EXOTIC FINE MODULI SPACES OF COHERENT SHEAVES

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Contents

1. Introduction

Let X be a smooth projective irreducible algebraic variety over $\mathbb{C}$. Let $\mathcal{S}$ be a nonempty set of isomorphism classes of coherent sheaves on X. A *fine moduli space for* $\mathcal{S}$ is an integral algebraic structure M on the set $\mathcal{S}$ (i.e. $\mathcal{S}$ is identified with the set of closed points of M), such that there exists a *universal sheaf*, at least locally : there is an open cover (U_i) of M and a coherent sheaf $\mathcal{F}_i$ on each $U_i \times X$, flat on U_i, such that for every $s \in U_i$, the fiber $\mathcal{F}_{is}$ is the sheaf corresponding to s, and $\mathcal{F}_i$ is a complete deformation of $\mathcal{F}_{is}$.

The most known fine moduli spaces appear in the theory of (semi-)stable sheaves. If X is a curve and r, d are integers with $r \geq 1$, then the moduli space $M_s(r,d)$ of stable vector bundles of rank r and degree d on X is a fine moduli space if and only if r and d are relatively prime (cf. [18], [6]). Similar results have been proved on surfaces (cf. [6], [21]).

An *exotic* fine moduli space of sheaves is one that does not come from the theory of (semi-)stable sheaves. We will give several examples of exotic fine moduli spaces in this paper. Sometimes the corresponding sheaves will not be simple, contrary to stable sheaves. The detailed proofs of the results given in this paper come mainly from [3], [5] and [8].

In section 2 we give the definitions and first properties of fine moduli spaces of sheaves. We prove that if $\mathcal{S}$ admits a fine moduli space, then the dimension of $\text{End}(E)$ is independent of $E \in \mathcal{S}$, and $\text{Aut}(E)$ acts trivially on $\text{Ext}^1(E,E)$ (by conjugation).

In section 3 we define *generic extensions* and build fine moduli spaces of such sheaves. They appear when one tries to construct moduli spaces of *prioritary sheaves* of given rank and Chern classes on $\mathbb{P}_2$ where there does not exist semi-stable sheaves of the same rank and Chern classes. In this case the sheaves are not simple. We give an example of fine moduli space where no global universal sheaf exists.

In section 4 we construct fine moduli spaces of very unstable sheaves when $\dim(X) \geq 3$. In this case also the sheaves are not simple.

In section 5 we give other examples of fine moduli spaces of sheaves, illustrating some of their properties :

- We give an example of a maximal fine moduli space of sheaves on $\mathbb{P}_2$ which is not projective.
- We show that there are simple vector bundles on $\mathbb{P}_2$ which cannot belong to an open family admitting a fine moduli space, but are limits of bundles in a fine moduli space.
- We give an example of two projective fine moduli spaces of sheaves on $\mathbb{P}_2$ which are distinct but share a nonempty open subset. One of them is the moduli space of stable sheaves of rank 6 and Chern classes -3, 8.

Notations : A *family of sheaves* on X parametrized by an algebraic variety T is a coherent sheaf $\mathcal{F}$ on $T \times X$, flat on T. If t is a closed point of T, let $\mathcal{F}_t$ denote the fiber of $\mathcal{F}$ over t.

We say that $\mathcal{F}$ is a *family of sheaves of* $\mathcal{S}$ if for every closed point t in T we have $\mathcal{F}_t \in \mathcal{S}$.

An algebraic vector bundle E on $\mathbb{P}_2$ is called *exceptional* if $\mathrm{End}(E) = \mathbb{C}$ and $\mathrm{Ext}^1(E, E) = \{0\}$. An exceptional vector bundle is stable (cf. [7]). Let Q denote the *universal quotient* on $\mathbb{P}_2$, i.e. if $\mathbb{P}_2 = \mathbb{P}(V)$ (lines in V), then we have a canonical exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{O} \otimes V \to Q \to 0$.

If S, T are algebraic varieties, p_S, p_T will denote the projections $S \times T \to S$, $S \times T \to T$ respectively.

If $f : S \to T$ is a morphism of algebraic varieties and $\mathcal{E}$ a coherent sheaf on $T \times X$, let $f^\sharp(\mathcal{E}) = (f \times I_X)^*(\mathcal{E})$.

2. Fine moduli spaces of sheaves

2.1. Definitions

Let $\mathcal{S}$ be a nonempty set of isomorphism classes of coherent sheaves on X. We say that $\mathcal{S}$ is *open* if for every family $\mathcal{F}$ of sheaves parametrized by an algebraic variety T, if t is a closed point of T such that $\mathcal{F}_t \in \mathcal{S}$ then the same is true for all closed points in a suitable open neighbourhood of t in T.

