Automatic Maps in Exotic Numeration Systems

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1 Introduction

Automatic sequences are one of the links between number theory and theoretical computer science; for example, see [17, 2]. A sequence is said to be k-automatic if, roughly speaking, its n-th term can be computed from the (classical) base-k expansion of n using a finite state machine.

Now suppose that one takes an integer base k, say $k = 3$, and that one replaces the familiar digits $\{0, 1, 2\}$ by other digits, say $\{-1, 0, 1\}$. (This example is called the *balanced ternary expansion* [29].) Then the ordinary notion of 3-automaticity is replaced by a seemingly new notion, as we consider a new set of digits. Actually, it happens that *this new notion is equivalent to the more familiar one.*

Still looking at the balanced ternary expansion, one sees that every integer in $\mathbb{Z}$ (not only in $\mathbb{N}$) has a unique representation in base 3 with digits $-1, 0, 1$, up to leading zeroes. Hence one could define in a natural way the notion of 3-automaticity for a sequence indexed by $\mathbb{Z}$. The question is then: is there a relationship between the 3-automaticity of the sequence $(u_n)_{n\in\mathbb{Z}}$ and the 3-automaticity of the sequences $(u_n)_{n\in\mathbb{N}}$ and $(u_{-n})_{n\in\mathbb{N}}$? We will prove here that *the sequence* $(u_n)_{n\in\mathbb{Z}}$ *is* 3*-automatic in this sense if and only if both sequences* $(u_n)_{n\in\mathbb{N}}$ *and* $(u_{-n})_{n\in\mathbb{N}}$ *are* 3*-automatic in the usual sense.*

In the same spirit, consider the expansion of integers in base $b = -3$, with digits $0, 1, 2$. This yields a notion of (-3)-automaticity for sequences indexed by $\mathbb{N}$, but also for sequences indexed by $\mathbb{Z}$. One of the results of this paper asserts for example that *a sequence indexed by* $\mathbb{Z}$ *is* (-3)*-automatic for the digits* $0, 1, 2$ *if and only if it is* 3*-automatic for the balanced ternary expansion, and this is also equivalent to the (ordinary)* 3*-automaticity of the sequences* $(u_n)_{n\in\mathbb{N}}$ and $(u_{-n})_{n\in\mathbb{N}}$.

Another motivation for defining the automaticity of sequences indexed by $\mathbb{Z}$ arises from the attempt to produce by finite automata the orbits of linear cellular automata which grow in both directions [6, 8, 5].

In what follows, we give a general framework for the automaticity of a map from a ring (or even a semiring) R with additional properties to a finite set V. Our work subsumes the examples given above, as well as the case of double automatic sequences in the sense of [36, 38, 37], (note that double sequences are also investigated in [11]), and the cases of sequences indexed by the Gaussian integers or by quadratic irrationals.

Here is a brief outline of the paper: in Section 2 we give the definition of a (D, b)-semiring, (i.e., a semiring with digits), (D, b)-automata, (D, b)-automatic sequences, and (D, b)-substitutions. In Section 3 we give the basic properties of the (D, b)-automatic sequences, in particular the equivalence of the definition by automata and the definition in terms of generalized substitutions. In Section 4 we study the effect of changing the set of digits and keeping the same base. In Section 5 we both change digits and multiply the base by a root of unity. In Section 6 we consider the situation where the semiring is embedded into a ring. In Section 7 and 8 we show how our notions of linked sets introduced in Sections 4 and 5 can be translated in terms of transducers, showing the relation between a "purely arithmetic" property and a "language-theoretic" property. Finally, in Section 9, we apply the preceding sections to many situations including sequences indexed by the semiring $\mathbb{N}$ with positive or negative bases, but also sequences "indexed by" (i.e., maps from) $\mathbb{Z}$, $\mathbb{Z}^2$, the Gaussian integers or a ring of quadratic algebraic integers. We also study the folded representation of integers and the revolving representation of Gaussian integers. Finally, we conclude with an intriguing open question.

We assume the reader is familiar with the basic concepts of language theory as given, for example, in [27, 18].

We point out the papers of Rauzy [34], of Honkala [25, 26], of Frougny [19, 20], and the recent work of Bruyère and Hansel [9, 10] which, although quite different from ours, nevertheless share some of the same spirit. We quote also the recent work of von Haeseler [24] who gives a different, although related, generalization of automatic sequences.

Finally we add that a short version of this paper appeared in [4], and that the reader can just look at the examples when the formalism in some definitions or proofs seems discouraging.

2 Definitions

2.1 (D, b)-semirings

Let R be a commutative semiring [18, p. 122], i.e., a set equipped with two operations, an addition and a multiplication, such that R is a commutative monoid with unit for both operations, with the property that multiplication is distributive with respect to addition. The identity elements for addition and multiplication are respectively denoted by 0 and 1. We first define digit representations for semirings.

Definition 1 *Let R be a semiring, let $b \in R$ and D be a finite subset of R containing 0. The semiring R is called a* (D, b)-semiring *if every element $r \in R \setminus \{0\}$ has a unique representation*

$$r = r_s b^s + \ldots + r_1 b + r_0,\ s \in \mathbb{N},\ r_i \in D,\ 0 \le i \le s,\ r_s \ne 0. \tag{1}$$

We call b a base *and the elements of the set D are called* digits.

Remark 1

1. The above property means that there exists a bijection (that we call the *digit-bijection*) between $R \setminus \{0\}$ and the subset $(D \setminus \{0\})D^*$ of the free monoid D^* generated by D. We will also stick to the usual convention that the representation of 0 is the empty word (in D^*).

2. Note that the hypothesis implies that: for every element $r \in R$ there exists a unique $d \in D$ and a unique $x \in R$ such that $r = xb + d$. Indeed we write

$$r = (r_s b^s + \cdots + r_1 b) + r_0 = (r_s b^{s-1} + \cdots + r_1)b + r_0.$$

Hence the existence of x (possibly equal to 0) and of d. Now suppose $r = xb + d = x'b + d'$, with d and d' in D. Decomposing x and x' as:

$$x = x_t b^t + \cdots + x_0,\quad x' = x'_{t'} b^{t'} + \cdots + x'_0,$$

one has

$$r = x_t b^{t+1} + \cdots + x_0 b + d = x'_{t'} b^{t'+1} + \cdots + x'_0 b + d'.$$

The uniqueness of the decomposition of r implies that $x = x'$ and $d = d'$. We then write $r \equiv d \bmod b$, and this notation is well-defined.

Note that the same proof shows that b can be simplified in any product ($bx = by \Longrightarrow x = y$) in any semiring having the property: there exist a subset D containing 0 and an element b such that for every element $r \in R$ there exists a unique $d \in D$ and a unique $x \in R$ such that $r = xb + d$, and in particular in any (D, b)-semiring.

Note also that if a semiring has the property that there exist an element b and a (finite) subset D such that every $r \in R$ can be written uniquely as $r = xb + d$ with $d \in D$, this does not imply that R is a (D, b)-ring: iterating this property to decompose r may never terminate. For example take for A any set; let $\mathcal{P}(A)$ be the set of its subsets. Then $\mathcal{P}(A)$ equipped with the operations Δ and $\cap$ is a ring with unit. Take $b = A$ and $D = \{A\}$. Then every element $X \in \mathcal{P}(A)$ can be uniquely decomposed as $X = ((A \setminus X) \cap A)\ \Delta\ A$.

Decomposing now $(A \setminus X)$ gives $(A \setminus X) = (X \cap A) \ \Delta \ A$ and the decomposition never terminates.

3. Our requirement that 0 be a member of D means that our results are not directly applicable to some common numeration systems, such as base-2 representation using digits $1, 2$.

Example 1

1. $R = \mathbb{N}$, $b \in \mathbb{N}$, $b \geq 2$, $D_b = \{0, \ldots, b-1\}$, the standard digit set.

2. $R = \mathbb{Z}$, $b \in \mathbb{Z}$, $|b| \geq 2$, D a basic digit set for the base b, i.e. a set satisfying the following conditions of Matula. (See [32, 31], where more general conditions are given.)

- $0 \in D$,
- D is a complete set of residues modulo b,
- let $\delta = \min D$ and $\Delta = \max D$, then every integer j satisfying
$$\frac{-\Delta}{b-1} \leq j \leq \frac{-\delta}{b-1}, \text{ if } b > 0, \qquad \text{or} \qquad \frac{-\delta b - \Delta}{b^2 - 1} \leq j \leq \frac{-\Delta b - \delta}{b-1}, \text{ if } b < 0,$$
has a representation $j = j_s b^s + \cdots + j_1 b + j_0$, with $j_k \in D$, $0 \leq k \leq s$.

3. Examples of the above Matula conditions are:

- $b = -2$, $D = \{0, 1\}$,
- $b = -2$, $D = \{0, -1\}$,
- $b \geq 3$, $D = \{0, 1, \ldots, b-2, -b^k + b - 1\}$, where $k \geq 1$,
- $b \leq -3$, $D = \{0, 1, \ldots, |b| - 2, -|b|^k + |b| - 1\}$, where $k \geq 1$,
- $b = k + \ell + 1$, where $k, \ell \geq 1$, $D = \{-k, -k+1, \ldots, -1, 0, 1, \ldots, \ell\}$,
- $b \leq -2$, $D_b = \{0, 1 \ldots, -b-1\}$.

4. If R is a sub-semiring of a (D, b)-semiring R', we sometimes adopt the natural convention to also call R a (D, b)-semiring. For example, one can replace $\mathbb{Z}$ by $\mathbb{N}$ in the examples above. We will come back to this point in Section 6.

5. $R = R_1 \times R_2$, where R_i are (D_i, b_i)-semirings, R is a (D, b)-semiring with the natural componentwise structure of a semiring, where $D = D_1 \times D_2$, $b = (b_1, b_2)$.

6. $R = \mathbb{Z}[i]$, the ring of Gaussian integers is a (D, b)-semiring for $b = -n + i$, $(n \in \mathbb{N}$, $n \geq 1)$, and $D = \{0, 1, \ldots, n^2\}$; see [33, 28].

7. $R = \mathbb{F}_{p^k}[X]$, the ring of polynomials with coefficients in the Galois field, $\mathbb{F}_{p^k}$, [12], $b = p(X)$, $d = \deg p(X) \geq 1$, $D = \{q(X) : \ q(X) \in \mathbb{F}_{p^k}[X], \ \deg q(X) \leq d - 1\}$.

8. For other examples see [29, 22, 23, 21, 35]. The third author [21, Thm. 1, p. 266] studied radix representations in quadratic fields as follows: let ρ be a quadratic integer and let $X^2 + p_1 X + p_0$ be its minimal polynomial. Let $D = \{0, 1, \ldots, |p_0| - 1\}$. Then $\mathbb{Z}[\rho]$ is a (D, ρ)-semiring if and only if $p_0 \geq 2$ and $-1 \leq p_1 \leq p_0$.

2.2 (D,b)-automata and (D,b)-automatic maps

In this section, we first define (D,b)-automata and (D,b)-automatic maps. The idea is to mimic the classical notions, but to feed the automata with digits in base (D,b) instead of digits in a k-expansion.

Definition 2 *Let R be a (D,b)-semiring. A (D,b)-automaton $\mathcal{A}$, (resp. a reverse (D,b)-automaton $\overline{\mathcal{A}}$) is a finite automaton with input maps the elements of the set D, i.e. $\mathcal{A} = (S, s_0, D, \tau, V)$, (resp. $\overline{\mathcal{A}} = (S, s_0, D, \tau, V)$) with*

- *a set of states: a finite set S,*
- *an initial state: $s_0 \in S$,*
- *input maps: $d : S \to S$, for any $d \in D$,*
- *an output set: a finite set V,*
- *an output map: $\tau : S \to V$.*

Let $r \in R$ be represented as in (1). We define a map $r : S \to S$ for the (D,b)-automaton $\mathcal{A}$ as, (see [18]),

$$r.s = r_k(r_{k-1}(\cdots(r_0)))(s), \tag{2}$$

and for the reverse-reading (D,b)-automaton $\overline{\mathcal{A}}$:

$$r \circ s = r_0(r_1(\cdots(r_k)))(s). \tag{3}$$

In both cases, by the usual convention, the empty word (representation of 0) applied to a state gives the same state.

Definition 3 *Let R be a (D,b)-semiring and V be a finite set. The map $f : R \to V$ is called (D,b)-automatic (resp. reverse (D,b)-automatic) if there exists a (D,b)-automaton $\mathcal{A} = (S, s_0, D, \tau, V)$, (resp. $\overline{\mathcal{A}} = (S, s_0, D, \tau, V)$), such that*

$$\forall r \in R, \ f(r) = \tau(r.s_0) \quad (resp.\ f(r) = \tau(r \circ s_0)).$$

Remark 2

In other words, to say that a map from the (D,b)-semiring R to the finite set V is (D,b)-automatic, means that the map from $(D \setminus \{0\})D^*$ to V can be computed (using the digit-bijection) by an automaton with input alphabet D and an output function taking its values in V.

Example 2

1. The ordinary b-automatic sequences [15, 13, 2] are (D_b, b)-automatic maps of the semiring $\mathbb{N}$, using the notation of Example 1.1.

2. The b-automatic double sequences, see [36, 38, 37, 3] are (D'_b, b')-automatic maps of the semiring $\mathbb{N}^2$, with $D'_b = (D_b)^2$, $b' = (b,b)$, b an integer ≥ 2.

3. Consider the Gaussian integers $\mathbb{Z}[i]$. It can be shown [28] that every element of $\mathbb{Z}[i]$ has a unique representation (up to leading zeroes) of the form $\sum_{0 \le j \le n} a_j(-1+i)^j$, where $a_j \in \{0,1\}$. Then one can consider the analogue of the Thue-Morse sequence for this ring, that is, the map that sends $a+bi$ to the sum of its "digits", modulo 2. This map was discussed by Salon [38]. It is a (D_2, b)-automatic map of the (D_2, b)-ring of Gaussian integers $\mathbb{Z}[i]$, where $b = -1+i$. See Section 9.4.

4. Let $R = \mathbb{N} \times \mathbb{N}$, $b = (2,3)$, $D = \{0,1\} \times \{0,1,2\}$. Then R is a (D,b)-semiring. Define $s_k(n)$ to be the sum, reduced modulo k, of the digits of n in its k-ary expansion. Let $u(m,n) = (s_2(m), s_3(n)) \in R$. Then the sequence $(u(m,n))_{m,n \in \mathbb{N}\times\mathbb{N}}$ is (D,b)-automatic. Note that this sequence is not automatic (in the sense of [36, 38, 37]): if it were a-automatic for some $a \ge 2$, then its components, the sequences $(s_2(n))_{n\in\mathbb{N}}$ and $(s_3(n))_{n\in\mathbb{N}}$, would also be a-automatic. As none of these sequences is ultimately periodic (see for instance [1]), Cobham's theorem [14] implies that a must be both a power of 2 and a power of 3.

2.3 The (D, b)-kernel of a sequence on a (D, b)-semiring

In this section, we generalize the notion of k-kernel of a sequence [36, 38, 37] to our more general setting.

Definition 4 *Let R be a semiring, let $b \in R$, let D be a finite subset of R containing 0. Let V be a finite set and let $f : R \to V$ be a map. The (D,b)-kernel $N_f(D,b)$ of the map f is the set of maps*

$$N_f(D,b) = \{\varphi : R \to V,\ \varphi(r) = f(b^k r + s),\ k \in \mathbb{N},\ s \in R_k\}, \tag{4}$$

where

$$R_k = R_k(D,b) = \{r \in R :\ r = r_{k-1}b^{k-1} + \cdots + r_1 b + r_0,\ r_i \in D,\ i = 0, \ldots, k-1\}. \tag{5}$$

Remark 3

1. By definition the set $N_f(D,b)$ is stable under the maps $g \longrightarrow g(br+d)$, for every $d \in D$, and contains f. It is the smallest (for inclusion) such set.

2. One can reformulate the above definition as follows. Let $c_d : R \to R$ be the map ("(b,d)-decimation") defined by

$$c_d(r) = br + d,$$

where (D,b) is fixed and $d \in D$. The map c_d induces a map $\overline{c_d} : V^R \to V^R$ by

$$\overline{c_d}(\psi) = \psi \circ c_d,$$

(here V^R is the set of all maps from R to V).
Furthermore, if $w = d_s d_{s-1} \cdots d_1 d_0 \in D^*$, one defines:

$$c_w = c_{d_0} c_{d_1} \cdots c_{d_s},$$

i.e.

$$c_w(r) = b^{s+1} r + d_s b^s + \cdots d_1 b + d_0.$$

The map c_w induces a map $\overline{c_w}$ as above, and $\overline{c_w} = \overline{c_{d_s}} \cdots \overline{c_{d_0}}$. Now $N_f(D,b) = \{\overline{c_w}(f) :\ w \in D^*\}$.

