Notes on cellular automata

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1 Introduction

Cellular automata were introduced by J. von Neumann (see [29]) after a suggestion of S. Ulam [147, p. 274]. They are a self-reproducing model, that was designed in order to answer the question "is it possible to construct robots that can construct identical robots, i.e., robots with the same "complexity"?". The model proposed by von Neumann gives a positive answer to this question. Another "philosophical" background is the production of order from chaos and the concept of "self-organization" (see for example [11], see also [125]).

It is of course tempting to see life itself behind self-reproduction. This might be the reason for the choice of many expressions in this theory: cells, living or dead structures, garden of Eden, game of Life...

2 The game of Life

The most popular example of cellular automaton is the so-called "Game of Life" introduced by Conway in the 70's. The reference given for example in [156, p. 66] is: J. H. Conway, 1970 unpublished. Other references are the articles of M. Gardner in 1970–1972 in *Scientific American*, see for example [68], and the books [19, ch. 25] and [30]. Note that this game is named after Conway and Golay in [120] (Golay's game has an underlying hexagonal tiling according to [69]).

This game is defined as follows. We have an infinite two-dimensional board whose elementary squares are called "cells". A cell can be "living" or "dead". The neighbors of a cell are defined to be the eight cells surrounding it. There is a regular time period. At each time step the cells are updated in a parallel way:

- if the number of living neighbors of a cell is exactly 2, then the cell keeps the same value (living or dead);
- if the number of living neighbors of a cell is exactly 3, then the cell will take the value "living" whatever the previous value was;
- if the number of living neighbors of a cell is any other number, then the cell will take the value "dead" whatever the previous value was.

In other words a cell "dies" either if it is "isolated" or in an "overcrowded" environment. It comes to life or survives if the number of living neighbors is just right.

Remark 1

- Instead of considering an infinite board, we can replace each elementary square by its south-west corner. The corner (instead of the cell) will take the values "living" or "dead". The board itself will thus be replaced by the square lattice $\mathbb{Z}^2$. We will use this alternative representation in the formal definition below.

- Typical problems studied in the Life game are: structures that "propagate" (the so called *gliders*), structures that give birth to structures identical to themselves (this question is close to the original problem), patterns that have a periodic time evolution, crashes between structures, existence of configurations without predecessors (called *gardens of Eden*), "computations" achieved by evolving patterns...

3 Definitions

We first begin with an intuitive definition. We start from a set that can be finite or infinite (this set is usually a grid). For each element v in this set we are given a finite number of elements of the set including the element v itself that are called the "neighbors" of v. We have a map from this set to a set of "states" (usually finite): this map is called a "configuration". The images of configurations can be seen as "observables". We also have an updating function called the local function (or local rule): to obtain the new value assigned to an element, we need the element itself, and the values of all its neighbors (we insist that the set of neighbors of an element contains this element). The update is done "in parallel": the new values are computed simultaneously.

Definition 1

- A *cellular automaton* on a set Γ is defined as follows: we are given for each element $v \in \Gamma$ (also called a *cell*) a finite subset $N(v)$ of Γ that contains v and is called the set of *neighbors* of the cell v. We are also given a set of values A (usually finite). Finally we are given for each $v \in \Gamma$ a *local map* F_v from $A^{N(v)}$ to A (where $A^{N(v)}$ is the set of maps from $N(v)$ to A). A *configuration* is an element of A^Γ, i.e., a map from Γ to A.

- The local maps F_v induce a *global map* F from the set of configurations to itself as follows. Let $\mathcal{C}$ be a configuration. We define the configuration $F(\mathcal{C})$ by:

$$\text{for each } v \in \Gamma, \;\; F(\mathcal{C})(v) := F_v(\mathcal{C}|_{N(v)})$$

  where $\mathcal{C}|_{N(v)}$ is the restriction of $\mathcal{C}$ to the set $N(v)$.

- The *time evolution* of the cellular automaton, starting from an initial configuration $\mathcal{C}$, is the orbit of $\mathcal{C}$ under F, i.e., the sequence of configurations

$$\mathcal{C}, \; F(\mathcal{C}), \; F(F(\mathcal{C})) = F^{(2)}(\mathcal{C}), \; \ldots, \; F^{(t)}(\mathcal{C}), \; \ldots$$

Most of the cellular automata that are studied are "homogeneous". This means intuitively that all the neighborhoods $N(v)$ and all the local functions F_v are "the same" for every v. We give a formal definition below (for a definition in terms of *regular* graphs, see for example [69, p. 20–21]).

Definition 2 A cellular automaton defined on the set Γ is said to be *homogeneous* if there exists a set of maps $\{\varphi_{v,w} : N(w) \to N(v); \; v, w \in \Gamma\}$ with the properties

- $\varphi_{v,w}(w) = v$ for all v, w in Γ,

- $\varphi_{v,v} = id_{N(v)}$ for all $v \in \Gamma$,
- $\varphi_{v,w} \circ \varphi_{w,z} = \varphi_{v,z}$ for all v, w, z in Γ,

and if there exists a $v_0 \in \Gamma$ such that the local maps F_v satisfy

$$\forall v \in \Gamma, \quad F_v = F_{v_0} \circ \varphi_{v_0,v}.$$

Note that the maps $\varphi_{v,w}$ are necessarily bijections, since the last two conditions imply that $\varphi_{v,w} \circ \varphi_{w,v} = id_{N(v)}$ for all v, w in Γ.