Suppose that $\mathcal{S}$ is open. A fine moduli space for $\mathcal{S}$ is given by

- an integral algebraic variety M,
- an open cover $(U_i)_{i \in I}$ of M,
- for every $i \in I$, a family of sheaves of $\mathcal{F}_i$ of $\mathcal{S}$ parametrized by U_i,

such that

(i) if $s \in U_i \cap U_j$ then $\mathcal{F}_{is} \simeq \mathcal{F}_{js}$,
(ii) for every E in $\mathcal{S}$ there exists one and only one $s \in M$ such that $E = \mathcal{F}_{is}$ if $s \in U_i$,

(iii) the sheaves $\mathcal{F}_i$ satisfy the *local universal property* : for every family $\mathcal{F}$ of sheaves of $\mathcal{S}$ parametrized by an algebraic variety T, there exists a morphism $f_{\mathcal{F}} : T \to M$ such that for every $i \in I$ and $t \in f_{\mathcal{F}}^{-1}(U_i)$ there exists a neighbourhood U of t such that $f_{\mathcal{F}}(U) \subset U_i$ and $f_{\mathcal{F}}^{\sharp}(\mathcal{F}_i)_{|U\times X} \simeq \mathcal{F}_{U\times X}$.

A fine moduli space for $\mathcal{S}$ is unique in the following sense : if $(M', (U'_j), (\mathcal{F}'_j))$ is another one then M, M' are canonically isomorphic and the families $\mathcal{F}_i$, $\mathcal{F}'_j$ are locally isomorphic.

We say that $\mathcal{S}$ admits a *globally defined fine moduli space* if I can be found with just one point, i.e. if there is a *universal sheaf* on the whole $M \times X$.

2.2. Endomorphisms

Let $\mathcal{S}$ be an open set of sheaves on X admitting a fine moduli space M.

2.2.1. Theorem : *For every $E \in \mathcal{S}$,* $\mathrm{Aut}(E)$ *acts trivially on* $\mathrm{Ext}^1(E,E)$ *and* $\dim(\mathrm{End}(E))$ *is independent of E.*

Proof. For simplicity we assume that there is a globally defined universal sheaf $\mathcal{E}$ on $M \times X$. Using locally free resolutions we find that there exists a morphism of vector bundles $\phi : A \to B$ on M such that for every $y \in M$ there is a canonical isomorphism $\mathrm{End}(\mathcal{E}_y) \simeq \ker(\phi_y)$. Now let $p = rk(\ker(\phi))$, it is the generic dimension of $\mathrm{Aut}(E)$, for $E \in \mathcal{S}$. Now let $E \in \mathcal{S}$ and $y \in M$ such that $\mathcal{E}_y \simeq E$. Let W be the image of $\ker(\phi)_y$ in A_y. Then we have $\dim(W) \leq p$.

Let $\mathcal{F}$ be a family of sheaves in $\mathcal{S}$ parametrized by an algebraic variety, and $s \in S$ such that $\mathcal{F}_s \simeq E$. Suppose that $\mathcal{F}$ is a complete family at s, and let $\psi \in \mathrm{Aut}(\mathcal{F})$. Then we have $\psi_s \in W$.

Now it remains to prove that for any $\sigma \in \mathrm{Aut}(E)$ there exists such a family $\mathcal{F}$ and $\psi \in \mathrm{Aut}(\mathcal{F})$ such that $\psi_y = \sigma$. This will prove that $\mathrm{Aut}(E) \subset W$, and finally that $\dim(\mathrm{End}(E)) = p$.

The preceding assertion can be proved using Quot schemes. □

2.2.2. Proposition : *If the sheaves in $\mathcal{S}$ are simple then there exists a universal sheaf $\mathcal{F}$ on $M \times X$. For every family $\mathcal{E}$ of sheaves of $\mathcal{S}$ parametrized by T there exists a line bundle L on T such that* $f_{\mathcal{E}}^{\sharp}(\mathcal{F}) \otimes p_T^*(L) \simeq \mathcal{E}$.

Proof. Suppose for simplicity that the sheaves in $\mathcal{S}$ are locally free. Then it is easy to build a *universal projective bundle* by glueing the projective bundles of all the local universal bundles, using the fact that the bundles of $\mathcal{S}$ are simple. Now this projective bundle comes from a universal bundle on $M \times X$ because a projective bundle which is banal on a nonempty open subset of the base is banal (cf. [12]). □