2.4 (D,b)-substitutions

We introduce here the notion of (D,b)-substitution. This notion is more complicated than the classical one, as it is less simple to view it "geometrically".

Definition 5 *Let R be a (D,b)-semiring and V be a finite set. A substitution is a pair of maps $(l,\overline{\sigma})$,*

$$l : V \to 2^D, \quad \overline{\sigma} : E = \{(v,\alpha),\ \alpha \in l(v)\} \to V,$$

where 2^D is the set of all subsets of D.

The substitution $(l,\overline{\sigma})$ is called a constant length substitution (more precisely a (D,b)-substitution) if l is the constant map $l(v) = D$ for all $v \in V$. In this case the set E defined above is equal to $V \times D$; hence $\overline{\sigma}$ induces a map $\sigma : V \to V^D$, given by $\sigma(v)(d) = \overline{\sigma}(v,d)$, $\forall (v,d) \in V \times D$. (As usual, A^B denotes the set of all maps from B to A.)

In what follows we consider only constant length substitutions σ and we make the following extra assumption: there exists $v_0 \in V$ with $\sigma(v_0)(0) = v_0$, (this hypothesis, in the usual case, means that the image of some letter begins by the letter itself, this is a necessary and sufficient condition for the existence of an infinite fixed point).

Now let $V_0^{R_k} = \{\varphi : \ R_k \to V,\ \varphi(0) = v_0\}$, $V_0^R = \{\varphi : \ R \to V,\ \varphi(0) = v_0\}$ and $V_0^\omega = \bigcup_k V_0^{R_k} \bigcup V_0^R$. On the set V_0^ω we consider the metric ϱ defined by

$$\varrho(\varphi_1,\varphi_2) = \begin{cases} 0 & \text{if } \varphi_1 = \varphi_2, \\ \frac{1}{k} & \text{if } k = \sup\{j : \ \varphi_1|R_j = \varphi_2|R_j\}, \text{ (note that such a } j \text{ is } \geq 1). \end{cases}$$

The space (V_0^ω, ϱ) is a complete metric space. The (D,b)-substitution induces a map

$$\hat{\sigma} : V_0^\omega \to V_0^\omega$$

defined for $\varphi \in V_0^{R_k}$ by

- $\hat{\sigma}(\varphi)(d) = \sigma(\varphi(0))(d) = \sigma(v_0)(d)$, for $d \in D$,
- $\hat{\sigma}(\varphi)(r_{k-1}b^{k-1} + \cdots + r_1 b + r_0) = \sigma(\varphi(r_{k-1}b^{k-2} + \cdots + r_2 b + r_1))(r_0)$ for $r_i \in D$, $i = 0, \ldots, k$.

If $\varphi \in V_0^R$ then, for every x in R, one writes x in base b, and defines $\hat{\sigma}(\varphi)(x)$ as above.

The map $\hat{\sigma} : (V_0^\omega, \varrho) \to (V_0^\omega, \varrho)$ is a contraction. Then, since (V_0^ω, ϱ) is a complete metric space, the Banach contraction principle implies that the map $\hat{\sigma}$ has a unique fixed point $\varphi_\sigma \in V_0^\omega$ (actually $\varphi_\sigma \in V_0^R$). We say that the map $\varphi_\sigma : R \to V$ is a fixed point of the substitution σ.

Remark 4

What precedes generalizes the "iteration" of a substitution and can also be seen as follows. The map $\sigma_1 := \sigma$ is a map from V to $V^D = V^{R_1}$, note that $\sigma_1(v_0) \in V_0^{R_1}$.

Then one defines $\sigma_2 : V_0^{R_1} \to V_0^{R_2}$ by $\sigma_2(\varphi)(r_1 b + r_0) = \sigma_1(\varphi(r_1))(r_0)$, and one defines similarly $\sigma_{j+1} : V_0^{R_j} \to V_0^{R_{j+1}}$ by $\sigma_{j+1}(\varphi)(r_j b^j + \cdots + r_1 b + r_0) = \sigma(\varphi(r_j b^{j-1} + \cdots + r_2 b + r_1))(r_0)$.

The first step of the iteration is $\sigma_1(v_0) \in V_0^{R_1}$. The second iteration is $\sigma_2\sigma_1(v_0) \in V_0^{R_2}, \ldots$, and the k-th step is $\sigma_k\sigma_{k-1}\cdots\sigma_2\sigma_1(v_0) \in V_0^{R_k}$.

Furthermore, denoting by π_k the "restriction" map from $V_0^{R_k}$ to $V_0^{R_{k-1}}$, we have :

$$\pi_k\sigma_k\sigma_{k-1}\cdots\sigma_2\sigma_1(v_0) = \sigma_{k-1}\cdots\sigma_2\sigma_1(v_0).$$

The fixed point $\varphi_\sigma : R \to V$, is defined on $R = \bigcup_{k=1}^{\infty} R_k$ by

$$\varphi_\sigma | R_k = \sigma_k\sigma_{k-1}\cdots\sigma_2\sigma_1(v_0).$$

We use the following notation:

$$\sigma^k(v_0) = \sigma_k\sigma_{k-1}\cdots\sigma_2\sigma_1(v_0).$$

The reader can translate the preceding comments in terms of projective limits, as the sequence $(V_0^{R_k}, \pi_k)_k$ is a projective sequence, and its projective limit is $V_0^R = \varprojlim(V_0^{R_k}, \pi_k)$.

Definition 6 *Let R be a (D,b)-semiring and A be a finite set. We say that the map $\psi : R \to A$ is generated by a (D,b)-substitution if there is a (D,b)-substitution $\sigma : V \to V^D$ and a map (output) $\tau : V \to A$ such that*

$$\psi(r) = \tau(\varphi_\sigma(r))$$

for $r \in R$, where $\varphi_\sigma : R \to V$ is the fixed point of the substitution σ.

Example 3

1. We first give a (D,b)-analogue of the Thue-Morse sequence. Let $b = -2$ and $D = \{0,1\}$. Then $\mathbb{Z}$ is a (D,b)-ring. For every $n \in \mathbb{Z}$, if $n = n_s(-2)^s + \cdots n_1(-2) + n_0$, with $n_j \in \{0,1\}$, we define $a_n = (n_s + \cdots + n_0) \bmod 2$. The sequence $(a_n)_{n\in\mathbb{Z}}$ (i.e. the map $a : \mathbb{Z} \to D$, $a(n) = a_n$) is a fixed point of the $(\{0,1\},-2)$-substitution $\sigma : D \to D^D$ defined by

$$\begin{array}{rclrcl} \sigma(0)(0) &=& 0, & \sigma(0)(1) &=& 1, \\ \sigma(1)(0) &=& 1, & \sigma(1)(1) &=& 1. \end{array}$$

The fixed point is constructed by iterating σ starting from 0. The k-th iterate $\sigma^k(0)$ defines a_n for $n \in R_k$:

first iteration, $\sigma(0)$,

$n \in R_1$	0	1
a_n	0	1

second iteration, $\sigma^2(0)$,

$n \in R_2$	-2	-1	0	1
a_n	1	0	0	1

third iteration, $\sigma^3(0)$,

$n \in R_3$	-2	-1	0	1	2	3	4	5
a_n	1	0	0	1	0	1	1	0

fourth iteration, $\sigma^4(0)$,

$n \in R_4$	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
a_n	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0

2. Our second example is the analogue of the 3-Thue-Morse sequence. Let $b = 3$ and $D = \{0, 1, -7\}$. Then $\mathbb{Z}$ is a (D, b)-ring. For every $n \in \mathbb{Z}$, if $n = n_s 3^s + \cdots + n_1 3 + n_0$, with $n_j \in \{0, 1, -7\}$, we define $a_n = (n_s + \cdots + n_0) \bmod 3$. The sequence $(a_n)_{n \in \mathbb{Z}}$ (i.e. the map $a : \mathbb{Z} \to \{0, 1, 2\}$, $a(n) = a_n$) is a fixed point of the $(\{0, 1, -7\}, 3)$-substitution $\sigma : \{0, 1, 2\} \to \{0, 1, 2\}^D$ defined by

$$\begin{array}{lllllllll}
\sigma(0)(0) & = & 0, & \sigma(0)(1) & = & 1, & \sigma(0)(-7) & = & 2, \\
\sigma(1)(0) & = & 1, & \sigma(1)(1) & = & 2, & \sigma(1)(-7) & = & 0, \\
\sigma(2)(0) & = & 2, & \sigma(2)(1) & = & 0, & \sigma(2)(-7) & = & 1.
\end{array}$$

The fixed point is constructed by iterating σ starting from 0. The k-th iterate $\sigma^k(0)$ defines a_n for $n \in R_k$:

first iteration, $\sigma(0)$,

$n \in R_1$	-7	0	1
a_n	2	0	1

second iteration, $\sigma^2(0)$,

$n \in R_2$	-28	-21	-20	-7	-4	0	1	3	4
a_n	1	2	0	2	0	0	1	1	2

We give below in Figures 1 and 2 first a $(\{0, 1\}, -2)$-automaton that generates the first sequence of Example 3, second a $(\{0, 1, -7\}, 3)$-automaton that generates the second sequence in the same example.

Remark 5

In this paper we do not consider the "(D, b)-regular maps", which are the natural analogue of the k-regular sequences [7].

3 Properties of (D, b)-automatic maps

In this section we show the equivalence between (D, b)-automaticity, generation by a (D, b)-substitution and finiteness of the (D, b)-kernel. This generalizes the corresponding result in the ordinary case [15]. We also give some properties of (D, b)-automatic maps.

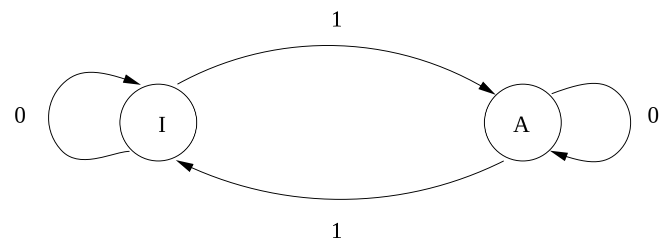

Figure 1: Output function: $\varphi(I) = 0$, $\varphi(A) = 1$.
A (direct or reverse) $(\{0,1\},-2)$-automaton for the sum of digits in base $(\{0,1\},-2)$, reduced modulo 2.
To compute the value of the term with index, say 3, one writes 3 in base $(\{0,1\},-2)$, obtaining $3 = 1(-2)^0 + 1(-2)^1 + 1(-2)^2$. One then feeds the automaton with the digits of 3, starting from the initial state I and following the arrows. After having read the digits, one stops in state A and the output function gives the value 1.

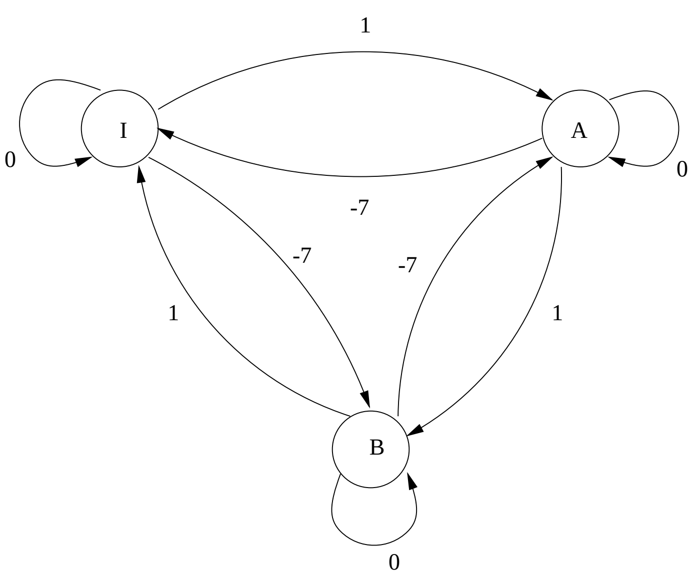

Figure 2: Output function: $\varphi(I) = 0$, $\varphi(B) = 1$, $\varphi(C) = 2$.
A (direct or reverse) $(\{0,1,-7\},3)$-automaton for the sum of digits in base $(\{0,1,-7\},3)$, reduced modulo 3.
To compute the value of the term with index, say -4, one writes -4 in base $(\{0,1,-7\},3)$, obtaining $-4 = -7(3^0) + 1(3^1)$. One then feeds the automaton with the digits of -4, starting from the initial state I and following the arrows. After having read the digits, one stops in state B and the output function gives the value 0.

Theorem 1 *Let R be a (D,b)-semiring and let V be a finite set. For a map $f : R \to V$, the following assertions are equivalent:*

- *f is (D,b)-automatic,*
- *f is reverse (D,b)-automatic,*
- *f is generated by a (D,b)-substitution,*
- *the (D,b)-kernel $N_f(D,b)$ of the map f is finite.*

Proof (sketch).

First note that it is obvious that a sequence is generated by a (D,b)-substitution if and only if it is reverse (D,b)-automatic. The equivalence between being (D,b)-automatic and being generated by (D,b)-substitution holds exactly as in Cobham's proof [15]. Finally, the equivalence with the finiteness of the (D,b)-kernel is proved following the methods given in [13], [2, pp. 251-252], or [36, pp. 4-06 – 4-08]. ∎

Theorem 2 *Let R be a (D,b)-semiring and let t be an integer ≥ 1. Then R is a (R_t, b^t)-semiring and a map from R to a finite set V is (D,b)-automatic if and only if it is (R_t, b^t)-automatic.*

Proof

The first part of the assertion is clear by grouping the terms in the base-b expansions into blocks of length t. To prove the second part, first notice that $N_f(R_t, b^t) \subseteq N_f(D,b)$. Now suppose $N_f(R_t, b^t)$ is finite, for some $t \geq 2$, and call $g^{(1)} = f, \ldots, g^{(n)}$ its elements. Then:

$$\forall s \in R_t,\ \forall i \in [1,n],\ \exists j \in [1,n],\ \forall r \in R\ \ g^{(i)}(b^t r + s) = g^{(j)}(r).$$

In other words,

$$\forall s \in D,\ \forall i \in [1,n],\ \exists j \in [1,n],\ \overline{c}_s(g^{(i)}) = g^{(j)}.$$

Define $\mathcal{F} = \{r \longrightarrow g^{(i)}(b^\alpha r + \beta) :\ i \in [1,n],\ \alpha \leq t-1,\ \beta \in R_\alpha\}$. This set is finite and contains f. It is stable under the (b,d)-decimations $\overline{c}_d$, $\forall d \in D$. Indeed let $\varphi(r) = g^{(i)}(b^\alpha r + \beta)$, $i \in [1,n]$, $\alpha \leq t-1$, $\beta \in R_\alpha$. Then $\overline{c}_d(\varphi)(r) = \varphi(br+d) = g^{(i)}(b^{\alpha+1} r + b^\alpha d + \beta)$.

If $\alpha \leq t-2$, then $\alpha + 1 \leq t-1$, $b^\alpha d + \beta \in R_{\alpha+1}$, hence $r \longrightarrow \varphi(br+d)$ belongs to $\mathcal{F}$.

If $\alpha = t-1$, then $\varphi(br+d) = g^{(i)}(b^t r + b^{t-1} d + \beta)$, $\beta \in R_{t-1}$. Write $\beta = \delta + bx$, where $\delta \in D$ and $x \in R_{t-2}$. Then $\varphi(br+d) = g^{(i)}(b(b^{t-1} r + b^{t-2} d + x) + \delta) = g^{(j)}(b^{t-1} r + b^{t-2} d + x)$ for some $j \in [1,n]$. Note that $b^{t-2} d + x$ belongs to R_{t-1}, which ends the proof. ∎

Theorem 3 *Let R' be a (D',b')-semiring, and let R be a subsemiring of R', such that there exists an integer $e \geq 1$ and a bijection θ from R^e to R' with the properties:*

- *the map θ is a morphism for addition;*
- *for every $a \in R$ and every $(r_1, r_2, \ldots, r_e) \in R^e$ one has $\theta(ar_1, ar_2, \ldots, ar_e) = a\theta(r_1, r_2, \ldots, r_e)$.*

If there exists an integer $t \geq 1$ such that $b'^t = b \in R$, then R^e is a $(\theta^{-1}(R'_t), (b, b, \ldots, b))$-semiring and a map f from R' to the finite set V is (D', b')-automatic if and only if the map $f \circ \theta$ from R^e to V is $(\theta^{-1}(R'_t), (b, b, \ldots, b))$-automatic.

