_π we obtain by restriction a rational map $r_l(j_\pi)$ on this special fiber. If the latter moreover is not contained in the zero set of j_π, then $r_l(j_\pi) \in \overline{\mathbb{F}}_l(M/M^0)^*$. In more algebraic words, notice that $\overline{\mathbb{Q}}_l(M/M^0)$ is the quotient field of $\overline{\mathbb{Z}}_l[M/M^0]$ which is a factorial ring, so that it makes sense speaking of the l-valuation of j_π. Saying the special fiber is not contained in the singular/zero locus of j_π is equivalent to saying that the l-valuation of j_π is zero. In this case we will say that j_π is l-regular.

On another hand, Vignéras has shown [25, II.5] that the Brauer-Nesbitt principle holds for p-adic groups, *i.e.* all finite type $\overline{\mathbb{Z}}_l G$-lattices in π have isomorphic semi-simplified reduction modulo

the maximal ideal of $\overline{\mathbb{Z}}_l$. The semi-simple finite length $\overline{\mathbb{F}}_l G$ representation thus obtained will be noted $r_l(\pi)$.

In this context we have the following compatibility result,

Proposition 8.2 *Going on with the foregoing notations, assume $\pi \in Irr_{\overline{\mathbb{Q}}_l}(M)$ is cuspidal. Then j_π is l-regular and for any $\sigma \in Irr_{\overline{\mathbb{F}}_l}(M)$ occurring in $r_l(\pi)$, we have $j_\sigma = r_l(j_\pi)$.*

Proof : First of all, lemma 6.11 and its proof provide us with an admissible model $\pi_{\mathcal{O}_E}$ of π over the ring of integers $\mathcal{O}_E$ of some finite subextension E of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$, such that the reduction modulo the maximal ideal of $\mathcal{O}_E$ of $\pi_{\mathcal{O}_E}$ be *absolutely semi-simple*. Let k_E be the residual field of $\mathcal{O}_E$. Then there is a surjective map $\phi_E : \pi_{\mathcal{O}_E} \longrightarrow \sigma_{k_E}$ where σ_{k_E} is a k_E-model of σ.

Now we apply lemma 7.2 in the following situation

- $R = \mathcal{O}_E[M/M^0]$, $R' = k_E[M/M^0]$ and f the reduction map.
- $\pi_R := \pi_{\mathcal{O}_E} \otimes_{\mathcal{O}_E} \mathcal{O}_E[M/M^0]$ and $\sigma_{R'} := \sigma_{k_E} \otimes_{k_E} k_E[M/M^0]$, where as usual M acts diagonally on the tensor product.
- $\phi : \pi_R \longrightarrow \sigma_{R'}$ is induced by ϕ_E.

By the discussion in 7.3, $\pi_R \otimes k'$ is $(P, \overline{P})$-regular for any parabolic subgroup P containing M. In view of this definition, π_R is certainly admissible. Moreover, its cuspidality and the geometric lemma insure that the representation $r_{\overline{P}}^M \circ i_P^G(\pi_R)$ is also R-admissible. By 2.6 iv), its base change to $E(M/M^0)$ has finite length.

So we may apply the lemma 7.2 to the pairs $(P, \overline{P})$ and $(\overline{P}, P)$. Let $\mathcal{P}$ be the kernel of f (generated by any uniformizer of E). The lemma implies firstly that the multiplication by j_π sends $i_P^G(\pi_{R_\mathcal{P}})$ into itself. Since $End_G(i_P^G(\pi_{R_\mathcal{P}}))$ is a finitely generated module over $R_\mathcal{P}$ (apply Frobenius reciprocity), it follows that j_π is integral over $R_\mathcal{P}$ and therefore lies in $R_\mathcal{P}$ since the latter is integrally closed. As a consequence j_π is l-regular, *i.e.* we can define $r_l(j_\pi)$, and the diagrams of lemma 7.2 show that $r_l(j_\pi) = j_\sigma$.

□

Remark : The hypothesis of cuspidality on π in proposition 8.2 was only used to insure the integrality of the exponents of the representation $r_{\overline{P}}^M \circ i_P^G(\pi)$. When G is a classical group over a p-adic field, 4.9 insures that this property of integrality holds for any l-integral π. Hence in this case, the cuspidality assumption is superfluous.

8.3 *Co-rank* 1 *case* : We recall here known facts on characteristic zero coefficients. We look at the special case where $M < G$ is maximal (and proper) and fix π an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of M. Let us note Ψ_π the stabilizer of π in $\Psi_{\overline{\mathbb{Q}}_l}(M)$. By its definition, the function j_π on $\Psi_{\overline{\mathbb{Q}}_l}(M)$ is invariant under translation by $\Psi_{\overline{\mathbb{Q}}_l}(G)\Psi_\pi$. So let us pick up an element $m_\pi \in M$ which generates the character group of the quotient torus $\Psi_{\overline{\mathbb{Q}}_l}(M)/\Psi_{\overline{\mathbb{Q}}_l}(G)\Psi_\pi$; we may thus write $j_\pi(\psi)$ as a rational function in the variable $\psi(m_\pi)$. Then, Silberger [21, 1.6] has shown that j_π has the following nice form (in case π is cuspidal). For $a \in \overline{\mathbb{Q}}_l^*$ write

$$u_-(X,a) := \frac{(X-a)(X^{-1}-a)}{(X-1)(X^{-1}-1)} \quad \text{and} \quad u_+(X,a) := \frac{(X+a)(X^{-1}+a)}{(X+1)(X^{-1}+1)}, \quad \text{both in } \overline{\mathbb{Q}}_l(X).$$

Then we have the following cases :

i) If j_π has no pole, then it is constant.

ii) If j_π has only one pole, say at ψ_0 then there is some non zero constant $c \in \overline{\mathbb{Q}}_l$ such that $j_\pi(\psi) = c.u_-(\psi\psi_0^{-1}(m_\pi), a)$ for some $a \in \overline{\mathbb{Q}}_l^*$.

iii) If j_π has more than one pole, then let ψ_0 be such a pole : there are constants $c, a, b \in \overline{\mathbb{Q}}_l^*$ such that $j_\pi(\psi) = c.u_-(\psi\psi_0^{-1}(m_\pi), a)u_+(\psi\psi_0^{-1}(m_\pi), b)$.