3. Moduli spaces of generic extensions

3.1. Generic extensions

Let E', E be coherent sheaves on X. The extensions of E by E' are parametrized by $\mathrm{Ext}^1(E,E')$. If $\sigma \in \mathrm{Ext}^1(E,E')$ let

$$(*) \qquad\qquad 0 \longrightarrow E' \longrightarrow F_\sigma \longrightarrow E \longrightarrow 0$$

be the corresponding extension. The group $G = \mathrm{Aut}(E) \times \mathrm{Aut}(E')$ acts obviously on $\mathrm{Ext}^1(E,E')$ and if $\sigma \in \mathrm{Ext}^1(E,E')$ and $g \in G$ we have $F_{g\sigma} = F_\sigma$. Let $\sigma \in \mathrm{Ext}^1(E,E')$. The tangent map at the identity of the orbit map

$$\begin{aligned} \Phi_\sigma : G &\longrightarrow \mathrm{Ext}^1(E,E') \\ g &\longmapsto g\sigma \end{aligned}$$

is

$$\begin{aligned} T_{\Phi_\sigma} : \mathrm{End}(E) \times \mathrm{End}(E') &\longrightarrow \mathrm{Ext}^1(E,E') \\ (\alpha,\beta) &\longmapsto \beta\sigma - \sigma\alpha \end{aligned}$$

We say that (*) is a *generic extension* if T_{Φ_σ} is surjective. In this case $G\sigma$ is the unique open orbit in $\mathrm{Ext}^1(E,E')$. It is the generic orbit. We call F_σ the *generic extension of E by E'* .

3.1.1. Lemma : *Suppose that* $\mathrm{Hom}(E',E) = \mathrm{Ext}^1(E',E) = \mathrm{Ext}^2(E,E') = \{0\}$ *and let* $\sigma \in \mathrm{Ext}^1(E,E')$ *such that F_σ is a generic extension. Then we have*

$$\mathrm{Ext}^1(F_\sigma,F_\sigma) \;\simeq\; \mathrm{Ext}^1(E',E') \oplus \mathrm{Ext}^1(E,E)$$

and an exact sequence

$$0 \longrightarrow \mathrm{Hom}(E,E') \longrightarrow \mathrm{End}(F_\sigma) \longrightarrow \ker(\Phi_\sigma) \longrightarrow 0.$$

Proof. This follows easily from a spectral sequence associated to the filtration $0 \subset E' \subset F_\sigma$. ☐

In other words, the only deformations of F_σ are generic extensions of deformations of E by deformations of E'.

3.2. Moduli spaces of generic extensions

Les $\mathcal{X}$, $\mathcal{X}'$ be open sets of sheaves admitting fine moduli spaces M, M'. Suppose that $\mathrm{Hom}(E',E) = \{0\}$ for every $E \in \mathcal{X}$, $E' \in \mathcal{X}'$. For simplicity we assume that they have globally defined universal sheaves $\mathcal{E}$, $\mathcal{E}'$ respectively. Let $Y \subset \mathcal{X} \times \mathcal{X}'$ be the set of pairs (E,E') such that

$$\mathrm{Ext}^1(E',E) = \mathrm{Ext}^2(E',E') = \mathrm{Ext}^2(E,E) = \mathrm{Ext}^2(E',E) = \mathrm{Ext}^i(E,E') = \{0\}$$

for $i \geq 2$ and that there exists a generic extension of E by E'. Let $\mathcal{Y}$ the set of such generic extensions. Then it is easy to see that the map $Y \to \mathcal{Y}$ associating to (E,E') the generic extension F_σ of E by E' is a bijection.

3.2.1. Theorem : *The set $\mathcal{Y}$ is open and admits a fine moduli spave N which is isomorphic to the open subset of $M \times M'$ consisting of pairs (s,t) such that $(\mathcal{E}_s, \mathcal{E}'_t) \in Y$.*

3.3. Prioritary sheaves

A coherent sheaf E on $\mathbb{P}_2 = \mathbb{P}(V)$ is called *prioritary* if it is torsionfree and $\mathrm{Ext}^2(E, E(-1)) = \{0\}$ (or equivalently $\mathrm{Hom}(E, E(-2)) = \{0\}$) . A semi-stable sheaf is prioritary but the converse is not true. A. Hirschowitz and Y. Laszlo have proved in [11] that the stack of prioritary sheaves of rank r and Chern classes c_1, c_2 is irreducible, and found when it is nonempty.

Let r, c_1, c_2 be integers, with $r \geq 1$, such that there are no semi-stable sheaves of rank r and Chern classes c_1, c_2, but there exist prioritary sheaves with these invariants. There are two cases :

(i) There exist distinct exceptional vector bundles E_1, E_2, E_3, and non negative integers m_1, m_2, m_3, with $m_1, m_2 > 0$, such that the generic prioritary sheaf of rank r and Chern classes c_1,c_2 is isomorphic to $(E_1 \otimes \mathbb{C}^{m_1}) \oplus (E_2 \otimes \mathbb{C}^{m_2}) \oplus (E_3 \otimes \mathbb{C}^{m_3})$.

(ii) There exist an exceptional vector bundle F, an integer $p > 0$ and a moduli space M of stable bundles M such that the generic prioritary sheaf of rank r and Chern classes c_1,c_2 is of the form $(F \otimes \mathbb{C}^p) \oplus E$, with $E \in M$.