Remark 6

Note that we do not suppose that θ is an isomorphism of semirings. Note also that in the case where R and R' are rings the conditions above mean that θ is an isomorphism of R-modules.

The structure of the semiring R^e is, of course, the structure induced by that of R.

Proof

Note first that we have for every $a \in R$ and $x \in R'$ the equality $\theta^{-1}(ax) = a\theta^{-1}(x)$, (just apply θ to both sides). Note also that from Theorem 2 above a map from R' to V is (D', b')-automatic if and only if it is (R'_t, b'^t)-automatic, where R'_t is, as usual, the set of all sums $\sum_{j=0}^{t-1} d'_j b'^j$, with $d'_j \in D'$. Hence we may suppose from now on that $t = 1$, hence $b = b'$ and $R'_t = D'$.

The base-b expansion is sent by θ to a digit expansion on R^e, with base $\underline{b} = (b, b, \ldots, b)$ and set of digits $\theta^{-1}(D')$, hence R^e is a $(\theta^{-1}(D'), \underline{b})$-semiring. The last step is to notice that a map φ from R' to the set V belongs to the (D', b)-kernel of f if and only if the map $\varphi \circ \theta$ belongs to the $(\theta^{-1}(D'), \underline{b})$-kernel of $f \circ \theta$. ■

Theorem 4 *Let u and v be two elements of the (D, b)-semiring R. Define the set $L_k = L_k(u, v)$ by $L_k = \{\alpha \in R : \exists \lambda, s \in R_k,\ b^k\alpha + \lambda = us + v\}$. Then, if f is a (D, b)-automatic map from R to the finite set V, and if the set $\bigcup_{k \geq 0} L_k$ is finite, then the map $r \longrightarrow f(ur + v)$ is also (D, b)-automatic.*

Proof

Define $\varphi(r) = f(ur + v)$, and let us consider the (D, b)-kernel of φ. A typical element in this kernel is a map $r \longrightarrow \varphi(b^k r + s) = f(u(b^k r + s) + v)$, $k \geq 0$, $s \in R_k$. Now consider the base-b expansion of $us + v$ and write it as: $us + v = b^k\alpha + \lambda$, where $\lambda \in R_k$, $\alpha \in R$. Hence we have:

$$\varphi(b^k r + s) = f(u(b^k r + s) + v) = f(b^k(ur + \alpha) + \lambda).$$

Hence the map $r \longrightarrow \varphi(b^k r + s)$ is the value at the point $ur + \alpha$ of an element of the (D, b)-kernel of f. Now α is independent of r and can take only a finite number of values as it belongs to $\bigcup_{k \geq 0} L_k$, which shows that the (D, b)-kernel of φ is finite. ■

Remark 7

The property used above that the set $\bigcup L_k$ is finite means that one can control the propagation of the carries when performing an addition or a multiplication by fixed elements u and v of the semiring R. We do not know of an example where the set $\bigcup_{k \geq 0} L_k$ is infinite.

4 Comparing (D, b)-automaticity and (D', b)-automaticity

The problem of digit set conversion for a fixed base has been addressed in various contexts (see [30]). In this section we study the effect of changing the set of digits (for a given base) on the automaticity of a map. For this purpose we introduce a new definition.

Definition 7 *Let R be a semiring and let D and D' be finite subsets of R. We say that D is E-linked to D' (in base b) and we write $(D, b) \xrightarrow{E} (D', b)$ if there exists a finite set E containing 0 such that $D' + E \subseteq D + bE$. Here, as usual, we put $X + \alpha Y = \{x + \alpha y : x \in X,\ y \in Y\}$ for $\alpha \in R$.*

Example 4

Let $R = \mathbb{N}$, and take $b = 3$. Then the sets $D = \{0, 1, 2\}$ and $D' = \{1, 2, 3, 4\}$ are E-linked to D, where $E = \{0, 1\}$.

Proposition 1 *Let R be a semiring, b in R and D, D', D'' finite subsets of R. Then D is E-linked to itself via the set $E = \{0\}$. If D is E-linked to D' and D' is F-linked to D'', then D is $(E + F)$-linked to D'':*

$$(D, b) \xrightarrow{E} (D', b) \text{ and } (D', b) \xrightarrow{F} (D'', b) \Longrightarrow (D, b) \xrightarrow{E+F} (D'', b).$$

If furthermore R is a ring *and if D and D' are complete residue systems modulo b, then D is E-linked to D' if and only if D' is $(-E)$-linked to D :*

$$(D, b) \xrightarrow{E} (D', b) \Longleftrightarrow (D', b) \xrightarrow{-E} (D, b).$$

In particular in the case where R is a ring the relation "to be linked to" (in base b) is an equivalence relation for the complete residue systems modulo b.

Proof

The first assertion, when R is only a semiring, is trivial, so we only give the proof for the case when R is a ring. Let R be a ring and $(D, b) \xrightarrow{E} (D', b)$. Let $d \in D$ and $e \in E$. Since D' is a complete residue system modulo b, there exists a unique $d_1 \in D'$ such that

$$d + (-e) \equiv d_1 \pmod{b}. \tag{1}$$

From the hypothesis there exist $d' \in D$ and $e_1 \in E$ such that

$$d_1 + e = d' + be_1.$$

Now from (1) follows that $d' = d$, as they are both in D and both congruent to $d_1 + e$ modulo b. Hence:

$$d + (-e) = d_1 + b(-e_1).$$

■

Now we give an *algorithm* to check whether two sets of digits for a given base b are E-linked and to construct the minimal set E such that $(D,b) \xrightarrow{E} (D',b)$. (Actually this procedure is not quite an algorithm in the usual sense, as it does not necessarily terminate. It does terminate if and only if the sets are linked.)

Theorem 5 *Let R be a semiring, $b \in R$ and $D', D \subseteq R$ containing 0. We suppose that for every $x \in R$, there exist a unique $d \in D$ and a unique $y \in R$ such that $x = d + by$. We define a map π from R to R by $\pi(x) = y$. We then define an increasing sequence of sets by*

$$E_1 = \pi(D'),\ E_2 = \pi(D' + E_1),\ \ldots,\ E_{n+1} = \pi(D' + E_n).$$

Then the digit set D is linked to the digit set D' if and only if there exists an n such that $E_{n+1} = E_n$, and then we have $(D,b) \xrightarrow{E} (D',b)$, where $E = E_n = \bigcup E_j$. This set E is minimal.

Proof

First note that all the sets E_j contain 0 as D' contains 0, hence the sequence of sets E_j is increasing.

If there exists a (finite) set E such that $(D,b) \xrightarrow{E} (D',b)$, then $D' \subseteq D' + E \subseteq D + bE$, hence $E_1 = \pi(D') \subseteq E$. Then $E_2 = \pi(D' + E_1) \subseteq \pi(D' + E) \subseteq \pi(D + bE) \subseteq E$. Hence, by induction on k, one has $E_k \subseteq E$ for every integer k. Finally E contains $\bigcup E_j$, and the sequence (E_j) is stationary.

Conversely if this sequence is stationary, the union of the E_j's yields a (finite) set E such that D is E-linked to D'. The minimality results from the first part of the proof. ∎

We now state a result which says that for a given base the automaticity does not depend on the set of digits provided the old digit set is E-linked to the new one. First, we need a lemma.

Lemma 1 *Let E be a finite set containing 0. Suppose that R is a semiring, $b \in R$ and let D and D' be two finite subsets of R both containing 0, where D is E-linked to D' in base b. Let f be a map from R to a finite set V. If the (D,b)-kernel of f is finite, then its (D',b)-kernel is also finite.*

Proof

We use here the (b,d)-*decimation* maps c_d and $\overline{c_d}$ defined in Remark 3. We also use the maps τ_d defined for $d \in R$ by $\forall r \in R,\ \tau_d(r) = r + d$. This map induces a d-*translation* map $\overline{\tau_d} : V^R \to V^R$ by $\overline{\tau_d}(\varphi) = \varphi \circ \tau_d$. From the hypothesis we have a finite set of maps $\mathcal{E} \subseteq V^R$ satisfying the following conditions:

- $f \in \mathcal{E}$,
- $\mathcal{E}$ is invariant under the maps $\overline{c_d} : V^R \to V^R$, $d \in D$, i.e. $\overline{c_d}(\varphi) \in \mathcal{E}$ for every $\varphi \in \mathcal{E}$ and $d \in D$.

Let us define a subset $\mathcal{E}'$ of V^R as follows.

$$\mathcal{E}' = \bigcup\{\overline{\tau_\ell}(\varphi): \ \varphi \in \mathcal{E}, \ \ell \in E\}.$$

The set $\mathcal{E}'$ is finite and contains the map f (remember that $0 \in E$). Let us show that it is invariant under the (b, d')-decimation maps $\overline{c_{d'}}$, for every $d' \in D'$. Let $\psi \in \mathcal{E}'$ and $d' \in D'$. Then $\psi = \overline{\tau_\ell}(\varphi)$ for some $\varphi \in \mathcal{E}$ and some $\ell \in E$. From the definition of the set E, there exist $d \in D$ and $\lambda \in E$ such that $d' + \ell = d + \lambda b$. Then

$$\overline{c_{d'}}(\psi) = \overline{\tau_\lambda c_d}(\varphi).$$

One then notices that $\overline{c_d}(\varphi)$ belongs to $\mathcal{E}$, hence $\overline{c_{d'}}(\psi)$ belongs to $\mathcal{E}'$. ■

Theorem 6 *Suppose that R is both a (D, b)-semiring and a (D', b)-semiring, where D is E-linked to D' in base b, then every map from R to a finite set V which is (D, b)-automatic is also (D', b)-automatic.*

Proof

It is a straightforward consequence of the above lemma and of Theorem 1. ■

Corollary 1 *If R is both a (D, b)-ring and a (D', b)-ring, if D and D' are complete residue systems modulo b, and if $(D, b) \xrightarrow{E} (D', b)$, then (D, b)-automaticity and (D', b)-automaticity are equivalent.*

Proof

This is a consequence of Theorem 6 and of Proposition 1. ■

5 Comparing (D, b)-automaticity and $(D', \beta b)$-automaticity, when $\beta^v = 1$

In this section, we compare automaticity in base b and base βb. As discussed in the Introduction, the original motivation is for the case $\beta = -1$. Since the case where β is a root of unity is not more complicated, we state our result in this general case.

Definition 8 *Let R be a semiring, b be an element of R and let β be a v-th root of unity in R (with $v > 1$). Let D and D' be two finite subsets of R. We say that (D, b) is linked to $(D', \beta b)$ via the sets $E_0, E_1, \ldots, E_{v-1}$ if there exist finite sets E_j containing 0 and satisfying the following hypothesis:*

$$\forall j \in [0, v-1], \ \beta^j D' + E_j \subseteq D + bE_{j+1}, \ (\text{where } E_v = E_0).$$

We use the notation:

$$(D, b) \xrightarrow{(E_0, E_1, \ldots, E_{v-1})} (D', \beta b).$$

Example 5

Let $b = 2$, $\beta = -1$, $D = \{0, 1\}$ and $D' = \{-1, 0, 1, 2\}$. Define the sets E and F by $E = \{-2, -1, 0, 1\}$ and $F = \{-2, -1, 0, 1\}$. Then

$$D' + E \subseteq D + 2F, \quad -D' + F \subseteq D + 2E,$$

in other words $(D, 2)$ is linked to $(D', -2)$ via the sets E, F.

Remark 8

Note that if v is taken equal to 1 above, then this definition coincides with Definition 7.

A semiring R with a non-trivial root of unity β is not necessarily a ring (take $R = \mathbb{N} \times \mathbb{Z}$ and $\beta = (1, -1)$). But if we make the extra assumption that every non-zero element in the semiring R can be simplified in a product,[1] then R is a ring. For define $a = 1 + \beta + \beta^2 + \cdots + \beta^{v-1}$; then $\beta a = a$, and hence $a = 0$. For every $x \in R$, let $x' = x(\beta + \beta^2 + \cdots + \beta^{v-1})$; then $x + x' = 0$.

Proposition 2 *Let R be a semiring, $b \in R$, β and γ be two roots of unity, (without loss of generality we may suppose $\beta^v = \gamma^v = 1$). Let D, D', D'' be finite subsets of R. If (D, b) is linked to $(D', \beta b)$ via the sets E_0, E_1, ..., E_{v-1} and if $(D', \beta b)$ is linked to $(D'', \gamma\beta b)$ via the sets F_0, F_1, ..., F_{v-1}, then (D, b) is linked to $(D'', \gamma\beta b)$ via the sets $E_j + \beta^j F_j$, (note that these sets are finite and contain 0):*

$$(D, b) \xrightarrow{(E_0, E_1, \ldots, E_{v-1})} (D', \beta b) \;\; \text{and} \;\; (D', \beta b) \xrightarrow{(F_0, F_1, \ldots, F_{v-1})} (D'', \gamma\beta b)$$

$$\Longrightarrow \;\; (D, b) \xrightarrow{(G_0, G_1, \ldots, G_{v-1})} (D, \gamma\beta b),$$

where $G_j = E_j + \beta^j F_j$.

If furthermore R is a ring, *if D and D' are complete residue systems modulo b and if (D, b) is linked to $(D', \beta b)$ via the sets E_0, E_1, ..., E_{v-1}, then $(D', \beta b)$ is linked to (D, b) via the sets $F_j = -\beta^{v-j} E_j$:*

$$(D, b) \xrightarrow{(E_0, E_1, \ldots, E_{v-1})} (D', \beta b) \Longleftrightarrow (D', \beta b) \xrightarrow{(F_0, F_1, \ldots, F_{v-1})} (D, b),$$

where $F_j = -\beta^{v-j} E_j$.

In particular in the case where R is a ring the relation

$$(D, b) \sim (D', b') \Longleftrightarrow \exists \beta \text{ root of unity such that } b' = \beta b, \text{ and } (D, b) \text{ is linked to } (D', \beta b)$$

is an equivalence relation for the pairs (D, b) where b is in R and D is a complete residue system modulo b, containing 0.

---

[1] Actually it would suffice to suppose that $\beta x = x \Longrightarrow x = 0$.

Proof

The first assertion is easy and left to the reader. For the second assertion we fix a j in $[0, v-1]$ and we want to prove that

$$\beta^{v-j} D + (-\beta^{v-j} E_j) \subseteq D' + \beta b(-\beta^{v-j-1} E_{j+1}).$$

Take $d \in D$, $e_j \in E_j$ and consider $\beta^{v-j} d - \beta^{v-j} e_j$. This element is congruent to some $d_1 \in D'$ modulo b, as D' is a complete residue system modulo b. But from the hypothesis follows that

$$\exists d' \in D,\ \exists e_{j+1} \in E_{j+1},\ \beta^j d_1 + e_j = d' + b e_{j+1}.$$

Hence $d = d'$, (they are both in D and both congruent to $\beta^j d_1 + e_j$ modulo b), which implies: $\beta^j d_1 + e_j = d + b e_{j+1}$. In other words:

$$\beta^{v-j} d + (-\beta^{v-j} e_j) = d_1 + (-\beta^{v-j} b e_{j+1}) = d_1 + \beta b f_{j+1},$$

where $f_{j+1} = -\beta^{v-j-1} e_{j+1}$ belongs to F_{j+1}.
The converse is deduced from this implication by interchanging D and D', replacing the base b by the base $B = \beta b$, and the root of unity β by β^{v-1}. ■

Proposition 3 *Let R_i ($i = 1, 2$) be a semiring, $b_i \in R_i$ and $\beta_i^{v_i} = 1$. We can assume without loss of generality that $v_1 = v_2$; we put $v = v_1 = v_2$.*

Let $D_i, D'_i \subseteq R_i$ be finite sets containing 0. *If we have for $i = 1, 2$*

$$(D_i, b_i) \xrightarrow{(E_0^{(i)}, E_1^{(i)}, \ldots, E_{v-1}^{(i)})} (D'_i, \beta_i b_i),$$

then

$$(D_1 \times D_2, (b_1, b_2)) \xrightarrow{(F_0, F_1, \ldots, F_{v-1})} (D'_1 \times D'_2, (\beta_1 b_1, \beta_2 b_2))$$

where $G_j = E_j^1 \times E_j^2$ for every $j \in [0, v-1]$.