In cases ii) and iii), there must be a non-trivial $s \in \mathcal{N}_G(M)/M$ such that $(\pi\psi_0^{-1})^s \simeq \pi\psi_0^{-1}$. Moreover the scalars $a, a^{-1}, -b$ and $-b^{-1}$ are mutually distinct.

With some additional assumptions, one can be more accurate about the constants which occur. First of all, if one believes in conjecture 1.1 (equivalently, 1.6.ii)) which is true at least for the classical groups considered in 4.9 , a and b should be l-adic units. By the compatibility result 8.2, it follows that c is a l-adic unit. If one wants to avoid using these expected facts, one may nevertheless assume that $a \in \mathbb{Z}_l$, after maybe replacing m_π by m_π^{-1}. Then, according as $b \in \mathbb{Z}_l$ or not, one gets from 8.2 that either c or b^2c is a l-adic unit.

Now, recall that for complex coefficients, the importance of the function j comes from Harish Chandra's Plancherel theorem. In particular, j_π controls to some extent the reducibility of the family of induced representations $i_M^G(\pi\psi)$. For example it is known [20, 5.4.2.4-5.4.5.2](and very easy to prove) that for $\psi(m_\pi).\psi_0^{-1}(m_\pi) \in \{a, a^{-1}, -b, -b^{-1}\}$, the representation $i_M^G(\pi\psi)$ has a square-integrable constituent.

A natural question is then : what's going on for positive characteristic coefficients ? Can we track the cuspidal subquotients with the behavior of j_π ? Here is a partial answer to these questions :

Proposition 8.4 *Let $M < G$ be maximal and π be an irreducible cuspidal $\overline{\mathbb{F}}_l$-representation of M with central character ω_π. We note $o_1(j_\pi)$ the order of vanishing of the function j_π at $\psi = 1$. Assume also $l \neq 2$ and fix $s \in \mathcal{N}_G(M) \setminus M$. Then we have*

i) If ω_π is regular, then j_π is regular at $\psi = 1$ (i.e. $o_1(j_\pi) \geqslant 0$) and $i_M^G(\pi)$ is reducible if and only if $o_1(j_\pi) \geqslant 1$

ii) If $\omega_\pi^s = \omega_\pi$ then $o_1(j_\pi) \geqslant -2$ is even, and the following holds :

(a) If $o_1(j_\pi) = -2$ then $i_M^G(\pi)$ is irreducible and $\pi \simeq \pi^s$.

(b) If $o_1(j_\pi) = 0$ then $i_M^G(\pi)$ is reducible if and only if $\pi \simeq \pi^s$ and is always semi-simple (hence has no cuspidal constituent).

(c) If $o_1(j_\pi) \geqslant 2$ then $i_M^G(\pi)$ is reducible.

iii) In any cases, if $i_M^G(\pi)$ has a cuspidal constituent (or equivalently has length $\geqslant 3$), then $o_1(j_\pi) \geqslant 2$.

Remarks :

- Suppose that s has no fixed point on the unramified orbit of π. Point ii)(a) implies that j_π has no pole. But then it follows from Silberger's form in characteristic zero and 8.2 that j_π has to be constant. Hence by points i) and ii)(b), the induced representations $i_M^G(\pi\psi)$ are always irreducible.

- Notice that for a $\overline{\mathbb{Q}}_l$-representation π as in the proposition, it follows from Silberger's form above that the possibilities for $o_1(j_\pi)$ are 0 or 1 for a regular central character ω_π, and -2 or 0 for a non-regular central character. Hence for a $\overline{\mathbb{F}}_l$-representation, proposition 8.2 implies that the possibilities for orders of poles and zeros listed in the above proposition are the only ones, under the assumptions made (especially $l \neq 2$).

- About the case $l = 2$: point i) is still true and proved as in the following proof. Point ii) and iii) need not be true as in the example of $SL(2)$ given in the end of the section. In fact for a classical group, it is generally predicted that $a, b \in p^{\frac{1}{2}\mathbb{N}^*}$. Modulo $l = 2$, we thus get $a, b = 1$, hence the j-function is constant.

Proof : Let us begin with the regular case $\omega_\pi \neq \omega_\pi^s$. Equivalently, the representation π is $(P, \overline{P})$ as well as $(\overline{P}, P)$-regular in the sense of 2.10. Then by 7.3-7.4 the rational intertwining operators $J_{\overline{P}|P}(\pi_{un})$ and $J_{P|\overline{P}}(\pi_{un})$ are regular at $\psi = 1$. Hence so is the rational function j_π and we have

$J_{\overline{P}|P}(\pi)J_{P|\overline{P}}(\pi) = j_\pi(1)$. It follows immediately that if $j_\pi(1) = 0$, at least one of the intertwining operators for π is not invertible. Since it is also non zero, either its kernel or its image provides a proper non-zero G-stable subspace of $i_M^G(\pi)$ which is therefore reducible. Conversely, if $\sigma \subset i_M^G(\pi)$ is a non-trivial proper subrepresentation, then by Frobenius reciprocity we must have $r_G^M(\sigma) = \pi$. In particular, by regularity of π, σ cannot be a sub-object of $\overline{i_M^G}(\pi)$. Therefore the intertwining operators for π cannot be isomorphisms and $j_\pi(1) = 0$.