In the first case there can be only one fine moduli space of prioritary sheaves of rank r and Chern classes c_1, c_2, consisting of one unique point.

In the second case we must assume that M is a fine moduli space of sheaves (in this case M is projective). Let μ denote the slope of sheaves in M. There is then a fine moduli space of prioritary sheaves of rank r and Chern classes c_1, c_2, consisting generic extensions $0 \to F \otimes \mathbb{C}^p \to \mathcal{E} \to E \to 0$ if $\mu \leq \mu(F)$ (resp. $0 \to E \to \mathcal{E} \to F \otimes \mathbb{C}^p \to 0$ if $\mu > \mu(F)$) . It is isomorphic to an open subset of M, and to the whole of M if p is sufficiently big. In general this moduli space will not contain all the prioritary sheaves of rank r and Chern classes c_1, c_2. We will treat only one example.

3.4. Prioritary sheaves of rank 8 and Chern classes -4, 11

According to [7] there are no semi-stable sheaves of rank 8 and Chern classes -4, 11. The generic prioritary sheaf with these invariants is of the form $(Q^* \otimes \mathbb{C}^2) \oplus E$, where E is a stable sheaf of rank 4 and Chern classes -2, 4.

Let $M(4,-2,4)$ be the moduli space of rank 4 semi-stable sheaves on $\mathbb{P}_2$ with Chern classes -2, 4. For these invariants semi-stability is equivalent to stability, hence $M(4,-2,4)$ is a fine moduli space of sheaves. It has the following description : we have

$$M(4,-2,4) \ \simeq \ \mathbb{P}(S^2V) \ \simeq \ \mathbb{P}_5.$$

If D is a line in S^2V we have a canonical morphism

$$\alpha_D : \mathcal{O}(-3) \longrightarrow \mathcal{O}(-1) \otimes (S^2V/D)$$

which is injective (as a morphism of sheaves) and the corresponding point in $M(4,-2,4)$ is $\mathrm{coker}(\alpha_D) = \mathcal{E}_D$. An easy computation shows that $\dim(\mathrm{Ext}^1(\mathcal{E}_D, Q^*)) = 0$ if D is not decomposable, 1 if $D = \mathbb{C}uv$, with $u \wedge v \neq \{0\}$ and 2 if $D = \mathbb{C}u^2$.

For $D \in \mathbb{P}(S^2V)$ let $0 \to Q^* \otimes \mathbb{C}^2 \to \mathcal{F}_D \to \mathcal{E}_D \to 0$ be the associated generic extension. We have $\dim(\mathrm{End}(\mathcal{F}_D)) = 5$. It follows from theorem 3.2.1 that the set of all sheaves $\mathcal{F}_D$ admits a fine modulo space $\mathbf{M}$ isomorphic to $\mathbb{P}_5$. In particular there are universal sheaves defined at least locally.

3.4.1. Proposition : *There is no universal sheaf on* $\mathbf{M}\times\mathbb{P}_2$ *.*

Proof. Suppose that there is a universal sheaf $\mathcal{G}$ on $\mathbf{M}\times\mathbb{P}_2$. Then $\mathbb{A}=p_{M*}(\mathcal{H}om(p_{\mathbb{P}_2}^*(Q^*),\mathcal{G}))$ is a rank 2 vector bundle on $\mathbf{M}$, and there exists a line bundle L on $\mathbf{M}$ and an exact sequence of sheaves on $\mathbf{M}\times\mathbb{P}_2$

$$0\longrightarrow p_{\mathbb{P}_2}^*(Q^*)\otimes p_{\mathbf{M}}^*(\mathbb{A})\longrightarrow\mathcal{G}\longrightarrow\mathcal{E}\otimes p_{\mathbf{M}}^*(L)\longrightarrow 0.$$

Let $\mathbb{B}=\mathcal{E}xt^1_{p_{\mathbf{M}}}(\mathcal{E},p_{\mathbb{P}_2}^*(Q^*))$. Let $P\subset\mathbf{M}$ denote the locus of elements $\mathbb{C}u^2$, $u\in V\backslash\{0\}$, which is isomorphic to $\mathbb{P}_2$. Then it is easy to see that $\mathbb{B}_{|P}$ is isomorphic to Q_P, the bundle Q on $P\simeq\mathbb{P}_2$.