If the set Γ is a graph or a lattice, we impose that a cellular automaton on Γ somehow "respects" the structure of Γ.

Definition 3

- A cellular automaton on a graph G is defined as a cellular automaton on the set of vertices of G such that the neighborhood of each vertex consists of the vertex itself and all vertices connected to it by an edge (see for example [141] for additive cellular automata on graphs).
- Let n be a positive integer. A cellular automaton on $\mathbb{Z}^n$ is defined as a cellular automaton on the graph whose vertices are the points of $\mathbb{Z}^n$, whose edges are those joining two points having all but one identical coordinates and the remaining coordinate of one of the points differ by ± 1 from the coordinate of the other point.

Remark 2

- We can define a cellular automaton on the *Cayley graph* of a finitely generated group (see for example [139, p. 47]). If a group Γ is generated by the elements $\gamma_1, \ldots, \gamma_d$, the Cayley graph of Γ is the graph whose vertices are the elements of Γ, whose edges are defined as follows: there is an edge between the elements v and w of Γ if and only if there exists a $i \in [1, d]$ such that $w = v\gamma_i$. This edge is then labeled γ_i.
- In the case of a cellular automaton on $\mathbb{Z}^2$ two neighborhoods are classically chosen, the von Neumann neighborhood and the Moore neighborhood (see Figure 1).
- We can also define cellular automata on a triangular lattice or on a hexagonal lattice: see Figure 2.

4 Another example: the 2-dimensional cellular automaton of Greenberg and Hastings

The Greenberg-Hastings cellular automaton was described in 1978 [84], see also [83, 82], as a simple model generating spatio-temporal structures similar to those that can be observed in the Belousov-Zaikin-Zhabotinsky oscillating chemical reaction (see for example [154, ch. 13]).

This cellular automaton consists of an infinite planar grid (the square lattice $\mathbb{Z}^2$), of two integers $N \geq 2$, and $e \in [1, N-2]$, and of the following local rules. Each "cell" can be in state $0, 1, \ldots, N-1$. The states $1, 2, \ldots, e$ are called *excited states*. The four *neighbors* of a cell are the cells situated north, south, west, and east (von Neumann's neighborhood). Now

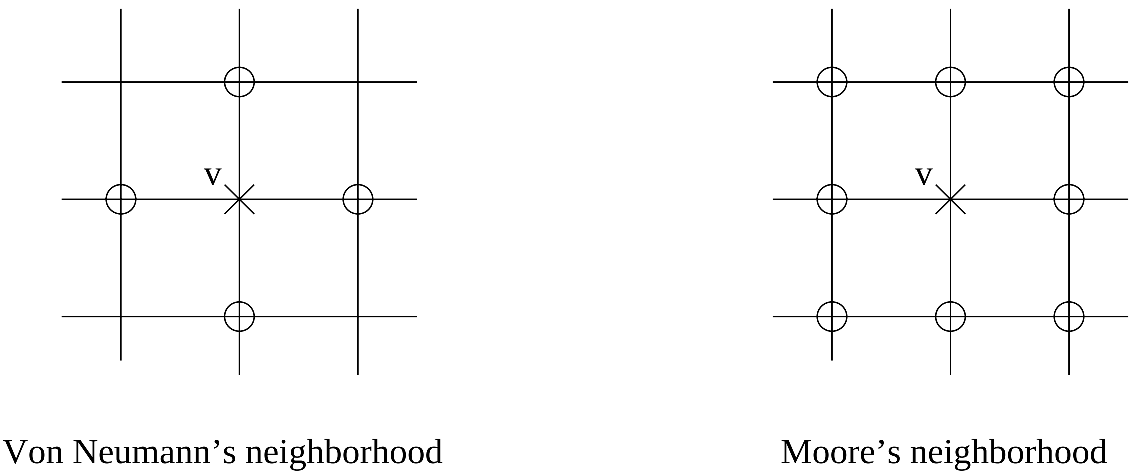

Figure 1: Two possible neighborhoods for cellular automata on $\mathbb{Z}^2$

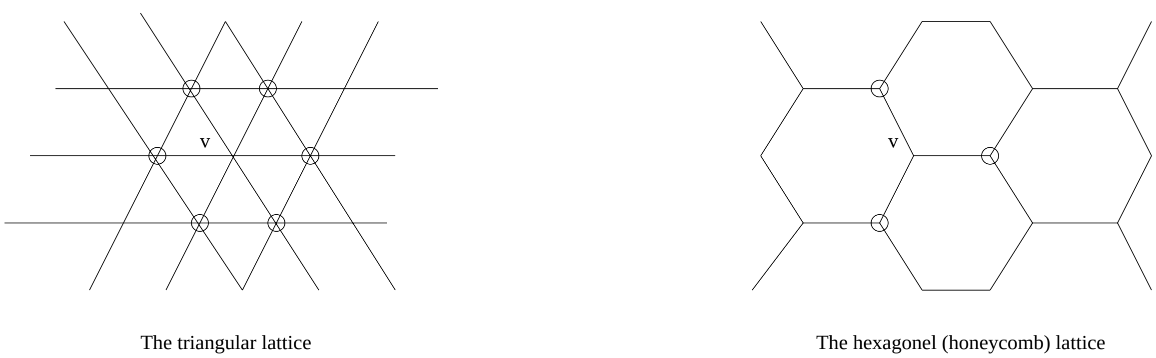

Figure 2: The triangular and the hexagonal lattices

- if a cell is in state k with $1 \leq k \leq N-2$, then its next state is $k+1$;
- if a cell is in state $N-1$, then its next state is 0;
- if a cell is in state 0,
  - if one of its neighbors is in an excited state, then its next state is 1,
  - if none of its neighbors is in an excited state, then its next state is 0.