Proof

Left to the reader. ■

Let us give, as previously, an *algorithm* (as mentioned above, this is not quite an algorithm in the usual sense) to check whether (D, b) is linked to $(D', \beta b)$ (where $\beta^v = 1$) via the sets E_0, E_1, ..., E_{v-1} and to construct the minimal sets E_j such that

$$(D, b) \xrightarrow{(E_0, E_1, \ldots, E_{v-1})} (D', \beta b).$$

Theorem 7 *Let R be a semiring, $b \in R$ and $D, D' \subseteq R$ finite sets containing* 0. *Let $\beta \in R$ such that $\beta^v = 1$. We suppose that, for every $x \in R$, there exist a unique $d \in D$ and a unique $y \in R$ such that $x = d + by$. We define a map π from R to R by $\pi(x) = y$. We then define a sequence $(G_n)_{n \in \mathbb{N}}$ of subsets of R by*

$$G_0 = \{0\},\ G_1 = \pi(D') = \pi(D' + G_0),\ G_2 = \pi(\beta D' + G_1),\ \ldots,\ G_{n+1} = \pi(\beta^n D' + G_n)\ \ldots$$

The following conditions are equivalent:

(i) for every j the sequence $(G_{vn+j})_{n\in\mathbb{N}}$ is stationary;

(ii) there exists a $j \in [0, v-1]$ such that the sequence $(G_{vn+j})_{n\in\mathbb{N}}$ is stationary;

(iii) there exists a $j \in \mathbb{N}$ such that the sequence $(G_{vn+j})_{n\in\mathbb{N}}$ is stationary;

(iv) there exist finite sets (containing 0) $A_0, A_1, \ldots, A_{v-1}$ such that

$$(D, b) \xrightarrow{(A_0,A_1,\ldots,A_{v-1})} (D', \beta b).$$

If one of the conditions is satisfied, the minimal sets satisfying (iv) are given by $A_j = \bigcup_{n\in\mathbb{N}} G_{vn+j}$.

Proof

The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are trivial. Now note the following simple properties:

- For every $n \in \mathbb{N}$ we have $0 \in G_n$.

  We have $0 \in D'$ and $\pi(0) = 0$, and so the result follows by induction on n.

- For every $j \in \mathbb{N}$ we have $G_j \subseteq G_{v+j}$.

  This is true for $j = 0$ and proved by induction on j, using $\beta^v = 1$.

- For every j the sequence $(G_{vn+j})_{n\in\mathbb{N}}$ is increasing.

  This is an immediate consequence of the previous remark.

Then we notice that (iii) $\Longrightarrow$ (i) is proved easily using the properties:

$$G_k \subseteq G_{v+k} \quad \text{and} \quad G_{vn+k+1} = \pi(\beta^{k+1}D' + G_{vn+k}).$$

Now let us prove that (i) $\Longrightarrow$ (iv). Define for every $j \in \mathbb{N}$ the set A_j by

$$A_j = \bigcup_{n\in\mathbb{N}} G_{vn+j}.$$

This set is finite and contains 0. Furthermore, for all $j \in \mathbb{N}$, we have:

$$\beta^j D' + A_j = \bigcup_{n\in\mathbb{N}} \left(\beta^j D' + G_{vn+j}\right).$$

Then by definition of π,

$$\beta^j D' + G_{vn+j} \subseteq D + b\pi(\beta^j D' + G_{vn+j}) = D + b\pi(\beta^{vn+j} D' + G_{vn+j}) = D + G_{vn+j+1} \subseteq D + bA_{j+1}.$$

Finally, we also observe that $A_v = \bigcup_n G_{v(n+1)}$. But as the sequence $(G_{vn})_{n\in\mathbb{N}}$ is increasing, this union is also equal to $\bigcup_n G_{vn} = A_0$.

The last point is to prove that that, if there exist sets A_i as in (iv), then $G_i \subseteq A_i$ for all i, where the sequence $(A_i)_{i\in\mathbb{N}}$ is defined by

$$\forall j \in [0, v-1],\ \forall n \in \mathbb{N},\ A_{vn+j} := A_j.$$

This will prove both the implication (iv) $\Longrightarrow$ (i) and the minimality of the A_i's.

From (iv) and from the above definition of A_i for every i, we easily deduce that

$$\forall j \in \mathbb{N},\ \beta^j D' + A_j \subseteq D + bA_{j+1}.$$

This implies that $\pi(\beta^j D' + A_j) \subseteq A_{j+1}$. But $G_0 = \{0\} \subseteq A_0$, hence, by induction on n, we have $G_n \subseteq A_n$ for every $n \in \mathbb{N}$. ■

Now we give a lemma which corresponds to Lemma 1.

Lemma 2 *Suppose that R is a semiring, b an element of R, β an element of R satisfying $\beta^v = 1$ with $v > 1$. Let D and D' be two finite subsets of R which contain both 0 and such that (D, b) and $(D', \beta b)$ are linked via the sets E_0, E_1, ..., E_{v-1} (finite and containing 0):*

$$(D, b) \xrightarrow{(E_0, E_1, \ldots, E_{v-1})} (D', \beta b).$$

Let f be a map from R to a finite set V. If the (D, b)-kernel of f is finite, then its $(D', \beta b)$-kernel is also finite.

Proof
Let

$$\mathcal{E}' = \bigcup_{j=0}^{v-1} \bigcup_{e_j \in E_j} \bar{\beta}^j \circ \overline{\tau_{e_j}}(\mathcal{E})$$

where $\bar{\beta} : V^R \to V^R$ is defined by $\bar{\beta}(\varphi) = \varphi(\beta r), r \in R$.
In other words,

$$\mathcal{E}' = \{\psi : \exists \varphi \in \mathcal{E},\ \exists j \in [0, v-1],\ \exists e_j \in E_j,\ \forall r \in R,\ \psi(r) = \varphi(\beta^j r + e_j)\}.$$

Taking $j = 0$ and $e_0 = 0$, we see that $f \in \mathcal{E}'$, as $f \in \mathcal{E}$. Now let us take $\psi \in \mathcal{E}'$, $d' \in D'$ and consider the map: $r \longrightarrow \psi(b\beta r + d')$. One has:

$$\exists \varphi \in \mathcal{E},\ \exists j \in [0, v-1],\ \exists e_j \in E_j,\ \forall r \in R,\ \psi(r) = \varphi(\beta^j r + e_j).$$

Hence

$$\psi(b\beta r + d') = \varphi(\beta^{j+1} br + \beta^j d' + e_j).$$

From the hypothesis, there exist $d \in D$ and $e_{j+1} \in E_{j+1}$ such that $\beta^j d' + e_j = d + be_{j+1}$.
Hence

$$\psi(b\beta r + d') = \varphi(b(\beta^{j+1} r + e_{j+1}) + d) = \overline{\beta}^{j+1} \overline{\tau_{e_{j+1}}}\ \overline{c_d}(\varphi)(r).$$

Hence the map $r \longrightarrow \psi(b\beta r + d')$ belongs to $\mathcal{E}'$.

Theorem 8 *Let R be both a (D, b)-semiring and a $(D', \beta b)$-semiring, where β is a v-th root of unity ($v \geq 1$). Suppose that*

$$(D, b) \xrightarrow{(E_0, E_1, \ldots, E_{v-1})} (D', \beta b).$$

Let f be a map from R to some finite set. Then if f is (D, b)-automatic, it is necessarily $(D', \beta b)$-automatic.

Proof

This is a consequence of Lemma 2 and Theorem 1. ■

Corollary 2 *If R is a ring, both a (D,b)-ring and a $(D',\beta b)$-ring, where $\beta^v=1$, if D and D' are complete residue systems modulo b, and if*

$$(D,b) \xrightarrow{(E_0,E_1,\ldots,E_{v-1})} (D',\beta b),$$

then (D,b)-automaticity and $(D',\beta b)$-automaticity are equivalent.

Proof

This is a consequence of Theorem 8 and Proposition 2. ■

Corollary 3 *Let R be both a (D,b)-ring and a $(D,\beta b)$-ring (where $\beta^v=1$) and $D=\beta D$. A map $f: R \to V$ is (D,b)-automatic if and only if it is $(D,\beta b)$-automatic.*

Proof

This is an immediate consequence of the previous corollary and the remark that, if $D=\beta D$, then

$$(D,b) \xrightarrow{(E_0,E_1,\ldots,E_{v-1})} (D,\beta b),$$

with $E_0=E_1=\cdots=\{0\}$. ■

Remark 9

The results in this section apply in particular when $\beta=-1$.

To conclude this section we notice that an apparently different condition on D and D' such that (D,b)-automaticity implies $(D',\beta b)$-automaticity can be deduced from Theorem 2. Indeed (D,b)-automaticity is equivalent from this theorem to (R_v,b^v)-automaticity, where $R_v=D+bD+b^2D+\cdots+b^{v-1}D$. In the same fashion, $(D',\beta b)$-automaticity is equivalent to $(R'_v,(\beta b)^v)$-automaticity, i.e. to (R'_v,b^v)-automaticity, where $R'_v=D'+\beta bD'+(\beta b)^2D'+\cdots(\beta b)^{v-1}D'$.

Hence a sufficient condition (see Theorem 6) which implies that every (D,b)-automatic function is also $(D',\beta b)$-automatic is that there exists a finite set E containing 0 such that $(R_v,b^v)\xrightarrow{E}(R'_v,(\beta b)^v)$ i.e.:

$$\Big(D'+\beta bD'+(\beta b)^2D'+\cdots+(\beta b)^{v-1}D'\Big)+E\subseteq\Big(D+bD+b^2D+\cdots+b^{v-1}D\Big)+b^vE.$$

However, our next theorem proves that this condition is actually equivalent to the condition previously given, i.e. (D,b) is linked to $(D',\beta b)$ via sets E_0, E_1, …, E_{v-1}.

Theorem 9 *Let R be a semiring, $b \in R$ and D and D' be two finite subsets of R containing 0. We suppose that every $r \in R$ can be uniquely written as $r = bx + d$, with $x \in R$ and $d \in D$. Let β be a root of unity, (with $\beta^v = 1$).*

There exists a finite set E containing 0 such that

$$\left((D + bD + b^2 D + \cdots + b^{v-1}D), b^v\right) \xrightarrow{E} \left((D' + (\beta b)D' + (\beta b)^2 D' + \cdots + (\beta b)^{v-1}D'), (\beta b)^v\right)$$

if and only if there exist finite sets E_0, E_1, …, E_{v-1} containing 0 such that

$$(D, b) \xrightarrow{(E_0, E_1, \ldots, E_{v-1})} (D', \beta b).$$

Proof

Suppose first that

$$(D, b) \xrightarrow{(E_0, E_1, \ldots, E_{v-1})} (D', \beta b).$$

Then we have $\beta^j D' + E_j \subseteq D + bE_{j+1}$, with $E_v = E_0$, for every $j \in [0, v-1]$. Define W_k for every $k \in [0, v-1]$, by

$$W_k = \sum_{j=k}^{v-1} (\beta b)^j D' + b^k E_k + \sum_{j=0}^{k-1} b^j D, \ (W_0 = \sum_{j=0}^{v-1} (\beta b)^j D' + E_0).$$

Let us show that $(W_k)_{k \leq v-1}$ is an increasing sequence. One has

$$(\beta b)^k D' + b^k E_k \subseteq b^k (D + bE_{k+1}),$$

hence:

$$W_k = \sum_{j=k}^{v-1} (\beta b)^j D' + b^k E_k + \sum_{j=0}^{k-1} b^j D \subseteq \sum_{j=k+1}^{v-1} (\beta b)^j D' + b^{k+1} E_{k+1} + \sum_{j=0}^{k} b^j D = W_{k+1}.$$

We now deduce:

$$W_0 \subseteq \cdots \subseteq W_{v-1} = (\beta b)^{v-1} D' + b^{v-1} E_{v-1} + \sum_{j=0}^{v-2} b^j D \subseteq b^{v-1}(D + bE_v) + \sum_{j=0}^{v-2} b^j D = \sum_{j=0}^{v-1} b^j D + b^v E_0,$$

which exactly means that

$$\left((D + bD + b^2 D + \cdots + b^{v-1}D), b^v\right) \xrightarrow{E_0} \left((D' + (\beta b)D' + (\beta b)^2 D' + \cdots + (\beta b)^{v-1}D'), (\beta b)^v\right).$$

Suppose now that

$$\left((D + bD + b^2 D + \cdots + b^{v-1}D), b^v\right) \xrightarrow{E} \left((D' + (\beta b)D' + (\beta b)^2 D' + \cdots + (\beta b)^{v-1}D'), (\beta b)^v\right).$$

Define the map π as previously (for every $x \in R$ there exist unique elements $d \in D$ and $y \in R$ with $x = d + by$, then $y = \pi(x)$). Define $E_0 = E$, $E_1 = \pi(D' + E)$, $E_2 = \pi(\beta D' + E_1)$, …, $E_{v-1} = \pi(\beta^{v-2} D' + E_{v-2})$.

By definition, for every $j \in [0, v-2]$, we have:

$$\beta^j D' + E_j \subseteq D + bE_{j+1},$$

hence we have to prove that $\beta^{v-1}D' + E_{v-1} \subseteq D + bE_0$.

Let us take $\beta^{v-1}d' + e \in \beta^{v-1}D' + E_{v-1}$. We prove by (finite) induction that, if $k = 1, 2, \ldots, v-1$, then there exist $d'_{k,1}, d'_{k,2}, \ldots, d'_{k,k} \in D'$, $\delta_{k,1}, \delta_{k,2}, \ldots, \delta_{k,k} \in D$ and $e_{v-k} \in E_{v-k}$ such that:

$$\sum_{j=2}^{k} \delta_{k,j} b^{v-j} + b^{v-1}(\beta^{v-1}d' + e) = \sum_{j=1}^{k} (\beta b)^{v-j} d'_{k,j} + b^{v-k} e_{v-k}.$$

This is clear for $k = 1$, take $d'_{1,1} = d'$ and $e_{v-1} = e$. Now suppose it is true for k, and remember that $E_{v-k} = \pi(\beta^{v-k-1}D' + E_{v-k-1})$. Hence there exist $\delta_{k+1,k+1} \in D$, $d'_{k+1,k+1} \in D'$ and $e_{v-k-1} \in E_{v-k-1}$ such that $\delta_{k+1,k+1} + be_{v-k} = \beta^{v-k-1}d'_{k+1,k+1} + e_{v-k-1}$.

Hence taking $\delta_{k+1,j} = \delta_{k,j}$ for $j \in [2,k]$ and $d'_{k+1,j} = d'_{k,j}$ for $j \in [2,k]$, we have:

$$\begin{aligned}
\sum_{j=2}^{k+1} \delta_{k+1,j} b^{v-j} + b^{v-1}(\beta^{v-1}d' + e) &= \sum_{j=1}^{k} (\beta b)^{v-j} d'_{k,j} + \delta_{k+1,k+1} b^{v-k-1} + b^{v-k} e_{v-k} \\
&= \sum_{j=1}^{k} (\beta b)^{v-j} d'_{k+1,j} + b^{v-k-1}(\beta^{v-k-1} d'_{k+1,k+1} + e_{v-k-1}) \\
&= \sum_{j=1}^{k+1} (\beta b)^{v-j} d'_{k+1,j} + b^{v-k-1} e_{v-k-1}.
\end{aligned}$$

Now take $k = v-1$, then we have:

$$\sum_{j=2}^{v-1} \delta_{v-1,j} b^{v-j} + b^{v-1}(\beta^{v-1}d' + e) = \sum_{j=1}^{v-1} (\beta b)^{v-j} d'_{v-1,j} + be_1.$$

But $e_1 \in E_1 = \pi(D' + E_0)$, hence $\exists \delta \in D$, $d'' \in D'$ and $e_0 \in E_0$ such that: $\delta + be_1 = d'' + e_0$. Hence:

$$\delta + \sum_{j=2}^{v-1} \delta_{v-1,j} b^{v-j} + b^{v-1}(\beta^{v-1}d' + e) = \sum_{j=1}^{v-1} (\beta b)^{v-j} d'_{v-1,j} + d'' + e_0.$$

Using the hypothesis, this last quantity belongs to $D + bD + b^2 D + \cdots b^{v-1}D + b^v E_0$. Hence there exist elements d_0, d_1, …, d_{v-1} in D and $e'_0 \in E_0$ such that:

$$\delta + \sum_{j=2}^{v-1} \delta_{v-1,j} b^{v-j} + b^{v-1}(\beta^{v-1}d' + e) = \sum_{j=0}^{v-1} d_j b^j + b^v e'_0.$$

Now looking at the sum "modulo b" we obtain $\delta = d_0$, then simplifying by b we obtain $\delta_{v-1,v-1} = d_1$ ... and finally

$$\beta^{v-1}d' + e = d_{v-1} + be'_0$$

which concludes the proof. ■

6 Automaticity on a semiring and its ring extension

Suppose R is a sub-semiring of a ring R'. If R' is a (D', b')-ring, we can consider by convention that R is a (D', b')-semiring. This has been noticed previously (see Example 1). Let $f : R \to V$, where V is a finite set. There is a natural way to say that f is (D', b')-automatic: if there exists an automaton with inputs D' which computes $f(x)$ from the (D', b')-expansion of x. This automaton is *a priori* defined for *all* inputs and not only for the selected words corresponding to expansions of elements of R. This is equivalent to saying that there exists a function $g : R' \to V$ that is (D', b')-automatic (in the sense previously defined) and such that $g|R = f$. How can this "new" automaticity be related to the "old" one? We give a proposition which covers the applications we discuss later on: for example a sequence $(u_n)_{n\in\mathbb{N}}$ is 2-automatic in the ordinary sense if and only if it is the restriction to $\mathbb{N}$ of a sequence indexed by $\mathbb{Z}$ which is $(\{0, 1\}, -2)$-automatic.

