

FRAGMENTED DEFORMATIONS OF PRIMITIVE MULTIPLE CURVES

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RESUME. A *primitive multiple curve* is a Cohen-Macaulay irreducible projective curve Y that can be locally embedded in a smooth surface, and such that Y_{red} is smooth.

This paper studies the deformations of Y to curves with smooth irreducible components, when the number of components is maximal (it is then the multiplicity n of Y).

We are particularly interested in deformations to n disjoint smooth irreducible components, which are called *fragmented deformations*. We describe them completely. We give also a characterization of primitive multiple curves having a fragmented deformation.

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1. INTRODUCTION

A *primitive multiple curve* is an algebraic variety Y over \mathbb{C} which is Cohen-Macaulay, such that the induced reduced variety $C = Y_{red}$ is a smooth projective irreducible curve, and that every closed point of Y has a neighborhood that can be embedded in a smooth surface. These curves have been defined and studied by C. Bănică and O. Forster in [1]. The simplest examples are infinitesimal neighborhoods of projective smooth curves embedded in a smooth surface (but most primitive multiple curves cannot be globally embedded in smooth surfaces, cf. [2], theorem 7.1).

Let Y be a primitive multiple curve with associated reduced curve C , and suppose that $Y \neq C$. Let \mathcal{I}_C be the ideal sheaf of C in Y . The *multiplicity* of Y is the smallest integer n such that $\mathcal{I}_C^n = 0$. We have then a filtration

$$C = C_1 \subset C_2 \subset \cdots \subset C_n = Y$$

where C_i is the subscheme corresponding to the ideal sheaf \mathcal{I}_C^i and is a primitive multiple curve of multiplicity i . The sheaf $L = \mathcal{I}_C/\mathcal{I}_C^2$ is a line bundle on C , called the *line bundle on C associated to Y* .

The deformations of double (i.e. of multiplicity 2) primitive multiple curves (also called *ribbons*) to smooth projective curves have been studied in [11]. In this paper we are interested in deformations of primitive multiple curves $Y = C_n$ of any multiplicity $n \geq 2$ to reduced curves having exactly n components which are smooth (n is the maximal number of components of deformations of Y). In this case the number of intersection points of two components is exactly $-\deg(L)$. We give some results in the general case (no assumption on $\deg(L)$) and treat more precisely the case $\deg(L) = 0$, i.e. deformations of Y to curves having exactly n disjoint irreducible components.

1.1. Motivation – Let $\pi : \mathcal{C} \rightarrow S$ be a flat projective morphism of algebraic varieties, P a closed point of S such that $\pi^{-1}(P) \simeq Y$, $\mathcal{O}_{\mathcal{C}}(1)$ a very ample line bundle on \mathcal{C} and P a polynomial in one variable with rational coefficients. Let

$$\tau : \mathcal{M}_{\mathcal{O}_{\mathcal{C}}(1)}(P) \longrightarrow S$$

be the corresponding relative moduli space of semi-stable sheaves (parametrizing the semi-stables sheaves on the fibers of π with Hilbert polynomial P with respect to the restriction of $\mathcal{O}_{\mathcal{C}}(1)$, cf. [15]).

We suppose first that there exists a closed point $s \in S$ such that \mathcal{C}_s is a smooth projective irreducible curve. Then in general τ is not flat (some other examples on non flat relative moduli spaces are given in [13]). The reason is that the generic structure of torsion free sheaves on Y is more complicated than on smooth curves, and some of these sheaves cannot be deformed to sheaves on the smooth fibers of π .

A coherent sheaf on a smooth algebraic variety X is locally free on some nonempty open subset of X . This is not true on Y . But a coherent sheaf E on Y is *quasi locally free* on some nonempty open subset of Y , i.e. on this open subset, E is locally isomorphic to a sheaf of the form $\bigoplus_{1 \leq i \leq n} m_i \mathcal{O}_{C_i}$, the sequence of non negative integers (m_1, \dots, m_n) being uniquely determined (cf. [3], [6]). It is not hard to see that if E can be extended to a coherent sheaf on \mathcal{C} , flat on S , then $R(E) = \sum_{i=1}^n i.m_i$ must be a multiple of n . For example, it is impossible to

deform the stable sheaf \mathcal{O}_{C_i} on Y in sheaves on the smooth fibers, if $1 \leq i < n$.

Now suppose that all the fibers $\pi^{-1}(s)$, $s \neq P$, are reduced with exactly n smooth components. I conjecture that (with suitable hypotheses) a torsion free coherent sheaf on Y can be extended to a coherent sheaf on \mathcal{C} , flat on S , using the fact that we allow coherent sheaves of the reducible fibers \mathcal{C}_s that have not the same rank on all the components. This would be a step in the study of the flatness of τ . For example (for suitable π), there exists a coherent sheaf \mathcal{E} on \mathcal{C} , flat on S , such that $\mathcal{E}_P = \mathcal{O}_{\mathcal{C}}$, and that for $s \neq P$, \mathcal{E}_s is the structural sheaf of an irreducible component of \mathcal{C}_s .

Moduli spaces of sheaves on reducible curves have been studied in [16], [17], [18].

1.2. Maximal reducible deformations – Let (S, P) be the germ of a smooth curve. Let Y be a primitive multiple curve of multiplicity $n \geq 2$ and $k > 0$ an integer. Let $\pi : \mathcal{C} \rightarrow S$ be a flat morphism, where \mathcal{C} is a reduced algebraic variety, such that

- For every closed point $s \in S$ such that $s \neq P$, the fiber \mathcal{C}_s has k irreducible components, which are smooth and transverse, and any three of these components have no common point.
- The fiber \mathcal{C}_P is isomorphic to Y .

We show that by making a change of variable, i.e. by considering a suitable germ (S', P') and a non constant morphism $\tau : S' \rightarrow S$, and replacing π with $\tau^*\mathcal{C} \rightarrow S'$, we can suppose that \mathcal{C} has exactly k irreducible components, inducing on every fiber \mathcal{C}_s , $s \neq P$ the k irreducible components of \mathcal{C}_s . In this case π is called a *reducible deformation of Y of length k* .

We show that $k \leq n$. We say that π (or \mathcal{C}) is a *maximal reducible deformation of Y* if $k = n$.

Suppose that π is a maximal reducible deformation of Y . We show that if \mathcal{C}' is the union of $i > 0$ irreducible components of \mathcal{C} , and $\pi' : \mathcal{C}' \rightarrow S$ is the restriction of π , then $\pi'^{-1}(P) \simeq C_i$, and π' is a maximal reducible deformation of C_i . Let $s \in S \setminus \{P\}$. We prove that the irreducible components of \mathcal{C}_s have the same genus as C . Moreover, if D_1, D_2 are distinct irreducible components of \mathcal{C}_s , then $D_1 \cap D_2$ consists of $-\deg(L)$ points.

1.3. Fragmented deformations (definition) – Let Y be a primitive multiple curve of multiplicity $n \geq 2$ and $\pi : \mathcal{C} \rightarrow S$ a maximal reducible deformation of Y . We call it a *fragmented deformation of Y* if $\deg(L) = 0$, i.e. if for every $s \in S \setminus \{P\}$, \mathcal{C}_s is the disjoint union of n smooth curves. In this case \mathcal{C} has n irreducible components $\mathcal{C}_1, \dots, \mathcal{C}_n$ which are smooth surfaces.

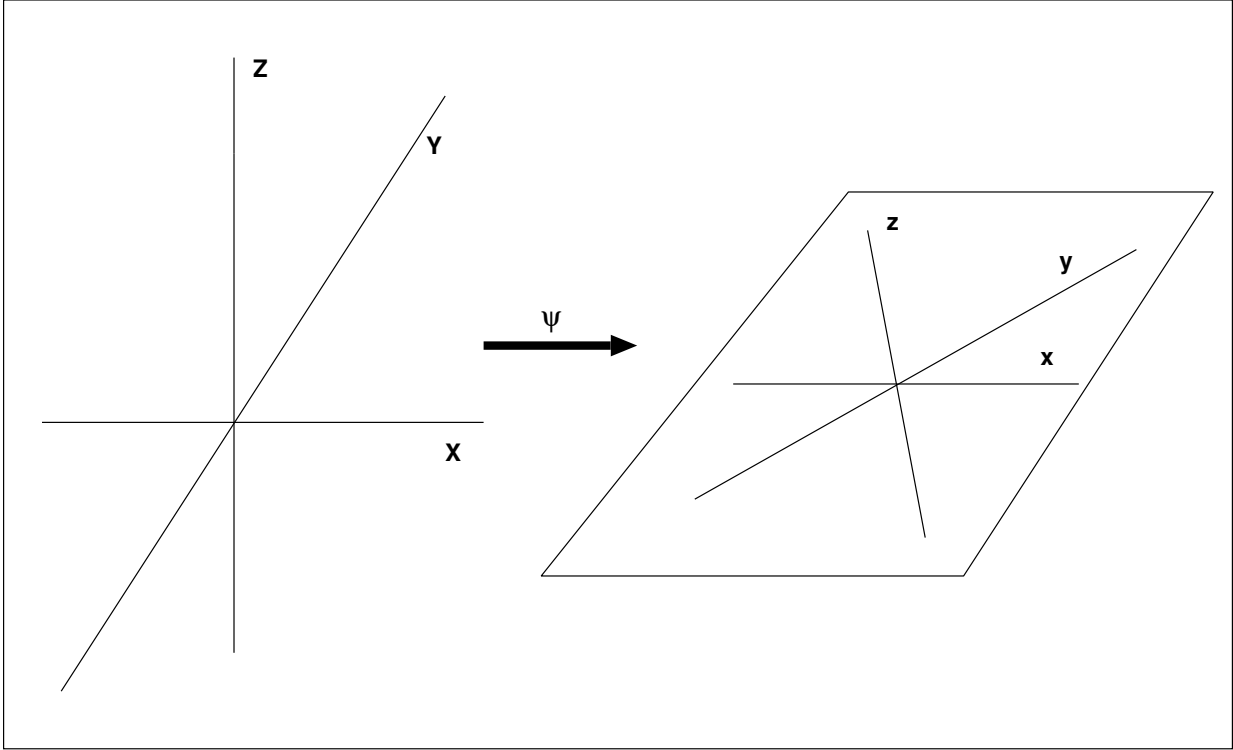
The variety \mathcal{C} appears as a particular case of a *gluing* of $\mathcal{C}_1, \dots, \mathcal{C}_n$ along C (cf. 4.1.5). We prove (proposition 4.1.6) that such a gluing \mathcal{D} is a fragmented deformation of a primitive multiple curve if and only if every closed point in C has a neighborhood in \mathcal{D} that can be embedded in a smooth variety of dimension 3. The simplest gluing is the trivial or *initial gluing* \mathcal{A} . An open subset U of \mathcal{A} (and \mathcal{C}) is given by open subsets U_1, \dots, U_n of $\mathcal{C}_1, \dots, \mathcal{C}_n$ respectively, having the same intersection with C , and

$$\mathcal{O}_{\mathcal{A}}(U) = \{(\alpha_1, \dots, \alpha_n) \in \mathcal{O}_{\mathcal{C}_1}(U \cap \mathcal{C}_1) \times \dots \times \mathcal{O}_{\mathcal{C}_n}(U \cap \mathcal{C}_n); \alpha_{1|C} = \dots = \alpha_{n|C}\},$$

and $\mathcal{O}_{\mathcal{C}}(U)$ appears as a subalgebra of $\mathcal{O}_{\mathcal{A}}(U)$, hence we have a canonical morphism $\mathcal{A} \rightarrow \mathcal{C}$.

We can view elements of $\mathcal{O}_{\mathcal{C}}(U)$ as n -tuples $(\alpha_1, \dots, \alpha_n)$, with $\alpha_i \in \mathcal{O}_{\mathcal{C}_i}(U \cap \mathcal{C}_i)$. In particular we can write $\pi = (\pi_1, \dots, \pi_n)$.

1.4. A simple analogy – Consider n copies of \mathbb{C} glued at 0. Two extreme examples appear: the trivial gluing \mathcal{A}_0 (the set of coordinate lines in \mathbb{C}^n), and a set \mathcal{C}_0 of n lines in \mathbb{C}^2 . We can easily construct a bijective morphism $\Psi : \mathcal{A}_0 \rightarrow \mathcal{C}_0$ sending each coordinate line to a line in the plane



But the two schemes are of course not isomorphic: the maximal ideal of 0 in \mathcal{A}_0 needs n generators, but 2 are enough for the maximal ideal of 0 in \mathcal{C}_0 .

Let $\pi_{\mathcal{C}_0} : \mathcal{C}_0 \rightarrow \mathbb{C}$ be a morphism sending each component linearly onto \mathbb{C} , and $\pi_{\mathcal{A}_0} = \pi_{\mathcal{C}_0} \circ \Psi : \mathcal{A}_0 \rightarrow \mathbb{C}$. The difference of \mathcal{A}_0 and \mathcal{C}_0 can be also seen by using the fibers of 0: we have

$$\pi_{\mathcal{C}_0}^{-1}(0) \simeq \text{spec}(\mathbb{C}[t]/(t^n)) \quad \text{and} \quad \pi_{\mathcal{A}_0}^{-1}(0) \simeq \text{spec}(\mathbb{C}[t_1, \dots, t_{n-1}]/(t_1, \dots, t_{n-1})^2).$$

Let \mathcal{D} be a general gluing of n copies of \mathbb{C} at 0, such that there exists a morphism $\pi : \mathcal{D} \rightarrow \mathbb{C}$ inducing the identity on each copy of \mathbb{C} . It is easy to see that we have $\pi^{-1}(0) \simeq \text{spec}(\mathbb{C}[t]/(t^n))$ if and only if some neighborhood of 0 in \mathcal{D} can be embedded in a smooth surface.

1.5. Fragmented deformations (main properties) – Let $\pi : \mathcal{C} \rightarrow S$ be a fragmented deformation of $Y = C_n$. Let $I \subset \{1, \dots, n\}$ be a proper subset, I^c its complement, and $\mathcal{C}_I \subset \mathcal{C}$ the subscheme union of the $\mathcal{C}_i, i \in I$. We prove (theorem 4.3.7) that the ideal sheaf $\mathcal{I}_{\mathcal{C}_I}$ of \mathcal{C}_I is isomorphic to $\mathcal{O}_{\mathcal{C}_{I^c}}$.

In particular, the ideal sheaf $\mathcal{I}_{\mathcal{C}_i}$ of \mathcal{C}_i is generated by a single regular function on \mathcal{C} . We show that we can find such a generator such that for $1 \leq j \leq n, j \neq i$, its j -th coordinate can be written as $\alpha_j \pi_j^{p_{ij}}$, with $p_{ij} > 0$ and $\alpha_j \in H^0(\mathcal{O}_S)$ such that $\alpha_j(P) \neq 0$. If $1 \leq j \leq n$ and $j \neq i$, we can then obtain a generator that can be written as

$$\mathbf{u}_{ij} = (u_1, \dots, u_m),$$

with

$$u_i = 0, \quad u_m = \alpha_{ij}^{(m)} \pi_m^{p_{im}} \text{ for } m \neq i, \quad \alpha_{ij}^{(j)} = 1.$$

The constants $\mathbf{a}_{ij}^{(m)} = \alpha_{ij|C}^{(m)} \in \mathbb{C}$ have interesting properties (propositions 4.5.2, 4.4.6). Let $p_{ii} = 0$ for $1 \leq i \leq n$. The symmetric matrix $(p_{ij})_{1 \leq i, j \leq n}$ is called the *spectrum* of π (or \mathcal{C}).

It follows also from the fact that $\mathcal{I}_{C_i} = (\mathbf{u}_{ij})$ that Y is a *simple* primitive multiple curve, i.e. the ideal sheaf of C in $Y = C_n$ is isomorphic to $\mathcal{O}_{C_{n-1}}$. Conversely, we show in theorem 4.7.1 that if Y is a simple primitive multiple curve, then there exists a fragmented deformation of Y .

We give in 4.4 and 4.5 a way to construct fragmented deformations by induction on n . This is used later to prove statements on fragmented deformations by induction on n .

1.6. n -stars and structure of fragmented deformations – An n -star of (S, P) is a gluing \mathbf{S} of n copies S_1, \dots, S_n of S at P , together with a morphism $\tau : \mathbf{S} \rightarrow S$ which is an identity on each S_i . All the n -stars have the same underlying Zariski topological space $S(n)$.

An n -star is called *oblate* if some neighborhood of P can be embedded in a smooth surface. This is the case if and only $\tau^{-1}(0) \simeq \text{spec}(\mathbb{C}[t]/(t^n))$.

Oblate n -stars are analogous to fragmented deformations but simpler. We provide a way to build oblate n -stars by induction on n .

Let $\pi : \mathcal{C} \rightarrow S$ be a fragmented deformation of $Y = C_n$. We associate to it an oblate n -star \mathbf{S} of S . Let \mathcal{C}^{top} be the Zariski topological space of \mathcal{C} . We have an obvious continuous map $\tilde{\pi} : \mathcal{C}^{top} \rightarrow S(n)$. For every open subset U of $S(n)$, $\mathcal{O}_{\mathbf{S}}(U)$ is the set of $(\alpha_1, \dots, \alpha_n) \in \mathcal{O}_{\mathcal{C}}(\tilde{\pi}^{-1}(U))$ such that $\alpha_i \in \mathcal{O}_{S_i}(U \cap S_i)$ for $1 \leq i \leq n$. We obtain also a canonical morphism $\mathbf{\Pi} : \mathcal{C} \rightarrow \mathbf{S}$. We prove (theorem 5.6.2) that $\mathbf{\Pi}$ is flat. Hence it is a flat family of smooth curves, with $\mathbf{\Pi}^{-1}(P) = C$. The converse is also true, i.e. starting from an oblate n -star of S and a flat family of smooth curves parametrized by it, we obtain a fragmented deformation of a multiple primitive curve of multiplicity n .

1.7. Fragmented deformations of double curves – Let $Y = C_2$ be a primitive double curve, C its associated smooth curve, $\pi : \mathcal{C} \rightarrow S$ a fragmented deformation of Y , of spectrum $\begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$, and $\mathcal{C}_1, \mathcal{C}_2$ the irreducible components of \mathcal{C} . For $i = 1, 2$, $q > 0$, let C_i^q be the infinitesimal neighborhood of order q of C in \mathcal{C}_i (defined by the ideal sheaf (π_i^q)). It is a primitive multiple curve of multiplicity q .

It follows from 4.3.5 that C_1^p and C_2^p are isomorphic, and C_1^{p+1}, C_2^{p+1} are two extensions of C_1^p in primitive multiple curves of multiplicity $p + 1$. According to [4] these extensions are parametrized by an affine space with associated vector space $H^1(C, T_C)$ (where T_C is the tangent bundle of C). Let $w \in H^1(C, T_C)$ be the vector from C_1^{p+1} to C_2^{p+1} .

Similarly, the primitive double curves with associated smooth curve C such that $\mathcal{I}_C \simeq \mathcal{O}_C$ are parametrized by $\mathbb{P}(H^1(C, T_C)) \cup \{0\}$ (cf. [2], [4]).

We prove in theorem 6.0.1 that the point of $\mathbb{P}(H^1(C, T_C)) \cup \{0\}$ corresponding to C_2 is $\mathbb{C}w$.

1.8. Notation: Let X be an algebraic variety and $Y \subset X$ a closed subvariety. We will denote by $\mathcal{I}_{Y,X}$ (or \mathcal{I}_Y if there is no risk of confusion) the ideal sheaf of Y in X .

2. PRELIMINARIES

2.1. LOCAL EMBEDDINGS IN SMOOTH VARIETIES

2.1.1. Proposition: *Let X be an algebraic variety, x a closed point of X and n a positive integer. Then the three following properties are equivalent:*

- (i) *There exist a neighborhood U of x and an embedding $U \subset Z$ in a smooth variety of dimension n .*
- (ii) *The $\mathcal{O}_{X,x}$ -module $m_{X,x}$ (maximal ideal of x) can be generated by n elements.*
- (iii) *We have $\dim_{\mathbb{C}}(m_{X,x}/m_{X,x}^2) \leq n$.*

Proof. It is obvious that (i) implies (ii), and (ii),(iii) are equivalent according to Nakayama's lemma. It remains to prove that (iii) implies (i).

Suppose that (iii) is true. There exist an integer N and an embedding $X \subset \mathbb{P}_N$. Let \mathcal{I}_X be the ideal sheaf of X in \mathbb{P}_N . Let p be the biggest integer such that there exists $f_1, \dots, f_p \in \mathcal{I}_{X,x}$ whose images in the \mathbb{C} -vector space $m_{\mathbb{P}_N,x}/m_{\mathbb{P}_N,x}^2$ are linearly independent. Then we have

$$\mathcal{I}_{X,x} \subset (f_1, \dots, f_p) + m_{\mathbb{P}_N,x}^2.$$

In fact, let $f \in \mathcal{I}_{X,x}$. Since p is maximal, the image of f in $m_{\mathbb{P}_N,x}/m_{\mathbb{P}_N,x}^2$ is a linear combination of those of f_1, \dots, f_p . Hence we can write

$$f = \sum_{i=1}^p \lambda_i f_i + g, \quad \text{with } \lambda_i \in \mathbb{C}, g \in m_{\mathbb{P}_N,x}^2,$$

and our assertion is proved. It follows that we have a surjective morphism

$$\alpha : \mathcal{O}_{X,x}/m_{X,x}^2 \longrightarrow \mathcal{O}_{\mathbb{P}_N,x}/((f_1, \dots, f_p) + m_{\mathbb{P}_N,x}^2).$$

We have

$$\dim_{\mathbb{C}}(\mathcal{O}_{X,x}/m_{X,x}^2) \leq n + 1, \quad \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{P}_N,x}/((f_1, \dots, f_p) + m_{\mathbb{P}_N,x}^2)) = N - p + 1.$$

Hence $N - p + 1 \leq n + 1$, i.e. $p \geq N - n$. We can take for Z a neighborhood of x in the subvariety of \mathbb{P}_N defined by f_1, \dots, f_{N-n} , which is smooth at x . \square

2.2. FLAT FAMILIES OF COHERENT SHEAVES

Let (S, P) be the germ of a smooth curve and $t \in \mathcal{O}_{S,P}$ a generator of the maximal ideal. Let $\pi : X \rightarrow S$ be a flat morphism. If \mathcal{E} is a coherent sheaf on X , \mathcal{E} is flat on S at $x \in \pi^{-1}(P)$ if and only if the multiplication by $t : \mathcal{E}_x \rightarrow \mathcal{E}_x$ is injective. In particular the multiplication by $t : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}$ is injective.