Now, let us show point iii) under the regularity assumption : we assume that $o_1(j_\pi) = 1$ and will prove that $i_M^G(\pi)$ has no cuspidal subquotient. Since we already know that $i_M^G(\pi)$ is reducible, it is equivalent to showing that it has length 2. Observe that $o_1(j_\pi)$ as well as the length of $i_M^G(\pi)$ are invariant under torsion by unramified characters of G. This makes us introduce the irreducible subvariety $\Psi_{\overline{\mathbb{F}}_l}(G).1 \subset \Psi_{\overline{\mathbb{F}}_l}(M)$ (orbit of the trivial character), its quotient field k and its local ring $\mathcal{O}$. Considering the representation $i_M^G(\pi \otimes k)$ (where as usual M acts diagonally), what we have to show is that it has length 2, as a kG-representation. Next, we consider the commutative diagram

$$\begin{array}{ccccc} i_M^G(\pi\otimes\mathcal{O}) & \xrightarrow{J_{\mathcal{O}}} & \overline{i_M^G}(\pi\otimes\mathcal{O}) & \xrightarrow{\overline{J}_{\mathcal{O}}} & i_M^G(\pi\otimes\mathcal{O}) \\ \downarrow & & \downarrow & & \downarrow \\ i_M^G(\pi\otimes k) & \xrightarrow{J_k} & \overline{i_M^G}(\pi\otimes k) & \xrightarrow{\overline{J}_k} & i_M^G(\pi\otimes k) \end{array}$$

which requires the following explanations : since $\Psi(G).1$ lies inside the regular locus of $J_{\overline{P}|P}(\pi_{un})$, the latter induces the morphism noted $J_{\mathcal{O}}$ in this diagram. Same thing for $\overline{J}_{\mathcal{O}}$. On another hand, J_k and $\overline{J}_k$ are the base changes of $J_{\overline{P}|P}(\pi)$ and $J_{P|\overline{P}}(\pi)$. Finally the vertical maps are the reduction modulo the maximal ideal of $\mathcal{O}$. The commutativity is a (easy) special case of lemma 7.2.

Now, the hypothesis $o_1(j_\pi) > 0$ is equivalent to the vanishing of the products $J_k\overline{J}_k = \overline{J}_k J_k = 0$, so that we have the following filtration in $i_M^G(\pi \otimes k)$:

$$0 \subsetneq im\ \overline{J}_k \subseteq ker\ J_k \subsetneq i_M^G(\pi \otimes k)$$

where the first and the last subquotients have already been observed to be irreducible. In turn the hypothesis $o_1(j_\pi) = 1$ is equivalent to the assertion $J_{\mathcal{O}}\overline{J}_{\mathcal{O}} = \overline{J}_{\mathcal{O}}J_{\mathcal{O}} = \lambda$ with $\lambda \in \mathcal{O}$ a generator of the maximal ideal of $\mathcal{O}$. Thus in this case, the inclusion $im\ \overline{J}_{\mathcal{O}} \subseteq J_{\mathcal{O}}^{-1}(\lambda.\overline{i_M^G}(\pi\otimes\mathcal{O}))$ is actually an equality. Hence on reduction modulo λ it remains an equality, and the composition series above has length 2.

Let us consider now the singular case $\omega_\pi = \omega_\pi^s$. By lemma 7.5, the poles of the rational intertwining operators lie on the $\Psi_{\overline{\mathbb{F}}_l}(G)$-orbits of such singular characters and have order $\leqslant 1$. So let us use the same notations $\mathcal{O}$ and k as in the proof of point i)(b) above. The regular function $\lambda :\ \psi \in \Psi_{\overline{\mathbb{F}}_l}(M) \mapsto \psi(m_0) - 1$, where m_0 is a generator of $(M \cap G^0)/M^0$, is a generator of the maximal ideal of $\mathcal{O}$. The poles of $J_{\overline{P}|P}(\pi_{un})$ being of order $\leqslant 1$, the following map is well defined

$$\lambda J_{\overline{P}|P}(\pi_{un}) :\ i_M^G(\pi\otimes\mathcal{O}) \longrightarrow \overline{i_M^G}(\pi\otimes\mathcal{O}).$$

We will call "residue" of $J_{\overline{P}|P}(\pi_{un})$ along λ and note $R_\lambda J_{\overline{P}|P} :\ i_M^G(\pi_k) \longrightarrow \overline{i_M^G}(\pi_k)$ the operator defined from the latter by reduction modulo λ. It is convenient to recall the Frobenius adjoint $\overline{r_G^M} \circ i_M^G(\pi_k) \longrightarrow \pi_k$ of $R_\lambda J_{\overline{P}|P}$ from the definitions : it is an endomorphism of $\overline{r_G^M} \circ i_M^G(\pi_k)$ with image in the canonical subspace $\pi_k \hookrightarrow \overline{r_G^M} \circ i_M^G(\pi_k)$ which is given by the action of an element in the group algebra $x \in k[A_M]$, of the form $x = c(a - \omega_{\pi_k}^s(a))$ where $c \in k$ is a non-zero constant, $a \in A_M$ (see the proof of lemma 7.5). Since $\omega_{\pi_k}^s = \omega_{\pi_k}$, the action of such an element vanishes if and only if the extension $\pi_k \hookrightarrow \overline{r_G^M} \circ i_M^G(\pi_k) \to \pi_k^s$ splits on A_M. Hence if it doesn't vanish, the

