SIMPLE SUBQUOTIENTS OF BIG PARABOLICALLY INDUCED REPRESENTATIONS OF p-ADIC GROUPS

JEAN-FRANÇOIS DAT

Abstract. This note is motivated by the problem of "uniqueness of supercuspidal support" in the modular representation theory of p-adic groups. We show that any counterexample to the same property for a finite reductive group lifts to a counterexample for the corresponding unramified p-adic group. To this end, we need to prove the following natural property : any simple subquotient of a parabolically induced representation is isomorphic to a subquotient of the parabolic induction of some simple subquotient of the original representation. The point is that we put no finiteness assumption on the orginal representation.

1. Main results

Let $\mathbb{G}$ be a connected reductive group over a p-adic field F and put $G = \mathbb{G}(F)$. Let R be a Noetherian ring where p is invertible. We denote by $\mathrm{Rep}_R(G)$ the category of all smooth RG-modules. If $P = \mathbb{P}(F)$ is a parabolic subgroup of G with Levi quotient $M = \mathbb{M}(F)$, we denote by $i_P : \mathrm{Rep}_R(M) \longrightarrow \mathrm{Rep}_R(G)$ the parabolic induction functor and by $r_P : \mathrm{Rep}_R(G) \longrightarrow \mathrm{Rep}_R(M)$ its left adjoint, the Jacquet functor. Recall that a smooth RG-module is called *cuspidal* if it is killed by all proper Jacquet functors. We will prove the following result.

Theorem 1.1. *Let V be any smooth RM-module and let π be a **simple** and **admissible** RG-subquotient of $i_P(V)$. Assume that **one** of the following holds.*

1. *V has level 0, **or***
2. *Bernstein's second adjunction holds for (G, R).*

Then there is a simple smooth RM-module σ such that π is isomorphic to a subquotient of $i_P(\sigma)$. Moreover, if V is cuspidal, σ can be chosen to be a subquotient of V.

Of course the result is well-known if V has finite length, and the point is that we don't require any finiteness property on V here. We haven't been able to find a general argument that would work for any functor between abelian categories with suitable assumptions (such as commuting with all limits and colimits). Instead, our argument turns out to be quite intricate, so that we thought it might be interesting to advertise the result and its proof.

Let us explain the hypothesis.

(1) "*V has level* 0" means that $V = \sum_x V^{G_x^+}$ where x runs over vertices of the semisimple Bruhat-Tits building of G and G_x^+ is the pro-radical of the associated parahoric group. The objects of level 0 form a direct summand subcategory

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$\mathrm{Rep}_R^0(G)$ of $\mathrm{Rep}_R(G)$, and we have $i_P(\mathrm{Rep}_R^0(M)) \subset \mathrm{Rep}_R^0(G)$ and $r_P(\mathrm{Rep}_R^0(G)) \subset \mathrm{Rep}_R^0(M)$, see [2, Prop. 6.3 ii)].

(2) By the sentence "*Bernstein's second adjunction holds for* (G, R)", we mean that for any pair $(Q, \overline{Q})$ of opposed parabolic subgroups of any Levi subgroup M of G, the functor $\delta_Q . r_{\overline{Q}}$ is right adjoint to the functor i_Q. Here δ_Q is the modulus character of Q. By [2, Thm 1.5] this property holds at least when G is a classical or linear group. Moreover, by [2, Prop. 6.3], this property also holds if we restrict these functors to the level 0 categories, without any hypothesis on G.

Let us now explain the typical application of this result that we have in mind. Suppose further that $\mathbb{G}$ extends to a connected reductive group over the integers $\mathcal{O}_F$ and that $\mathbb{P}$ extends to a parabolic subgroup over $\mathcal{O}_F$ (in other words, fix an hyperspecial point x in the appartment associated to some maximal torus of $\mathbb{P}$). Put $\bar{G} := \mathbb{G}(k_F)$ and $\bar{P} := \mathbb{P}(k_F)$ and let Z_G be the center of G. Let $\bar{\sigma}$, resp. $\bar{\pi}$, be a cuspidal irreducible $\overline{\mathbb{F}}_\ell$-representation of $\bar{M}$, resp. of $\bar{G}$. Write $\mathrm{ind}_{G_x}^G(\bar{\pi})$ for the compact induction from $G_x := \mathbb{G}(\mathcal{O}_F)$ of the inflation of $\bar{\pi}$ to G_x. Then the set Π of simple quotients of $\mathrm{ind}_{G_x}^G(\bar{\pi})$ is an unramified orbit of level 0 cuspidal irreducible representations of G. Similarly we get from $\bar{\sigma}$ an unramified orbit Σ of cuspidal irreducible representations of M.