2.2.1. Lemma: *Let \mathcal{E} be a coherent sheaf on X flat on S . Then, for every open subset U of X , the restriction $\mathcal{E}(U) \rightarrow \mathcal{E}(U \setminus \pi^{-1}(P))$ is injective.*

Proof. Let $s \in \mathcal{E}(U)$ whose restriction to $U \setminus \pi^{-1}(P)$ vanishes. We must show that $s = 0$. By covering U with smaller open subsets we can suppose that U is affine: $U = \text{spec}(A)$. Hence $U \setminus \pi^{-1}(P) = \text{spec}(A_t)$. Let $M = \mathcal{E}(U)$, it is an A -module. We have $\mathcal{E}|_U = \widetilde{M}$ and $\mathcal{E}(U \setminus \pi^{-1}(P)) = M_t$. Hence if the restriction of s to $U \setminus \pi^{-1}(P)$ vanishes, there exists an integer

$n > 0$ such that $t^n s = 0$. Since the multiplication by t is injective (because \mathcal{E} is flat on S), we have $s = 0$. \square

Let \mathcal{E} be a coherent sheaf on X flat on S . Let $\mathcal{F} \subset \mathcal{E}|_{X \setminus \pi^{-1}(P)}$ be a subsheaf. For every open subset U of X we denote by $\overline{\mathcal{F}}(U)$ the subset of $\mathcal{F}(U \setminus \pi^{-1}(P))$ of elements that can be extended to sections of \mathcal{E} on U . If $V \subset U$ is an open subset, the restriction $\mathcal{F}(U \setminus \pi^{-1}(P)) \rightarrow \mathcal{F}(V \setminus \pi^{-1}(P))$ induces a morphism $\overline{\mathcal{F}}(U) \rightarrow \overline{\mathcal{F}}(V)$.

2.2.2. Proposition: $\overline{\mathcal{F}}$ is a subsheaf of \mathcal{E} , and $\mathcal{E}/\overline{\mathcal{F}}$ is flat on S .

Proof. To prove the first assertion, we must show that if U is an open subset of X and $(U_i)_{i \in I}$ is an open cover of U , then

- (i) If $s \in \overline{\mathcal{F}}(U)$ is such that for every i we have $s|_{U_i} = 0$, then $s = 0$.
- (ii) For every $i \in I$ let $s_i \in \overline{\mathcal{F}}(U_i)$. Then if for all i, j we have $s_i|_{U_{ij}} = s_j|_{U_{ij}}$, then there exists $s \in \overline{\mathcal{F}}(U)$ such that for every $i \in I$ we have $s|_{U_i} = s_i$.

This follows easily from lemma 2.2.1.

Now we prove that $\mathcal{E}/\overline{\mathcal{F}}$ is flat on S . Let $x \in \pi^{-1}(P)$ and $u \in (\mathcal{E}/\overline{\mathcal{F}})_x$ such that $tu = 0$. We must show that $u = 0$. Let $v \in \mathcal{E}_x$ over u . Then we have $tv \in \overline{\mathcal{F}}_x$. Let U be a neighborhood of x such that tv comes from $w \in \overline{\mathcal{F}}(U)$. This means that $w|_{U \setminus \pi^{-1}(P)} \in \mathcal{F}(U \setminus \pi^{-1}(P))$. Since t is invertible on $U \setminus \pi^{-1}(P)$ we can write $w = tw'$, with $w' \in \mathcal{F}(U \setminus \pi^{-1}(P))$. We have then $w' = v$ on $U \setminus \pi^{-1}(P)$. Hence $v \in \overline{\mathcal{F}}_x$ and $u = 0$. \square

2.3. PRIMITIVE MULTIPLE CURVES

(cf. [1], [2], [3], [4], [6], [7], [8], [10]).

Let C be a smooth connected projective curve. A *multiple curve with support C* is a Cohen-Macaulay scheme Y such that $Y_{red} = C$.

Let n be the smallest integer such that $Y = C^{(n-1)}$, $C^{(k-1)}$ being the k -th infinitesimal neighborhood of C , i.e. $\mathcal{I}_{C^{(k-1)}} = \mathcal{I}_C^k$. We have a filtration $C = C_1 \subset C_2 \subset \dots \subset C_n = Y$ where C_i is the biggest Cohen-Macaulay subscheme contained in $Y \cap C^{(i-1)}$. We call n the *multiplicity* of Y .

We say that Y is *primitive* if, for every closed point x of C , there exists a smooth surface S , containing a neighborhood of x in Y as a locally closed subvariety. In this case, $L = \mathcal{I}_C/\mathcal{I}_{C_2}$ is a line bundle on C and we have $\mathcal{I}_{C_j} = \mathcal{I}_X^j$, $\mathcal{I}_{C_j}/\mathcal{I}_{C_{j+1}} = L^j$ for $1 \leq j < n$. We call L the line bundle on C *associated* to Y . Let $P \in C$. Then there exist elements y, t of $m_{S,P}$ (the maximal ideal of $\mathcal{O}_{S,P}$) whose images in $m_{S,P}/m_{S,P}^2$ form a basis, and such that for $1 \leq i < n$ we have $\mathcal{I}_{C_i,P} = (y^i)$.

The simplest case is when Y is contained in a smooth surface S . Suppose that Y has multiplicity n . Let $P \in C$ and $f \in \mathcal{O}_{S,P}$ a local equation of C . Then we have $\mathcal{I}_{C_i,P} = (f^i)$ for $1 < j \leq n$, in particular $\mathcal{I}_{Y,P} = (f^n)$, and $L = \mathcal{O}_C(-C)$.

We will write $\mathcal{O}_n = \mathcal{O}_{C_n}$ and we will see \mathcal{O}_i as a coherent sheaf on C_n with schematic support C_i if $1 \leq i < n$.

If \mathcal{E} is a coherent sheaf on Y one defines its *generalized rank* $R(\mathcal{E})$ and *generalized degree* $\text{Deg}(\mathcal{E})$ (cf. [6], 3-): take any filtration of \mathcal{E}

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$$

by subsheaves such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is concentrated on C for $1 \leq i \leq n$, then

$$R(\mathcal{E}) = \sum_{i=1}^n \text{rk}(\mathcal{E}_i/\mathcal{E}_{i-1}) \quad \text{and} \quad \text{Deg}(\mathcal{E}) = \sum_{i=1}^n \text{deg}(\mathcal{E}_i/\mathcal{E}_{i-1}).$$

Let $\mathcal{O}_Y(1)$ be a very ample line bundle on Y . Then the Hilbert polynomial of \mathcal{E} is

$$P_{\mathcal{E}}(m) = R(\mathcal{E}) \text{deg}(\mathcal{O}_C(1))m + \text{Deg}(\mathcal{E}) + R(\mathcal{E})(1 - g)$$

(where g is the genus of C).

We deduce from proposition 2.1.1:

2.3.1. Proposition: *Let Y be a multiple curve with support C . Then Y is a primitive multiple curve if and only if $\mathcal{I}_C/\mathcal{I}_C^2$ is zero, or a line bundle on C .*

2.3.2. Parametrization of double curves - In the case of double curves, D. Bayer and D. Eisenbud have obtained in [2] the following classification: if Y is of multiplicity 2, we have an exact sequence of vector bundles on C

$$0 \longrightarrow L \longrightarrow \Omega_{Y|C} \longrightarrow \omega_C \longrightarrow 0$$

which is split if and only if Y is the *trivial curve*, i.e. the second infinitesimal neighborhood of C , embedded by the zero section in the dual bundle L^* , seen as a surface. If Y is not trivial, it is completely determined by the line of $\text{Ext}_{\mathcal{O}_C}^1(\omega_C, L)$ induced by the preceding exact sequence. The non trivial primitive curves of multiplicity 2 and of associated line bundle L are therefore parametrized by the projective space $\mathbb{P}(\text{Ext}_{\mathcal{O}_C}^1(\omega_C, L))$.

2.4. SIMPLE PRIMITIVE MULTIPLE CURVES

Let C be a smooth projective irreducible curve, $n \geq 2$ an integer and C_n a primitive multiple curve of multiplicity n and associated reduced curve C . Then the ideal sheaf \mathcal{I}_C of C in C_n is a line bundle on C_{n-1} .

We say that C_n is *simple* if $\mathcal{I}_C \simeq \mathcal{O}_{n-1}$.

In this case the line bundle on C associated to C_n is \mathcal{O}_C . The following result is proved in [8] (théorème 1.2.1):

2.4.1. Theorem: *Suppose that C_n is simple. Then there exists a flat family of smooth projective curves $\tau : \mathcal{C} \rightarrow \mathbb{C}$ such that $\tau^{-1}(0) \simeq C$ and that C_n is isomorphic to the n -th infinitesimal neighborhood of C in \mathcal{C} .*

3. REDUCIBLE REDUCED DEFORMATIONS OF PRIMITIVE MULTIPLES CURVES

3.1. CONNECTED COMPONENTS

Let (S, P) be the germ of a smooth curve and $t \in \mathcal{O}_{S,P}$ a generator of the maximal ideal. Let $n > 0$ be an integer and $Y = C_n$ a projective primitive multiple curve of multiplicity n .

Let $k > 0$ be an integer. Let $\pi : \mathcal{C} \rightarrow S$ be a flat morphism, where \mathcal{C} is a reduced algebraic variety, such that

- For every closed point $s \in S$ such that $s \neq P$, the fiber \mathcal{C}_s has k irreducible components, which are smooth and transverse, and any three of these components have no common point.
- The fiber \mathcal{C}_P is isomorphic to C_n .

It is easy to see that the irreducible components of \mathcal{C} are reduced surfaces.

Let Z be the open subset of $\mathcal{C} \setminus \mathcal{C}_P$ of points z belonging to only one irreducible component of $\mathcal{C}_{\pi(z)}$. Then the restriction of $\pi : Z \rightarrow S \setminus \{P\}$ is a smooth morphism. For every $s \in S \setminus \{P\}$, let $\mathcal{C}'_s = \mathcal{C}_s \cap Z$. It is the open subset of smooth points of \mathcal{C}_s .

Let $z \in Z$ and $s = \pi(z)$. There exist a neighborhood (for the Euclidean topology) U of s , isomorphic to \mathbb{C} , and a neighborhood V of z such that $V \simeq \mathbb{C}^2$, $\pi(V) = U$, the restriction of $\pi : V \rightarrow U$ being the projection $\mathbb{C}^2 \rightarrow \mathbb{C}$ on the first factor. We deduce easily from that the following facts:

- let $s \in S \setminus \{P\}$ and C_1 an irreducible component of \mathcal{C}_s . Let $z_1, z_2 \in C_1 \cap Z$. Then there exist neighborhoods (in Z , for the Euclidean topology) U_1, U_2 of z_1, z_2 respectively, such that if $y_1 \in U_1, y_2 \in U_2$ are such that $\pi(y_1) = \pi(y_2)$, then y_1 and y_2 belong to the same irreducible component of $\mathcal{C}_{\pi(y_1)}$.
- for every continuous map $\sigma : [0, 1] \rightarrow S \setminus \{P\}$ and every $z \in Z$ such that $\sigma(0) = \pi(z)$ there exists a lifting of σ , $\sigma' : [0, 1] \rightarrow Z$ such that $\sigma'(0) = z$. Moreover, if $\sigma'' : [0, 1] \rightarrow Z$ is another lifting of σ such that $\sigma''(0) = z$, then $\sigma'(1)$ and $\sigma''(1)$ are in the same irreducible component of $\mathcal{C}_{\sigma(1)}$. More generally, if we only impose that $\sigma''(0)$ is in the same irreducible component of $\mathcal{C}_{\sigma(0)}$ as z , then $\sigma'(1)$ and $\sigma''(1)$ are in the same irreducible component of $\mathcal{C}_{\sigma(1)}$.

3.1.1. Lemma: *Let $\sigma_0, \sigma_1 : [0, 1] \rightarrow S \setminus \{P\}$ be two continuous maps such that $\sigma_0(0) = \sigma_1(0)$, $s = \sigma_0(1) = \sigma_1(1)$. Suppose that they are homotopic. Let σ'_0, σ'_1 be liftings $[0, 1] \rightarrow Z$ of σ_0, σ_1 respectively, such that $\sigma'_0(0) = \sigma'_1(0)$. Then $\sigma'_0(1)$ and $\sigma'_1(1)$ belong to the same irreducible component of \mathcal{C}'_s .*

Proof. Let

$$\Psi : [0, 1] \times [0, 1] \longrightarrow S \setminus \{P\}$$

be an homotopy:

$$\Psi(0, t) = \sigma_0(t), \quad \Psi(1, t) = \sigma_1(t), \quad \Psi(t, 0) = \sigma_0(0), \quad \Psi(t, 1) = \sigma_0(1)$$

for $0 \leq t \leq 1$. For every $u \in [0, 1]$ and $\epsilon > 0$ let $I_{u,\epsilon} = [u - \epsilon, u + \epsilon] \cap [0, 1]$. By using the local structure of $\pi|_Z$ for the Euclidean topology it is easy to see that for every $u \in [0, 1]$, there exists

an $\epsilon > 0$ such that the restriction of Ψ

$$I_{u,\epsilon} \times [0, 1] \longrightarrow S \setminus \{P\}$$

can be lifted to a morphism

$$\Psi' : I_{u,\epsilon} \times [0, 1] \longrightarrow Z$$

such that $\Psi'(t, 0) = \sigma'_0(0)$ for every $t \in I_{u,\epsilon}$. It follows that if $I_{u,\epsilon} = [a_{u,\epsilon}, b_{u,\epsilon}]$, then $\Psi'(a_{u,\epsilon}, 1)$ and $\Psi'(b_{u,\epsilon}, 1)$ are in the same irreducible component of $\mathcal{C}'_{\sigma_0(1)}$. We have just to cover $[0, 1]$ with a finite number of intervals $I_{u,\epsilon}$ to obtain the result. \square

Let $s \in S \setminus \{P\}$, D_1, \dots, D_k be the irreducible components of \mathcal{C}'_s and $x_i \in D_i$ for $1 \leq i \leq k$. Let σ be a loop of $S \setminus \{P\}$ with origin s , defining a generator of $\pi_1(S \setminus \{P\})$. Let i be an integer such that $1 \leq i \leq k$. The liftings $\sigma' : [0, 1] \rightarrow Z$ of σ such that $\sigma'(0) = x_i$ end up at a component D_j which does not depend on x_i . Hence we can write

$$j = \alpha_{\mathcal{C}}(i).$$

3.1.2. Lemma: $\alpha_{\mathcal{C}}$ is a permutation of $\{1, \dots, k\}$.

Proof. Suppose that $i \neq j$ and $\alpha_{\mathcal{C}}(i) = \alpha_{\mathcal{C}}(j)$. By inverting the paths we find liftings of paths from $D_{\alpha_{\mathcal{C}}(i)}$ to D_i and D_j . This contradicts lemma 3.1.1. \square

Let $p > 0$ be an integer such that $\alpha_{\mathcal{C}}^p = I_{\{1, \dots, k\}}$. Let t be a generator of the maximal ideal of $\mathcal{O}_{S,P}$, K the field of rational functions on S and $K' = K(t^{1/p})$. Let S' be the germ of the curve corresponding to K' , $\theta : S' \rightarrow S$ canonical the morphism and P' the unique point of $\theta^{-1}(P)$. Let $\mathcal{D} = \theta^*(\mathcal{C})$. We have therefore a cartesian diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\rho} & S' \\ \downarrow \Theta & & \downarrow \theta \\ \mathcal{C} & \xrightarrow{\pi} & S \end{array}$$

where ρ is flat, and for every $s' \in S'$, Θ induces an isomorphism $\mathcal{D}_{s'} \simeq \mathcal{C}_{\theta(s')}$. We have

$$\alpha_{\mathcal{D}} = I_{\{1, \dots, k\}}.$$

Let $Z' \subset \mathcal{D}$ be the complement of the union of $\rho^{-1}(P')$ and of the singular points of the curves $\mathcal{D}_{s'}$, $s' \neq P'$ (hence $Z' = \Theta^{-1}(Z)$).

3.1.3. Proposition: *The open subset Z' has exactly k irreducible components Z'_1, \dots, Z'_k . Let $\overline{Z'_1}, \dots, \overline{Z'_k}$ be their closures in \mathcal{D} . Then for every $s' \in S' \setminus \{P'\}$, the $Z'_i \cap \mathcal{D}_{s'}$, $1 \leq i \leq k$, are the irreducible components of $\mathcal{D}_{s'}$ minus the intersection points with the other components, and the $\overline{Z'_i} \cap \mathcal{D}_{s'}$ are the irreducible components of $\mathcal{D}_{s'}$.*

3.1.4. Definition: *Let $k > 0$ be an integer. A reducible deformation of length k of C_n is a flat morphism $\pi : \mathcal{C} \rightarrow S$, where \mathcal{C} is a reduced algebraic variety, such that*

- *For every closed point $s \in S$, $s \neq P$, the fiber \mathcal{C}_s has k irreducible components, which are smooth and transverse, and any three of these components have no common point.*

- The fiber \mathcal{C}_P is isomorphic to C_n .
- We have $\alpha_{\mathcal{C}} = I_{\{1, \dots, k\}}$.

3.2. MAXIMAL REDUCIBLE DEFORMATIONS

Let (S, P) be the germ of a smooth curve and $t \in \mathcal{O}_{S, P}$ a generator of the maximal ideal. Let $n > 0$ be an integer and $Y = C_n$ a projective primitive multiple curve of multiplicity n , with underlying smooth curve C . We denote by g the genus of C and L the line bundle on C associated to C_n .

Let $\pi : \mathcal{C} \rightarrow S$ be a reducible deformation of length k of C_n . Let Z_1, \dots, Z_k be the closed subvarieties of $\pi^{-1}(S \setminus \{P\})$ such that for every $s \in S \setminus \{P\}$, Z_{1s}, \dots, Z_{ks} are the irreducible components of \mathcal{C}_s (cf. prop. 3.1.3).

For $1 \leq i \leq k$, we denote by \mathcal{J}_i the ideal sheaf of $Z_1 \cup \dots \cup Z_i$ in $\pi^{-1}(S \setminus \{P\})$. This sheaf is flat on $S \setminus \{P\}$, and we have

$$0 = \mathcal{J}_k \subset \mathcal{J}_{k-1} \subset \dots \subset \mathcal{J}_1 \subset \mathcal{O}_{\pi^{-1}(S \setminus \{P\})} .$$

The quotients $\mathcal{O}_{\pi^{-1}(S \setminus \{P\})} / \mathcal{J}_1$, $\mathcal{J}_i / \mathcal{J}_{i+1}$, $1 \leq i < k$, are also flat on $S \setminus \{P\}$. We obtain the filtration of sheaves on \mathcal{C}

$$0 = \overline{\mathcal{J}}_k \subset \overline{\mathcal{J}}_{k-1} \subset \dots \subset \overline{\mathcal{J}}_1 \subset \mathcal{O}_{\mathcal{C}} .$$

(cf. 2.2). According to proposition 2.2.2 the quotients $\mathcal{O}_{\mathcal{C}} / \overline{\mathcal{J}}_1$ and $\overline{\mathcal{J}}_i / \overline{\mathcal{J}}_{i+1}$, $1 \leq i < n$, are flat on S . We have $\mathcal{O}_{\pi^{-1}(S \setminus \{P\})} / \mathcal{J}_1 = \mathcal{O}_{Z_1}$. We denote by \mathbf{X}_i the closed subvariety of \mathcal{C} corresponding to the ideal sheaf $\overline{\mathcal{J}}_i$.

Similarly we consider the ideal sheaf \mathcal{J}'_i of $Z_{i+1} \cup \dots \cup Z_n$ on $\pi^{-1}(S \setminus \{P\})$, the associated ideal sheaf $\overline{\mathcal{J}}'_i$ on \mathcal{C} and the corresponding subvariety \mathbf{X}'_i .

3.2.1. Proposition: We have $k \leq n$.

Proof. Let $\mathcal{E}_0 = \mathcal{O}_{\mathcal{C}} / \overline{\mathcal{J}}_1$ and $\mathcal{E}_i = \overline{\mathcal{J}}_i / \overline{\mathcal{J}}_{i+1}$ for $1 \leq i < n$. The sheaves \mathcal{E}_{iP} are not concentrated on a finite number of points. To see this we use a very ample line bundle $\mathcal{O}(1)$ on \mathcal{C} . The Hilbert polynomial of \mathcal{E}_{iP} is the same as that of \mathcal{E}_{is} , $s \neq P$, hence it is not constant. So we have $R(\mathcal{E}_i) \geq 1$ ($R(\mathcal{E}_i)$ is the generalized rank of \mathcal{E}_i , cf. 2.3), and since

$$(1) \quad n = R(\mathcal{O}_{C_n}) = \sum_{i=0}^k R(\mathcal{E}_{iP}) ,$$

we have $k \leq n$. □

3.2.2. Definition: We say that π (or \mathcal{C}) is a maximal reducible deformation of C_n if $k = n$.

3.2.3. Theorem: Suppose that \mathcal{C} is a maximal reducible deformation of C_n . Then we have, for $1 \leq i < n$

$$\overline{\mathcal{J}}_{i, P} = \mathcal{I}_{C_i, C_n}$$

and \mathbf{X}_i is a maximal reducible deformation of C_i .

Proof. Let $\mathcal{O}_{\mathcal{C}}(1)$ be a very ample line bundle on \mathcal{C} .

Let Q be a closed point of C . Let $z \in \mathcal{O}_{n,Q}$ be an equation of C and $x \in \mathcal{O}_{n,Q}$ over a generator of the maximal ideal of Q in $\mathcal{O}_{C,Q}$. Let $\mathbf{z}, \mathbf{x} \in \mathcal{O}_{\mathcal{C},Q}$ be over z, x respectively. The maximal ideal of $\mathcal{O}_{n,Q}$ is (x, z) . The maximal ideal of $\mathcal{O}_{\mathcal{C},Q}$ is generated by $\mathbf{z}, \mathbf{x}, t$. It follows from proposition 2.1.1 that there exist a neighborhood U of Q in \mathcal{C} and an embedding $j : U \rightarrow \mathbb{P}_3$. We can assume that the restriction of j to $\overline{Z_1} \cap U$ is induced by the morphism $\phi : \mathbb{C}[X, Z, T] \rightarrow \mathcal{O}_{\overline{Z_1},Q}$ of \mathbb{C} -algebras which associates x, z, t to X, Z, T respectively.