From the above extension we deduce a section of $\mathbb{B}\otimes L\otimes\mathbb{A}^*$ inducing an isomorphism $Q_P=\mathbb{B}_{|P}\simeq\mathbb{A}_{|P}\otimes L_{|P}*$, since it gives a generic extension at each point of P. But Q_P cannot be the restriction of a vector bundle on $\mathbf{M}$ since its determinant is not the restriction of a line bundle on $\mathbf{M}$. This prooves that the universal sheaf $\mathcal{G}$ does not exist. □

4. Fine moduli spaces of wide extensions

4.1. Introduction

Let $\mathcal{O}_X(1)$ be a very ample line bundle on X. We consider extensions of sheaves

$$(*)\qquad\qquad 0\longrightarrow G\longrightarrow\mathcal{E}\longrightarrow F\longrightarrow 0$$

where $G,\mathcal{E}$ are locally free and F torsionfree. We want to see what happens when $\mu(G)\gg\mu(\mathcal{E})$, say let $G=G_0(p)$ and let p tend to infinity. If we want non trivial extensions F must not be locally free. The correct settings are as follows : F is torsionfree and $T=F^{**}/F$ is pure of codimension 2.

Using the dual of $(*)$ we obtain the exact sequence

$$0\longrightarrow F^*\longrightarrow\mathcal{E}^*\longrightarrow G^*\xrightarrow{\lambda}\mathcal{E}xt^1(F,\mathcal{O})\longrightarrow 0.$$

We have $\mathcal{E}xt^1(F,\mathcal{O})\simeq\mathcal{E}xt^2(T,\mathcal{O})=\widetilde{T}$, which pure of codimension 2, with the same support as T.

We suppose now that $\operatorname{Ext}^i(F^{**},G)=\{0\}$ if $i\geq 1$ (which is true if $G=G_0(p)$ with $p\gg 0$) . Then from $(*)$ we deduce that $\operatorname{Ext}^1(F,G)\simeq\operatorname{Ext}^2(T,G)=\operatorname{Hom}(G^*,\widetilde{T})$. It is easy to see that the element of $\operatorname{Ext}^1(F,G)$ associated to $(*)$ is λ.

Let $\Gamma=\ker(\lambda)$. We have an exact sequence

$$(**)\qquad\qquad 0\longrightarrow F^*\longrightarrow\mathcal{E}^*\longrightarrow\Gamma\longrightarrow 0,$$

$\operatorname{Ext}^1(\Gamma,F^*)\simeq\operatorname{Hom}(F^{**},T)$, and the element of $\operatorname{Ext}^1(\Gamma,F^*)$ associated to $(**)$ is the quotient map $F^{**}\to T$.

It follows that to obtain $\mathcal{E}$ we need only the following data : two vector bundles F^{**}, G^*, a sheaf T, and two surjective morphisms $F^{**} \to T$ and $G^* \to \widetilde{T}$.

If the preceding hypotheses are satisfied we call $(*)$ (and $(**)$, $\mathcal{E}$, $\mathcal{E}^*$) *wide extensions* .

4.2. Fine moduli spaces of wide extensions

We suppose here that $\dim(X) > 2$. We start with two open families $\mathcal{X}$, $\mathcal{Y}$ of simple vector bundles on X admitting fine moduli spaces M, N, such that

– If $E \in \mathcal{X}$ (or $\mathcal{Y}$) then the trace map $\mathrm{Ext}^2(E,E) \to H^2(\mathcal{O}_X)$ is an isomorphism.
– If $A, B \in \mathcal{X}$ (or $\mathcal{Y}$) then $\mathrm{Hom}(A,B) = \{0\}$.

The family $\mathcal{X}$ will contain the F^{**} of 4.1, and $\mathcal{Y}$ the G^*.

Let $\mathcal{Z}$ be an open family of simple pure codimension 2 sheaves on X having a smooth fine moduli space Z such that the sheaves in $\mathcal{Z}$ are *perfect*, i.e. if $T \in \mathcal{Z}$ then $\mathcal{E}xt^q(T, \mathcal{O}_X) = 0$ for $q > 2$ (this is the case if T is a vector bundle on a locally complete intersection subvariety of X of codimension 2).

We suppose that

– For every $\mathbb{F} \in \mathcal{X}$, $\mathbb{G} \in \mathcal{Y}$ we have $\mathrm{Ext}^i(\mathbb{F}, \mathbb{G}^*) = \{0\}$ if $i \geq 1$, and $\mathrm{Ext}^i(\mathbb{G}^*, \mathbb{F}) = \{0\}$ if $i < \dim(X)$.
– For every $\mathbb{F} \in \mathcal{X}$, $\mathbb{G} \in \mathcal{Y}$, $T \in \mathbb{Z}$ and surjective morphisms $\mu : \mathbb{F} \to T$, $\lambda : \mathbb{G} \to \widetilde{T}$ we have $\mathrm{Ext}^i(\ker(\mu), \mathbb{G}^*) = \mathrm{Ext}^i(\ker(\lambda), \mathbb{F}^*) = \{0\}$ if $i \geq 2$, $\mathrm{Ext}^i(\mathbb{F}, T) = \mathrm{Ext}^i(\mathbb{G}, \widetilde{T}) = \{0\}$ if $i \geq 1$, and $H^0(\mathbb{F} \otimes \widetilde{T}) = H^0(\mathbb{G} \otimes T) = \{0\}$.

