Essentially tame $\mathbb{Z}[\frac{1}{p}]$-representations of p-adic groups

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Contents

1 Main results

Let $\mathbf{G}$ be a connected reductive group defined over a p-adic field F and put $G := \mathbf{G}(F)$. For any commutative ring, we denote by $\mathrm{Rep}_R(G)$ the category of smooth RG-modules. We write ${}^L\mathbf{G} = \hat{\mathbf{G}} \rtimes W_F$ for the Weil form of the L-group of $\mathbf{G}$ over $\mathbb{C}$. Let $\Phi(G) \subset H^1(W'_F, \hat{\mathbf{G}})$ be the set of admissible Langlands parameters. For any subgroup $K_F \subset W_F$, we denote by $\Phi(K_F, G)$ the image of $\Phi(G)$ by the restriction map $H^1(W'_F, \hat{\mathbf{G}}) \longrightarrow H^1(K_F, \hat{\mathbf{G}})$.

The conjectural Langlands correspondence for G and its expected compatibility with parabolic induction joint to Bernstein's decomposition theorem imply that there should be a decomposition of the abelian category $\mathrm{Rep}_{\mathbb{C}}(G)$ as a direct product indexed by parameters from the wild inertia subgroup P_F

$$\mathrm{Rep}_{\mathbb{C}}(G) = \prod_{\phi \in \Phi(P_F, G)} \mathrm{Rep}_{\mathbb{C}}^{\phi}(G)$$

where the simple objects of $\mathrm{Rep}_{\mathbb{C}}^{\phi}(G)$ should be all irreducible representations whose Langlands parameter φ_π satisfy $\varphi_{\pi|P_F} \sim \phi$. In particular, to any $\phi \in \Phi(P_F, G)$ should be associated an idempotent e_ϕ in the center $\mathfrak{Z}_{\mathbb{C}}(G)$ of the category $\mathrm{Rep}_{\mathbb{C}}(G)$. In [9] we have advertized the idea that such an idempotent should be ℓ-integral for all $\ell \neq p$. Let us denote by R the subring of $\mathbb{C}$

generated by all p-power roots of unity and $\frac{1}{p}$. Then we expect that $e_\phi \in \mathfrak{Z}_R(G)$, or equivalently that the above decomposition comes from a decomposition

$$\mathrm{Rep}_R(G) = \prod_{\phi \in \Phi(P_F, G)} \mathrm{Rep}_R^\phi(G). \tag{1.0.1}$$

One case where this summand is well understood is when $\phi = 1$ is trivial. Then $\mathrm{Rep}_{\mathbb{C}}^\phi(G)$ is the depth 0 factor of $\mathrm{Rep}_{\mathbb{C}}(G)$ and it is well known to be defined over R, and even over $\mathbb{Z}[\frac{1}{p}]$. Actually we have the following description of this summand :

$$\mathrm{Rep}_R^1(G) = \left\{ V \in \mathrm{Rep}_R(G),\, V = \sum_{x \in \mathcal{B}} V^{G_{x,0+}} \right\}$$

where $\mathcal{B}$ denotes the Bruhat-Tits building and $G_{x,0+}$ is the pro-p-radical of the parahoric subgroup attached to x.

In [9] we have also speculated that for more general ϕ, the summand category $\mathrm{Rep}_R^\phi(G)$ should be equivalent to the depth 0 category of a certain possibly non-connected reductive group associated with the centralizer $C_{\hat{\mathbf{G}}}(\phi)$ of the image $\phi(P_F)$ in $\hat{\mathbf{G}}$. This speculation can be made more precise when

i) $C_{\hat{\mathbf{G}}}(\phi)$ is connected, and

ii) ϕ has an extension φ to W_F such that conjugation by $\varphi(W_F)$ preserves a pinning of $C_{\hat{\mathbf{G}}}(\phi)$.

Indeed, i) allows us to define a tamely ramified reductive group $\mathbf{G}_\phi$ over F with $\hat{\mathbf{G}}_\phi = C_{\hat{\mathbf{G}}}(\phi)$, while any φ as in ii) provides us with an L-homomorphism ${}^L\mathbf{G}_\phi \xrightarrow{\xi_\varphi} {}^L\mathbf{G}$. Moreover, the inclusion $Z(\hat{\mathbf{G}})^{W_F} \subset Z(C_{\hat{\mathbf{G}}}(\phi))^{\varphi(W_F)}$ induces a map $H^1(F, \mathbf{G}_\phi) \xrightarrow{h} H^1(F, \mathbf{G})$ via Kottwitz' homomorphism (see 2.1.4), and we expect the existence of an equivalence of categories

$$\prod_{\alpha \in \ker(h)} \mathrm{Rep}_R^1(G_\phi^\alpha) \xrightarrow{\sim} \mathrm{Rep}_R^\phi(G) \tag{1.0.2}$$

that interpolates the usual transfer of L-packets, via the L-homomorphism ${}^L\mathbf{G}_\phi \xrightarrow{\xi_\varphi} {}^L\mathbf{G}$. Here, $\mathbf{G}_\phi^\alpha$ stands for "the" pure inner form of $\mathbf{G}$ associated to α. This comprises the existence of a decomposition

$$\mathrm{Rep}_R^\phi(G) = \prod_{\alpha \in \ker(h)} \mathrm{Rep}_R^{\phi,\alpha}(G) \tag{1.0.3}$$

which is expected to be compatible with the extended Langlands correspondence in the sense that an irreducible $\pi \in \mathrm{Rep}_{\mathbb{C}}^\phi(G)$ with extended parameter $(\varphi_\pi, \varepsilon_\pi)$ should lie in $\mathrm{Rep}_R^{\phi,\alpha}(G)$ if and only if the restriction of ε_π to $Z(C_{\hat{\mathbf{G}}}(\phi))^{\varphi(W_F)}$ is α-isotypic (recall that ε_π is a character of $C_{\hat{\mathbf{G}}}(\varphi_\pi) = C_{\hat{\mathbf{G}}}(\phi)^{\varphi_\pi(W_F)}$).

In Section 2 of this paper we associate a Serre subcategory $\mathrm{Rep}_R^\phi(G)$ of $\mathrm{Rep}_R(G)$ together with a decomposition (1.0.3) to any parameter $\phi \in \Phi(P_F, \mathbf{G})$ whose centralizer $C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup (and under some mild conditions on p). Note that this hypothesis forces $\mathbf{G}$ to split over a tamely ramified extension. The Serre subcategories that we construct are pairwise orthogonal, and actually the Serre subcategory $\mathrm{Rep}_R^{e.t.}(G)$ that they generate is a direct product of all these $\mathrm{Rep}_R^{\phi}(G)$. Our constructions are much inspired by Kaletha's paper [11], where we also found solutions to a lot of technical issues, and ultimately they rely on Yu's theory of "generic characters". Informally, $\mathrm{Rep}_R^{e.t.}(G)$ can be thought as the subcategory "generated" by Yu's characters. It turns out that if p does not divide the order of the absolute Weyl group of $\mathbf{G}$, then any $\phi \in \Phi(P_F, \mathbf{G})$ satisfies the hypothesis. Correspondingly, a result of Fintzen insures that in this situation we indeed have $\mathrm{Rep}_R^{e.t.}(G) = \mathrm{Rep}_R(G)$ so that we get a complete decomposition (1.0.1) of $\mathrm{Rep}_R(G)$ in this case. We also prove some compatibility properties of our constructions with isogenies and parabolic induction. For example, if $\mathbf{P}$ is a parabolic F-subgroup of $\mathbf{G}$ with Levi $\mathbf{M}$ and ϕ comes from $\phi_M \in \Phi(P_F, \mathbf{M})$ via some dual embedding ${}^L\mathbf{M} \hookrightarrow {}^L\mathbf{G}$, then the parabolic induction functor i_P takes $\mathrm{Rep}_R^{\phi_M}(M)$ into $\mathrm{Rep}_R^{\phi}(G)$ and, moreover, induces an equivalence of categories whenever $C_{\hat{\mathbf{G}}}(\phi) \subset \hat{\mathbf{M}}$. We do not address the problem of compatibility of our decomposition with Langlands correspondence since the latter is not yet established in this context, but we note that it is compatible with Kaletha's correspondence for regular supercuspidal representations [11].

In section 3, under the same hypothesis on ϕ, we will construct equivalences of categories as in (1.0.2).

1.1 Notation

todo

2 From parameters to subcategories

2.1 Levi-center embeddings and duality

In this subsection, $\mathbf{G}$ is a general connected reductive group over F. From the construction of the dual group $\hat{\mathbf{G}}$ we have a bijection between $\mathbf{G}$-conjugacy classes of *maximal* tori embeddings $\mathbf{S} \hookrightarrow \mathbf{G}$ and $\hat{\mathbf{G}}$-conjugacy classes of *maximal* tori embeddings $\hat{\mathbf{S}} \hookrightarrow \hat{\mathbf{G}}$. We seek a generalization of this for embeddings of tori as connected center of a Levi subgroup.

2.1.1 *Lemma. — Let* $\mathbf{S}$ *be a torus contained in* $\mathbf{G}$. *The following are equivalent*

i) $\mathbf{S}$ *is the connected center of a Levi subgroup of* $\mathbf{G}$.

ii) $\mathbf{S} = C_{\mathbf{G}}(C_{\mathbf{G}}(\mathbf{S}))^{\circ}$

iii) there is a maximal torus $\mathbf{T}$ *of* $\mathbf{G}$ *containing* $\mathbf{S}$ *and a Levi subroot system* $\Phi' \subset \Phi(\mathbf{T}, \mathbf{G})$ *such that* $\mathbf{S} = \left(\bigcap_{\alpha\in\Phi'} \ker(\alpha)\right)^{\circ}$.

Proof. Standard. □

2.1.2 *Definition.–* A *Levi-center embedding* in $\mathbf{G}$ is a pair $(\mathbf{S}, \iota)$ with $\mathbf{S}$ a torus and $\iota : \mathbf{S} \hookrightarrow \mathbf{G}$ an embedding such that $\iota(\mathbf{S})$ satisfies the properties of the last lemma.

2.1.3 *Duality.* — Let $\hat{\mathbf{S}}$ be a complex algebraic torus. The algebraic group $\hat{\mathbf{G}}$ acts by conjugation on Levi-center embeddings $\hat{\iota} : \hat{\mathbf{S}} \hookrightarrow \hat{\mathbf{G}}$. Our aim is to attach to a conjugacy class $\{\hat{\iota}\}$ of such embeddings, a "dual" conjugacy class $\{\iota\}$ of Levi-center embeddings in the $\overline{F}$-algebraic group $\mathbf{G}$.

The stabilizer $\hat{\mathbf{G}}_{\hat{\iota}}$ of $\hat{\iota}$ for the adjoint action of $\hat{\mathbf{G}}$ is the centralizer $C_{\hat{\mathbf{G}}}(\hat{\iota}(\hat{\mathbf{S}}))$ of the torus $\hat{\iota}(\hat{\mathbf{S}})$, which is a Levi subgroup of $\hat{\mathbf{G}}$. Its cocenter $\hat{\mathbf{G}}_{\hat{\iota},\mathrm{ab}}$ is a torus which *only depends on the conjugacy class* $\{\hat{\iota}\}$ *of* $\hat{\iota}$, in the sense that for $\hat{\iota}, \hat{\iota}'$ we have a canonical isomorphism $\hat{\mathbf{G}}_{\hat{\iota},\mathrm{ab}} \xrightarrow{\sim} \hat{\mathbf{G}}_{\hat{\iota}',\mathrm{ab}}$ given by conjugation under any $\hat{g} \in \hat{\mathbf{G}}$ that conjugates $\hat{\iota}$ to $\hat{\iota}'$. Let us denote by $\hat{\mathbf{S}}_{\{\hat{\iota}\}} := \lim_{\hat{\iota}} \hat{\mathbf{G}}_{\hat{\iota},\mathrm{ab}}$ this common torus. Since $\hat{\iota}(\hat{\mathbf{S}})$ is the connected center of $\hat{\mathbf{G}}_{\hat{\iota}}$, the embedding $\hat{\iota}$ induces an isogeny $\hat{\mathbf{S}} \longrightarrow \hat{\mathbf{G}}_{\hat{\iota},\mathrm{ab}}$ which is compatible with the analogous isogeny $\hat{\mathbf{S}} \longrightarrow \hat{\mathbf{G}}_{\hat{\iota}',\mathrm{ab}}$ through the isomorphism $\hat{\mathbf{G}}_{\hat{\iota},\mathrm{ab}} \xrightarrow{\sim} \hat{\mathbf{G}}_{\hat{\iota}',\mathrm{ab}}$ for any $\hat{\iota}' \in \{\hat{\iota}\}$. Therefore, this defines an isogeny $\hat{\mathbf{S}} \longrightarrow \hat{\mathbf{S}}_{\{\hat{\iota}\}}$ which, again, *only depends on* $\{\hat{\iota}\}$. Concretely, the (finite) kernel $H_{\{\hat{\iota}\}} := \ker(\hat{\mathbf{S}} \longrightarrow \hat{\mathbf{G}}_{\hat{\iota},\mathrm{ab}})$ is independent of $\hat{\iota}$ in $\{\hat{\iota}\}$ and we have $\hat{\mathbf{S}}_{\{\hat{\iota}\}} = \hat{\mathbf{S}}/H_{\{\hat{\iota}\}}$.

Now let us choose a maximal torus $\hat{\mathbf{T}}$ in $\hat{\mathbf{G}}_{\hat{\iota}}$. The isogeny $\hat{\mathbf{S}} \longrightarrow \hat{\mathbf{S}}_{\{\hat{\iota}\}}$ factors as $\hat{\mathbf{S}} \xrightarrow{\hat{\iota}} \hat{\mathbf{T}} \xrightarrow{\hat{\pi}} \hat{\mathbf{S}}_{\{\hat{\iota}\}}$ and identifies

$$\hat{\iota} : \hat{\mathbf{S}} \xrightarrow{\sim} \left(\bigcap_{\alpha \in \Sigma_{\hat{\iota}}} \ker(\alpha) \right)^{\circ} \subset \hat{\mathbf{T}} \text{ and } \hat{\pi} : \hat{\mathbf{T}} / \left(\sum_{\alpha \in \Sigma_{\hat{\iota}}} \mathrm{im}(\alpha^{\vee}) \right) \xrightarrow{\sim} \hat{\mathbf{S}}_{\{\hat{\iota}\}},$$

where $\Sigma_{\hat{\iota}}$ is the root system $\Sigma(\hat{\mathbf{T}}, \hat{\mathbf{G}}_{\hat{\iota}})$. Hence the dual isogeny $\mathbf{S}_{\{\hat{\iota}\}} \longrightarrow \mathbf{S}$ factors through the dual torus $\mathbf{T}$ giving isomorphisms

$$\pi : \mathbf{S}_{\{\hat{\iota}\}} \xrightarrow{\sim} \left(\bigcap_{\alpha \in \Sigma_{\hat{\iota}}} \ker(\alpha^{\vee}) \right)^{\circ} \subset \mathbf{T} \text{ and } \hat{\hat{\iota}} : \mathbf{T} / \left(\sum_{\alpha \in \Sigma_{\hat{\iota}}} \mathrm{im}(\alpha) \right) \xrightarrow{\sim} \mathbf{S},$$

where $\alpha^{\vee}$, resp. α, is seen as a character, resp. a cocharacter, of $\mathbf{T}$. Now recall that the embedding $\hat{\mathbf{T}} \hookrightarrow \hat{\mathbf{G}}$ gives rise to a canonical $\mathbf{G}$-conjugacy class of embeddings $\mathbf{T} \hookrightarrow \mathbf{G}$. Choose such a "dual embedding" j. By construction it identifies $\Sigma(\hat{\mathbf{T}}, \hat{\mathbf{G}})$ with $\Sigma(\mathbf{T}, \mathbf{G})^{\vee}$. In particular, we see from point iii) in Lemma 2.1.1 that the composition $\iota : \mathbf{S}_{\{\hat{\iota}\}} \xrightarrow{\pi} \mathbf{T} \hookrightarrow \mathbf{G}$ is a Levi-center embedding such that, by construction, the Levi subgroup $\mathbf{G}_{\iota} = C_{\mathbf{G}}(\iota(\mathbf{S}_{\{\hat{\iota}\}}))$ is dual to $\hat{\mathbf{G}}_{\hat{\iota}}$.

Lemma. — The $\mathbf{G}$*-conjugacy class* $\{\iota\}$ *of* ι *only depends on the* $\hat{\mathbf{G}}$*-conjugacy class* $\{\hat{\iota}\}$*.*

Proof. Let $\hat{\iota}'$ be conjugate to $\hat{\iota}$, let $\hat{\mathbf{T}}'$ be a maximal torus in $\hat{\mathbf{G}}_{\hat{\iota}'}$ and let $j' : \mathbf{T}' \hookrightarrow \mathbf{G}$ be a choice of dual embedding. Since all maximal tori of $\hat{\mathbf{G}}_{\hat{\iota}}$ are conjugate, there is an element $\hat{g} \in \hat{\mathbf{G}}$ which conjugates $\hat{\iota}$ to $\hat{\iota}'$ and $\hat{\mathbf{T}}$ to $\hat{\mathbf{T}}'$. Then we have a commutative diagram

$$\begin{tikzcd} \hat{\mathbf{S}} \arrow[r, "\hat{\iota}"] \arrow[d, "\hat{\iota}'"'] & \hat{\mathbf{T}} \arrow[d, "\hat{\pi}"] \arrow[dl, "\mathrm{Ad}_{\hat{g}}"', "\sim"] \\ \hat{\mathbf{T}}' \arrow[r, "\hat{\pi}'"'] & \hat{\mathbf{S}}_{\{\hat{\iota}\}} \end{tikzcd} .$$

It follows that on the dual side we get $\pi = \hat{\mathrm{Ad}}_{\hat{g}} \circ \pi'$, where $\hat{\mathrm{Ad}}_{\hat{g}}$ is the isomorphism $\mathbf{T}' \xrightarrow{\sim} \mathbf{T}$ dual to $\mathrm{Ad}_{\hat{g}}$. On the other hand, $j \circ \hat{\mathrm{Ad}}_{\hat{g}}$ is a dual embedding of $\mathbf{T}'$ into $\mathbf{G}$, hence there is $g \in \mathbf{G}$ such that $\mathrm{Ad}_g \circ j' = j \circ \hat{\mathrm{Ad}}_{\hat{g}}$. It follows that g conjugates the embedding $\iota' = j' \circ \pi'$ to the embedding $\iota = j \circ \pi$. $\square$

Although we won't need it in this paper, note that we can play the game in the other direction and get the following result.

Proposition. — The above construction sets up a bijection between $\mathbf{G}$*-conjugacy classes of Levi-center embeddings in* $\mathbf{G}$ *and* $\hat{\mathbf{G}}$*-conjugacy classes of Levi-center embeddings in* $\hat{\mathbf{G}}$.

2.1.4 *Rationality. —* We now assume that the complex torus $\hat{\mathbf{S}}$ is endowed with a finite action of W_F. Then W_F acts on the set of Levi-center embeddings $\hat{\iota} : \hat{\mathbf{S}} \longrightarrow \hat{\mathbf{G}}$ by the formula ${}^{\gamma}\hat{\iota} := \gamma_{\hat{\mathbf{S}}} \circ \hat{\iota} \circ \gamma_{\hat{\mathbf{G}}}^{-1}$. *We further assume that the conjugacy class* $\{\hat{\iota}\}$ *is* W_F*-stable.*

In this case, the finite subgroup $\hat{H}_{\{\hat{\iota}\}}$ of $\hat{\mathbf{S}}$ is W_F-stable, its quotient torus $\hat{\mathbf{S}}_{\{\hat{\iota}\}}$ is therefore also equipped with a finite action of W_F, allowing to define an F-structure on the dual torus $\mathbf{S}_{\{\hat{\iota}\}}$. We also define a quasi-split F-group $\mathbf{G}_{\{\hat{\iota}\}}$ as follows. First note that we have an action of the L-group ${}^L\mathbf{G} = \hat{\mathbf{G}} \rtimes W_F$ on Levi-center embeddings given by ${}^{(\hat{g},\gamma)}\hat{\iota} := \mathrm{Ad}_{\hat{g}} \circ {}^{\gamma}\hat{\iota}$, and that the $\hat{\mathbf{G}}$-conjugacy class $\{\hat{\iota}\}$ is W_F-stable if and only if the stabilizer $({}^L\mathbf{G})_{\hat{\iota}}$ surjects to W_F through the projection ${}^L\mathbf{G} \longrightarrow W_F$, so that we get a short exact sequence $\hat{\mathbf{G}}_{\hat{\iota}} \hookrightarrow ({}^L\mathbf{G})_{\hat{\iota}} \twoheadrightarrow W_F$. It follows that the conjugation action $({}^L\mathbf{G})_{\hat{\iota}} \longrightarrow \mathrm{Aut}(\hat{\mathbf{G}}_{\hat{\iota}})$ induces an "outer" action

$$W_F \longrightarrow \mathrm{Out}(\hat{\mathbf{G}}_{\hat{\iota}}) = \mathrm{Aut}(\psi_0(\hat{\mathbf{G}}_{\hat{\iota}})),$$

where ψ_0 denotes the based root datum associated to a reductive group. For any conjugate $\hat{\iota}'$, this outer action is compatible with the canonical isomorphism $\psi_0(\hat{\mathbf{G}}_{\hat{\iota}}) \xrightarrow{\sim} \psi_0(\hat{\mathbf{G}}_{\hat{\iota}'})$ induced by conjugation under any $\hat{g}$ such that $\hat{\iota}' = \mathrm{Ad}_{\hat{g}} \circ \hat{\iota}$. Further, this outer action is *finite* since it induces the given action on the connected center $\hat{\mathbf{S}}$ of $\hat{\mathbf{G}}_{\hat{\iota}}$. Therefore, there is a quasi-split F-group $\mathbf{G}_{\{\hat{\iota}\}}$ endowed with a W_F-equivariant isomorphism $\alpha : \psi_0(\mathbf{G}_{\{\hat{\iota}\}}) \xrightarrow{\sim} \psi_0(\hat{\mathbf{G}}_{\hat{\iota}})^{\vee}$. This pair is unique up to isomorphism and its automorphism group is $\hat{\mathbf{G}}_{\{\hat{\iota}\},\mathrm{ad}}(F)$. Thanks to α, we have F-rational isomorphisms

$$\mathbf{G}_{\{\hat{\iota}\},\mathrm{ab}} \xrightarrow{\sim} \mathbf{S} \;\text{ and }\; \mathbf{S}_{\{\hat{\iota}\}} \xrightarrow{\sim} Z(\mathbf{G}_{\{\hat{\iota}\}})^{\circ}.$$

Moreover, and again thanks to α, we also have a map

$$H^1(F, \mathbf{G}_{\{\hat{\iota}\}}) \longrightarrow H^1(F, \mathbf{G}),$$

defined through "Kottwitz duality" [14, Prop. 6.4] by the inclusion $Z(\hat{\mathbf{G}})^{W_F} \subset Z(\hat{\mathbf{G}}_{\hat{\iota}})^{W_F}$ where the action of W_F on $Z(\hat{\mathbf{G}}_{\hat{\iota}})$ is deduced from the conjugation action of $({}^L\mathbf{G})_{\hat{\iota}}$. [1]

2.1.5 *Proposition. — Assume that* $\mathbf{G}$ *is quasi-split. Then the* $\mathbf{G}$*-conjugacy class of Levi-center embeddings* $\{\mathbf{S}_{\{\hat{\iota}\}} \hookrightarrow \mathbf{G}\}$ *dual to* $\{\hat{\iota}\}$ *contains an* F*-rational embedding* ι *whose stabilizer* $\mathbf{G}_{\iota}$ *is "naturally" isomorphic to* $\mathbf{G}_{\{\hat{\iota}\}}$*. Moreover, the map* $H^1(F, \mathbf{G}_{\iota}) \longrightarrow H^1(F, \mathbf{G})$ *induced by the inclusion* $\mathbf{G}_{\iota} \subset \mathbf{G}$ *coincides with the map defined above.*

Here, "naturally" isomorphic means that there is a natural isomorphism unique up to inner automorphism, or equivalently that there is a natural W_F-equivariant isomorphism between the associated based root data.

[1] We could also write $Z({}^L\mathbf{G})$ for $Z(\hat{\mathbf{G}})^{W_F}$ and $Z(({}^L\mathbf{G})_{\hat{\iota}})$ for $Z(\hat{\mathbf{G}}_{\hat{\iota}})^{W_F}$ since the center of W_F is trivial.

Proof. Let us choose a maximal torus $\hat{\mathbf{T}} \subset \hat{\mathbf{G}}_{\hat{\iota}}$ and a Borel subgroup $\hat{\mathbf{B}}$ of $\hat{\mathbf{G}}_{\hat{\iota}}$ that contains $\hat{\mathbf{T}}$. Let $\mathcal{T}_{\hat{\mathbf{B}}}$ be the normalizer of the Borel pair $(\hat{\mathbf{T}}, \hat{\mathbf{B}})$ in $({}^L\mathbf{G})_{\hat{\iota}}$. Since $\hat{\mathbf{G}}_{\hat{\iota}}$ acts transitively on the set of its Borel pairs, we see that the map $\mathcal{T}_{\hat{\mathbf{B}}} \longrightarrow W_F$ is surjective, and we have a short exact sequence $\hat{\mathbf{T}} \hookrightarrow \mathcal{T}_{\hat{\mathbf{B}}} \twoheadrightarrow W_F$. In particular, the conjugation action of $\mathcal{T}_{\hat{\mathbf{B}}}$ on $\hat{\mathbf{T}}$ factors through an action of W_F on $\hat{\mathbf{T}}$. By construction, this action preserves the "based root datum" $\psi_{(\hat{\mathbf{T}},\hat{\mathbf{B}})} = (X^*(\hat{\mathbf{T}}), \Delta(\hat{\mathbf{T}}, \hat{\mathbf{B}}), X_*(\hat{\mathbf{T}}), \Delta(\hat{\mathbf{T}}, \hat{\mathbf{B}})^\vee)$ of $\hat{\mathbf{G}}_{\hat{\iota}}$ associated to the Borel pair $(\hat{\mathbf{T}}, \hat{\mathbf{B}})$ and the induced action $W_F \longrightarrow \mathrm{Aut}(\psi_{(\hat{\mathbf{T}},\hat{\mathbf{B}})})$ coincides with the outer action $W_F \longrightarrow \mathrm{Aut}(\psi_0(\hat{\mathbf{G}}_{\hat{\iota}}))$ defined above the proposition, through the canonical isomorphism $\psi_{(\hat{\mathbf{T}},\hat{\mathbf{B}})} = \psi_0(\hat{\mathbf{G}}_{\hat{\iota}})$.

In particular the action of W_F on $\hat{\mathbf{T}}$ is finite and induces the given action of W_F on $\hat{\mathbf{S}}$, so that the whole factorization $\hat{\mathbf{S}} \xrightarrow{\hat{\iota}} \hat{\mathbf{T}} \xrightarrow{\hat{\pi}} \hat{\mathbf{S}}_{\{\hat{\iota}\}}$ is W_F-equivariant. Therefore, the dual $\overline{F}$-torus $\mathbf{T}$ carries an F-structure, and the dual morphism $\mathbf{S}_{\{\hat{\iota}\}} \xrightarrow{\pi} \mathbf{T}$ is defined over F. But since $\mathcal{T}_{\hat{\mathbf{B}}}$ surjects to W_F, any W_F-conjugate of the embedding $\hat{\mathbf{T}} \subset \hat{\mathbf{G}}$ is also $\hat{\mathbf{G}}$-conjugate to it. In other words, the $\hat{\mathbf{G}}$-conjugacy class of this embedding is W_F-stable. It follows that the dual $\mathbf{G}$-conjugacy class of embeddings $\mathbf{T} \hookrightarrow \mathbf{G}$ is also Galois stable. Since $\mathbf{G}$ is quasisplit, ... tells us that there is a dual embedding $j : \mathbf{T} \hookrightarrow \mathbf{G}$ defined over F. Then the composite $\iota = j \circ \pi$ is also defined over F.

Now, the stabililizer $\mathbf{G}_\iota = C_{\mathbf{G}}(\iota(\mathbf{S}_{\{\hat{\iota}\}}))$ is also defined over F and is an F-subgroup of $\mathbf{G}$, with $j(\mathbf{T})$ a maximal torus defined over F. By construction, the W_F action on $\hat{\mathbf{T}}$ preserves the basis $\Delta(\hat{\mathbf{T}}, \hat{\mathbf{B}})$ of the root system $\Sigma(\hat{\mathbf{T}}, \hat{\mathbf{G}}_{\hat{\iota}})$, therefore the F-structure on $\mathbf{T}$ preserves a basis of the root system $\Sigma(j(\mathbf{T}), \mathbf{G}_\iota)$, which points to some Borel subgroup $\mathbf{B}$ of $\mathbf{G}_\iota$ defined over F and containing $j(\mathbf{T})$. The associated based root datum $\psi_{j(\mathbf{T}),\mathbf{B}}$ of $\mathbf{G}_\iota$ is then W_F-equivariantly dual to $\psi_{(\hat{\mathbf{T}},\hat{\mathbf{B}})}$, and this provides a W_F-equivariant isomorphism $\psi_0(\mathbf{G}_\iota) \xrightarrow{\sim} \psi_0(\hat{\mathbf{G}}_{\hat{\iota}})$, hence a whole class of F-rational isomorphisms $\mathbf{G}_\iota \xrightarrow{\sim} \mathbf{G}_{\{\hat{\iota}\}}$ modulo inner automorphisms.