Corollary 1.2. *Assume that $\bar{\pi}$ is isomorphic to a $\overline{\mathbb{F}}_\ell\bar{G}$-subquotient of $i_{\bar{P}}(\bar{\sigma})$. Then any $\pi \in \Pi$ is isomorphic to a $\overline{\mathbb{F}}_\ell G$-subquotient of $i_P(\sigma)$ for an appropriate $\sigma \in \Sigma$.*

We note that the converse also holds since, denoting by G_x^+ the kernel of $G_x \twoheadrightarrow \bar{G}$, we have $\pi^{G_x^+} \simeq_{G_x} \bar{\pi}$ for all $\pi \in \Pi$, while $i_P(\sigma)^{G_x^+} \simeq_{G_x} i_{\bar{P}}(\bar{\sigma})$ for all $\sigma \in \Sigma$.

Proof. Pick some $\pi \in \Pi$. By construction, π is a quotient of $\mathrm{ind}_{G_x}^G(\bar{\pi})$, hence it is a subquotient of $\mathrm{ind}_{G_x}^G(i_{\bar{P}}(\bar{\sigma}))$. On the other hand we have an isomorphism $\mathrm{ind}_{G_x}^G(i_{\bar{P}}(\bar{\sigma})) \simeq i_P(\mathrm{ind}_{M_x}^M(\bar{\sigma}))$ (this follows e.g. from Proposition 6.2 (i) and Corollary 3.6 (ii) of [2]). Since $\mathrm{ind}_{M_x}^M(\bar{\sigma})$ is cuspidal and has level 0, Theorem 1.1 insures the existence of a simple $\overline{\mathbb{F}}_\ell M$-subquotient ρ of $\mathrm{ind}_{M_x}^M(\bar{\sigma})$ such that π occurs as a $\overline{\mathbb{F}}_\ell G$-subquotient in $i_P(\rho)$. Now ρ is automatically a $\overline{\mathbb{F}}_\ell[Z_M]M$-subquotient of $\mathrm{ind}_{M_x}^M(\bar{\sigma})$, and the latter is an admissible and noetherian $\overline{\mathbb{F}}_\ell[Z_M]M$-module. Let χ be the central character of ρ and let I_χ be the kernel of χ in $\overline{\mathbb{F}}_\ell[Z_M]$. Then, as in Lemma 2.2 below, the Artin-Rees lemma implies that ρ is isomorphic to a subquotient of $\mathrm{ind}_{M_x}^M(\bar{\sigma})/I_\chi\mathrm{ind}_{M_x}^M(\bar{\sigma})$. But the latter is also $\mathrm{ind}_{M_x Z_M}^M(\bar{\sigma}_\chi)$ for an appropriate extension $\bar{\sigma}_\chi$ of $\bar{\sigma}$ to $M_x Z_M$. Hence it is simple and belongs to Σ. $\square$

This answers a question asked to me by V. Sécherre, motivated by a long-standing problem in modular representation theory of p-adic groups on the "uniqueness" of the "supercuspidal support". Given an irreducible π occurring as a subquotient of $i_P(\sigma)$ and $i_{P'}(\sigma')$ with σ and σ' *super*cuspidal, then are (M, σ) and (M', σ') G-conjugate ? The same problem arises for finite reductive groups, and the last corollary tells us that if this property is violated in $\bar{G}$, then so it is in G. This uniqueness property was proved by Vignéras for $\mathbb{G} = \mathrm{GL}_n$ [6, IV.6.1] and is absolutely crucial to her work on the mod-ℓ Langlands correspondence [7] as well as to Emerton and Helm's subsequent developments [3], [4]. It is also true for inner forms of GL_n, by [5]. However, it has been observed very recently by Olivier Dudas, that this property is violated for $\mathrm{Sp}_8(\mathbb{F}_q)$ whenever ℓ divides $q^2 + 1$. Therefore, the last corollary says it is also violated for $\mathrm{Sp}_8(F)$ in the same setting.