action of x induces and isomorphism between π_k^s and π_k, and the extension has to be non-trivial. In particular, in this case $R_\lambda J_{\overline{P}|P}$ is an isomorphism (it corresponds to the natural identification $i_M^G(\pi_k) \xrightarrow{\sim} \overline{i_M^G}(\pi_k^s)$ through the isomorphism $\pi_k^s \xrightarrow{\sim} \pi_k$ provided by the action of $c(a - \omega_{\pi_k}^s(a))$). Summing up these considerations, we see that we are in one of the two following exclusive situations :

i) $J_{\overline{P}|P}(\pi_{un})$ is regular at π (or equivalently at π_k), $R_\lambda J_{\overline{P}|P} = 0$ and the extension $\pi_k \hookrightarrow \overline{r_G^M} \circ i_M^G(\pi_k) \to \pi_k^s$ splits on A_M.

ii) $J_{\overline{P}|P}(\pi_{un})$ is singular at π (or equivalently at π_k), $R_\lambda J_{\overline{P}|P}$ is an isomorphism, the extension $\pi_k \hookrightarrow \overline{r_G^M} \circ i_M^G(\pi_k) \to \pi_k^s$ is non-trivial, and $\pi_k \simeq \pi_k^s$.

Moreover, we are in the situation i) for $J_{\overline{P}|P}$ if and only if we are in the corresponding situation i) for $J_{P|\overline{P}}$: this follows from the criterion of non-triviality of the extension $\pi_k \hookrightarrow \overline{r_G^M} \circ i_M^G(\pi_k) \to \pi_k^s$, together with the fact that the endofunctors $\overline{r_G^M} \circ i_M^G$ and $s \circ r_G^M \circ \overline{i_M^G} \circ s$ of the categories of smooth M-representations are isomorphic.

Now we claim that in situation ii), the induced representation $i_M^G(\pi)$ is irreducible. Assume the contrary and let σ_k be an irreducible subrepresentation of $i_M^G(\pi_k)$. Then $r_G^M(\sigma_k) = \pi_k$ and by Casselman's reciprocity σ_k is also isomorphic to a quotient of $\overline{i_M^G}(\pi_k)$. This provides us with a non-trivial non-invertible morphism $\overline{i_M^G}(\pi_k) \longrightarrow i_M^G(\pi_k)$. Since $R_\lambda J_{P|\overline{P}}$ is another such morphism, but invertible, we get $dim_k(Hom_G(\overline{i_M^G}(\pi_k), i_M^G(\pi_k)) \geqslant 2$, which contradicts the non-triviality of the extension $\pi_k \hookrightarrow \overline{r_G^M} \circ i_M^G(\pi_k) \to \pi_k$. Whence the claim.

We apply this to the case ii)(a) of the proposition, namely when $o_1(j_\pi) = -2$. Equivalently, $j_\pi \in \lambda^{-2}\mathcal{O}^* \subset \overline{\mathbb{F}}_l(M/M^0)$. Letting $r \in k^*$ be the image of $\lambda^2 j_\pi$ modulo $\lambda\mathcal{O}$, it follows that we have the product identities $R_\lambda J_{\overline{P}|P} R_\lambda J_{P|\overline{P}} = R_\lambda J_{P|\overline{P}} R_\lambda J_{\overline{P}|P} = r \in k^*$ and in particular the residues of intertwining operators are non-zero. Hence we are in situation ii) above and $i_M^G(\pi)$ is irreducible.

From now on, we assume that $o_1(j_\pi) \neq -2$, which implies by the form of j_π given by Silberger and our assumption $l \neq 2$, that $o_1(j_\pi) \geqslant 0$. Then the products $R_\lambda J_{\overline{P}|P} R_\lambda J_{P|\overline{P}} = R_\lambda J_{P|\overline{P}} R_\lambda J_{\overline{P}|P}$ are zero. Hence at least one of the residue operator is not invertible, and by what has been said above it is actually zero and so is the other one. Hence we are in situation i) above. In particular, $J_{\overline{P}|P}(\pi_{un})$ is regular at π and specializes to an intertwining operator $J_{\overline{P}|P}(\pi_k)$. The latter is non-trivial since, by regularity and specialization, its Frobenius adjoint has to be a retraction of the embedding $\pi_k \hookrightarrow \overline{r_G^M} \circ i_M^G(\pi_k)$. Moreover we have the product formulas $J_{\overline{P}|P}(\pi_k) J_{P|\overline{P}}(\pi_k) = J_{P|\overline{P}}(\pi_k) J_{\overline{P}|P}(\pi_k) = c \in k$.