Proposition 4 *Let R' be a ring and let R be a sub-semiring of R'. Let $\beta \in R'$ be a root of unity, (say $\beta^v = 1$, $v \geq 1$). Let $b \in R$, $D \subseteq R$, $D' \subseteq R'$ be such that R is a (D, b)-semiring and R' is a $(D', \beta b)$-ring. We suppose that:*

$$(D, b) \xrightarrow{(E_0, E_1, \ldots, E_{v-1})} (D', \beta b).$$

Let $f : R \to V$, where V is a finite set,

(i) if f is $(D', \beta b)$-automatic, then f is (D, b)-automatic,

(ii) if f is (D, b)-automatic and if the set

$$\mathcal{R} = \bigcup_{k=0}^{\infty} \bigcup_{s \in R_k} \left\{ r \in R' \setminus R : \; b^k r + s \in R \right\}$$

is finite (as usual, $R_k = D + bD + \cdots + b^{k-1}D$), then f is $(D', \beta b)$-automatic. More precisely any function $g : R' \to V$ defined by $g|R = f$ and g is constant on $R' \setminus R$ is $(D', \beta b)$-automatic.

Proof

(i) There exists a function $g : R' \to V$ such that $g|R = f$ and g is $(D', \beta b)$-automatic. Hence the kernel $N_g(D', \beta b)$ is finite (Theorem 1). As R' is a ring it results from Proposition 2 and Lemma 2 that the kernel $N_g(D, b)$ is finite. Hence $N_f(D, b) = \{h|R : \; h \in N_g(D, b)\}$ is also finite, and f is (D, b)-automatic (Theorem 1).

(ii) Suppose now that f is (D, b)-automatic, hence $N_f(D, b)$ is finite. Let v be any element in V and define g by $g|R = f$ and $g(x) = v$ for all $x \in R' \setminus R$. Then the set

$$\mathcal{N} = \{h : R' \to V : \; h|R \in N_f(D, b), \; h|(R' \setminus R) = v\}$$

is also finite. (Note that a map $h \in \mathcal{N}$ is constant on $R' \setminus R$ and the constant for a given map can be any element in V.)

Let $h \in N_g(D, b)$, then $h(x) = g(b^k x + s)$ for some $k \in \mathbb{N}$ and some $s \in R_k$. Define h' by $h'|R = h$ and $h'|(R' \setminus R) = v$. Then $h' \in \mathcal{N}$. But the definition of the set $\mathcal{R}$ implies that h and h' can only differ on $\mathcal{R}$. Since this set is finite, and the functions take only finitely many values, we deduce that $N_g(D, b)$ is finite. Hence $N_g(D', \beta b)$ is also finite (Lemma 2). ∎

Now we give a sufficient condition on a ring with digits to ensure that a map f is automatic if and only if the map $x \to f(-x)$ is automatic. We begin with a lemma.

Lemma 3 *Let R' be a (D',b')-ring. We suppose that $(D',b') \xrightarrow{E} (-D',b')$. Let $\alpha \in R'$ such that $\exists k \geq 0$, $\exists \lambda, s \in R'_k = D' + b'D' + \cdots + (b')^{k-1}D'$ with $(b')^k\alpha + \lambda + s \in E$, then $\alpha \in E$.*

Proof

There is nothing to prove if $k = 0$. Let us prove by induction on k that the result holds for $k \geq 1$.

For $k = 1$, there exist $a \in E$, λ and s in $R'_1 = D'$ such that

$$b'\alpha + \lambda + s = a.$$

Hence:

$$b'\alpha + \lambda = -s + a \in -D' + E \subseteq D' + b'E.$$

This implies the existence of $d' \in D'$ and $a' \in E$ with:

$$b'\alpha + \lambda = d' + b'a'.$$

The uniqueness of decomposition in base b' and the consequence that b' can be simplified in a product imply: $\alpha = a' \in E$.

Now suppose the result is true for k, and let λ and s be in R'_{k+1} such that

$$(b')^{k+1}\alpha + \lambda + s = a \in E.$$

Writing the (D',b')-expansions of λ and s, we deduce:

$$(b')^{k+1}\alpha + \sum_{j=0}^{k} \lambda_j (b')^j = a - s_0 - \sum_{j=1}^{k} s_j (b')^j,$$

where the λ_j's and the s_j's are in D'.

But $-D'+E \subseteq D'+b'E$, hence there exist $\delta \in D'$ and $a' \in E$ such that $-s_0+a = \delta+b'a'$. Therefore we have

$$(b')^{k+1}\alpha + \sum_{j=0}^{k} \lambda_j (b')^j = \delta + b'a' - \sum_{j=1}^{k} s_j (b')^j.$$

Reducing this equation modulo b' yields $\lambda_0 = \delta$, since λ_0 and δ are congruent modulo b' and belong both to D'. Now, simplifying by the base b we obtain:

$$(b')^k\alpha + \sum_{j=1}^{k} \lambda_j (b')^{j-1} + \sum_{j=1}^{k} s_j (b')^{j-1} = a' \in E,$$

in other words, $(b')^k\alpha + \lambda' + s'$ belongs to E for a λ' and a s' in R'_k. The induction hypothesis implies $\alpha \in E$. ∎

Proposition 5 *Let R' be a (D',b')-ring. We suppose that $(D',b') \xrightarrow{E} (-D',b')$. Let $f : R' \to V$, where V is a finite set. Then f is (D',b')-automatic if and only if the map $\tilde{f} : x \longrightarrow f(-x)$ is (D',b')-automatic.*

Proof

We apply Theorem 4 with $u = -1$ and $v = 0$, and Lemma 3. If $\alpha \in L_k = L_k(-1,0)$ (with the notations of Theorem 4), then there exist λ and s in R'_k such that

$$(b')^k \alpha + \lambda + s = 0.$$

As $0 \in E$, this implies $\alpha \in E$, and hence $\bigcup_{k \geq 0} L_k \subseteq E$ is finite. ■

Let R be a semiring in which the property $x + y = x + z \Longrightarrow y = z$ holds. Applying the same construction as for the extension of the semiring of non-negative integers $\mathbb{N}$ to the ring of integers $\mathbb{Z}$ we extend R to the ring $\bar{R}$ as follows: $\bar{R}$ is the quotient of the semiring $R \times R$ by the equivalence relation

$$(u,v) \sim (u',v') \Longleftrightarrow u + v' = u' + v.$$

An embedding of R in $\bar{R}$ is induced by the map $u \longmapsto (u,0)$, $u \in R$. We call $\bar{R}$ the ring extension of R.

Remark 10

For $R = \mathbb{N}$, $\bar{R} = \mathbb{Z}$; for $R = \mathbb{N} \times \mathbb{Z}$, $\bar{R} = \mathbb{Z} \times \mathbb{Z}$.

Theorem 10 *Let R be a semiring in which the property $x + y = x + z \Longrightarrow y = z$ holds. Let $\bar{R}$ be its ring extension. Let $b \in R$ and $D \subseteq R$ such that R is a (D,b)-semiring and $\bar{R}$ is a $(D,-b)$-ring. For a map f from $\bar{R}$ to a finite set V we define the map $\tilde{f}$ from $\bar{R}$ to V by $\tilde{f}(x) = f(-x)$. We also define the maps f^+ and f^- by $f^+ = f|R$ and $f^- = \tilde{f}|R$. We make the following assumptions:*

- *A1. $(D,b) \xrightarrow{(E,F)} (D,-b)$;*
- *A2. there exists $a \in R$ such that $-(R+a) \cap R = \emptyset$ and $-(R+a) \cup R = \bar{R}$;*
- *A3. the set $\mathcal{R} = \bigcup_{k=0}^{\infty} \bigcup_{s \in R_k} \left\{ r \in \overline{R} \setminus R : \ b^k r + s \in R \right\}$ is finite;*
- *A4. the set $\bigcup_{k \geq 0} \{\alpha \in R : \ \exists \lambda, \mu \in R_k, \ b^k \alpha + \lambda = \mu + a\}$ is finite; (the element a is the one in A2).*
- *A5. the set $\bigcup_{k \geq 0} \{\alpha \in \bar{R} : \ \exists \lambda, \mu \in \bar{R}_k, \ (-b)^k \alpha + \lambda = \mu + a\}$ is finite; (the element a is the one in A2).*

Then a map f from $\bar{R}$ to a finite set V is $(D,-b)$-automatic if and only if the maps f^+ and f^- (from R to V) are (D,b)-automatic.

Remark 11

Before giving the proof we note that (this will be proved in the section of applications) the semiring $\mathbb{N}$ and its extension $\mathbb{Z}$, as well as the semiring $\mathbb{N} \times \mathbb{Z}$ and its extension $\mathbb{Z}^2$ satisfy assumptions A1 to A4 for any base.

Proof

We first notice that $(D,-b) \xrightarrow{A} (-D,-b)$ for some finite set A containing 0. Indeed the hypothesis $(D,b) \xrightarrow{(E,F)} (D,-b)$ implies the existence of two finite sets E and F, containing 0, such that:

$$\begin{array}{rcccccc} D & + & E & \subseteq & D & + & bF \\ -D & + & F & \subseteq & D & + & bE. \end{array}$$

Hence: $-D-E+F \subseteq -D-bF+F \subseteq D+bE-bF = D-b(F-E)$, which gives the result with $A = F-E$.
From Proposition 5 we now know that a map g from $\bar{R}$ to a finite set V is $(D,-b)$-automatic if and only if the map $\tilde{g}$ is $(D,-b)$-automatic.

First, suppose that f is (D,b)-automatic, then $\tilde{f}$ is also $(D,-b)$-automatic. Hence the kernels $N_f(D,-b)$ and $N_{\tilde{f}}(D,-b)$ are finite (Theorem 1). As $\bar{R}$ is a ring and as $(D,b) \xrightarrow{(E,F)} (D,-b)$ we have (Proposition 2) $(D,-b) \xrightarrow{(-E,F)} (D,b)$. Hence the kernels $N_f(D,b)$ and $N_{\tilde{f}}(D,b)$ are also finite, (Lemma 2). But $N_{f^+} = \{h|R : h \in N_f(D,b)\}$ and $N_{f^-} = \{h|R : h \in N_{\tilde{f}}(D,b)\}$, so these two kernels are finite, which implies that f^+ and f^- are (D,b)-automatic.

Next, suppose that f^+ and f^- are (D,b)-automatic. From Assumption A4 and Theorem 4 the map $x \longrightarrow f^-(x+a)$ is also (D,b)-automatic. Now define the maps F_1 and F_2 as follows:

$$F_1(x) = \begin{cases} f(x) & \text{if } x \in R, \\ v & \text{otherwise,} \end{cases} \qquad \text{and} \qquad F_1(x) = \begin{cases} f^-(x+a) & \text{if } x \in R, \\ v & \text{otherwise,} \end{cases}$$

where v is a fixed element of V.

These maps are $(D,-b)$-automatic from Proposition 4, using Assumption A3. Hence the map $\tilde{F}_2$ is also $(D,-b)$-automatic. We deduce that $x \longrightarrow \tilde{F}_2(x+a)$ is also $(D,-b)$-automatic using Assumption 5 and Theorem 4.

Choose now any operation $*$ on V with the property that $x * v = x$ for each $x \in V$. Then

$$f(x) = F_1(x) * \tilde{F}_2(x+a),$$

which easily implies the $(D,-b)$-automaticity of f. ■

The following theorem is proved as above and we skip its proof.

Theorem 11 *Let R_i, $i = 1,2$, be two semirings in which the property $x+y = x+z \Longrightarrow y = z$ holds. Let $\bar{R}_i$ their ring extensions. Let $b_i \in R_i$ and $D_i \subseteq R_i$ such that R_i is a (D_i, b_i)-semiring and $\bar{R}_i$ is a $(D_i, -b_i)$-ring. We make the hypothesis that R_1 and R_2 satisfy*

Assumptions 1 to 5 of Theorem 10. Then a map $f : \bar{R}_1 \times \bar{R}_2 \longrightarrow V$ (where V is a finite set) is $(D_1 \times D_2, (-b_1, -b_2))$-automatic if and only if the maps $f_{\varepsilon_1,\varepsilon_2} : R_1 \times R_2 \to V$ are $(D_1 \times D_2, (b_1, b_2))$-automatic for all choices of $\varepsilon_1, \varepsilon_2 = \pm 1$, where $f_{\varepsilon_1,\varepsilon_2}(r_1, r_2) = f(\varepsilon_1 r_1, \varepsilon_2 r_2)$, for every $(r_1, r_2) \in R_1 \times R_2$.

7 E-linked digit sets and transducers

In this section we show that the condition of being linked for two digit sets (related to a given base) is equivalent to the existence of a transducer mapping one representation to the other. The transducers in this paper will always be, unless otherwise indicated, what is sometimes called 1-uniform transducers: that is, each input letter results in exactly one output letter. More precisely let us give a formal definition.

Definition 9 *A transducer T is given by $T = (S, s_0, A, B, \sigma, \tau)$ where S, A, B are finite sets, where σ and τ are maps defined on $S \times A$ and with values in S, resp. B:*

$$\sigma : S \times A \longrightarrow S, \quad \tau : S \times A \longrightarrow B.$$

The elements of S are called the states of the transducer, s_0 being the initial state. The sets A and B are called respectively the input alphabet and the output alphabet. The maps σ and τ are called respectively the transition map and the output map.

The maps σ and τ are extended to maps $\sigma : S \times A^* \longrightarrow S$, $\tau : S \times A^* \longrightarrow B$ as follows :

$$\sigma(s, a_1 \dots a_n) = \sigma(\sigma(s, a_1 \dots a_{n-1}), a_n),$$

$$\tau(s, a_1 \dots a_n) = \tau(\sigma(s, a_1 \dots a_{n-1}), a_n).$$

Usually a transducer is represented by a directed multigraph with labelled arrows. The vertices of the graph are the states (elements of the set S). An arrow with label a/b connects the state s_1 to the state s_2 if $\sigma(s_1, a) = s_2$ and $\tau(s_1, a) = b$.

Notation

If an arrow with label a/b connects the state s_1 to the state s_2, i.e., if $\sigma(s_1, a) = s_2$ and $\tau(s_1, a) = b$, we write

$$s_1 \xrightarrow{a/b} s_2.$$

The transducer T maps A^* to B^* : $a_1 \dots a_n \longmapsto b_1 \dots b_n$ where $b_i = \tau(s_0, a_1 \dots a_i)$. For this notion the reader can look at [18] Chapter X.

Theorem 12 *Let R be a (D, b)- and (D', b)-semiring. Then there exists a finite set E containing 0 such that $(D, b) \xrightarrow{E} (D', b)$ if and only if there exists a finite state transducer mapping the (D', b)-representation of each element of R to its (D, b)-representation.*

Proof

Let us first suppose that $(D,b) \xrightarrow{E} (D',b)$, i.e. that $D' + E \subseteq D + bE$, where E is a finite set containing 0. We define a transducer $T = (S, s_0, D', D, \sigma, \tau)$ as follows: $S = E$, $s_0 = 0$ and $\sigma(e, d') = e_1$, $\tau(e, d') = d$ (i.e., with the above notation $e \xrightarrow{d'/d} e_1$) if and only if $d' + e = d + be_1$, with e and e_1 belonging to E.

Now take $x \in R$ and write it in base (D', b) *with an infinite number of leading zeroes,*

$$x = d'_0 + d'_1 b + \cdots + d'_t b^t + 0 b^{t+1} + \cdots$$

Using the property $D' + E \subseteq D + bE$, we write:

$$\begin{aligned} d'_0 &= d_0 + be_1, \\ d'_1 + e_1 &= d_1 + be_2, \\ &\cdots \\ d'_t + e_t &= d_t + be_{t+1}. \end{aligned}$$

where the e_j are in E.