2. Proofs

Since our arguments use Bernstein and Zelevinski's geometric lemma in [1], it will be more convenient to normalize the parabolic functors. So we will assume that R contains a square root of p if necessary, and the notation i_P and r_P will denote *normalized* parabolic functors. Of course it would be possible to write the argument with the ordinary functors, at the cost of introducing a bunch of modulus characters in the formulas.

2.1. **Extra endomorphisms.** Let M be a Levi subgroup of G, and let us denote by Z_M the center of M. Any smooth RM-module V is canonically a smooth $R[Z_M]M$-module, and any morphism of smooth RM-modules is $R[Z_M]$-linear. In other words we have a canonical morphism of rings $R[Z_M] \longrightarrow \mathrm{End}\left(\mathrm{Id}_{\mathrm{Rep}_R(M)}\right)$.

By functoriality, $R[Z_M]$ also acts on the induced representation $i_P(V)$, and we will restrict this action to the subring $B_M := R[Z_M]^{N_G(M)}$ of functions on Z_M that are invariant under the normalizer $N_G(M)$ of M in G. Hence we may view $i_P(V)$ as a smooth B_MG-module. The reason why we consider this subring B_M is the following remarkable property.

Fact. [2, Lemma 4.8]. *Let P and Q be two parabolic subgroups with a common Levi factor M, and let V, W be two smooth* ***cuspidal*** *RM-modules. Then any RG-morphism $i_P(V) \longrightarrow i_Q(W)$ is automatically B_M-linear*[1].

2.2. **A particular case.** We first note that in the setting of Theorem 1.1, V can be assumed to be finitely generated as a RM-module without loss of generality. Moreover, in this case, $i_P(V)$ is then a finitely generated RG-module. Indeed, in the context of assumption (2) (second adjointness) this is Lemma 4.6 of [2], while in the context of assumption (1) (level 0) this follows more directly from Proposition 6.2 (i) and Corollary 3.6 (ii) of [2], which imply that $i_P(\mathrm{ind}_{M_x^+}^M R) \simeq \mathrm{ind}_{G_x^+ P_x}^G R$ for any point x in the building of M inside that of G.

Lemma. *Assume that V is a cuspidal and finitely generated RM-module. Let W be a B_MG-subquotient of $i_P(V)$ and let I be the annihilator ideal of W in B_M. Then every simple RG-subquotient of W is isomorphic to a RG-subquotient of $i_P(V/IV)$.*

Proof. By [2, lemme 4.2], our hypothesis on V implies that it is an admissible $R[Z_M]M$-module, meaning that for any open pro-p-subgroup H of M, the $R[Z_M]$-module V^H is finitely generated. As a consequence of the Mackey formula, the induced representation $i_P(V)$ is in turn an *admissible* $R[Z_M]G$-module. Actually, for any open pro-p-subgroup H of G, the action of $R[Z_M]$ on $i_P(V)^H$ factors over a finitely generated, and therefore noetherian, R algebra. Since B_M is a ring of invariants under some finite group action on $R[Z_M]$, it follows that $i_P(V)^H$ is a noetherian B_M-module.

Now let W be as in the lemma, and let U be a B_MG-submodule of $i_P(V)$ such that $U \twoheadrightarrow W$. As recalled above, our running hypothesis imply that $i_P(V)$ is finitely generated as a RG-module, so it is generated by its H-invariants for some open pro-p-subgroup of G. Moreover, level decompositions tell us that we may choose H such that any RG-submodule of $i_P(V)$ is also generated by its H-invariants. By noetheriannity of $i_P(V)^H$ as a B_M-module, the Artin-Rees lemma gives us an

[1] Although it is not clearly stated in *loc. cit.*, a look at the proof there shows that it is important to use normalized parabolic functors here. Otherwise, some twists appear.