The first two items describe a reaction rule, while the last one describes a diffusion rule. This model is simple in that the diffusion rule and the reaction rule do not act together for a same cell at the same time. The combinatorial study of this cellular automaton yields the following results (see [83, 84, 82, 6]).

Theorem 1 *Let us suppose that the initial configuration has only a finite number of cells in nonzero states, then*

- *there exists a period $T \geq 1$ such that the configurations of any bounded region of the plane form a periodic sequence that is T-periodic from some time on. (This period can be 1. In this case any bounded region of the plane has only cells in state 0 after some time, the structure is called* dead. *The structure is called* living *otherwise;)*
- *if $N \geq 5$ and if $e \geq 2$, any structure that is living has period N;*
- *outside some bounded region of the plane (depending on the initial condition) the* wavefronts, *i.e., the set of cells in state 1, are composed of "circular-like parallel" curves. The distance between two "consecutive" such curves is N, and they are expanding with speed 1.*

The reader will find more motivations for this model and more results in [84, 83, 82, 6, 56, 61, 62, 81, 79].

5 Cellular automata as global maps

5.1 Cellular automata as special maps on the set of configurations

As we have seen above a cellular automaton can be seen as a map on the set of configurations, and its time evolution can be seen as the orbit of the initial configuration under this map. We will see in this section that the cellular automata are exactly the maps on the set of configurations that are *continuous* and *homogeneous*.

More precisely we restrict ourselves to cellular automata Γ defined on $\mathbb{Z}^n$ ($n \geq 1$) and with values in some finite set A. We recall that the set of configurations is the set $A^{\mathbb{Z}^n}$ of all maps from $\mathbb{Z}^n$ to A. This set is equipped with the following metric: the distance $d(\mathcal{C}_1, \mathcal{C}_2)$ between two configurations $\mathcal{C}_1$ and $\mathcal{C}_2$ is $1/k$ if $\mathcal{C}_1$ and $\mathcal{C}_2$ agree on all points of $\mathbb{Z}^n$ inside the cube $I_k = \{(a_1, \ldots, a_n);\ \max_{i=1}^n \{|a_i|\} < k\}$ and they disagree for at least one point $(a_1, \ldots, a_n)$ such that $\max_{i=1}^n \{|a_i|\} = k$. The topology associated with this metric is the product topology on $A^{\mathbb{Z}^n}$ that is a *compact* metric space (i.e., all sequences of configurations have an accumulation point): actually $A^{\mathbb{Z}^n}$ is homeomorphic to the ternary Cantor set. The *shifts* on the space $A^{\mathbb{Z}^n}$ are the translations $\tau_{(a_1,\ldots,a_n)}$ defined by

$$\tau_{(a_1,\ldots,a_n)}(\mathcal{C})(b_1, \ldots, b_n) := \mathcal{C}(a_1+b_1, \ldots, a_n+b_n).$$

A map F on the set of configurations $A^{\mathbb{Z}^n}$ is said to commute with all shifts, if $F \circ \tau_{(a_1,\ldots,a_n)} = \tau_{(a_1,\ldots,a_n)} \circ F$ for all $(a_1, \ldots, a_n) \in \mathbb{Z}^n$. Of course this property is equivalent to saying that F commutes with all shifts for which all a_i's but one are equal to 0, and the remaining a_i is equal to 1.

The following fundamental theorem can be found in the seminal paper [91].

Theorem 2 *The cellular automata defined on $\mathbb{Z}^n$ ($n \geq 1$) and with values in the finite set A are exactly the continuous maps of $A^{\mathbb{Z}^n}$ that commute with all shifts.*

Remark 3 The proof of this result uses essentially the compactness of the set $A^{\mathbb{Z}^n}$.

5.2 Invertibility of global maps defined by cellular automata

Cellular automata are interesting in that the set of values A is finite, and that the (usually simple) local rule extends to a global map: but the global map might be quite complicated. We recall that the global map is said to be *injective* if for any two different configurations $\mathcal{C}_1 \neq \mathcal{C}_2$ we have $F(\mathcal{C}_1) \neq F(\mathcal{C}_2)$. The map F is said to be *onto* or *surjective* if, denoting as usual by A^Γ the set of configurations, we have $F(A^\Gamma) = A^\Gamma$ (in other words every configuration is the image under the global map of some configuration, that is a predecessor sometimes called *father*). We recalled above that a *garden of Eden* is a configuration without predecessors for the global map. Another way of saying this is that there exists a garden of Eden if and only if the global map F is not surjective.