It remains to check that the map $H^1(F, \mathbf{G}_\iota) \longrightarrow H^1(F, \mathbf{G})$ induced by coincides with the map $H^1(F, \mathbf{G}_{\{\hat{\iota}\}}) \longrightarrow H^1(F, \mathbf{G})$ defined above the proposition through any such isomorphism. We have Kottwitz' isomorphisms $\xi_{\mathbf{G}} : H^1(F, \mathbf{G}) \xrightarrow{\sim} \pi_0(Z(\hat{\mathbf{G}})^{W_F})^*$ and $\xi_{\mathbf{G}_\iota} : H^1(F, \mathbf{G}_\iota) \xrightarrow{\sim} \pi_0(Z(\widehat{\mathbf{G}_\iota})^{W_F})^*$ and a canonical W_F-equivariant isomorphism $Z(\widehat{\mathbf{G}_\iota}) = Z(\hat{\mathbf{G}}_{\hat{\iota}})$, so the question is a matter of compatibility of the Kottwitz isomorphisms with the inclusion maps $\mathbf{G}_\iota \subset \mathbf{G}$ on one side, and $Z(\hat{\mathbf{G}}) \subset Z(\hat{\mathbf{G}}_{\hat{\iota}})$ on the other side. This compatibiliy easily follows from Kottwitz' argument in [14, Prop 6.4] (and this must be well known). Indeed, assume first that $\mathbf{G}_{\mathrm{der}}$ is simply connected, so that also $\mathbf{G}_{\iota,\mathrm{der}}$ is simply connected. Then $\xi_{\mathbf{G}}$ and $\xi_{\mathbf{G}_\iota}$ factor as follows

$$\begin{tikzcd}
\xi_{\mathbf{G}_\iota} : H^1(F, \mathbf{G}_\iota) \arrow[r, "\sim"] \arrow[d] & H^1(F, \mathbf{G}_{\iota,\mathrm{ab}}) \arrow[r, "\sim"] \arrow[d] & \pi_0(Z(\widehat{\mathbf{G}_\iota})^{W_F})^* \arrow[d] \\
\xi_{\mathbf{G}} : H^1(F, \mathbf{G}) \arrow[r, "\sim"] & H^1(F, \mathbf{G}_{\mathrm{ab}}) \arrow[r, "\sim"] & \pi_0(Z(\hat{\mathbf{G}})^{W_F})^*
\end{tikzcd}$$

where the first square is obviously commutative (since it is obtained by applying $H^1(F, -)$ to a commutative diagram of algebraic F-groups) and the second square is also commutative since it boils down to local duality for tori, which is functorial. Now, to tackle the general case, Kotwittz considers a central extension $\mathbf{H}$ of $\mathbf{G}$ by an anistropic torus $\mathbf{Z}$ such that $\mathbf{H}_{\mathrm{der}}$ is simply connected. Then the fibre product $\mathbf{H}_\iota = \mathbf{G}_\iota \times_{\mathbf{H}} \mathbf{G}$ is a central extension of $\mathbf{G}_\iota$ by $\mathbf{Z}$

with simply connected derived subgroup. We have just seen that the diagram

$$\begin{tikzcd} H^1(F,\mathbf{H}_\iota) \arrow[r, "\sim", "\xi_{\mathbf{H}_\iota}"'] \arrow[d] & \pi_0(Z(\widehat{\mathbf{H}}_\iota)^{W_F})^* \arrow[d] \\ H^1(F,\mathbf{H}) \arrow[r, "\sim", "\xi_{\mathbf{H}}"'] & \pi_0(Z(\hat{\mathbf{H}})^{W_F})^* \end{tikzcd}$$

is commutative. It is moreover equivariant for the action of $H^1(F,\mathbf{Z})$ given as usual on the first column and through $\pi_0(\hat{\mathbf{Z}}^{W_F})^*$ on the right column. But Kottwitz shows that the diagram we are interested in (with $\mathbf{G}$'s instead of $\mathbf{H}$'s) is obtained form this one by modding out by this action. Therefore this diagram is commutative too. $\square$

When $\mathbf{G}$ is not quasi-split, there may be no F-rational Levi-center embedding $\iota : \mathbf{S}_{\{\hat\iota\}} \hookrightarrow \mathbf{G}$ dual to $\{\hat\iota\}$. Let us call $\{\hat\iota\}$ *relevant to* $\mathbf{G}$ if there exists such an F-rational embedding ι. We will make a connection with the notion of relevance of [5, 3]. To this aim, consider the centralizer $\mathcal{M}_{\hat\iota}$ in ${}^L\mathbf{G}$ of the torus $Z(({}^L\mathbf{G})_{\hat\iota})^\circ$. It contains $({}^L\mathbf{G})_{\hat\iota}$, hence it surjects to W_F and by [5, Lemma 3.5], it is a Levi subgroup of ${}^L\mathbf{G}$ in the sense of *loc. cit.*

2.1.6 *Proposition. — The conjugacy class $\{\hat\iota\}$ is relevant to $\mathbf{G}$ if and only if $\mathcal{M}_{\hat\iota}$ is relevant to $\mathbf{G}$ in the sense of [5, 3.4]. Moreover, in this case, the centralizer $\mathbf{G}_\iota$ of any F-rational Levi-center embedding $\iota : \mathbf{S}_{\{\hat\iota\}} \hookrightarrow \mathbf{G}$ dual to $\{\hat\iota\}$ is an inner form of $\mathbf{G}_{\{\hat\iota\}}$.*

Proof. Assume first that $\{\hat\iota\}$ is relevant and let $\iota : \mathbf{S}_{\{\hat\iota\}} \hookrightarrow \mathbf{G}$ be an F-rational dual embedding. Choose a maximal F-torus $\mathbf{T}$ of $\mathbf{G}_\iota$, and a dual embedding $\hat\jmath : \hat{\mathbf{T}} \hookrightarrow \hat{\mathbf{G}}$ that extends $\hat\iota$. Its stabilizer $({}^L\mathbf{G})_{\hat\jmath}$ in ${}^L\mathbf{G}$ is contained in $({}^L\mathbf{G})_{\hat\iota}$, hence $Z(({}^L\mathbf{G})_{\hat\jmath})^\circ$ contains $Z(({}^L\mathbf{G})_{\hat\iota})^\circ$ and therefore the Levi subgroup $\mathcal{M}_{\hat\jmath} := C_{{}^L\mathbf{G}}(Z(({}^L\mathbf{G})_{\hat\jmath})^\circ)$ of ${}^L\mathbf{G}$ is contained in $\mathcal{M}_{\hat\iota}$. Since any Levi subgroup of ${}^L\mathbf{G}$ that contains a relevant Levi subgroup is relevant, we are left to show that $\mathcal{M}_{\hat\jmath}$ is relevant. Now observe that, by definition, $({}^L\mathbf{G})_{\hat\jmath}$ is an extension of W_F by $\hat{\mathbf{T}}$ such that the action of W_F on $\hat{\mathbf{T}}$ induced by conjugation is the one inherited from the F-structure on $\mathbf{T}$. In particular we have $Z(({}^L\mathbf{G})_{\hat\jmath})^\circ = \hat\jmath(\hat{\mathbf{T}}^{W_F,\circ})$, and we see that

$$\Sigma(\hat{\mathbf{T}},\mathcal{M}_{\hat\jmath}^\circ) = \left\{\alpha^\vee \in \Sigma(\hat{\mathbf{T}},\hat{\mathbf{G}}), \langle\alpha^\vee, X_*(\hat{\mathbf{T}})^{W_F}\rangle = 0\right\}.$$

We claim that for $\alpha \in \Sigma(\mathbf{T},\mathbf{G})$ we have $\langle\alpha^\vee, X_*(\hat{\mathbf{T}})^{W_F}\rangle = 0 \Leftrightarrow \langle\alpha, X_*(\mathbf{T})^{W_F}\rangle = 0$. Indeed, let $W_{F,\alpha}$ be the finite subgroup of $\mathrm{Aut}_{\mathbb{Q}}(X_*(\hat{\mathbf{T}})_{\mathbb{Q}})$ generated by the image of W_F and s_α. Then $\langle\alpha^\vee, X_*(\hat{\mathbf{T}})^{W_F}\rangle = 0 \Leftrightarrow \dim_{\mathbb{Q}}(X_*(\hat{\mathbf{T}})_{\mathbb{Q}}^{W_F}) = \dim_{\mathbb{Q}}(X_*(\hat{\mathbf{T}})_{\mathbb{Q}}^{W_{F,\alpha}})$ which by duality is equivalent to $\dim_{\mathbb{Q}}(X_*(\mathbf{T})_{\mathbb{Q}}^{W_F}) = \dim_{\mathbb{Q}}(X_*(\mathbf{T})_{\mathbb{Q}}^{W_{F,\alpha}})$ hence to $\langle\alpha, X_*(\mathbf{T})^{W_F}\rangle = 0$. Now, denoting by $\mathbf{T}^{\rm split}$ the maximal split subtorus of $\mathbf{T}$, we get $\Sigma(\hat{\mathbf{T}},\mathcal{M}_{\hat\jmath}^\circ) = \{\alpha \in \Sigma(\mathbf{T},\mathbf{G}),\ \alpha_{|\mathbf{T}^{\rm split}} \equiv 1\}^\vee$ and it follows that $\mathcal{M}_{\hat\jmath}$ is dual to the F-Levi subgroup $C_{\mathbf{G}}(\mathbf{T}^{\rm split})$ of $\mathbf{G}$ and is therefore relevant.

Conversely, assume now that $\mathcal{M}_{\hat\iota}$ is relevant. After replacing $\hat\iota$ by a conjugate, we may assume that $\mathcal{M}_{\hat\iota}$ is a standard Levi subgroup of ${}^L\mathbf{G}$, and in particular of the form $\hat{\mathbf{M}}_{\hat\iota} \rtimes W_F$ for some W_F-stable Levi subgroup $\hat{\mathbf{M}}_{\hat\iota}$ of $\hat{\mathbf{G}}$. Since $\mathcal{M}_{\hat\iota}$ is relevant to $\mathbf{G}$, $\hat{\mathbf{M}}_{\hat\iota} \rtimes W_F$ is the L-group of some F-Levi subgroup $\mathbf{M}_{\hat\iota}$ of $\mathbf{G}$. On the other hand, $\hat\iota$ factors through $\hat{\mathbf{M}}_{\hat\iota}$ and provides a Levi-center embedding for this group. Since $({}^L\mathbf{G})_{\hat\iota}$ is contained in $\mathcal{M}_{\hat\iota}$, the stabilizer

$({}^L\mathbf{M}_{\hat\iota})_{\hat\iota} = ({}^L\mathbf{G})_{\hat\iota}$ surjects to W_F so that the $\hat{\mathbf{M}}_{\hat\iota}$-conjugacy class of $\hat\iota$ is W_F-stable. So we are now left to show that $\hat\iota$ is relevant for $\mathbf{M}_{\hat\iota}$. Equivalently, we may and will restrict to the case where $\mathcal{M}_{\hat\iota} = {}^L\mathbf{G}$, that is $Z(({}^L\mathbf{G})_{\hat\iota})^\circ = Z({}^L\mathbf{G})^\circ$.

We will now reduce further to the case where $\mathbf{G}$ is an adjoint group. To this aim, denote by $\pi : \mathbf{G} \longrightarrow \mathbf{G}_{\mathrm{ad}}$ the adjoint quotient map (defined over F) and by $\hat\pi : \widehat{\mathbf{G}_{\mathrm{ad}}} = \hat{\mathbf{G}}_{\mathrm{sc}} \longrightarrow \hat{\mathbf{G}}$ its dual map (W_F-equivariant). Consider the connected fibre product $\hat{\mathbf{S}}_{\mathrm{ad}} := (\hat{\mathbf{S}} \times_{\hat{\mathbf{G}}} \widehat{\mathbf{G}_{\mathrm{ad}}})^\circ$. This is a torus with finite W_F-action and the second projection $\hat\iota_{\mathrm{ad}} : \hat{\mathbf{S}}_{\mathrm{ad}} \longrightarrow \widehat{\mathbf{G}_{\mathrm{ad}}}$ is a Levi-center embedding whose stabilizer $(\hat{\mathbf{G}}_{\mathrm{ad}})_{\hat\iota_{\mathrm{ad}}}$ is the inverse image $\pi^{-1}(\hat{\mathbf{G}}_{\hat\iota})$ of that of $\hat\iota$. Moreover, if we write an element $\hat g \in \hat{\mathbf{G}}$ in the form $\hat g = \hat z \hat\pi(\hat h)$ according to the decomposition $\hat{\mathbf{G}} = Z(\hat{\mathbf{G}})\hat\pi(\hat{\mathbf{G}}_{\mathrm{sc}})$, then we see that $({}^{\hat g}\hat\iota)_{\mathrm{ad}} = {}^{\hat h}(\hat\iota_{\mathrm{ad}})$. It follows that $\{\hat\iota\}$ determines a $\widehat{\mathbf{G}_{\mathrm{ad}}}$-conjugacy class $\{\hat\iota_{\mathrm{ad}}\}$. Since $\hat\pi$ is W_F-equivariant, $\{\hat\iota_{\mathrm{ad}}\}$ is W_F-stable, and its stabilizer $({}^L\mathbf{G}_{\mathrm{ad}})_{\hat\iota_{\mathrm{ad}}}$ is the preimage of $({}^L\mathbf{G})_{\hat\iota}$ along $\pi \rtimes \mathrm{Id}_{W_F}$. Also $\hat\pi$ induces a W_F-equivariant morphism $\hat{\mathbf{S}}_{\{\hat\iota_{\mathrm{ad}}\}} \longrightarrow \hat{\mathbf{S}}_{\{\hat\iota\}}$ which, dually, induces an F-morphism $\mathbf{S}_{\{\hat\iota\}} \longrightarrow \mathbf{S}_{\{\hat\iota_{\mathrm{ad}}\}}$. Now we claim that

$$\{\hat\iota\} \textit{ is relevant to } \mathbf{G} \textit{ if and only if } \{\hat\iota_{\mathrm{ad}}\} \textit{ is relevant to } \mathbf{G}_{\mathrm{ad}}.$$

Indeed, suppose there is an F-rational Levi-center embedding $\iota_{\mathrm{ad}} : \mathbf{S}_{\{\hat\iota_{\mathrm{ad}}\}} \hookrightarrow \mathbf{G}_{\mathrm{ad}}$ in $\mathbf{G}_{\mathrm{ad}}$ dual to $\{\hat\iota_{\mathrm{ad}}\}$. Then consider the torus $\mathbf{S} := (\mathbf{S}_{\{\hat\iota_{\mathrm{ad}}\}} \times_{\mathbf{G}_{\mathrm{ad}}} \mathbf{G})^\circ$. The second projection provides an F-rational Levi-center embedding $\iota : \mathbf{S} \hookrightarrow \mathbf{G}$ and we need to prove it is dual to $\{\hat\iota\}$. This is a problem over $\bar F$ and we need to go through the duality procedure of 2.1.3. So let us choose a maximal torus $\hat{\mathbf{T}}$ in $\hat{\mathbf{G}}_{\hat\iota}$ with dual $\mathbf{T}$ over $\bar F$. It provides a maximal torus $\widehat{\mathbf{T}_{\mathrm{ad}}} = \hat\pi^{-1}(\hat{\mathbf{T}})$ in $(\widehat{\mathbf{G}_{\mathrm{ad}}})_{\hat\iota_{\mathrm{ad}}}$ whose dual we denote by $\mathbf{T}_{\mathrm{ad}}$. Also $\hat\pi$ provides a dual morphism $\mathbf{T} \xrightarrow{\pi} \mathbf{T}_{\mathrm{ad}}$. Now choose an embedding $j : \mathbf{T} \hookrightarrow \mathbf{G}$ dual to $\hat{\mathbf{T}} \subset \hat{\mathbf{G}}$ and that factors through $\mathbf{G}_\iota$. Then $\pi \circ j$ factors over an embedding $j_{\mathrm{ad}} : \mathbf{T}_{\mathrm{ad}} \hookrightarrow \mathbf{G}_{\mathrm{ad}}$ dual to $\widehat{\mathbf{T}_{\mathrm{ad}}} \subset \widehat{\mathbf{G}_{\mathrm{ad}}}$ and that factors through $(\mathbf{G}_{\mathrm{ad}})_{\iota_{\mathrm{ad}}}$. As in 2.1.3, the embedding ι_{ad} identifies $\mathbf{S}_{\{\hat\iota_{\mathrm{ad}}\}}$ to the subtorus $\left(\bigcap_{\alpha\in\Sigma_{\hat\iota_{\mathrm{ad}}}} \ker(\alpha^\vee)\right)^\circ$ of $\mathbf{T}_{\mathrm{ad}}$ involving the subroot system $\Sigma_{\hat\iota_{\mathrm{ad}}}$ of $\Sigma(\widehat{\mathbf{T}_{\mathrm{ad}}}, \widehat{\mathbf{G}_{\mathrm{ad}}})$. This subroot system coincides wit $\Sigma_{\hat\iota}$ through the canonical identification $\Sigma(\widehat{\mathbf{T}_{\mathrm{ad}}}, \widehat{\mathbf{G}_{\mathrm{ad}}}) = \Sigma(\hat{\mathbf{T}}, \hat{\mathbf{G}})$. Now our definition of $\mathbf{S}$ and ι show that ι identifies $\mathbf{S}$ to the subtorus $\left(\bigcap_{\alpha\in\Sigma_{\hat\iota}} \ker(\alpha^\vee)\right)^\circ$ of $\mathbf{T}$, hence ι is dual to $\{\hat\iota\}$ as desired. The other implication is seen in a similar way but we omit the proof since we do not need here.

So we are now left to prove that if $\mathbf{G}$ is an adjoint group and $Z(({}^L\mathbf{G})_{\hat\iota})^\circ = \{1\}$ then $\{\hat\iota\}$ is relevant. Since $\mathbf{G}$ is adjoint, there is $\eta \in H^1(\Gamma_F, \mathbf{G})$ such that the associated pure inner form $\mathbf{G}_\eta$ over F is quasi-split. Then $\eta^{-1} \in H^1(\Gamma_F, \mathbf{G}_\eta)$ and we have $(\mathbf{G}_\eta)_{\eta^{-1}} = \mathbf{G}$. Through Kottwitz duality we can view η^{-1} as a character of the finite group $Z({}^L\mathbf{G})$. Since $Z(({}^L\mathbf{G})_{\hat\iota})^\circ = \{1\}$ we may extend η^{-1} to a character of the finite group $Z(({}^L\mathbf{G})_{\hat\iota})$ that we denote by ζ^{-1}. Going through Kottwitz duality again, we get a cohomology class $\zeta^{-1} \in H^1(F, \mathbf{G}_{\{\hat\iota\}})$. Now by Proposition 2.1.5 there is an F-rational Levi-center embedding $\iota : \mathbf{S}_{\{\hat\iota\}} \hookrightarrow \mathbf{G}_\eta$ with a natural F-rational isomorphism $\mathbf{G}_{\eta,\iota} \simeq \mathbf{G}_{\{\hat\iota\}}$. Let us choose a 1-cocycle $\zeta^{-1} : \Gamma_F \longrightarrow \mathbf{G}_{\eta,\iota}$ that represents the cohomology class ζ^{-1}. Then ι is still F-rational for the F-structure of $\mathbf{G}_\eta$ twisted by ζ^{-1}, *i.e.* ι is an F-rational Levi-embedding $\mathbf{S}_{\{\hat\iota\}} \hookrightarrow (\mathbf{G}_\eta)_{\zeta^{-1}}$. However, we know by Proposition 2.1.5 that the map $H^1(F, \mathbf{G}_{\eta,\iota}) \longrightarrow H^1(F, \mathbf{G}_\eta)$ is induced by the inclusion $Z({}^L\mathbf{G}) \subset Z(({}^L\mathbf{G})_{\hat\iota})$ through Kottwitz duality. Therefore we have $\zeta^{-1} = \eta^{-1}$ in $H^1(F, \mathbf{G}_\eta)$, so that $(\mathbf{G}_\eta)_{\zeta^{-1}} \simeq \mathbf{G}$ and ι finally provides the desired F-rational Levi-center embedding into $\mathbf{G}$.

We now turn to the second assertion of the proposition. Our argument has provided one ι with centralizer $\mathbf{G}_\iota$ an inner form of $\mathbf{G}_{\{\hat\iota\}}$. The fact that this property remains true for all

F-rational embedding dual to $\{\hat{\iota}\}$ follows from the discussion above Lemma 2.1.7 below. $\square$

Now that we have studied the existence of F-rational dual Levi-center embeddings, we may try to classify all of them. Obviously $\mathbf{G}(F)$ acts by conjugation on these F-rational embeddings. So, let us fix one of them, ι and let ι' be another one. Then pick some $g \in \mathbf{G}$ such that $\iota' = \mathrm{Ad}_g \circ \iota$. Then for any $\gamma \in \Gamma_F$ we also have $\iota' = {}^\gamma\iota' = \mathrm{Ad}_{\gamma(g)} \circ {}^\gamma\iota = \mathrm{Ad}_{\gamma(g)} \circ \iota$, so that $g^{-1}\gamma(g) \in \mathbf{G}_\iota$. We then see that

- $(\gamma \mapsto g^{-1}\gamma(g)) \in Z^1(F, \mathbf{G}_\iota)$ and its image $\eta_{\iota,\iota'}$ in $H^1(F, \mathbf{G}_\iota)$ is independent of the choice of g.
- Ad_g is an inner twisting $\mathbf{G}_\iota \xrightarrow{\sim} \mathbf{G}_{\iota'}$ with associated inner cocycle $\gamma \mapsto g^{-1}\gamma(g)$.

2.1.7 *Lemma. — The map $\iota' \mapsto \eta_{\iota,\iota'}$ induces a bijection between the set of $\mathbf{G}(F)$-conjugacy classes of F-rational embeddings in $\{\iota\}$ and* $\ker(H^1(F, \mathbf{G}_\iota) \longrightarrow H^1(F, \mathbf{G}))$.

Proof. Indeed,it is easily seen that $\eta_{\iota,\iota'}$ only depends on the $\mathbf{G}(F)$ conjugacy class of ι', and by construction it lies in the above kernel. Conversely, let η belong to this kernel. Then it can be represented by a 1-cocycle of the form $\gamma \mapsto g^{-1}\gamma(g)$ for some $g \in \mathbf{G}(\overline{F})$, and the embedding $\iota' = \mathrm{Ad}_g \circ \iota$ is thus F-rational. This element g is not unique, but any other one is of the form hgk with $h \in \mathbf{G}(F)$ and $k \in \mathbf{G}_\iota$ and thus leads to a $\mathbf{G}(F)$-conjugate rational embedding. We thus have constructed the inverse map. $\square$

2.2 Levi factorization of a parameter

In this subsection, we fix a wild inertial parameter $\phi \in P_F \longrightarrow {}^L\mathbf{G}$.

2.2.1 *The group $\mathbf{L}_\phi$.* — The centralizer $\hat{\mathbf{L}}_\phi := C_{\hat{\mathbf{G}}}(Z(C_{\hat{\mathbf{G}}}(\phi))^\circ)$ of the connected center $Z(C_{\hat{\mathbf{G}}}(\phi))^\circ$ of $C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup of $\hat{\mathbf{G}}$ which contains $C_{\hat{\mathbf{G}}}(\phi)$. If $\varphi : W_F \longrightarrow {}^L\mathbf{G}$ extends ϕ, then the conjugation action Ad_φ of W_F on $C_{\hat{\mathbf{G}}}(\phi)$ preserves its connected center and therefore also $\hat{\mathbf{L}}_\phi$. Since for any other extensions φ' the ratio $\varphi^{-1}\varphi'$ takes values in $C_{\hat{\mathbf{G}}}(\phi)$, the outer action $W_F \xrightarrow{\mathrm{Ad}_\varphi} \mathrm{Out}(\hat{\mathbf{L}}_\phi)$ is independent of the choice of φ. We know from [9, Lemma 2.1.1] that this action is finite. Hence we may denote by $\mathbf{L}_\phi$ a quasi-split group over F endowed with a W_F-equivariant isomorphism $\psi_0(\mathbf{L}_\phi) \xrightarrow{\sim} \psi_0(\hat{\mathbf{L}}_\phi)^\vee$. Note that $\psi_0(\hat{\mathbf{L}}_\phi)$ only depends on the conjugacy class of ϕ in the sense that if ϕ' is conjugate to ϕ, there is a canonical isomorphism $\psi_0(\hat{\mathbf{L}}_\phi) \xrightarrow{\sim} \psi_0(\hat{\mathbf{L}}_{\phi'})$ given by any $\hat{g}$ that conjugates ϕ to ϕ'. Note also that the inclusion $Z(\hat{\mathbf{L}}_\phi)^{W_F} \subset Z(\hat{\mathbf{G}})^{W_F}$ induces by Kottwitz duality a map $H^1(F, \mathbf{L}_\phi) \longrightarrow H^1(F, \mathbf{G})$. We put

$$H^1(F, \mathbf{L}_\phi, \mathbf{G}) := \ker(H^1(F, \mathbf{L}_\phi) \longrightarrow H^1(F, \mathbf{G})).$$

2.2.2 *The group $\mathcal{L}_\phi$ and the L-group of $\mathbf{L}_\phi$.* — Consider the subgroup $\mathcal{L}_\phi := \hat{\mathbf{L}}_\phi \cdot \varphi(W_F)$ of ${}^L\mathbf{G}$. As the notation suggests, it is independent of the choice of a parameter φ extending ϕ. It sits in a split exact sequence $\hat{\mathbf{L}}_\phi \hookrightarrow \mathcal{L}_\phi \twoheadrightarrow W_F$ and we may ask whether it is isomorphic to ${}^L\mathbf{L}_\phi$. To this aim, fix a pinning ε_ϕ of $\hat{\mathbf{L}}_\phi$ and consider the stabilizer $\mathcal{L}_{\phi,\varepsilon_\phi}$ of ε_ϕ in $\mathcal{L}_\phi$. It sits in an exact sequence $Z(\hat{\mathbf{L}}_\phi) \hookrightarrow \mathcal{L}_{\phi,\varepsilon_\phi} \twoheadrightarrow W_F$.

Lemma. — *The extension $Z(\hat{\mathbf{L}}_\phi) \hookrightarrow \mathcal{L}_{\phi,\varepsilon_\phi} \twoheadrightarrow W_F$ splits continuously, and the set of its splittings $W_F \longrightarrow \mathcal{L}_{\phi,\varepsilon_\phi}$ is principal homogeneous under $Z^1(W_F, Z(\hat{\mathbf{L}}_\phi))$.*

Proof. Only the existence of a splitting requires a proof, the second assertion being easy. Recall first that, by [9, Lemma 2.1.1], the extension under consideration comes from a finite quotient of W_F. By Langland's lemma 4 in [16], the image of $H^2_{cts}(\Gamma_F, Z(\hat{\mathbf{L}}_\phi)^\circ) \longrightarrow H^2(W_F, Z(\hat{\mathbf{L}}_\phi)^\circ)$ is $\{1\}$. This reduces the problem to showing that the extension

$$\pi_0(Z(\hat{\mathbf{L}}_\phi)) \hookrightarrow \mathcal{L}_{\phi,\varepsilon_\phi}/Z(\hat{\mathbf{L}}_\phi)^\circ \twoheadrightarrow W_F$$

splits. This in turn follows from the argument in Kaletha's lemma 5.2.5 in [11]. In order to explain this, we may assume that the pinning $\varepsilon_\phi = (\hat{\mathbf{T}}, \hat{\mathbf{B}}_\phi, \{X_{\hat{\alpha}}\}_{\hat{\alpha}\in\Delta(\hat{\mathbf{T}},\hat{\mathbf{B}}_\phi)})$ is the restriction of a W_F-stable pinning $\varepsilon = (\hat{\mathbf{T}}, \hat{\mathbf{B}}, \{X_{\hat{\alpha}}\}_{\hat{\alpha}\in\Delta(\hat{\mathbf{T}},\hat{\mathbf{B}})})$ of $\hat{\mathbf{G}}$ (after conjugating (ϕ, ε_ϕ) by some appropriate $\hat{g} \in \hat{\mathbf{G}}$). Then the exact sequence of the lemma is a pull-back of the exact sequence

$$Z(\hat{\mathbf{L}}_\phi) \hookrightarrow \mathcal{N}_{{}^L\mathbf{G}}(\hat{\mathbf{L}}_\phi)_{\varepsilon_\phi} \twoheadrightarrow (\Omega(\hat{\mathbf{T}}, \hat{\mathbf{G}}) \rtimes W_F)_{\varepsilon_\phi}$$

where the index ε_ϕ indicates the stabilizer of the pinning ε_ϕ. Now, using Tits' liftings with respect to ε, we have a map $\Omega(\hat{\mathbf{T}}, \hat{\mathbf{G}}) \rtimes W_F \longrightarrow \mathcal{N}_{{}^L\mathbf{G}}(\hat{\mathbf{T}})$ which by restriction provides in turn a map $(\Omega(\hat{\mathbf{T}}, \hat{\mathbf{G}}) \rtimes W_F)_{\varepsilon_\phi} \longrightarrow \mathcal{N}_{{}^L\mathbf{G}}(\hat{\mathbf{L}}_\phi)_{\varepsilon_\phi}$. The content of Kaletha's study of the Tits liftings in the proof of [11, Lemma 5.2.5] is that the composed map

$$(\Omega(\hat{\mathbf{T}}, \hat{\mathbf{G}}) \rtimes W_F)_{\varepsilon_\phi} \longrightarrow \mathcal{N}_{{}^L\mathbf{G}}(\hat{\mathbf{L}}_\phi)_{\varepsilon_\phi}/Z(\hat{\mathbf{L}}_\phi)^\circ$$

is a homomorphism. $\square$

Let $\psi : W_F \longrightarrow \mathcal{L}_{\phi,\varepsilon_\phi}$ be a continuous splitting as in the lemma. We get an isomorphism of extensions $\mathrm{Id} \times \psi : {}^L\mathbf{L}_\phi \xrightarrow{\sim} \mathcal{L}_\phi$, where the L-group is formed by using the section $\mathrm{Out}(\hat{\mathbf{L}}_\phi) \hookrightarrow \mathrm{Aut}(\hat{\mathbf{L}}_\phi)$ associated to ε_ϕ. Then $\varphi_L := (\mathrm{Id} \times \psi)^{-1} \circ \varphi$ is a Langlands parameter for $\mathbf{L}_\phi$, whose restriction to P_F we denote by $\phi_L \in \Phi(P_F, \mathbf{L}_\phi)$. We thus get a factorization of ϕ

$$\phi : P_F \xrightarrow{\phi_L} {}^L\mathbf{L}_\phi \xrightarrow{\xi_\psi} {}^L\mathbf{G}$$

with ξ_ψ the composition of $\mathrm{Id} \times \psi$ and the inclusion $\mathcal{L}_\phi \subset {}^L\mathbf{G}$. Then we see that ξ_ψ induces an isomorphism $C_{\hat{\mathbf{L}}_\phi}(\phi_L) \xrightarrow{\sim} C_{\hat{\mathbf{G}}}(\phi)$, which makes it fall into the framework of [9, Expectation 1.3.2], which predicts (at least when $\mathbf{G}$ is quasi-split) the existence of an equivalence of categories $\prod_{\eta\in H^1(F,\mathbf{L}_\phi,\mathbf{G})} \mathrm{Rep}^{\phi_L}(L_{\phi,\eta}) \xrightarrow{\sim} \mathrm{Rep}^\phi(G)$ where $\mathbf{L}_{\phi,\eta}$ is the pure inner form of $\mathbf{L}_\phi$ associated to η. Interestingly, this set $H^1(F, \mathbf{L}_\phi, \mathbf{G})$ and the associated pure inner forms of $\hat{\mathbf{L}}_\phi$ also appear when we try to go from $\hat{\mathbf{L}}_\phi$ to twisted Levi subgroups of $\mathbf{G}$.