integer n such that $I^n i_P(V)^H \cap U^H \subset IU^H$. Hence we also have $I^n i_P(V) \cap U \subset IU$, and it follows that W is a subquotient of $i_P(V)/I^n i_P(V)$, which is isomorphic to $i_P(V/I^nV)$ since i_P commutes to colimits. Now if π is a simple subquotient of W, it is isomorphic to a subquotient of some $i_P(I^kV/I^{k+1}V)$. But I^kV is a quotient of a finite sum of copies of V, hence $I^kV/I^{k+1}V$ is a quotient of a finite sum of copies of V/IV, and finally π occurs as a subquotient of $i_P(V/IV)$. □

Proposition. *Assume that V is a cuspidal and finitely generated RM-module. Let π be a simple admissible RG-subquotient of a* ***cuspidal*** *B_MG-subquotient W of $i_P(V)$. Then, there is a simple RM-subquotient σ of V such that π is isomorphic to a subquotient of $i_P(\sigma)$.*

Proof. Put $B_G := R[Z_G]$, a subring of B_M. The key point is that, by [2, lemme 4.10], W is an admissible B_G-module. In particular, the ring $\mathrm{End}_{RG}(W)$ is finite over B_G, and so is its subring B_M/I, with the notation of the previous lemma. Hence V/IV and $i_P(V/IV)$ are also admissible B_GG-modules. Now π, being an RG-subquotient, is also a B_GG-subquotient of $i_P(V/IV)$. Let J be the annihilator of π in B_G. The quotient ring B_G/J is then a finite field extension of the image $\overline{R}$ of R in $\mathrm{End}_{RG}(\pi)$ (note that $\overline{R}$ is a field because π is assumed to be R-admissible, so that the division ring $\mathrm{End}_{RG}(\pi)$ is finite over R). As in the proof of the last lemma, π occurs as a subquotient of $i_P(V/(I+J)V)$. But $V/(I+J)V$ is admissible over B_G/J, hence also over the field $\overline{R}$. Being admissible and finitely generated over a field, it has finite length (bounded by the dimension of $[V/(I+V)V]^H$ for H well chosen). □

2.3. **A functorial filtration.** We recall here a functorial filtration from [2, Lemma 4.7], which follows from the second adjointness property of parabolic functors. In the setting (1) of Theorem 1.1, denote by $\mathcal{C}$ the category $\mathrm{Rep}^0_R(G)$, and in the setting (2) of the same theorem, put $\mathcal{C} = \mathrm{Rep}_R(G)$. We will define a filtration of the identity functor $\mathrm{Id}_{\mathcal{C}}$. To this aim, we fix a minimal parabolic subgroup P_0 and a maximal split torus S in P_0. A parabolic subgroup P is then called *standard* if $P \supset P_0$, and in this case we denote by $\overline{P}$ the opposite parabolic subgroup with respect to S. We fix a total ordering $P_0 \subset \cdots \subset P_g = G$ of the set of standard parabolic subgroups that refines the inclusion relation. Then for $i = 0, \cdots, g$ we define an increasing filtration of $\mathrm{Id}_{\mathcal{C}}$ in the abelian category of endofunctors of $\mathcal{C}$ by

$$\mathcal{F}_i := \sum_{j=0}^{i} \mathrm{im}\left(i_{P_i} \circ r_{\overline{P}_i} \xrightarrow{adj} \mathrm{Id}_{\mathcal{C}} \right).$$

By Lemma 4.17 of [2], the last graded piece $\mathcal{F}_g(V)/\mathcal{F}_{g-1}(V)$ is the maximal cuspidal quotient V^{cusp} of an object $V \in \mathcal{C}$, and for all $i = 0, \cdots, g$, the graded piece

$$\mathcal{G}_i(V) := \mathcal{F}_i(V)/\mathcal{F}_{i-1}(V)$$

is a quotient of $i_{P_i}(r_{\overline{P}_i}(V)^{\mathrm{cusp}})$.

2.4. **The main step.** Our aim here is to prove the following apparently weaker form of Theorem 1.1.

Theorem. *Let V be a* ***cuspidal*** *RM-module and let π be a simple admissible* ***cuspidal*** *RG-subquotient of $i_P(G)$. Then there is a simple RM-subquotient of V such that π is isomorphic to a subquotient of $i_P(\sigma)$.*

Proof. We may assume that P is standard as in the last paragraph. Our proof is by induction on the corank of P in G. Note that when this corank is 0, there is nothing to prove. So let us assume that $P \neq G$.