Assume now that $o_1(j_\pi) = 0$, or equivalently that c is non-zero in k. We begin proving the equivalence in statement ii)(b) ; if $i_M^G(\pi_k)$ is reducible and σ_k is an irreducible subrepresentation, then σ_k is also, via $J_{\overline{P}|P}(\pi_k)$, a subrepresentation of $i_M^G(\pi_k^s)$, so that we have $r_G^M(\sigma_k) = \pi_k = \pi_k^s$. Conversely if $\pi_k = \pi_k^s$ and since the extension $\pi_k \hookrightarrow \overline{r_G^M} \circ i_M^G(\pi_k) \to \pi_k^s$ is split, we have $dim_k(End_G(i_M^G(\pi_k))) = 2$, so that $i_M^G(\pi_k)$ is reducible. To see that it is semi-simple, we fix an isomorphism $I_s : \pi_k^s \xrightarrow{\sim} \pi_k$ and note A_s the corresponding isomorphism $\overline{i_M^G}(\pi_k) = i_M^G(\pi_k^s) \xrightarrow{\sim} i_M^G(\pi_k)$. It follows from 7.16 that $J_{P|\overline{P}}(\pi_k) = A \circ J_{\overline{P}|P}(\pi_k) \circ A$. We are thus lead to define $J := A \circ J_{\overline{P}|P} \in End_G(i_M^G(\pi_k))$. From the product formula it satisfies $J^2 = c$. To unravel the definition of J, consider the extension

$$0 \longrightarrow \pi_k \xrightarrow{\sim} \pi_k^s \longrightarrow r_G^M \circ i_M^G(\pi_k) \longrightarrow \pi_k \longrightarrow 0.$$

The right hand map corresponds to the identity morphism of $i_M^G(\pi_k)$, while J corresponds to some splitting of the left hand map. In particular $\{J, Id\}$ is a basis of $End_G(i_M^G(\pi_k))$. Since J satisfies the separable[2] equation $J^2 = c$, it follows that $i_M^G(\pi)$ is semi-simple. □

[2]Here the assumption $l \neq 2$ is unavoidable, see the $SL(2)$ example in the end of the section

The result of this proposition is quite unsatisfactory, because point iii) does not claim an equivalence between having a cuspidal subquotient and having $o_1(j_\pi) \geqslant 2$. However with more assumptions, we can get the equivalence thanks to Shahidi's results.

Let us first recall the notion of genericity. Let U_0 be the unipotent radical of P_0 and $\chi : U_0 \longrightarrow k^*$ be a smooth k-valued character of U_0 which is non-degenerate in the sense that its stabilizer under the natural action of M_0 is in the center of G. A representation $\pi \in Irr_k(G)$ is called χ-generic if there is an embedding $\pi \hookrightarrow Ind_{U_0}^G(\chi)$ (smooth induction with no support assumption), or equivalently, if $Hom_{U_0}(\pi, \chi) \neq 0$.

Lemma 8.5 *Let $\chi : U_0 \longrightarrow \overline{\mathbb{F}}_l^*$ be a non-degenerate character and $\pi \in Irr_{\overline{\mathbb{F}}_l}(G)$ be χ-generic. Then any l-integral representation $\widetilde{\pi} \in Irr_{\overline{\mathbb{Q}}_l}(G)$ such that $r_l(\widetilde{\pi})$ "contains" π, is $\widetilde{\chi}$-generic for any lifting $\widetilde{\chi} : N \longrightarrow \overline{\mathbb{Z}}_l$.*

Proof : We start with a very general remark. Let R be any commutative ring containing p^*, U be a unipotent p-adic group, and $\chi : U \longrightarrow R^*$ a smooth character. Then the functor

$$\begin{array}{rcl} Mod_R(U) & \rightarrow & R-\text{mod} \\ M & \mapsto & M_\chi := M/\langle nm - \chi(n)m, m \in M\rangle \end{array}$$

is exact. Indeed, it is the quotient of the identity functor by the subfunctor $M \mapsto M(\chi) := \langle nm - \chi(n)m, m \in M\rangle$, thus it suffices to show that the latter is exact. Let $(U_n)_{n\in\mathbb{N}}$ be an increasing sequence of compact open subgroups of U, and e_n the idempotent of $\mathcal{H}_R(U)$ associated to $\chi_{|U_n}$ (which exists because p is invertible in R). Then $M(\chi) = \varinjlim_{n\in\mathbb{N}} ker\, e_n$ and this exhibits $M \mapsto M(\chi)$ as an inductive limit of exact functors.

Let us apply this to the setting $U := U_0$, $R := \overline{\mathbb{Z}}_l$ and $\chi := \widetilde{\chi}$. Let $\omega \subset V_{\widetilde{\pi}}$ be a stable $\overline{\mathbb{Z}}_l$-lattice whose reduction is semi-simple, as in lemma 6.11. Hence we have a G-equivariant epimorphism $\omega \longrightarrow V_\pi$, which induces an epimorphism $\omega_{\widetilde{\chi}} \longrightarrow \pi_\chi$ by right exactness of $M \mapsto M_{\widetilde{\chi}}$. Moreover, since ω is l-torsion free, so $\omega_{\widetilde{\chi}}$ is, by left exactness of $M \mapsto M_{\widetilde{\chi}}$.

Now if π is χ-generic, the foregoing discussion shows that $0 \neq \omega_{\widetilde{\chi}} \hookrightarrow \widetilde{\pi}_{\widetilde{\chi}}$, thus $\widetilde{\pi}_{\widetilde{\chi}} \neq 0$ and $\widetilde{\pi}$ is $\widetilde{\chi}$-generic.

□

Proposition 8.6 *Assume G is quasi-split over a p-adic number field. Let $M < G$ be maximal and π be an irreducible supercuspidal $\overline{\mathbb{F}}_l$ representation which is $\chi_{|U_0\cap M}$-generic for some non-degenerate character $\chi : U_0 \longrightarrow \overline{\mathbb{F}}_l^*$. Assume also $l \neq 2$. Then the following conditions are equivalent :*

i) $o_1(j_\pi) \geqslant 2$.

ii) $i_M^G(\pi)$ *has a cuspidal subquotient.*

iii) There is a cuspidal l-integral $\widetilde{\pi} \in Irr_{\overline{\mathbb{Q}}_l}(M)$ whose reduction "contains" π, and such that $i_M^G(\widetilde{\pi})$ has a discrete series constituent whose reduction contains a generic cuspidal component.