An unexpected result was given by G. A. Hedlund in 1969 [91, th. 5.14] for cellular automata on $\mathbb{Z}^n$ (remember that the inverse of a continuous bijective map on a compact set must be continuous):

Theorem 3 *A cellular automaton on $\mathbb{Z}^n$ with values in the finite set A that is injective is necessarily surjective (hence bijective and hence a homeomorphism of $A^{\mathbb{Z}^n}$).*

Surjective cellular also have a special structure. The following result was proved independently by A. Gleason and by G. A. Hedlund (see [72] and [91, Theorem 11.5]).

Theorem 4 *Each configuration of a surjective cellular automaton on $\mathbb{Z}^n$ with values in the finite set A has only finitely many predecessors. The number of predecessors might vary from one configuration to the other, but is bounded.*

Actually if a cellular automaton on $\mathbb{Z}^n$ with values in the finite set A is not surjective, G. A. Hedlund [91, Theorem 5.13] proved that there exists a configuration with uncountably many predecessors.

6 Cellular automata as dynamical systems

We can consider cellular automata as *dynamical systems*. We recall that a *topological dynamical system* consists on a compact set X, together with a continuous map $f : X \to X$ (about dynamical systems, see for example [9, ch. 1, p. 154–173 and ch. 3, p. 197–227]). We will restrict ourselves to compact *metric* sets (i.e., compact sets whose topology is given by a distance). We just saw in Section 5.1 above that a cellular automaton, say on $\mathbb{Z}^n$, can be considered as a continuous map on the (metric compact) set of configurations.

In classical topological dynamics, one of the main focuses is the study of "repetitiveness" (existence of periodic points, chain recurrence, non-wandering sets, center of the dynamical system...) and of "attracting" or "repelling" invariant sets (that are related to stability properties of the dynamical system).

6.1 Periodic points

For a dynamical system (X, f), a point $x_0 \in X$ is called *periodic* if there exists an integer $n \geq 1$ such that $f^n(x_0) = x_0$. Then the orbit of x_0 under f, i.e. the set $\{f^k(x_0);\ k \geq 0\}$ is finite. Its cardinality is called the *period* of the point x_0. A point $x_0 \in X$ is called *ultimately periodic* if there exists two integers $k \geq 0$ and $n \geq 1$, such that $f^{n+k}(x_0) = f^k(x_0)$. Then the orbit of x_0 is finite; if k is the smallest integer ≥ 0 with the property that there exists $n \geq 1$ such that $f^{n+k}(x_0) = f^k(x_0)$, then the set $\{f^j(x_0);\ 0 \leq j \leq k-1\}$ is called the *transient* part of the orbit of x_0. The smallest n associated with this k is called the period of x_0.

Periodic points and generalizations (almost periodic points, recurrent points) for dynamical systems arising from cellular automata were studied for example in [91, 98]. Periodic points of *cylindrical* cellular automata (i.e., cellular automata defined on the Cayley graph of $\mathbb{Z}/m\mathbb{Z}$), as well as relations between spatial and temporal periods were studied in [116], see also [103, 105].

6.2 Attractors

An attractor of a dynamical system (X, f) is a compact invariant set that intuitively "attracts" all points in some neighborhood (i.e., iterating the map f from one of these points gives points that "converge" to a limit belonging to the attractor. Such an attractor is also "Lyapunov stable", i.e., every orbit under the map f starting sufficiently close to it remains in a neighborhood of the attractor. These orbits ultimately converge to the attractor. Among the possible definitions of attractors (see for example [118] [50, p. 201–214], [41]) we choose the definition given by Conley [41] in 1978.

Definition 4 The set $\mathcal{A} \subset X$ is an *attractor* of the dynamical system (X, f) if there is an open neighborhood $\mathcal{U}$ of $\mathcal{A}$ such that $f(\overline{\mathcal{U}}) \subset \mathcal{U}$ and $\mathcal{A} = \cap_{n=0}^{\infty} f^n(\mathcal{U})$ (where $\overline{\mathcal{U}}$ is the closure of $\mathcal{U}$). The open set $\mathcal{B}(\mathcal{A}) := \cup_{n=0}^{\infty} f^{-n}(\mathcal{U})$ is called the *basin of attraction* of $\mathcal{A}$.

Remark 4 Note that the hypotheses imply that the set $\mathcal{A}$ is compact and that the set $\mathcal{B}(\mathcal{A})$ is also the set of points x such that all limit points in their orbits belong to $\mathcal{A}$. In other words $\mathcal{B}(\mathcal{A})$ does not depend on the open set $\mathcal{U}$ such that $f(\overline{\mathcal{U}}) \subset \mathcal{U}$ and $\mathcal{A} = \cap_{n=0}^{\infty} f^n(\mathcal{U})$.

It is not always possible to have a complete description of all attractors for a given dynamical system (for positive results see [41]). For the case of a cellular automaton the following result holds [97] (see also [99, 21, 111]).