Let $U \subset i_P(V)$ be an RG-submodule such that $U \twoheadrightarrow \pi$. A priori U need not be B_M-stable. Let us thus consider $\widetilde{U} := B_M.U = \sum_{b \in B_M} b.U \subset i_P(V)$, and also the following RG-submodule of U

$$U_M := \mathrm{im}\left(i_P \circ r_{\overline{P}}(U) \xrightarrow{Adj} U\right) \subset U \subset i_P(V).$$

Then we have the following properties :

- *U_M is B_MG-stable inside $i_P(V)$, hence it is contained in $\widetilde{U}$.* This follows indeed from Fact 2.1.
- *π is an RG subquotient of $\widetilde{U}/U_M$.* Indeed, U_M lies in the kernel of $U \twoheadrightarrow \pi$ because π is cuspidal.
- *$r_{P'}(\widetilde{U}/U_M) = 0$ for all standard P' which is associate to P.* Indeed, we certainly have $r_{\overline{P}}(U/U_M) = 0$ by construction, hence also $r_{\overline{P}}(b.U/U_M) = 0$ for all $b \in B_M$ and thus $r_{\overline{P}}(\widetilde{U}/U_M) = 0$. But by [2, cor. 3.9 (ii)], this is equivalent to $r_P(\widetilde{U}/U_M) = 0$. If P has corank 1 we are done since any associate P' is either P or conjugate to $\overline{P}$. In general, there is a sequence $P^0 = P, \cdots, P^r = P'$ of standard parabolic subgroups associate to P and such that P^i, P^{i+1} have both corank 1 in some standard parabolic subgroup Q^i. Then [2, cor. 3.9 (ii)] shows inductively that $r_{P^i}(\widetilde{U}/U_M) = 0$ for all i and in particular $r_{P'}(\widetilde{U}/U_M) = 0$.

Hence we have realized π as a RG-subquotient of a B_MG-subquotient $W := \widetilde{U}/U_M$ of $i_M(V)$ which is killed by r_P. When P has corank 1 in G, W is cuspidal hence Proposition 2.2 provides us with the desired σ. When P has corank > 1, we apply the functorial filtration of subsection 2.3 to W. Note that, being functorial, it is in particular B_M-stable, so that all graded pieces $\mathcal{G}_i(W)$ are B_MG-subquotients of $i_P(V)$. We claim that if the graded piece $\mathcal{G}_i(W)$ is non zero, then P_i has strictly smaller corank than P in G. Indeed, the Geometric Lemma of [1] tells us that $i_P(V)$ is killed by all r_Q with Q a standard parabolic subgroup not containing an associate of P. Since W is also killed by all $r_{P'}$ for P' associate to P, the claim follows.

Now pick some i such that π occurs as a RG-subquotient of $\mathcal{G}_i(W)$. If $i = g$ then, $\mathcal{G}_g(W)$ being cuspidal and B_M-stable, Proposition 2.2 provides us with the desired σ. So let us suppose $i < g$, so that $\mathcal{G}_i(W)$ is a quotient of $i_{P_i}(r_{\overline{P}_i}(W)^{\mathrm{cusp}})$. Then the induction hypothesis provides us with a simple RM_i-subquotient σ_i of $r_{\overline{P}_i}(W)^{\mathrm{cusp}}$ such that π is isomorphic to a subquotient of $i_{P_i}(\sigma_i)$. This σ_i is a subquotient of $r_{\overline{P}_i}(i_P(V))$, hence by Bernstein and Zelevinski's geometric lemma, there is some element w in the Weyl group W_S such that $w(M) \subset M_i$ and σ_i occurs as a subquotient of $i_{{}^wP \cap M_i}({}^wV)$. Applying again the induction hypothesis, now to the parabolic subgroup ${}^wP \cap M_i$ which has corank in M_i strictly smaller than that of P in G, we get a simple RM-subquotient σ of V such that σ_i is isomorphic to a subquotient of $i_{{}^wP \cap M_i}({}^w\sigma)$. It follows that our original π is isomorphic to a subquotient of $i_{({}^wP \cap M_i)U_{P_i}}({}^w\sigma) = i_{P'}(\sigma)$ where $P' = (P \cap {}^{w^{-1}}M_i)U_{{}^{w^{-1}}P_i}$ is a (semi-standard) parabolic subgroup with Levi component M. But now, Lemma 4.13 of [2] tells[2] us that π is also a subquotient of $i_P(\sigma)$. $\square$