Let $\mathbf{F}$ (resp. $\mathbf{G}$, $\mathbf{T}$) be universal sheaves on $M \times X$ (resp. $N \times X$, $Z \times X$). Let $\mathcal{F} = \mathcal{H}om(p_M^\sharp(\mathbf{F}), \mathbf{T})$, Let $\mathcal{G} = \mathcal{H}om(p_M^\sharp(\mathbf{G}), \widetilde{\mathbf{T}})$, which are vector bundles on $M \times N \times Z$. For every $(m,n,z) \in M \times N \times Z$ we have $\mathcal{F}_{(m,n,z)} = \mathrm{Hom}(\mathbf{F}_m, \mathbf{T}_z)$, $\mathcal{G}_{(m,n,z)} = \mathrm{Hom}(\mathbf{G}_m, \widetilde{\mathbf{T}_z})$. Let $\mathcal{F}^{surj} \subset \mathcal{F}$ (resp. $\mathcal{G}^{surj} \subset \mathcal{G}$) be the open subset corresponding to surjective morphisms, and $\mathcal{S}$ the set of wide extensions defined by surjective morphisms $\mathbb{F} \to T$, $\mathbb{G} \to \widetilde{T}$, with $\mathbb{F} \in \mathcal{X}$, $\mathbb{G} \in \mathcal{Y}$, $T \in \mathcal{Z}$.

4.2.1. Theorem : *The set $\mathcal{S}$ is an open family of sheaves. It has a fine moduli space* $\mathbf{M}$ *which is canonically isomorphic to* $\mathbb{P}(\mathcal{F}^{surj}) \times_{M \times N \times Z} \mathbb{P}(\mathcal{G}^{surj})$.

4.3. Examples

We give two examples of fine moduli spaces of wide extensions on $\mathbb{P}_3$. Let $M_{\mathbb{P}_3}(0,1)$ denote the fine moduli space of stable rank-2 vector bundles with Chern classes 0,1, i.e. *null correlation bundles.* It is canonically isomorphic to an open subset of $\mathbb{P}_5$.

4.3.1. *Rank 3 bundles*

Let $n > 4$ be an integer. We consider wide extensions

$$0 \longrightarrow E(n) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_\ell \longrightarrow 0$$

where $E \in M_{\mathbb{P}_3}(0,1)$, ℓ is a line in $\mathbb{P}_3$ and $\mathcal{I}_\ell$ its ideal sheaf. The bundle $\mathcal{E}$ has rank 3 and Chern classes $2n$, n^2+2, $2n+2$. It follows from theorem 4.2.1 that the set of such extensions admits a fine moduli space $\mathbf{M}$ of dimension $2n+14$. We have

$$\dim(\mathrm{End}(\mathcal{E})) = \frac{n(n+2)(n+4)}{3} + 1, \qquad \dim(\mathrm{Ext}^2(\mathcal{E},\mathcal{E}) = 2n+10.$$

Note that $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})$ does not vanish, but that $\mathbf{M}$ is smooth.

4.3.2. *Rank 4 bundles*

Let m, n be positive integers, with $n > \mathrm{Max}(m,4)$. We consider wide extensions

$$0 \longrightarrow E(n) \longrightarrow \mathcal{E} \longrightarrow \ker(\pi) \longrightarrow 0$$

where $E \in M_{\mathbb{P}_3}(0,1)$ and π is a surjective morphism $E' \to \mathcal{O}_\ell(m)$, ℓ beeing a line in $\mathbb{P}_3$ and $E' \in M_{\mathbb{P}_3}(0,1)$. The bundle $\mathcal{E}$ has rank 4 and Chern classes $2n$, n^2+3, $4n-2m+2$. It follows from theorem 4.2.1 that the set of such extensions admits a fine moduli $\mathbf{M}$ space of dimension $2n+20$. In this case also $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})$ does not vanish, but $\mathbf{M}$ is smooth.

5. Further examples

5.1. Maximal fine moduli spaces

Let $\mathcal{S}$ be an open set of sheaves admitting a fine moduli space M . Then $\mathcal{S}$ (or M) is called *maximal* if there does not exist an open set of sheaves $\mathcal{T}$, such that $\mathcal{S} \subset \mathcal{T}$, $\mathcal{S} \neq \mathcal{T}$, admitting a fine moduli space. Of course if M is projective then $\mathcal{S}$ and M are maximal. We will give an example that shows that the converse is not true.