Theorem 5 *Any cellular automaton on $\mathbb{Z}^n$, considered as a dynamical system, satisfies exactly one of the following conditions:*

- *there exists a (unique) minimal attractor that is contained in every other attractor;*
- *there is a (unique) minimal quasi-attractor (a* quasi-attractor *is the intersection of a sequence of attractors $\mathcal{A} = \cap_{n=1}^{\infty} \mathcal{A}_n$) contained in every attractor;*

- *there exist two disjoint attractors. In this case the dynamical system associated with the cellular automaton has uncountably many minimal quasi-attractors.*

Several other properties as cellular automata viewed as dynamical systems are of interest: expansiveness, equicontinuity, transitivity, sensitivity. We survey quickly some of these properties. The dynamical systems below will be *topological dynamical systems*, i.e., topological dynamical systems whose topology is associated with a distance (this is always the case for cellular automata as we have seen above).

6.3 Expansiveness and permutivity

A topological dynamical system (X, f) (resp. (X, f) with some measure μ on X) is called *expansive at* $x_0 \in X$ (resp. μ*-expansive at* $x_0 \in X$) if there exists a constant $\delta > 0$, called the *constant of expansiveness at* x_0, with the property that for every point (resp. μ-almost every point) $y \neq x_0$ there exists $k \geq 0$ such that the distance between $f^k(y)$ and $f^k(x_0)$ is at least δ. The dynamical system is called *expansive* (resp. μ*-expansive*) if it is expansive (resp. μ-expansive) at every $x \in X$.

For example any one-dimensional shift is expansive (the one-dimensional shift on a finite set A is the map defined on the set of two-sided sequences on A by $(u_n)_{n\in\mathbb{Z}} \to (u_{n+1})_{n\in\mathbb{Z}}$). Actually the expansive maps are "close" to shifts: the following theorem is due to G. A. Hedlund [91, Th. 2.1, p. 322].

Theorem 6 *If the topological dynamical system (X, f) is expansive, then it is isomorphic to a* subshift*, i.e., there exists a one-dimensional shift σ and an imbedding ι from X to $A^{\mathbb{Z}}$ such that the set $\iota(X)$ is shift-invariant, and $\sigma \circ \iota = \iota \circ f$.*

For a class of cellular automata we have a more precise result. A cellular automaton is *permutive* if its local function depends "in an essential way" of the values of the leftmost and the rightmost neighbors (see a precise definition in [91, p. 332–333]). R. H. Gilman proved in 1988 [71] the following theorem.

Theorem 7 *Any permutive cellular automata is expansive. Furthermore the corresponding dynamical system is isomorphic to a one-dimensional one-sided shift (i.e., to the dynamical system $(A^{\mathbb{N}}, \sigma)$, where A is a finite set, and σ is the map defined on $A^{\mathbb{N}}$ by $(u_n)_{n\geq 0} \to (u_{n+1})_{n\geq 0}$).*

It is interesting to note that *n-dimensional cellular automata are never expansive for $n \geq 2$*, [134].

6.4 Equicontinuity (Lyapunov stability)

A topological dynamical system (X, f) is called *equicontinuous (or Lyapunov stable) at* $x_0 \in X$ if for every $\varepsilon > 0$ there exists a neighborhood of x_0 such that for every y in this neighborhood all iterations $f^i(x_0)$ and $f^i(y)$ are ε-close. The dynamical system is called *equicontinuous* if it is equicontinuous at every $x \in X$.

Before stating a result about equicontinuity for cellular automata with values in a finite set A, we define for such an automaton the *uniform Bernoulli measure* μ on the configuration space as the product measure defined on A by $\mu(\{x\}) = 1/\#A$ for each $x \in A$, see [42, p. 4] for example. The following theorem is due to R. H. Gilman [71].

Theorem 8 *Every cellular automaton with values in a finite set A satisfies one of the following conditions (where μ is the Bernoulli measure):*

- *either for every $\varepsilon > 0$ there exists a compact shift-invariant subset Y_ε of the set of configurations such that $\mu(Y_\varepsilon) > 1 - \varepsilon$ and the dynamical system obtained by restricting the cellular automaton to Y_ε is equicontinuous,*
- *or the cellular automaton is μ-expansive.*

6.5 Sensitivity, transitivity

Two more properties are important in the study of topological dynamical systems: sensitivity and transitivity.

A topological dynamical system (X, f) is called *sensitive at* $x_0 \in X$ if it is not equicontinuous at x_0. It is called *sensitive* if it is sensitive for all $x \in X$.

Sensitivity of a dynamical system is an evidence for its "chaotic behavior". Another property associated with a chaotic behavior is (topological) transitivity:

A topological dynamical system (X, f) is called *transitive (or topologically mixing)* if for every two non-empty open subsets $\mathcal{U}$ and $\mathcal{V}$ of X, some iteration $f^k(\mathcal{U})$ intersects $\mathcal{V}$ (see [50, p. 49]).

There are different definitions of a "chaotic dynamical system". One definition is due to R. L. Devaney in 1989 [50, p. 50]:

A topological dynamical system (X, f) is called *chaotic* if it is transitive, sensitive, and if the set of all periodic points is dense in X. It was observed that the sensitivity follows from the other two properties [14].

These and other properties for cellular automata seen as dynamical systems were investigated for example in [133, 75, 39, 22, 109, 59, 124, 114, 33, 32].