2.2.3 *Twisted Levi subgroups of* $\mathbf{G}$. — With the outer action map, also the action maps $W_F \xrightarrow{\mathrm{Ad}_\varphi} \mathrm{Aut}(Z(\hat{\mathbf{L}}_\phi)^\circ)$ and $W_F \xrightarrow{\mathrm{Ad}_\varphi} \mathrm{Aut}(\hat{\mathbf{L}}_{\phi,\mathrm{ab}})$ are independent of the choice of φ. Moreover the existence of φ tells us that the $\hat{\mathbf{G}}$-conjugacy class of the embedding $Z(\hat{\mathbf{L}}_\phi)^\circ \subset \hat{\mathbf{G}}$ is W_F-stable.

Notation.– We denote by $\mathbf{S}_\phi$ the F-torus dual to the complex torus $\hat{\mathbf{L}}_{\phi,\mathrm{ab}}$ with its W_F-action, and by I_ϕ the $\mathbf{G}$-conjugacy class of Levi-center embeddings $\mathbf{S}_\phi \hookrightarrow \mathbf{G}$ which is "dual" to the Levi-center embedding $Z(\hat{\mathbf{L}}_\phi)^\circ \subset \hat{\mathbf{G}}$ in the sense of 2.1.3.

Proposition. — Assume that $\mathbf{G}$ *is quasi-split.*

i) There is an F*-rational embedding* $\iota \in I_\phi$ *such that* $\mathbf{G}_\iota$ *is naturally isomorphic to* $\mathbf{L}_\phi$*.*

ii) For any other F*-rational* $\iota' \in I_\phi$ *there is* $\eta_{\iota,\iota'} \in H^1(F, \mathbf{L}_\phi)$ *such that* $\mathbf{G}_{\iota'}$ *is naturally isomorphic to the pure inner form* $\mathbf{L}_{\phi,\eta_{\iota,\iota'}}$*.*

iii) The map $\iota' \mapsto \eta_{\iota,\iota'}$ *induces a bijection between the set of* G*-conjugacy classes of* F*-rational embeddings in* I_ϕ *and the set* $H^1(F, \mathbf{L}_\phi, \mathbf{G})$*.*

Proof. Follows from Proposition 2.1.5 and Lemma 2.1.7. □

When $\mathbf{G}$ is not quasi-split, we need a relevance condition on ϕ. Namely, we call ϕ *relevant to* $\mathbf{G}$ if it is the restriction of a Langlands parameter $\varphi' : W'_F \longrightarrow {}^L\mathbf{G}$ that is relevant to $\mathbf{G}$ in the sense of [5, 8.2].

Proposition. — i) If ϕ *is relevant to* $\mathbf{G}$*, then* I_ϕ *contains an* F*-rational embedding* ι*.*

ii) The converse is true, provided that there is a parameter $\varphi : W_F \longrightarrow {}^L\mathbf{G}$ *extending* ϕ *and preserving an epinglage of* $\hat{\mathbf{L}}_\phi$*.*

iii) If ι *is an* F*-rational embedding in* I_ϕ*, then* $\mathbf{G}_\iota$ *is an inner form of* $\mathbf{L}_\phi$*. Moreover, to any other* F*-rational embedding* $\iota' \in I_\phi$ *is attached an element* $\eta_{\iota,\iota'} \in H^1(F, \mathbf{G}_\iota)$ *such that* $\mathbf{G}_{\iota'}$ *is isomorphic to the pure inner form* $\mathbf{G}_{\iota,\eta_{\iota,\iota'}}$*, and such that the map* $\iota' \mapsto \eta_{\iota,\iota'}$ *induces a bijection between* $H^1(F, \mathbf{G}_\iota, \mathbf{G})$ *and the set of* $\mathbf{G}(F)$*-conjugacy classes of rational embeddings in* I_ϕ*.*

Proof. i) By Proposition 2.1.6, it suffices to prove that the centralizer $\mathcal{M}_\phi$ of the connected center of the centralizer of the embedding $Z(\hat{\mathbf{L}}_\phi)^\circ \subset \hat{\mathbf{G}}$ in ${}^L\mathbf{G}$, which is a Levi-subgroup of ${}^L\mathbf{G}$, is relevant to $\mathbf{G}$. By definition of the W_F-action on $Z(\hat{\mathbf{L}}_\phi)$, the centralizer of the embedding $Z(\hat{\mathbf{L}}_\phi)^\circ \subset \hat{\mathbf{G}}$ in ${}^L\mathbf{G}$ is $\mathcal{L}_\phi$. Hence $\mathcal{M}_\phi$ is the centralizer of the torus $Z(\mathcal{L}_\phi)^\circ$ and in particular contains $\mathcal{L}_\phi$. Now fix a relevant Langlands parameter $\varphi' : W'_F \longrightarrow {}^L\mathbf{G}$ that extends ϕ. We claim that $\varphi'(W'_F) \subset \mathcal{L}_\phi$. Indeed, the inclusion $\varphi'(W_F) \subset \mathcal{L}_\phi$ holds by definition, and the inclusion $\varphi'(\mathrm{SL}_2) \subset \mathcal{L}_\phi$ holds too since $\varphi'(\mathrm{SL}_2)$ is contained in $C_{\hat{\mathbf{G}}}(\phi)$, hence commutes with $Z(C_{\hat{\mathbf{G}}}(\phi))^\circ$ and is thus contained in $\hat{\mathbf{L}}_\phi$. It follows that φ' factors through $\mathcal{M}_\phi$ and, by our hypothesis, that $\mathcal{M}_\phi$ is relevant to $\mathbf{G}$.

ii) Let us assume the existence of an F-rational ι in I_ϕ. By Proposition 2.1.6, the Levi subgroup $\mathcal{M}_\phi$ of ${}^L\mathbf{G}$ defined above is then relevant to $\mathbf{G}$. We have just seen that any Langlands parameter φ' that extends ϕ factors through $\mathcal{M}_\phi$. We will construct such a φ' that does not factor through any proper Levi subgroup of $\mathcal{M}_\phi$. Then this φ' will be relevant, as desired. Note that if φ' factors through a Levi subgroup $\mathcal{M}$ then $Z(\mathcal{M})^\circ \subset C_{\hat{\mathbf{G}}}(\varphi')$. Therefore it will be sufficient to construct φ' such that $C_{\hat{\mathbf{G}}}(\varphi')^\circ = Z(\mathcal{M}_\phi)^\circ$. Now, observe that if $\varphi : W_F \longrightarrow {}^L\mathbf{G}$ is any Weil parameter extending ϕ, then we may use a principal SL_2-subgroup of the reductive group $C_{\hat{\mathbf{G}}}(\varphi)^\circ$ to extend further φ to a Langlands parameter φ' such that $C_{\hat{\mathbf{G}}}(\varphi')^\circ = Z(C_{\hat{\mathbf{G}}}(\varphi)^\circ)^\circ$. So the problem now becomes to find φ such that $Z(C_{\hat{\mathbf{G}}}(\varphi)^\circ)^\circ = Z(\mathcal{M}_\phi)^\circ$. Recall that $C_{\hat{\mathbf{G}}}(\phi)$ is contained in $\hat{\mathbf{L}}_\phi$, so that $C_{\hat{\mathbf{G}}}(\varphi) = (\hat{\mathbf{L}}_\phi)^{\varphi(W_F)}$. But the next lemma tells us that if φ preserves an épinglage of $\hat{\mathbf{L}}_\phi$, then we have $Z(C_{\hat{\mathbf{G}}}(\varphi)^\circ)^\circ = Z(\hat{\mathbf{L}}_\phi)^{W_F,\circ} = Z(\mathcal{L}_\phi)^\circ$. Since $Z(\mathcal{L}_\phi)^\circ \subset Z(\mathcal{M}_\phi)^\circ$, this implies $Z(C_{\hat{\mathbf{G}}}(\varphi)^\circ)^\circ = Z(\mathcal{L}_\phi)^\circ = Z(\mathcal{M}_\phi)^\circ$, as desired.

iii) This follows from Lemma 2.1.7 and the paragraph thereabove. □

Lemma. — Let $\mathbf{H}$ *be a complex reductive group and* Γ *a group acting on* $\mathbf{H}$ *and preserving an épinglage of* $\mathbf{H}$*. Then* $Z(\mathbf{H}^{\Gamma,\circ})^\circ = Z(\mathbf{H})^{\Gamma,\circ}$.

Proof. The inclusion $Z(\mathbf{H}^{\Gamma,\circ})^\circ \supset Z(\mathbf{H})^{\Gamma,\circ}$ is clear. To get the other inclusion it is enough to show that $Z(\mathbf{H}^{\Gamma,\circ}) \subset Z(\mathbf{H})$. Observe that any isogeny $\mathbf{H}' \longrightarrow \mathbf{H}$ is Γ-equivariant for the action of Γ on $\mathbf{H}'$ obtained by lifting a Γ-stable épinglage from $\mathbf{H}$ to $\mathbf{H}'$ and identifying $\mathrm{Out}(\mathbf{H}) = \mathrm{Out}(\mathbf{H}')$. In such a situation, the image of $(\mathbf{H}')^\Gamma$ has finite index in $\mathbf{H}^\Gamma$ so that the statement of the lemma is true for $\mathbf{H}$ if and only if it is true for $\mathbf{H}'$. Since this statement is clear for tori, the isogeny $\mathbf{H}' = \mathbf{H}_{\mathrm{sc}} \times Z(\mathbf{H}) \longrightarrow \mathbf{H}$ allows us to reduce to the case where $\mathbf{H}$ is semi-simple and simply connected. Then Γ permutes the set of simple factors of $\mathbf{H}$, so we may restrict to the case with one orbit, and then restrict to a simple factor with the action of its stabilizer. Hence we may assume that $\mathbf{H}$ is simple and replace Γ by its image in $\mathrm{Out}(\mathbf{H})$ which is either $\mathbb{Z}/2\mathbb{Z}$ or S_3. At this point we could conclude with a case by case inspection. But we can also invoke Steinberg's Thm 8.1 in ..., which insures that $\mathbf{H}^\Gamma = \mathbf{H}^{\Gamma,\circ}$ is a reductive group with maximal torus $\mathbf{T}^\Gamma = \mathbf{T}^{\Gamma,\circ}$, where $\mathbf{T}$ is part of a Γ-stable épinglage. In particular $Z(\mathbf{H}^\Gamma)^\circ \subset \mathbf{T}^\Gamma$. Now let $(\mathbf{T}, \mathbf{B}, (X_\alpha)_{\alpha\in\Delta(\mathbf{T},\mathbf{B})})$ be a Γ-stable épinglage of $\mathbf{H}$, where X_α is a non-zero element of the weight α subspace in the Lie algebra $\mathfrak{h}$ of H. Then $Z(\mathbf{H}^\Gamma)^\circ$ must act trivially on the elements $\sum_{\gamma\in\Gamma} X_{\gamma\alpha} \in \mathfrak{h}^\Gamma$ for $\alpha \in \Delta(\mathbf{T}, \mathbf{B})$. These elements are non-zero (here, compared to Steinberg's result, we need the fact that Γ preserves the épinglage and not only the pair $(\mathbf{T}, \mathbf{B})$), therefore we have $Z(\mathbf{H}^\Gamma)^\circ \subset \bigcap_{\alpha\in\Delta} \ker(\alpha) = Z(\mathbf{H})$. $\square$

2.2.4 *Lemma. — Assume that* $\mathbf{G}$ *is tamely ramified. Then* $\mathbf{L}_\phi$ *is tamely ramified, the subgroup* $1 \times P_F$ *of* ${}^L\mathbf{G}$ *is contained in* $\mathcal{L}_{\phi,\varepsilon_\phi}$ *and there is a splitting* $\psi : W_F \hookrightarrow \mathcal{L}_{\phi,\varepsilon_\phi}$ *which is* tame *in the sense that* $\psi_{|P_F} = 1 \times \mathrm{Id}$.

Proof. We can write $\phi = \hat{\phi} \times \mathrm{Id}$ with $\hat{\phi} : P_F \longrightarrow \hat{\mathbf{G}}$ a homomorphism. Then $C_{\hat{\mathbf{G}}}(\phi) = C_{\hat{\mathbf{G}}}(\hat{\phi})$ so that $\hat{\phi}(P_F) \subset \hat{\mathbf{L}}_\phi$. Since $1 \times P_F$ acts trivially on $\hat{\mathbf{G}}$, it follows that the action of $\phi(P_F)$ on $\hat{\mathbf{L}}_\phi$ is inner, hence $\mathbf{L}_\phi$ is tamely ramified. Moreover, since $\phi(P_F) \subset \mathcal{L}_\phi$ by construction, we get that $1 \times P_F \subset \mathcal{L}_\phi$, and because this group acts trivially on $\hat{\mathbf{G}}$, we even have $1 \times P_F \subset \mathcal{L}_{\phi,\varepsilon_\phi}$. Now, the extension $\mathcal{L}_{\phi,\varepsilon_\phi}$ considered above is the pullback of the extension $Z(\hat{\mathbf{L}}_\phi) \hookrightarrow \mathcal{L}_{\phi,\varepsilon_\phi}/(1 \times P_F) \twoheadrightarrow W_F/P_F$ by the projection $W_F \twoheadrightarrow W_F/P_F$ and we need to show that the latter extension splits. By ..., we know that for any complex torus $\hat{\mathbf{S}}$ with a finite action of W_F/P_F we have $H^2_{cts}(W_F/P_F, \hat{\mathbf{S}}) = \{1\}$ (an alternative argument relying on Langlands' lemma 4 in [16] can be found in the proof of [11, Lemma 5.2.5]). On the other hand, the same argument as in Lemma 2.2.2 shows that the extension $\pi_0(Z(\hat{\mathbf{L}}_\phi)) \hookrightarrow \mathcal{L}_{\phi,\varepsilon_\phi}/(1 \times P_F)Z(\hat{\mathbf{L}}_\phi)^\circ \twoheadrightarrow W_F/P_F$ splits. Indeed, it suffices to replace ${}^L\mathbf{G}$ by its quotient $\mathbf{G} \rtimes (W_F/P_F)$. $\square$

2.2.5 *Lemma. — Assume that* $C_{\hat{\mathbf{G}}}(\phi)$ *is a Levi subgroup of* $\hat{\mathbf{G}}$*. Then* $\mathbf{G}$ *is tamely ramified,* $C_{\hat{\mathbf{G}}}(\phi) = \hat{\mathbf{L}}_\phi$*, the subgroup* $\phi(P_F)$ *is contained in* $\mathcal{L}_{\phi,\varepsilon_\phi}$ *and* $\hat{\phi}(P_F) \subset Z(\hat{\mathbf{L}}_\phi)$*. Moreover, the following are equivalent :*

i) There is a splitting $\varphi : W_F \hookrightarrow \mathcal{L}_{\phi,\varepsilon_\phi}$ *that extends* ϕ.

ii) There is a 1-cocycle $\hat{\varphi} : W_F \longrightarrow Z(\hat{\mathbf{L}}_\phi)$ *that extends* $\hat{\phi}$.

iii) The image $\hat\phi(\mathcal{E}) \in H^2(W_F/P_F, \pi_0(Z(\hat{\mathbf{L}}_\phi)))$ *of the canonical extension* $\mathcal{E} = [W_F/\overline{[P_F,P_F]}] \in H^2(W_F/P_F, P_F^{\mathrm{ab}})$ *vanishes.*

Proof. The equality $C_{\hat{\mathbf{G}}}(\phi) = \hat{\mathbf{L}}_\phi$ is clear by definition of $\hat{\mathbf{L}}_\phi$. The inclusion $\phi(P_F) \subset \mathcal{L}_\phi$ holds by construction, and since $\phi(P_F)$ centralizes $\hat{\mathbf{L}}_\phi$, it normalizes ε_ϕ, whence the inclusion $\phi(P_F) \subset \mathcal{L}_{\phi,\varepsilon_\phi}$. Actually, $\phi(P_F)$ centralizes any maximal torus of $\hat{\mathbf{L}}_\phi$, so a $\hat{\mathbf{G}}$-conjugate of $\phi(P_F)$ centralizes a reference maximal torus $\hat{\mathbf{T}}$ of $\hat{\mathbf{G}}$ (i.e. a part of a W_F-stable pinning ε of $\hat{\mathbf{G}}$). But since $\Omega(\hat{\mathbf{T}},\hat{\mathbf{G}}) \rtimes_\varepsilon \mathrm{Out}(\hat{\mathbf{G}}) \hookrightarrow \mathrm{Aut}(\hat{\mathbf{T}})$, the centralizer of $\hat{\mathbf{T}}$ in ${}^L\mathbf{G}$ is $\hat{\mathbf{T}} \times \ker(W_F \longrightarrow \mathrm{Out}(\hat{\mathbf{G}}))$. It follows that P_F acts trivially on $\hat{\mathbf{T}}$, hence that $\mathbf{G}$ is tamely ramified. Now, with $1 \times P_F$ and $\phi(P_F)$, also $\hat\phi(P_F)$ centralizes $\hat{\mathbf{L}}_\phi$, hence $\hat\phi(P_F) \subset Z(\hat{\mathbf{L}}_\phi)$.

Now, since $\mathbf{G}$ is tamely ramified, Lemma 2.2.4 provides us with a splitting $\psi : W_F \hookrightarrow \mathcal{L}_{\phi,\varepsilon_\phi}$ such that $\psi_{|P_F} = 1 \times \mathrm{Id}$. Therefore, if φ is as in item i), we can write it in the form $\varphi = \hat\varphi \cdot \psi$ and $\hat\varphi$ is as in item ii). Conversely, the same formula shows the equivalence $i) \Leftrightarrow ii)$. Now, let us prove the equivalence $i) \Leftrightarrow iii)$. Since $\phi(P_F) \subset \mathcal{L}_{\phi,\varepsilon_\phi}$, the extension $\mathcal{L}_{\phi,\varepsilon_\phi}$ considered above is a pullback of the extension $Z(\hat{\mathbf{L}}_\phi) \hookrightarrow \mathcal{L}_{\phi,\varepsilon_\phi}/\phi(P_F) \twoheadrightarrow W_F/P_F$. Therefore, there is a splitting as in i) if and only if the latter extension splits. As in the previous lemma, we know that $H^2(W_F/P_F, Z(\hat{\mathbf{L}}_\phi)^\circ) = \{1\}$ so we are left to study when the extension $\pi_0(Z(\hat{\mathbf{L}}_\phi)) \hookrightarrow \mathcal{L}_{\phi,\varepsilon_\phi}/\phi(P_F)Z(\hat{\mathbf{L}}_\phi)^\circ \twoheadrightarrow W_F/P_F$ splits. Choose a set theoretical section $s : W_F/P_F \longrightarrow W_F$ and denote by σ the composition $W_F/P_F \xrightarrow{s} W_F \xrightarrow{\psi} \mathcal{L}_{\phi,\varepsilon_\phi} \xrightarrow{\pi} \mathcal{L}_{\phi,\varepsilon_\phi}/\phi(P_F)Z(\hat{\mathbf{L}}_\phi)^\circ$. Then for all $\bar v, \bar w \in W_F/P_F$ we have in $\pi_0(Z(\hat{\mathbf{L}}_\phi))$

$$\begin{aligned}\sigma(\bar v\bar w)\sigma(\bar w)^{-1}\sigma(\bar v)^{-1} &= \pi(\psi(s(\bar v\bar w)s(\bar w)^{-1}s(\bar v)^{-1})) = \pi(1 \times s(\bar v\bar w)s(\bar w)^{-1}s(\bar v)^{-1}) \\ &= \hat\phi(s(\bar v\bar w)s(\bar w)^{-1}s(\bar v)^{-1})^{-1}.\end{aligned}$$

□

2.2.6 *Remark.* — Condition iii) is certainly satisfied if $\hat\phi(P_F) \subset Z(C_{\hat{\mathbf{G}}}(\phi))^\circ$, hence in particular when p does not divide the order of $\pi_0(Z(\hat{\mathbf{L}}_\phi))$, which in turn is satisfied if p does not divide $|\pi_0(Z(\hat{\mathbf{G}}))|$ since $Z(\hat{\mathbf{L}}_\phi) = Z(\hat{\mathbf{L}}_\phi)^\circ Z(\hat{\mathbf{G}})$. Note also that $|\pi_0(Z(\hat{\mathbf{G}}))| = |\pi_1(\mathbf{G}_{\mathrm{der}})|$ since the semi-simple group $\hat{\mathbf{G}}/Z(\hat{\mathbf{G}})^\circ$ is dual to $\mathbf{G}_{\mathrm{der}}$.

2.2.7 *The category* $\mathrm{Rep}_R^\phi(G'_\phi)$. — Let us assume that $C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup and that the equivalent conditions of Lemma 2.2.5 are satisfied. In accordance with our notation in the introduction and in [9], we write $\mathbf{G}_\phi = \mathbf{L}_\phi$ and we denote by $\mathbf{G}'_\phi$ an inner form of $\mathbf{G}_\phi$. Then Borel's construction in [5, 10.2] associates to any $\hat\varphi$ as in ii) of Lemma 2.2.5 a character

$$\check\varphi : G'_\phi = \mathbf{G}'_\phi(F) \longrightarrow \mathbb{C}^\times$$

Any other choice $\hat\varphi'$ differs from $\hat\varphi$ by a cocycle $\hat\delta \in Z^1(W_F, Z(\hat{\mathbf{G}}_\phi))$ such that $\hat\delta_{|P_F} = 1 \times \mathrm{Id}$. We then have $\check\varphi' = \check\varphi\check\delta$ with $\check\delta$ a depth 0 character of G'_ϕ (see Lemma 2.4.1). It follows that for any $x \in \mathcal{B}(\mathbf{G}'_\phi, F)$, the restriction $(\check\varphi)_{|G'_{\phi,x,0+}}$ is independent of the choice of $\hat\varphi$. Accordingly, the category $\check\varphi \otimes \mathrm{Rep}^1_{\mathbb{C}}(G'_\phi)$ is independent of this choice too. Now, the expected compatibility between Langlands correspondence and twisting naturally leads us to put $\mathrm{Rep}^\phi_{\mathbb{C}}(G'_\phi) := \check\varphi \otimes \mathrm{Rep}^1_{\mathbb{C}}(G'_\phi)$.

It is defined over the ring $R=\mathbb{Z}[\mu_{p^\infty},1/p]$ by the following formula

$$\mathrm{Rep}^{\phi}_R(G'_\phi)=\left\{V\in\mathrm{Rep}_R(G'_\phi), V=\sum_{x\in\mathcal{B}(\mathbf{G}'_\phi,F)} e^\phi_x V\right\}$$

where $e^\phi_x\in RG'_{\phi,x,0+}$ is the idempotent associated to the restriction of any $\check\varphi$ to $G'_{\phi,x,0+}$.

2.2.8 *Levi-center embeddings and root systems.* — Assume again that $C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup and that the equivalent conditions of Lemma 2.2.5 are satisfied. Fix an F-rational Levi-center embedding $\iota:\mathbf{S}_\phi\hookrightarrow\mathbf{G}$ in the set I_ϕ of Proposition 2.2.3. Then, as in the last paragraph, we get a class of characters $\check\varphi:\ \mathbf{G}_\iota(F)\longrightarrow\mathbb{C}^\times$ modulo depth 0 characters, associated to ϕ.

Let $\mathbf{S}$ be any tamely ramified maximal F-torus of $\mathbf{G}$ containing $\iota(\mathbf{S}_\phi)$, and let $E\supset F$ be a tamely ramified Galois extension that splits $\mathbf{S}$. We then have a norm map $N_{E|F}:\mathbf{S}(E)\to\mathbf{S}(F)$ and an inclusion $\mathbf{S}(F)\subset\mathbf{G}_\iota(F)$.

Lemma. — For any character $\check\varphi$ of $\mathbf{G}_\iota(F)$ associated to ϕ, the root system of $\mathbf{G}_\iota$ with respect to $\mathbf{S}$ is given by

$$\Sigma(\mathbf{S},\mathbf{G}_\iota)=\left\{\alpha\in\Sigma(\mathbf{S},\mathbf{G}),\ \check\varphi(N_{E|F}(\alpha^\vee(E^\times_{0+})))=\{1\}\right\}.$$

Proof. The inclusion $\mathbf{S}\subset\mathbf{G}_\iota$ gives rise to a W_F-stable conjugacy class of maximal torus embeddings $\hat{\mathbf{S}}\hookrightarrow C_{\hat{\mathbf{G}}}(\phi)$. Fix any such embedding and identify $\hat{\mathbf{S}}$ to a subtorus of $C_{\hat{\mathbf{G}}}(\phi)$ thanks to this choice. Then, through the bijection $\alpha\leftrightarrow\alpha^\vee,\Sigma(\mathbf{S},\mathbf{G})\leftrightarrow\Sigma(\hat{\mathbf{S}},\hat{\mathbf{G}})$, the subset $\Sigma(\mathbf{S},\mathbf{G}_\iota)$ corresponds to $\Sigma(\hat{\mathbf{S}},C_{\hat{\mathbf{G}}}(\phi))$, by the construction in 2.1.3.

On the other hand, $\hat{\mathbf{S}}$ contains $Z(C_{\hat{\mathbf{G}}}(\phi))$ and $\hat\varphi$ factors through $\hat{\mathbf{S}}$, giving a Langlands parameter that we still denote by $\hat\varphi\in Z^1(W_F,\hat{\mathbf{S}})$. This is the Langlands parameter of the character $\check\varphi:\mathbf{S}(F)\hookrightarrow\mathbf{G}_\iota(F)\longrightarrow\mathbb{C}^\times$. Then the Langlands parameter of the character $\check\varphi\circ N_{E|F}:\mathbf{S}(E)\longrightarrow\mathbb{C}^\times$ is $\hat\varphi_{|W_E}$. Accordingly, the character $\check\varphi\circ N_{E|F}\circ\alpha^\vee:\ E^\times\longrightarrow\mathbb{C}^\times$ coincides via the local class field reciprocity to the character $\alpha^\vee\circ\hat\varphi$ of W_E (where $\alpha^\vee$ is first seen as a cocharacter of $\mathbf{S}$, then as a character of $\hat{\mathbf{S}}$). Its restriction to $E^\times_{0+}$ is therefore trivial if and only if $\alpha^\vee\circ\hat\phi$ is a trivial character of $P_E=P_F$, which is equivalent to $\alpha^\vee$ being a root of $\hat{\mathbf{S}}$ in the centralizer $C_{\hat{\mathbf{G}}}(\phi)$, as desired. $\square$

2.3 Ramification groups and twisted Levi sequences

We denote by I^r_F, $r\in\mathbb{R}_+$, the ramification subgroups of the Galois group Γ_F in the upper numbering. We also put $I^{r+}_F:=\overline{\bigcup_{s>r}I^s_F}$. So we have $I^0_F=I_F$ and $I^{0+}_F=P_F$. Following ... we use the notation $\widetilde{\mathbb{R}}:=\mathbb{R}\sqcup\{r+,r\in\mathbb{R}\}$, which is ordered by letting $r<r+<s$ for any $r<s\in\mathbb{R}$.

We will assume from now on that the following hypothesis is satisfied :

(H1): the connected *centralizer of an abelian p-group of $\hat{\mathbf{G}}$ is a Levi subgroup.*

This is a rather mild hypothesis. By [3, Prop. A.7], it is satisfied if p is good for $\mathbf{G}$, *i.e.* if $p>2$ in types B_n,C_n and D_n, if $p>3$ in type G_2, E_6 and E_7, and if $p>5$ in type E_8.

We also fix an admissible $\phi : P_F \longrightarrow \mathbf{G}$ and we assume that $C_{\hat{\mathbf{G}}}(\phi)$ is a Levi subgroup.

By Lemma 2.2.5, this implies that $\mathbf{G}$ is tamely ramified and that $\hat{\phi}(P_F)$ is a finite abelian p-group contained in the center of $C_{\hat{\mathbf{G}}}(\phi) = \hat{\mathbf{G}}_\phi = \hat{\mathbf{L}}_\phi$. Actually, when p is prime to $|\pi_1(\mathbf{G}_{\mathrm{der}})|$ and under $(H1)$, this is equivalent to $\hat{\phi}(P_F)$ being abelian, due to [4, Cor. 2.9].

Recall that $\mathbf{S}_\phi$ denotes the F-torus that is dual to $\hat{\mathbf{S}}_\phi = \hat{\mathbf{G}}_{\phi,\mathrm{ab}}$ with its canonical Galois action. We are going to define a filtration of $\mathbf{S}_\phi$ by F-subtori.

2.3.1 *The groups $\mathbf{G}_{\phi,r}$ and $\mathbf{S}_{\phi,r}$.* — Fix $r \in \widetilde{\mathbb{R}}_{>0}$. We put $\hat{\mathbf{G}}_{\phi,r} := C_{\hat{\mathbf{G}}}(\phi(I_F^r))^\circ$ and $\hat{\mathbf{S}}_{\phi,r} := \hat{\mathbf{G}}_{\phi,r,\mathrm{ab}}$. By our running hypothesis, $\hat{\mathbf{G}}_{\phi,r}$ is a Levi subgroup of $\mathbf{G}$ that contains $C_{\hat{\mathbf{G}}}(\phi)$. Therefore, the group $\mathcal{G}_{\phi,r} := \hat{\mathbf{G}}_{\phi,r} \cdot \varphi(W_F)$ does not depend on the choice of an extension of ϕ to W_F and sits in an exact sequence $\hat{\mathbf{G}}_{\phi,r} \hookrightarrow \mathcal{G}_{\phi,r} \twoheadrightarrow W_F$ which provides a canonical and finite outer action $W_F \longrightarrow \mathrm{Out}(\hat{\mathbf{G}}_{\phi,r})$ and thus defines a quasi-split reductive F-group $\mathbf{G}_{\phi,r}$. Since $\hat{\phi}(P_F)$ is contained in $C_{\hat{\mathbf{G}}}(\phi)$ hence also in $\hat{\mathbf{G}}_{\phi,r}$, the outer action factors over W_F/P_F and accordingly $\mathbf{G}_{\phi,r}$ is tamely ramified. Also this outer action descends to $\hat{\mathbf{S}}_{\phi,r}$, providing a dual tamely ramified F-torus $\mathbf{S}_{\phi,r}$ with a canonical isomorphism $\mathbf{S}_{\phi,r} \xrightarrow{\sim} Z(\mathbf{G}_{\phi,r})^\circ$.

More importantly, the inclusion $\hat{\mathbf{G}}_\phi \subset \hat{\mathbf{G}}_{\phi,r}$ induces a W_F-equivariant epimorphism $\hat{\mathbf{S}}_\phi \twoheadrightarrow \hat{\mathbf{S}}_{\phi,r}$, which on the dual side induces an F-rational embedding $\mathbf{S}_{\phi,r} \hookrightarrow \mathbf{S}_\phi$. Note that the latter embedding only depends on ϕ, and on no other choice.