Since \mathcal{C} is reduced, U is an open subset of a reduced hypersurface of \mathbb{P}_3 having n irreducible components, corresponding to $\overline{Z_1}, \dots, \overline{Z_n}$. It is then clear that \mathbf{X}_i , being the smallest subscheme of \mathcal{C} containing $Z_1 \setminus C, \dots, Z_i \setminus C$, is the union in U of the first i hypersurface components.

Since $j(\overline{Z_1})$ is a hypersurface, the kernel of ϕ is a principal ideal generated by the equation F of the image of Z_1 .

Recall that $\mathcal{O}_n = \mathcal{O}_{C_n} = (\mathcal{O}_{\mathcal{C}})_P$. We have $R(\mathcal{O}_n/\overline{\mathcal{J}}_{1,P}) = 1$ according to (1). Hence there exists a nonempty open subset V of C_n such that $(\mathcal{O}_n/\overline{\mathcal{J}}_{1,P})|_V$ is a line bundle on $V \cap C$. It follows that the projection $\mathcal{O}_n \rightarrow \mathcal{O}_C$ vanishes on $\overline{\mathcal{J}}_{1,P}|_V$. Since \mathcal{O}_C is torsion free this projection vanishes everywhere on $\overline{\mathcal{J}}_1$, i.e. $\overline{\mathcal{J}}_{1,P} \subset \mathcal{I}_{C,C_n}$, with equality on V .

The sheaf $\mathcal{E}_0 = \mathcal{O}_{\mathcal{C}}/\overline{\mathcal{J}}_1$ is the structural sheaf of $\overline{Z_1}$, and the projection $\overline{Z_1} \rightarrow S$ is a flat morphism. For every $s \in S \setminus \{P\}$, $(\overline{Z_1})_s$ is a smooth curve. The fiber $(\overline{Z_1})_P$ consists of C and a finite number of embedded points. There exist flat families of curves whose general fiber is smooth and the special fiber consists of an integral curve and some embedded points (cf. [12], III, Example 9.8.4). We will show that this cannot happen in our case, i.e. we have $\overline{\mathcal{J}}_{1,P} = \mathcal{I}_{C,C_n}$.

Let $\mathbf{m} = (X, Z, T) \subset \mathbb{C}[X, Z, T]$, and \mathbf{m}_{Z_1} the maximal ideal of $\mathcal{O}_{\overline{Z_1},Q}$. The ideal of $(\overline{Z_1})_P$ in $\mathcal{O}_{n,Q}$ contains z^q and $x^p z$ (for suitable minimal integers $p \geq 0, q > 0$), with $p > 0$ if and only if Q is an embedded point. Hence the ideal of $\overline{Z_1}$ in $\mathcal{O}_{\mathcal{C},Q}$ contains elements of type $\mathbf{x}^p \mathbf{z} - t\alpha, \mathbf{z}^q - t\beta$, with $\alpha, \beta \in \mathcal{O}_{\mathcal{C},Q}$.

Let $\widehat{\mathcal{O}_{\overline{Z_1},Q}}$ be the completion of $\mathcal{O}_{\overline{Z_1},Q}$ with respect to \mathbf{m}_{Z_1} and

$$\widehat{\phi} : \mathbb{C}((X, Z, T)) \longrightarrow \widehat{\mathcal{O}_{\overline{Z_1},Q}}$$

the morphism deduced from ϕ . We can also see $\widehat{\mathcal{O}_{\overline{Z_1},Q}}$ as the completion with respect to (X, Z, T) of $\mathcal{O}_{\overline{Z_1},Q}$ seen as a $\mathbb{C}[X, Z, T]$ -module. It follows that $\ker(\widehat{\phi}) = (F)$ (cf. [9], lemma 7.15). Note that $\widehat{\phi}$ is surjective (this is why we use completions). Let $\alpha, \beta \in \mathbb{C}((X, Y, Z))$ be such that $\widehat{\phi}(\alpha) = \alpha, \widehat{\phi}(\beta) = \beta$. So we have

$$X^p Z - T\alpha, Z^q - T\beta \in \ker(\widehat{\phi}).$$

Hence there exist $A, B \in \mathbb{C}((X, Z, T))$ such that $X^p Z - T\alpha = AF, Z^q - T\beta = BF$. We can write in a unique way

$$A = A_0 + TA_1, B = B_0 + TB_1, F = F_0 + TF_1,$$

with $A_0, B_0, F_0 \in \mathbb{C}((X, Z))$ and $A_1, B_1, F_1 \in \mathbb{C}((X, Z, T))$, and we have

$$A_0 F_0 = X^p Z, B_0 F_0 = Z^q.$$

Since F is not invertible, it follows that F_0 is of the form $F_0 = cZ$, with $c \in \mathbb{C}((X, Z, T))$ invertible. So we have $F = cZ + TF_1$. It follows that $z \in (t)$ in $\widehat{\mathcal{O}_{\overline{Z}_1, Q}}$. This implies that this is also true in $\mathcal{O}_{\overline{Z}_1, Q}$: in fact the assertion in $\widehat{\mathcal{O}_{\overline{Z}_1, Q}}$ implies that

$$z \in \bigcap_{n \geq 0} ((t) + \mathfrak{m}_{Z_1}^n)$$

in $\mathcal{O}_{\overline{Z}_1, Q}$, and the latter is equal to (t) according to [14], vol. II, chap. VIII, theorem 9. Hence $z \in (t)$ in $\mathcal{O}_{\overline{Z}_1, Q}$, i.e. $p = 0$ and Q is not an embedded point. So there are no embedded points. This implies that $\overline{\mathcal{J}}_{1P} = \mathcal{I}_{C, C_n}$. Similarly, if I_j denotes the ideal sheaf of \overline{Z}_j for $1 \leq j \leq n$, we have $I_{j, P} = \mathcal{I}_{C, C_n}$. Since the restriction of $\pi : \overline{Z}_j \rightarrow S$ is flat, the curves $\mathcal{E}_{j, s}$, $s \neq P$, have the same genus as C , and the same Hilbert polynomial with respect to $\mathcal{O}_C(1)$.

Now we show that \mathbf{X}'_1 is a maximal reducible deformation of C_{n-1} . We need only to show that $\mathbf{X}'_{1, P} = C_{n-1}$. As we have seen, for $2 \leq j \leq n$, a local equation of \overline{Z}_j at any point $Q \in C$ induces a generator u_j of $\mathcal{I}_{C, C_n, Q}$. Hence $u = \prod_{2 \leq j \leq n} u_j$ is a generator of $\mathcal{I}_{C_{n-1}, C_n, Q}$. But $u = 0$ on \mathbf{X}'_1 . It follows that $\mathbf{X}'_{1, P} \subset C_{n-1}$. But the Hilbert polynomial of $\mathcal{O}_{C_{n-1}}$ is the same as that of the structural sheaves of the fibers of the flat morphism $\mathbf{X}'_1 \rightarrow S$ over $s \neq P$, hence the same as $\mathcal{O}_{\mathbf{X}'_{1, P}}$. Hence $\mathbf{X}'_{1, P} = C_{n-1}$.

The theorem 3.2.3 is then easily proved by induction on n . □

3.2.4. Corollary: *Let $s \in S \setminus \{P\}$ and D_1, D_2 be two irreducible components of \mathcal{C}_s . Then D_1 is of genus g and $D_1 \cap D_2$ consists of $-\deg(L)$ points.*

Proof. According to theorem 3.2.3, there exists a flat family of smooth curves \mathbf{C} parametrized by S such that $\mathbf{C}_P = C$ and $\mathbf{C}_s = D_1$. So the genus of D_1 is equal to that of C .

Let us prove the second assertion. Again according to theorem 3.2.3 we can suppose that $n = 2$. We have then $\chi(\mathcal{C}_s) = \chi(C_2) = 2\chi(C) + \deg(L)$. Let x_1, \dots, x_N be the intersection points of D_1 and D_2 . We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{D_2}(-x_1 - \dots - x_N) \longrightarrow \mathcal{O}_{\mathcal{C}_s} \longrightarrow \mathcal{O}_{D_1} \longrightarrow 0.$$

Whence $\chi(\mathcal{O}_{\mathcal{C}_s}) = \chi(D_1) + \chi(D_2) - N = 2\chi(\mathcal{O}_C) - N$ (according to the first assertion). Whence $N = -\deg(L)$. □

3.2.5. It follows from the previous results that if $\pi : \mathcal{C} \rightarrow S$ is a maximal reducible deformation of C_n , then we have

- (i) $\deg(L) \leq 0$.
- (ii) \mathcal{C} has exactly n irreducible components $\mathcal{C}_1, \dots, \mathcal{C}_n$.
- (iii) For $1 \leq i \leq n$, the restriction of π , $\pi_i : \mathcal{C}_i \rightarrow S$ is a flat morphism, and $\pi_i^{-1}(P) = C$.
- (iv) For every nonempty subset $I \subset \{1, \dots, n\}$, let \mathcal{C}_I be the union of the \mathcal{C}_i such that $i \in I$, and m the number of elements of I . Then the restriction of π , $\pi_I : \mathcal{C}_I \rightarrow S$ is a maximal reducible deformation of C_m .

The following is immediate, and shows that we need only to consider maximal reducible deformations parametrized by a neighborhood of 0 in \mathbb{C} :

3.2.6. Proposition: *Let $t \in \mathcal{O}_S(P)$ be a generator of the maximal ideal, and $\pi : \mathcal{C} \rightarrow S$ a maximal reducible deformation of C_n . Let $S' \subset S$ be an open neighborhood of P where t is defined and $\mathcal{C}' = \pi^{-1}(U)$, $V = t(U)$. Then $\pi' = t \circ \pi : \mathcal{C}' \rightarrow V$ is a maximal reducible deformation of C_n .*

4. FRAGMENTED DEFORMATIONS OF PRIMITIVE MULTIPLE CURVES

The fragmented deformations of primitive multiple curves are particular cases of reducible deformations.

In this chapter (S, P) denotes the germ of a smooth curve. Let $t \in \mathcal{O}_{S,P}$ be a generator of the maximal ideal of P . We can suppose that t is defined on the whole of S , and that the ideal sheaf of P in S is generated by t .

4.1. FRAGMENTED DEFORMATIONS AND GLUING

Let $n > 0$ be an integer and $Y = C_n$ a projective primitive multiple curve of multiplicity n .

4.1.1. Definition: *Let $k > 0$ be an integer. A general fragmented deformation of length k of C_n is a flat morphism $\pi : \mathcal{C} \rightarrow S$ such that for every point $s \neq P$ of S , the fiber \mathcal{C}_s is a disjoint union of k projective smooth irreducible curves, and such that \mathcal{C}_P is isomorphic to C_n .*

We have then $k \leq n$. If $k = n$ we say that π (or \mathcal{C}) is a *general maximal fragmented deformation* of C_n . We suppose in the sequel that it is the case.

The line bundle on C associated to C_n is \mathcal{O}_C (by proposition 3.2.4).

Let $p > 0$ be an integer. Let K be the field of rational functions on S and $K' = K(t^{1/p})$. Let S' be the germ of curve corresponding to K' , $\theta : S' \rightarrow S$ the canonical morphism and P' the unique point of $\theta^{-1}(P)$. Let $\mathcal{D} = \theta^*(\mathcal{C})$. So we have a cartesian diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\rho} & S' \\ \downarrow \Theta & & \downarrow \theta \\ \mathcal{C} & \xrightarrow{\pi} & S \end{array}$$

where ρ is flat, and for every $s' \in S'$, Θ induces an isomorphism $\mathcal{D}_{s'} \simeq \mathcal{C}_{\theta(s')}$.

4.1.2. Proposition: *For a suitable choice of p , \mathcal{D} has exactly n irreducible components $\mathcal{D}_1, \dots, \mathcal{D}_n$, and for every point $s \neq P'$ of S' , $\mathcal{D}_{1s}, \dots, \mathcal{D}_{ns}$ are the irreducible components of \mathcal{D}_s , for $1 \leq i \leq n$ the restriction of $\rho : \mathcal{D}_{is} \rightarrow S'$ is flat, and $\mathcal{D}_{P'} = C_n$.*

(See proposition 3.1.3)

4.1.3. Definition: *A fragmented deformation of C_n is a general maximal fragmented deformation of length n of C_n having n irreducible components.*

We suppose in the sequel that \mathcal{C} is a fragmented deformation of C_n , union of n irreducible components $\mathcal{C}_1, \dots, \mathcal{C}_n$.

4.1.4. Proposition: *Let $I \subset \{1, \dots, n\}$ be a nonempty subset having m elements. Let $\mathcal{C}_I = \cup_{i \in I} \mathcal{C}_i$. Then the restriction of $\pi, \mathcal{C}_I \rightarrow S$, is flat, and the fiber \mathcal{C}_{IP} is canonically isomorphic to C_m .*

(See 3.2.5)

In particular there exists a filtration of ideal sheaves

$$0 \subset \mathcal{I}_1 \subset \dots \subset \mathcal{I}_{n-1} \subset \mathcal{O}_{\mathcal{C}}$$

such that for $1 \leq i < n$ and $s \in S \setminus \{P\}$, \mathcal{I}_{is} is the ideal sheaf of $\cup_{j=i}^n \mathcal{C}_{js}$, and that \mathcal{I}_{iP} is that of C_{n-i} .

4.1.5. Definition: *For $1 \leq i \leq n$, let $\pi_i : \mathcal{C}_i \rightarrow S$ be a flat family of smooth projective irreducible curves, with a fixed isomorphism $\pi_i^{-1}(P) \simeq C$. A gluing of $\mathcal{C}_1, \dots, \mathcal{C}_n$ along C is an algebraic variety \mathcal{D} such that*

- for $1 \leq i \leq n$, \mathcal{C}_i is isomorphic to a closed subvariety of \mathcal{D} , also denoted by \mathcal{C}_i , and \mathcal{D} is the union of these subvarieties.
- $\coprod_{1 \leq i \leq n} (\mathcal{C}_i \setminus C)$ is an open subset of \mathcal{D} .
- There exists a morphism $\pi : \mathcal{D} \rightarrow S$ inducing π_i on \mathcal{C}_i , for $1 \leq i \leq n$.
- The subvarieties $C = \pi_i^{-1}(P)$ of \mathcal{C}_i coincide in \mathcal{D} .

For example the previous fragmented deformation \mathcal{C} of C_n is a gluing of $\mathcal{C}_1, \dots, \mathcal{C}_n$ along C .

All the gluings of $\mathcal{C}_1, \dots, \mathcal{C}_n$ along C have the same underlying Zariski topological space.

Let \mathcal{A} be the *initial gluing* of the \mathcal{C}_i along C . It is an algebraic variety whose points are the same as those of \mathcal{C} , i.e.

$$\left(\prod_{i=1}^n \mathcal{C}_i \right) / \sim \quad ,$$

where \sim is the equivalence relation: if $x \in \mathcal{C}_i$ and $y \in \mathcal{C}_j$, $x \sim y$ if and only if $x = y$, or if $x \in \mathcal{C}_{iP} \simeq C$, $y \in \mathcal{C}_{jP} \simeq C$ and $x = y$ in C . The structural sheaf is defined by: for every open subset U of \mathcal{A}

$$\mathcal{O}_{\mathcal{A}}(U) = \{(\alpha_1, \dots, \alpha_n) \in \mathcal{O}_{\mathcal{C}_1}(U \cap \mathcal{C}_1) \times \dots \times \mathcal{O}_{\mathcal{C}_n}(U \cap \mathcal{C}_n); \alpha_{1|C} = \dots = \alpha_{n|C}\}.$$

For every gluing \mathcal{D} of $\mathcal{C}_1, \dots, \mathcal{C}_n$, we have an obvious dominant morphism $\mathcal{A} \rightarrow \mathcal{D}$. It follows that the sheaf of rings $\mathcal{O}_{\mathcal{D}}$ can be seen as a subsheaf of $\mathcal{O}_{\mathcal{A}}$.

The fiber $D = \mathcal{A}_P$ is not a primitive multiple curve (if $n > 2$): if $\mathcal{I}_{C,D}$ denotes the ideal sheaf of C in D we have $\mathcal{I}_{C,D}^2 = 0$, and $\mathcal{I}_{C,D} \simeq \mathcal{O}_C \otimes \mathbb{C}^{n-1}$.

4.1.6. Proposition: *Let \mathcal{D} be a gluing of $\mathcal{C}_1, \dots, \mathcal{C}_n$. Then $\pi^{-1}(P)$ is a primitive multiple curve if and only if for every closed point x of C , there exists a neighborhood of x in \mathcal{D} that can be embedded in a smooth variety of dimension 3.*

Proof. Suppose that $\pi^{-1}(P)$ is a primitive multiple curve. Then $\mathcal{I}_C/(\mathcal{I}_C^2 + (\pi))$ is a principal module at x : suppose that the image of $u \in m_{\mathcal{D},x}$ is a generator. The module $m_{\mathcal{D},x}/\mathcal{I}_C$ is also principal (since it is the maximal ideal of x in C): suppose that the image of $v \in m_{\mathcal{D},x}$ is a generator. Then the images of u, v, π generate $m_{\mathcal{D},x}/m_{\mathcal{D},x}^2$, so according to proposition 2.1.1, we can locally embed \mathcal{D} in a smooth variety of dimension 3.

Conversely, suppose that a neighborhood of $x \in C$ in \mathcal{D} is embedded in a smooth variety Z of dimension 3. The proof of the fact that $\pi^{-1}(P)$ is Cohen-Macaulay is similar to that of theorem 3.2.3. We can suppose that π is defined on Z . We have $\pi|_{\mathcal{C}_1} = \pi_1 \notin m_{\mathcal{C}_1,x}^2$, so $\pi \notin m_{Z,x}^2$. It follows that the surface of Z defined by π is smooth at x , and that we can locally embed $\pi^{-1}(P)$ in a smooth surface. Hence $\pi^{-1}(P)$ is a primitive multiple curve. \square

4.2. FRAGMENTED DEFORMATIONS OF LENGTH 2

Let $\pi : \mathcal{C} \rightarrow S$ be a fragmented deformation of C_2 . So \mathcal{C} has two irreducible components $\mathcal{C}_1, \mathcal{C}_2$. Let \mathcal{A} be the initial gluing of \mathcal{C}_1 and \mathcal{C}_2 along C . For every open subset U of \mathcal{C} , U is also an open subset of \mathcal{A} and $\mathcal{O}_{\mathcal{C}}(U)$ is a sub-algebra of $\mathcal{O}_{\mathcal{A}}(U)$. For $i = 1, 2$, let $\pi_i : \mathcal{C}_i \rightarrow S$ be the restriction of π . We will also denote $t \circ \pi$ by π , and $t \circ \pi_i$ by π_i . So we have $\pi = (\pi_1, \pi_2) \in \mathcal{O}_{\mathcal{C}}(\mathcal{C})$.

Let \mathcal{I}_C be the ideal sheaf of C in \mathcal{C} . Since $C_2 = \pi^{-1}(P)$ we have $\mathcal{I}_C^2 \subset \langle (\pi_1, \pi_2) \rangle$.

Let $m > 0$ be an integer, $x \in C$, $\alpha_1 \in \mathcal{O}_{\mathcal{C}_1,x}$, $\alpha_2 \in \mathcal{O}_{\mathcal{C}_2,x}$. We denote by $[\alpha_1]_m$ (resp. $[\alpha_2]_m$) the image of α_1 (resp. α_2) in $\mathcal{O}_{\mathcal{C}_1,x}/(\pi_1^m)$ (resp. $\mathcal{O}_{\mathcal{C}_2,x}/(\pi_2^m)$).

4.2.1. Proposition: 1 – *There exists an unique integer $p > 0$ such that $\mathcal{I}_C/\langle (\pi_1, \pi_2) \rangle$ is generated by the image of $(\pi_1^p, 0)$.*

2 – *The image of $(0, \pi_2^p)$ generates $\mathcal{I}_C/\langle (\pi_1, \pi_2) \rangle$.*

3 – *For every $x \in C$, $\alpha \in \mathcal{O}_{\mathcal{C}_1,x}$ and $\beta \in \mathcal{O}_{\mathcal{C}_2,x}$, we have $(\pi_1^p \alpha, 0) \in \mathcal{O}_{\mathcal{C},x}$ and $(0, \pi_2^p \beta) \in \mathcal{O}_{\mathcal{C},x}$.*

Proof. Let $x \in C$ and $u = (\pi_1 \alpha, \pi_2 \beta)$ whose image is a generator of $\mathcal{I}_C/\langle (\pi_1, \pi_2) \rangle$ at x ($\mathcal{I}_C/\langle (\pi_1, \pi_2) \rangle$ is a locally free sheaf of rank 1 of $\mathcal{O}_{\mathcal{C}}$ -modules). Let $\beta_0 \in \mathcal{O}_{\mathcal{C}_1,x}$ be such that $(\beta_0, \beta) \in \mathcal{O}_{\mathcal{C},x}$. Then the image of

$$u - (\pi_1, \pi_2)(\beta_0, \beta) = (\pi_1(\alpha - \beta_0), 0)$$

is also a generator of $\mathcal{I}_C/\langle (\pi_1, \pi_2) \rangle$ at x . We can write it $(\pi_1^p \lambda, 0)$, where λ is not a multiple of π_1 .

Now we show that p is the smallest integer q such that $(\mathcal{I}_C/\langle (\pi_1, \pi_2) \rangle)_x$ contains the image of an element of the form $(\pi_1^q \mu, 0)$, with μ not divisible by π_1 . We can write

$$(\pi_1^q \mu, 0) = (u_1, u_2)(\pi_1^p \lambda, 0) + (v_1, v_2)(\pi_1, \pi_2)$$

with $(u_1, u_2), (v_1, v_2) \in \mathcal{O}_{\mathcal{C},x}$. So we have $v_2 = 0$, hence $(v_1, v_2) \in \mathcal{I}_{\mathcal{C},x}$. So we can write (v_1, v_2) as the sum of a multiple of $(\pi_1^p \lambda, 0)$ and a multiple of (π_1, π_2) . Finally we obtain $(\pi_1^q \mu, 0)$ as

$$(\pi_1^q \mu, 0) = (u_{12}, u_{22})(\pi_1^p \lambda, 0) + (v_{11}, 0)(\pi_1, \pi_2)^2.$$

In the same way we see that $(\pi_1^q \mu, 0)$ can be written as

$$(\pi_1^q \mu, 0) = (u_{1p}, u_{2p})(\pi_1^p \lambda, 0) + (v_{1p}, 0)(\pi_1, \pi_2)^p,$$

which implies immediately that $q \geq p$.