Moreover, these conditions hold only if $q^{2r} = 1$ in $\overline{\mathbb{F}}_l$, where r is the order of the stabilizer $\Psi_{\widetilde{\pi}}$ of $\widetilde{\pi}$ in $\Psi_{\overline{\mathbb{Q}}_l}(M)$.

Remark : All the conditions in this proposition are fulfilled in the case of linear groups. Thus for general linear groups, this reproves results of Vignéras, by different means. However, the results of [25] and [24] are much more precise and describe the reducibility of parabolic induction from any Levi subgroup. Recasting these results in terms of modular Plancherel measures on one hand, and taking into account the main result of [14] on the other hand, we are lead to the following natural guess : *For any G (say over a p-adic number field), any Levi M, any cuspidal $\pi \in Irr_{\overline{\mathbb{F}}_l}(M)$ and assuming l prime to $|\mathcal{N}_G(M)/M|$, the induced representation $i_M^G(\pi)$ has cuspidal subquotients if*

and only if the function j_π has a zero at 1 *of order greater than the parabolic rank of M.* Assuming this guess true, the next guess would be that such zeros always occur at specializations of maximal order zeros of the j-function of a lifting of π. And then the ultimate one would be that the cuspidal subquotient thus obtained occur in the reduction modulo l of a (ordinary) discrete series... Much work in perspective ! It would be interesting to guess some statement taking into account those l dividing $|\mathcal{N}_G(M)/M|$.

Proof : (Sketch) We will rely on Shahidi's paper [19], especially theorem 8.1. The implication *iii)* $\Rightarrow$ *ii)* is obvious and the implication *ii)* $\Rightarrow$ *i)* was proved in proposition 8.4. We will prove *i)* $\Rightarrow$ *iii)*, so we assume $o_1(j_\pi) \geqslant 2$. In this case, the proof of proposition 8.4 shows that the intertwining operators $J_{\overline{P}|P}(\pi_{un})$ and $J_{P|\overline{P}}(\pi_{un})$ are regular at π and specialize to non-trivial operators $J_{P|\overline{P}}(\pi)$ and $J_{\overline{P}|P}(\pi)$ such that $J_{P|\overline{P}}(\pi)J_{\overline{P}|P}(\pi) = J_{\overline{P}|P}(\pi)J_{P|\overline{P}}(\pi) = 0$. Thus we have a filtration

$$0 \subsetneq im\,(J_{P|\overline{P}}(\pi)) \subseteq ker\,(J_{\overline{P}|P}(\pi)) \subsetneq i_M^G(\pi)$$

where the extremal subquotients, namely *im* $J_{P|\overline{P}}(\pi)$ and *coim* $J_{\overline{P}|P}(\pi)$ are irreducible and non-cuspidal. The problem is therefore to prove that the middle $\subseteq$ is a $\subsetneq$. Our strategy is to observe first that $i_M^G(\pi)$ must have a χ-generic constituent. Then we will show that neither *im* $J_{P|\overline{P}}(\pi)$ nor *coim* $J_{\overline{P}|P}(\pi)$ can be generic. This will imply that there must be something (necessarily cuspidal) in the middle. In the course of the proof we will exhibit it as a piece of the reduction of some discrete series.

First step : $i_M^G(\pi)$ has a χ-generic constituent

Equivalently, we have to show that the $\overline{\mathbb{F}}_l$-vector space $Hom_G\,(i_M^G(\pi), Ind_{U_0}^G(\chi))$ is non-zero. By [25, I.4.13 and I.5.11] this vector space is isomorphic to $Hom_G\,(ind_{U_0}^G(\chi^{-1}), i_M^G(\pi^\vee))$ where $ind_{U_0}^G$ stands for smooth induction with compact supports. By Frobenius reciprocity, the latter space is also $Hom_M\,(r_G^M \circ ind_{U_0}^G(\chi^{-1}), \pi^\vee)$. But a geometric-lemma-like argument together with the non-degeneracy of χ show that $r_G^M \circ ind_{U_0}^G(\chi^{-1}) \simeq ind_{U_0\cap M}^M(\chi_{|U_0\cap M}^{-1})$. Therefore our first space is isomorphic to $Hom_M\,(\pi, Ind_{U_0\cap M}^M(\chi_{|U_0\cap M}))$ which is non-zero by assumption.

Second step : let $\widetilde{\pi} \in Irr_{\overline{\mathbb{Q}}_l}(M)$ containing some $\overline{\mathbb{Z}}_l M$-lattice ω with semi-simple reduction and equipped with a non-zero morphism $\omega \longrightarrow \pi$. If $J_{\overline{P}|P}(\widetilde{\pi}_{un})$ is regular at $\widetilde{\pi}$ then $J_{\overline{P}|P}(\widetilde{\pi})(i_P^G(\omega)) \subset i_{\overline{P}}^G(\omega)$ and the following square is commutative :

$$\begin{array}{ccc} i_P^G(\omega) & \xrightarrow{J_{\overline{P}|P}(\widetilde{\pi})} & i_{\overline{P}}^G(\omega) \\ \downarrow & & \downarrow \\ i_P^G(\pi) & \xrightarrow{J_{\overline{P}|P}(\pi)} & i_{\overline{P}}^G(\pi) \end{array}$$

The reduction of ω to $\overline{\mathbb{F}}_l$ will be noted $\overline{\omega}$. By assumption it is a semi-simple $\overline{\mathbb{F}}_l$-representation containing π. Since $j_{\widetilde{\pi}}$ induces the j-function of each constituent π' of $\overline{\omega}$, we have $o_1(j_{\pi'}) = o_1(j_\pi) \geqslant 2$.