2.3.2 *Lemma.* — *Let $\iota : \mathbf{S}_\phi \hookrightarrow \mathbf{G}$ be an F-rational Levi-center embedding in the set I_ϕ of Proposition 2.2.3, and let $\mathbf{S}$ be a maximal F-torus of $\mathbf{G}_\iota$ split by some tamely ramified Galois extension E of F. Then for any character $\check{\varphi}$ of $\mathbf{G}_\iota(F)$ associated to ϕ as in Lemma 2.2.5, we have*

$$\Sigma(\mathbf{S}, C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi,r}))) = \left\{\alpha \in \Sigma(\mathbf{S},\mathbf{G}),\ \check{\varphi}(N_{E|F}(\alpha^\vee(E_r^\times))) = \{1\}\right\}.$$

Proof. As in the proof of Lemma 2.2.8, fix a dual embedding $\hat{\mathbf{S}} \subset C_{\hat{\mathbf{G}}}(\phi)$. Then, by the construction in 2.1.3, the bijection $\alpha \leftrightarrow \alpha^\vee, \Sigma(\mathbf{S},\mathbf{G}) \leftrightarrow \Sigma(\hat{\mathbf{S}},\hat{\mathbf{G}})$ takes $\Sigma(\mathbf{S}, C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi,r})))$ to $\Sigma(\hat{\mathbf{S}}, C_{\hat{\mathbf{G}}}(\phi(I_F^r))^\circ) = \{\alpha^\vee \in \Sigma(\hat{\mathbf{S}},\hat{\mathbf{G}}), \alpha^\vee \circ \hat{\phi}(I_F^r) = \{1\}\}$. It remains to follow the proof of [11, Lemma 3.7.8]. Indeed $\alpha^\vee \circ \hat{\varphi}_{|W_E}$ corresponds to $\check{\varphi} \circ N_{E|F} \circ \alpha^\vee$ via the local class field reciprocity $E^\times \xrightarrow{\sim} W_E^{\mathrm{ab}}$, while the latter also takes $E_r^\times$ to the image of $I_E^r = I_F^r$ in W_E^{ab}. The lemma follows. $\square$

We also have a factorization $H^1(F,\mathbf{G}_\phi) \longrightarrow H^1(F,\mathbf{G}_{\phi,r}) \longrightarrow H^1(F,\mathbf{G})$ defined through Kottwitz duality by the inclusions $Z(\hat{\mathbf{G}})^{W_F} \subset Z(\hat{\mathbf{G}}_{\phi,r})^{\varphi(W_F)} \subset Z(\hat{\mathbf{G}}_\phi)^{\varphi(W_F)}$. We will still denote by η the image in $H^1(F,\mathbf{G}_{\phi,r})$ of some $\eta \in H^1(F,\mathbf{G}_\phi)$.

2.3.3 *Lemma.* — *Assume that $\mathbf{G}$ is quasi-split and that $\mathbf{G}_\iota$ is isomorphic to $\mathbf{G}_{\phi,\eta}$. Then $C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi,r}))$ is isomorphic to $\mathbf{G}_{\phi,r,\eta}$.*

Proof. By Proposition 2.1.5, there is an F-rational embedding $\iota_r : \mathbf{S}_{\phi,r} \hookrightarrow \mathbf{G}$ dual to the inclusion $Z(C_{\hat{\mathbf{G}}}(\phi(I_F^r))^\circ)^\circ \subset \hat{\mathbf{G}}$ and whose centralizer $\mathbf{G}_{\iota_r}$ is quasi-split. By the same argument, there is now an F-rational embedding $\iota^r : \mathbf{S}_\phi \hookrightarrow \mathbf{G}_{\iota_r}$ dual to $Z(C_{\hat{\mathbf{G}}}(\phi))^\circ \subset C_{\hat{\mathbf{G}}}(\phi(I_F^r))^\circ$, and whose centralizer $(\mathbf{G}_{\iota_r})_{\iota^r}$ is quasi-split. Then the composition $\iota_0 := \iota^r \circ \iota_r$ is an embedding in I_ϕ such that both $\mathbf{G}_{\iota_0}$ and $C_{\mathbf{G}}(\iota_0(\mathbf{S}_{\phi,r}))$ are quasi-split, hence are respectively isomorphic to $\mathbf{G}_\phi$

and $\mathbf{G}_{\phi,r}$. Now by Lemma 2.1.7, there is an F-rational embedding ι in I_ϕ such that $\eta = \eta_{\iota_0,\iota} \in H^1(F, \mathbf{G}_{\iota_0})$. By the definition of $\eta_{\iota_0,\iota}$ given above Lemma 2.1.7, we see that $C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi,r}))$ is the pure inner form of $C_{\mathbf{G}}(\iota_0(\mathbf{S}_{\phi,r}))$ associated to the image of η in $H^1(F, C_{\mathbf{G}}(\iota_0(\mathbf{S}_{\phi,r})))$, hence it is isomorphic to $\mathbf{G}_{\phi,r,\eta}$. □

2.3.4 *The twisted Levi sequence associated to ϕ and ι.* — We denote by $0 < r_0 < \cdots < r_{d-1}$ the jumps of the decreasing filtration $(\mathbf{S}_{\phi,r})_{r>0}$ of $\mathbf{S}_\phi$. Namely we have

$$\{r_0, \cdots, r_{d-1}\} = \{r > 0,\, \mathbf{S}_{\phi,r+} \subsetneq \mathbf{S}_{\phi,r}\} = \{r > 0, C_{\hat{\mathbf{G}}}(\phi(I_F^{r+}))^\circ \supsetneq C_{\hat{\mathbf{G}}}(\phi(I_F^r))^\circ\}.$$

Note that $\mathbf{S}_{\phi,r} = \mathbf{S}_\phi$ for $r \leqslant r_0$ while $\mathbf{S}_{\phi,r} = Z(\mathbf{G})^\circ$ for $r > r_{d-1}$. We also put $r_{-1} := 0$ and $r_d := \mathrm{depth}(\phi) := \inf\{r > 0, \phi(I_F^r) = \{1\}\}$, which satisfies $r_d \geqslant r_{d-1}$.

Now fix an F-rational Levi-center embedding $\iota : \mathbf{S}_\phi \hookrightarrow \mathbf{G}$ in I_ϕ. In order to simplify the notation a bit, we put

$$\mathbf{G}_\iota^i := C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi,r_i})) = C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi,r_{i-1}+})) \text{ for } i = 0, \cdots, d-1 \text{ and } \mathbf{G}_\iota^d := \mathbf{G}$$

We thus get a tamely ramified twisted Levi sequence in $\mathbf{G}$

$$\vec{\mathbf{G}}_\iota := \left(\mathbf{G}_\iota = \mathbf{G}_\iota^0 \subset \cdots \subset \mathbf{G}_\iota^d = \mathbf{G}\right).$$

2.4 Characters and idempotents

In this section, we will use Borel's procedure in [5, 0.2] to construct certain characters of $\mathbf{G}_\iota^i(F)$ that are suitable to apply Yu's procedure in [21] and get characters of certain open pro-p subgroups of G. Yu's work involves the group side analogue of the ramification filtration, namely the Moy-Prasad filtrations [18]. For each point x in the building $\mathcal{B}(\mathbf{G}, F)$ we thus have a filtration $(G_{x,r} = \mathbf{G}(F)_{x,r})_{r\geqslant 0}$ of the stabilizer $G_x = \mathbf{G}(F)_x$ by open normal subgroups. If we put $G_{x,r+} := \bigcup_{s>r} G_{x,s}$, then $G_{x,0+}$ is known to be the pro-p-radical of the parahoric group $G_{x,0}$. We will need the following relation between both filtrations, which follows easily from Yu's [22, Theorem 7.10].

2.4.1 *Lemma.* — *Let $\mathbf{G}$ be a tamely ramified reductive group over F and let $\theta : \mathbf{G}(F) \longrightarrow \mathbb{C}^\times$ be the character associated to some $\hat{\varphi} \in H^1(W_F, Z(\hat{\mathbf{G}}))$. Then θ is trivial on $\mathbf{G}_{\mathrm{sc}}(F)$ and for any $x \in \mathcal{B}(\mathbf{G}, F)$ and any $r \in \widetilde{\mathbb{R}}_{\geqslant 0}$ we have $\theta_{|G_{x,r}} \equiv 1 \Leftrightarrow \hat{\varphi}_{|I_F^r} \equiv 1$.*

Proof. We need to go through Borel's procedure in [5, 10.2]. So let $\tilde{\mathbf{G}} \longrightarrow \mathbf{G}$ be a z-extension, i.e. a central extension of $\mathbf{G}$ by an induced torus $\mathbf{Z}$ whose derived subgroup is simply connected. On the dual side we get a W_F-equivariant embedding of $Z(\hat{\mathbf{G}})$ into the torus $Z(\hat{\tilde{\mathbf{G}}})$. Pushing $\hat{\varphi}$ by this embedding we get a Langlands parameter for the tamely ramified torus $\tilde{\mathbf{G}}_{\mathrm{ab}}$, whence a character $\tilde{\theta}$ of $\tilde{\mathbf{G}}_{\mathrm{ab}}(F)$. By [22, Thm 7.10] we have $\tilde{\theta}_{|\tilde{\mathbf{G}}_{\mathrm{ab}}(F)_r} \equiv 1 \Leftrightarrow \tilde{\varphi}_{|I_F^r} \equiv 1$. Now, θ is defined as follows. The map $\tilde{\mathbf{G}}(F) \longrightarrow \mathbf{G}(F)$ is surjective and the character $\tilde{\theta} : \tilde{\mathbf{G}}(F) \longrightarrow \tilde{\mathbf{G}}_{\mathrm{ab}}(F) \longrightarrow \mathbb{C}^\times$ is trivial on the kernel $\mathbf{Z}(F)$ of this map and on $\tilde{\mathbf{G}}_{\mathrm{der}}(F) = \mathbf{G}_{\mathrm{sc}}(F)$. Therefore $\tilde{\theta}$ descends to the desired character θ of $\mathbf{G}(F)$, which is trivial on (the image of) $\mathbf{G}_{\mathrm{sc}}(F)$. Now, for any $x \in \mathcal{B}(\mathbf{G}, F)$ and any $\tilde{x} \in \mathcal{B}(\tilde{\mathbf{G}}, F)$ above x, Lemma 3.5.3 of [11] tells us that the maps $\tilde{\mathbf{G}}(F)_{\tilde{x},r} \longrightarrow \tilde{\mathbf{G}}_{\mathrm{ab}}(F)_r$ and $\tilde{\mathbf{G}}(F)_{\tilde{x},r} \longrightarrow \mathbf{G}(F)_{x,r}$ are both surjective. This implies the equivalence claimed in the lemma. □

2.4.2 *Remark.* — Conversely, *any character $\theta : \mathbf{G}(F) \longrightarrow \mathbb{C}^\times$ that is trivial on $\mathbf{G}_{\mathrm{sc}}(F)$ comes from some $\hat{\varphi} \in H^1(W_F, Z(\hat{\mathbf{G}}))$ via Borel's procedure.* Indeed, with the notation of the above proof, the surjectivity of $\tilde{\mathbf{G}}(F) \longrightarrow \mathbf{G}(F)$ allows one to inflate θ to a character $\widetilde{\theta}$ of $\tilde{\mathbf{G}}(F)$ that is trivial on $\mathbf{Z}(F)\tilde{\mathbf{G}}_{\mathrm{der}}(F)$. In particular $\tilde{\theta}$ factors through the surjective map $\tilde{\mathbf{G}}(F) \longrightarrow \tilde{\mathbf{G}}_{\mathrm{ab}}(F)$, giving a character of $\tilde{\mathbf{G}}_{\mathrm{ab}}(F)$ which, by Langlands' correspondence for tori, comes from some $\hat{\varphi} \in H^1(W_F, Z(\hat{\tilde{\mathbf{G}}}))$. But the pushforward of $\hat{\varphi}$ into $H^1(W_F, \hat{\mathbf{Z}})$ has to be trivial, hence $\hat{\varphi}$ comes from $H^1(W_F, Z(\hat{\mathbf{G}}))$.

Recall the definitions of $\hat{\mathbf{G}}_{\phi,r}$ and $\mathcal{G}_{\phi,r}$ from 2.3.1, and let us choose a pinning $\varepsilon_{\phi,r}$ of $\hat{\mathbf{G}}_{\phi,r}$ and consider the stabilizer $\mathcal{G}_{\phi,r,\varepsilon_{\phi,r}}$ of this pinning in $\mathcal{G}_{\phi,r}$, which sits in an exact sequence $Z(\hat{\mathbf{G}}_{\phi,r}) \hookrightarrow \mathcal{G}_{\phi,r,\varepsilon_{\phi,r}} \twoheadrightarrow W_F$. Observe that $1 \times P_F$ and $\phi(I_F^r)$ are contained in $\mathcal{G}_{\phi,r,\varepsilon_{\phi,r}}$.

2.4.3 *Lemma.* — *There exists a splitting $\psi_r : W_F \longrightarrow \mathcal{G}_{\phi,r,\varepsilon_{\phi,r}}$ of the above sequence such that $\psi_{r|I_F^r} = 1 \times \mathrm{Id}$. Moreover the following hypothesis are equivalent.*

i) There is a splitting $\varphi_r : W_F \longrightarrow \mathcal{G}_{\phi,r,\varepsilon_{\phi,r}}$ such that $\varphi_{r|I_F^r} = \phi_{|I_F^r}$.

ii) There is $\hat{\varphi}_r \in Z^1(W_F, Z(\hat{\mathbf{G}}_{\phi,r}))$ such that $\hat{\varphi}_{r|I_F^r} = \hat{\phi}_{|I_F^r}$.

iii) The image $\hat{\phi}(\mathcal{E}_r) \in H^2(W_F/I_F^r, \pi_0(Z(\hat{\mathbf{G}}_{\phi,r})))$ of the canonical extension $\mathcal{E}_r = [W_F/\overline{[I_F^r, I_F^r]}] \in H^2(W_F/I_F^r, I_F^{r,\mathrm{ab}})$ vanishes.

Further, these hypothesis are satisfied if p does not divide $|\pi_0(Z(\hat{\mathbf{G}}))| = |\pi_1(\mathbf{G}_{\mathrm{der}})|$.

Proof. Thanks to Lemma 2.4.4 below, the first assertion is proved as in Lemma 2.2.4, while the equivalence between the three hypothesis is proved as in Lemma 2.2.5. The last assertion follows from Remark 2.2.6. □

2.4.4 *Lemma.* — *Let $\mathbf{S}$ be a tamely ramified torus and $r \in \widetilde{\mathbb{R}}_{>0}$. Then the image of $H^2_{cts}(\Gamma_F/I_F^r, \hat{\mathbf{S}})$ in $H^2(W_F/I_F^r, \hat{\mathbf{S}})$ is trivial.*

Proof. Start with $\eta \in H^2_{cts}(\Gamma_F/I_F^r, \hat{\mathbf{S}})$ and let $\bar{\eta}$ denote its image in $H^2(W_F/I_F^r, \hat{\mathbf{S}})$. By definition of continuous cohomology, η comes from an element $\eta \in H^2(\Gamma_{E/F}, \hat{\mathbf{S}})$ with E a finite extension that splits $\mathbf{S}$ and that is r-ramified in the sense that I_F^r maps to $\{1\}$ in $\Gamma_{E/F}$. We may and will assume that $\mathbf{S}$ is also split by the maximal tamely ramified subextension E^{tr} of E over F. As in the proof of [16, Lemma 4], we can choose an exact sequence $\hat{\mathbf{S}} \hookrightarrow \hat{\mathbf{S}}_1 \twoheadrightarrow \hat{\mathbf{S}}_2$ with $\mathbf{S}_1$ an induced torus for $\Gamma_{E^{tr}/F}$. Note that each $\mathbf{S}_i$ is then tamely ramified. Let us look at the exact sequence

$$H^1(W_F/I_F^r, \hat{\mathbf{S}}_1) \longrightarrow H^1(W_F/I_F^r, \hat{\mathbf{S}}_2) \longrightarrow H^2(W_F/I_F^r, \hat{\mathbf{S}}) \longrightarrow H^2(W_F/I_F^r, \hat{\mathbf{S}}_1)$$

Since $H^2(\Gamma_{E/F}, \hat{\mathbf{S}}_1) = \{1\}$ (because $\mathbf{S}_1$ is an induced torus also for E/F), the image of $\bar{\eta}$ in $H^2(W_F/I_F^r, \hat{\mathbf{S}}_1)$ is trivial. So if we can prove that the first map is surjective, we infer that $\bar{\eta}$ itself is trivial. But by [22, Theorem 7.10] the local Langlands correspondence identifies $H^1(W_F/I_F^r, \hat{\mathbf{S}}_i)$ with the group of characters of the group $\mathbf{S}_i(F)/\mathbf{S}_i(F)_r$. Moreover, by [11, Lemma 3.1.1] the dual embedding $\mathbf{S}_2 \hookrightarrow \mathbf{S}_1$ satisfies $\mathbf{S}_2(F)_r = \mathbf{S}_2(F) \cap \mathbf{S}_1(F)_r$. So this dual embedding induces an injective map $\mathbf{S}_2(F)/\mathbf{S}_2(F)_r \hookrightarrow \mathbf{S}_1(F)/\mathbf{S}_1(F)_r$ which shows the surjectivity of the map $H^1(W_F/I_F^r, \hat{\mathbf{S}}_1) \longrightarrow H^1(W_F/I_F^r, \hat{\mathbf{S}}_2)$. □

2.4.5 *Characters.* — We now fix an F-rational embedding $\iota \in I_\phi$ and we take up the notation $\vec{\mathbf{G}}_\iota$ of 2.3.4. From now on, we will make the following additional hypothesis :

$$\textit{(H2): } p \textit{ does not divide } |\pi_0(Z(\hat{\mathbf{G}}))| = |\pi_1(\mathbf{G}_{\mathrm{der}})|.$$

Thanks to this hypothesis, Lemma 2.4.3 insures the existence of a 1-cocycle $\hat{\varphi}_i : W_F \longrightarrow Z(\hat{\mathbf{G}}_{\phi,r_{i-1}+})$ that extends $\hat{\phi}_{|I_F^{r_{i-1}+}}$ for each $i = 0, \cdots, d$. Since $\mathbf{G}_\iota^i$ is an inner form of $\mathbf{G}_{\phi,r_{i-1}+}$, Borel's procedure [5, 10.2] associates to $\hat{\varphi}_i$ a character $\check{\varphi}_i : \mathbf{G}_\iota^i(F) \longrightarrow \mathbb{C}^\times$. Then, Lemma 2.4.1 has the following consequences, for any $x \in \mathcal{B}(\mathbf{G}_\iota^i, F)$:

i) the restriction $(\check{\varphi}_i)_{|\mathbf{G}_\iota^i(F)_{x,r_{i-1}+}}$ only depends on $\hat{\phi}_{|I_F^{r_{i-1}+}}$, and not on the choice of $\hat{\varphi}_i$,

ii) for all $j \geqslant i$ we have $(\check{\varphi}_i)_{|\mathbf{G}_\iota^i(F)_{x,r_{j-1}+}} = (\check{\varphi}_j)_{|\mathbf{G}_\iota^i(F)_{x,r_{j-1}+}}$,

iii) the character $\psi_i := \check{\varphi}_i \check{\varphi}_{i+1}^{-1}$ of $\mathbf{G}_\iota^i(F)$ is trivial on $\mathbf{G}_\iota^i(F)_{x,r_i+}$ (we put $\check{\varphi}_{d+1} = 1$).

2.4.6 *The subset $\mathcal{B}_\iota$ of the building.* — We write $\mathcal{B}$ for the building $\mathcal{B}(\mathbf{G}, F)$. If $\mathbf{S}$ is a maximal F-torus of $\mathbf{G}$ that splits over some tamely ramified finite extension E of F, we put $\mathcal{A}(\mathbf{G}, \mathbf{S}, F) := \mathcal{A}(\mathbf{G}, \mathbf{S}, E) \cap \mathcal{B}(\mathbf{G}, F)$, the intersection holding in $\mathcal{B}(\mathbf{G}, E)$. As the notation suggests, this does not depend on the choice of E. But in contrast to what the notation may suggest, this need not be an appartment of $\mathcal{B}$, unless $\mathbf{S}$ has maximal F-split rank. Now we associate to ι the following subset of $\mathcal{B}$:

$$\mathcal{B}_\iota := \bigcup_{\mathbf{S} \subset \mathbf{G}_\iota} \mathcal{A}(\mathbf{G}, \mathbf{S}, F)$$

where $\mathbf{S}$ runs over tamely ramified maximal F-tori of $\mathbf{G}_\iota$. Recall that there is a canonical class of "toral" embeddings $\mathcal{B}(\mathbf{G}_\iota, F) \hookrightarrow \mathcal{B}$ modulo translations by $X_*(\mathbf{S}_\iota)_{\mathbb{R}}^{W_F}$. The set $\mathcal{B}_\iota$ is also the common image of all these toral embeddings. We could have restricted the above union to maximally F-split (and tamely ramified) F-tori of $\mathbf{G}_\iota$, thanks to [21, Lemma 2.1]. For such a maximally split torus, the subset $\mathcal{A}(\mathbf{G}, \mathbf{S}, F)$ is an appartment of $\mathcal{B}(\mathbf{G}_\iota, F)$, but it is not an appartment of $\mathcal{B}$ unless $\mathbf{G}_\iota$ is an F-Levi subgroup.

2.4.7 *A construction of Yu's.* — Let us fix $x \in \mathcal{B}_\iota$. For each $i = 0, \cdots, d$, the intersection

$$G_{\iota,x,r}^i := \mathbf{G}_\iota^i(F)_{x,r} := G_{x,r} \cap \mathbf{G}_\iota^i(F)$$

is the Moy-Prasad group associated to r and the preimage of x by any toral embedding $\mathcal{B}(\mathbf{G}_\iota^i, F) \hookrightarrow \mathcal{B}$. Note that $G_{\iota,x,r}^i$ normalizes $G_{\iota,x,s}^j$ whenever $i \leqslant j$, so that we can define an open subgroup of $G_{x,0+}$ by

$$\vec{G}_{\iota,x}^{++} := G_{\iota,x,0+}^0 G_{\iota,x,r_0+}^1 \cdots G_{\iota,x,r_{d-1}+}^d.$$

By property ii) of 2.4.5, the characters $(\check{\varphi}_i)_{i=0,\cdots,d}$ "glue" to a character $\check{\phi}_{\iota,x}^{++}$ of $\vec{G}_{\iota,x}^{++}$. By property i) of 2.4.5, $\check{\phi}_{\iota,x}^{++}$ *only depends on* ϕ and not on the choice of the characters $\check{\varphi}_i$. Now let us consider the following bigger open subgroup of $G_{x,0+}$:

$$\vec{G}_{\iota,x}^{+} := G_{\iota,x,0+}^0 G_{\iota,x,(r_0/2)+}^1 \cdots G_{\iota,x,(r_{d-1}/2)+}^d.$$

In [21, §4], Yu describes a "canonical" way to build a character $\check{\phi}^{+}_{\iota,x}$ of $\vec{G}^{+}_{\iota,x}$ that extends $\check{\phi}^{++}_{\iota,x}$, starting from the characters ψ_i of 2.4.5. To explain this, we first observe that $\check{\phi}^{++}_{\iota,x}$ is also the product $\prod_{i=0}^{d}(\psi^{++}_{i,x})_{|\vec{G}^{++}_{\iota,x}}$ where $\psi^{++}_{i,x}$ denotes the unique character of the group $G^{i}_{\iota,x,0+}G_{x,r_i+}$ that extends both $\psi_{i|G^{i}_{\iota,x,0+}}$ and the trivial character of G_{x,r_i+}. Similarly, Yu defines $\check{\phi}^{+}_{\iota,x}$ as a product

$$\check{\phi}^{+}_{\iota,x} := \prod_{i=0}^{d}(\psi^{+}_{i,x})_{|\vec{G}^{+}_{\iota,x}}$$

where $\psi^{+}_{i,x}$ is a certain character of $G^{i}_{\iota,x,0+}G_{x,(r_i/2)+}$ that extends $\psi_{i|G^{i}_{\iota,x,0+}}$.

Yu's construction of $\psi^{+}_{i,x}$ uses the Moy-Prasad filtrations $(\mathfrak{g}_{x,r})_{r\in\mathbb{R}}$ on the Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathbf{G})(F)$. We adopt Yu's notation $G_{x,(r/2)+:r+}$ for the quotient group $G_{x,(r/2)+}/G_{x,r+}$. This group is abelian and Moy and Prasad have defined a canonical isomorphism $\mathfrak{g}_{x,(r/2)+:r+} \xrightarrow{\sim} G_{x,(r/2)+:r+}$. Now the Lie subalgebra $\mathfrak{g}^{i}_{\iota} = \mathrm{Lie}(\mathbf{G}^{i}_{\iota})(F)$ of $\mathfrak{g}$ has a canonical complement, namely the sum $\mathfrak{n}^{i}_{\iota}$ of non-zero weight spaces of $\iota(\mathbf{S}_{\phi,r_i})$ acting on $\mathfrak{g}$ through the adjoint representation. This induces a decomposition

$$\mathfrak{g}_{x,(r/2)+:r+} = \mathfrak{g}^{i}_{\iota,x,(r/2)+:r+} \oplus \mathfrak{n}^{i}_{\iota,x,(r/2)+:r+}.$$

Thanks to this decomposition, any character ψ of $G^{i}_{\iota,x,(r/2)+:r+}$ can be extended to a character $\widetilde{\psi}$ of $G_{x,(r/2)+:r+}$ by letting it be trivial on $\mathfrak{n}^{i}_{\iota,x,(r/2)+:r+}$. In particular we get from $\psi_{i|G^{i}_{\iota,x,(r_i/2)+}}$ a character $\widetilde{\psi}_i$ of $G_{x,(r_i/2)+}$ which, in turn, can be glued with $\psi_{i|G^{i}_{\iota,x,0+}}$ to yield the desired character $\psi^{+}_{i,x}$ of $G^{i}_{\iota,x,0+}G_{x,(r_i/2)+}$. Note that this character depends on the choices of $\check{\varphi}_i$ and $\check{\varphi}_{i+1}$ and, a priori, also the restriction $(\psi^{+}_{i,x})_{|\vec{G}_{\iota,x}}$ depends on these choices. However we have the following independence result.

2.4.8 *Lemma.* — *The character $\check{\phi}^{+}_{\iota,x}$ only depends on ϕ, ι, x, and not on the choice of $\check{\varphi}_i$.*

Proof. We first note that the map $\psi \mapsto \widetilde{\psi}$ described above is obviously multiplicative in ψ, and has the following property : if ξ is a character of G of depth $\leqslant r$, then $\widetilde{\xi_{|G^{i}_{\iota,x,(r/2)+}}} = \xi_{|G_{x,(r/2)+}}$. Indeed, ξ is trivial on root subgroups of G hence $\xi_{|G_{x,(r/2)+}}$ has to be trivial on $\mathfrak{n}^{i}_{\iota,x,(r/2)+:r+}$. More generally, if ξ_j is a character of G^j for some $j \geqslant i$, then $(\xi_{j|G^{i}_{\iota,x,(r/2)+}})_{|G^{j}_{\iota,x,(r/2)+}} = \xi_{j|G^{j}_{\iota,x,(r/2)+}}$ hence also $\widetilde{\xi_{j|G^{i}_{\iota,x,(r/2)+}}} = \widetilde{\xi_{j|G^{j}_{\iota,x,(r/2)+}}}$. So we may unambiguously denote this character by $\widetilde{\xi}_j$.

Let us now check that the product $\prod_{i=1}^{d}(\psi^{+}_{i,x})_{|\vec{G}^{+}_{\iota,x}}$ is independent of the choices of cocycles $\hat{\varphi}_i$. So let $(\hat{\varphi}'_i)_{i=0,\cdots,d}$ be another choice of cocycles leading to characters $\psi'^{+}_{i,x}$, and write $\check{\varphi}'_i = \check{\varphi}_i\xi_i$. Then ξ_i is a character of G_i of depth $\leqslant r_{i-1}$, hence $\xi_{i|G^{i}_{\iota,x,(r_{i-1}/2)+}}$ extends to a character $\widetilde{\xi}_i$ of $G_{x,(r_{i-1}/2)+}$ according to the procedure described above the lemma. Then we see that for all $i, j \leqslant d$ we have

$$(\psi'^{+}_{i,x})_{|G^{j}_{\iota,x,(r_{j-1}/2)+}} = \begin{cases} (\psi^{+}_{i,x})_{|G^{j}_{\iota,x,(r_{j-1}/2)+}} \cdot (\xi_i\xi_{i+1}^{-1})_{|G^{j}_{\iota,x,(r_{j-1}/2)+}} & \text{if } j \leqslant i \\ (\psi^{+}_{i,})_{|G^{j}_{\iota,x,(r_{j-1}/2)+}} \cdot (\widetilde{\xi}_i\widetilde{\xi}_{i+1}^{-1})_{|G^{j}_{\iota,x,(r_{j-1}/2)+}} & \text{if } j > i \end{cases}$$

where we agree that $\xi_{d+1}=1$. Taking products we get for all $j=0,\cdots,d$

$$\prod_{i=0}^{d}(\psi'^{+}_{i,x})_{|G^j_{x,(r_{j-1}/2)+}} = \prod_{i=0}^{d}(\psi^{+}_{i,x})_{|G^j_{x,(r_{j-1}/2)+}} \cdot (\widetilde{\xi}_j)^{-1}_{|G^j_{x,(r_{j-1}/2)+}}(\xi_j)_{|G^j_{x,(r_{j-1}/2)+}} = \prod_{i=0}^{d}(\psi^{+}_{i,x})_{|G^j_{x,(r_{j-1}/2)+}}$$

as desired. □

Since $\vec{G}^{+}_{\iota,x}$ is a pro-p-group, the smooth character $\check{\phi}^{+}_{\iota,x}$ takes values in the ring R of ...

2.4.9 *Definition.–* We denote by $e^{\phi}_{\iota,x}$ the idempotent of the Hecke algebra $\mathcal{H}_R(G)$ associated to $\check{\phi}^{+}_{\iota,x}$. By construction, it is supported on $\vec{G}^{+}_{\iota,x}$.

From the construction of $\vec{G}^{+}_{\iota,x}$ and $e^{\phi}_{\iota,x}$, and in particular the fact that the latter does not depend on any further choice than ι, x and ϕ, we see that

$$\forall g\in G,\ {}^{g}\vec{G}^{+}_{\iota,x}=\vec{G}^{+}_{g\iota,gx} \text{ and } {}^{g}e^{\phi}_{\iota,x}=e^{\phi}_{g\iota,gx}, \tag{2.4.10}$$

where $g\iota$ is a short notation for $\mathrm{Int}_g\circ\iota$. Note that $\vec{G}^{+}_{\iota,x}$ also depends on ϕ, and we write $\vec{G}^{\phi,+}_{\iota,x}$ whenever we want to emphasize this dependence.