But e_{t+1} can be written in base (D, b) as $e_{t+1} = \sum_{j=0}^{T} \alpha_j b^j$. Define

$$\begin{aligned} d_{t+1} &= \alpha_0, & e_{t+2} &= \textstyle\sum_{j=1}^{T} \alpha_j b^{j-1}, \\ d_{t+2} &= \alpha_1, & e_{t+3} &= \textstyle\sum_{j=2}^{T} \alpha_j b^{j-2}, \\ \cdots \\ d_{t+T+1} &= \alpha_T, & e_{t+T+2} &= 0, \end{aligned}$$

and $e_j = 0$ for every $j \geq t + T + 2$.

Then $x = d_0 + d_1 b + \cdots + d_{t+1} b^{t+1} + \cdots + d_{t+T+1} b^{t+T+1}$ and

$$\begin{aligned} e_{t+1} &= d_{t+1} + be_{t+2} \\ e_{t+2} &= d_{t+2} + be_{t+3} \\ &\cdots \end{aligned}$$

This gives, from the very definition of the transducer, the arrows:

$$0 \xrightarrow{d'_0/d_0} e_1 \xrightarrow{d'_1/d_1} e_2 \xrightarrow{d'_2/d_2} \cdots \xrightarrow{d'_t/d_t} e_{t+1} \xrightarrow{0/d_{t+1}} e_{t+2} \xrightarrow{0/d_{t+2}} \cdots \xrightarrow{0/d_{t+T+1}} 0 \xrightarrow{0/0} 0 \xrightarrow{0/0} 0 \cdots$$

This means that this transducer maps $(d'_0, d'_1, \ldots, d'_t, 0, 0, \ldots)$ to $(d_0, d_1, \ldots, d_{t+T+1}, 0, 0, \ldots)$, where $x = \sum d'_j b^j = \sum d_k b^k$.

Suppose now that there exists a finite state transducer $T = (S, s_0, D', D, \sigma, \tau)$ mapping the (D', b)-representation of each element of R to its (D, b)-representation.

First we define a *(content-)map* $c : S \to R$. Let $s \in S$. Feed the transducer, starting from the state s, with an infinite sequence of zeroes:

$$s \xrightarrow{0/a_0} . \xrightarrow{0/a_1} \cdots \xrightarrow{0/a_n} . \cdots$$

Only a finite number of the terms of the corresponding output sequence $(a_n)_{n \in \mathbb{N}}$ are not zero. To see this take a path in the transition graph of the transducer T connecting the initial state s_0 with s:

$$s_0 \xrightarrow{d'_0/d_0} s_1 \xrightarrow{d'_1/d_1} s_2 \xrightarrow{d'_2/d_2} \cdots \xrightarrow{d'_k/d_k} s$$

Hence we have:

$$s_0 \xrightarrow{d'_0/d_0} s_1 \xrightarrow{d'_1/d_1} s_2 \xrightarrow{d'_2/d_2} \cdots \xrightarrow{d'_k/d_k} s \xrightarrow{0/a_0} . \xrightarrow{0/a_1} \cdots \xrightarrow{0/a_n} . \cdots$$

The hypothesis implies that the (D, b)-representation of the element $r' = d'_k b^k + \cdots + d'_1 b + d'_0$, ($d_j \in D$ for $j = 0, \ldots, k$) is given by $r' = \sum_{j=0}^{k} d_j b^j + b^{k+1} \sum a_i b^i$. This implies that only a finite number of the a_i's are not zero. Hence the quantity $c(s) = \sum_{i=0}^{\infty} a_i b^i$ is well defined. Furthermore its definition shows that it clearly does not depend on the path chosen between s_0 and s. In other words, for every state s and for every path

$$s_0 \xrightarrow{d'_0/d_0} s_1 \xrightarrow{d'_1/d_1} s_2 \xrightarrow{d'_2/d_2} \cdots \xrightarrow{d'_\ell/d_\ell} s$$

connecting s_0 to s we have:

$$\sum_{j=0}^{\ell} d'_j b^j = \sum_{i=0}^{\ell} d_i b^i + b^{\ell+1} c(s).$$

Let $E = c(S)$. This is a finite subset of R containing 0 (since there is a loop labelled 0/0 at the initial state and hence $c(s_0) = 0$). We prove that D' is E-linked with D, i.e. that $D' + E \subseteq D + bE$.

Let $d \in D, e = c(s) \in E$. Consider the arrow $s \xrightarrow{d'/d} s_1$ in the transducer and add it to a path joining s_0 to s:

$$s_0 \xrightarrow{d'_0/d_0} s_1 \xrightarrow{d'_1/d_1} s_2 \xrightarrow{d'_2/d_2} \cdots \xrightarrow{d'_k/d_k} s \xrightarrow{d'/d} s_1$$

One has:

$$d'_0 + d'_1 b + \cdots + d'_k b^k = d_0 + d_1 + \cdots + d_k b^k + b^{k+1} c(s)$$

and

$$d'_0 + d'_1 b + \cdots + d'_k b^k + d' b^{k+1} = d_0 + d_1 + \cdots + d_k b^k + d b^{k+1} + b^{k+2} c(s_1).$$

Hence

$$(d_0 + d_1 + \cdots + d_k b^k + b^{k+1} c(s)) + d' b^{k+1} = d_0 + d_1 + \cdots + d_k b^k + d b^{k+1} + d b^{k+1} + b^{k+2} c(s_1).$$

It follows from the uniqueness of the decomposition in base (D, b) that $bx = by \Longrightarrow x = y$, hence, comparing these two (D, b)-expansions and using this cancellation property for b, we deduce the following:

$$d' + c(s) = d + bc(s_1).$$

Hence $d' + c(s)$ belongs to $D + bE$. ∎

8 $(E_0, \ldots, E_{v-1})$-linked sets and transducers

In this section, we generalize the result of the previous section to the case of two bases (D, b) and $(D', \beta b)$ which are $(E_0, \ldots, E_{v-1})$-linked, (as usual, $\beta^v = 1$).

Theorem 13 *Let R be both a (D, b)-semiring and a $(D', \beta b)$-semiring. As usual, β is a v-th root of unity, $(v \geq 2)$. We make the extra assumption that β satisfies what we call "the β-hypothesis":*

$$\beta^u x + y = x + y, \; u \neq 0 \pmod{v} \Longrightarrow x = 0.$$

Then there exist sets E_0, E_1, …, E_{v-1} such that

$$(D, b) \xrightarrow{(E_0, E_1, \ldots, E_{v-1})} (D', \beta b)$$

if and only if there exists a finite state transducer mapping the $(D', \beta b)$-representation of each element of R to its (D, b)-representation.

Proof

Let us first suppose that $(D, b) \xrightarrow{(E_0, E_1, \ldots, E_{v-1})} (D', \beta b)$, i.e. that for every $j \in [0, v-1]$ we have $\beta^j D' + e_j \subseteq D + bE_{j+1}$, (where $E_v = E_0$).

We define a transducer $T = (S, s_0, D', D, \sigma, \tau)$ as follows:

$$S = \bigcup_{j=0}^{v-1} \{j\} \times E_j = \{(j, e_j) : \; j \in [0, v-1], \; e_j \in E_j\},$$

$s_0 = (0, 0)$ and the input and output maps (arrows) σ and τ are given by $(j, e_j) \xrightarrow{d'/d} (j + 1, e_{j+1})$, (where $j + 1$ is taken modulo v), if and only if $\beta^j d' + e_j = d + b e_{j+1}$, where $e_j \in E_j$ and $e_{j+1} \in E_{j+1}$. As previously, any x in R is written in base $(D', \beta b)$ with infinitely many leading zeroes as

$$x = d'_0 + d'_1 \beta b + \cdots + d'_t (\beta b)^t + 0 (\beta b)^{t+1} + \cdots$$

Then, using the property $\beta^j D' + e_j \subseteq D + bE_{j+1}$, we write:

$$\begin{aligned}
d'_0 \quad &= d_0 + b e_1, \; e_1 \in E_1 \\
\beta d'_1 + e_1 \quad &= d_1 + b e_2, \; e_2 \in E_2 \\
&\cdots \\
\beta^t d'_t + e_t \quad &= d_t + b e_{t+1}, \; e_{t+1} \in E_{t+1}.
\end{aligned}$$

where the sequence $(E_i)_{i \in \mathbb{N}}$ is defined from $(E_i)_{i \leq v-1}$ by periodicity, $(E_{i+v} = E_i$ for every $i)$, and where $e_j \in E_j$.

One has $x = d'_0 + d'_1 \beta b + \cdots + d'_t (\beta b)^t = d_0 + b d_1 + \cdots + b^t d_t + b^{t+1} e_{t+1}$. But e_{t+1} can be written in base (D, b) as $e_{t+1} = \sum_{j=0}^{T} \alpha_j b^j$. Define

$$\begin{aligned}
d_{t+1} &= \alpha_0, & e_{t+2} &= \textstyle\sum_{j=1}^{T} \alpha_j b^{j-1}, \\
d_{t+2} &= \alpha_1, & e_{t+3} &= \textstyle\sum_{j=2}^{T} \alpha_j b^{j-2}, \\
&\cdots \\
d_{t+T+1} &= \alpha_T, & e_{t+T+2} &= 0,
\end{aligned}$$

and $e_j = 0$ for every $j \geq t+T+2$.

Then $x = d_0 + d_1 b + \cdots + d_{t+1} b^{t+1} + \cdots + d_{t+T+1} b^{t+T+1}$ and

$$\begin{aligned} e_{t+1} &= d_{t+1} + b e_{t+2} \\ e_{t+2} &= d_{t+2} + b e_{t+3} \\ &\cdots \end{aligned}$$

This gives, from the very definition of the transducer, the arrows:

$$(0,0) \xrightarrow{d'_0/d_0} (1,e_1) \xrightarrow{d'_1/d_1} (2,e_2) \xrightarrow{d'_2/d_2} \cdots \xrightarrow{d'_t/d_t} (t+1, e_{t+1}) \xrightarrow{0/d_{t+1}} (t+2, e_{t+2}) \xrightarrow{0/d_{t+2}} \cdots$$

$$\cdots \xrightarrow{0/d_{t+T+1}} (t+T+2, 0) \xrightarrow{0/0} (t+T+3, 0) \xrightarrow{0/0} \cdots$$

where the "first components" of the states have to be taken modulo v.
This means that this transducer maps $(d'_0, d'_1, \ldots, d'_t, 0, 0, \ldots)$ to $(d_0, d_1, \ldots, d_{t+T+1}, 0, 0, \ldots)$, where $x = \sum d'_j (\beta b)^j = \sum d_k b^k$.

Suppose now that there exists a finite state transducer $T = (S, s_0, D', D)$ mapping the $(D', \beta b)$-representation of each element of R to its (D, b)-representation.

The content map $c(s)$ of each state s is defined as previously and it has the following property: for every state s and for every path

$$s_0 \xrightarrow{d'_0/d_0} s_1 \xrightarrow{d'_1/d_1} s_2 \xrightarrow{d'_2/d_2} \cdots \xrightarrow{d'_\ell/d_\ell} s$$

connecting s_0 to s we have:

$$\sum_{j=0}^{\ell} d'_j (\beta b)^j = \sum_{i=0}^{\ell} d_i b^i + b^{\ell+1} c(s).$$

Now we define the "parity map" $p : S \to \{0, 1, \ldots v-1\}$ by taking a path joining s_0 to s, say

$$s_0 \xrightarrow{d'_0/d_0} s_1 \xrightarrow{d'_1/d_1} s_2 \xrightarrow{d'_2/d_2} \cdots \xrightarrow{d'_k/d_k} s,$$

then

$$p(s) \equiv k+1 \pmod{v}.$$

We have to prove that the definition of the parity map p is correct, i.e., it does not depend on the choice of the path.

Suppose there are two paths connecting the initial state s_0 with the state s in the transition graph of the transducer T:

$$s_0 \xrightarrow{d'_0/d_0} s_1 \xrightarrow{d'_1/d_1} s_2 \xrightarrow{d'_2/d_2} \cdots \xrightarrow{d'_k/d_k} s$$

and

$$s_0 \xrightarrow{d'_0/d_0} s_1 \xrightarrow{d'_1/d_1} s_2 \xrightarrow{d'_2/d_2} \cdots \xrightarrow{d'_\ell/d_\ell} s.$$

Let $s \xrightarrow{d'/d} s_1$ with $d' \neq 0$. One has:

$$d'_0 + (\beta b)d'_1 + \cdots + (\beta b)^k d'_k = d_0 + bd_1 + \cdots + b^k d_k + b^{k+1}c(s)$$

and

$$d'_0 + (\beta b)d'_1 + \cdots + (\beta b)^k d'_k + (\beta b)^{k+1}d' = d_0 + bd_1 + \cdots + b^k d_k + b^{k+1}d + b^{k+2}c(s_1),$$

hence:

$$d_0 + bd_1 + \cdots + b^k d_k + b^{k+1}c(s) + (\beta b)^{k+1}d' = d_0 + bd_1 + \cdots + b^k d_k + +b^{k+1}d + b^{k+2}c(s_1).$$

As (D, b) is a base, we can simplify by b and we get:

$$c(s) + \beta^{k+1}d' = d + bc(s_1).$$

But the same proof with the other path gives

$$c(s) + \beta^{\ell+1}d' = d + bc(s_1),$$

which implies

$$c(s) + \beta^{k+1}d' = c(s) + \beta^{\ell+1}d'.$$

Let $\ell = k + u$. Then

$$\beta^{v-k-1}c(s) + d' = \beta^{v-k-1}c(s) + \beta^u d'.$$

The β-hypothesis implies that $u \equiv 0 \pmod{v}$.

Define now $S_j = \{s \in S : \ p(s) = j \pmod{v}\}$, and $E_j = c(S_j)$. We now prove that (D, b) is $(E_0, \ldots, E_{v-1})$-linked to $(D', \beta b)$.

Let $e_n \in E_n$, let $s \in S_n$ such that $c(s) = e_n$ and let $d' \in D'$. There exists in the transducer a path of length $L \equiv n \pmod{v}$ connecting s_0 to s, say

$$s_0 \xrightarrow{d'_0/d_0} s_1 \xrightarrow{d'_1/d_1} s_2 \xrightarrow{d'_2/d_2} \cdots \xrightarrow{d'_{L-1}/d_{L-1}} s.$$

Let $s \xrightarrow{d'/d} s_1$, which gives the path

$$s_0 \xrightarrow{d'_0/d_0} s_1 \xrightarrow{d'_1/d_1} s_2 \xrightarrow{d'_2/d_2} \cdots \xrightarrow{d'_{L-1}/d_{L-1}} s \xrightarrow{d'/d} s_1.$$

As above we obtain: $c(s) + \beta^L d' = d + bc(s_1)$. Furthermore $p(s_1) \equiv L + 1 \pmod{v}$, hence $c(s_1)$ belongs to E_{L+1}. Finally the condition $E_v = E_0$ results from the very definition of the E_j's. ■

Remark 12

It follows from the results in this section and the previous one that the theorems in this paper deduced from the linked set property can be rephrased in terms of transducers. However, we proved these theorems in a purely arithmetic fashion; furthermore, we provided the "algorithms" that explicitly calculate the finite sets associated with the linked set property, and hence the corresponding transducers.

9 Applications

We begin this section with what we call the "carry-condition": if a (D,b)-semiring has this property, there is a relatively simple way to prove that two sets are linked. Then we apply the previous theorems to the case of one- and two-dimensional sequences, to the case of maps on the Gaussian integers and to maps on a ring of quadratic algebraic integers. We finally give results for the folded representation of integers and for the revolving representation of Gaussian integers.

9.1 The carry-condition

We first give a sufficient condition for two digit sets to be E-linked.

Definition 10 *Let R be a (D,b)-semiring and as previously*

$$R_t = R_t(D,b) = \{r: \ r = d_{t-1}b^{t-1} + \cdots + d_1 b + d_0, \ d_j \in D, \ 0 \le j \le t-1\}.$$

We say that R satisfies the (D,b)-carry condition if and only if for every $t > 0$ and every $x, y \in R_t(D,b)$ we have $x + y \in R_{t+1}(D,b)$.

Remark 13

This condition means that one can easily control the propagation of carries when adding two elements of the semiring.