Let us note $R := \overline{\mathbb{Z}}_l[M/M^0]$ and consider the prime ideals Λ corresponding to the reduction map $\overline{\mathbb{Z}}_l \longrightarrow \overline{\mathbb{F}}_l$, and $\mathcal{P}$ corresponding to the trivial character $M/M^0 \longrightarrow \overline{\mathbb{Z}}_l$. Their sum $\mathcal{M} := \mathcal{P}+\Lambda$ is a maximal ideal. Lemma 7.2 tells us that the operator $J_{\overline{P}|P}(\widetilde{\pi}_{un})$ is regular along Λ (meaning that $J_{\overline{P}|P}(\widetilde{\pi}_{un})$ sends $i_P^G(\omega \otimes R_\Lambda)$ into $i_{\overline{P}}^G(\omega \otimes R_\Lambda)$ and that it induces $J_{\overline{P}|P}(\overline{\omega}_{un})$ after reduction modulo Λ). Let us note $\overline{R} := R/\Lambda = \overline{\mathbb{F}}_l[M/M^0]$ and $\overline{\mathcal{P}}$ the image of $\mathcal{P}$ in $\overline{R}$. We also know by the proof of proposition 8.4 that $J_{\overline{P}|P}(\overline{\omega}_{un})$ is regular at $\psi = 1$, that is, it sends $i_P^G(\overline{\omega}_{\overline{R}_{\overline{\mathcal{P}}}})$ into $i_{\overline{P}}^G(\overline{\omega}_{\overline{R}_{\overline{\mathcal{P}}}})$ and by definition induces $J_{\overline{P}|P}(\overline{\omega})$ on reduction modulo $\overline{\mathcal{P}}$. It follows that $J_{\overline{P}|P}(\widetilde{\pi}_{un})$ is regular along $\mathcal{M}$. Actually we already knew from our assumption that it is regular along $\mathcal{P}$ and induces $J_{\overline{P}|P}(\pi)$ on reduction modulo $\mathcal{P}$. What we get from the regularity along $\mathcal{M}$, is that the latter

$J_{\overline{P}|P}(\pi)$ is regular along Λ, *i.e.* it sends $i_P^G(\omega)$ into $i_{\overline{P}}^G(\omega)$ and induces $J_{\overline{P}|P}(\overline{\omega})$ after reduction modulo Λ. Composing with the map $\overline{\omega} \longrightarrow \pi$, we get the second step.

Third step : there exist $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ as in the statement of the second step, such that im $J_{P|\overline{P}}(\widetilde{\pi}_1)$ and coim $J_{\overline{P}|P}(\widetilde{\pi}_2)$ are non-generic, while im $J_{P|\overline{P}}(\widetilde{\pi}_2)$ and coim $J_{\overline{P}|P}(\widetilde{\pi}_1)$ are generic discrete series.

By corollary 6.9 we may choose some cuspidal l-integral $\widetilde{\pi} \in Irr_{\overline{\mathbb{Q}}_l}(M)$ whose reduction contains π. Then according to proposition 8.2 and our hypothesis $o_1(j_\pi) \neq 0$, the j-function $j_{\widetilde{\pi}}$ of $\widetilde{\pi}$ cannot be constant hence it is either of type 8.3 ii) or of type 8.3 iii).

We assume first that $j_{\widetilde{\pi}}$ is of type 8.3 ii), *i.e.* there are $c, a \in \overline{\mathbb{Q}}_l^*$ and $\psi_0 \in \Psi_{\overline{\mathbb{Q}}_l}(M)$, such that $j_{\widetilde{\pi}}(\psi) = c.u_-(\psi\psi_0^{-1}(m_\pi), a)$. By our assumption on the group, c and a are l-adic units. Since moreover we have $(\widetilde{\pi}\psi_0^{-1})^s \simeq \widetilde{\pi}\psi_0^{-1}$, $\psi_0(m_\pi)$ is also a l-adic unit and we may assume ψ_0 to be integrally valued. We will note $\overline{a}, \overline{c}$ and $\overline{\psi_0}$ the respective reductions of the latter. Then by proposition 8.2, our assumption $o_1(j_\pi) \geqslant 2$ is equivalent to $\overline{\psi}_0(m_\pi) = \overline{a} = \overline{a}^{-1}$. So let us choose two unramified characters $\psi_1, \psi_2 : M/M^0 \longrightarrow 1 + \Lambda$ (where Λ is the maximal ideal of $\overline{\mathbb{Z}}_l$) such that $(\psi_1\psi_0^{-1})(m_\pi) = a$ and $(\psi_2\psi_0^{-1})(m_\pi) = a^{-1}$. We put $\widetilde{\pi}_i := \widetilde{\pi}\psi_i$. Then $o_1(j_{\widetilde{\pi}_i}) = 1$ hence each $i_M^G(\widetilde{\pi}_i)$ is reducible with one discrete series constituent. By lemma 8.5, both $\widetilde{\pi}_i$ are generic, hence by [19, 8.1], the discrete series constituent is generic and the other constituent is not generic. We need to know more precisely who is a sub and who is a quotient. By [19, 8.1], we know that $a = p^{\pm r}$ or $a = p^{\pm \frac{r}{2}}$. In particular, we may assume to fix ideas that $a \in p^{\frac{1}{2}\mathbb{N}^*}$. In this case the subrepresentation of $i_M^G(\widetilde{\pi}_1)$ is the non-generic constituent, whereas the subrepresentation of $i_M^G(\widetilde{\pi}_2)$ is the discrete series constituent.