2.5 Intertwining

We keep the data ϕ, $\iota\in I_\phi$, and $x\in\mathcal{B}_\iota$ of the previous section and we now introduce the pro-p-group

$$\vec{G}_{\iota,x} := G^{0}_{\iota,x,0+}G^{1}_{\iota,x,(r_0/2)}\cdots G^{d}_{\iota,x,(r_{d-1}/2)},$$

which normalizes $\vec{G}^{+}_{\iota,x}$. We will write $\vec{G}^{\phi}_{\iota,x}$ when we want to emphasize the dependence on ϕ. The aim of this section is to prove the following result.

2.5.1 *Proposition. — i) The group $\vec{G}^{\phi}_{\iota,x}$ centralizes $e^{\phi}_{\iota,x}$.*
ii) If (ϕ',ι',x') is another triple of the same nature, then

$$e^{\phi}_{\iota,x}e^{\phi'}_{\iota',x'}\neq 0 \Rightarrow \left(\phi\simeq\phi' \text{ and } \vec{G}^{\phi}_{\iota,x}\cdot\iota\cap\vec{G}^{\phi'}_{\iota',x'}\cdot\iota'\neq\emptyset\right). \tag{2.5.2}$$

On the left hand side of (2.5.2), $e^{\phi}_{\iota,x}e^{\phi'}_{\iota',x'}$ is a product of distributions in $\mathcal{H}_R(G)$. Note that $e^{\phi}_{\iota,x}e^{\phi'}_{\iota',x'}\neq 0$ if and only if $(\check{\phi}^{+}_{\iota,x})_{|\vec{G}^{+}_{\iota,x}\cap\vec{G}^{+}_{\iota',x'}} = (\check{\phi}'^{+}_{\iota',x'})_{|\vec{G}^{+}_{\iota,x}\cap\vec{G}^{+}_{\iota',x'}}$. On the right hand side, $\vec{G}^{\phi}_{\iota,x}\cdot\iota$ denotes the $\vec{G}^{\phi}_{\iota,x}$-orbit of the embedding ι inside I_ϕ. Under the equivalence $\phi\simeq\phi'$, we have $\mathbf{S}_\phi=\mathbf{S}_{\phi'}$ and $I_\phi=I_{\phi'}$, so that the intersection makes sense and is taken inside I_ϕ.

The outline of the proof is the following : thanks to a lemma of Kaletha we prove that the characters ψ_i of 2.4.5 iii) are generic of depth r_i, in the sense of Yu in [21, §9]. This generic condition is precisely what allows to control the intertwining as in ii).

2.5.3 *Strata and intertwining.* — We denote by $\mathfrak{g}^*$ the dual of the Lie algebra $\mathfrak{g}$. In order to simplify the notation we merely write $\mathfrak{g}$ for $\mathfrak{g}(F)$ and $\mathfrak{g}^*$ for $\mathfrak{g}^*(F)$ if there is no ambiguity. if $\mathcal{L}$ is any lattice in $\mathfrak{g}$, we put $\mathcal{L}^\bullet = \{f \in \mathfrak{g}^*, \langle f, \mathcal{L}\rangle \subset (F)_{0+}\}$. Then, following Moy and Prasad, we write $\mathfrak{g}^*_{x,-r} := (\mathfrak{g}_{x,r+})^\bullet$.

Let us fix a character $\Psi : F \longrightarrow R^\times$ of depth 0, and recall Adler's version of the Moy-Prasad isomorphism $\varphi_{x,r+} : \mathfrak{g}_{x,(r/2)+}/\mathfrak{g}_{x,r+} \xrightarrow{\sim} G_{x,(r/2)+}/G_{x,r+}$ from [1, 1.6.6]. Any group J between $G_{x,r+}$ and $G_{x,(r/2)+}$ corresponds to a lattice $\mathfrak{j}$ between $\mathfrak{g}_{x,r+}$ and $\mathfrak{g}_{x,(r/2)+}$. A character ψ of J is said to be *realized* by an element $X \in \mathfrak{g}^*_{x,-r}$ if we have $\psi(h) = \Psi(\langle X, \varphi^{-1}_{x,r+}(h)\rangle)$ for all $h \in J$. Such an X is not uniquely determined by ψ, but the stratum $X + \mathfrak{j}^\bullet$ is. The following result is certainly well known to the specialists, but we couldn't find a reference in this generality.

Lemma. — Let x, r, J, ψ be as above, and let x', r', J', ψ' be an other tuple of the same nature. Suppose that ψ is realized by some $X \in \mathfrak{g}^*_{x,-r}$ and that ψ' is realized by some $X' \in \mathfrak{g}^*_{x',-r'}$. Then we have $\psi_{|J\cap J'} = \psi'_{|J\cap J'}$ if and only if $(X + \mathfrak{j}^\bullet) \cap (X' + \mathfrak{j}'^\bullet) \neq \emptyset$.

Proof. Let us choose an appartment that contains both x and x' and denote by $\mathbf{S}$ the corresponding maximal F-slit torus. Choose a maximal F-torus $\mathbf{T}$ that contains $\mathbf{S}$ and let E be a Galois splitting field of $\mathbf{T}$. Then Adler defines in [1, §1.5] a "mock" exponential map $\varphi_T : \mathfrak{u} \subset \mathfrak{g}(E) \longrightarrow \mathbf{G}(E)$ where $\mathfrak{u}$ is an open subset of $\mathfrak{g}(E)$ that contains all $\mathfrak{g}_{y,0+}$ for y in $A(\mathbf{T}, E)$. It is an homeomorphism onto an open subset U, and the restriction to each $\mathfrak{g}_{y,r+}$ is an homeomorphism onto $G_{y,r+}$. Now, by definition, both $\varphi_{x,r+}$ and $\varphi_{x',r'+}$ are induced by restriction from φ_T. It follows that $\varphi_T^{-1}(J) = \mathfrak{j}$ and, since $\Psi(\langle X, \mathfrak{g}_{x,r+}\rangle) = 1$, that $\psi(j) = \Psi(\langle X, \varphi_T^{-1}(j)\rangle)$ for all $j \in J$. Similarly $\varphi_T^{-1}(J') = \mathfrak{j}'$ and $\psi(j') = \Psi(\langle X', \varphi_T^{-1}(j')\rangle)$ for all $j' \in J'$. Therefore, we have $\psi_{|J\cap J'} = \psi'_{|J\cap J'}$ if and only if $\Psi(\langle X, Y\rangle) = \Psi(\langle X', Y\rangle)$ for all $Y \in \mathfrak{j} \cap \mathfrak{j}'$. This is equivalent to $X - X' \in (\mathfrak{j} \cap \mathfrak{j}')^\bullet = \mathfrak{j}^\bullet + \mathfrak{j}'^\bullet$, which in turn is equivalent to $(X + \mathfrak{j}^\bullet) \cap (X' + \mathfrak{j}'^\bullet) \neq \emptyset$. $\square$

2.5.4 *Generic elements.* — Following Yu, we denote by $\mathfrak{z}^*$ the image of $(\mathfrak{g}_{\mathrm{ab}})^*$ in $\mathfrak{g}^*$, which is also the space of invariants under the coadjoint action of G. Since $\mathbf{G}_{\mathrm{ab}}$ is a torus, there is a canonical filtration $(\mathfrak{z}^*_r)_{r\in\mathbb{R}}$. It satisfies $\mathfrak{z}^*_{-r} = \mathfrak{g}^*_{x,-r} \cap \mathfrak{z}^*$ for all x.

Now let $\mathbf{L}$ be a tamely ramified twisted Levi subgroup of $\mathbf{G}$. We may identify $\mathfrak{l}^* = \mathrm{Lie}^*(\mathbf{L})$ with the weight 0 subspace of $\mathfrak{g}^*$ for the coadjoint action of the connected center of $\mathbf{L}$. Then also $\mathfrak{z}^*_L$ becomes a subspace of $\mathfrak{g}^*$.

An element $X \in \mathfrak{z}^*_{\mathbf{L},-r}$ is called $\mathbf{G}$-*generic*[2] if $X \notin \mathfrak{z}^*_{\mathbf{L},-r+}$ and for some (equivalently, any) maximal torus $\mathbf{S} \subset \mathbf{L}$ and any $\alpha \in \Sigma(\mathbf{S}, \mathbf{G}) \setminus \Sigma(\mathbf{S}, \mathbf{L})$ we have $v(\langle X, H_\alpha\rangle) = -r$, where E is a splitting field of $\mathbf{S}$, the valuation v extends that of F, and $H_\alpha \in \mathfrak{s}(E)$ is the canonical element associated to α. Note that we do not exclude the case $\mathbf{L} = \mathbf{G}$, where only the first condition $X \notin \mathfrak{z}^*_{\mathbf{L},-r+}$ is non-empty.

For $x \in \mathcal{B}(\mathbf{L}, F)$ define $\mathfrak{j}_{x,r} := \mathfrak{l}_{x,r} \oplus \mathfrak{n}_{x,r/2}$ and $\mathfrak{j}^+_{x,r} := \mathfrak{l}_{x,r} \oplus \mathfrak{n}_{x,(r/2)+}$. Here $\mathfrak{n}$ denotes the sum of the non invariant eigenspaces in $\mathfrak{g}$ under the adjoint action of the center of $\mathbf{L}$. These lattices correspond to subgroups $J_{x,r}$ and $J^+_{x,r}$ between $G_{x,r}$ and $G_{x,r/2}$. With Yu's notation in [21] we would write $J_{x,r} = (L, G)_{x,(r,r/2)}$ and $J^+_{x,r} = (L, G)_{x,(r,r/2+)}$. As in the previous paragraph, the element X defines a character ψ_x of $J^+_{x,r}$ that is trivial on $G_{x,r+}$. Moreover this character is

[2] Here we only recall condition GE1 of [21, §8] since under our hypothesis (H1) and (H2), [21, Lemma 8.1] show that condition GE2 is implied by GE1

centralized by $J_{x,r}$ since $[J_{x,r}, J_{x,r}^+] \subset G_{x,r+}$. The following lemma follows from an adaptation to our setting of Yu's arguments in [21, §8].

Lemma. — Let $(\mathbf{L}, X, r, x)$ be as above and let $(\mathbf{L}', X', r', x')$ be another tuple of the same nature (so in particular X' is generic). If $(X + (\mathfrak{j}_{x,r}^+)^\bullet) \cap (X' + (\mathfrak{j}_{x',r'}^+)^\bullet) \neq \emptyset$, then $r = r'$ and there are $g \in J_{x,r}$ and $g' \in J_{x',r'}$ such that ${}^g\mathbf{L} = {}^{g'}\mathbf{L}'$ and ${}^gX - {}^{g'}X' \in \mathfrak{z}^*_{{}^g\mathbf{L},-r+}$.

Proof. Let us first show that $r = r'$. Indeed, since $(\mathfrak{j}_{x,r}^+)^\bullet \subset \mathfrak{g}^*_{x,-r+}$, the strata $X + \mathfrak{g}^*_{x,-r+}$ and $X' + \mathfrak{g}_{x',-r'+}$ have a non empty intersection. But r is the *depth* $d(X)$ of the non nilpotent element X in the sense of [2, §3.3], and by [2, Lemma 3.3.7], this depth function is constant on the coset $X + \mathfrak{g}^*_{x,-r+}$. Similarly, the depth function is constant equal to r' on the coset $X' + \mathfrak{g}^*_{x',-r'+}$ hence we get that $r = r'$.

Suppose now that $\mathbf{L} = \mathbf{L}' = \mathbf{G}$. Then we have $X, X' \in \mathfrak{z}^*_{-r}$ and $(X - X' + \mathfrak{g}^*_{x,-r+}) \cap \mathfrak{g}^*_{x',-r+} \neq \emptyset$. This non-empty intersection implies that $d(X - X') > -r$, hence $X - X' \in \mathfrak{z}^*_{-r+}$.

Let us return to the general case. The intersection $(X + (\mathfrak{j}_{x,r}^+)^\bullet) \cap (X' + (\mathfrak{j}_{x',r}^+)^\bullet)$ is open in $\mathfrak{g}^*$. Since it is assumed to be non empty, it contains a regular semi-simple element Y, so that the centralizer $\mathbf{S}$ of Y is a maximal torus of $\mathbf{G}$. Now Lemma 8.6 of [21] provides us with an element $g \in G_{x,r/2}$ such that ${}^{g^{-1}}Y \in X + \mathfrak{l}^*_{x,-r+}$. It follows that $\mathbf{S} \subset {}^g\mathbf{L}$ and that we have $v(\langle Y, H_\alpha\rangle) \geqslant -r$ for $\alpha \in \Sigma(\mathbf{S}, \mathbf{G})$, with equality if and only if $\alpha \notin \Sigma(\mathbf{S}, {}^g\mathbf{L})$. Similarly, there is an element $g' \in G_{x',r/2}$ such that ${}^{g'^{-1}}Y \in X' + \mathfrak{l}'^*_{x',-r+}$ and it follows that $\mathbf{S} \subset {}^{g'}\mathbf{L}'$ and that we have $v(\langle Y, H_\alpha\rangle) = -r$ if and only if $\alpha \notin \Sigma(\mathbf{S}, {}^{g'}\mathbf{L}')$.

We thus obtain the equality $\Sigma(\mathbf{S}, {}^g\mathbf{L}) = \Sigma(\mathbf{S}, {}^{g'}\mathbf{L}')$. This implies ${}^g\mathbf{L} = {}^{g'}\mathbf{L}'$, and also $({}^gX + {}^g\mathfrak{l}^*{}_{x,-r+}) \cap ({}^{g'}X' + {}^g\mathfrak{l}^*{}_{x',-r+}) \neq \emptyset$. This situation is similar to the case $\mathbf{L} = \mathbf{L}' = \mathbf{G}$ treated above hence we conclude that ${}^gX - {}^{g'}X' \in \mathfrak{z}^*_{{}^g\mathbf{L},-r+}$. □

Let us resume the setting associated to (ϕ, ι). Taking invariants under the coadjoint action of $\iota(\mathbf{S}_{\phi,r_i})$ we get an increasing sequences of subspaces $\mathfrak{g}_\iota^{0*} \subset \cdots \subset \mathfrak{g}_\iota^{d*} = \mathfrak{g}^*$ whose Moy-Prasad filtrations are induced from those of $\mathfrak{g}^*$. We also denote by $\mathfrak{z}_\iota^{i*}$ the image of $(\mathfrak{g}^i_{\iota,\mathrm{ab}})^*$ in $\mathfrak{g}_\iota^{i*}$ and in $\mathfrak{g}^*$. Finally, recall the character ψ_i of $\mathbf{G}^i_\iota(F)$ defined in 2.4.5 for each $i = 0, \cdots, d-1$.

2.5.5 *Lemma. — There is a $\mathbf{G}_\iota^{i+1}$-generic element $X_i \in \mathfrak{z}^{i*}_{\iota,-r_i}$ that represents $\psi_{i|G^i_{\iota,x,r_i}}$ for all $x \in \mathcal{B}(\mathbf{G}^i_\iota, F)$. (In particular ψ_i has depth r_i).*

Proof. Let $\mathbf{S}$ be a maximal F-torus of $\mathbf{G}_\iota$ split by some tamely ramified Galois extension E. By Lemma 2.3.2, for all roots α in $\Sigma(\mathbf{S}, \mathbf{G}_\iota^{i+1}) \setminus \Sigma(\mathbf{S}, \mathbf{G}^i_\iota)$ we have $\check{\varphi}_0(N_{E|F}(\alpha^\vee(E^\times_{r_i}))) \neq \{1\}$. We note that $\check{\varphi}_0(N_{E|F}(\alpha^\vee(E^\times_{r_i}))) = \check{\varphi}_i(N_{E|F}(\alpha^\vee(E^\times_{r_i})))$. Indeed, this follows from the fact that, if x is any point in $\mathcal{B}(\mathbf{G}_\iota, F) \cap A(\mathbf{S}, E)$, we have $N_{E|F}(\alpha^\vee(E^\times_{r_i})) \subset \mathbf{G}_\iota(F)_{x,r_i}$ and, by 2.4.5 ii), $\check{\varphi}_{0|\mathbf{G}_\iota(F)_{x,r_i}} = \check{\varphi}_{i|\mathbf{G}_\iota(F)_{x,r_i}}$ (since $r_i > r_{i-1}+$). On the other hand we certainly have $\check{\varphi}_{i+1}(N_{E|F}(\alpha^\vee(E^\times_{r_i}))) = \{1\}$ for such a root, since $N_{E|F}(\alpha^\vee(E^\times)) \subset (\mathbf{G}_\iota^{i+1})_{\mathrm{sc}}(F)$. It follows that the character ψ_i of $\mathbf{G}^i_\iota(F)$ has depth r_i and the hypothesis of [11, Lemma 3.7.5] are satisfied. This lemma asserts that ψ_i is $\mathbf{G}_\iota^{i+1}$-generic in the sense recalled above. More precisely, when $\mathbf{G}_{\mathrm{der}}$ is simply connected, the argument there provides directly an element X_i as desired. In general, (but still assuming (H1) and (H2)), the argument there together with [11, Lemma 3.5.2] shows that for each point x, there is some $\mathbf{G}_\iota^{i+1}$-generic $X_i(x) \in \mathfrak{z}^{i*}_{-r_i}$ that represents $\psi_{i|G^i_{\iota,x,r_i}}$. However it follows from [10, Lemma 2.51] that we may choose X_i uniformly. □

For $i = 1, \cdots, d$ and $x \in \mathcal{B}_\iota$, let us introduce the open compact subgroup $J^i_{\iota,x}$ of G^i_ι that lies in between $G^i_{\iota,x,r_{i-1}}$ and $G^i_{\iota,x,r_{i-1}/2}$ and corresponds to the lattice

$$\mathfrak{j}^i_{\iota,x} = \mathfrak{g}^{i-1}_{\iota,x,r_{i-1}} \oplus (\mathfrak{n}^{i-1}_\iota \cap \mathfrak{g}^i_\iota)_{x,r_{i-1}/2}$$

of $\mathfrak{g}^i_\iota$ through the Moy-Prasad isomorphism. This is the group $J^i_{\iota,x} = (G^{i-1}_\iota, G^i_\iota)_{x,r_{i-1},r_{i-1}/2}$ in Yu's notation. Similarly we get $J^{i+}_{\iota,x}$ by replacing $r_{i-1}/2$ by $r_{i-1}/2+$.

Recall the character $\psi^+_{i,x}$ of the group $G^i_{\iota,x,0+}G_{x,r_i/2+}$ defined in 2.4.7. By construction, the restriction $(\psi^+_{i-1,x})_{|J^{i+}_{\iota,x}}$ is represented by the element X_{i-1} provided by Lemma 2.5.5.

2.5.6 *Proof of Proposition 2.5.1.* — i) We have $\vec{G}_{\iota,x} = G^0_{\iota,x,0+}\prod_i J^i_{\iota,x}$, so it is enough to prove that for each i and j, the group $J^i_{\iota,x}$ centralizes the character $\psi^+_{j,x}$. When $j \neq i-1$ this is immediate since $J^i_{\iota,x}$ is contained in the group $G^j_{\iota,x,0+}G_{\iota,x,r_j/2+}$ where $\psi^+_{j,x}$ is defined. When $j = i-1$, this follows from $[J^i_{\iota,x}, G^{i-1}_{\iota,0+}G_{r_{i-1}/2+}] \subset G_{r_{i-1}+} \subset \ker(\psi^+_{i-1,x})$.

ii) We will prove (2.5.2) by an inductive argument.

We start with two triples (ϕ, ι, x) and (ϕ', ι', x') such that

$$(\check{\phi}_{\iota,x})_{|\vec{G}^+_{\iota,x}\cap\vec{G}^+_{\iota',x'}} = (\check{\phi}'_{\iota',x'})_{|\vec{G}^+_{\iota,x}\cap\vec{G}^+_{\iota',x'}}. \tag{2.5.7}$$

Here we lighten the notation by omitting the exponent $+$ of $\check{\phi}^+_{\iota,x}$. We will decorate with a symbol $'$ all objects pertaining to the triple (ϕ', ι', x'). In particular the jumps of the filtration $\mathbf{S}_{\phi',r}$ are denoted by $r'_0, \cdots, r'_{d'}$. Recall that $r'_{d'}$ is the depth of the character ϕ'.

We first reduce to the case where $r_{d-1} = r_d$. Indeed, it suffices to replace ϕ by $\phi \cdot (\varphi_d^{-1})_{|P_F}$ and ϕ' by $\phi' \cdot (\varphi_d^{-1})_{|P_F}$. This operation does not affect (2.5.7) and does not change $\vec{G}_{\iota,x}$ nor $\vec{G}_{\iota,x}$. We are thus left to prove the conclusion of (2.5.2) for these new ϕ and ϕ'. We now have $r_d = r_{d-1}$ as desired, but $r'_{d'-1}$ and $r'_{d'}$ might be distinct a priori.

Since $r_d = r_{d-1}$, the character $\check{\phi}_{\iota,x}$ is trivial on $G_{x,r_{d-1}+}$ and $(\check{\phi}_{\iota,x})_{|J^{d+}_{\iota,x}} = (\psi^+_{d-1,x})_{|J^{d+}_{\iota,x}}$ is represented by the generic element X_{d-1} of Proposition 2.5.5. On the other hand, we have a priori two possibilities for $\check{\phi}'_{\iota',x'}$:

- either $r'_{d'} > r'_{d'-1}$ and $(\check{\phi}'_{\iota',x'})_{|G_{x',r'_{d'}}}$ is represented by some generic element $X'_{d'} \in \mathfrak{z}^*_{-r'_{d'}}$.
- or $r'_{d'} = r'_{d'-1}$ and $(\check{\phi}'_{\iota',x'})_{|J^{d'+}_{\iota',x'}}$ is represented by $X'_{d'-1}$ as provided by Proposition 2.5.5.

The first case is actually impossible. Indeed by (2.5.7) the characters $\check{\phi}_{\iota,x}$ and $\check{\phi}'_{\iota',x'}$ coincide on $J^{d+}_{\iota,x} \cap G_{x',r'_{d'}}$. So in the setting of the first case, Lemma 2.5.3 and Lemma 2.5.4 imply that $\mathbf{G}^{d-1}_\iota$ is conjugate to $\mathbf{G}^{d'}_{\iota'} = \mathbf{G}$ which is absurd.

So we are in the second case, and by (2.5.7) the characters $\check{\phi}_{\iota,x}$ and $\check{\phi}'_{\iota',x'}$ coincide on $J^{d+}_{\iota,x} \cap J^{d'+}_{\iota',x'}$. Then, Lemma 2.5.3 and Lemma 2.5.4 tell us that $r_{d-1} = r'_{d'-1}$ and provide elements $j \in J^d_{\iota,x}$ and $j' \in J^{d'}_{\iota',x'}$ such that ${}^j\mathbf{G}^{d-1}_\iota = {}^{j'}\mathbf{G}^{d'-1}_{\iota'}$. Note that ${}^j\mathbf{G}^{d-1}_\iota = \mathbf{G}^{d-1}_{j\iota}$. Since $j \in \vec{G}^\phi_{\iota,x}$, statement i) of the proposition and (2.4.10) show that $e^\phi_{\iota,x} = e^\phi_{j\iota,x}$. Therefore it is sufficient to prove the conclusion of (2.5.2) for the triples $(\phi, j\iota, x)$ and $(\phi', j'\iota', x')$. In other words, we may and will assume that $\mathbf{G}^{d-1}_\iota = \mathbf{G}^{d'-1}_{\iota'}$.

Let us put $\mathbf{H} := \mathbf{G}_\iota^{d-1} = \mathbf{G}_{\iota'}^{d'-1}$. The admissible parameter $\hat{\phi} : P_F \longrightarrow \hat{\mathbf{G}}$ factors through $\hat{\mathbf{H}}$, giving an admissible parameter of $\mathbf{H}$ denoted by $\phi^{|H}$ (note that an extension of ϕ to a Langlands parameter of $\mathbf{H}$ is provided by φ_{d-1}). Its associated F-torus $\mathbf{S}_{\phi^{|H}}$ is equal to $\mathbf{S}_\phi$ and the Levi-center embedding $\iota : \mathbf{S}_\phi \hookrightarrow \mathbf{G}$ factors through $\mathbf{H}$. Moreover, the associated set $\mathcal{B}_\iota \subset \mathcal{B}(\mathbf{H}, F)$ is the same as the one considered sofar. We thus get a triple $(\phi^{|H}, \iota, x)$ pertaining to $\mathbf{H}$, whence groups $\vec{H}_{\iota,x}$ and $\vec{H}_{\iota,x}^+$ and a character $\check{\phi}_{\iota,x}^{|H}$. Actually we simply have

$$\vec{H}_{\iota,x}^+ = G_{\iota,x,0+}^0 \cdots G_{\iota,x,(r_{d-2}/2)+}^{d-1} \subset \vec{G}_{\iota,x}^+ \text{ and } \check{\phi}_{\iota,x}^{|H} = (\check{\phi}_{\iota,x})_{|\vec{H}_{\iota,x}^+}.$$

Similarly we have a triple $({\phi'}^{|H}, \iota', x')$ pertaining to $\mathbf{H}$, groups $\vec{H}_{\iota',x'} \subset \vec{G}_{\iota',x'}$ and $\vec{H}_{\iota',x'}^+ \subset \vec{G}_{\iota',x'}^+$, as well as a character $\check{\phi}'^{|H}_{\iota',x'}$ of $\vec{H}_{\iota',x'}^+$ which coincides with the restriction of $\check{\phi}'_{\iota',x'}$. In particular (2.5.7) implies that the characters $\check{\phi}_{\iota,x}^{|H}$ and $\check{\phi}'^{|H}_{\iota',x'}$ coincide on the intersection $\vec{H}_{\iota,x}^+ \cap \vec{H}_{\iota',x'}^+$. Suppose now that the conclusion of (2.5.2) is known for the triples $(\phi^{|H}, \iota, x)$ and $({\phi'}^{|H}, \iota', x')$. It then implies that the same conclusion holds for the triples (ϕ, ι, x) and (ϕ', ι', x').

It follows that in order to finish the proof of (2.5.2), we may argue by induction, for example on the number $n(\phi, \mathbf{G}) = \dim(\mathbf{S}_\phi) - \dim(Z(\mathbf{G}))$. It remains however to initiate the induction process by considering the case $n(\phi, \mathbf{G}) = 0$. In this case we have $d = 0$, $\vec{G}_{\iota,x} = G_{x,0+}$ and $\check{\phi}_{\iota,x}$ is the restriction of a character $\check{\varphi}_0$ of G. As we have done above, we may multiply both ϕ and ϕ' by $(\check{\varphi}_0^{-1})_{|P_F}$ so as to get ϕ trivial. Then we need to show that ϕ' is trivial too, or equivalently that it has depth $r'_{d'} = 0$. However if ϕ' had depth $r'_{d'} > 0$, then $(\check{\phi}'_{\iota',x'})_{|G_{x',r'_{d'}}}$ would be represented by a generic element $X'_{d'}$ of depth $-r'_{d'}$. Since the depth function of [2] is constant on the stratum $X'_{d'} + \mathfrak{g}^*_{x',-r'_{d'}+}$, the latter cannot intersect the stratum $\mathfrak{g}^*_{x,0+}$, hence by Lemma 2.5.3 we would get a contradiction with (2.5.2). $\square$

2.5.8 *A weak converse to (2.5.2).* — Fix ϕ, $\iota \in I_\phi$ and two points $x, x' \in \mathcal{B}_\iota$. For $r \in \tilde{\mathbb{R}}_{>0}$ we put $G_{x,x',r} := G_{x,r} \cap G_{x',r}$ and $G_{\iota,x,x',r}^i = G_\iota^i \cap G_{x,x',r} = G_{\iota,x,r}^i \cap G_{\iota,x',r}^i$. Since $G_{\iota,x,x',(r_{i-1}/2)+}^i$ normalizes $G_{\iota,x,x',(r_{j-1}/2)+}^j$ for $i \leqslant j$ we may consider the group

$$\vec{G}_{\iota,x,x'}^+ := G_{\iota,x,x',0+}^0 G_{\iota,x,x',(r_0/2)+}^1 \cdots G_{\iota,x,x',(r_{d-1}/2)+}^d,$$

which is contained in $\vec{G}_{\iota,x}^+ \cap \vec{G}_{\iota,x'}^+$.

Lemma. — We have $(\check{\phi}_{\iota,x}^+)_{|\vec{G}_{\iota,x,x'}^+} = (\check{\phi}_{\iota,x'}^+)_{|\vec{G}_{\iota,x,x'}^+}$.