6.6 Cellular automata as measurable dynamical systems

A *measurable* dynamical system is a triple (X, μ, f) consisting of a set X with a probability measure μ, and a function $f : X \to X$ that is measure-preserving, i.e., for all $Y \subset X$ we have $\mu(Y) = \mu(f^{-1}(Y))$, see for example [42, ch. 1]. A measurable dynamical system is called *ergodic* if every subset of X invariant under f has measure 0 or 1, see for example [42].

The following result can be found in [91, p. 327] (see also [119, p. 368]).

Theorem 9 *The dynamical system associated with a one-dimensional cellular automaton equipped with the Bernoulli measure is a measurable dynamical system if and only if it is surjective.*

6.7 Complements

Ergodicity as well as other properties of cellular automata viewed as measurable dynamical systems were investigated in particular in [152, 112, 136, 133, 21, 109, 130, 32].

Dynamical properties of cylindrical automata were studied in [149, 104] by using techniques of finite fields.

7 Cellular automata as universal computers. Decidability questions

Cellular automata can be used as “computers”. This was first observed by J. von Neumann [29], who constructed a 2-dimensional cellular automaton that can simulate arbitrary Turing machines, i.e., that is “computation-universal”. Simpler and more particular cellular automata (e.g., one-dimensional, totalistic, reversible) were proved later to be computation-universal [138, 80, 100]. Universality results for other cellular automata can be found in [76, 77, 79]. Finally, Conway’s Game of Life is also a computation-universal cellular automaton [19].

Another question about cellular automata and computing or algorithms is the following. Cellular automata are “finite machines”: they have a finite number of states and are completely determined by their local generating functions (local rules) that are maps between finite sets. It is then quite natural to ask the following question: given a property of the global map (e.g., surjectivity, injectivity), is there an algorithm that can tell in finite time, given the local rule of an arbitrary cellular automaton, whether the automaton has this property? For one-dimensional cellular automata there exist algorithms that can determine whether the global map is surjective or injective [8]. In dimension $D \geq 2$ the question whether a cellular automaton is surjective (resp. injective) is undecidable [107]. The proofs of these results are based on a reduction of the question to the tiling problem in the plane (the “domino problem”) due to R. Berger in 1966 [18]. Other papers on cellular automata, decidability, and tilings are for example [55, 1, 117].

8 One-dimensional cellular automata and Wolfram’s classification

The most popular cellular automata are those defined on $\mathbb{Z}$, usually with homogeneous and symmetrical neighborhoods (and often “additive” see Section 9 below). Many papers were devoted to these cellular automata (see [156] and all references therein). In particular S. Wolfram proposed in 1984 [155] (reprinted in [156, p. 115–157]) a phenomelogical/empirical *classification* of one-dimensional cellular automata in four classes. Only a few rigorous results are known in this direction (see for example [54, 49, 142], see also [85, 108, 17]).

9 Additive and linear cellular automata

Usually “linear” maps are easier to study than general maps. We hence give the following definitions.

Definition 5 We are given a cellular automaton on the set Γ with values in the finite set A. Suppose that A is equipped with an operation $+$ such that $(A, +)$ is a commutative group. For each $v \in \Gamma$ we denote by $+$ the componentwise addition on $A^{N(v)}$. The cellular automaton is said to be *additive* if the local map F_v (and hence the global map F) is additive, i.e., if for all $v \in \Gamma$, and for all f and g in $A^{N(v)}$ we have $F_v(f+g) = F_v(f) + F_v(g)$.

Definition 6 We are given a cellular automaton on the set Γ with values in the finite set A. Suppose that A is equipped with two operations $+$ and $\times$ such that $(A, +, \times)$ is a ring. The set $A^{N(v)}$ is equipped with the usual module structure on A. The cellular automaton is said to be *linear* if all local maps F_v (and hence the global map F) are linear, i.e., if for all $v \in \Gamma$, for all f and g in $A^{N(v)}$, and for all $a \in A$ we have $F_v(f + ag) = F_v(f) + aF_v(g)$.

Remark 5

- In the literature these two notions are often considered as the same: this is clearly true if A is the ring $\mathbb{Z}/m\mathbb{Z}$ for some integer $m \geq 2$, which is the case that is usually studied.

- Let us consider a cellular automaton on $\mathbb{Z}$ taking its values in the ring $\mathbb{Z}/m\mathbb{Z}$ for some integer $m \geq 2$. Let us also restrict ourselves (although it is not necessary) to *finite* configurations (i.e., configurations with finitely many nonzero values). We can associate with each finite configuration a *Laurent polynomial*, i.e., the sum of a polynomial in X and a polynomial in $1/X$ as follows: if the cell indexed by $n \in \mathbb{Z}$ has the value a_n, the polynomial is $\sum_{n \in \mathbb{Z}} a_n X^n$ (the sum is finite). It is then not hard to see that the local rule can be expressed by multiplying the Laurent polynomial of such a configuration by a fixed Laurent polynomial. For example the Pascal triangle modulo $d \geq 2$ can be seen as a cellular automaton defined on $\mathbb{Z}$, the neighbors of a cell are the cell itself and the cell immediately on its left. The local rule is additive: the new value of a cell is the sum, taken modulo d, of its value and of the value of its left neighbor. The initial configuration consists of a 1 at the cell indexed by 0 and 0 everywhere else. Going from a configuration to its image under the global map is exactly multiplying modulo d its Laurent polynomial by $(1 + X)$.