It follows that p does not depend on x and that $\mathcal{I}_C/\langle(\pi_1, \pi_2)\rangle$ is a subsheaf of $\langle(\pi_1^p, 0)\rangle/\langle(\pi_1^{p+1}, 0)\rangle \simeq \mathcal{O}_C$. Since $\mathcal{I}_C/\langle(\pi_1, \pi_2)\rangle$ is of degree 0 by (by corollary 3.2.4) it follows that $\mathcal{I}_C/\langle(\pi_1, \pi_2)\rangle \simeq \langle(\pi_1^p, 0)\rangle/\langle(\pi_1^{p+1}, 0)\rangle$, from which we deduce assertion 1- of proposition 4.2.1. The second assertion comes from the fact that $(0, \pi_2^p) = \pi^p - (\pi_1^p, 0)$.

To prove the third, we use the fact that there exists $\alpha' \in \mathcal{O}_{\mathcal{C}_2, x}$ such that $(\alpha, \alpha') \in \mathcal{O}_{\mathcal{C}, x}$ (because $\mathcal{C}_1 \subset \mathcal{C}$). Hence $(\pi_1^p, 0)(\alpha, \alpha') = (\pi_1^p \alpha, 0) \in \mathcal{O}_{\mathcal{C}, x}$. Similarly, we obtain that $(0, \pi_2^p \beta) \in \mathcal{O}_{\mathcal{C}, x}$. \square

According to the proof of proposition 4.2.1, for every $x \in C$, p is the smallest integer q such that there exists an element of $\mathcal{O}_{\mathcal{C}, x}$ of the form $(\pi_1^q \alpha, 0)$ (resp. $(0, \pi_2^q \alpha)$), with $\alpha \in \mathcal{O}_{\mathcal{C}_1, x}$ (resp. $\alpha \in \mathcal{O}_{\mathcal{C}_2, x}$) not vanishing on C .

Let $x \in C$ and $\alpha_1 \in \mathcal{O}_{\mathcal{C}_1, x}$. Since $\mathcal{C}_1 \subset \mathcal{C}$ there exists $\alpha_2 \in \mathcal{O}_{\mathcal{C}_2, x}$ such that $(\alpha_1, \alpha_2) \in \mathcal{O}_{\mathcal{C}, x}$. Let $\alpha'_2 \in \mathcal{O}_{\mathcal{C}_2, x}$ such that $(\alpha_1, \alpha'_2) \in \mathcal{O}_{\mathcal{C}, x}$. We have then $(0, \alpha_2 - \alpha'_2) \in \mathcal{O}_{\mathcal{C}, x}$. So there exists $\alpha \in \mathcal{O}_{\mathcal{C}_2, x}$ such that $\alpha_2 - \alpha'_2 = \pi_2^p \alpha$. It follows that the image of α_2 in $\mathcal{O}_{\mathcal{C}_2, x}/(\pi_2^p)$ is uniquely determined. Hence we have:

4.2.2. Proposition: *There exists a canonical isomorphism*

$$\Phi : \mathcal{C}_1^{(p)} \longrightarrow \mathcal{C}_2^{(p)}$$

between the infinitesimal neighborhoods of order p of \mathcal{C}_1 and \mathcal{C}_2 (i.e. $\mathcal{O}_{\mathcal{C}_i^{(p)}} = \mathcal{O}_{\mathcal{C}_i}/(\pi_i^p)$), such that for every $x \in C$, $\alpha_1 \in \mathcal{O}_{\mathcal{C}_1, x}$ and $\alpha_2 \in \mathcal{O}_{\mathcal{C}_2, x}$, we have $(\alpha_1, \alpha_2) \in \mathcal{O}_{\mathcal{C}, x}$ if and only if $\Phi_x([\alpha_1]_p) = [\alpha_2]_p$. For every $\alpha \in \mathcal{O}_{\mathcal{C}_1, x}$ we have $\Phi_x(\alpha)|_C = \alpha|_C$, and $\Phi_x(\pi_1) = \pi_2$.

The simplest case is $p = 1$. In this case $\Phi : C \rightarrow C$ is the identity and $\mathcal{C} = \mathcal{A}$ (the *initial* gluing).

4.2.3. Converse - Recall that \mathcal{A} denotes the initial gluing of $\mathcal{C}_1, \mathcal{C}_2$ (cf. 4.1.5). Let $\Phi : \mathcal{C}_1^{(p-1)} \rightarrow \mathcal{C}_2^{(p-1)}$ be an isomorphism inducing the identity on C and such that $\Phi(\pi_1) = \pi_2$. We define a subsheaf of algebras \mathcal{U}_Φ of $\mathcal{O}_{\mathcal{A}}$: $\mathcal{U}_\Phi = \mathcal{O}_{\mathcal{A}}$ on $\mathcal{A} \setminus C$, and for every point x of C

$$\mathcal{U}_{\Phi, x} = \{(\alpha_1, \alpha_2) \in \mathcal{O}_{\mathcal{C}_1, x} \times \mathcal{O}_{\mathcal{C}_2, x} ; \Phi_x([\alpha_1]_p) = [\alpha_2]_p\}.$$

It is easy to see that \mathcal{U}_Φ is the structural sheaf of an algebraic variety \mathcal{A}_Φ , that the inclusion $\mathcal{U}_\Phi \subset \mathcal{O}_{\mathcal{A}}$ defines a dominant morphism $\mathcal{A} \rightarrow \mathcal{A}_\Phi$ inducing an isomorphism between the underlying topological spaces (for the Zariski topology), and that the composed morphisms $\mathcal{C}_i \subset \mathcal{A} \rightarrow \mathcal{A}_\Phi$, $i = 1, 2$, are immersions. Moreover, the morphism $\pi : \mathcal{A} \rightarrow S$ factorizes through \mathcal{A}_Φ :

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A}_\Phi \xrightarrow{\pi_\Phi} S \\ & \searrow \pi & \nearrow \end{array}$$

and $\pi_\Phi : \mathcal{A}_\Phi \rightarrow S$ is flat.

For $2 \leq i \leq p$, let $\Phi^{(i)} : \mathcal{C}_1^{(i)} \rightarrow \mathcal{C}_2^{(i)}$ be the isomorphism induced by Φ .

4.2.4. Proposition: $\pi_\Phi^{-1}(P)$ is a primitive double curve.

Proof. Let x be a closed point of C . We first show that $\mathcal{I}_{\mathcal{C}_2, x}^2 \subset (\pi)$. Let $u = (\pi_1 \alpha, \pi_2 \beta) \in \mathcal{I}_{\mathcal{C}, x}$. Let $\beta' \in \mathcal{O}_{\mathcal{C}_2, x}$ be such that $\Phi_x([\alpha]_p) = [\beta']_p$. We have then $v = (\alpha, \beta') \in \mathcal{O}_{\mathcal{C}, x}$. We have

$u - \pi v = (0, \pi_2(\beta - \beta')) \in \mathcal{O}_{C,x}$. Therefore $[\pi_2(\beta - \beta')]_p = \Phi_x(0) = 0$. Hence $\pi_2(\beta - \beta') \in (\pi_2^p)$. We can then write

$$u = \pi v + (0, \pi_2^p \gamma).$$

Let $u' \in \mathcal{I}_{C,x}$, that can be written as $u' = \pi v' + (0, \pi_2^p \gamma')$. We have then

$$uu' = \pi.(\pi v v' + (0, \pi_2 \gamma')v + (0, \pi_2 \gamma)v' + (0, \pi_2^{2p-1} \gamma \gamma')) \in (\pi).$$

It remains to show that $\mathcal{I}_{C,x}/(\pi) \simeq \mathcal{O}_{C,x}$. We have

$$\begin{aligned} \mathcal{I}_{C,x} &= \{(\pi_1 \alpha, \pi_2 \beta) \in \mathcal{O}_{C_1,x} \times \mathcal{O}_{C_2,x}; \Phi_x([\pi_1 \alpha]_p) = [\pi_2 \beta]_p\} \\ &= \{(\pi_1 \alpha, \pi_2 \beta) \in \mathcal{O}_{C_1,x} \times \mathcal{O}_{C_2,x}; \Phi_x^{(p-1)}([\alpha]_{p-1}) = [\beta]_{p-1}\}, \\ (\pi)_x &= \{(\pi_1 \alpha, \pi_2 \beta) \in \mathcal{O}_{C_1,x} \times \mathcal{O}_{C_2,x}; \Phi_x([\alpha]_p) = [\beta]_p\}. \end{aligned}$$

So if $(\pi_1 \alpha, \pi_2 \beta) \in \mathcal{I}_{C,x}$, we have $w = \Phi_x([\alpha]_p) - [\beta]_p \in (\pi_2^{p-1})_x / (\pi_2^p)_x \simeq \mathcal{O}_{C,x}$. Hence we have a morphism of $\mathcal{O}_{C,x}$ -modules

$$\begin{aligned} \lambda : \mathcal{I}_{C,x} &\longrightarrow \mathcal{O}_{C,x} \\ (\pi_1 \alpha, \pi_2 \beta) &\longmapsto w \end{aligned}$$

whose kernel is $(\pi)_x$. We have now only to show that λ is surjective, which follows from the fact that $\lambda(\pi_1^p, 0) = 1$. \square

4.3. SPECTRUM OF A FRAGMENTED DEFORMATION AND IDEALS OF SUB-DEFORMATIONS

Let $\pi : \mathcal{C} \rightarrow S$ be a fragmented deformation of C_n , $\mathcal{C}_1, \dots, \mathcal{C}_n$ the irreducible components of \mathcal{C} . For $1 \leq i \leq n$, let $\pi_i = \pi|_{\mathcal{C}_i}$. As in 4.2, we denote also $t \circ \pi_i$ by π_i . Let $I = \{i, j\}$ be a subset of $\{1, \dots, n\}$, with $i \neq j$. Then $\pi : \mathcal{C}_I \rightarrow S$ is a fragmented deformation of C_2 . According to 4.2 there exists a unique integer $p > 0$ such that $\mathcal{I}_{C,\mathcal{C}_I}/(\pi)$ is generated by the image of $(\pi_i^p, 0)$ (and also by the image of $(0, \pi_j^p)$). Recall that p is the smallest integer q such that $\mathcal{I}_{C,\mathcal{C}_I}$ contains a non zero element of the form $(\pi_i^q \lambda, 0)$ (or $(0, \pi_j^q \mu)$), with $\lambda|_{C} \neq 0$ (resp. $\mu|_{C} \neq 0$). Let

$$p_{ij} = p_{ji} = p,$$

and $p_{ii} = 0$ for $1 \leq i \leq n$. The symmetric matrix $(p_{ij})_{1 \leq i, j \leq n}$ is called the *spectrum* of \mathcal{C} .

4.3.1. Generators of $(\mathcal{I}_C^p + (\pi))/(\mathcal{I}_C^{p+1} + (\pi))$ - Let $i, j \in \{1, \dots, n\}$ be such that $i \neq j$. Let $x \in C$. Since $\mathcal{C}_{\{i,j\}} \subset \mathcal{C}$ there exists an element $\mathbf{u}_{ij} = (u_m)_{1 \leq m \leq n}$ of $\mathcal{O}_{C,x}$ such that $u_i = 0$ and $u_j = \pi_j^{p_{ij}}$. According to proposition 4.1.4, the image of \mathbf{u}_{ij} generates $\mathcal{I}_C/(\mathcal{I}_C^2 + (\pi))$ at x .

According to proposition 4.2.1 and the fact that the image of \mathbf{u}_{ij} generates $\mathcal{I}_{C,\mathcal{C}_{ij},x}/(\mathcal{I}_{C,\mathcal{C}_{ij},x}^2 + (\pi))$, for every integer m such that $m \neq i, j$ and that $1 \leq m \leq n$, u_m is of the form $u_m = \alpha_{ij}^{(m)} \pi_m^{p_{im}}$, with $\alpha_{ij}^{(m)} \in \mathcal{O}_{C_m,x}$ invertible. Let $\alpha_{ij}^{(i)} = 0$ and $\alpha_{ij}^{(j)} = 1$.

4.3.2. Proposition: $1 - \alpha_{ij|C}^{(m)}$ is a non zero constant, uniquely determined and independent of x .

2 - Let $\mathbf{a}_{ij}^{(m)} = \alpha_{ij|C}^{(m)} \in \mathbb{C}$. Then we have, for all integers i, j, k, m, q such that $1 \leq i, j, k, m, q \leq n$, $i \neq j$, $i \neq k$

$$\mathbf{a}_{ik}^{(m)} \mathbf{a}_{ij}^{(q)} = \mathbf{a}_{ik}^{(q)} \mathbf{a}_{ij}^{(m)}.$$

In particular we have $\mathbf{a}_{ij}^{(m)} = \mathbf{a}_{ik}^{(m)} \mathbf{a}_{ij}^{(k)}$ and $\mathbf{a}_{ij}^{(m)} \mathbf{a}_{im}^{(j)} = 1$.

Proof. Let \mathbf{u}'_{ij} have the same properties as \mathbf{u}_{ij} . Then $\mathbf{v} = \mathbf{u}'_{ij} - \mathbf{u}_{ij} \in \mathcal{I}_{C,x}^2 + (\pi)$. So the image of \mathbf{v} in $\mathcal{O}_{C_{im},x}$ belongs to $\mathcal{I}_{C,C_{im},x}^2 + (\pi)$. It follows that the m -th component of \mathbf{v} is a multiple of $\pi_m^{p_{im}+1}$. Hence $\alpha_{ij|C}^{(m)}$ is uniquely determined. It follows that when x varies the $\alpha_{ij|C}^{(m)}$ can be glued together and define a global section of \mathcal{O}_C , which must be a constant. This proves 1-.

Now we prove 2-. There exists $u \in \mathcal{O}_{C,x}$ such that the k -th component of u is $\alpha_{ij}^{(k)}$, and u is invertible. Then the image of $(v_m) = \frac{\mathbf{u}_{ij}}{u}$ generates $\mathcal{I}_C/(\mathcal{I}_C^2 + (\pi))$, and $v_k = 1$. Hence according to 1-, we have $v_{m|C} = \mathbf{a}_{ik}^{(m)}$, i.e.

$$\frac{\mathbf{a}_{ij}^{(m)}}{\mathbf{a}_{ij}^{(k)}} = \mathbf{a}_{ik}^{(m)}.$$

We have the same equality with q instead of m , whence 2- is easily deduced. \square

Let p be an integer such that $1 \leq p < n$, and $(i_1, j_1), \dots, (i_p, j_p)$ p pairs of distinct integers of $\{1, \dots, n\}$. Then the image of $\prod_{m=1}^p \mathbf{u}_{i_m j_m}$ is a generator of $(\mathcal{I}_C^p + (\pi))/(\mathcal{I}_C^{p+1} + (\pi))$.

Let $I \subset \{1, \dots, n\}$ be a nonempty subset, distinct from $\{1, \dots, n\}$. Let $i \in \{1, \dots, n\} \setminus I$. Let

$$\mathbf{u}_{I,i} = \prod_{j \in I} \mathbf{u}_{ji}.$$

Recall that $C_I = \cup_{j \in I} C_j \subset C$.

4.3.3. Proposition: *The ideal sheaf of C_I is generated by $\mathbf{u}_{I,i}$ at x .*

Proof. According to proposition 4.1.6 there exists an embedding of a neighborhood of x in a smooth variety of dimension 3. In this variety each C_i is a smooth surface defined by a single equation. The ideal of the union of the C_i , $i \in I$ is the product of these equations. \square

4.3.4. Proposition: *Let i, j, k be distinct integers such that $1 \leq i, j, k \leq n$. Then if $p_{ij} < p_{jk}$, we have $p_{ik} = p_{ij}$.*

Proof. We can come down to the case $n = 3$ by considering $C_{\{i,j,k\}}$. We can suppose that $p_{23} \leq p_{12} \leq p_{13}$, and we must show that $p_{23} = p_{12}$. We have

$$\mathbf{u}_{21} = (\pi_1^{p_{12}}, 0, \alpha_{21}^{(3)} \pi_3^{p_{23}}), \quad \mathbf{u}_{31} = (\pi_1^{p_{13}}, \alpha_{31}^{(2)} \pi_2^{p_{23}}, 0).$$

So

$$\mathbf{u}_{31} - \pi^{p_{13}-p_{12}} \mathbf{u}_{21} = (0, \alpha_{31}^{(2)} \pi_2^{p_{23}}, -\alpha_{21}^{(3)} \pi_3^{p_{23}+p_{13}-p_{12}}) \in \mathcal{O}_{C,x}.$$

Taking the image of this element in $\mathcal{O}_{C_{12},x}$, we see that $p_{23} \geq p_{12}$, hence $p_{23} = p_{12}$. \square

4.3.5. Proposition: 1 – *Let i, j be distinct integers such that $1 \leq i, j \leq n$. Then we have $\mathcal{I}_{C,x} = (\mathbf{u}_{ij}) + (\pi)$.*

2 – *Let $v = (v_m)_{1 \leq m \leq n} \in \mathcal{I}_{C,x}$ such that v_i is a multiple of π_i^p , with $p > 0$. Then we have $v \in (\mathbf{u}_{ij}) + (\pi^p)$.*

Proof. Let $N = 1 + \max_{1 \leq k \leq n}(q_i)$, where $q_i = \sum_{j=1}^n p_{ij}$. For every integer j such that $1 \leq j \leq n$ we have $(0, \dots, 0, \pi_j^{q_j}, 0, \dots, 0) \in \mathcal{O}_C(\mathcal{C})$. Hence $\mathcal{I}_C^N \subset (\pi)$. We will show by induction on k that $\mathcal{I}_{C,x} \subset (\mathbf{u}_{ij}) + (\pi) + \mathcal{I}_{C,x}^k$. Taking $k = N$ we obtain 1-.

For $k = 1$ it is obvious. Suppose that it is true for $k - 1 \geq 1$. It is enough to prove that $\mathcal{I}_{C,x}^{k-1} \subset (\mathbf{u}_{ij}) + (\pi) + \mathcal{I}_{C,x}^k$. Let $w_1, \dots, w_{k-1} \in \mathcal{I}_{C,x}$. Since the image of \mathbf{u}_{ij} generates $\mathcal{I}_{C,x}/(\mathcal{I}_{C,x}^2 + (\pi))$, we can write w_p as

$$w_p = \lambda_p \mathbf{u}_{ij} + \pi \mu_p + \nu_p,$$

with $\lambda_p, \mu_p \in \mathcal{O}_{C,x}$ and $\nu_p \in \mathcal{I}_{C,x}^2$. So we have

$$w_1 \cdots w_{k-1} = \lambda \mathbf{u}_{ij} + \pi \mu + \nu,$$

with $\lambda, \mu \in \mathcal{O}_{C,x}$ and $\nu \in \mathcal{I}_{C,x}^{2k-2}$. Since $2k - 2 \geq k$, we have $w_1 \cdots w_{k-1} \in (\mathbf{u}_{ij}) + (\pi) + \mathcal{I}_{C,x}^k$. This proves 1-.

We prove 2- by induction on p . The case $p = 1$ follows 1-. Suppose that it is true for $p - 1 \geq 1$. So we can write v as

$$v = \lambda \mathbf{u}_{ij} + \pi^{p-1} \mu,$$

with $\lambda, \mu \in \mathcal{O}_{C,x}$. We can write v_i as $v_i = \alpha \pi^p$. So we have $\alpha \pi_i^p = \pi_i^{p-1} \mu_i$, whence $\mu_i = \alpha \pi_i$. Hence $\mu \in \mathcal{I}_{C,x}$. According to 1- we can write μ as $\mu = \theta \mathbf{u}_{ij} + \pi \tau$, with $\theta, \tau \in \mathcal{O}_{C,x}$. So

$$v = (\lambda + \pi^{p-1} \theta) \mathbf{u}_{ij} + \pi^p \tau,$$

which proves the result for p . □

4.3.6. *The ideal sheaves \mathcal{I}_{C_I}* – Recall that $I \subset \{1, \dots, n\}$ is a nonempty subset, distinct from $\{1, \dots, n\}$. For every subset J of $\{1, \dots, n\}$, let $J^c = \{1, \dots, n\} \setminus J$ and $\mathcal{O}_J = \mathcal{O}_{C_J}$. It follows from proposition 4.3.3 that \mathcal{I}_{C_I} is a line bundle on C_{I^c} .

From now on, we suppose that $S \subset \mathbb{C}$ and $P = 0$ (cf. proposition 3.2.6).

4.3.7. Theorem: *We have $\mathcal{I}_{C_I} \simeq \mathcal{O}_{I^c}$.*

Proof. By induction on n . If $n = 2$ the result follows from proposition 4.2.1 and the fact that $S \subset \mathbb{C}$. Suppose that it is true for $n - 1 \geq 2$. We will prove that it is true for n by induction on the number of elements q of I^c . Suppose first that $q = 1$ and let i be the unique element of I^c . Then according to proposition 4.3.3, \mathcal{I}_{C_I} is generated by $(0, \dots, 0, \pi_i^{q_i}, 0, \dots, 0)$, so the result is true in this case. Suppose that it is true if $1 \leq q < k < n$, and that $q = k$. Let $K = \{1, \dots, n - 1\}$. We can assume that $I \subset K$.