2.5. **End of the proof of Theorem 1.1.** We will first remove the assumption that π is cuspidal in Theorem 2.4. So let π be any simple admissible subquotient of $i_P(V)$. There is a standard parabolic subgroup P_i such that $r_{P_i}(\pi)$ is cuspidal. Let π_i be a simple RM_i-submodule of $r_{P_i}(\pi)$. It is a subquotient of $r_{P_i}(i_P(V))$ hence, as in the last proof, there is some $w \in W_S$ such that ${}^wM \subset M_i$ and π_i is a RM_i-subquotient of $i_{{}^wP\cap M_i}({}^wV)$. Then Theorem 2.4 provides us with a simple RM-subquotient σ of V such that π_i occurs as a subquotient of $i_{{}^wP\cap M_i}({}^wV)$. It follows that π occurs as a subquotient of $i_{P'}(\sigma)$ with $P' = (P \cap {}^{w^{-1}}M_i)U_{{}^{w^{-1}}P_i}$, hence also of $i_P(\sigma)$ by [2, Lemme 4.13].

Finally we remove the assumption of cuspidality for V, at the cost of a less precise result. To this aim we apply to V the filtration of subsection 2.3 (or rather its avatar relative to M). So π is isomorphic to a subquotient of some $i_P(\mathcal{G}_i(V))$, with $P_i \supset P$ and $\mathcal{G}_i(V)$ a quotient of $i_{P_i\cap M}(r_{\overline{P}_i\cap M}(V)^{\mathrm{cusp}})$. Therefore π is a subquotient of $i_{P_i}(r_{\overline{P}_i\cap M}(V)^{\mathrm{cusp}})$ and by what we have proven sofar, there is a simple RM_i-subquotient ρ of $r_{\overline{P}_i\cap M}(V)$ such that π occurs as a subquotient in $i_{P_i}(\rho)$. Since the latter has finite length, there certainly is a simple subquotient σ of $i_{P_i\cap M}(\rho)$ such that π occurs as a subquotient of $i_P(\sigma)$. However, this argument does not imply that σ is a subquotient of V a priori, only that it is a subquotient of $i_{P_i\cap M} \circ r_{\overline{P}_i\cap M}(V)$.

References

[1] I.N. Bernstein and A.V. Zelevinski. Induced representations on reductive p-adic groups. *Ann.Sci.Ec.Norm.Sup*, 10:441–472, 1977.

[2] J.-F. Dat. Finitude pour les représentations lisses de groupes p-adiques. *J. Inst. Math. Jussieu*, 8(2):261–333, 2009.

[3] M. Emerton and D. Helm. The local Langlands correspondence for GL_n in families. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(4):655–722, 2014.

[4] D. Helm. The Bernstein center of the category of smooth $W(k)[\mathrm{GL}_n(F)]$-modules. *Forum Math. Sigma*, 4:11, 98, 2016.

[5] A. Mínguez and V. Sécherre. Représentations lisses modulo ℓ de $\mathrm{GL}_m(D)$. *Duke Math. J.*, 163(4):795–887, 2014.

[6] M.-F. Vignéras. Induced R-representations of p-adic reductive groups. *Selecta Math. (N.S.)*, 4(4):549–623, 1998.

[7] M.-F. Vignéras. Correspondance de Langlands semi-simple pour $GL(n, F)$ modulo $\ell \neq p$. *Invent. Math.*, 144:177–223, 2001.

Université Pierre et Marie Curie, Institut de Mathématiques de Jussieu-PRG, 4 place Jussieu, 75005 Paris, France

E-mail address: jean-francois-dat@imj-prg.fr

Ecole Normale Supérieure, Département de Mathématiques et applications, 45 rue d'Ulm, 75005 Paris, France

[2]Again, it is important to use normalized induction here