9.1 First properties of additive cellular automata

We survey in this section properties of additive cellular automata, in particular in the case where the states are integers modulo some natural number. The reader is referred to [102, 10, 130, 114, 32].

To state some results let us consider a D-dimensional cellular automaton with local generating function $f : (\mathbb{Z}/m\mathbb{Z})^s \to \mathbb{Z}/m\mathbb{Z}$ defined by

$$f(x_1, x_2, \ldots, x_s) = \sum_{i=1}^{s} \lambda_i x_i \bmod m,$$

where $f(x_1, x_2, \ldots, x_s)$ is the updated value of the cell at position 1 with value x_1, and $x_2, \ldots, x_s$ are the values of the (other) neighbors. Let F be the global map associated with this cellular automaton.

Theorem 10 *Let $\mathcal{A}$ be an additive cellular automaton as above, then the following conditions are equivalent:*

- *$\mathcal{A}$ is transitive,*

- *$\mathcal{A}$ is ergodic,*

- *F is surjective, and for all $n > 0$, the map $Id - F^n$ is surjective,*

- $\gcd(m, \lambda_1, \lambda_2, \ldots, \lambda_s) = 1$.

Theorem 11 *Let $\mathcal{A}$ be an additive cellular automaton as above. Then*

(i) *F is surjective if and only if* $\gcd(m, \lambda_2, \lambda_3, \ldots, \lambda_s) = 1$.

(ii) *$\mathcal{A}$ is sensitive if and only if there exists a prime number p such that $p|m$ and* $p \nmid \gcd(\lambda_1, \lambda_2, \ldots, \lambda_s)$.

(iii) $\mathcal{A}$ is equicontinuous if and only if every prime divisor p of m divides $\gcd(\lambda_1, \lambda_2, \ldots, \lambda_s)$.

Theorem 12 *Let $\mathcal{A}$ be a one-dimensional additive cellular automaton, with local rule f given by $f(x_{-r}, x_{-r+1}, \ldots, x_0, \ldots, x_{r-1}, x_r) = \sum_{i=-r}^{r} a_i x_i$. (This is the updated value of the cell in state x_0 while the other x_i's are the values of the neighbors.)*

Then $\mathcal{A}$ is expansive if and only if $\gcd(m, a_{-r}, a_{-r+1}, \ldots, a_{-1}, a_1, \ldots, a_{r-1}, a_r) = 1$.

9.2 Additive cellular automata and fractals

It has been observed that the time evolution patterns starting from an initial configuration exhibit self-similar properties for many cellular automata. The most popular way of representing this phenomenon is to look at a one-dimensional cellular automaton and to draw a two-dimensional figure by drawing the successive configurations (at time $0, 1, 2 \ldots$) one above the other (see Figure 3). Many examples can be found in [156] and [132]. The occurrence of self-similar patterns is already apparent if we restrict to additive one-dimensional cellular automata. Actually the self-reproduction, which is the main idea behind the invention of cellular automata includes a kind of self-similarity [29]. For a mathematical rigorous understanding of the self-similar structure of the time evolution patterns for (one-dimensional) additive cellular automata starting from finite configurations, the most useful tool is *rescaling* or *renormalization* [153]. The self-similarity of one-dimensional additive cellular automata with values in the integers modulo prime powers was deciphered in [87, 143, 88] using tools from fractal geometry and iterated function systems (see [16] and [123]). We finally quote [15] which is a recent paper dealing with fractals generated by (non-necessarily additive) cellular automata.

9.3 Additive cellular automata and finite automata

Cellular automata are one class among the many different types of "automata" that are studied. Another class consists of the so-called *finite automata*. We are not going to give details about this class (see for example [94]); nevertheless, after briefly discussing the concept of *automatic sequences*, we will study relations between patterns generated by additive cellular automata and automatic sequences.

Definition 7

- Let A be an *alphabet* (finite set). The *free monoid* or *monoid of words* generated by A is the set of finite sequences with values in A (including the empty word denoted by ε) equipped with the *concatenation* law: the concatenation of the word $a_1 a_2 \ldots a_k$ with the word $b_1 b_2 \ldots b_\ell$ (where the letters a_i and b_j all belong to A), is by definition the word $(a_1 a_2 \ldots a_k).(b_1 b_2 \ldots b_\ell) := a_1 a_2 \ldots a_k b_1 b_2 \ldots b_\ell$. The free monoid generated by A is denoted by A^*. The *length* of a word is the number of letters it contains (and the length of the empty word is equal to 0).

- Let $d \geq 2$ be an integer, and let A be an *alphabet*. A *d-morphism* on A is a map φ that sends each element of A to a word of length d in A^*. This map is extended to A^* "morphically" (i.e., the image by φ of the word $a_1 a_2 \ldots a_k$ is by definition the word $\varphi(a_1).\varphi(a_2) \ldots \varphi(a_k)$). This map can be extended to infinite sequences $(u_n)_{n \geq 0}$ on A by concatenation: the image by φ of the sequence $u_0 u_1 \ldots$ is the sequence $\varphi(u_0)\varphi(u_1) \ldots$ (so that for example the first d letters of this sequence are the letters of the image of u_0 by φ).