According to proposition 4.3.3, we have, for every $x \in C$, $\mathcal{I}_{C_I,x} \simeq \mathcal{O}_{I^c,x}$. We have $\mathcal{I}_{C_K} \subset \mathcal{I}_{C_I}$, and $\mathcal{I}_{C_K} \simeq \mathcal{O}_{\{n\}}$. We have

$$\mathcal{I}_{C_I}/\mathcal{I}_{C_K} = \mathcal{I}_{C_I, C_K}$$

(the ideal sheaf of C_I in C_K). From the first induction hypothesis we have

$$\mathcal{I}_{C_I, C_K} \simeq \mathcal{O}_{(I \cup \{n\})^c}.$$

So we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\{n\}} \longrightarrow \mathcal{I}_{\mathcal{C}_I} \longrightarrow \mathcal{O}_{I^c \setminus \{n\}} \longrightarrow 0.$$

Now we will compute $\text{Ext}_{\mathcal{O}_{\mathcal{C}}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}})$. According to [6], 2.3, we have an exact sequence

$$0 \longrightarrow \text{Ext}_{\mathcal{O}_{I^c}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) \longrightarrow \text{Ext}_{\mathcal{O}_{\mathcal{C}}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) \longrightarrow \text{Hom}(\text{Tor}_{\mathcal{O}_{\mathcal{C}}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{I^c}), \mathcal{O}_{\{n\}}).$$

Since $\text{Tor}_{\mathcal{O}_{\mathcal{C}}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{I^c})$ is concentrated on $\mathcal{C}_{I^c \setminus \{n\}}$, we have

$$\text{Hom}(\text{Tor}_{\mathcal{O}_{\mathcal{C}}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{I^c}), \mathcal{O}_{\{n\}}) = \{0\}.$$

So we have

$$\text{Ext}_{\mathcal{O}_{\mathcal{C}}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) = \text{Ext}_{\mathcal{O}_{I^c}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}).$$

Let \mathcal{J} denote the ideal sheaf of $\mathcal{C}_{\{n\}}$ in \mathcal{C}_{I^c} . The ideal sheaf of $\mathcal{C}_{I^c \setminus \{n\}}$ is generated by $\mathbf{w} = (0, \dots, 0, \pi_n^m)$, with $m = \sum_{i \in I^c \setminus \{n\}} p_{in}$. So we have an exact sequence of sheaves on \mathcal{C}_{I^c}

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_{I^c} \xrightarrow{\alpha} \mathcal{O}_{I^c} \longrightarrow \mathcal{O}_{I^c \setminus \{n\}} \longrightarrow 0,$$

where α is the multiplication by \mathbf{w} . By the induction hypothesis there exists a surjective morphism $\mathcal{O}_{I^c} \rightarrow \mathcal{J}$, so we get a locally free resolution of $\mathcal{O}_{I^c \setminus \{n\}}$

$$\mathcal{O}_{I^c} \longrightarrow \mathcal{O}_{I^c} \xrightarrow{\alpha} \mathcal{O}_{I^c} \longrightarrow \mathcal{O}_{I^c \setminus \{n\}} \longrightarrow 0,$$

that can be used to compute $\mathcal{E}xt_{\mathcal{O}_{I^c}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}})$. It follows easily that

$$\mathcal{E}xt_{\mathcal{O}_{I^c}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) \simeq \mathcal{O}_{\{n\}}/(\pi_n^m).$$

We have $\text{Hom}(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) = 0$, hence

$$\begin{aligned} \text{Ext}_{\mathcal{O}_{I^c}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) &\simeq H^0(\mathcal{E}xt_{\mathcal{O}_{I^c}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}})) \\ &\simeq H^0(\mathcal{O}_{\{n\}}/(\pi_n^m)) \\ &\simeq H^0(\mathcal{O}_S/(\pi_n^m)) \\ &\simeq \mathbb{C}[\pi_n]/(\pi_n^m). \end{aligned}$$

We will now describe the sheaves \mathcal{E} such that there exists an exact sequence

$$(2) \quad 0 \longrightarrow \mathcal{O}_{\{n\}} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{I^c \setminus \{n\}} \longrightarrow 0.$$

Let $\nu \in \mathbb{C}[\pi_n]/(\pi_n^m)$ be associated to this exact sequence, and $\bar{\nu} \in H^0(\mathcal{O}_S)$ over ν . Let

$$\begin{aligned} \tau : \mathcal{O}_{\{n\}} &\longrightarrow \mathcal{O}_{\{n\}} \oplus \mathcal{O}_{I^c} \\ u &\longmapsto (\bar{\nu}u, \mathbf{w}u) \end{aligned}$$

Then according to the preceding resolution of $\mathcal{O}_{I^c \setminus \{n\}}$ and the construction of extensions (cf. [5], 4.2), we have $\mathcal{E} \simeq \text{coker}(\tau)$. It is easy to see that if $\nu = -1$ then $\mathcal{E} \simeq \mathcal{O}_{I^c}$. If ν is invertible, then we have also $\mathcal{E} \simeq \mathcal{O}_{I^c}$, because the corresponding extension can be obtained from the one corresponding to $\nu = -1$ by multiplying the left morphism of the exact sequence by ν .

A similar construction can be done for extensions of $\mathcal{O}_{I^c, x}$ -modules (for every $x \in \mathcal{C}$)

$$0 \longrightarrow \mathcal{O}_{\{n\}, x} \longrightarrow V \longrightarrow \mathcal{O}_{I^c \setminus \{n\}, x} \longrightarrow 0.$$

These extensions are classified by $\mathcal{O}_{\{n\}, x}/(\pi_n^m)$, and $\mathcal{O}_{I^c, x}$ corresponds to -1 .

Conversely we consider extensions

$$0 \longrightarrow \mathcal{O}_{\{n\},x} \xrightarrow{\lambda} \mathcal{O}_{I^c,x} \xrightarrow{\mu} \mathcal{O}_{I^c \setminus \{n\},x} \longrightarrow 0.$$

Using the facts that $\text{Hom}(\mathcal{O}_{\{n\},x}, \mathcal{O}_{I^c,x})$ is generated by the multiplication by \mathbf{w} and $\text{Hom}(\mathcal{O}_{I^c,x}, \mathcal{O}_{I^c \setminus \{n\},x})$ by the restriction morphism, it is easy to see that λ, μ are unique up to multiplication by an invertible element of $\mathcal{O}_{I^c,x}$. Hence the elements of $\text{Ext}_{\mathcal{O}_{I^c,x}}^1(\mathcal{O}_{I^c \setminus \{n\},x}, \mathcal{O}_{\{n\},x})$ corresponding to the preceding extensions are exactly the invertible elements of $\mathcal{O}_{\{n\},x}/(\pi_n^m)$.

It follows that the extensions (2) where \mathcal{E} is locally free correspond to invertible elements of $\mathbb{C}[\pi_n]/(\pi_n^m)$, and we have seen that in this case we have $\mathcal{E} \simeq \mathcal{O}_{I^c}$. Hence we have $\mathcal{I}_{\mathcal{C}_I} \simeq \mathcal{O}_{I^c}$ and theorem 4.3.7 is proved. \square

4.3.8. Corollary: *The ideal sheaf of \mathcal{C}_I is globally generated by an element \mathbf{u}_I such that for every integer i such that $1 \leq i \leq n$ and $i \notin I$, the i -th coordinate of \mathbf{u}_I belongs to $H^0(\mathcal{O}_S)$.*

4.4. PROPERTIES OF THE FRAGMENTED DEFORMATIONS

We use the notations of 4.3.

Let i be an integer such that $1 \leq i \leq n$ and $J_i = \{1, \dots, n\} \setminus \{i\}$. We denote by \mathcal{B} the image of $\mathcal{O}_{\mathcal{C}}$ in $\prod_{1 \leq j \leq n} \mathcal{O}_{\mathcal{C}_j}/(\pi_j^{q_j})$; it is a sheaf of \mathbb{C} -algebras on C . Let \mathcal{B}_i be the image of $\mathcal{O}_{\mathcal{C}_{J_i}}$ in $\prod_{1 \leq j \leq n, j \neq i} \mathcal{O}_{\mathcal{C}_j}/(\pi_j^{q_j})$; it is also a sheaf of \mathbb{C} -algebras on C . For every point x of C and every $\alpha = (\alpha_m)_{1 \leq m \leq n}$ in $\prod_{1 \leq j \leq n} \mathcal{O}_{\mathcal{C}_j,x}$, we denote by $b_i(\alpha)$ its image in $\prod_{1 \leq j \leq n, j \neq i} \mathcal{O}_{\mathcal{C}_j,x}$ (obtained by forgetting the i -th coordinate of α).

If p, k are positive integers, with $k \leq n$, $x \in C$ and $\alpha \in \mathcal{O}_{\mathcal{C}_k,x}$, let $[\alpha]_p$ denote the image of α in $\mathcal{O}_{\mathcal{C}_k,x}/\pi_k^p$.

4.4.1. Proposition: *There exists a morphism of sheaves of algebras on C*

$$\Phi_i : \mathcal{B}_i \longrightarrow \mathcal{O}_{\mathcal{C}_i}/(\pi_i^{q_i})$$

such that for every point x of C and all $(\alpha_m)_{1 \leq m \leq n, m \neq i} \in \mathcal{O}_{\mathcal{C}_{J_i},x}$, $\alpha_i \in \mathcal{O}_{\mathcal{C}_i,x}$, we have $\alpha = (\alpha_m)_{1 \leq m \leq n} \in \mathcal{O}_{\mathcal{C},x}$ if and only if $\Phi_{i,x}(b_i(\alpha)) = [\alpha_i]_{q_i}$.

Proof. Let $(\alpha_m)_{1 \leq m \leq n, m \neq i} \in \mathcal{O}_{\mathcal{C}_{J_i},x}$. Since $\mathcal{C}_{J_i} \subset \mathcal{C}$, there exists $\alpha_i \in \mathcal{O}_{\mathcal{C}_i,x}$ such that $(\alpha_m)_{1 \leq m \leq n} \in \mathcal{O}_{\mathcal{C},x}$. If $\alpha'_i \in \mathcal{O}_{\mathcal{C}_i,x}$ has the same property, we have $(0, \dots, 0, \alpha_i - \alpha'_i, 0, \dots, 0) \in \mathcal{I}_{J_i,x}$. So according to proposition 4.3.3, we have $[\alpha_i]_{q_i} = [\alpha'_i]_{q_i}$. Hence we have a well defined morphism of algebras $\theta_x : \mathcal{O}_{\mathcal{C}_{J_i},x} \rightarrow \mathcal{O}_{\mathcal{C}_{J_i},x}/(\pi_i^{q_i})$ sending $(\alpha_m)_{1 \leq m \leq n, m \neq i}$ to $[\alpha_i]_{q_i}$. If $j \in J_i$, we have, according to proposition 4.3.3, $\theta_x(0, \dots, 0, \pi_j^{q_j}, 0, \dots, 0) = 0$. Hence θ_x induces a morphism of algebras $\mathcal{B}_{i,x} \rightarrow \mathcal{O}_{\mathcal{C}_i,x}/(\pi_i^{q_i})$. \square

The morphism Φ_i has the following properties: for every point x of C

- (i) For every $\alpha = (\alpha_m)_{1 \leq m \leq n, m \neq i} \in \mathcal{B}_{i,x}$, we have $\Phi_{i,x}(\alpha)|_C = \alpha_m|_C$ for $1 \leq m \leq n$, $m \neq i$.
- (ii) We have $\Phi_{i,x}((\pi_m)_{1 \leq m \leq n, m \neq i}) = \pi_i$.

- (iii) Let $j, k \in \{1, \dots, n\}$ be such that i, j, k are distinct. Let \mathbf{v} be the image of \mathbf{u}_{jk} in \mathcal{B}_i . Then there exists $\lambda \in \mathcal{O}_{\mathcal{C}_i, x}^*$ such that $\Phi_{i, x}(\mathbf{v}) = \lambda \pi_i^{p_{ij}}$.
- (iv) Let j be an integer such that $1 \leq j \leq n$ and $j \neq i$. Let \mathbf{v} be the image of \mathbf{u}_{ij} in \mathcal{B}_i, x . Then we have $\ker(\Phi_{i, x}) = (\mathbf{v})$.

4.4.2. Converse - Let \mathcal{C}' be a gluing of $\mathcal{C}_1, \dots, \mathcal{C}_{i-1}, \mathcal{C}_{i+1}, \dots, \mathcal{C}_n$ along C , which is a fragmented deformation of a primitive multiple curve of multiplicity $n-1$. Let $(p_{jk})_{1 \leq j, k \leq n, j, k \neq i}$ be the spectrum of \mathcal{C}' . Let $p_{ij}, 1 \leq j \leq n, j \neq i$ be positive integers, and $p_{ii} = 0$. For $1 \leq j \leq n$, let $q_j = \sum_{1 \leq k \leq n} p_{kj}$.

Let \mathcal{B}_i be the image of $\mathcal{O}_{\mathcal{C}'}$ in $\prod_{1 \leq j \leq n, j \neq i} \mathcal{O}_{\mathcal{C}_j} / (\pi_j^{q_j})$ and

$$\Phi_i : \mathcal{B}_i \longrightarrow \mathcal{O}_{\mathcal{C}_i} / (\pi_i^{q_i})$$

a morphism of sheaves of algebras on C satisfying properties (i), (ii), (iii) above. Let \mathcal{A} be the subsheaf of algebras of \mathcal{A} defined by: $\mathcal{A} = \mathcal{A}$ on $\mathcal{A}_{top} \setminus C$, and for every point x of C , and every $\alpha = (\alpha_m)_{1 \leq m \leq n} \in \prod_{m=1}^n \mathcal{O}_{\mathcal{C}_m, x}$, $\alpha \in \mathcal{A}_x$ if and only if $b_i(\alpha) \in \mathcal{B}_i, x$ and $\Phi_{i, x}(b_i(\alpha)) = [\alpha_i]_{q_i}$.

It is easy to see that \mathcal{A} is the structural sheaf of a gluing of C_1, \dots, C_n along C , which is a fragmented deformation of a primitive multiple curve of multiplicity n , and that $\mathcal{C}' = \mathcal{A}_{\{1, \dots, i-1, i+1, \dots, n\}}$.

We give now some applications of the preceding construction.

4.4.3. Corollary: *Let N be an integer such that $N \geq \max_{1 \leq i \leq n} (q_i)$. Let $x \in C$, $\beta \in \mathcal{O}_{\mathcal{C}_1, x} \times \dots \times \mathcal{O}_{\mathcal{C}_n, x}$ and $u \in \mathcal{O}_{\mathcal{C}, x}$ such that $u|_C \neq 0$. Suppose that $[\beta u]_N \in \mathcal{O}_{\mathcal{C}, x} / (\pi^N)$. Then we have $[\beta]_N \in \mathcal{O}_{\mathcal{C}, x} / (\pi^N)$.*

Proof. By induction on n . It is obvious if $n = 1$. Suppose that the lemma is true for $n-1$. Let $I = \{1, \dots, n-1\}$. So we have $[\beta|_{\mathcal{C}_I \times \dots \times \mathcal{C}_{n-1}}]_N \in \mathcal{O}_{\mathcal{C}_I, x} / (\pi_1, \dots, \pi_{n-1})^N$ by the induction hypothesis. Let γ (resp. v) be the image of β (resp. u) in \mathcal{B}_n . To show that $[\beta]_N \in \mathcal{O}_{\mathcal{C}, x} / (\pi^N)$ it is enough to verify that

$$\Phi_n(\gamma) = [\beta_n]_{q_n}.$$

We have $\Phi_n(\gamma v) = [\beta_n u_n]_{q_n}$ because $[\beta u]_N \in \mathcal{O}_{\mathcal{C}, x} / (\pi^N)$, and $\Phi_n(v) = [u_n]_{q_n}$ because $u \in \mathcal{O}_{\mathcal{C}, x}$. So we have

$$\Phi_n(\gamma)[u_n]_{q_n} = \Phi_n(\gamma)\Phi_n(v) = \Phi_n(\gamma v) = [\beta_n u_n]_{q_n} = [\beta_n]_{q_n}[u_n]_{q_n}.$$

Since $u|_C \neq 0$, $[u_n]_{q_n}$ is not a zero divisor in $\mathcal{O}_{\mathcal{C}_n, x} / (\pi_n^{q_n})$, so we have $\Phi_n(\gamma) = [\beta_n]_{q_n}$. \square

4.4.4. Corollary: *Let $\mathbf{q} = \max_{1 \leq i \leq n} (q_i)$ and p the number of integers i such that $1 \leq i \leq n$ and $q_i = \mathbf{q}$. Then we have $p \geq 2$.*

Proof. Suppose that $q_i = \mathbf{q}$. Then we have $\pi_i^{q_i-1} \neq 0$ in $\mathcal{O}_{\mathcal{C}_i} / (\pi_i^{q_i})$. Since $\pi_i = \Phi_i((\pi_m)_{1 \leq m \leq n, m \neq i})$, we have $(\pi_m^{q_i-1})_{1 \leq m \leq n, m \neq i} \neq 0$ in \mathcal{B}_i . So we cannot have $q_m < q_i$ for all the $m \neq i$. \square

Let i be an integer such that $1 \leq i \leq n$,

$$\mathcal{H} = \prod_{1 \leq j \leq n} (\pi_j^{q_j-1})/(\pi_j^{q_j}) \simeq \mathcal{O}_C^n \quad (\text{resp. } \mathcal{H}_i = \prod_{1 \leq j \leq n, j \neq i} (\pi_j^{q_j-1})/(\pi_j^{q_j}) \simeq \mathcal{O}_C^{n-1}).$$

It is an ideal sheaf of $\prod_{1 \leq j \leq n} \mathcal{O}_{C_j}/(\pi_j^{q_j})$ (resp. $\prod_{1 \leq j \leq n, j \neq i} \mathcal{O}_{C_j}/(\pi_j^{q_j})$). Let $\mathcal{J} = \mathcal{H} \cap \mathcal{B}$ (resp. $\mathcal{J}_i = \mathcal{H}_i \cap \mathcal{B}_i$), which is an ideal sheaf of \mathcal{B} (resp. \mathcal{B}_i).

4.4.5. Proposition: *There exists a unique $\lambda(\mathcal{C}) = (\lambda_1, \dots, \lambda_n) \in \mathbb{P}_n(\mathbb{C})$ such that for every $\mathbf{u} = (u_j)_{1 \leq j \leq n} \in \mathcal{H}$, we have $\mathbf{u} \in \mathcal{J}$ if and only if $\lambda_1 u_1 + \dots + \lambda_n u_n = 0$. The λ_i are all non zero.*

Proof. We have $(\pi_m)_{1 \leq m \leq n, m \neq i} \cdot \mathcal{J}_i = 0$. Hence $\pi_i \Phi_i(\mathcal{J}_i) = 0$ and $\Phi_i(\mathcal{J}_i) \subset (\pi_i^{q_i-1})/(\pi_i^{q_i})$. The restriction of $\Phi_i, \mathcal{J}_i \rightarrow (\pi_i^{q_i-1})/(\pi_i^{q_i})$ is a morphism $(n-1)\mathcal{O}_C \rightarrow \mathcal{O}_C$ of vector bundles on C . The existence of $(\lambda_1, \dots, \lambda_n)$ follows from that.

If $\lambda_i = 0$, we have $(0, \dots, 0, \pi_i^{q_i-1}, 0, \dots, 0) \in \mathcal{O}_C(\mathcal{C})$. This is impossible because according to proposition 4.3.3, $(0, \dots, 0, \pi_i^{q_i}, 0, \dots, 0)$ generates the ideal sheaf of \mathcal{C}_{J_i} in \mathcal{C} . \square

For all distinct integers i, j such that $1 \leq i, j \leq n$, let $I_{ij} = \{1, \dots, n\} \setminus \{i, j\}$. Then according to proposition 4.3.3, $\mathbf{u}_{I_{ij}}$ generates the ideal sheaf of $\mathcal{C}_{I_{ij}}$. We have $\mathbf{u}_{I_{ij}} = (b_k)_{1 \leq k \leq n}$, with $b_k = 0$ if $k \neq i, j$, $b_i = \pi_i^{q_i-p_{ij}}$ and

$$b_j = \left(\prod_{1 \leq m \leq n, m \neq i, j} \alpha_{mi}^{(j)} \right) \cdot \pi_j^{q_j-p_{ij}}.$$

So we have $\pi^{p_{ij}-1} \mathbf{u}_{I_{ij}} \in \mathcal{J}_i$, which gives the equation

$$(3) \quad \frac{\lambda_i}{\lambda_j} = - \prod_{1 \leq m \leq n, m \neq i, j} \mathbf{a}_{mi}^{(j)}.$$

4.4.6. Proposition: *For all distinct integers i, j, k such that $1 \leq i, j, k \leq n$, we have*

$$\mathbf{a}_{ki}^{(j)} = -\mathbf{a}_{ik}^{(j)} \mathbf{a}_{ji}^{(k)}.$$

Proof. We need only to treat the case $n = 3$, and we get the preceding formula by writing that $\frac{\lambda_1}{\lambda_3} = \frac{\lambda_1}{\lambda_2} \cdot \frac{\lambda_2}{\lambda_3}$, and by using (3). \square

4.4.7. Proposition: *Let $(\alpha_1 \pi_1^{m_1}, \dots, \alpha_n \pi_n^{m_n}) \in \mathcal{O}_{C,x}$, with $\alpha_1, \dots, \alpha_n$ invertible. Let $M = m_1 + \dots + m_n$. then*

$$\left(\frac{1}{\alpha_1} \pi_1^{M-m_1}, \dots, \frac{1}{\alpha_n} \pi_n^{M-m_n} \right) \in \mathcal{O}_{C,x}.$$

Proof. By induction on n . It is obvious for $n = 1$. Suppose that it is true for $n - 1 \geq 1$. Let $I = \{1, \dots, n - 1\}$. Then $(\alpha_1 \pi_1^{m_1}, \dots, \alpha_{n-1} \pi_{n-1}^{m_{n-1}}) \in \mathcal{O}_{C_{I,x}}$. Hence, by the induction hypothesis, we have

$$\left(\frac{1}{\alpha_1} \pi_1^{M-m_1-m_n}, \dots, \frac{1}{\alpha_{n-1}} \pi_{n-1}^{M-m_{n-1}-m_n} \right) \in \mathcal{O}_{C_{I,x}}.$$

So there exists $\gamma \in \mathcal{O}_{\mathcal{C}_n, x}$ such that

$$u = \left(\frac{1}{\alpha_1} \pi_1^{M-m_1-m_n}, \dots, \frac{1}{\alpha_{n-1}} \pi_{n-1}^{M-m_{n-1}-m_n}, \gamma \right) \in \mathcal{O}_{\mathcal{C}, x}.$$

Multiplying by $(\alpha_1 \pi_1^{m_1}, \dots, \alpha_n \pi_n^{m_n})$ we see that $(\pi_1^{M-m_n}, \dots, \pi_{n-1}^{M-m_n}, \gamma \alpha_n \pi_n^{m_n}) \in \mathcal{O}_{\mathcal{C}, x}$. Subtracting π^{M-m_n} , we find that $(0, \dots, 0, \gamma \alpha_n \pi_n^{m_n} - \pi_n^{M-m_n}) \in \mathcal{O}_{\mathcal{C}, x}$. There exists $\alpha \in \mathcal{O}_{\mathcal{C}, x}$ such that the n -th coordinate of α is α_n , and α is invertible. It follows that $v = (0, \dots, 0, \gamma \pi_n^{m_n} - \frac{1}{\alpha_n} \pi_n^{M-m_n}) \in \mathcal{O}_{\mathcal{C}, x}$. Now we have

$$\pi^{m_n} u - v = \left(\frac{1}{\alpha_1} \pi_1^{M-m_1}, \dots, \frac{1}{\alpha_n} \pi_n^{M-m_n} \right) \in \mathcal{O}_{\mathcal{C}, x}.$$

□

4.4.8. Corollary: *Let $V \subset U$ be open subsets of \mathcal{C} , and suppose that $U \cap C \neq \emptyset$. Let $\alpha \in \mathcal{O}_{\mathcal{C}}(V)$ and $\beta \in \mathcal{O}_{\mathcal{A}}(U)$ such that $\beta|_V = \alpha$. Then $\beta \in \mathcal{O}_{\mathcal{C}}(U)$.*

(Recall that \mathcal{A} is the *initial gluing* of $\mathcal{C}_1, \dots, \mathcal{C}_n$ (cf. 4.1.5)).