Proof. With the notation of 2.4.7, it suffices to show that for each $i = 0, \cdots, d$ we have $(\psi_{i,x}^+)_{|G_{\iota,x,x',0+}^i G_{x,x',(r_i/2)+}} = (\psi_{i,x'}^+)_{|G_{\iota,x,x',0+}^i G_{x,x',(r_i/2)+}}$. On one hand we have $(\psi_{i,x}^+)_{|G_{\iota,x,x',0+}^i} = (\psi_i)_{|G_{\iota,x,x',0+}^i} = (\psi_{i,x'}^+)_{|G_{\iota,x,x',0+}^i}$. On the other hand, both $(\psi_{i,x}^+)_{|G_{x,x',(r_i/2)+}}$ and $(\psi_{i,x'}^+)_{|G_{x,x',(r_i/2)+}}$ are obtained from $(\psi_i)_{|G_{\iota,x,x',(r_i/2)+}^i}$ via the Moy-Prasad-Adler isomorphism $\mathfrak{g}_{x,x',(r_i/2)+:r_i+} \xrightarrow{\sim} G_{x,x',(r_i/2)+:r_i+}$ (induced by φ_T as in the proof of Lemma 2.5.3) by trivially extending characters along the decomposition

$$\mathfrak{g}_{x,x',(r_i/2)+:r_i+} = \mathfrak{g}_{\iota,x,x',(r_i/2)+:r_i+}^i \oplus \mathfrak{n}_{\iota,x,x',(r_i/2)+:r_i+}^i.$$

$\square$

2.5.9 *The Heisenberg property.* — By [21, Lemma 1.3], we know that the quotient group $\vec{G}_{\iota,x}/\vec{G}^+_{\iota,x}$ is abelian, hence the derived subgroup $[\vec{G}_{\iota,x}, \vec{G}_{\iota,x}]$ is contained in $\vec{G}^+_{\iota,x}$ and we have a map

$$\vec{G}_{\iota,x} \times \vec{G}_{\iota,x} \longrightarrow \mu_{p^\infty},\ (g,h) \mapsto \check{\phi}^+_{\iota,x}(ghg^{-1}h^{-1})$$

Since $\vec{G}_{\iota,x}$ centralizes the character $\check{\phi}^+_{\iota,x}$, this map descends to a map

$$\theta :\ \vec{G}_{\iota,x}/\vec{G}^+_{\iota,x} \times \vec{G}_{\iota,x}/\vec{G}^+_{\iota,x} \longrightarrow \mu_{p^\infty}.$$

Proposition. — *The group $\vec{G}_{\iota,x}/\vec{G}^+_{\iota,x}$ has exponent p and the map θ defines a perfect anti-symmetric pairing on this group, taking values in μ_p.*

Proof. By [21, Lemma 1.3], we know that $\vec{G}_{\iota,x}/\vec{G}^+_{\iota,x}$ is isomorphic to its Lie algebra counterpart $\vec{\mathfrak{g}}_{\iota,x}/\vec{\mathfrak{g}}^+_{\iota,x}$. By construction, the latter decomposes as

$$\vec{\mathfrak{g}}_{\iota,x}/\vec{\mathfrak{g}}^+_{\iota,x} = \bigoplus_{i=1}^{d} (\mathfrak{g}^i_\iota \cap \mathfrak{n}^{i-1}_\iota)_{x,(r_{i-1}/2):(r_{i-1}/2)+}$$

which is a direct sum of vector spaces over the residue field of F, hence has exponent p. The equality $[g, hh'] = [g,h] \cdot {}^h[g,h']$ shows that the map θ is $\mathbb{Z}$-bilinear, hence the image of this map is contained in the only subgroup μ_p of μ_{p^∞} of exponent p. Note that in the above decomposition of $\vec{\mathfrak{g}}_{\iota,x}/\vec{\mathfrak{g}}^+_{\iota,x}$, the summand $(\mathfrak{g}^i_\iota \cap \mathfrak{n}^{i-1}_\iota)_{x,(r_{i-1}/2):(r_{i-1}/2)+} = \mathfrak{j}^i_{\iota,x}/\mathfrak{j}^{i+}_{\iota,x}$ identifies to the image in $\vec{G}_{\iota,x}/\vec{G}^+_{\iota,x}$ of the subgroup $J^i_{\iota,x}$, so that another way to write this decomposition is

$$\vec{G}_{\iota,x}/\vec{G}^+_{\iota,x} = \prod_{i=1}^{d} \left(J^i_{\iota,x}/J^{i+}_{\iota,x}\right).$$

Now, recall the factorization $\check{\phi}^+_{\iota,x} = \prod_{k=0}^{d} (\psi^+_{k,x})_{|\vec{G}^+_{\iota,x}}$ of 2.4.7 and let $i,j \in \{1,\cdots,d\}$ and $k \in \{0,\cdots,d\}$. We claim that $[J^i_{\iota,x}, J^j_{\iota,x}] \subset \ker \psi^+_{k,x}$ unless $i = j = k+1$. When $k+1 \neq i,j$, this follows from the fact that both $J^i_{\iota,x}$ and $J^j_{\iota,x}$ are contained in the group $G^k_{\iota,0+}G_{r_k/2+}$ where $\psi^+_{k,x}$ is defined. When $k+1 = i$ and $i \neq j$, this follows from the inclusion $[J^i_{\iota,x}, J^j_{\iota,x}] \subset [J^i_{\iota,x}, G^k_{\iota,0+}G_{r_k/2+}] \subset G_{r_k+} \subset \ker(\psi^+_{k,x})$, and similarily for $k+1 = j \neq i$.

As a consequence, the last displayed decomposition is orthogonal for the bilinear form θ, and the restriction of θ to the summand $J^i_{\iota,x}/J^{i+}_{\iota,x}$ is given by $\theta(\bar{j},\bar{h}) = \psi^+_{i-1,x}(jhj^{-1}h^{-1})$. By [21, Lemma 11.1] the latter bilinear form on $J^i_{\iota,x}/J^{i+}_{\iota,x}$ is non-degenerate, hence so is θ. □

2.6 Systems of idempotents

In this section we fix a parameter ϕ and a $\mathbf{G}(F)$-conjugacy class $I \subset I_\phi$ of F-rational embeddings $\mathbf{S}_\phi \hookrightarrow \mathbf{G}$. For $\iota \in I$ and $x \in \mathcal{B}_\iota$ we simply write $e_{\iota,x}$ for $e^\phi_{\iota,x}$.

2.6.1 *The idempotents associated to ϕ and I.* — For $x \in \mathcal{B}(\mathbf{G}, F)$, we put $I_x := \{\iota \in I, x \in \mathcal{B}_\iota\}$. If $\iota, \iota' \in I_x$, we declare that $\iota' \sim_x \iota$ if $\iota' \in \vec{G}_{\iota,x} \cdot \iota$. Using (2.4.10) we see that if $\iota' \in \vec{G}_{\iota,x} \cdot \iota$, we have $\vec{G}_{\iota',x} = \vec{G}_{\iota,x}$, whence the transitivity and symmetry of the relation $\sim_x$, which is thus an equivalence relation on I_x.

Lemma. — *For $\iota', \iota \in I_x$ we have* $e_{\iota,x} e_{\iota',x} = \begin{cases} e_{\iota,x} & \textit{if } \iota \sim_x \iota' \\ 0 & \textit{else.} \end{cases}$

Proof. Suppose that $e_{\iota,x} e_{\iota',x} \neq 0$. By ii) of Proposition 2.5.1 this implies that $\vec{G}_{\iota',x} \cdot \iota' \cap \vec{G}_{\iota,x} \cdot \iota \neq \emptyset$ hence it follows that $\iota' \sim_x \iota$. Then, by (2.4.10) and i) of Proposition 2.5.1, we have $e_{\iota',x} = e_{\iota,x}$. □

Since there is a finite number of idempotents $e \in \mathcal{H}_R(G)$ that are supported on G_x and such that $e \cdot e_{G_{x,r_\phi+}} = e$, the lemma shows that the sum

$$e_x = e^{\phi}_{I,x} := \sum_{\iota \in I_x/\sim_x} e_{\iota,x}$$

is finite and defines an idempotent of $\mathcal{H}_R(G)$ supported on $G_{x,0+}$. This idempotent only depends on ϕ, I, and x. It is non zero if and only if I_x is non empty. Also we have by construction the equivariance property

(2.6.2) $$\forall x \in \mathcal{B}, \forall g \in G, e_{gx} = {}^g e_x.$$

In particular e_x is a central idempotent in $\mathcal{H}_R(G_x)$.

Lemma. — *Fix $x, x' \in \mathcal{B}$, put $I_{x,x'} := I_x \cap I_{x'}$ and endow this set with the equivalence relation $\iota \sim_{x,x'} \iota' \Leftrightarrow (\iota \sim_x \iota'$ and $\iota \sim_{x'} \iota')$. Then we have*

(2.6.3) $$e_x e_{x'} = \sum_{\iota \in I_{x,x'}/\sim_{x,x'}} e_{\iota,x} e_{\iota,x'}.$$

Proof. Denote by $\bar{\iota} := \vec{G}_{\iota,x} \cdot \iota$ the $\sim_x$-equivalence class of $\iota \in I_x$, and similarly for $\iota' \in I_{x'}$. By ii) of Proposition 2.5.1 we have $\bar{\iota} \cap \bar{\iota}' \neq \emptyset$ whenever $e_{\iota,x} e_{\iota',x'} \neq 0$. Hence $e_x e_{x'} = \sum_{\bar{\iota} \cap \bar{\iota}' \neq \emptyset} e_{\bar{\iota},x} e_{\bar{\iota}',x'}$. Now, the intersection $\bar{\iota} \cap \bar{\iota}'$ in I is contained in $I_{x,x'}$ and, if it is non-empty, it is actually a $\sim_{x,x'}$-equivalence class. We thus have a map $(\bar{\iota}, \bar{\iota}') \mapsto \bar{\iota} \cap \bar{\iota}'$,

$$\left\{ (\bar{\iota}, \bar{\iota}') \in I_{x/\sim_x} \times I_{x'/\sim_{x'}}, \bar{\iota} \cap \bar{\iota}' \neq \emptyset \right\} \longrightarrow I_{x,x'/\sim_{x,x'}}.$$

This map is clearly surjective since the $\sim_{x,x'}$-class of $\iota_0 \in I_{x,x'}$ is the image of $(\bar{\iota}_0, \bar{\iota}'_0)$ and it is also obviously injective. □

2.6.4 *A telescopic identity.* — We aim at proving a telescopic identity when moving along a geodesic. In the following lemma, we fix $\iota \in I$ and two points $x, x' \in \mathcal{B}_\iota$. Since $\mathcal{B}_\iota$ is convex, it contains the segment $[x, x']$.

Lemma. — *Suppose $x'' \in [x, x']$. Then $e_{\iota,x} e_{\iota,x'} = e_{\iota,x} e_{\iota,x''} e_{\iota,x'}$ (product in $\mathcal{H}_R(G)$).*

Proof. As in [20, §1.27], the line (x, x') determines a pair of F-rational opposed parabolic subgroups of any of the G^i_ι's. By the associated Iwahori decomposition of $G^i_{\iota,x,r+}$ of *loc. cit.* we see that $G^i_{\iota,x'',(r_{i-1}/2)+} = G^i_{\iota,x,x'',(r_{i-1}/2)+} G^i_{\iota,x'',x',(r_{i-1}/2)+}$ for each i (recall the notation of 2.5.8). Using the fact that $G^j_{\iota,x'',(r_{j-1}/2)+}$ is normalized by $G^i_{\iota,x,x'',(r_{i-1}/2)+}$ and $G^i_{\iota,x'',x',(r_{i-1}/2)+}$ for $i \leqslant j$, we see by induction that

$$\begin{aligned} \vec{G}^+_{\iota,x''} &= G^0_{\iota,x,x'',0+} \cdots G^d_{\iota,x,x'',(r_{d-1}/2)+} G^d_{\iota,x'',x',(r_{d-1}/2)+} \cdots G^0_{\iota,x'',x',0+} \\ &= \vec{G}^+_{\iota,x,x''} \vec{G}^+_{\iota,x'',x'}. \end{aligned}$$

Now denote by $e_{\iota,x,x''}$ and $e_{\iota,x'',x'}$ the idempotents associated respectively to $(\check{\phi}^+_{\iota,x''})_{|\vec{G}^+_{\iota,x,x''}}$ and $(\check{\phi}^+_{\iota,x''})_{|\vec{G}^+_{\iota,x'',x'}}$. We get a factorization $e_{\iota,x''} = e_{\iota,x,x''}e_{\iota,x'',x'}$ in $\mathcal{H}_R(G_{x''})$. By Lemma 2.5.8, it follows that $e_{\iota,x}e_{\iota,x''}e_{\iota,x'} = e_{\iota,x}e_{\iota,x,x''}e_{\iota,x'',x'}e_{\iota,x'} = e_{\iota,x}e_{\iota,x'}$. □

Proposition. — Let $x, x' \in \mathcal{B}$ and $x'' \in [x, x']$. Suppose that x and x'' lie in the closure of a facet of $\mathcal{B}$. Then $e_x e_{x''} e_{x'} = e_x e_{x'}$.

Proof. Since x and x'' lie in the closure of some F-facet, $G_{x''}$ contains $G_{x,0+}$, and it follows that $e_{x''}$ commutes with all $e_{\iota,x}$ since the former is central in $\mathcal{H}_R(G_{x''})$ and the latter are supported on $G_{x,0+}$. This commutation property provides the first and fourth equalities in the following computation.

$$\begin{aligned} e_x e_{x''} e_{x'} = e_{x''} e_x e_{x'} &= e_{x''}\left(\sum_{\iota\in I_{x,x'}/\sim_{x,x'}} e_{\iota,x}e_{\iota,x'}\right) = e_{x''}\left(\sum_{\iota\in I_{x,x'}/\sim_{x,x'}} e_{\iota,x}e_{\iota,x''}e_{\iota,x'}\right) \\ &= \sum_{\iota\in I_{x,x'}/\sim_{x,x'}} e_{\iota,x}e_{x''}e_{\iota,x''}e_{\iota,x'} = \sum_{\iota\in I_{x,x'}/\sim_{x,x'}} e_{\iota,x}e_{\iota,x''}e_{\iota,x'} \\ &= \sum_{\iota\in I_{x,x'}/\sim_{x,x'}} e_{\iota,x}e_{\iota,x'} = e_x e_{x'}. \end{aligned}$$

In the second and the last equalities, we use (2.6.3). In the third and sixth equalities we have used the last lemma, and in the fith one we use Lemma 2.6.1. □

2.6.5 *E-Facets.* — We aim at finding a polysimplicial G-equivariant structure on $\mathcal{B}$ such that e_x only depends on the facet it belongs to. Simple examples show that the usual Bruhat-Tits structure on $\mathcal{B}$ will not work. Instead, we will consider the trace on $\mathcal{B}$ of the Bruhat-Tits structure on $\mathcal{B}(\mathbf{G}, E)$ for a suitable tamely ramified extension of F.

Lemma. — There is a tamely ramified Galois extension E of F that splits a maximal F-torus of $\mathbf{G}_\iota$ (for any $\iota \in I$) and such that $\{r_0/2, \cdots, r_{d-1}/2\} \subset v(E^\times)$, where v is the unique valuation on E that extends the normalized valuation of F.

Proof. Let E be a tamely ramified splitting field of some maximal F-torus $\mathbf{S}$ in $\mathbf{G}_\iota$. Lemma 2.3.2 shows that for $0 \leqslant i \leqslant d-1$, the real number r_i is a jump of the filtration on $E^\times$, hence belongs to $v(E^\times) = \frac{1}{e}\mathbb{Z}$ with e the ramification index of E/F. After replacing E by a quadratic ramified extension if necessary, we get that $r_i/2 \in v(E^\times)$ for each i, as desired. Then the Galois closure of this E meets the requirements of the lemma. □

Let E be as in the lemma. Recall that the reduced building $\mathcal{B}^{\mathrm{red}}(\mathbf{G}, E)$ carries a canonical polysimplicial structure. The inverse image in $\mathcal{B}(\mathbf{G}, E)$ of a polysimplex of $\mathcal{B}^{\mathrm{red}}(\mathbf{G}, E)$ will be called a facet of $\mathcal{B}(\mathbf{G}, E)$. Recall also [15, Thm 2.1.1] that there is a canonical embedding $\mathcal{B}(\mathbf{G}, F) \hookrightarrow \mathcal{B}(\mathbf{G}, E)$, so we may and will identify $\mathcal{B}$ to a subset of $\mathcal{B}(\mathbf{G}, E)$. There is also a canonical action of $\mathrm{Gal}(E/F)$ on $\mathcal{B}(\mathbf{G}, E)$ and by a result of Rousseau [19] we have $\mathcal{B} = \mathcal{B}(\mathbf{G}, E)^{\mathrm{Gal}(E/F)}$, because E/F is tamely ramified. The intersection with $\mathcal{B}$ of a facet of $\mathcal{B}(\mathbf{G}, E)$ will be called an "E-facet of $\mathcal{B}$". We thus get a partition of $\mathcal{B}$ into "E-facets". We will denote by $\mathcal{F}_E(x)$ the E-facet of $\mathcal{B}$ that contains x.

Proposition. — *Let E be as in the previous lemma. Then we have*

$$\forall x, x' \in \mathcal{B},\ \ \mathcal{F}_E(x) = \mathcal{F}_E(x') \Rightarrow e_x = e_{x'}. \tag{2.6.6}$$

Proof. Fix $x, x' \in \mathcal{B}$ with $\mathcal{F}_E(x) = \mathcal{F}_E(x')$.

We first show that $I_x = I_{x'}$. To this aim, it suffices to show that for any $\iota \in I_x$, we have $\mathcal{F}_E(x) \subset \mathcal{B}_\iota$. Indeed, denote by $\tilde{\mathcal{F}}_E(x)$ the facet of $\mathcal{B}(\mathbf{G}, E)$ that contains x. Since $\mathbf{G}_\iota$ is a split Levi subgroup over E, the facet $\tilde{\mathcal{F}}_E(x)$ is contained in the image $\mathcal{B}_\iota(E)$ of any toral embedding $\mathcal{B}(\mathbf{G}_\iota, E) \hookrightarrow \mathcal{B}(\mathbf{G}, E)$. Therefore $\mathcal{F}_E(x) = \tilde{\mathcal{F}}_E(x) \cap \mathcal{B} \subset \mathcal{B}_\iota(E)^{\mathrm{Gal}(E/F)} = \mathcal{B}_\iota$, as desired.

Now we are left to show that for $\iota \in I_x$ we have $e_{\iota,x} = e_{\iota,x'}$. By [20, Prop 1.1], the groups $\mathbf{G}(E)_{x,r+}$ only depend on the facet containing x provided that $r \in v(E^\times)$ (in *loc.cit.* the valuation is normalized by $v(E^\times) = \mathbb{Z}$). Recall also from [21, §2] that $G_{x,r+} = G \cap \mathbf{G}(E)_{x,r+}$. It follows that for each $i = 0, \cdots, d$ the group $G^i_{\iota,x,(r_{i-1}/2)+} = G^i_\iota \cap \mathbf{G}(E)_{x,(r_{i-1}/2)+}$ only depends on the E-facet $\mathcal{F}_E(x)$. Therefore we get $\vec{G}^+_{\iota,x} = \vec{G}^+_{\iota,x'}$. Moreover, with the notation of 2.5.8, we also have $\vec{G}^+_{\iota,x} = \vec{G}^+_{\iota,x,x'}$, so that Lemma 2.5.8 shows that $\check{\phi}^+_{\iota,x} = \check{\phi}^+_{\iota,x'}$ hence also $e_{\iota,x} = e_{\iota,x'}$. □

2.6.7 *Proposition.* — *Let E be an extension of F as in 2.6.5. Suppose x, x' are contained in the closure of an E-facet of $\mathcal{B}$. Then for any $x'' \in]x, x'[$ we have $e_x e_{x'} = e_{x''} = e_{x'} e_x$.*

Proof. Note first that in this situation we have $x, x' \in \overline{\mathcal{F}_E(x'')}$.

Let us show that $I_{x,x'} = I_{x''}$. Indeed, the inclusion $I_{x,x'} \subset I_{x''}$ follows from the convexity of $\mathcal{B}_\iota$ in $\mathcal{B}$, while the other inclusion follows from its closedness, since we have seen in the last proof that $\mathcal{F}_E(x'') \subset \mathcal{B}_\iota$ for any $\iota \in I_{x''}$.

Now, let us fix $\iota \in I_{x,x'}$. By [20, Prop 1.1], the groups $\mathbf{G}^i_\iota(E)_{x,r+}$ and $\mathbf{G}^i_\iota(E)_{x',r+}$ are contained in $\mathbf{G}^i_\iota(E)_{x'',r+}$ for each i, provided that $r \in v(E^\times)$. By taking Galois fixed elements, it follows that $G^i_{\iota,x,(r_{i-1}/2)+}$ and $G^i_{\iota,x',(r_{i-1}/2)+}$ are contained in $G^i_{\iota,x'',(r_{i-1}/2)+}$. With the notation of 2.5.8, we infer that $\vec{G}^+_{\iota,x,x''} = \vec{G}^+_{\iota,x} \subset \vec{G}^+_{\iota,x''}$ and $\vec{G}^+_{\iota,x'',x'} = \vec{G}^+_{\iota,x'} \subset \vec{G}^+_{\iota,x''}$. Lemma 2.5.8 then implies that $e_{\iota,x}e_{\iota,x''} = e_{\iota,x''}$ and $e_{\iota,x''}e_{\iota,x'} = e_{\iota,x''}$. We conclude thanks to Lemma 2.6.4 that $e_{\iota,x}e_{\iota,x'} = e_{\iota,x}e_{\iota,x''}e_{\iota,x'} = e_{\iota,x''}$. Using (2.6.3) we thus get the formula

$$e_x e_{x'} = \sum_{\iota \in I_{x''}/\sim_{x,x'}} e_{\iota,x''}, \text{ to compare with } e_{x''} = \sum_{\iota \in I_{x''}/\sim_{x''}} e_{\iota,x''}.$$

For $\iota_1, \iota_2 \in I_{x''}$, Lemma 2.6.1 tells us that

$$\begin{aligned} &\iota_1 \sim_{x''} \iota_2 \Leftrightarrow e_{\iota_1,x''} = e_{\iota_2,x''} \\ &\iota_1 \sim_{x,x'} \iota_2 \Leftrightarrow (e_{\iota_1,x} = e_{\iota_2,x} \text{ and } e_{\iota_1,x'} = e_{\iota_2,x'}) \end{aligned}$$

The equality $e_{\iota,x}e_{\iota,x'} = e_{\iota,x''}$ proved just above shows that $\iota_1 \sim_{x,x'} \iota_2 \Rightarrow \iota_1 \sim_{x''} \iota_2$. On the other hand, Lemma 2.6.1 also shows that

$$\begin{aligned} &\iota_1 \sim_{x''} \iota_2 \Leftrightarrow e_{\iota_1,x''}e_{\iota_2,x''} \neq 0 \\ &\iota_1 \sim_{x,x'} \iota_2 \Leftrightarrow (e_{\iota_1,x}e_{\iota_2,x} \neq 0 \text{ and } e_{\iota_1,x'}e_{\iota_2,x'} \neq 0) \end{aligned}$$

This time, the equalities $e_{\iota,x}e_{\iota,x'} = e_{\iota,x''} = e_{\iota,x'}e_{\iota,x}$ show that $\iota_1 \sim_{x''} \iota_2 \Rightarrow \iota_1 \sim_{x,x'} \iota_2$. □

2.7 The category $\mathrm{Rep}_R^{\phi,I}(G)$

We now construct the category attached to a parameter $\phi : P_F \longrightarrow {}^L\mathbf{G}$ and a G-conjugacy class $I \subset I_\phi$ of F-rational embeddings $\iota : \mathbf{S}_\phi \hookrightarrow \mathbf{G}$. If V is any smooth RG-module, it has an action of the Hecke algebra $\mathcal{H}_R(G)$ and in particular the idempotents $e_x = e_{I,x}^\phi$ act on it. This subsection is mainly devoted to the proof of the following theorem.

2.7.1 *Theorem. — The subcategory* $\mathrm{Rep}_R^{\phi,I}(G)$ *of* $\mathrm{Rep}_R(G)$ *defined by*

$$\mathrm{Rep}_R^{\phi,I}(G) := \left\{ V \in \mathrm{Rep}_R(G),\, V = \sum_{x\in\mathcal{B}} e_{I,x}^\phi V \right\}$$

is a Serre subcategory of $\mathrm{Rep}_R(G)$*, stable under arbitrary colimits, and generated by the following compact projective object of* $\mathrm{Rep}_R(G)$

$$P^{\phi,I} := \bigoplus_{x\in\Delta_0} \bigoplus_{\iota\in I_x/\sim_x} \mathrm{ind}_{\vec{G}_{\iota,x}^+}^G(\check{\phi}_{\iota,x}^+),$$

where Δ_0 *denotes the set of* e*-vertices of a* 1*-chamber* Δ *of* $\mathcal{B}$*. Moreover, any object* $V \in \mathrm{Rep}_R^{\phi,I}(G)$ *is canonically and functorially an extension*

$$V^{\phi,I} \hookrightarrow V \twoheadrightarrow V_{\phi,I} \tag{2.7.7}$$

where $V^{\phi,I} \in \mathrm{Rep}_R^{\phi,I}(G)$ *and* $V_{\phi,I}$ *has no subquotient that belongs to* $\mathrm{Rep}_R^{\phi,I}(G)$*.*

The strategy is of course to put ourselves in a position where we can apply [17, Thm 3.1], or at least closely follow its proof. This reference is concerned with systems of idempotents associated to vertices (more generally to polysimplices) in the reduced building $\mathcal{B}' := \mathcal{B}(\mathbf{G}_{\mathrm{ad}}, F)$, while we have constructed idempotents associated to points of $\mathcal{B}$. However our idempotent e_x only depends on the image of x in $\mathcal{B}'$, so that we actually have idempotents associated to points of $\mathcal{B}'$. Unfortunately, these idempotents are not constant on F-facets, but only on E-facets for some Galois extension E of F as in 2.6.5.

2.7.2 *The* e^{th}*-subdivision of* $\mathcal{B}'$*. —* Fix an integer $e \geqslant 1$. We define a subdivision of the polysimplicial structure on $\mathcal{B}'$ in the following way.

Start with an appartment A and define an e-wall to be an affine hyperplane of the form $\varphi^{-1}(\frac{k}{e}t_1 + \frac{e-k}{e}t_2)$ where φ is an affine root on A, $t_1, t_2 \in \mathbb{R}$ are such that $\varphi^{-1}(t_1)$ and $\varphi^{-1}(t_2)$ are walls of A, and k is an integer between 0 and e. In particular, 1-walls are the usual walls and are also e-walls for any $e \geqslant 1$, and moreover any e-wall is parallel to some 1-wall. We thus get an enlarged collection of hyperplanes, leading to a refined partition of A into facets, that we call e-facets. We note that if o is a special point of A, then the e-walls of A are the images of the walls by the homothety of ratio $1/e$ centered at o. Indeed, for any affine root φ on A, it follows from [6, (6.2.16)] that the set of all $t \in \mathbb{R}$ such that $\varphi^{-1}(t + \varphi(o))$ is a wall is a discrete subgroup of $\mathbb{R}$. As a consequence, the e-facets are the images of the usual facets by the same homothety.

If A' is another appartment, we define e-walls and e-facets in the same way. Then for any $g \in G$, the action of g on $\mathcal{B}'$ takes an e-wall of A to an e-wall of gA. In particular, if F is an

e-facet of A which intersects A', then $F \subset A'$ and F is an e-facet of A'. Indeed, there is some $g \in G$ with $A' = gA$ and such that g fixes the facet $F_1 \in \mathcal{B}'$ that contains F. This allows to define unambiguously the e-facets of $\mathcal{B}'$, and we get a polysimplicial structure on $\mathcal{B}'$ which is preserved by the action of G. We will call it the e^{th}-subdivision of $\mathcal{B}'$.

Lemma. — Le E be a Galois tamely ramified extension of F with ramification index $e = e(E/F)$. Suppose that E splits $\mathbf{G}$ and that its maximal unramified subextension E_0 quasi-splits $\mathbf{G}$. Then the e^{th}-subdivision of $\mathcal{B}'$ refines the partition of $\mathcal{B}'$ into E-facets, i.e. *any E-facet is a union of e-facets.*

Proof. Given a maximal F-split torus $\mathbf{T}$ of $\mathbf{G}$, [7, Cor. 5.1.12] insures that we can find a maximal E_0-split F-torus $\mathbf{T}_0$ that contains $\mathbf{T}$. Then the centralizer $\mathbf{S} = C_{\mathbf{G}}(\mathbf{T_0})$ of $\mathbf{T}_0$ is a maximal F-torus of $\mathbf{G}$, and is split by E. In this situation there are inclusions of appartments $A = A(\mathbf{G}, \mathbf{T}, F) \subset A_0 = A(\mathbf{G}, \mathbf{T}_0, E_0) \subset A_S = A(\mathbf{G}, \mathbf{S}, E)$ and each subspace is obtained by taking suitable Galois invariants. By [7, Thm 5.1.20 iii)], the walls of the appartment A are exactly the non-trivial intersections of A with the walls of the appartement A_0. Moreover, by [7, 4.2.4] each wall of A_0 is the intersection of A_0 with a wall of A_S. Conversely, the intersection of a wall of A_S with A_0, when non-trivial, may not be a wall of A_0 but, at least, is parallel to a wall of A. More precisely, fix an origin o which is a special point in A_0 (e.g. that comes from a Chevalley-Steinberg system as in [7, 4.2.3]) and let a be a non-divisible root of $\mathbf{T}_0$ in $\mathbf{G}$, and let $E_0 \subset E_a \subset E$ be the associated extension (denoted by L_a in *loc. cit.*). Denote by $\Gamma_a \subset \mathbb{R}$ the set of real numbers v such that $\{x \in A_0, a(x) = v\}$ is a wall of A_0. Then by [7, 4.2.21] we have $\Gamma_a = v(E_a^\times)$ (the valuation lattice of E_a) if $2a$ is not a root, and $\Gamma_a = \frac{1}{2}v(E_a^\times)$ if $2a$ is a root.

On the other hand, let $\Gamma_{a,E} \subset \mathbb{R}$ be the set of real numbers v such that $\{x \in A_0, a(x) = v\}$ is the intersection of A_0 with a wall of A_S. If $2a$ is not a root and $v \in \Gamma_{a,E}$, then there is a root α of $\mathbf{S}$ in $\mathbf{G}$ that restricts to a and such that $\{x \in A_S, \alpha(x) = v\}$ is a wall of A_S, hence $v \in v(E^\times)$. If $2a$ is a root, then either there is α as above and then $v \in v(E^\times)$, or there are α, α' as above with $\alpha + \alpha'$ a root, and $\{x \in A_S, (\alpha + \alpha')(x) = 2v\}$ is a wall of A_S, in which case $v \in \frac{1}{2}v(E)^\times$. It follows that we have $\Gamma_{a,E} = v(E^\times)$ if $2a$ is not a root, and $\Gamma_{a,E} = \frac{1}{2}v(E^\times)$ if $2a$ is a root. In any case, for all non-divisible roots a of $\mathbf{T}_0$ we have $\Gamma_{a,E} = \frac{1}{e(E/E_a)}\Gamma_a$.

Since $e = e(E/E_0)$ is a common multiple of all $e(E/E_\alpha)$, the above discussion shows that the e^{th}-subdivision of the polysimplicial structure on A_0 refines the one that comes from A_S. Since the polysimplicial structure on A_0 induces the one on A (again by Thm 5.1.20 iii) of [7]), it follows that the e^{th} subdivision of the polysimplicial structure on A refines its partition into E-facets. $\square$

For $x \in \mathcal{B}'$ we denote by $\mathcal{F}_e(x)$ the unique e-facet of $\mathcal{B}'$ that contains x. Further, we denote by $\mathcal{B}'_{d/e}$ the set of d-dimensional e-facets. For $d = 0$ we also speak of "e-vertices". A family $x_1, \cdots, x_r$ of e-vertices are called "adjacent" if they are contained in the closure of an e-facet. Then there is a unique e-facet $\mathcal{F}_e(x_1, \cdots, x_r)$ with this property and which is minimal for the order defined by $\mathcal{F}' \preceq \mathcal{F} \Leftrightarrow \overline{\mathcal{F}'} \subset \mathcal{F}$.

If x, x' are two points in $\mathcal{B}$, they are contained in a common appartment A. The intersection of all affine roots (*i.e.* half spaces associated to walls) of A that contain x and x' is known to be independent of the choice of A. It is called the "combinatorial convex hull" of x an x' and we will denote it by $\mathcal{H}_1(x, x')$. It is a union of facets. Similarly we denote by $\mathcal{H}_e(x, x')$ the intersection of all e-half spaces (corresponding to e-walls) of A that contain x and x'. This is again independent of A and a union of e-facets. Obviously $[x,x'] \subset \mathcal{H}_e(x,x') \subset \mathcal{H}_1(x,x')$.