Let n be a positive integer. Let Z be a finite subscheme of $\mathbb{P}_2$ such that $h^0(\mathcal{O}_Z) = \frac{n(n+1)}{2}$, and $\mathcal{I}_Z$ be its ideal sheaf. Then it follows easily from the Beilinson spectral sequence that Z is not contained in a curve of degree $n-1$ if and only if there is an exact sequence

$$0 \longrightarrow \mathcal{O}(-n-1)\otimes\mathbb{C}^n \longrightarrow \mathcal{O}(-n)\otimes\mathbb{C}^{n+1} \longrightarrow \mathcal{I}_Z \longrightarrow 0.$$

Let $W = \mathrm{Hom}(\mathcal{O}(-n-1)\otimes\mathbb{C}^n, \mathcal{O}(-n)\otimes\mathbb{C}^{n+1})$. We consider the action of the reductive group $G = \mathrm{SL}(n)\times\mathrm{SL}(n+1)$ on $\mathbb{P}(W)$, with the obvious linearization. If $f \in W\backslash\{0\}$, then $\mathbb{C}f$ is semi-stable if and only if $\mathbb{C}f$ is stable, if and only if for any two linear subspaces $H \subset \mathbb{C}^n$, $K \subset \mathbb{C}^{n+1}$, of dimension $p > 0$, $f(\mathcal{O}(-n-1)\otimes H)$ is not contained in $\mathcal{O}(-n)\otimes K$. Let $\mathcal{N}_n$ denote the geometric quotient of the open set of stable points of $\mathbb{P}(W)$ by G. Then $\mathcal{N}_n$ is a smooth projective variety of dimension $n(n+1)$. There exist a *universal morphism*

$$\Phi : p_{\mathbb{P}_2}^*(\mathcal{O}(-n-1))\otimes p_{\mathcal{N}_n}^*(\mathbf{H}) \longrightarrow p_{\mathbb{P}_2}^*(\mathcal{O}(-n))\otimes p_{\mathcal{N}_n}^*(\mathbf{K}),$$

$\mathbf{H}$ (resp. $\mathbf{K}$) beeing a rank n (resp. $n+1$) vector bundle on $\mathcal{N}_n$.

Let $U \subset \mathcal{N}_n$ be the open subset corresponding to injective morphisms of sheaves, and $\mathcal{U}$ the set of sheaves $\mathrm{coker}(\Phi_u)$, with $u \in U$.

5.1.1. Theorem : *The set $\mathcal{U}$ admits a fine moduli space isomorphic to U , with universal sheaf* $\mathrm{coker}(\Phi_{|U})$. *Moreover U is maximal, and if $n \geq 5$ then $U \neq \mathcal{N}_n$.*

Hence if $n \geq 5$, U is a maximal non projective fine moduli space.

Let $M(1,0,n(n+1)/2)$ denote the moduli space of rank 1 stable sheaves with Chern classes 1, $n(n+1)/2$. It is a smooth projective variety which is canonically isomorphic to $\mathrm{Hilb}^{n(n+1)/2}(\mathbb{P}_2)$. The intersection $M(1,0,n(n+1)/2) \cap U$ corresponds to length $n(n+1)/2$ subschemes of $\mathbb{P}_2$ which are not contained in a curve of degree $n-1$, and to morphisms of vector bundles ϕ in W such that the set of points $x \in \mathbb{P}_2$ such that ϕ_x is not injective is finite.

5.2. Admissible sheaves

A coherent sheaf E on X is called *admissible* if there exists an open set of sheaves containing E and admitting a fine moduli space. If E is admissible then it follows from theorem 2.2.1 that $\mathrm{Aut}(E)$ acts trivially on $\mathrm{Ext}^1(E,E)$ by conjugation. The converse is not true. We will show that when $X = \mathbb{P}_2$ then there exists simple sheaves E such that $\mathrm{Ext}^2(E,E) = \{0\}$ and which can be deformed into stable admissible sheaves.

Let n be a positive integer. We consider morphisms of vector bundles on $\mathbb{P}_2$

$$\mathcal{O} \otimes \mathbb{C}^{2n-1} \longrightarrow Q \otimes \mathbb{C}^{2n+1}.$$

Let $W = \mathrm{Hom}(\mathcal{O} \otimes \mathbb{C}^{2n-1}, Q \otimes \mathbb{C}^{2n+1})$. The reductive group $G = \mathrm{SL}(2n-1) \times \mathrm{SL}(2n+1)$ acts on $\mathbb{P}(W)$, and there is an obvious linearization of this action. According to [1] a morphism $\phi \in W$ is semi-stable if and only if it is stable, if and only if it is injective (as a morphism of sheaves) and $\mathrm{coker}(\phi)$ is stable. We obtain in this way an isomorphism

$$\mathbb{P}(W)^s/G \;\simeq\; M(2n+3, 2n+1, (n+1)(2n+1)),$$

where $\mathbb{P}(W)^s$ denotes the set of stable points in $\mathbb{P}(W)$ and $M(2n+3, 2n+1, (n+1)(2n+1))$ the fine moduli space of stable sheaves of rank $2n+3$ and Chern classes $2n+1$, $(n+1)(2n+1)$.