Proof. This can be proved easily by induction on n , using proposition 4.4.1. □

4.5. CONSTRUCTION OF FRAGMENTED DEFORMATIONS

Consider a fragmented deformation

$$\pi = \pi^{[n-1]} = (\pi_1, \dots, \pi_{n-1}) : \mathcal{C}^{[n-1]} \longrightarrow S$$

of \mathcal{C}_{n-1} , with $n-1$ irreducible components $\mathcal{C}_1, \dots, \mathcal{C}_{n-1}$. Let $(p_{ij}^{[n-1]})_{1 \leq i, j < n}$ be its spectrum. For $1 \leq i < n$, let $q_i^{[n-1]} = \sum_{1 \leq j < n} p_{ij}^{[n-1]}$. We denote by $\mathcal{I}_C^{[n-1]}$ the ideal sheaf of C in $\mathcal{C}^{[n-1]}$. Let

$$\lambda(\mathcal{C}^{[n-1]}) = (\lambda_1, \dots, \lambda_{n-1}).$$

Let $p_{1n}, \dots, p_{n-1, n}$ be positive integers, $q_i = q_i^{[n-1]} + p_{in}$ for $1 \leq i < n$, and $q_n = p_{1n} + \dots + p_{n-1, n}$. Let $\mathbf{u} \in H^0(\mathcal{I}_C^{[n-1]})$ whose image generates $\mathcal{I}_C^{[n-1]} / ((\mathcal{I}_C^{[n-1]})^2 + (\pi))$, of the form

$$\mathbf{u} = (\beta_1 \pi_1^{p_{1n}}, \dots, \beta_{n-1} \pi_{n-1}^{p_{n-1, n}}),$$

with $\beta_i \in H^0(\mathcal{O}_S)$ invertible for $1 \leq i < n$.

Let $\mathcal{B}^{[n-1]}$ be the image of $\mathcal{O}_{\mathcal{C}^{[n-1]}}$ in $\mathcal{O}_{\mathcal{C}_1} / (\pi_1^{q_1}) \times \dots \times \mathcal{O}_{\mathcal{C}_{n-1}} / (\pi_{n-1}^{q_{n-1}})$. We will also denote by \mathbf{u} the image of \mathbf{u} in $\mathcal{B}^{[n-1]}$. Let $\mathcal{Q} = \mathcal{B}^{[n-1]} / (\mathbf{u})$, $\rho : \mathcal{B}^{[n-1]} \rightarrow \mathcal{Q}$ the projection and $\pi_n = \rho(\pi)$.

4.5.1. Proposition: *We have $\pi_n^{q_n} = 0$.*

Proof. According to proposition 4.4.7 we have for every $x \in C$

$$v = \left(\frac{1}{\beta_1} \pi_1^{q_1 - p_{1n}}, \dots, \frac{1}{\beta_{n-1}} \pi_{n-1}^{q_{n-1} - p_{n-1, n}} \right) \in \mathcal{O}_{\mathcal{C}^{[n-1]}, x}.$$

Hence $\pi_n^{q_n} = v \mathbf{u} \in (\mathbf{u})$ in $\mathcal{O}_{\mathcal{C}^{[n-1]}, x}$, and $\pi_n^{q_n} = 0$. □

Now we suppose that we have $p_{ij} = p_{kl}$, for $1 \leq i, j, k, l \leq n$, $i \neq j$, $k \neq l$. This is the case for example if all the components \mathcal{C}_i are transverse in \mathcal{C} (i.e. $p_{ij} = 1$, $1 \leq i, j \leq n$, $i \neq j$).

4.5.2. Proposition: 1 – We have $\pi_n^{q_n-1} = 0$ if and only if

$$\frac{\lambda_1}{\beta_{1|C}} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1|C}} = 0.$$

We suppose now that $\frac{\lambda_1}{\beta_{1|C}} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1|C}} \neq 0$. Let $x \in C$. Then

2 – For every $\epsilon \in \mathcal{B}_x^{[n-1]}$ such that $\epsilon|_C \neq 0$, we have $\pi_n^{q_n-1}\epsilon \notin (\mathbf{u})$.

3 – For every $\eta \in \mathcal{B}_x^{[n-1]}/(\mathbf{u})$, and every integer k such that $1 \leq k < q_n$, we have $\pi_n^k \eta = 0$ if and only if η is a multiple of $\pi_n^{q_n-k}$.

4 – $\mathcal{B}_x^{[n-1]}/(\mathbf{u})$ is a flat $\mathbb{C}[\pi_n]/(\pi_n^{q_n})$ -module.

Proof. We have $\pi_n^{q_n-1} = 0$ if and only if $(\pi_1^{q_n-1}, \dots, \pi_{n-1}^{q_n-1}) \in (\mathbf{u})$ in $\mathcal{B}_x^{[n-1]}$. We have, in $\mathcal{O}_{\mathcal{C}_{1x}} \times \cdots \times \mathcal{O}_{\mathcal{C}_{n-1,x}}$,

$$(\pi_1^{q_n-1}, \dots, \pi_{n-1}^{q_n-1}) = (\beta_1 \pi_1^{p_{1n}}, \dots, \beta_{n-1,n} \pi_{n-1}^{p_{n-1,n}}) \cdot \left(\frac{1}{\beta_1} \pi_1^{q_1^{[n-1]-1}}, \dots, \frac{1}{\beta_p} \pi_1^{q_{n-1}^{[n-1]-1}} \right),$$

and $\pi_n^{q_n-1} = 0$ if and only if there exist $\eta \in \mathcal{O}_{\mathcal{C}^{[n-1],x}}$, $a_i \in \mathcal{O}_{\mathcal{C}_i,x}$, $1 \leq i < n$, such that

$$(\pi_1^{q_n-1}, \dots, \pi_{n-1}^{q_n-1}) = \eta \mathbf{u} + (a_1 \pi_1^{q_1}, \dots, a_{n-1} \pi_{n-1}^{q_{n-1}}).$$

This equality is equivalent to

$$\left(\frac{1}{\beta_1} \pi_1^{q_1^{[n-1]-1}}, \dots, \frac{1}{\beta_{n-1}} \pi_1^{q_{n-1}^{[n-1]-1}} \right) - \eta = \left(\frac{a_1}{\beta_1} \pi_1^{q_1^{[n-1]}}, \dots, \frac{a_{n-1}}{\beta_{n-1}} \pi_{n-1}^{q_{n-1}^{[n-1]}} \right).$$

Since for $1 \leq i < n$, we have $(0, \dots, 0, \pi_i^{q_i^{[n-1]}}, 0, \dots, 0) \in \mathcal{O}_{\mathcal{C}^{[n-1],x}}$, we have $\pi_n^{q_n-1} = 0$ if and only if

$$\left(\frac{1}{\beta_1} \pi_1^{q_1^{[n-1]-1}}, \dots, \frac{1}{\beta_{n-1}} \pi_1^{q_{n-1}^{[n-1]-1}} \right) \in \mathcal{O}_{\mathcal{C}^{[n-1],x}}.$$

So the result of 1- follows from the definition of $\lambda(\mathcal{C}^{[n-1]})$ (cf. prop. 4.4.5), 2- is an easy consequence.

Now we prove 3-, by induction on k . Suppose that it is true for $k = 1$, and that $\pi_n^k \eta = 0$, with $2 \leq k < q_n$. We have $\pi_n^{k-1} \cdot \pi_n \eta = 0$, so according to the induction hypothesis, $\pi_n \eta$ is a multiple of $\pi_n^{q_n-k+1}$: $\pi_n \eta = \pi_n^{q_n-k+1} \lambda$. So $\pi_n(\eta - \pi_n^{q_n-k} \lambda) = 0$. Since 3- is true for $k = 1$, we can write $\eta - \pi_n^{q_n-k} \lambda = \pi_n^{q_n-1} \epsilon$, i.e. $\eta = \pi_n^{q_n-k} (\lambda + \pi_n^{k-1} \epsilon)$, and 3- is true for k .

It remains to prove 3- for $k = 1$. Suppose that $\pi_n \eta = 0$ (with $\eta \neq 0$). We can write η as $\eta = \pi_n^m \theta$, where θ is not a multiple of π_n , and $0 \leq m < q_n$. Let $\bar{\theta} \in \mathcal{B}_x^{[n-1]}$ be over θ . Since $\mathcal{I}_C^{[n-1]} = (\mathbf{u}) + (\pi)$ according to proposition 4.3.5, the condition “ θ is not a multiple of π_n ” is equivalent to $\bar{\theta} \notin \mathcal{I}_{\mathcal{C},x}^{[n-1]}$. We have $\pi^{m+1} \bar{\theta} \in (\mathbf{u})$, so according to 2-, we have $m+1 \geq q_n$, which proves 3- for $k = 1$. The last assertion is an easy consequence of 3-. \square

4.5.3. Example: Let N be an integer, $s \in H^0(\mathcal{O}_S)$ invertible, and k, l integers such that $1 \leq k, l < n$, $k \neq l$. Suppose that for every integer i such that $1 \leq i < n$ and $i \neq k$ we have $N > p_{ik}^{[n-1]}$ and $N \geq q_i^{[n-1]} - q_k^{[n-1]} + p_{ik}^{[n-1]}$. We take $\mathbf{u} = \mathbf{u}_{kl} - s\pi^N$. We have then $\beta_i = \alpha_{kl}^{(i)}$ if $i \neq k$, and $\beta_k = -s$. The condition $\frac{\lambda_1}{\beta_{1|C}} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1|C}} \neq 0$ is fulfilled if and only if

$$\sum_{1 \leq i < n, i \neq k} \frac{\lambda_i}{\mathbf{a}_{kl}^{(i)}} - \frac{\lambda_k}{s|C} \neq 0.$$

4.5.4. Construction of fragmented deformations – Suppose that $\frac{\lambda_1}{\beta_{1|C}} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1|C}} \neq 0$. From proposition 4.5.2, 4-, it is easy to prove that

- There exists a flat morphism of algebraic varieties $\tau : Y \rightarrow \text{spec}(\mathbb{C}[\pi_n]/(\pi_n^{q_n}))$ with a canonical isomorphism of sheaves of $\mathbb{C}[\pi_n]/(\pi_n^{q_n})$ -algebras $\mathcal{O}_Y \simeq \mathcal{Q}$, such that $\tau^{-1}(*) = C$ (where $*$ is the closed point of $\text{spec}(\mathbb{C}[\pi_n]/(\pi_n^{q_n}))$).
- There exist a family of smooth curves \mathcal{C}_n and a flat morphism $\pi_n : \mathcal{C}_n \rightarrow S$ extending τ (recall that S is a germ). Hence Y is the inverse image of the subscheme of \mathcal{C}_n corresponding to the ideal sheaf $(\pi_n^{q_n})$. The existence of \mathcal{C}_n can be proved using Hilbert schemes of curves in projective spaces. Of course \mathcal{C}_n need not be unique.

We obtain a gluing \mathcal{C} of $\mathcal{C}_1, \dots, \mathcal{C}_n$ by defining the sheaves of algebras $\mathcal{O}_{\mathcal{C}}$ (on the Zariski topological space corresponding to the initial gluing \mathcal{A}) as in 4.4.2, using for Φ_n the quotient morphism $\mathcal{B}^{[n-1]} \rightarrow \mathcal{Q}$. It is easy to see that $\pi^{-1}(P)$ is a primitive multiple curve C_n of multiplicity n extending C_{n-1} , hence \mathcal{C} is a fragmented deformation of C_n .

4.5.5. Remark: 1 – The multiple curve C_n depends on the choice of the family \mathcal{C}_n extending the family Y parametrized by $\text{spec}(\mathbb{C}[\pi_n]/(\pi_n^{q_n}))$.

2 – The multiple curve C_{n-1} is completely defined by $\mathcal{B}^{[n-1]}$, because $(\pi_1^{q_1}) \times \cdots \times (\pi_{n-1}^{q_{n-1}}) \subset (\pi)$. But it is not enough to know $\mathcal{B}^{[n-1]}$ and \mathbf{u} to define C_n . In fact we need $\mathcal{O}_{C_i}/(\pi_i^{q_i+1})$, $1 \leq i \leq n$.

4.6. BASIC ELEMENTS

We use the notations of 4.3 and 4.4.

Let $\mathbf{m} = (m_1, \dots, m_n)$ be an n -tuple of positive integers, and

$$\mathbf{\Pi}^{\mathbf{m}} = (\pi_1^{m_1}) \times \cdots \times (\pi_n^{m_n}).$$

4.6.1. Definition: Let $x \in C$. An element u of $\mathcal{O}_{C,x}$ is called basic at order \mathbf{m} if there exist polynomials $P_1, \dots, P_n \in \mathbb{C}[X]$ such that

$$u \equiv (P_1(\pi_1), \dots, P_n(\pi_n)) \pmod{\mathbf{\Pi}^{\mathbf{m}}}.$$

If $u = (P_1(\pi_1), \dots, P_n(\pi_n))$, we say that u is basic.

Let $\mathbf{q} = (q_1, \dots, q_n)$. Then according to corollary 4.4.8, if u is basic at order \mathbf{q} , then for every $y \in C$, we have $(P_1(\pi_1), \dots, P_n(\pi_n)) \in \mathcal{O}_{C,y}$. So $(P_1(\pi_1), \dots, P_n(\pi_n))$ is defined on a neighborhood of C .

4.6.2. Lemma: *Let $u, v, w \in \mathcal{O}_{C,x}$ such that $w = uv$ and $w \neq 0$. Suppose that u and w are basic at every order. Then v is basic at every order.*

Proof. Let N be a positive integer such that $N \gg 0$ and $\mathbf{N} = (N, \dots, N)$. Suppose that $w \equiv (Q_1(\pi_1), \dots, Q_n(\pi_n)) \pmod{(\pi^N)}$, where $Q_1, \dots, Q_n \in \mathbb{C}[X]$. Let $\mathbf{m} = (m_1, \dots, m_n)$ be an n -tuple of positive integers, and $v = (v_i)_{1 \leq i \leq n}$. Suppose that

$$u \equiv (P_1(\pi_1), \dots, P_n(\pi_n)) \pmod{\mathbf{\Pi}^{\mathbf{N}}}$$

Then we have

$$Q_i(\pi_i) \equiv P_i(\pi_i) \cdot v_i \pmod{(\pi_i^N)}$$

for $1 \leq i \leq n$. We can write $P_i(X)$ as $P_i(X) = X^{n_i} R_i(X)$, where $R_i(X) \in \mathbb{C}[X]$ is such that $R_i(0) \neq 0$. Then $Q_i(X)$ is also divisible by X^{n_i} : $Q_i(X) = X^{n_i} S_i(X)$, and we have in $\mathcal{O}_{\mathbf{A}x}$:

$$S_i(\pi_i) \equiv R_i(\pi_i) \cdot v_i \pmod{(\pi_i^{N'})}$$

for some integer $N' \gg 0$. We can write $R_i(X) = a_i \cdot (1 - X \cdot T_i(X))$, with $a_i \in \mathbb{C}^*$, $T_i \in \mathbb{C}(X)$. We have then

$$v_i \equiv \frac{S_i(\pi_i)}{a_i} \sum_{p=1}^{m_i-1} (\pi_i T_i(\pi_i))^p \pmod{\mathbf{\Pi}^{\mathbf{m}}}.$$

□

For $1 \leq i \leq n$, let $\mathbf{u}_{(i)} = ((u_{(i)j})_{1 \leq j \leq n})$ be a generator of the ideal sheaf \mathcal{I}_{C_i} of C_i in \mathcal{C} , such that for $1 \leq j \leq n$, $u_{(i)j} \in \mathbb{C}[\pi_j]$ (cf. corollary 4.3.8).

4.6.3. Proposition: *Let $v \in \mathcal{O}_{C,x}$. then v is basic at every order if and only if for every n -tuple \mathbf{m} of positive integers, there exist an integer $q > 0$ and $P_1, \dots, P_q \in \mathbb{C}[X]$ such that*

$$v \equiv \sum_{1 \leq j \leq q} P_j(\pi) \cdot \mathbf{u}_{(i)}^j \pmod{\mathbf{\Pi}^{\mathbf{m}}}.$$

Proof. We use the notations of the proof of lemma 4.6.2. Suppose that $v = (v_j)_{1 \leq j \leq n}$ is basic at every order. Let N be a positive integer and $\mathbf{N} = (N, \dots, N)$. We will prove by induction on $q \geq 0$ that we can write v as

$$(4) \quad v \equiv \sum_{0 \leq j \leq q} P_j(\pi) \cdot \mathbf{u}_{(i)}^j + \gamma_q \mathbf{u}_{(i)}^{q+1} \pmod{\mathbf{\Pi}^{\mathbf{N}}}$$

with $P_0, \dots, P_q \in \mathbb{C}[X]$, and $\gamma_q \in \mathcal{O}_{C,x}$. This proves proposition 4.6.3 if q and N are big enough.

For $q = 0$, we have $v_i \equiv P(\pi_i) \pmod{\pi_i^N}$, for some $P \in \mathbb{C}[X]$, and we can take $P_0 = P$. Suppose that the result is true for q and that we have (4). Since $v - \sum_{1 \leq j \leq q} P_j(\pi) \cdot \mathbf{u}_{(i)}^j$ is basic at any

order, using the same method as in the proof of lemma 4.6.2, we see that γ_q is basic at order \mathbf{N}' , where $\mathbf{N}' = (N', \dots, N')$, for some integer $N' \gg 0$. As in the case $q = 0$ we have

$$\gamma_q \equiv P_{q+1}(\pi) + \mathbf{u}_{(i)} \cdot \gamma_{q+1} \pmod{\mathbf{\Pi}^{\mathbf{N}'},}$$

with $P_{q+1} \in \mathbb{C}[X]$. Hence

$$v \equiv \sum_{0 \leq j \leq q+1} P_j(\pi) \cdot \mathbf{u}_{(i)}^j + \gamma_{q+1} \mathbf{u}_{(i)}^{q+2} \pmod{\mathbf{\Pi}^{\mathbf{N}}}$$

□

4.6.4. Proposition: *Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{O}_{\mathcal{C},x}$ be such that there exists $P_1, \dots, P_{n-1} \in \mathbb{C}[X]$ such that, for $1 \leq i \leq n-1$, we have $\alpha_i \equiv P_i(\pi_i) \pmod{(\pi_i^{q_i})}$. Then there exists $P_n \in \mathbb{C}[X]$ such that $\alpha_n \equiv P_n(\pi_n) \pmod{(\pi_n^{q_n})}$, i.e. α is a basic element of order \mathbf{q} .*

Proof. By induction on n . The case $n = 2$ is an easy consequence of proposition 4.2.2. Suppose that $n \geq 3$ and that the result is true for $n-1$.

By subtracting multiples of $(0, \dots, 0, \pi_i^{q_i}, 0, \dots, 0)$ we may assume that for $1 \leq i \leq n-1$, $\alpha_i \in \mathbb{C}[\pi_i]$. By subtracting a regular function on a neighborhood of C in \mathcal{C} , and a multiple of $(\pi_1^{q_1}, 0, \dots, 0)$ we may also assume that $\alpha_1 = 0$. The ideal sheaf of \mathcal{C}_1 is generated by $\mathbf{u}_{(1)}$. We can then write $\alpha = \beta \mathbf{u}_{(1)}$, with $\beta = (\beta_i)_{1 \leq i \leq n} \in \mathcal{O}_{\mathcal{C},x}$. We have

$$(\alpha_2, \dots, \alpha_{n-1}) = (\beta_2, \dots, \beta_{n-1}) \cdot (u_{(1)2}, \dots, u_{(1)n-1}),$$

hence by lemma 4.6.2, $(\beta_2, \dots, \beta_{n-1})$ is a basic element at any order. By the induction hypothesis, there exists $Q \in \mathbb{C}[X]$ such that $\beta_n \equiv Q(\pi_n) \pmod{(\pi_n^{q_n - p_{1n}})}$. Since $u_{(1)n}$ is a multiple of $\pi_n^{p_{1n}}$ (from the definition of p_{1n}), it follows that $\alpha_n \equiv u_{(1)n} Q(\pi_n) \pmod{(\pi_n^{q_n})}$. □

4.7. SIMPLE PRIMITIVE CURVES AND FRAGMENTED DEFORMATIONS

Let C_n be a primitive multiple curve of multiplicity n and associated smooth curve C . Let \mathcal{I}_C be the ideal sheaf of C in C_n . It is obvious from proposition 4.3.5, 1-, that if there exists a fragmented deformation of C_n , then we have $\mathcal{I}_{C, C_n} \simeq \mathcal{O}_{C_{n-1}}$, i.e. C_n is *simple* (cf. 2.4). Conversely we have

4.7.1. Theorem: *Let C_n be a simple primitive multiple curve of multiplicity n . Then there exists a fragmented deformation of C_n .*

Proof. According to theorem 2.4.1, there exists a flat family of smooth projective curves $\tau : \mathcal{C} \rightarrow \mathbb{C}$ such that $\tau^{-1}(0) \simeq C$ and that C_n is isomorphic to the n -th infinitesimal neighborhood of C in \mathcal{C} . Let $\rho_n : \mathbb{C} \rightarrow \mathbb{C}$ be the map defined by $\rho_n(z) = z^n$, and $\theta = \rho_n \circ \tau : \mathcal{C} \rightarrow \mathbb{C}$. It is a flat morphism, $\theta^{-1}(0) = C_n$, and for every $z \neq 0$ in the image of τ , $\theta^{-1}(z)$ is a disjoint union of n smooth irreducible curves. We can then apply the process of proposition 3.1.3 to obtain the desired fragmented deformation: it is $\mathcal{C} \times_{\mathbb{C}} \mathbb{C}$

$$\begin{array}{ccc} \mathcal{C} \times_{\mathbb{C}} \mathbb{C} & \xrightarrow{\pi} & \mathbb{C} \\ \downarrow & & \downarrow \rho_n \\ \mathcal{C} & \xrightarrow{\theta} & \mathbb{C} \end{array}$$

□

4.7.2. Remark: let (p_{ij}) be the spectrum of the fragmented deformation constructed in the proof of theorem 4.7.1. Then it is easy to see that $p_{ij} = 1$ for $1 \leq i, j \leq n$, $i \neq j$. If $x \in C$, then

$(\mathbb{C} \times_{\mathbb{C}} \mathbb{C})_x = \mathcal{O}_{\mathbb{C},x} \otimes_{\mathcal{O}_{\mathbb{C},x}} \mathcal{O}_{\mathbb{C},x}$, and if $t = I_{\mathbb{C}} \in \mathcal{O}_{\mathbb{C},x}$, we have for $1 \leq k \leq n$

$$(\pi_1, \dots, \pi_{k-1}, 0, \pi_{k+1}, \dots, \pi_n) = \frac{1}{n-1} (1 \otimes t - e^{\frac{2ki\pi}{n}} (t \otimes 1)) .$$

5. STARS OF A CURVE

5.1. DEFINITIONS

Let S be a smooth irreducible curve, and $P \in S$ (we can also take for (S, P) the germ of a smooth curve). Let n be a positive integer.