2.7.3 *Lemma.* — *Let e be the ramification index of an extension E as in 2.6.5. Then the idempotents $(e_x)_{x\in\mathcal{B}'}$ have the following properties.*

i) for all $x,x' \in \mathcal{B}'$ we have $\mathcal{F}_e(x) = \mathcal{F}_e(x') \Rightarrow e_x = e_{x'}$.

ii) If x,x' are adjacent e-vertices, then $e_x e_{x'} = e_{x'} e_x = e_{\mathcal{F}_e(x,x')}$.

iii) If x,x',x'' are three e-vertices with $x' \in \mathcal{H}_e(x,x'')$ and x' adjacent to x, then $e_x e_{x'} e_{x''} = e_x e_{x''}$.

In particular the system $(e_x)_{x\in\mathcal{B}'_{0/e}}$ is consistent in the sense of [17, Def. 2.1].

Proof. Thanks to Lemma 2.7.2, i) follows from (2.6.6) and ii) follows from Proposition 2.6.7. Now let $y \in]x,x']$ be sufficiently closed to x so that $]x,y]$ is contained in an e-facet $\mathcal{F}$. We have $x \in \overline{\mathcal{F}}$ and $\overline{\mathcal{F}} \subset \mathcal{H}_e(x,x'')$, and moreover, as in Lemma 2.9 of [17], $\mathcal{F}$ is the unique maximal facet with these two properties. Indeed the geometrical properties of the polysimplicial structure of $\mathcal{B}'$ used in the proof of this lemma 2.9 are satisfied by its e^{th}-subdivision, since the latter is homothetic to the former. It follows in particular that the facet $\mathcal{F}_e(x,x')$ is contained in $\overline{\mathcal{F}}$. Now, using Propositions 2.6.4 and 2.6.7 we get $e_x e_{x''} = e_x e_{\mathcal{F}} e_{x''} = e_x e_{x'} e_{\mathcal{F}} e_{x''} = e_{x'} e_x e_{\mathcal{F}} e_{x''} = e_{x'} e_x e_{x''} = e_x e_{x'} e_{x''}$. ☐

We now check that the proof of Thm 2.4 of [17] can be adapted to our setting. Let $V \in \mathrm{Rep}_R(G)$ be a smooth RG-module. It defines a coefficient system $\mathcal{F} \mapsto \mathcal{V}(\mathcal{F}) := e_{\mathcal{F}} V$ over $\mathcal{B}'_{\bullet/e}$. The transition maps $e_{\mathcal{F}} V \longrightarrow e_{\mathcal{F}'} V$ for $\mathcal{F}' \subset \overline{\mathcal{F}}$ are just given by inclusion thanks to the identity $e_{\mathcal{F}} e_{\mathcal{F}'} = e_{\mathcal{F}}$ in such a situation. After choosing an orientation of $BT'_{\bullet/e}$ we may form the cellular chain complex $\mathcal{C}(\mathcal{B}'_{\bullet/e}, \mathcal{V})$ whose homology we denote by $H_*(\mathcal{B}'_{\bullet/e}, \mathcal{V}))$. More generally, for any polysimplicial subcomplex Σ of $\mathcal{B}'_{\bullet/e}$ we have a chain complex and its homology $H_*(\Sigma, \mathcal{V})$.

2.7.4 *Lemma.* — *For any convex polysimplicial subcomplex Σ of $\mathcal{B}'_{\bullet/e}$, we have $H_0(\Sigma,\mathcal{V}) = \sum_{x\in\Sigma_0} e_x V$ and $H_n(\Sigma,\mathcal{V}) = 0$ for $n > 0$.*

Proof. We review the different steps of Meyer and Solleveld's proof.

Step 1. Prove it when Σ is a polysimplex. The argument below Lemma 2.18 of *loc. cit.* relies directly on properties i) ii) iii) of Lemma 2.7.3 and works without any change.

Step. 2. Divide and conquer method : suppose Σ is *finite* and is the union of two convex subcomplexes Σ_+ and Σ_- with convex intersection Σ_0, then if the statement holds for Σ_+, Σ_- and Σ_0, it holds for Σ. This reduction step follows from Theorem 2.12 of *loc. cit.*, which asserts that the distribution $e_\Sigma := \sum_{\mathcal{F}\subset\Sigma} (-1)^{\dim(\mathcal{F})} e_{\mathcal{F}}$ is an idempotent such that $e_\Sigma e_{\mathcal{F}} = e_{\mathcal{F}} e_\Sigma = e_{\mathcal{F}}$ for all $\mathcal{F} \subset \Sigma$. This theorem in turn follows from Lemmas 2.8 and 2.9 and Proposition 2.2 of *loc. cit.* But, provided properties i) ii) iii) of Lemma 2.7.3, all these statements are concerned with the geometry of combinatorial convex hulls in an appartment, hence they still hold for any subdivision as in our case.

Step. 3. Prove that if Σ is *finite* and not a polysimplex, then it can be split as in Step 2. Here the argument has to be completed a bit. Suppose first that Σ is contained in an appartment. Then there is an e-wall whose two associated open half-spaces do intersect non trivially Σ. Simply take $\Sigma_\pm$ to be the intersection with the closed half spaces, and Σ_0 the intersection with the wall. Now suppose that Σ is not contained in a single appartment. Then we may find an F-chamber Δ that intersects non-trivially Σ but does not contain it. Pick an appartement A that contains Δ and a wall of A that supports a face of Δ and intersects non-trivially Σ. It corresponds to some affine root a and we can use the retraction on A centered at Δ exactly as on p.140 of *loc.cit.*

Step. 4. The statement is now known for Σ finite. It follows for Σ infinite by writing it as the union of an increasing sequence of finite convex subcomplexes Σ_n, which can always be done. Indeed, the chain complex Σ is the direct limit of the chain complexes of the Σ_n. □

2.7.5 *Proof of Theorem 2.7.1.* — If $x \in \mathcal{B}$, there is an e-vertex y that lies in the closure of the e-facet $\mathcal{F}_e(x)$. Then it follows from Proposition 2.6.7 that $e_x = e_y e_x$, thus we see that $V \in \mathrm{Rep}_R^{\phi,I}(G)$ if and only if $\sum_{x\in\mathcal{B}_{0/e}} e^\phi_{I,x}V$. Therefore, thanks to the case $\Sigma = \mathcal{B}_{\bullet/e}$ of the last lemma, the proof of Thm 3.1 of [17] adapts immediately to show that the category $\mathrm{Rep}_R^{\phi,I}(G)$ is a Serre category stable under arbitrary colimits and generated as claimed in the theorem. Further let V be any smooth RG-module and put $V^{\phi,I} := \sum_{x\in\mathcal{B}} e_x V$. We certainly have $V^{\phi,I} \in \mathrm{Rep}_R^{\phi,I}(G)$, and we see that the quotient $V_{\phi,I} := V/V^{\phi,I}$ is killed by all e_x so that no non-zero subquotient of $V_{\phi,I}$ belongs to $\mathrm{Rep}_R^{\phi,I}(G)$.

2.7.6 *Remark.* — Let $(\Sigma_n)_{n\in\mathbb{N}}$ be an increasing sequence of convex polysimplicial subcomplexes of $\mathcal{B}$ such that $\mathcal{B} = \bigcup_n \Sigma_n$. We have already recalled that $e_{\Sigma_n} := \sum_{\mathcal{F}\subset\Sigma_n}(-1)^{\dim(\mathcal{F})}e_{\mathcal{F}}$ is an idempotent such that $e_{\mathcal{F}}e_{\Sigma_n} = e_{\mathcal{F}}$ for all $\mathcal{F} \subset \Sigma_n$. It follows that $e_{\Sigma_{n+1}}e_{\Sigma_n} = e_{\Sigma_n}$ and that for all $V \in \mathrm{Rep}_R(G)$ we have

$$\sum_{x\in\mathcal{B}} e_x V = \bigcup_n e_{\Sigma_n} V.$$

2.8 Some properties of $\mathrm{Rep}_R^{\phi,I}(G)$

2.8.1 *The direct factor problem.* — We strongly believe that the category $\mathrm{Rep}_R^{\phi,I}(G)$ is actually a direct factor of $\mathrm{Rep}_R(G)$, but this does not follow formally from the Meyer-Solleveld theory, nor from Yu's theory. The problem is to show that the extension (2.7.7) splits canonically, and more precisely that the subspace $\bigcap_x \ker(e_x|V)$ of V maps onto $V_{\phi,I}$.

What is missing is the existence of injective cogenerators in $\mathrm{Rep}_R^{\phi,I}(G)$, or equivalently, of sufficiently many projective representations killed by all $e^\phi_{I,x}$. It seems that in order to bypass these problems, one needs some exhaustion result, e.g. as the ones proved by Kim [13] and Fintzen [...] (see below), or the one provided by the Bushnell-Kutzko-Stevens types theory. However, if one restricts attention to admissible objects over a complete local ring, then a duality trick implies the desired splitting.

Proposition. — Let $\mathcal{R}$ be a complete local R-algebra, and let V be an admissible smooth $\mathcal{R}G$-module (meaning that for any open compact subgroup H of G, the $\mathcal{R}$-module V^H is noetherian). Then the extension (2.7.7) splits. In other words, the admissible category $\mathrm{Adm}_{\mathcal{R}}^{\phi,I}(G)$ is a direct factor of $\mathrm{Adm}_{\mathcal{R}}(G)$.

Proof. Let $\mathcal{E}$ be a Matlis module over $\mathcal{R}$ (i.e. an injective hull of the residue field), and extend the Matlis duality functor to smooth $\mathcal{R}G$-modules by putting $V^* := \mathrm{Hom}_{\mathcal{R}}(V, \mathcal{E})^\infty$. Since p is invertible in $\mathcal{R}$, we have $(V^*)^H = \mathrm{Hom}_{\mathcal{R}}(V^H, \mathcal{E})$ for any open pro-p-subgroup. Therefore, the usual Matlis duality theorem for noetherian $\mathcal{R}$-modules implies that for an admissible $\mathcal{R}G$-module, the canonical map $V \longrightarrow V^{**}$ is an isomorphism, and induces isomorphisms $W \mapsto (W^\perp)^\perp$ for each $\mathcal{R}G$-submodule W of V.

Now, let e_x^* be the image of e_x by the anti-involution $g \mapsto g^{-1}$ on $\mathcal{H}_R(G)$. Note that the system of idempotents $(e_x^*)_{x\in\mathcal{B}_{0/e}}$ is the one attached to the pair $(\bar{\phi}, I)$ where $^-$ denotes the automorphism of ${}^L\mathbf{G}$ induced by complex conjugation. Let us put $eV := \sum_{x\in\mathcal{B}} e_x V$ and $e^*V^* := \sum_{x\in\mathcal{B}} e_x^* V^*$. Then we see that $(e^*V^*)^\perp = \bigcap_{x\in\mathcal{B}} \ker(e_x|V)$ and $(eV)^\perp = \bigcap_{x\in\mathcal{B}} \ker(e_x^*)$. By biduality we have $(\bigcap_{x\in\mathcal{B}} \ker(e_x|V))^\perp = e^*V^*$ and it follows that $V = eV \oplus \bigcap_{x\in\mathcal{B}} \ker(e_x|V)$ as desired. □

2.8.2 *Proposition.* (Disjonction) — *Let (ϕ', I') and (ϕ, I) be two distinct pairs as in 2.7.1. Then the categories $\mathrm{Rep}_R^{\phi',I'}(G)$ and $\mathrm{Rep}_R^{\phi,I}(G)$ are orthogonal in the sense that for all objects $V \in \mathrm{Rep}_R^{\phi,I}(G)$ and $V' \in \mathrm{Rep}_R^{\phi',I'}(G')$ we have* $\mathrm{Ext}^*_{RG}(V, V') = \mathrm{Ext}^*_{RG}(V', V) = \{0\}$.

Proof. By (2.5.2) we have $e^{\phi'}_{I',x'} e^{\phi}_{I,x} = 0$ for all $x', x \in \mathcal{B}$. It follows that $e^{\phi'}_{I',x'} V = 0$ for all $V \in \mathrm{Rep}_R^{\phi,I}(G)$, hence also $\mathrm{Hom}_{RG}(V', V) = 0$ for all $V' \in \mathrm{Rep}_R^{\phi',I'}(G')$. Using projective resolutions inside $\mathrm{Rep}_R^{\phi',I'}(G')$, we also get that $\mathrm{Ext}^*_{RG}(V', V) = 0$. □

2.8.3 *The "essentially tame" subcategory.* — Introduce the subcategory

$$\mathrm{Rep}_R^{\mathrm{et}}(G) := \left\{ V \in \mathrm{Rep}_R(G), V = \sum_{\phi,\iota,x} e^{\phi}_{\iota,x} V \right\}.$$

Here "et" stands for "essentially tame" in order to stick to Bushnell and Henniart's terminology, although we fear that this is a bit misleading. Morally (and under the hypothesis (H1) and (H2)), this subcategory should capture all the representations associated to Langlands parameters that are trivial on the derived subgroup $[P_F, P_F]$. Whatsoever, this category is a Serre subcategory generated by projective objects and closed under arbitrary colimits, and the proposition tells us that it decomposes as a direct product

$$\mathrm{Rep}_R^{\mathrm{et}}(G) = \prod_{(\phi,I)} \mathrm{Rep}_R^{\phi,I}(G).$$

As mentioned above, we don't know in general whether it is a direct factor subcategory, but we may hope with further hypothesis that it is the entire category. Recall that our construction of $\mathrm{Rep}_R^{\phi,I}(G)$ applies to any ϕ such that $\hat{\phi}(P_F)$ is abelian, under the hypothesis (H1) and (H2), that is, assuming that p is good for $\mathbf{G}$ and does not divide $|\pi_1(\mathbf{G}_{\mathrm{der}})|$. With further (and much stronger) hypothesis, one can insure that actually any $\phi \in \Phi(P_F, \mathbf{G})$ has abelian image.

Lemma. — Suppose that p does not divide the order of the absolute Weyl group of $\mathbf{G}$. Then any finite p-group of $\hat{\mathbf{G}}$ is abelian.

Proof. Let $P \subset \hat{\mathbf{G}}$ be a non-abelian finite p-group. Then P contains an abelian normal but non-central subgroup A. For example, one can take A to be generated by the center $Z(P)$ and any non-central element of the second-center (the inverse image of the center of $P/Z(P)$). Since p is in particular good for $\hat{\mathbf{G}}$, the connected centralizer of A is a proper Levi subgroup $\hat{\mathbf{M}}$, which is normalized by P. But P is not contained in $\hat{\mathbf{M}}$ since A is not central in P, therefore P has a non trivial image in the group $N_{\hat{\mathbf{G}}}(\hat{\mathbf{M}})/\hat{\mathbf{M}}$, which is a subquotient of the Weyl group of $\hat{\mathbf{G}}$. □

Under the assumption of this lemma, it is thus very tempting to believe that all representations are essentially tame in the sense introduced above. Thanks to a recent result of J. Fintzen, this turns out to be true.

2.8.4 *Theorem.* — *Suppose that p does not divide the order of the absolute Weyl group of $\mathbf{G}$. Then $\mathrm{Rep}_R^{\mathrm{et}}(G) = \mathrm{Rep}_R(G)$.*

Proof. Let us start with an outline of what we have to do. Fintzen has shown that any simple smooth RG-module contains a character attached to some Yu datum (see below), while we have considered only Yu data arising from parameters ϕ. So we have to show that these particular Yu data are actually sufficient. The needed ingredients for this are essentially contained in Kaletha's paper [11], which builds on previous work of Hakim and Murnaghan in [10].

Fintzen's result. Observe first that in order to apply Yu's construction described in 2.4.7, all that is needed is a triple $(\vec{\mathbf{G}}, \vec{\psi}, x)$, that we will refer to as a "Yu datum", in which $\vec{\mathbf{G}} = (\mathbf{G}_0 \subset \cdots \subset \mathbf{G}_d)$ is a tame twisted Levi sequence, $x \in \mathcal{B}$ lies in the image of a toral embedding $\mathcal{B}(\mathbf{G}_0, F) \hookrightarrow \mathcal{B}$, and $\vec{\psi} = (\psi_i)_{i=0,\cdots d}$ is a collection of characters[3] $\psi_i : \mathbf{G}_i(F) \longrightarrow \mathbb{C}^\times$ such that

– the first $d-1$ depths form an increasing sequence $r_{-1} := 0 < r_0 < \cdots < r_{d-1}$

– either $\psi_d = 1$ (we then put $r_d = r_{d-1}$) or ψ_d has depth $r_d > r_{d-1}$.

To such a Yu datum, the procedure of 2.4.7 attaches a pair $(\vec{G}_x^+, \vec{\psi}_x)$ consisting of an open pro-p-subgroup and a character of this subgroup. This pair defines in turn an idempotent $e(\vec{\mathbf{G}}, \vec{\psi}, x)$ in $\mathcal{H}_R(G)$, supported on $\vec{G}_x^+$. Fintzen's theorem can be stated as follows.

For any RG-module V, there is a generic Yu datum $(\vec{\mathbf{G}}, \vec{\psi}, x)$ such that $e(\vec{\mathbf{G}}, \vec{\psi}, x)V \neq 0$.

Here, a Yu datum is called *generic* if for all $i < d$, the restriction $(\psi_i)_{|G^i_{x,r_i}}$ is represented by a $\mathbf{G}^{i+1}$-generic element (as in 2.5.5). With this result in hand, we are left to prove that for any generic Yu datum there is a parameter ϕ, an embedding $\iota \in I_\phi$ and a point $y \in \mathcal{B}_\iota$ such that $e^\phi_{\iota,y} = e(\vec{\mathbf{G}}, \vec{\psi}, x)$.

From Yu data to parameters. Following Kaletha, we will say that a Yu datum $(\vec{\mathbf{G}}, \vec{\psi}, x)$ is *normalized* if ψ_i is trivial on $(\mathbf{G}_i)_{\mathrm{sc}}(F)$ for all i. Using Hakim and Murnaghan's concept of refactorization, Kaletha proves in [11, Lemma 3.6.2] that for any Yu datum $(\vec{\mathbf{G}}, \vec{\psi}, x)$ there is a normalized Yu datum $(\vec{\mathbf{G}}, \vec{\psi}', x)$ such that $e(\vec{\mathbf{G}}, \vec{\psi}, x) = e(\vec{\mathbf{G}}, \vec{\psi}', x)$.

We may thus restrict our attention to a normalized Yu datum $(\vec{\mathbf{G}}, \vec{\psi}, x)$. Then the character $\varphi_i := \prod_{k=i}^d (\psi_k)_{|\mathbf{G}_i(F)}$ of $\mathbf{G}_i(F)$ is also trivial on $(\mathbf{G}_i)_{\mathrm{sc}}(F)$ for all i. By Remark 2.4.2, such a

[3] In the literature the standard notation for these characters is ϕ_i but in this paper the letter ϕ has already been dedicated to parameters

character φ_i comes from some element $\hat{\varphi}_i \in H^1(W_F, Z(\hat{\mathbf{G}}_i))$. By construction, we see that $(\hat{\varphi}_0)_{|I_F^{r_i}} = (\hat{\varphi}_i)_{|I_F^{r_i}}$ in $H^1(I_F^{r_i}, Z(\hat{\mathbf{G}}_0))$. Let us put

$$\hat{\phi}_0 := (\hat{\varphi}_0)_{|P_F} \in H^1(P_F, Z(\hat{\mathbf{G}}_0)).$$

As in 2.1.3 there is a canonical conjugacy class of embeddings $Z(\hat{\mathbf{G}}_0) \hookrightarrow \hat{\mathbf{G}}$ (even though $Z(\hat{\mathbf{G}}_0)$ may not be connected), which are P_F-equivariant since P_F acts trivially on both sides, thus allowing us to pushforward $\hat{\phi}_0$ to some $\hat{\phi} \in H^1(P_F, \hat{\mathbf{G}})$.

We claim that $\hat{\phi} \in \Phi(P_F, \mathbf{G})$. Indeed, choose any tamely ramified maximal torus $\mathbf{S}$ in $\mathbf{G}_0$. There is a canonical embedding $Z(\hat{\mathbf{G}}_0) \hookrightarrow \hat{\mathbf{S}}$ that allows us to pushforward $\hat{\varphi}_0$ into $H^1(W_F, \hat{\mathbf{S}})$, giving the Langlands parameter of the character $(\varphi_0)_{|\mathbf{S}(F)}$. Further, there is also a canonical W_F-stable conjugacy class of embeddings $\hat{\mathbf{S}} \hookrightarrow \hat{\mathbf{G}}$. Any such embedding κ can be extended to a tamely ramified L-embedding ${}^L\kappa : {}^L\mathbf{S} \hookrightarrow {}^L\mathbf{G}$ (see the proof of [11, Prop. 5.2.4]). Then the restriction of ${}^L\kappa \circ \hat{\varphi}_0$ to P_F is $\hat{\phi}$, showing that $\hat{\phi}$ arises by restriction of a plain Langlands parameter. In particular we have a W_F-equivariant epimorphism of tori $\hat{\mathbf{S}} \twoheadrightarrow \hat{\mathbf{S}}_\phi$ and, dually, an F-rational embedding $\iota : \mathbf{S}_\phi \hookrightarrow \mathbf{S} \hookrightarrow \mathbf{G}$ (which by proposition 2.2.3 implies that ϕ is relevant).

What we have done sofar is to associate a pair (ϕ, ι) to any normalized Yu datum $(\vec{\mathbf{G}}, x, \vec{\psi})$. We now assume that this Yu datum is generic, in order to be able to recover it from (ϕ, ι). We keep the notation $\mathbf{S}, \hat{\mathbf{S}}, {}^L\kappa$ introduced above and work with the representative $\phi = ({}^L\kappa \circ \hat{\varphi}_0)_{|P_F}$.

The point now is that, letting E be a tame extension that splits $\mathbf{S}$, then for any $r \in \mathbb{R}$ such that $r_{i-1} < r \leqslant r_i$ we have

$$\left\{\alpha \in \Sigma(\mathbf{S}, \mathbf{G}), \varphi_0(N_{E/F}(\alpha^\vee(E_r^\times))) = \{1\}\right\} = \Sigma(\mathbf{S}, \mathbf{G}^i).$$

Indeed, the case $i = 0$ is addressed in the proof of Lemma 3.6.9 of [11] and the same proof applies to $i < d$. As in the proof of Lemma 2.3.2, it follows that the identification between $\Sigma(\mathbf{S}, \mathbf{G})$ and $\Sigma(\hat{\mathbf{S}}, \hat{\mathbf{G}})^\vee$ identifies $\Sigma(\mathbf{S}, \mathbf{G}_i)$ and $\Sigma(\hat{\mathbf{S}}, C_{\hat{\mathbf{G}}}(\phi(I_F^r)))^\vee$ for $r_{i-1} < r \leqslant r_i$. In other words we have $\mathbf{G}^i = C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi,r}))$ and, in particular $\mathbf{G}^i = \mathbf{G}^i_\iota$.

Going through the definitions, it is now clear that $e(\vec{\mathbf{G}}, \vec{\psi}, x) = e^\phi_{\iota,x}$. $\square$

2.8.5 *Compatibility with isogenies.* — To any isogeny $f : \mathbf{G}' \longrightarrow \mathbf{G}$ is associated a canonical conjugacy class of dual isogenies $\hat{f} : \hat{\mathbf{G}} \longrightarrow \hat{\mathbf{G}}'$. Moreover if f is defined over F then any such dual isogeny can be extended to a morphism of L-groups ${}^L\mathbf{G} \longrightarrow {}^L\mathbf{G}'$. We thus get a well defined transfer map $f^* : \Phi(P_F, \mathbf{G}) \mapsto \Phi(P_F, \mathbf{G}')$. Note that our hypothesis (H1) and (H2) hold for $\mathbf{G}'$ since they are assumed to hold for $\mathbf{G}$. If we fix a dual isogeny $\hat{f}$ and a morphism $\phi : P_F \longrightarrow \hat{\mathbf{G}}$, we get an isogeny of Levi subgroups $C_{\hat{\mathbf{G}}}(\phi) \longrightarrow C_{\hat{\mathbf{G}}'}(\hat{f} \circ \phi)$ with kernel $\ker(\hat{f})$ which dually provides a conjugacy class of F-rational isogenies $\mathbf{G}_{f^*\phi} \longrightarrow \mathbf{G}_\phi$ with kernel $\ker(f)$ together with an F-rational isogeny $\mathbf{S}_{f^*\phi} \longrightarrow \mathbf{S}_\phi$. Now any $\iota \in I_\phi$ induces an isomorphism $\mathbf{S}_\phi \xrightarrow{\sim} Z(\mathbf{G}_\iota)^\circ$, and the discussion in 2.1.3 shows that this isomorphism lifts uniquely to an isomorphism $\mathbf{S}_{f^*\phi} \xrightarrow{\sim} Z(f^{-1}(\mathbf{G}_\iota))^\circ$, thus providing an element $f^*\iota \in I_{f^*\phi}$. In this way we get a bijection $I_\phi = I_{f^*\phi}$ that respects F-rationality, but the G'-conjugacy is a priori coarser than the G-conjugacy.

Proposition. — In this setting, the pull-back functor $f^* : \mathrm{Rep}_R(G) \longrightarrow \mathrm{Rep}_R(G')$ *takes* $\mathrm{Rep}_R^{\phi,I}(G)$ *into* $\prod_{I' \subset f^*I} \mathrm{Rep}_R^{f^*\phi,I'}(G')$.

Proof. Let us identify the Bruhat-Tits buldings of $\mathbf{G}$ and $\mathbf{G}'$. Then, from the definitions we see that $\mathcal{B}_\iota = \mathcal{B}_{f^*\iota}$ for any $\iota \in I_\phi$. Moreover, for $x \in \mathcal{B}_\iota$, we have a surjection $\vec{G'}^+_{f^*\iota,x} \twoheadrightarrow \vec{G}^+_{\iota,x}$ and the construction of the characters shows that $(\check{f^*\phi})^+_{f^*\iota,x}$ is the pull back of $\check{\phi}^+_{\iota,x}$. The claim then follows from the definition of the categories under consideration. $\square$

2.8.6 *Compatibility with parabolic induction.* — Here we suppose that ϕ comes from an F-Levi subgroup $\mathbf{M}$ of $\mathbf{G}$. More precisely, we assume that we have a factorization $\phi : P_F \xrightarrow{\phi_M} {}^L\mathbf{M} \longrightarrow {}^L\mathbf{G}$ where ϕ_M is an *admissible* parameter of $\mathbf{M}$ (recall this means that ϕ_M admits an extension to a genuine relevant Langlands parameter $W'_F \longrightarrow {}^L\mathbf{M}$).

In this context we have $\hat{\mathbf{M}}_{\phi_M} = \hat{\mathbf{M}} \cap \hat{\mathbf{G}}_\phi$, whence a W_F-equivariant surjection $\hat{\mathbf{S}}_{\phi_M} \twoheadrightarrow \hat{\mathbf{S}}_\phi$, which on the dual side induces an injection of F-tori $\mathbf{S}_\phi \hookrightarrow \mathbf{S}_{\phi_M}$. It follows that any F-rational Levi-center embedding $\iota_M : \mathbf{S}_{\phi_M} \hookrightarrow \mathbf{M}$ in the set I_{ϕ_M} induces an embedding $\iota : \mathbf{S}_\phi \hookrightarrow \mathbf{G}$ in the set I_ϕ. Obviously, M-conjugate embeddings lead to G-conjugate embeddings, so that any choice of an M-conjugacy class $I_M \subset I_{\phi_M}$ points to a G-conjugacy class $I \subset I_\phi$.

Theorem. — Let (ϕ_M, I_M) and (ϕ, I) be as above, and let $\mathbf{P}$ be a parabolic F-subgroup of $\mathbf{G}$ with Levi component $\mathbf{M}$. Then, denoting by i_P the associated parabolic induction functor, we have

$$i_P\left(\mathrm{Rep}_R^{\phi_M,I_M}(M)\right) \subset \mathrm{Rep}_R^{\phi,I}(G).$$

Proof. Denote by $\mathcal{B}_M$ the image of any toral embedding $\mathcal{B}(\mathbf{M},F) \hookrightarrow \mathcal{B}(\mathbf{G},F)$. If $\iota_M \in I_M$ induces $\iota \in I$ then we have $\mathcal{B}_{\iota_M} = \mathcal{B}_\iota \cap \mathcal{B}_M$. In view of the projective generator given in Theorem 2.7.1, it is sufficient to prove that for each $\iota_M \in I_M$ and $x \in \mathcal{B}_\iota \cap \mathcal{B}_M$ we have

$$i_P\left(\mathrm{ind}^M_{\vec{M}^+_{\iota_M,x}}\left(\check{\phi_M}^+_{\iota_M,x}\right)\right) \in \mathrm{Rep}_R^{\phi,I}(G).$$

Actually we will prove that $i_P\left(\mathrm{ind}^M_{\vec{M}^+_{\iota_M,x}}\left(\check{\phi_M}^+_{\iota_M,x}\right)\right)$ is a subquotient of $\mathrm{ind}^G_{\vec{G}^+_{\iota,x}}(\check{\phi}^+_{\iota,x})$. To this aim, we will need the following lemma.

Lemma. — Let $\bar{\mathbf{P}}$ be the opposite parabolic subgroup of $\mathbf{P}$ with respect to $\mathbf{M}$ and denote by $\bar{\mathbf{U}}$ and $\mathbf{U}$ their unipotent radicals.