Let us now consider the case where $j_{\widetilde{\pi}}$ is of type 8.3 iii). The constants c, b, a again are l-adic units. Using [19, 8.1] we may assume to fix ideas that $a, b \in p^{\frac{1}{2}\mathbb{N}^*}$. Now we leave the reader check that up to changing a and b, our assumption $o_1(j_\pi) \geqslant 2$ implies that one of the following equalities occur on reduction to $\overline{\mathbb{F}}_l$:

i) $\overline{a} = \overline{a}^{-1} \neq \pm 1$, ii) $\overline{a} = -\overline{b} \neq \pm 1$, *or* iii) $\overline{a} = -\overline{b}^{-1} \neq \pm 1$.

Cases i) and iii) are treated exactly as in the foregoing argument. Besides, case ii) is actually impossible : we observe that [19, 8.1] implies (up to changing a and b) that either $a = b$ or $a^2 = b$. The first case is impossible since we obtain $\overline{a} = -\overline{a}$, and so the second one is, since we obtain $\overline{a} = -\overline{a}^2$, *i.e.* $\overline{a} = -1$.

Fourth step : end of the proof

Let $\widetilde{\pi}$ be as in the statement of the second step. Then by this second step, *im* $J_{P|\overline{P}}(\pi)$, resp. *coim* $J_{\overline{P}|P}(\pi)$, is a constituent of the reduction of *im* $J_{P|\overline{P}}(\widetilde{\pi})$, resp. *coim* $J_{\overline{P}|P}(\widetilde{\pi})$. Applying this to $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ of the third step, we get from lemma 8.5 that neither *im* $J_{P|\overline{P}}(\pi)$ nor *coim* $J_{\overline{P}|P}(\pi)$ can be generic. Hence there must be a generic cuspidal constituent and the latter occurs in the reduction of the unique generic constituent of $i_M^G(\widetilde{\pi}_1)$, resp. $i_M^G(\widetilde{\pi}_2)$, which is a discrete series.

Besides it follows from [19], 8.1 and 8.2, that the constants a and b in Silberger's form recalled in 8.3 equal $q^{\pm r}$ or $q^{\pm \frac{r}{2}}$ (according to the value of the indexes i in thm 8.1. of [19]) ; this implies the last assertion of the proposition. □

Example : $G = SL(2)$, M is the diagonal torus, P is the group of upper triangular matrices and $\pi = 1$ is the trivial representation. To express the j function we may put $m_\pi = m := \begin{pmatrix} \varpi & \\ & \varpi^{-1} \end{pmatrix}$ for some uniformizer ϖ of F. Over $\mathbb{C}$ the j function has the form

$$j_1(\psi) = \frac{1}{q^2} \frac{(\psi(m) - q)(\psi(m) - q^{-1})}{(\psi(m) - 1)(\psi(m) - 1)}.$$

As an illustration of the last two propositions, we have the following discussion, when $l \neq 2$, for the reducibility and the composition series of the induced representations $i_P^G(\psi)$ for any unramified $\overline{\mathbb{F}}_l$-character of T :

i) If $q \neq \pm 1 \bmod l$, then $i_P^G(\psi)$ is

irreducible	if $\psi(m) \neq -1, q, q^{-1}$,
reducible semi-simple	if $\psi(m) = -1$,
reducible non-split of length 2	if $\psi(m) = q$ or q^{-1}.

This case essentially behaves as the classical complex case.

ii) If $q = 1 \bmod l$, then $i_P^G(\psi)$ is

irreducible	if $\psi(m) \neq \pm 1$,
reducible semi-simple	if $\psi(m) = \pm 1$.

Notice that in this case the j-function is constant on $\Psi_{\overline{\mathbb{F}}_l}(M)$.

iii) If $q = -1 \bmod l$, then $i_P^G(\psi)$ is

irreducible	if $\psi(m) \neq -1$,
reducible of length $\geqslant 3$	if $\psi(m) = -1$.

Actually in the latter case, the length of $i_M^G(\psi)$ is 4 or 6. As a matter of fact, in the analogous situation for $GL(2)$, Vigneras has shown that the length is 3. let us note π the irreducible cuspidal subquotient which appears in this $GL(2)$-case. Since $F^*SL(2,F)$ has index 4 in $GL(2,F)$, the length of the restriction of π to $SL(2,F)$ is $1, 2$ or 4. On another hand, the representation $i_M^G(\psi)$ is the reduction of the $\overline{\mathbb{Q}}_l$-representation $i_M^G(m \mapsto -1)$ which is known to be the direct sum of two stably conjugate representations. Hence it must have even length.

Let us assume now $l = 2$. Then the j-function is constant. In the regular case $\psi(m) \neq 1$, the representation $i_M^G(\psi)$ is irreducible (see the third remark below proposition 8.4). In the case $\psi(m) = 1$, let us return to the end of the proof of proposition 8.4 where some endomorphism J of $i_M^G(\psi)$ was defined. Since $\{J, \mathrm{Id}\}$ is a basis of $End_G(i_M^G(\psi))$ and $J^2 - c = (J-c)^2 = 0$ for some scalar c, the representation $i_M^G(\psi)$ is *not* semi-simple. Actually the trivial representation is both a sub and a quotient, and there is some cuspidal part "in the middle". By the same remark as above, the length is 4 or 6.

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