5.2.1. Proposition : *There are injective unstable morphisms of vector bundles $\phi \in W$ such that $E = \mathrm{coker}(\phi)$ is simple. The vector bundle E is not admissible.*

Note that E can be deformed in stable bundles in $M(2n+3, 2n+1, (n+1)(2n+1))$ (which are admissible).

5.3. Deformations of fine moduli spaces of stable sheaves

We consider here rank 6 coherent sheaves on $\mathbb{P}_2$ with Chern classes -3, 8. For these invariants semi-stability is equivalent to stability, hence the moduli space $M(6,-3,8)$ of stable sheaves of rank 6 and Chern classes -3, 8 is a fine moduli space. It is a smooth projective variety of dimension 16. We will construct another smooth projective fine moduli space $\mathbf{M}$ which has a nonempty intersection with $M(6,-3,8)$.

Stable sheaves of rank 6 and Chern classes -3, 8 are related to morphisms

$$\mathcal{O}(-3) \otimes \mathbb{C}^2 \longrightarrow \mathcal{O}(-2) \oplus (\mathcal{O}(-1) \otimes \mathbb{C}^7).$$

Let W be the vector space of such morphisms. We consider the action of the non reductive group

$$G \;=\; \mathrm{Aut}(\mathcal{O}(-3) \otimes \mathbb{C}^2) \times \mathrm{Aut}(\mathcal{O}(-2) \oplus (\mathcal{O}(-1) \otimes \mathbb{C}^7))$$

on $\mathbb{P}(W)$. Let

$$H \;=\; \left\{ (I, \begin{pmatrix} I & 0 \\ \alpha & I \end{pmatrix}) \quad ; \quad \alpha \in \mathrm{Hom}(\mathcal{O}(-2), \mathcal{O}(-1) \otimes \mathbb{C}^7) \right\}$$

be the unipotent subgroup of G, and

$$G_{red} \;=\; \mathrm{Aut}(\mathcal{O}(-3) \otimes \mathbb{C}^2) \times \mathrm{Aut}(\mathcal{O}(-2)) \times \mathrm{Aut}(\mathcal{O}(-1) \otimes \mathbb{C}^7),$$

which is a reductive subgroup of G . It is possible to construct geometric quotients of G-invariant open subsets of $\mathbb{P}(W)$ using a notion of semi-stability, depending on the choice of a rational number t such that $0 < t < 1$. Let $\phi \in W$ be a non zero morphism. Then the point $\mathbb{C}\phi$ of $\mathbb{P}(W)$ is called *semi-stable* (resp. *stable*) with respect to t if

– $\mathrm{im}(\phi)$ is not contained in $\mathcal{O}(-1) \otimes \mathbb{C}^7$,
– For every proper linear subspace $D \subset \mathbb{C}^7$, $\mathrm{im}(\phi)$ is not contained in $\mathcal{O}(-2) \oplus (\mathcal{O}(-1) \otimes D)$.
– For every 1-dimensional linear subspace $L \subset \mathbb{C}^2$, if $K \subset \mathbb{C}$, $D \subset \mathbb{C}^7$ are linear subspaces such that $\phi(\mathcal{O}(-3) \otimes L) \subset (\mathcal{O}(-2) \otimes K) \oplus (\mathcal{O}(-1) \otimes D)$, then we have

$$t \dim(K) + \frac{1-t}{7} \dim(D) \;\geq \frac{1}{2}$$

(resp. $>$) .

Let $\mathbb{P}(W)^{ss}(t)$ (resp. $\mathbb{P}(W)^{s}(t)$) denote the open set of semi-stable (resp. stable) points of $\mathbb{P}(W)$ with respect to t.

According to [8], [4], if $t > \frac{3}{10}$ then there exists a good quotient $\mathbb{P}(W)^{ss}(t)//G$ and a geometric quotient $\mathbb{P}(W)^{s}(t)/G$. In this range $t = \frac{1}{2}$ is the only value such that $\mathbb{P}(W)^{s}(t) \neq \mathbb{P}(W)^{ss}(t)$. For all t such that $\frac{3}{10} < t < \frac{1}{2}$ we obtain the same geometric quotient $\mathbf{M}_1$, and for t such that $\frac{1}{2} < t < 1$ the geometric quotient $\mathbf{M}_2$. These two quotients are smooth projective varieties.

According to [2] we have an isomorphism $M(6, -3, 8) \simeq \mathbf{M}_1$. To obtain it we associate to a morphism its cokernel.

The other moduli space $\mathbf{M}_2$ is also a fine moduli space of sheaves. The corresponding open set of sheaves consists of the cokernels of the morphisms parametrized by $\mathbf{M}_2$. All these sheaves are torsion free and have at most 2 singular points. The two fine moduli spaces are distinct and have a common dense open subset.

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