- *i) $\vec{G}^+_{\iota,x}$ has the Iwahori decomposition property with respect to $P, \bar{P}$, i.e., the multiplication map $(U \cap \vec{G}^+_{\iota,x}) \times (M \cap \vec{G}^+_{\iota,x}) \times (\bar{U} \cap \vec{G}^+_{\iota,x}) \longrightarrow \vec{G}^+_{\iota,x}$ is a bijection. Moreover, we have $(\check{\phi}^+_{\iota,x})_{|U\cap\vec{G}^+_{\iota,x}} \equiv 1 \equiv (\check{\phi}^+_{\iota,x})_{|\bar{U}\cap\vec{G}^+_{\iota,x}}$.*
- *ii) We have $M \cap \vec{G}^+_{\iota,x} = \vec{M}^+_{\iota_M,x}$ and $(\check{\phi}^+_{\iota,x})_{|M\cap\vec{G}^+_{\iota,x}} = \check{\phi_M}^+_{\iota_M,x}$.*
- *iii) The bilinear form θ of Proposition 2.5.9 induces a perfect pairing between $(U\cap\vec{G}_{\iota,x})/(U\cap \vec{G}^+_{\iota,x})$ and $(\bar{U} \cap \vec{G}_{\iota,x})/(\bar{U} \cap \vec{G}^+_{\iota,x})$.*

Proof. i) Since $Z(\mathbf{G}_\iota)^\circ = \iota(\mathbf{S}_\phi) \subset \mathbf{M}$, we are in the setting described above Proposition 9.3 of [8]. There, it is asserted that the Iwahori decomposition holds and, moreover, that we have for $H = M, U$ or $\bar{U}$, the equalities $H \cap \vec{G}^+_{\iota,x} = \vec{H}^+_{\iota,x} := (H \cap G^0_{\iota,x,0+}) \cdots (H \cap G^d_{\iota,x,(r_{d-1}/2)+})$. Since no proof is provided in *loc. cit.*, let us supply some details. First of all, by the properties of the big cell $\mathbf{U}\mathbf{M}\bar{\mathbf{U}}$ it suffices to prove that we have the equality $\vec{G}^+_{\iota,x} = \vec{U}^+_{\iota,x}\vec{M}^+_{\iota,x}\vec{\bar{U}}^+_{\iota,x}$. But for each $i = 0, \cdots, d$, the intersections $\mathbf{P} \cap \mathbf{G}^i_\iota$ and $\bar{\mathbf{P}} \cap \mathbf{G}^i_\iota$ are a pair of opposite parabolic subgroups of $\mathbf{G}^i_\iota$ with intersection $\mathbf{M} \cap \mathbf{G}^i_\iota$. Indeed, this follows from the fact that $Z(\mathbf{M})^\circ$ is contained in $\mathbf{G}_\iota$, hence also in $\mathbf{G}^i_\iota$. Then, because x belongs to $\mathcal{B}_{\mathbf{M}\cap\mathbf{G}^i_\iota} \subset \mathcal{B}_{\mathbf{G}^i_\iota}$, we know that the Moy-Prasad group $G^i_{\iota,x,r_{i-1}/2+}$ has the Iwahori decomposition $G^i_{\iota,x,(r_{i-1}/2)+} = (U \cap G^i_{\iota,x,(r_{i-1}/2)+})(M \cap G^i_{\iota,x,(r_{i-1}/2)+})(\bar{U} \cap G^i_{\iota,x,(r_{i-1}/2)+})$. Now, using the fact that for $i < j$, the group $\bar{U} \cap G^i_{\iota,x,(r_{i-1}/2)+}$ normalizes $G^j_{\iota,x,(r_{j-1}/2)+}$ and the group $\bar{M} \cap G^i_{\iota,x,(r_{i-1}/2)+}$ normalizes $U \cap G^j_{\iota,x,(r_{j-1}/2)+}$, the desired decomposition follows inductively.

Now it remains to check that $\check{\phi}^+_{\iota,x}$ is trivial on $\vec{U}^+_{\iota,x}$ and on $\vec{\bar{U}}^+_{\iota,x}$. Going back to the construction of $\check{\phi}^+_{\iota,x}$ it suffices to show that for each $i = 0, \cdots, d$, the character $\psi^+_{i,x}$ of Paragraph 2.4.7 is trivial on the group $U \cap (G^i_{\iota,x,0+}G_{x,(r_i/2)+})$ (and similarly with $\overline{U}$). By the same argument as above, this group is $(U \cap G^i_{\iota,x,0+})(U \cap G_{x,(r_i/2)+})$. On one hand, the restriction of $\psi^+_{i,x}$ to $U \cap G^i_{\iota,x,0+}$ is trivial since it is also the restriction of the character ψ_i of G^i_ι and $U \cap G^i_\iota$ is contained in the derived subgroup of G^i_ι. On the other hand, the restriction of $\psi^+_{i,x}$ to $G_{x,(r_i/2)+}$ is the character denoted by $\widetilde{\psi}_i$ in 2.4.7, which extends trivially $(\psi_i)_{|G^i_{\iota,x,(r_i/2)+}}$ according to the decomposition $\mathfrak{g}_{x,(r_i/2)+:r_i+} = \mathfrak{g}^i_{\iota,x,(r_i/2)+:r_i+} \oplus \mathfrak{n}^i_{\iota,x,(r_i/2)+:r_i+}$ and via the Moy-Prasad isomorphism. Recall that the latter decomposition is induced by $\mathfrak{g} = \mathfrak{g}^i_\iota \oplus \mathfrak{n}^i_\iota$ where $\mathfrak{g}^i_\iota$, resp. $\mathfrak{n}^i_\iota$, is the trivial eigenspace, resp. the sum of all non-trivial eigenspaces, of $\iota(\mathbf{S}_{\phi,r_i})$ acting on $\mathfrak{g}$. On the other hand, the image $\mathfrak{u}_{x,(r_i/2)+:r_i+}$ of $U \cap G_{x,(r_i/2)+}$ in $\mathfrak{g}_{x,(r_i/2)+:r_i+}$ is induced by the Lie algebra $\mathfrak{u}$ of $\mathbf{U}$ which is a sum of (non-trivial) eigenspaces for $Z(\mathbf{M})^\circ$ acting on $\mathfrak{g}$. The point is now that these two tori commute since $\mathbf{M}$ contains $\iota(\mathbf{S}_{\phi,r_i})$. Therefore, $\mathfrak{u}$ is stable under $\iota(\mathbf{S}_{\phi,r_i})$ and we have $\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{g}^i_\iota) \oplus (\mathfrak{u} \cap \mathfrak{n}^i_\iota)$. Correspondingly, $\mathfrak{u}_{x,(r_i/2)+:r_i+}$ decomposes as the direct sum of $\mathfrak{u}_{x,(r_i/2)+:r_i+} \cap \mathfrak{n}^i_{\iota,x,(r_i/2)+:r_i+}$ and $\mathfrak{u}_{x,(r_i/2)+:r_i+} \cap \mathfrak{g}^i_{\iota,x,(r_i/2)+:r_i+}$. We have already seen that ψ_i is trivial on the latter intersection, and by definition $\widetilde{\psi}_i$ is trivial on the former one, so we can conclude.

ii) The equality $\hat{\mathbf{M}}_{\phi_M} = \hat{\mathbf{M}} \cap \hat{\mathbf{G}}_\phi$ provides an isomorphisn $\hat{\mathbf{S}}_{\phi_M} \xrightarrow{\sim} \hat{\mathbf{S}}_\phi \times_{\hat{\mathbf{G}}_{\mathrm{ab}}} \hat{\mathbf{M}}_{\mathrm{ab}}$, which dually provides two inclusions $\mathbf{S}_\phi \subset \mathbf{S}_{\phi_M}$ and $Z(\mathbf{M})^\circ \subset \mathbf{S}_{\phi_M}$ such that $\mathbf{S}_{\phi_M} = \mathbf{S}_\phi Z(\mathbf{M})^\circ$. In particular, if $\iota_M \in I_M$ induces $\iota \in I_\phi$ through the first embedding we have $\iota_M(\mathbf{S}_{\phi_M}) = \iota(\mathbf{S}_\phi)Z(\mathbf{M})^\circ$ and therefore $\mathbf{M}_{\iota_M} = \mathbf{M} \cap \mathbf{G}_\iota$. Similarly, for any $r > 0$ we have $\iota_M(\mathbf{S}_{\phi_M,r}) = \iota(\mathbf{S}_{\phi,r})Z(\mathbf{M})^\circ$ and therefore $C_{\mathbf{M}}(\iota_M(\mathbf{S}_{\phi_M,r})) = \mathbf{M} \cap C_{\mathbf{G}}(\iota(\mathbf{S}_{\phi,r}))$. It follows in particular that the set of jumps $r'_{-1} = 0 < r'_0 < \cdots < r'_{d'-1}$ of the decrasing filtration $(\mathbf{S}_{\phi_M,r})_r$ is a subset of the set of jumps $r_{-1} = 0 < r_0 < \cdots < r_{d-1}$ of the filtration $(\mathbf{S}_{\phi,r})_r$. For $0 \leqslant k < d'$ write i_k for the unique integer between 0 and $d-1$ such that $r'_k = r_{i_k}$, and put $i_{-1} = -1$. Then, with the notation of 2.4.7, we have $M \cap G^i_{\iota,x,r+} = M^k_{\iota_M,x,r+}$ for all $i = 0, \cdots, d$ and k such that $i_{k-1} < i \leqslant i_k$, and in particular we see that $M \cap G^i_{\iota,x,(r_{i-1}/2)+} \subset M^k_{\iota_M,x,(r'_{k-1}/2)+}$ with equality if $i = i_{k-1} + 1$. Therefore, starting from the decomposition that we have seen in the proof of i), we get

$$\begin{aligned} M \cap \vec{G}^+_{\iota,x} &= (M \cap G^0_{\iota,x,0+})(M \cap G^1_{\iota,x,(r_0/2)+}) \cdots (M \cap G^d_{\iota,x,(r_{d-1}/2)+}) \\ &= M^0_{\iota_M,0+}M^1_{\iota_M,(r'_0/2)+} \cdots M^{d'}_{\iota_M,(r'_{d'-1}/2)+} = \vec{M}^+_{\iota_M,x} \end{aligned}$$

Let us turn to characters. By definition, $\check{\phi_M}^+_{\iota_M,x}$ is a product $\prod_{k=0}^{d'}(\psi^+_{M,k})_{|\vec{M}^+_{\iota_M,x}}$ with $\psi^+_{M,k}$ a certain character of $M^k_{\iota_M,x,0+}M_{x,(r'_k/2)+}$, while $(\check{\phi}^+_{\iota,x})_{|\vec{M}^+_{\iota_M,x}}$ is a product $\prod_{i=0}^{d}(\psi^+_i)_{|\vec{M}^+_{\iota_M,x}}$ with ψ^+_i

a certain character of $G^i_{\iota,x,0+}G_{x,(r_i/2)+}$. Note that if $i_{k-1} < i \leqslant i_k$, we have

$$M \cap G^i_{\iota,x,0+}G_{x,(r_i/2)+} = (M \cap G^i_{\iota,x,0+})M_{x,(r_i/2)+} = M^k_{\iota_M,x,0+}M_{x,(r_i/2)+} \supset M^k_{\iota_M,x,0+}M_{x,(r'_k/2)+}.$$

Therefore, it will suffice to prove that $\psi^+_{M,k} = \prod_{i=i_{k-1}+1}^{i_k}(\psi^+_i)_{|M^k_{\iota_M,x,0+}M_{x,(r'_k/2)+}}$. Recall that $(\psi^+_i)_{|G^i_{\iota,x,0+}}$ is the restriction of a character $\check{\varphi}_i\check{\varphi}^{-1}_{i+1}$ of G^i_ι that depends on the choice of $\hat{\varphi}_i \in H^1(W_F, Z(\hat{\mathbf{G}}_{\phi,r_{i-1}+}))$ extending $\hat{\phi}_{|I_F^{r_{i-1}+}}$. Similarly, $(\psi^+_{M,k})_{|M^k_{\iota_M,x,0+}}$ is the restriction of a character $\check{\varphi}_{M,k}\check{\varphi}^{-1}_{M,k+1}$ that depends on the choice of $\hat{\varphi}_{M,k} \in H^1(W_F, Z(\hat{\mathbf{M}}_{\phi_M,r'_{k-1}+}))$ extending $\hat{\phi_M}_{|I_F^{r'_{k-1}+}}$. By Lemma 2.4.8, these choices eventually do not matter, so we may choose $\hat{\varphi}_{M,k}$ to be the push-forward of $\hat{\varphi}_{i_{k-1}+1}$ via the inclusion $Z(\hat{\mathbf{G}}_{\phi,r_{i_{k-1}}+}) \subset Z(\hat{\mathbf{M}}_{\phi_M,r'_{k-1}+})$. In this way, we insure that the characters $\psi^+_{M,k}$ and $\prod_{i=i_{k-1}+1}^{i_k}\psi^+_i$ coincide on $M^k_{\iota_M,x,0+}$. It then remains to see that they also coincide on $M_{x,(r'_k/2)+}$. For this, we proceed as in the proof of i) above in order to check that Yu's extension procedures over M and G are compatible. As above, the key point is that for all $i_{k-1} < i \leqslant i_k$, the decomposition $\mathfrak{g}_{x,(r_i/2)+:r_i+} = \mathfrak{g}^i_{\iota,x,(r_i/2)+:r_i+} \oplus \mathfrak{n}^i_{\iota,x,(r_i/2)+:r_i+}$ intersected with the trace of $\mathfrak{m}$ yields back the corresponding decomposition $\mathfrak{m}_{x,(r_i/2)+:r_i+} = \mathfrak{m}^k_{\iota,x,(r_i/2)+:r_i+} \oplus \mathfrak{n}^{M,k}_{\iota,x,(r_i/2)+:r_i+}$. We leave the details to the reader.

iii) We have $[(U\cap\vec{G}_{\iota,x}),(P\cap\vec{G}_{\iota,x})] \subset (U\cap\vec{G}^+_{\iota,x})$. Since $\check{\phi}^+_{\iota,x}$ is trivial on $U\cap\vec{G}^+_{\iota,x}$, we see that $P\cap G_{\iota,x}$ is orthogonal to $U\cap G_{\iota,x}$ for the bilinear form θ. From the Iwahori decomposition and the non-degeneracy of θ, it follows that $\bar{U}\cap G^+_{\iota,x}$ cannot be orthogonal to $U\cap G^+_{\iota,x}$, and item iii) follows. ☐

We now resume the proof of the proposition. Denote by $e^M_{\iota,x}$ the idempotent of $\mathcal{H}_R(M_x)$ associated to $\check{\phi}^{\ +}_{M_{\iota_M,x}} = (\check{\phi}^+_{\iota,x})_{|\vec{M}^+_{\iota_M,x}}$. We can see it as an idempotent in the R-algebra RG_x of R-valued distributions (not necessarily smooth) on G_x. This R-algebra also contains the averaging idempotents $e_{\vec{U}^+_{\iota,x}}$ and $e_{\vec{\bar{U}}^+_{\iota,x}}$ associated to the the pro-p subgroups $\vec{U}^+_{\iota,x} = U\cap\vec{G}^+_{\iota,x}$ and $\vec{\bar{U}}^+_{\iota,x} = \bar{U}\cap\vec{G}^+_{\iota,x}$ of G_x, and the last lemma implies the following equality in RG_x :

$$e_{\iota,x} = e_{\vec{U}^+_{\iota,x}}e^M_{\iota,x}e_{\vec{\bar{U}}^+_{\iota,x}} = e_{\vec{U}^+_{\iota,x}}e_{\vec{\bar{U}}^+_{\iota,x}}e^M_{\iota,x}.$$

On the other hand, RG_x also contains the averaging idempotents e_{U_x}, $e_{U^+_x}$, $e_{\bar{U}_x}$, $e_{\bar{U}^+_x}$ associated to the pro-p subgroups $U_x := U\cap G_x$, $U^+_x := U\cap G_{x,0+}$, $\bar{U}_x := \bar{U}\cap G_x$ and $\bar{U}^+_x := U\cap G_{x,0+}$ and, by [8, Prop. 9.3], we have the property

$$e_{U^+_x}e_{\bar{U}_x}e^M_{\iota,x} \in RG_xe_{U_x}e_{\bar{U}_x}e^M_{\iota,x}$$

where we can also exchange the roles of P and $\bar{P}$. Thanks to this property, it follows from [8, Cor. 3.6 (ii)] that

$$i_P\left(\mathrm{ind}^M_{\vec{M}^+_{\iota,x}}\left(\check{\phi}^{M+}_{\iota,x}\right)\right) \simeq \mathrm{ind}^G_{G_x}\left(\mathcal{H}_R(G_x)e_{U_x}e_{\bar{U}^+_x}e^M_{\iota,x}\right) = \mathcal{H}_R(G)e_{U_x}e_{\bar{U}^+_x}e^M_{\iota,x}.$$

Now consider the representation $\mathcal{H}_R(G)e_{U_x}e_{\iota,x} \in \mathrm{Rep}_R(G)$. On one hand it is contained in $\mathcal{H}_R(G)e_{\iota,x} = \mathrm{ind}^G_{\vec{G}^+_{\iota,x}}(\check{\phi}^+_{\iota,x})$. On the other hand, multiplying on the right by $e_{\bar{U}^+_x}$ gives a surjection

$$\mathcal{H}_R(G)e_{U_x}e_{\iota,x} \twoheadrightarrow \mathcal{H}_R(G)e_{U_x}e_{\iota,x}e_{\bar{U}^+_x} = \mathcal{H}_R(G)e_{U_x}e_{\bar{U}^+_x}e^M_{\iota,x}.$$

Thus we see that $i_P\left(\mathrm{ind}_{M^+_{\iota,x}}^{M}\left(\check\phi^{M+}_{\iota,x}\right)\right)$ is a subquotient of $\mathrm{ind}_{G^+_{\iota,x}}^{G}(\check\phi^+_{\iota,x})$ hence belongs to $\mathrm{Rep}_R^{\phi,I}(G)$. $\square$

2.8.7 *Corollary.* — *Suppose that p does not divide the order of the absolute Weyl group of* $\mathbf{G}$*. Let* $\mathbf{P}$ *be a parabolic F-subgroup of* $\mathbf{G}$ *with Levi factor* $\mathbf{M}$ *and denote by* r_P *the associated Jacquet functor.*

i) We have $r_P(\mathrm{Rep}_R^{\phi,I}(G))\subset\prod_{(\phi_M,I_M)\mapsto(\phi,I)}\mathrm{Rep}_R^{\phi_M,I_M}(M)$.

ii) If $C_{\hat{\mathbf{G}}}(\phi)\subset\hat{\mathbf{M}}$ *then* i_P *induces an equivalence of categories* $\mathrm{Rep}_R^{\phi_M,I_M}(M)\xrightarrow{\sim}\mathrm{Rep}_R^{\phi,I}(G)$ *with quasi inverse the composition of* r_P *and the projection onto the* (ϕ_M,I_M)*-factor.*

Proof. i) Our hypothesis is inherited by Levi subgroups of $\mathbf{G}$ so that, by Theorem 2.8.4, we have the two decompositions $\mathrm{Rep}_R(G)=\prod\mathrm{Rep}_R^{\phi,I}(G)$ and $\mathrm{Rep}_R(M)=\prod\mathrm{Rep}_R^{\phi_M,I_M}(M)$. Therefore i) follows from Proposition 2.8.6 by Frobenius reciprocity.

ii) Let us denote by $\tilde r_P$ the composition of r_P with the projection on $\mathrm{Rep}_R^{\phi_M,I_M}(M)$. We will first show that $\tilde r_P\circ i_P$ is isomorphic to the identity functor on $\mathrm{Rep}_R^{\phi_M,I_M}(M)$ so that, in particular, i_P is fully faithful on $\mathrm{Rep}_R^{\phi_M,I_M}(M)$.

To this aim, recall that Frobenius reciprocity is given by a natural transform $r_P\circ i_P\longrightarrow\mathrm{Id}$ which is an epimorphism in the category of additive endofunctors of $\mathrm{Rep}_R(M)$ and whose kernel is described by the Mackey formula as follows : there is a filtration indexed by double cosets $P\dot wP$ in $G\setminus P$ whose graded pieces are of the form $\mathcal{F}_{\dot w}:=\mathrm{Ad}_{\dot w}\circ i_{P\cap M^w}\circ r_{M\cap P^w}$. Here, we have chosen representatives $\dot w$ in the rational normalizer $N_G(T)$ of a maximally split maximal torus $\mathbf{T}$ of $\mathbf{M}$ and w is the image of $\dot w$ in $W(\mathbf{T},\mathbf{G})$. In this situation, $\mathbf{M}\cap\mathbf{P}^w$ is a parabolic F-subgroup of $\mathbf{M}$ with Levi component $\mathbf{M}\cap\mathbf{M}^w$, while $\mathbf{P}\cap\mathbf{M}^w$ is a parabolic subgroup of $\mathbf{M}^w$ with the same Levi component $\mathbf{M}\cap\mathbf{M}^w$. It then follows from the last proposition and point i) above that

$$\mathcal{F}_{\dot w}\left(\mathrm{Rep}_R^{\phi_M,I_M}(M)\right)\subset\prod_{(\phi_w,I_w)\mapsto(\phi_M,I_M)}\mathrm{Ad}_{\dot w}\left(\mathrm{Rep}_R^{\phi_w,I_w}(M^w)\right)$$

where the product is over pairs (ϕ_w,I_w) relative to $\mathbf{M}\cap\mathbf{M}^w$ that map to (ϕ_M,I_M), and whose pushforward to $\mathbf{M}^w$ we still denote by (ϕ_w,I_w).

Let us draw the dual picture. We may assume that $\hat{\mathbf{M}}$ contains a reference maximal torus $\hat{\mathbf{T}}$ in $\hat{\mathbf{G}}$ (part of a W_F-stable épinglage of $\hat{\mathbf{G}}$). We have a duality between $\mathbf{T}$ and $\hat{\mathbf{T}}$ that exchanges roots and coroots. This induces an isomorphism $w\mapsto\hat w$, $W_{\mathbf{G}}(\mathbf{T})\xrightarrow{\sim}W_{\hat{\mathbf{G}}}(\hat{\mathbf{T}})$. Let us choose a lift $\hat{\dot w}$ of w in the normalizer $N_{\hat{\mathbf{G}}}(\hat{\mathbf{T}})$. Then $\hat{\mathbf{M}}^{\hat{\dot w}}$ is a Levi subgroup of $\hat{\mathbf{G}}$ that is dual to $\mathbf{M}$ and $\mathrm{Ad}_{\hat{\dot w}^{-1}}$ is a dual isogeny (actually isomorphism) to $\mathrm{Ad}_{\dot w}$. Therefore the last inclusion can be rewritten

$$\mathcal{F}_{\dot w}\left(\mathrm{Rep}_R^{\phi_M,I_M}(M)\right)\subset\prod_{(\phi_w,I_w)\mapsto(\phi_M,I_M)}\left(\mathrm{Rep}_R^{\mathrm{Ad}_{\hat{\dot w}^{-1}}(\phi_w,I_w)}(M)\right)$$

with the same convention as above. Now let $\phi_w:P_F\longrightarrow\hat{\mathbf{M}}\cap\hat{\mathbf{M}}^{\hat{\dot w}}$ be a parameter for $\mathbf{M}\cap\mathbf{M}^w$ whose pushforward to $\hat{\mathbf{M}}$ represents ϕ_M. Assume that $\mathrm{Ad}_{\hat{\dot w}^{-1}}(\phi_w)$ also represents ϕ_M. Then there is some $\hat m\in\hat{\mathbf{M}}$ such that $\mathrm{Ad}_{\hat m\hat{\dot w}^{-1}}(\phi_{\hat w})=\phi_{\hat w}$, i.e. $\hat m\hat{\dot w}^{-1}\in C_{\hat{\mathbf{G}}}(\phi_w)$. By our assumption, this implies that $\hat{\dot w}\in\hat{\mathbf{M}}$ hence $\dot w\in\mathbf{M}$, which contradicts the fact that $P\dot wP\neq P$. This means that the projection of $\mathcal{F}_{\dot w}\left(\mathrm{Rep}_R^{\phi_M,I_M}(M)\right)$ on $\mathrm{Rep}_R^{\phi_M,I_M}(M)$ is zero, and finally we have proven that that the natural transform $r_P\circ i_P\longrightarrow\mathrm{Id}$ induces an isomorphism $\tilde r_P\circ i_P\xrightarrow{\sim}\mathrm{Id}_{\mathrm{Rep}^{\phi_M,I_M}(M)}$.

Now, to conclude that $\tilde{r}_P$ and i_P are quasi-inverse equivalences of categories, it suffices to prove that $\tilde{r}_P$ is conservative on $\mathrm{Rep}_R^{\phi,I}(G)$. To this aim, we will use Theorem 6.3 of [12] in a particular setting. We start by reducing our statement to the "minimal" case. Observe that the hypothesis $C_{\hat{\mathbf{G}}}(\phi) \subset \hat{\mathbf{M}}$ is equivalent to the equality $C_{\hat{\mathbf{G}}}(\phi) = C_{\hat{\mathbf{M}}}(\phi_M)$ and therefore to the equality $\mathbf{S}_\phi = \mathbf{S}_{\phi_M}$. Fix an embedding ι_M in I_M and denote by ι its image in I. Then we have $\mathbf{G}_\iota = \mathbf{M}_{\iota_M} \subset \mathbf{M}$, or equivalently $Z(\mathbf{M}) \subset \iota(\mathbf{S}_\phi)$. It follows that the centralizer $\mathbf{L} := C_{\mathbf{G}}(\iota(\mathbf{S}_\phi^{\mathrm{split}}))$ of the split component of the center of $\mathbf{G}_\iota$ is an F-Levi subgroup of $\mathbf{G}$ contained in $\mathbf{M}$. It is dual to the Levi subgroup ${}^L\mathbf{L} := C_{{}^L\mathbf{G}}(Z(C_{\hat{\mathbf{G}}}(\phi))^{W_F,\circ})$ of ${}^L\mathbf{G}$ (and of ${}^L\mathbf{M}$). Since any extension $\varphi : W_F \longrightarrow {}^L\mathbf{G}$ of ϕ has to factor through ${}^L\mathbf{L}$, we see that ϕ (and also ϕ_M) comes from an admissible parameter $\phi_L \in \Phi(P_F, \mathbf{L})$. By construction we have $\mathbf{S}_{\phi_L} = \mathbf{S}_{\phi_M} = \mathbf{S}_\phi$ so that ι_M can be considered as an element of I_{ϕ_L}. Denote by I_L its conjugacy class under L. Now let $\mathbf{Q}$ be any parabolic F-subgroup of $\mathbf{G}$ with Levi component $\mathbf{L}$ and contained in $\mathbf{P}$. Denote by $\tilde{r}_Q$ the composition of r_Q with the projection onto $\mathrm{Rep}_R^{\phi_L,I_L}(G)$. Since $\tilde{r}_P$ factors $\tilde{r}_Q$, it is enough to prove that $\tilde{r}_Q$ is conservative on $\mathrm{Rep}_R^{\phi,I}(G)$. Therefore we may, and will, assume that $\mathbf{M} = \mathbf{L} = C_{\mathbf{G}}(Z(\mathbf{G}_\iota)^{\circ,\mathrm{split}})$.

For $i = 0, \cdots, d$ define $\mathbf{M}^i = \mathbf{G}_\iota^i \cap \mathbf{M} := C_{\mathbf{G}_\iota^i}(\iota(\mathbf{S}_\phi^{\mathrm{split}}))$. We then are in a special case of the setting in [12, §2.4], namely the case where $M^0 = G^0$ in the notation there. Now, Theorem 6.3 of [12] tells us that for any object $V \in \mathrm{Rep}_R(G)$ the map $e_{\iota,x}V \longrightarrow e_{\iota,x}^M r_P(V)$ is injective, provided that x is the image of a point $y \in \mathcal{B}(\mathbf{G}_\iota, F)$ by a "$\vec{s}$-generic" embedding with respect to y in the sense of [12, 6.1(iii)]. In this case we will say for short that x is "$\vec{s}$-generic". So, in particular, we get that if x is $\vec{s}$-generic and $e_{\iota,x}V$ is non-zero, then $\tilde{r}_P(V)$ is non-zero. In order to finish the proof that $\tilde{r}_P$ is conservative, it remains thus to show that if $V \in \mathrm{Rep}_R^{\phi,I}(G)$ is non-zero, there is an $\vec{s}$-generic point x of $\mathcal{B}_\iota$ such that $e_{\iota,x}V$ is non-zero. By definition of $\mathrm{Rep}_R^{\phi,I}(G)$ there is at least a point $y \in \mathcal{B}_\iota$ such that $e_{\iota,y}V \neq 0$. If y is $\vec{s}$-generic we are done, so suppose it is not and fix an element $\gamma \in X_*(\iota(\mathbf{S}_\phi^{\mathrm{split}}))$ such that $\langle \alpha, \gamma \rangle > 0$ for all root of $\iota(\mathbf{S}_\phi^{\mathrm{split}})$ in $\mathrm{Lie}(\mathbf{U})$ and put $y(t) := y + t\gamma \in \mathcal{B}_\iota$ for $t \in \mathbb{R}$ (recall that $\mathcal{B}_\iota$ has a translation action by $X_*(\iota(\mathbf{S}_\phi^{\mathrm{split}}))_{\mathbb{R}}$). There is $\varepsilon > 0$ such that $e_{\iota,y(t)}$ and $e_{\iota,y(-t)}$ are constant on the open interval $]0, \varepsilon[$. Denote by $y+$, resp. $y-$ any point of the form $y(t)$, resp. $y(-t)$ for $t \in]0, \varepsilon[$. Then, as in the proof of [12, Thm 6.3] we have the following properties :

– $y+$ and $y-$ are $\vec{s}$-generic

– $U \cap \vec{G}_{\iota,y} = U \cap \vec{G}_{\iota,y-} = U \cap \vec{G}^+_{\iota,y-} \supset U \cap \vec{G}^+_{\iota,y} = U \cap \vec{G}_{\iota,y+} = U \cap \vec{G}^+_{\iota,y+}$.

– $\bar{U} \cap \vec{G}_{\iota,y} = \bar{U} \cap \vec{G}_{\iota,y+} = \bar{U} \cap \vec{G}^+_{\iota,y+} \supset \bar{U} \cap \vec{G}^+_{\iota,y} = \bar{U} \cap \vec{G}_{\iota,y-} = \bar{U} \cap \vec{G}^+_{\iota,y-}$.

– $M \cap \vec{G}_{\iota,y} = M \cap \vec{G}_{\iota,y-} = M \cap \vec{G}_{\iota,y+} = \vec{M}_{\iota,y}$ and $M \cap \vec{G}^+_{\iota,y} = M \cap \vec{G}^+_{\iota,y-} = M \cap \vec{G}^+_{\iota,y+} = \vec{M}^+_{\iota,y}$.

Here, there is a subtlety when comparing to the corresponding equalities in the proof of [12, Thm 6.3]. While our $\vec{G}^+_{\iota,y}$ is one of their $\mathcal{K}_+(t)$, recall that our $\vec{G}_{\iota,y}$ is not exactly the corresponding $\mathcal{K}(t)$, but rather its pro-p-radical, since the G^0_ι contribution to $\vec{G}_{\iota,y}$ is $G^0_{\iota,x,0+}$ and not $G^0_{\iota,x,0}$. However, the intersections $U \cap \vec{G}_{\iota,y}$ and $\bar{U} \cap \vec{G}_{\iota,y}$ coincide with the corresponding $U \cap \mathcal{K}(t)$ and $\bar{U} \cap \mathcal{K}(t)$, because in our setting we have $M^0 = G^0$, so that $U \cap G^0 = \{1\} = \bar{U} \cap G^0$.

By items i) and ii) of Lemma 2.8.6 we deduce that $e_{\iota,y-} = e_{\vec{U}_{\iota,y}} e_{\iota,y}$, where $\vec{U}_{\iota,y} = U \cap \vec{G}_{\iota,y}$ and $e_{\vec{U}_{\iota,y}}$ is the corresponding averaging idempotent. Moreover, thanks to item iii) of the same lemma, the calculation in [8, §5.18] shows that $e_{\iota,y} \in \mathcal{H}_R(\vec{G}_{\iota,y}) e_{\iota,y-} \mathcal{H}_R(\vec{G}_{\iota,y})$. In particular, the non-vanishing of $e_{\iota,y}V$ implies that of $e_{\iota,y-}V$ and, since $y-$ is $\vec{s}$-generic, this concludes the proof that $\tilde{r}_P$ is conservative. $\square$

3 Equivalences of categories

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