Lemma 4 *Let R be a (D,b)-semiring and let D' be a finite set. If R satisfies the (D,b)-carry condition, then (D,b) is linked to (D',b).*

Proof

Let t_0 be such that $D' \subseteq R_{t_0}(D,b)$ and define $E = R_{t_0}$. For $d' \in D'$, $e \in E$ we have

$$d' + e = d_{t_0+1}b^{t_0+1} + \cdots + d_1 b + d_0, \ d_j \in D, \ 0 \le j \le t_0 + 1.$$

Then $d' + e = d_0 + be_1$ where $e_1 = d_{t_0+1}b^{t_0} + \cdots + d_2 b + d_1$, i.e.

$$D' + E \subseteq D + bE.$$

■

Example 6

Let $b \ge 2$ be an integer and let $D = \{0, 1, \ldots, b-1\}$. Then $\mathbb{N}$ satisfies the (D,b)-carry condition and $\mathbb{Z}$ satisfies the $(D,-b)$-carry condition.

In the next section, we consider the case of $\mathbb{N}$ and $\mathbb{Z}$ and we need a second lemma which proves slightly more in this particular case.

Lemma 5 *Let $b \geq 2$, let $\varepsilon = \pm 1$. If $D, D' \subseteq \mathbb{Z}$ are complete residue systems modulo b, then*

$$(D, \varepsilon b) \xrightarrow{E} (D', \varepsilon b) \text{ and } (D, \varepsilon b) \xrightarrow{(F,G)} (D', -\varepsilon b),$$

for some finite sets E, F and G which contain 0.

Proof

Let us first show that $(D, \varepsilon b) \xrightarrow{E} (D', \varepsilon b)$, for some finite set E containing 0. We use Theorem 5 and the algorithm it gives. If A is a subset of $\mathbb{Z}$ we write $|A| = \sup_{x \in A} |x|$. For the map π and the sets $E_1 = \pi(D')$, $E_2 = \pi(D' + E_1)$, ..., which have been defined in Theorem 5, we prove by induction on n that

$$|E_n| \leq (|D| + |D'|) \sum_{j=1}^{n} b^{-j}.$$

First, if $y \in E_1$, then $y = \pi(x)$ for some $x \in D'$, hence there exists $d \in D$ satisfying $x = d + b\pi(x) = d + by$. This gives

$$b|y| \leq |x - d| \leq |x| + |d| \leq |D| + |D'|.$$

Now suppose that $|E_n| \leq (|D| + |D'|) \sum_{j=1}^{n} b^{-j}$, and let $z \in E_{n+1} = \pi(D' + E_n)$. There exist $\delta \in D$, $\delta' \in D'$ and $e_n \in E_n$ with $bz + \delta = \delta' + e_n$. hence:

$$b|z| \leq |e_n| + |D| + |D'| \leq (|D| + |D'|)(1 + \sum_{j=1}^{n} b^{-j}),$$

which gives:

$$|z| \leq (|D| + |D'|)(\sum_{j=1}^{n+1} b^{-j}).$$

This implies that for every n we have $|E_n| \leq \frac{|D|+|D'|}{b-1}$, hence the sequence $(E_n)_n$ is stationary, which implies that $(D, \varepsilon b)$ is linked to $(D', \varepsilon b)$.

The proof that $(D, \varepsilon b) \xrightarrow{(E,F)} (D', -\varepsilon b)$ for some finite sets E and F containing 0 is exactly the same using Theorem 7 instead of Theorem 5.

The last claim in this lemma is proved analogously. ■

9.2 Automaticity of unidirectional and bidirectional sequences

In this section, we consider the automaticity of sequences $f : \mathbb{N} \to V$ and $f : \mathbb{Z} \to V$, where V is a finite set. Let $b \in \mathbb{N}$, $b \geq 2$ and $D_b = \{0, 1, \ldots, b-1\}$. As stated in Example 1, $\mathbb{N}$ is a (D_b, b)-semiring and $\mathbb{Z}$ is a $(D_b, -b)$-ring.

Theorem 14 *Let $b \geq 2$, $D_b = \{0, 1, \ldots, b-1\}$ and $D' \subseteq \mathbb{Z}$. We suppose $\mathbb{Z}$ is a $(D', -b)$-ring. Let $f = (f_n)_{n \in \mathbb{N}}$ be a sequence with values in some finite set V. Then the following properties are equivalent:*

(i) f is (D_b, b)-automatic, i.e. b-automatic in the ordinary sense;

(ii) f is $(D', -b)$-automatic.

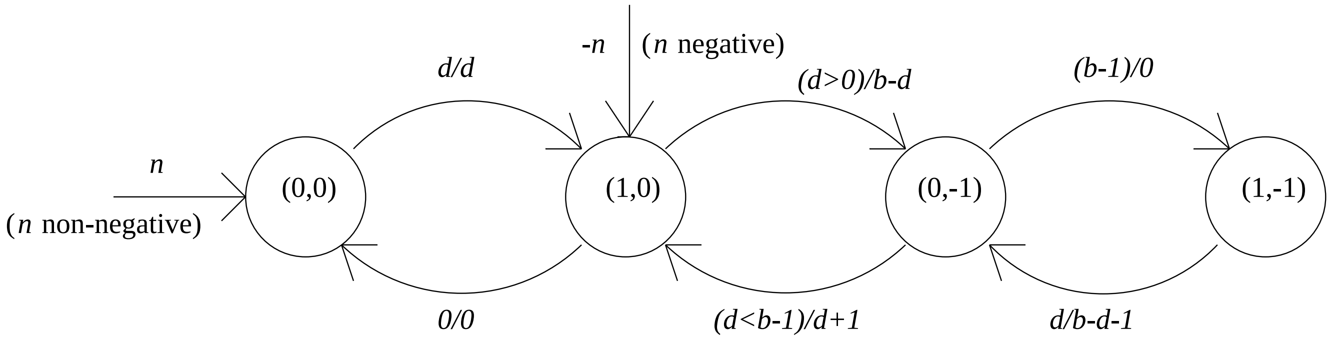

Figure 3: Transducer converting from base-b to base-$(-b)$ representation.

Proof

The result is a consequence of Lemma 5 and Proposition 4, with $R = \mathbb{N}$, $R' = \mathbb{Z}$ and $\beta = -1$. The finiteness of the set $\mathcal{R}$ is proved as above. ■

Example 7

We now give a transducer which converts the ordinary base-b representation of a positive integer n into the base-$(-b)$ representation. Both digit sets are assumed to be equal to $\{0, 1, 2, \ldots, b-1\}$. In Theorem 13, we put $D = D' = D_b = \{0, 1, 2, \ldots, b-1\}$, $\beta = -1$, and $v = 2$. From Theorem 7 we obtain $E_0 = E_1 = \{-1, 0\}$. It follows that the transducer has four states, which are given by

$$S = \{s_0 = (0,0), (0,-1), (+1,0), (+1,-1)\}.$$

The transducer is represented in Figure 3. Note that the number of states is independent of b. We also remind the reader that Theorem 13 produces a transducer which converts representations assumed to have an infinite number of trailing zeroes; however, in the transducer given below, it is easy to see that at most two trailing zeroes in the input are needed.

As Figure 3 indicates, the base-$(-b)$ representation for $n < 0$ may be obtained by feeding the transducer with the base-b representation of $-n$, but starting in state $(1, 0)$.

We know from Theorem 14 that any $(-b)$-automatic sequence must be b-automatic. We illustrate this in a special case; namely, the analogue $(a_n)_{n\geq 0}$ in base -2 (with digits $\{0, 1\}$) of the well-known Thue-Morse sequence already quoted in Example 3. This sequence is defined as follows: write $n = \sum_{i\geq 0} d_i(-2)^i$, where $d_i \in \{0, 1\}$, and define $s_{-2}(n) = \sum_{i\geq 0} d_i$. Define $a_n = s_{-2}(n) \bmod 2$. Then $(a_n)_{n\geq 0}$ is 2-automatic. Here is a brief table of the sequences $(s_{-2}(n))_{n\geq 0}$ and $(a_n)_{n\geq 0}$:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$s_{-2}(n)$	0	1	2	3	1	2	3	4	2	3	4	5	3	4	2	3
a_n	0	1	0	1	1	0	1	0	0	1	0	1	1	0	0	1

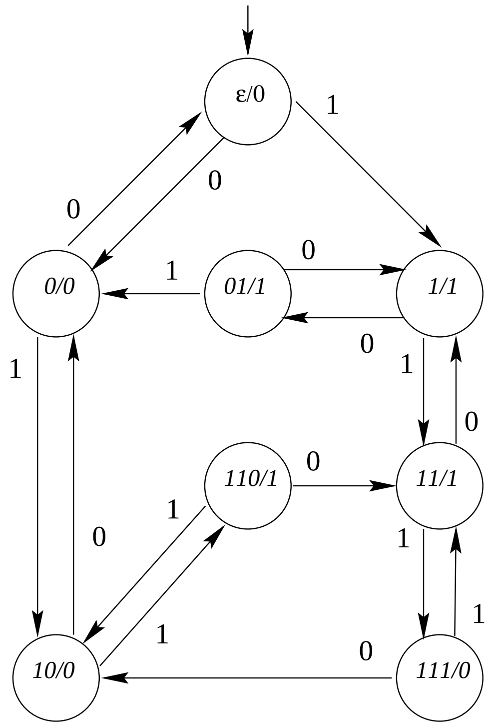

Figure 4: Usual 2-automaton for the Thue-Morse analogue a_n.

We leave it to the reader to show that the 2-kernel of the sequence $(a_n)_{n\geq 0}$ is given by

$$\{(a_n), (a_{2n}), (a_{2n+1}), (a_{4n+1}), (a_{4n+2}), (a_{4n+3}), (a_{8n+6}), (a_{8n+7})\}$$

and that the identities $a_{4n} = a_n$, $a_{8n+1} = a_{8n+3} = a_{2n+1}$, $a_{8n+2} = a_{8n+5} = a_{2n}$, $a_{16n+6} = a_{16n+15} = a_{4n+3}$, $a_{16n+7} = a_{16n+14} = a_{4n+2}$ hold for $n \geq 0$. Using these facts, we may easily construct the automaton in Figure 4. Each state is labelled w/x, where w is a string, and x is an output symbol. Such a state represents the subsequence $(a_{2^{|w|}n+[w]_2})_{n\geq 0}$, where $[w]_2$ gives the value of the string w when expressed in base 2. To compute a_n, feed the automaton with the base-2 digits of n, starting with the least significant bit. Upon finishing the string, you reach a state labelled w/x. Now output x.

In fact, one can say even more: the sequence $(s_{-2}(n))_{n\geq 0}$ is 2-regular in the sense of Allouche and Shallit [7].

Theorem 15 *Let $b \geq 2$, $\varepsilon = \pm 1$. Let $D_b = \{0, 1, \ldots, b-1\}$. Let D' be a subset of $\mathbb{Z}$. We suppose that $\mathbb{Z}$ is a $(D', \varepsilon b)$-ring. Let f be a map from $\mathbb{Z}$ to a finite set V. Define f^+ and f^- from $\mathbb{N}$ to V by $f^+(x) = f(x)$ and $f^-(x) = f(-x)$ for all $x \in \mathbb{N}$. Then the following conditions are equivalent:*

(i) f is $(D_b, -b)$-automatic;

(ii) f^+ and f^- are (D_b, b)-automatic, i.e. b-automatic in the ordinary sense;

(iii) f is $(D', \varepsilon b)$-automatic.

Such a (bidirectional) sequence f is called b-automatic.

Proof

The equivalence between (i) and (ii) is a consequence of Theorem 10 with $R = \mathbb{N}$, $\overline{R} = \mathbb{Z}$ and $D = D_b$:

A1 is satisfied from Lemma 5,

A2 is satisfied with $a = 0$.

To check A3, let $r \leq 0$ and $s = d_0 + d_1 b + \cdots + d_{k-1} b^{k-1} \in R_k$ such that $b^k r + s \geq 0$. Then:

$$0 \leq -r \leq \frac{s}{b^k} \leq |D| (\sum_{j=1}^{k} b^{-j}) \leq \frac{|D|}{b-1},$$

hence $\mathcal{R} = \bigcup_{k=0}^{\infty} \bigcup_{s \in R_k} \left\{ r \in \mathbb{Z} \setminus \mathbb{N} : \; b^k r + s \in \mathbb{N} \right\}$ is finite.

A4 and A5 are checked similarly.

Using lemma 5 above we see that $(D_b, -b)$ is linked to $(D', \varepsilon b)$ in $\mathbb{Z}$. Hence (i) and (iii) are equivalent by Corollary 1. ■

Example 8

Using Theorem 15 the reader can prove properties we gave in the introduction: *a sequence $(f(x))_{x \in \mathbb{N}}$ with values in a finite set is $(\{-1, 0, 1\}, 3)$-automatic if and only if is 3-automatic in the usual sense.* Furthermore *a sequence $(f(x))_{x \in \mathbb{Z}}$ is $(\{0, 1, 2\}, -3)$-automatic if and only if both sequences $(f(x))_{x \in \mathbb{N}}$ and $(f(-x))_{x \in \mathbb{N}}$ are 3-automatic in the usual sense.*

Example 9

We now give a transducer which converts the ordinary base-b representation of a positive integer n into its (D, b)-representation, where $D = \{-k, -k+1, \ldots, -1, 0, 1, \ldots, \ell\}$, with $k, \ell \geq 1$ and $k + \ell = b - 1$. In Theorem 12, we put $D' = D_b = \{0, 1, 2, \ldots, b-1\}$, $D = \{-k, -k+1, \ldots, 0, 1, \ldots, \ell\}$. From Theorem 5 we obtain $E = \{0, 1\}$. It follows that the transducer has two states, which are given by

$$S = \{s_0 = 0, 1\}.$$

The transducer is represented in Figure 5. Note that the number of states is again independent of b.

9.3 Double sequences

In this section we study maps from $\mathbb{N} \times \mathbb{N}$ to a finite set V, or maps from $\mathbb{N} \times \mathbb{Z}$ to V or maps from $\mathbb{Z} \times \mathbb{Z}$ to V.

Theorem 16 *Let $b \in \mathbb{N}$, $b \geq 2$ and $D_1, D_2 \subseteq \mathbb{Z}$. We suppose that $\mathbb{Z}$ is both a $(D_1, -b)$-ring and a $(D_2, -b)$-ring. Then a map $f : \mathbb{Z}^2 \to V$ is $(D_1 \times D_2, (-b, -b))$-automatic if and only if the maps $f_{\varepsilon_1, \varepsilon_2} : \mathbb{Z}^2 \to V$, with $\varepsilon_1, \varepsilon_2 = \pm 1$ are (D_b, b)-automatic, where $f_{\varepsilon_1, \varepsilon_2}(n_1, n_2) = f(\varepsilon_1 n_1, \varepsilon_2 n_2)$, for $(n_1, n_2) \in \mathbb{N}$.*

Proof

Use Theorem 11. ■

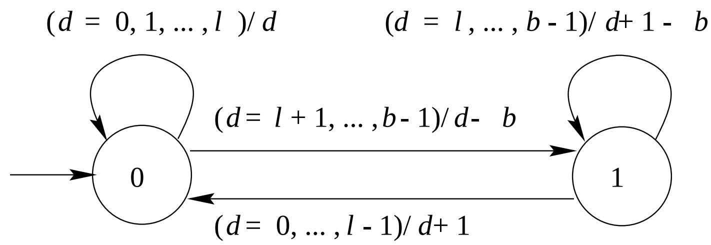

Figure 5: Transducer converting the positive integers from base-b to $(\{-k, \ldots, \ell\}, b)$-representation ($k+\ell = b-1$).

9.4 Automaticity of maps on Gaussian integers

In this section, we consider the ring $\mathbb{Z}[i]$ of Gaussian integers. This ring is a (D, b)-ring for $D = \{0, 1\}$, $b = -1+i$, see Example 1. As in Theorem 3, we use the bijection $\theta : \mathbb{Z}^2 \to \mathbb{Z}[i]$ defined by $\theta(m, n) = m + ni$, where $m, n \in \mathbb{Z}$.