5.1.1. Definition: An n -star (or more simply, a star) of (S, P) is an algebraic variety \mathcal{S} such that

- (i) \mathcal{S} is the union of n irreducible components S_1, \dots, S_n , with fixed isomorphisms $S_i \simeq S$, $1 \leq i \leq n$.
- (ii) For $1 \leq i < j \leq n$, $S_i \cap S_j$ has only one closed point, namely P .
- (iii) There exists a morphism $\pi : \mathcal{S} \rightarrow S$, such that for $1 \leq i \leq n$, the restriction $\pi|_{S_i} : S_i \rightarrow S$ is the isomorphism $S_i \simeq S$ of (i).

All the n -stars of (S, P) have the same underlying Zariski topological space $S(n)$ and set of closed points. The latter is $(\bigcup_{1 \leq i \leq n} \widehat{S}_i) / \sim$, where \widehat{S}_i is the set of closed points of S_i , and the equivalence relation \sim is defined by: for $x \in \widehat{S}_i$ and $y \in \widehat{S}_j$, $x \sim y$ if and only if $i = j$ and $x = y$, or $x = P \in \widehat{S}_i$ and $y = P \in \widehat{S}_j$. An open subset of \mathcal{S} is defined by open subsets U_1 of S_1, \dots, U_n of S_n , such that for $1 \leq i < j \leq n$, we have $P \in U_i$ if and only if $P \in U_j$.

The *initial* star \mathcal{S}_0 of (S, P) is defined as follows: for every open subset U of $S(n)$, $\mathcal{O}_{\mathcal{S}_0}(U)$ is the set of $(\alpha_1, \dots, \alpha_n) \in \mathcal{O}_{S_1}(U \cap S_1) \times \dots \times \mathcal{O}_{S_n}(U \cap S_n)$ such that if $P \in U$ then $\alpha_1(P) = \dots = \alpha_n(P)$.

For every n -star \mathcal{S} of (S, P) , there is a unique dominant morphism $\mathcal{S}_0 \rightarrow \mathcal{S}$ inducing the identity on each component. So $\mathcal{O}_{\mathcal{S},P}$ is a subring of $\mathcal{O}_{\mathcal{S}_0,P}$.

Note that (iii) is equivalent to

- (iii)' For every $\alpha \in \mathcal{O}_{S,P}$, we have $(\alpha, \dots, \alpha) \in \mathcal{O}_{\mathcal{S},P}$.

5.1.2. Definition: An oblate n -star (or more simply, an oblate star) of (S, P) is an n -star \mathcal{S} such that some neighborhood of P in \mathcal{S} can be embedded in a smooth surface.

5.1.3. Proposition: An n -star \mathcal{S} is oblate if and only if $\pi^{-1}(P) \simeq \text{spec}(\mathbb{C}[X]/(X^n))$.

(cf. prop. 4.1.6).

Let $I \subset \{1, \dots, n\}$ be a nonempty subset. Let $\mathcal{S}^{(I)} = \bigcup_{i \in I} S_i \subset \mathcal{S}$. If \mathcal{S} is oblate then $\mathcal{S}^{(I)}$ is oblate too.

5.2. PROPERTIES OF OBLATE STARS

Let \mathcal{S} be an oblate n -star of S . Recall that t denotes a generator of the maximal ideal of P in S . We will denote this generator on $S_i \subset \mathcal{S}$ by t_i . We will also denote by π the element $t \circ \pi$ of the maximal ideal of P in \mathcal{S} . Let \mathcal{I}_P be the ideal sheaf of P in \mathcal{S} .

We begin with 2-stars:

5.2.1. Proposition: *Suppose that $n = 2$. Then*

- 1 – *There exists a unique integer $p > 0$ such that $\mathcal{I}_{P,P}/(\pi)$ is generated by the image of $(t_1^p, 0)$.*
- 2 – *The image of $(0, t_2^p)$ is also a generator of $\mathcal{I}_{P,P}/(\pi)$.*
- 3 – *$(0, t_2^p)$ (resp. $(t_1^p, 0)$) is a generator of the ideal sheaf of S_1 (resp. S_2) at P .*
- 4 – *$\mathcal{O}_{S^{(2)},P}$ consists of pairs $(\alpha, \beta) \in \mathcal{O}_{S,P} \times \mathcal{O}_{S,P}$ such that $\alpha - \beta \in (t^p)$.*

(cf. prop. 4.2.1 and 4.2.2).

Now suppose that $n \geq 2$. Let $I = \{i, j\} \subset \{1, \dots, n\}$, with $i \neq j$. Then $S_i \cup S_j \subset \mathcal{S}$ is a 2-star of S . Hence by proposition 5.2.1 there exists a unique integer $p_{ij} > 0$ such that $\mathcal{I}_{P,P}/(\pi)$ (on $S_i \cup S_j$) is generated by the image of $(t_i^{p_{ij}}, 0)$ (and also by the image of $(0, t_j^{p_{ij}})$). Let $p_{ii} = 0$. Then the symmetric matrix $(p_{ij})_{1 \leq i, j \leq n}$ is called the *spectrum* of \mathcal{S} .

There exists an element $v_{ij} = (\nu_m)_{1 \leq m \leq n}$ such that $\nu_i = 0$ and $\nu_j = t_j^{p_{ij}}$. For every integer m such that $1 \leq m \leq n$, $m \neq i, j$, there exists an invertible element $\beta_{ij}^{(m)} \in \mathcal{O}_{S,P}$ such that $\nu_m = \beta_{ij}^{(m)} t_m^{p_{im}}$. Let $\beta_{ij}^{(i)} = 0$, $\beta_{ij}^{(j)} = 1$.

5.2.2. Proposition: *Let $\mathbf{b}_{ij}^{(m)} = \beta_{ij}^{(m)}(P) \in \mathbb{C}$. Then we have, for all integers i, j, k, m, q such that $1 \leq i, j, k, m, q \leq n$, $i \neq j$, $i \neq k$*

$$\mathbf{b}_{ik}^{(m)} \mathbf{b}_{ij}^{(q)} = \mathbf{b}_{ik}^{(q)} \mathbf{b}_{ij}^{(m)}.$$

In particular we have $\mathbf{b}_{ij}^{(m)} = \mathbf{b}_{ik}^{(m)} \mathbf{b}_{ij}^{(k)}$ and $\mathbf{b}_{ij}^{(m)} \mathbf{b}_{im}^{(j)} = 1$.

For all distinct integers i, j, k such that $1 \leq i, j, k \leq n$, we have

$$\mathbf{b}_{ki}^{(j)} = -\mathbf{b}_{ik}^{(j)} \mathbf{b}_{ji}^{(k)}.$$

(cf. prop. 4.3.2 and 4.4.6).

Let p be an integer such that $1 \leq p < n$, and $(i_1, j_1), \dots, (i_p, j_p)$ p pairs of distinct integers of $\{1, \dots, n\}$. Then the image of $\prod_{m=1}^p \mathbf{v}_{i_m j_m}$ is a generator of $(\mathcal{I}_{P,P}^p + (\pi))/(\mathcal{I}_{P,P}^{p+1} + (\pi))$.

Let $I \subset \{1, \dots, n\}$ be a nonempty subset, distinct from $\{1, \dots, n\}$. Let $i \in \{1, \dots, n\} \setminus I$. Let

$$\mathbf{v}_{I,i} = \prod_{j \in I} \mathbf{v}_{ji}.$$

5.2.3. Proposition: *The ideal sheaf of $\mathcal{S}^{(I)}$ in \mathcal{S} is generated by $\mathbf{v}_{I,i}$ at P .*

(cf. prop. 4.3.3).

Note that if $I = \{1, \dots, n\} \setminus \{i\}$ then $\mathbf{v}_{I, i|S_j} = 0$ if $j \neq i$, and $\mathbf{v}_{I, i|S_i} = t_i^{q_i}$, with $q_i = \sum_{1 \leq j \leq n} p_{ij}$.

Let i be an integer such that $1 \leq i \leq n$ and $J_i = \{1, \dots, n\} \setminus \{i\}$. Let \mathcal{K}_i be the image of $\mathcal{O}_{\mathcal{S}}$ in $\prod_{1 \leq j \leq n, j \neq i} \mathcal{O}_{S_j}/(t_j^{q_j})$. We can view \mathcal{K}_i as a \mathbb{C} -algebra. For every $\alpha = (\alpha_m) \in \mathcal{O}_{\mathcal{S}, P}$, let $k_i(\alpha)$ be the image of α in \mathcal{K}_i .

5.2.4. Proposition: *There exists a morphism of \mathbb{C} -algebras*

$$\Psi_i : \mathcal{K}_i \longrightarrow \mathcal{O}_{S_i, P}/(t_i^{q_i})$$

such that for every $(\alpha_m)_{1 \leq m \leq n, m \neq i} \in \mathcal{O}_{\mathcal{S}^{(J_i)}, P}$, $\alpha_i \in \mathcal{O}_{S_j, P}$, we have $\alpha = (\alpha_m)_{1 \leq m \leq n} \in \mathcal{O}_{\mathcal{S}, P}$ if and only if $\Psi_i(k_i(\alpha)) = [\alpha_i]_{q_i}$.

(cf. prop. 4.4.1).

The morphism Ψ_i has the following properties:

- (i) For every $(\alpha_m)_{1 \leq m \leq n, m \neq i} \in \mathcal{O}_{\mathcal{S}^{(J_i)}, P}$, we have $\Psi_i(\alpha)(P) = \alpha_m(P)$ for $1 \leq m \leq n, m \neq i$.
- (ii) We have $\Psi_i((t_m)_{1 \leq m \leq n, m \neq i}) = t_i$.
- (iii) Let $j, k \in \{1, \dots, n\}$ be such that i, j, k are distinct. Let \mathbf{w} be the image of \mathbf{v}_{jk} in \mathcal{B}_i . Then there exists $\lambda \in \mathcal{O}_{S_i, P}^*$ such that $\Psi_i(\mathbf{w}) = \lambda t_i^{p_{ij}}$.
- (iv) Let j be an integer such that $1 \leq j \leq n$ and $j \neq i$. Let \mathbf{w} be the image of \mathbf{v}_{ij} in \mathcal{K}_i . Then we have $\ker(\Psi_i) = (\mathbf{w})$.

5.2.5. Converse – Let $\mathcal{S}^{[n-1]}$ be a $(n-1)$ -star of S , with components S_1, \dots, S_{n-1} , of spectrum $(p_{jk})_{1 \leq j, k \leq n-1}$. Let $p_{nj} = p_{jn}$, $1 \leq j < n$ be positive integers, and $p_{nn} = 0$. For $1 \leq j \leq n$, let $q_j = \sum_{1 \leq k \leq n} p_{kj}$.

Let S_n be another copy of S . Let \mathcal{K}_n be the image of $\mathcal{O}_{\mathcal{S}^{[n-1]}}$ in $\prod_{1 \leq j \leq n-1} \mathcal{O}_{S_j}/(t_j^{q_j})$ and

$$\Psi_n : \mathcal{K}_n \longrightarrow \mathcal{O}_{S_n}/(t_n^{q_n})$$

a morphism of \mathbb{C} -algebras satisfying properties (i), (ii), (iii) above. Let \mathcal{K} be the subsheaf of algebras of $\mathcal{O}_{\mathcal{S}_0}$ defined by: $\mathcal{K} = \mathcal{O}_{\mathcal{S}_0}$ on $\mathcal{S}_0 \setminus \{P\}$, and for every $\alpha = (\alpha_m)_{1 \leq m \leq n} \in \mathcal{O}_{\mathcal{S}_0, P}$, $\alpha \in \mathcal{K}_P$ if and only if $\Psi_n(\alpha') = [\alpha_n]_{q_n}$ (where α' is the image of $(\alpha_m)_{1 \leq m \leq n-1}$ in \mathcal{K}_n).

It is easy to see that \mathcal{K} is the structural sheaf of an oblate n -star of S .

Let $\mathcal{H} = \prod_{1 \leq j \leq n} (t_j^{q_j-1})/(t_j^{q_j}) \simeq \mathbb{C}^n$ and \mathcal{K} be the image of $\mathcal{O}_{\mathcal{S}}$ in $\prod_{1 \leq j \leq n} \mathcal{O}_{S_j}/(t_j^{q_j})$. We can view \mathcal{K} as a \mathbb{C} -algebra. Let $\mathcal{J} = \mathcal{H} \cap \mathcal{K}$.

5.2.6. Proposition: *There exists a unique $\lambda(\mathcal{S}) = (\lambda_1, \dots, \lambda_n) \in \mathbb{P}_n(\mathbb{C})$ such that for every $\mathbf{u} = (u_j)_{1 \leq j \leq n} \in \mathcal{H}$, we have $\mathbf{u} \in \mathcal{J}$ if and only if $\lambda_1 u_1 + \dots + \lambda_n u_n = 0$. The λ_i are all non zero.*

(cf. prop. 4.4.5).

For all distinct integers i, j such that $1 \leq i, j \leq n$, we have

$$\frac{\lambda_i}{\lambda_j} = - \prod_{1 \leq m \leq n, m \neq i, j} \mathbf{b}_{mi}^{(j)}.$$

5.3. CONSTRUCTION OF OBLATE STARS OF A CURVE

Consider an oblate $(n-1)$ -star of S , $\mathcal{S}^{[n-1]}$, with $n-1$ irreducible components S_1, \dots, S_{n-1} , copies of S . Let $(p_{ij}^{[n-1]})_{1 \leq i, j < n}$ be its spectrum. For $1 \leq i < n$, let $q_i^{[n-1]} = \sum_{1 \leq j < n} p_{ij}^{[n-1]}$. We

denote by $\mathcal{I}_P^{[n-1]}$ the ideal of P in $\mathcal{O}_{\mathcal{S}^{[n-1]}, P}$. Let $\lambda(\mathcal{S}^{[n-1]}) = (\lambda_1, \dots, \lambda_{n-1})$.

Let $p_{1n}, \dots, p_{n-1, n}$ be positive integers, $q_i = q_i^{[n-1]} + p_{in}$ for $1 \leq i < n$, and $q_n = p_{1n} + \dots + p_{n-1, n}$. Let $\mathbf{u} \in \mathcal{I}_{P, P}^{[n-1]}$ whose image generates $\mathcal{I}_P^{[n-1]} / ((\mathcal{I}_P^{[n-1]})^2 + (\pi))$, of the form

$$\mathbf{u} = (\beta_1 t_1^{p_{1n}}, \dots, \beta_{n-1} t_{n-1}^{p_{n-1, n}}),$$

with $\beta_i \in \mathcal{O}_{S_i, P}$ invertible for $1 \leq i < n$.

Let $\mathcal{K}^{[n-1]}$ be the image of $\mathcal{O}_{\mathcal{S}^{[n-1]}}$ in $\mathcal{O}_{S_1} / (t_1^{q_1}) \times \dots \times \mathcal{O}_{S_{n-1}} / (t_{n-1}^{q_{n-1}})$. We will also denote by \mathbf{u} the image of \mathbf{u} in $\mathcal{K}^{[n-1]}$. Let $\mathcal{Q} = \mathcal{K}^{[n-1]} / (\mathbf{u})$, $\rho : \mathcal{K}^{[n-1]} \rightarrow \mathcal{Q}$ the projection and $t_n = \rho(\pi)$.

5.3.1. Proposition: 1 – We have $t_n^{q_n} = 0$.

2 – We have $t_n^{q_n-1} = 0$ if and only if

$$\frac{\lambda_1}{\beta_1(P)} + \dots + \frac{\lambda_{n-1}}{\beta_{n-1}(P)} = 0.$$

We suppose now that $\frac{\lambda_1}{\beta_1(P)} + \dots + \frac{\lambda_{n-1}}{\beta_{n-1}(P)} \neq 0$. Then

3 – For every $\epsilon \in \mathcal{K}^{[n-1]}$ such that $\epsilon(P) \neq 0$, we have $t_n^{q_n-1} \epsilon \notin (\mathbf{u})$.

4 – For every $\eta \in \mathcal{K}^{[n-1]} / (\mathbf{u})$, and every integer k such that $1 \leq k < q_n$, we have $t_n^k \eta = 0$ if and only if η is a multiple of $t_n^{q_n-k}$.

5 – $\mathcal{K}^{[n-1]} / (\mathbf{u})$ is a flat $\mathbb{C}[t_n] / (t_n^{q_n})$ -module.

(cf. prop. 4.5.2).

5.3.2. Construction of stars of a curve – Suppose that $\frac{\lambda_1}{\beta_1(P)} + \dots + \frac{\lambda_{n-1}}{\beta_{n-1}(P)} \neq 0$. From proposition 5.3.1, 5-, it is easy to prove, using 5.2.5, that there is a unique oblate n -star \mathcal{S} such that $\mathcal{S}^{[n-1]}$ is the union $\bigcup_{1 \leq i \leq n-1} S_i$ in \mathcal{S} and Ψ_n is the quotient map $\mathcal{K}_n = \mathcal{K}^{[n-1]} \rightarrow \mathcal{Q}$.

5.4. MORPHISMS OF STARS

Recall that if \mathcal{S} is an oblate n -star of S , then we have a canonical inclusion of sheaves of algebras (on the underlying topological space $S(n)$ of \mathcal{S}) $\mathcal{O}_{\mathcal{S}} \subset \mathcal{O}_{\mathcal{S}_0}$.

Let $\mathcal{S}, \mathcal{S}'$ be oblate n -stars of S , with irreducible components S_1, \dots, S_n , and $f : \mathcal{S} \rightarrow \mathcal{S}'$ a morphism inducing the identity on all the components. Such a morphism exists if and only if $\mathcal{O}_{\mathcal{S}'} \subset \mathcal{O}_{\mathcal{S}}$, and in this case f is unique and is induced by the previous inclusion. Let (p_{ij}) (resp. (p'_{ij})) be the spectrum of \mathcal{S} (resp. \mathcal{S}').

5.4.1. Proposition: *We have $p_{ij} \leq p'_{ij}$ for $1 \leq i, j \leq n$. If f is not the identity morphism then there exist i, j such that $p_{ij} < p'_{ij}$.*

Proof. Let $I = \{i, j\}$. Then f induces a morphism $\mathcal{S}^{(I)} \rightarrow \mathcal{S}'^{(I)}$. So we have $\mathcal{O}_{\mathcal{S}'^{(I)}, P} \subset \mathcal{O}_{\mathcal{S}^{(I)}, P}$. From proposition 5.2.1, 4-, it follows that $p_{ij} \leq p'_{ij}$.

Suppose now that $p'_{ij} = p_{ij}$ for $1 \leq i, j \leq n$. We must prove that $\mathcal{S} = \mathcal{S}'$, i.e. that $\mathcal{O}_{\mathcal{S}', P} = \mathcal{O}_{\mathcal{S}, P}$. This is done by induction on n . For $n = 2$ it is obvious. Suppose that it is true for $n - 1$. Let $I = \{1, \dots, n - 1\}$. Then f induces a morphism $f_{n-1} : \mathcal{S}^{(I)} \rightarrow \mathcal{S}'^{(I)}$. It follows from the induction hypothesis that $\mathcal{S}^{(I)} = \mathcal{S}'^{(I)}$. Since the integers q_i are the same for \mathcal{S} and \mathcal{S}' , the algebras \mathcal{K}_n for \mathcal{S} and \mathcal{S}' (cf. proposition 5.2.4) are also the same. Now let $\alpha \in \mathcal{O}_{\mathcal{S}, P}$, and let $\beta \in \mathcal{K}_n$ be the image of α . Let $\alpha' \in \mathcal{O}_{\mathcal{S}', P}$ be such that its image in \mathcal{K}_n is also β . Then $\alpha - \alpha'$ belongs to the ideal generated by the $(0, \dots, 0, t_i^{q_i}, 0, \dots, 0)$, $1 \leq i \leq n$, which is included in $\mathcal{O}_{\mathcal{S}', P}$. Hence $\alpha \in \mathcal{O}_{\mathcal{S}', P}$. \square

5.4.2. Lemma: *Suppose that f is not the identity morphism. Then there exist an ideal $\mathcal{I} \subset \mathcal{O}_{\mathcal{S}', P}$ and $u \in \mathcal{I}, v \in \mathcal{O}_{\mathcal{S}, P}$ such that*

$$u \otimes v \neq 0 \quad \text{in} \quad \mathcal{I} \otimes_{\mathcal{O}_{\mathcal{S}', P}} \mathcal{O}_{\mathcal{S}, P}$$

and $uv = 0$.

Proof. Let $q_1 = \sum_{i=1}^n p_{1i}$, $q'_1 = \sum_{i=1}^n p'_{1i}$. According to proposition 5.4.1 we can assume that $q_1 < q'_1$. Let u be a generator of the ideal of S_1 in $\mathcal{O}_{\mathcal{S}', P}$ and $\mathcal{I} = (u)$. Let $v = (t_1^{q_1}, 0, \dots, 0)$. We have $uv = 0$. We have to prove that $u \otimes v \neq 0$. We need only to find an $\mathcal{O}_{\mathcal{S}', P}$ -module M and a $\mathcal{O}_{\mathcal{S}', P}$ -bilinear map

$$\phi : \mathcal{I} \otimes_{\mathcal{O}_{\mathcal{S}', P}} \mathcal{O}_{\mathcal{S}, P} \longrightarrow M$$

such that $\phi(u \otimes v) \neq 0$. We take $M = \mathcal{O}_{S_1, P} / (t_1^{q'_1})$, which is a quotient of $\mathcal{O}_{\mathcal{S}'}$. It is easy to verify that

$$\phi : ((\lambda_i)_{1 \leq i \leq n} u, (w_i)_{1 \leq i \leq n}) \longmapsto \lambda_1 w_1 \pmod{t_1^{q'_1}}$$

is well defined, bilinear, and that $\phi(u \otimes v) \neq 0$. \square

5.4.3. Corollary: *Suppose that f is not the identity morphism. Let Y be an algebraic variety and $g : Y \rightarrow S$ a morphism such that $g^* : \mathcal{O}_{\mathcal{S}, P} \rightarrow \mathcal{O}_{Y, P}$ is injective. Then $f \circ g : Y \rightarrow \mathcal{S}'$ is not flat.*

Proof. We use the notations of the proof of lemma 5.4.2. We have a commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_{\mathcal{S},P} & \xrightarrow{g^*} & \mathcal{O}_{Y,P} \\
\downarrow \lambda_S & & \downarrow \lambda_Y \\
\mathcal{I} \otimes_{\mathcal{O}_{\mathcal{S}',P}} \mathcal{O}_{\mathcal{S},P} & \xrightarrow{I_{\mathcal{I}} \otimes g^*} & \mathcal{I} \otimes_{\mathcal{O}_{\mathcal{S}',P}} \mathcal{O}_{Y,P} \\
\downarrow \mu_S & & \downarrow \mu_Y \\
\mathcal{O}_{\mathcal{S},P} & \xrightarrow{g^*} & \mathcal{O}_{Y,P}
\end{array}$$

where $\lambda_S(\alpha) = u \otimes \alpha$, $\mu_S(u \otimes \alpha) = u\alpha$, and λ_Y, μ_Y are defined similarly. It follows that $\mu_Y(u \otimes g^*v) = 0$. We will show that $u \otimes g^*v \neq 0$, and this will imply that $f \circ g$ is not flat. Let $w = (t_1^{q_1}, 0, \dots, 0)$. Then we have $\mathcal{I} \simeq \mathcal{O}_{\mathcal{S}',P}/(w)$, and from the exact sequence of $\mathcal{O}_{\mathcal{S}',P}$ -modules $0 \rightarrow (w) \rightarrow \mathcal{O}_{\mathcal{S}',P} \rightarrow \mathcal{I} \rightarrow 0$ we deduce that $\ker(\lambda_Y) = (w) \cdot \mathcal{O}_{Y,P}$. Suppose that $u \otimes g^*v = 0$. Then g^*v is a multiple of w : $g^*v = w \cdot a$, for some $a \in \mathcal{O}_{Y,P}$. But we have $w = g^* \pi^{q_1 - q_1} v$. Hence $g^*v \cdot (1 - g^* \pi^{q_1 - q_1}) = 0$. Since $1 - g^* \pi^{q_1 - q_1}$ is invertible, we have $g^*v = 0$, which is false since g^* is injective. Hence $u \otimes g^*v \neq 0$. \square

5.5. STRUCTURE OF IDEALS

Let \mathcal{S} be an oblate n -star of S .

5.5.1. Proposition: *Let $\mathcal{I} \subset \mathcal{O}_{\mathcal{S},P}$ be a proper ideal. Then*

1 - *There exists a positive integer k such that $k \leq n$ and a filtration by ideals*

$$\{0\} = \mathcal{I}_{k+1} \subset \mathcal{I}_k \subset \dots \subset \mathcal{I}_1 = \mathcal{I}$$

such that, for $1 \leq i \leq k$ there exists a positive integer j such that $j \leq n$ and an isomorphism $\mathcal{I}_i/\mathcal{I}_{i+1} \simeq \mathcal{O}_{S_j,P}$ of $\mathcal{O}_{\mathcal{S},P}$ -modules.

2 - *If $\mathcal{I}_i/\mathcal{I}_{i+1} \simeq \mathcal{O}_{S_j,P}$, then $\mathcal{I}_{i+1} \subset \mathcal{I}_{S_j}$ and $\mathcal{I}_i \not\subset \mathcal{I}_{S_j}$.*

Proof. We prove **1-** by induction on n . The case $n = 1$ is trivial. Suppose that $n > 1$ and that the result is true for $n - 1$. Let \mathcal{J}_1 be the ideal sheaf of $S_1 \subset \mathcal{S}$, and $\mathcal{S}' = S_2 \cup \dots \cup S_{n-1} \subset \mathcal{S}$. We can view \mathcal{J}_1 as an ideal of $\mathcal{O}_{\mathcal{S}',P}$. We can suppose that $\mathcal{I} \not\subset \mathcal{O}_{\mathcal{S}',P}$, i.e that some element of \mathcal{I} has a nonzero first coordinate. Let m be the smallest positive integer such that \mathcal{I} contains an element u of the form

$$u = (t^m, \alpha_2, \dots, \alpha_n).$$

Then every element v of \mathcal{I} can be written as

$$v = \lambda u + v',$$

with $\lambda \in \mathcal{O}_{\mathcal{S},P}$ and $v' \in \mathcal{J}_1 \cap \mathcal{I}$, and the first coordinate of λ is uniquely determined. It follows that $\mathcal{I}/(\mathcal{J}_1 \cap \mathcal{I}) \simeq \mathcal{O}_{S_1,P}$. We can apply the recurrence hypothesis to the ideal $\mathcal{J}_1 \cap \mathcal{I}$ of $\mathcal{O}_{\mathcal{S}',P}$ and get a filtration of it, from which we deduce the filtration of \mathcal{I} . This proves **1-** for n .

Now we prove **2-**. Let $\alpha \in \mathcal{O}_{\mathcal{S},P} \setminus \mathcal{I}_{S_j}$. Let $u \in \mathcal{I}_i$ be over a generator of $\mathcal{I}_i/\mathcal{I}_{i+1}$. Then the image of αu in $\mathcal{I}_i/\mathcal{I}_{i+1}$ is not zero, i.e. $\alpha u \notin \mathcal{I}_{i+1}$. Hence $\alpha \notin \mathcal{I}_{i+1}$, and $\mathcal{I}_{i+1} \subset \mathcal{I}_{S_j}$. Let

$v_i = (0, \dots, 0, t_i^{q_i}, 0, \dots, 0) \in \mathcal{O}_{\mathcal{S},P}$. Then the image of $v_i u$ in $\mathcal{I}_i/\mathcal{I}_{i+1}$ is not zero, hence $u \notin \mathcal{I}_{S_j}$ and $\mathcal{I}_i \not\subset \mathcal{I}_{S_j}$. \square

5.6. STAR ASSOCIATED TO A FRAGMENTED DEFORMATION

We keep the notations of chapter 4. Let $n \geq 2$ be an integer, $\pi : \mathcal{C} \rightarrow S$ a fragmented deformation of C_n , and $\mathcal{C}_1, \dots, \mathcal{C}_n$ the irreducible components of \mathcal{C} .

Recall that $S(n)$ is the underlying (Zariski) topological space of any n -star of S . Let \mathcal{C}^{top} be the underlying topological space of \mathcal{C} . We have an obvious continuous map $\pi : \mathcal{C}^{top} \rightarrow S(n)$. Let \mathcal{A}_n be the sheaf of algebras on $S(n)$ defined by: for every open subset U of $S(n)$, $\mathcal{A}_n(U)$ is the algebra of $(\alpha_1, \dots, \alpha_n) \in \mathcal{O}_{\mathcal{C}}(\pi^{-1}(U))$ such that $\alpha_i \in \mathcal{O}_{S_i}(U \cap S_i)$ for $1 \leq i \leq n$.

According to corollary 4.4.8, for every $x \in \mathcal{C}$, $\mathcal{A}_{n,P}$ is the algebra of $(\alpha_1, \dots, \alpha_n) \in \mathcal{O}_{\mathcal{C},x}$ such that $\alpha_i \in \mathcal{O}_{S_i,P}$ for $1 \leq i \leq n$.

5.6.1. Proposition: *The sheaf \mathcal{A}_n is the structural sheaf of an oblate n -star of S .*

Proof. By induction on n . The case $n = 1$ is obvious. Suppose that $n > 1$ and that the result is true for $n - 1$. Let $\mathcal{C}' = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_{n-1} \subset \mathcal{C}$, and \mathcal{A}_{n-1} the corresponding oblate $(n - 1)$ -star of S . Let

$$\Phi_n : \mathcal{B}_n \longrightarrow \mathcal{O}_{\mathcal{C}_n}/(\pi_n^{q_n})$$

be the morphism of proposition 4.4.1. According to proposition 4.6.4, Φ_n induces a morphism

$$\Psi_n : \mathcal{K}_n \longrightarrow \mathcal{O}_{S_n,P}/(t_n^{q_n}).$$

By the definitions of \mathcal{A}_n and Φ_n , if $u = (\alpha_1, \dots, \alpha_n) \in \mathcal{O}_{S_1,P} \times \dots \times \mathcal{O}_{S_1,P}$, then $u \in \mathcal{A}_{n,P}$ if and only if $\Psi_n(u') = v$, where u' (resp. v) is the image of u in \mathcal{K}_n (resp. $\mathcal{O}_{S_n,P}/(t_n^{q_n})$). The result follows then from 5.2.5. \square

We denote by $\mathcal{S}(\mathcal{C})$ (or more simply \mathcal{S}) the oblate n -star corresponding to \mathcal{A}_n , so $\mathcal{O}_{\mathcal{S}(\mathcal{C})} = \mathcal{A}_n$. From the definition of \mathcal{A}_n we get a canonical morphism

$$\mathbf{\Pi} : \mathcal{C} \longrightarrow \mathcal{S}$$

such that $\mathbf{\Pi}|_{\mathcal{C}_i} = \pi_i : \mathcal{C}_i \rightarrow S_i$ for $1 \leq i \leq n$.

5.6.2. Theorem: *The morphism $\mathbf{\Pi}$ is flat.*

Proof. We need only to prove that $\mathbf{\Pi}$ is flat at any point x of \mathcal{C} . Let $\mathcal{I} \subset \mathcal{O}_{\mathcal{S},P}$ be a proper ideal. We have to show that the canonical morphism of $\mathcal{O}_{\mathcal{S},P}$ -modules

$$\tau = \tau_{\mathcal{I}} : \mathcal{O}_{\mathcal{C},x} \otimes_{\mathcal{O}_{\mathcal{S},P}} \mathcal{I} \longrightarrow \mathcal{O}_{\mathcal{C},x}$$

is injective. According to proposition 5.5.1 there is a filtration by ideals

$$\{0\} = \mathcal{I}_{k+1} \subset \mathcal{I}_k \subset \dots \subset \mathcal{I}_1 = \mathcal{I}$$

such that, for $1 \leq i \leq k$ there exists a positive integer j such that $j \leq n$ and an isomorphism $\mathcal{I}_i/\mathcal{I}_{i+1} \simeq \mathcal{O}_{S_j,P}$ of $\mathcal{O}_{\mathcal{S},P}$ -modules. We will prove the injectivity of τ by induction on k .

Recall that for $1 \leq j \leq n$, $\mathcal{I}_{S_j, P} = \mathcal{I}_{S_j, \mathcal{S}, P}$ is a principal ideal, generated by an element u_j which is also a generator of $\mathcal{I}_{\mathcal{C}_j, x} = \mathcal{I}_{\mathcal{C}_j, \mathcal{C}, x}$ (cf. corollary 4.3.8 and proposition 5.2.3), and that the only zero coordinate of u_j is the j -th.

Suppose that $k = 1$, so \mathcal{I} is isomorphic to $\mathcal{O}_{S_j, P}$ for some j . Let u be a generator of \mathcal{I} and $w \in \mathcal{O}_{\mathcal{C}, x} \otimes_{\mathcal{O}_{S, P}} \mathcal{I}$, that can be written as $w = v \otimes u$, $v \in \mathcal{O}_{\mathcal{C}, x}$. Suppose that $\tau(v \otimes u) = vu = 0$. Since \mathcal{I} is annihilated by $\mathcal{I}_{S_j, P}$, we have $\mathcal{I} \subset ((0, \dots, 0, t_j^{q_j}, 0, \dots, 0))$. Since $vu = 0$, the j -th component of v is zero, i.e. $v \in \mathcal{I}_{\mathcal{C}_j, x}$. Hence v is a multiple of u_j : $v = \alpha u_j$. We have then

$$\begin{aligned} w &= \alpha u_j \otimes u \\ &= \alpha \otimes u_j u \quad (\text{because } u_j \in \mathcal{O}_{S, P}) \\ &= 0 \quad (\text{because } u_j u = 0) . \end{aligned}$$

Hence τ is injective.

Suppose that the result is true for $k - 1 \geq 1$ and that the filtration of \mathcal{I} is of length k . According to proposition 5.5.1, **1-**, we have $\mathcal{I}/\mathcal{I}_2 \simeq \mathcal{O}_{S_j, P}$ for some j . Let $u \in \mathcal{I}$ be such that its image in $\mathcal{I}/\mathcal{I}_2$ is a generator, and $w \in \mathcal{O}_{\mathcal{C}, x} \otimes_{\mathcal{O}_{S, P}} \mathcal{I}$ such that $\tau(w) = 0$. We can write w as $w = \alpha \otimes v + \beta \otimes u$, with $\alpha, \beta \in \mathcal{O}_{\mathcal{C}, x}$ and $v \in \mathcal{I}_2$. Since $\alpha v + \beta u = 0$, we have $\beta u \in \mathcal{O}_{\mathcal{C}, x} \mathcal{I}_2$, and $\mathcal{O}_{\mathcal{C}, x} \mathcal{I}_2 \subset \mathcal{I}_{\mathcal{C}_j}$ by proposition 5.5.1, **2-**, i.e. the j -th coordinate of βu is zero. By proposition 5.5.1, **2-**, the j -th coordinate of u does not vanish, hence the j -th coordinate of β is zero, i.e. $\beta \in \mathcal{I}_{\mathcal{C}_j}$. Hence β is a multiple of u_j : $\beta = \gamma u_j$. We have then

$$\beta \otimes u = \gamma u_j \otimes u = \gamma \otimes u_j u,$$

and $u_j u \in \mathcal{I}_2$ (because its image in $\mathcal{I}/\mathcal{I}_2$ vanishes). It follows that w is the image of an element w' of $\mathcal{O}_{\mathcal{C}, x} \otimes_{\mathcal{O}_{S, P}} \mathcal{I}_2$. We have $\tau_{\mathcal{I}_2}(w') = 0$, hence by the induction hypothesis $w' = 0$. It follows that we have also $w = 0$. \square

5.6.3. Remark: If \mathcal{S}' is an oblate n -star of S , and if $\Pi' : \mathcal{C} \rightarrow \mathcal{S}'$ is a flat morphism compatible with the projections to S , then we have $\mathcal{S}' = \mathcal{S}(\mathcal{C})$ and $\Pi' = \Pi$. This is an easy consequence of corollary 5.4.3.

5.6.4. Converse - Let $\pi : \mathcal{S} \rightarrow S$ be an oblate n -star of S . Let $\Pi : \mathcal{C} \rightarrow \mathcal{S}$ be a flat morphism such that for every closed point $s \in \mathcal{S}$, $\Pi^{-1}(s)$ is a smooth irreducible projective curve. Let $C = \Pi^{-1}(P)$ and $\tau = \pi \circ \Pi : \mathcal{C} \rightarrow S$. Then $C_n = \tau^{-1}(P)$ is a primitive multiple curve of multiplicity n and associated smooth curve C , and \mathcal{C} is a fragmented deformation of C_n . This is an easy consequence of proposition 4.1.6.

6. CLASSIFICATION OF FRAGMENTED DEFORMATIONS OF LENGTH 2

Let $\pi : \mathcal{C} \rightarrow \mathbb{C}$ be a fragmented deformation of length 2. The corresponding double curve C_2 is $\pi^{-1}(0)$. Suppose that the spectrum of \mathcal{C} is $\begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$. This means that the infinitesimal neighborhoods of order p of C in \mathcal{C}_1 and \mathcal{C}_2 are isomorphic, i.e. we have an isomorphism of sheaves of algebras on C

$$\Phi : \mathcal{O}_{\mathcal{C}_1}/(\pi_1^p) \longrightarrow \mathcal{O}_{\mathcal{C}_2}/(\pi_2^p) ,$$

and for every point x of C , we have

$$\mathcal{O}_{C,x} = \{(\alpha_1, \alpha_2) \in \mathcal{O}_{\mathcal{C}_1,x} \times \mathcal{O}_{\mathcal{C}_2,x} ; \alpha_2 \pmod{\pi_2^p} = \Phi(\alpha_1 \pmod{\pi_1^p})\} .$$

Let C_i^k denote the infinitesimal neighborhood of order k of C in \mathcal{C}_i , $i = 1, 2$, $k > 0$. It is a primitive multiple curve of multiplicity k and associated smooth curve C , and we have $C_1^p = C_2^p$. Hence C_1^{p+1} and C_2^{p+1} appear as extensions of C_1^p in primitive multiple curves of multiplicity $p+1$. According to [4] and [8] these extensions are classified by $H^1(C, T_C)$ (T_C being the tangent sheaf on C). More precisely, we say that two such extensions D, D' are *isomorphic* if there exists an isomorphism $D \simeq D'$ leaving C_1^p invariant. Then if \mathcal{H} is the set of isomorphism classes of such extensions, a bijection $\lambda : H^1(C, T_C) \rightarrow \mathcal{H}$ is defined in [4], such that $\lambda(0) = C_1^{p+1}$.

On the other hand, it follows from [2], [4] that the primitive double curves with associated smooth curve C and associated line bundle \mathcal{O}_C are classified by $\mathbb{P}(H^1(C, T_C)) \cup \{0\}$.

6.0.1. Theorem: *The point of $\mathbb{P}(H^1(C, T_C)) \cup \{0\}$ corresponding to C_2 is $\mathbb{C} \cdot \lambda^{-1}(C_2^{p+1})$.*

Proof. According to [4], there exists an open covering $(U_i)_{i \in I}$ of C such that for $k = 1, 2$, the open subset of C_k^{p+1} corresponding to U_i is isomorphic to $U_i \times \text{spec}(C[t]/(t^{p+1}))$. Here t is π_1 on \mathcal{C}_1 and π_2 on \mathcal{C}_2 . We obtain then cocycles $(\theta_{ij}^{(k)})_{i,j \in I}$, where $\theta_{ij}^{(k)}$ is an automorphism of $U_{ij} \times \text{spec}(C[t]/(t^{p+1}))$. We can also suppose that $\omega_{C|U_i}$ is trivial, for every $i \in I$. Let $dx_{ij} = dx$ be a generator of $\omega_C(U_{ij})$. Since the ideal sheaf of C in C_k^{p+1} is the trivial sheaf on C_k^p , we can write, using the notations of [4], $\theta_{ij}^{(k)} = \phi_{\mu_{ij}^{(k)}, 1}$, with $\mu_{ij}^{(k)} \in \mathcal{O}_C(U_{ij})[t]/(t^p)$, i.e. for every $\alpha \in \mathcal{O}_C(U_i)$, we have, at the level of regular functions

$$\theta_{ij}^{(k)}(\alpha) = \sum_{m=0}^p \frac{1}{m!} (\mu_{ij}^{(k)} t)^m \frac{d^m \alpha}{dx^m} ,$$

and $\theta_{ij}^{(k)}(t) = t$. Since $C_1^p = C_2^p$ we can suppose that $\mu_{ij}^{(1)} \equiv \mu_{ij}^{(2)} \pmod{t^{p-1}}$. Hence $\tau_{ij} = \mu_{ij}^{(2)} - \mu_{ij}^{(1)} \in (t^{p-1})/(t^p) \simeq \mathcal{O}_C(U_i)$. The family (τ_{ij}) is (in some sense) a cocycle representing $\lambda^{-1}(C_2^{p+1})$ (cf. [4], [8]).

We have $(\pi_1^{p+1}) + (\pi_2^{p+1}) \subset (\pi)$ in \mathcal{O}_C . Hence $C_2 = \pi^{-1}(0)$ is contained in the subscheme Z of \mathcal{C} corresponding to the ideal sheaf $(\pi_1^{p+1}) + (\pi_2^{p+1})$. We have

$$\begin{aligned} \mathcal{O}_Z(U_{ij}) &= \{(\alpha_1, \alpha_2) \in \mathcal{O}_{\mathcal{C}_1}(U_{ij})/(t^{p+1}) \times \mathcal{O}_{\mathcal{C}_2}(U_{ij})/(t^{p+1}) ; \Phi(\alpha_1 \pmod{t^p}) = \alpha_2 \pmod{t^p}\} \\ &= \{(\alpha_1, \alpha_2) \in \mathcal{O}_C(U_{ij})[t]/(t^{p+1}) \times \mathcal{O}_C(U_{ij})[t]/(t^{p+1}) ; \alpha_1 \equiv \alpha_2 \pmod{t^p}\} . \end{aligned}$$

To obtain $\mathcal{O}_{C_2}(U_{ij})$, we have just to quotient by $\pi = (t, t)$, and we obtain

$$\mathcal{O}_{C_2}(U_{ij}) = \mathcal{O}_Z(U_{ij})/(t, t) \simeq \mathcal{O}_C(U_{ij})[z]/(z^2) ,$$

the last isomorphism being

$$(a_0 + a_1t + \cdots + a_{p-1}t^{p-1} + \alpha t^p, a_0 + a_1t + \cdots + a_{p-1}t^{p-1} + \beta t^p) \mapsto \alpha_0 + (\beta - \alpha)z.$$

Now we can make explicit the automorphism of $\mathcal{O}_C(U_{ij})[z]/(z^2)$ induced by θ_{ij} (these isomorphisms will define the cocycle corresponding to C_2). It is easy to see that this isomorphism is $\phi_{\tau_{ij},1}$, which proves theorem 6.0.1. \square

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Note : This text is a version of

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with the following modifications :

- an unnecessary hypothesis was removed from definition 3.2.2.
- a supplementary hypothesis was added before proposition 4.5.2.

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