Theorem 17 *A map $f : \mathbb{Z}[i] \to V$ is $(\{0,1\}, -1+i)$-automatic if and only if the maps $(f \circ \theta)_{\varepsilon_1, \varepsilon_2} : \mathbb{N}^2 \to V$ are $(\{0,1\}, 2)$-automatic, where $(f \circ \theta)_{\varepsilon_1, \varepsilon_2}(m, n) = (f \circ \theta)(\varepsilon_1 m, \varepsilon_2 n)$, for $\varepsilon_1, \varepsilon_2 = \pm 1$, and $m, n \in \mathbb{N}$.*

Proof

Theorem 3 applied to $R' = \mathbb{Z}[i]$, $R = \mathbb{Z}$, $b = -1+i$ and the bijective map θ together with the property $(-1+i)^4 = -4$ shows that the map f is $(\{0,1\}, -1+i)$-automatic if and only if $f \circ \theta$ is $(\theta^{-1}(R'_4), (-4, -4))$-automatic, where, as usual, $R'_4 = \{\alpha + \beta(-1+i) + \gamma(-1+i)^2 + \delta(-1+i)^3 : \ \alpha, \beta, \gamma, \delta \in \{0, 1\}\}$. Theorem 16 shows that this is equivalent to saying that the maps $(f \circ \theta)_{\varepsilon_1, \varepsilon_2}$ from $\mathbb{N} \times \mathbb{N}$ to V are 4-automatic, which is equivalent to saying they are 2-automatic (Theorem 2). ■

Example 10

One can also prove Theorem 17 by explicitly constructing a 2-uniform transducer which converts the base-2 expansion of a Gaussian integer to its expansion in base $-1+i$. We do this below. This transducer is not obtained directly as a consequence of our theorems on linked sets, but one could in principle rephrase the theorems on $\mathbb{Z}^2$ in terms of transducers.

Our transducer $(Q, \Sigma, \Delta, \delta, \tau, q_0)$ has 32 states

$$Q = \{0, 1\} \times \{1, -1\} \times \{1, -1\} \times \{0, 1\} \times \{0, 1\}.$$

Each state is of the form $\{d, s_x, s_y, c_x, c_y\}$, where d governs whether the first entry (x) or 2nd entry (y) is the real part; s_x governs the sign of x; s_y governs the sign of y; c_x is the carry bit for x; and c_y is the carry bit for y.

The initial state chosen depends on the sign of the real and imaginary parts of the input, and is given as follows:

$$\begin{array}{lcc} x \geq 0,\ y \geq 0 & : & q_0 = (0,1,1,0,0); \\ x < 0,\ y \geq 0 & : & q_0 = (0,-1,1,0,0); \\ x \geq 0,\ y < 0 & : & q_0 = (0,1,-1,0,0); \\ x < 0,\ y < 0 & : & q_0 = (0,-1,-1,0,0). \end{array}$$

The transducer is then fed simultaneously with the absolute values of x and y, starting with the least significant bits, and it outputs the base-$(-1+i)$ representation of $x+iy$, starting with the least significant bit.

This transducer converts $z = x + iy$ from base-2 to base-$(-1+i)$. We write

$$\begin{array}{ll} |x| & = \sum_{0 \leq r} a_j 2^j; \\ |y| & = \sum_{0 \leq r} b_j 2^j. \end{array}$$

and the input to the transducer is a sequence of pairs of bits :

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \cdots \begin{bmatrix} a_r \\ b_r \end{bmatrix}$$

The output is $e_0 e_1 \ldots e_r$ where $\sum_{0 \leq j \leq r} e_j (-1+i)^j$.

The idea is for the transducer to keep track of z_j, where $z_0 = z$ and

$$z_{j+1} = \frac{z_j - t_j}{-2i}.$$

Here t_j is defined as follows, where $z_j = x_j + y_j i$ and

$$t_j = \begin{cases} 0 & \text{if} \quad x_j \equiv y_j \equiv 0 \pmod 2; \\ i & \text{if} \ \ x_j \equiv 0,\ y_j \equiv 1 \pmod 2; \\ 1 & \text{if} \ \ x_j \equiv 1,\ y_j \equiv 0 \pmod 2; \\ -1+i & \text{if} \ \ x_j \equiv 1,\ y_j \equiv 1 \pmod 2. \end{cases}$$

We define

$$\delta((0, s_x, s_y, c_x, c_y), \begin{bmatrix} a_s \\ b_s \end{bmatrix}) = (1, s'_x, s'_y, c'_x, c'_y)$$

where

$$(s'_x, s'_y) = \begin{cases} (1,-1) & \text{if} \ \ (s_x, s_y) = (1,1); \\ (1,1) & \text{if} \ \ (s_x, s_y) = (1,-1); \\ (-1,-1) & \text{if} \ \ (s_x, s_y) = (-1,1); \\ (-1,1) & \text{if} \ \ (s_x, s_y) = (-1,-1); \end{cases}$$

and

$$\frac{(a_s + c_x)s_x + (b_s + c_y)s_y i - t_s}{-2i} = c'_x s'_x i + c'_y s'_y.$$

We also define

$$\delta((1, s_x, s_y, c_x, c_y), \begin{bmatrix} a_s \\ b_s \end{bmatrix}) = (0, s'_x, s'_y, c'_x, c'_y),$$

where

$$(s'_x, s'_y) = \begin{cases} (-1,1) & \text{if } (s_x,s_y)=(1,1); \\ (-1,-1) & \text{if } (s_x,s_y)=(1,-1); \\ (1,1) & \text{if } (s_x,s_y)=(-1,1); \\ (1,-1) & \text{if } (s_x,s_y)=(-1,-1); \end{cases}$$

and

$$\frac{(a_s+c_x)s_x i + (b_s+c_y)s_y - t_s}{-2i} = c'_x s'_x + c'_y s'_y i.$$

It can be verified that $0 \leq c'_x, c'_y \leq 1$. The output is then

$$\tau((0,s_x,s_y,c_x,c_y), \begin{bmatrix} a_s \\ b_s \end{bmatrix}) = \begin{cases} 00 & \text{if } a_s+c_x \equiv 0;\ b_s+c_y \equiv 0; \\ 11 & \text{if } a_s+c_x \equiv 0;\ b_s+c_y \equiv 1; \\ 10 & \text{if } a_s+c_x \equiv 1;\ b_s+c_y \equiv 0; \\ 01 & \text{if } a_s+c_x \equiv 1;\ b_s+c_y \equiv 1; \end{cases}$$

and

$$\tau((1,s_x,s_y,c_x,c_y), \begin{bmatrix} a_s \\ b_s \end{bmatrix}) = \begin{cases} 00 & \text{if } a_s+c_x \equiv 0;\ b_s+c_y \equiv 0; \\ 11 & \text{if } a_s+c_x \equiv 1;\ b_s+c_y \equiv 0; \\ 10 & \text{if } a_s+c_x \equiv 0;\ b_s+c_y \equiv 1; \\ 01 & \text{if } a_s+c_x \equiv 1;\ b_s+c_y \equiv 1. \end{cases}$$

The reader can verify, for example, that this machine transduces the base-2 representation of $1-i$ to its base-$(-1+i)$ representation, 111010.

9.5 Automaticity of maps on a ring of quadratic algebraic integers

In this section we consider two more applications of the theorems in the preceding sections.

Let $m \in \mathbb{N}$, $m \geq 2$, and $-m \equiv -1, 2 \pmod 4$. Then the ring $\mathbb{Z}[i\sqrt{m}]$ is a $(D_m, i\sqrt{m})$-ring; see [21, p. 266]. Let $\theta : \mathbb{Z}^2 \to \mathbb{Z}[i\sqrt{m}]$ be the map defined by $\theta(k,l) = k + il\sqrt{m}$, where $k, l \in \mathbb{Z}$.

Theorem 18 *Let $m \in \mathbb{N}$, $m \geq 2$ and $-m \equiv -1, 2 \pmod 4$. A map $f : \mathbb{Z}[i\sqrt{m}] \to V$ is $(D_m, i\sqrt{m})$-automatic if and only if the maps $(f \circ \theta)_{\varepsilon_1,\varepsilon_2} : \mathbb{N}^2 \to V$ are (D_m, m)-automatic, where $(f \circ \theta)_{\varepsilon_1,\varepsilon_2}(k,l) = (f \circ \theta(\varepsilon_1 k, \varepsilon_2 l)$, where $\varepsilon_1, \varepsilon_2 = \pm 1$.*

Proof

Same as the proof of Theorem 17. ■

Finally, we consider the case $m \equiv 1 \pmod 4$ only for $m = -3$. The ring of algebraic integers L in $D(i\sqrt{3})$ is a $(\{0,1,\zeta\}, i\sqrt{3})$-ring for $\zeta = (1 + i\sqrt{3})/2$, [35]. The ring L is isomorphic as an abelian group to $\mathbb{Z} \oplus \mathbb{Z}\zeta$. Define a map $\theta : \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}\zeta$ by $\theta(k,l) = k + l\zeta$. The same proof as for Theorem 17 gives

Theorem 19 *Let L be the ring of algebraic integers of $Q(i\sqrt{3})$. Then a map $f : L \to V$ is $((0,1,\zeta), i\sqrt{3})$-automatic if and only if the maps $(f \circ \theta)_{\varepsilon_1,\varepsilon_2} : \mathbb{N}^2 \to V$ are $(D_3, 3)$-automatic, where $(f \circ \theta)_{\varepsilon_1,\varepsilon_2}(k,l) = (f \circ \theta)(\varepsilon_1 k, \varepsilon_2 l)$, where $k, l \in \mathbb{N}$ and $\varepsilon_1, \varepsilon_2 = \pm 1$.*

9.6 Folded representation of integers and revolving representation of Gaussian integers

There are examples of radix representations of integers or Gaussian integers which are not covered by the notion of (D, b)-representations. The problem is that the set of digits is redundant and therefore the radix representations are not unique. Nevertheless these representations are quite natural. For such (exotic) radix representations there is an obvious notion of automatic map: a map which is produced by a finite automaton with input alphabet the corresponding set of digits. The natural question is how this notion of automaticity is connected with the corresponding notion we considered in the previous sections. In this section, without going into details, we consider the folded representation of integers and the revolving representation of Gaussian integers introduced in [16].

Folded representation of integers

The digit set is $D = \{0, 1, \bar{1}\}$, the base is $b = 2$. Here $\bar{1} = -1$. Every non-zero integer $n \in \mathbb{Z}$ has exactly two folded representations

$$n = n_s 2^s + \cdots + n_1 2 + n_0,\ n_j \in D,$$

a positive one in which the rightmost non-zero digit is 1 and a negative one in which the rightmost non-zero digit is $\bar{1}$, with the condition that the non-zero digits in either representation alternate $1, \bar{1}$, (see [16], I, Theorem 1, p. 70).

Revolving representation of Gaussian integers

The digit set is $D = \{0, 1, \bar{1}, i, \bar{i}\}$, the base is $b = i + 1$. Here $\bar{1} = -1, \bar{i} = -i$. Every non-zero Gaussian integer r has exactly four revolving representations:

$$r = r_s(1+i)^s + \cdots + r_1(1+i) + r_0,\ r_j \in D.$$

For each integer, one of these representations has the property that the rightmost non-zero digit takes the value $1, \bar{1}, i, \bar{i}$ and that in the word $r_s \cdots r_1 r_0$ from left to right non-zero digits follow the cycle $(i, 1, \bar{i}, \bar{1})$, (see [16], Theorem 2, p. 71).

For both folded and revolving representations the digit sets are redundant modulo b. The base $b = 1 + i$ for the revolving representation is not a base for the digit set $\{0, 1\}$ for Gaussian integers, (see [28]).

Here we give more general definition of radix representation including both the preceding examples and the (D, b)-representations we discussed earlier.

Let R be a subsemiring of the semiring R', D a finite subset of R' containing 0, $b \in R$. We consider digit representations of elements of R with the elements of D as digits and b as base. The precise definition is as follows. Let $D[x]$ be the set of polynomials with coefficients in D and $\mathcal{P}_D$ be a given subset in it containing 0. The semiring R is called a $(\mathcal{P}_D, b)$-semiring if there exists $k \in \mathbb{N}$, $k \geq 1$ such that for every element $r \in R$, $r \neq 0$, there are exactly k polynomials $p_1(x), \ldots, p_k(x) \in \mathcal{P}_D$ such that

$$p_j(b) = r,\ j = 1, \ldots, k.$$

For $\mathcal{P}_D = D[x]$ and $k = 1$, R is a (D, b)-semiring. The ring of integers $\mathbb{Z}$ is a $(\mathcal{P}_D, 2)$-ring with $D = \{0, 1, \bar{1}\}$, $k = 2$ and $\mathcal{P}_D$ defined by

$$p(x) = p_s x^s + \cdots + p_1 x + p_0 \in \mathcal{P}_{\mathcal{D}}$$

if and only if $p_j \in D$ and in the word $p_s \cdots p_1 p_0$ non-zero elements $1, \bar{1}$ alternate. The ring of Gaussian integers $\mathbb{Z}[i]$ is a $(\mathcal{P}_D, 1+i)$ ring with $D = \{0, 1, \bar{1}, i, \bar{i}\}$, $k = 4$ and the polynomial $p(x) \in \mathcal{P}_D$ if and only if $p_j \in D$ and in the word $p_s \cdots p_1 p_0$ read from left to right non-zero elements $1, \bar{1}, i, \bar{i}$ follow the cycle $(i, 1, \bar{i}, \bar{1})$.

Let R be a $(\mathcal{P}_D, b)$-semiring for a given $k \in \mathbb{N}$, $k \geq 1$ and

$$\mathcal{P}_D \setminus \{0\} = \bigcup_{j=1}^{k} \mathcal{P}_D^j, \quad \mathcal{P}_D^j \cap \mathcal{P}_D^\ell = \emptyset, \quad j \neq \ell$$

and for every non-zero $r \in R$ there is a $p_j(x) \in \mathcal{P}_D^j$ such that $p_j(b) = r$.

Folded and revolving representation satisfy this condition. For folded representation

$$\mathcal{P}_D \setminus \{0\} = \mathcal{P}_D^1 \cup \mathcal{P}_D^{\bar{1}}$$

and $p(x) \in \mathcal{P}_D^a$ if and only if $p(x) \in \mathcal{P}_D$ and the rightmost non-zero coefficient of the polynomial $p(x)$ is $a = 1, \bar{1}$.

For the revolving representation

$$\mathcal{P}_D \setminus \{0\} = \bigcup_{a \in D} \mathcal{P}_D^a,$$

and $p(x) \in \mathcal{P}_D^a$ if and only if $p(x) \in \mathcal{P}_D$ and the rightmost non-zero coefficient of the polynomial $p(x)$ is $a = 1, \bar{1}, i, \bar{i}$.

Let D^* be the set of all finite words written with the digits D and

$$\mathcal{P}_D^* = \{w \in D : \ w = w_s \cdots w_0, \ p(x) = w_s x^s + \cdots + w_0 \in \mathcal{P}_D\}.$$

The sets $\mathcal{P}_D^{j,*}$, $1 \leq j \leq k$ are defined in a similar way.

The map $f : R \to V$ is called $(\mathcal{P}_D^j, b)$-automatic if there is a finite automaton $\mathcal{A} = (S, s_0, D, \tau, V)$ such that

$$f(r) = \tau(w \circ s_0), \quad r \in R, \quad w = w_s \ldots w_0 \in \mathcal{P}_D^j$$

and

$$r = w_s b^s + \cdots + w_1 b + w_0.$$

Then we have the notions of automatic maps of $\mathbb{Z}$ corresponding to the positive (negative) folded representation of integers and similarly for the maps on $\mathbb{Z}[i]$ corresponding to fixed a-revolving representation (the rightmost non-zero digit is $a = 1, \bar{1}, i, \bar{i}$).

Finally, we give without proof the following two theorems:

Theorem 20 *A map $f : \mathbb{Z} \to V$ is automatic with respect to the positive (negative) folded representation if and only if the maps $f^\varepsilon : \mathbb{N} \to V$, with $\varepsilon = \pm 1$ are 2-automatic, where $f^\varepsilon(n) = f(\varepsilon n)$, $n \in \mathbb{N}$.*

Theorem 21 *A map $f : \mathbb{Z}[i] \to V$ is automatic with respect to the a-revolving representation, $a = 1, \bar{1}, i, \bar{i}$ if and only if the maps $(f \circ \theta)_{\varepsilon_1, \varepsilon_2} : \mathbb{N}^2 \to V$ are $(D_2 \times D_2, (2, 2))$-automatic, where $(f \circ \theta)_{\varepsilon_1, \varepsilon_2}(m, n) = f(\varepsilon_1 m, \varepsilon_2 n)$, where $m, n \in \mathbb{N}$, for $\varepsilon_1, \varepsilon_2 = \pm 1$.*

9.7 Open question

We conclude the paper with a question we have not been able to resolve. Is it necessarily true that that if a sequence of Gaussian integers $(a_n)_{n \geq 0}$ is $(-k+i)$-automatic ($k \geq 2$), then the associated double sequence formed by the real and imaginary parts of a_n is b-automatic for some $b \geq 2$? Since it is not hard to see that no nontrivial power of $-k+i$ can ever be a (rational) integer when $k \geq 2$, the proper generalization of Cobham's theorem [14] would settle this question in the negative.

Acknowledgments We thank the three referees for their careful reading of the manuscript and their valuable comments and suggestions.

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