- Let $d \geq 2$ be an integer, let A be an *alphabet*. Let φ be a d-morphism on A. A sequence $(u_n)_{n \geq 0}$ with values in A is called a *fixed point* of φ if it is equal to its image under φ.

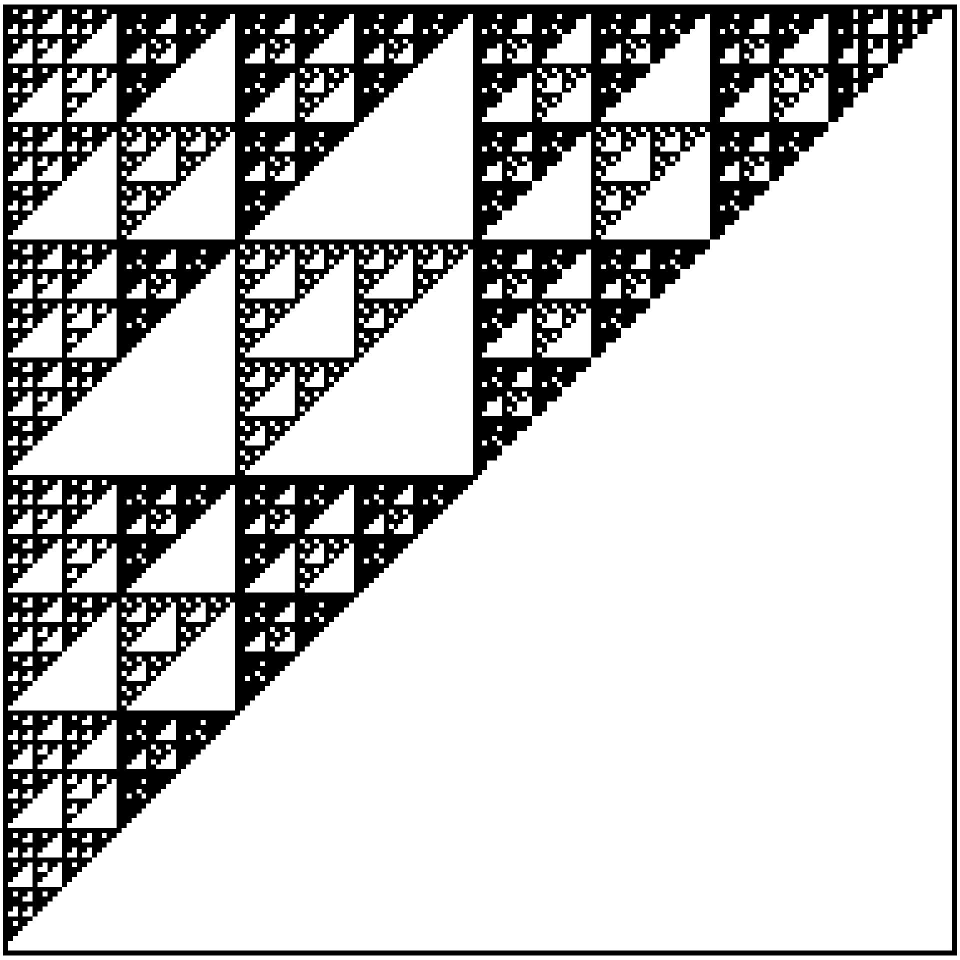

Figure 3: The Pascal triangle modulo 4

- Let $d \geq 2$ be an integer. A sequence $(u_n)_{n\geq 0}$ on the alphabet A is called d-automatic, if there exist an alphabet B, a sequence $(v_n)_{n\geq 0}$ with values in B, a d-morphism on B, and a map θ from B to A, such that the sequence $(v_n)_{n\geq 0}$ is a fixed point of φ, and for every $n \geq 0$ we have $u_n = \theta(v_n)$. (In other words the sequence $(u_n)_{n\geq 0}$ is the pointwise image of a fixed point of a d-morphism.)

Example 1

The celebrated Prouhet-Thue-Morse sequence (see for example [7] for a survey on this sequence) is the fixed point beginning with 0 of the 2-morphism defined on $\{0, 1\}$ by $\varphi(0) = 01$, $\varphi(1) = 10$, so that this sequence is equal to

$$0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,0\,\ldots$$

The Rudin-Shapiro sequence (see for example [5] and references therein) is the image by the map $a \to +1$, $b \to +1$, $c \to -1$, $d \to -1$ of the fixed point of the morphism φ defined on the alphabet $\{a, b, c, d\}$ by $\varphi(a) = ab$, $\varphi(b) = ac$, $\varphi(c) = db$, $\varphi(d) = dc$. This fixed point is equal to

$$a\ b\ a\ c\ a\ b\ d\ b\ a\ b\ a\ c\ d\ c\ \ldots$$

so that the Rudin-Shapiro sequence is equal to (replacing $+1$ by $+$ and -1 by $-$)

$$+\ +\ +\ -\ +\ +\ -\ +\ +\ +\ +\ -\ -\ -\ \ldots$$

Definition 8

- Let A be an alphabet, and let $d \geq 2$ be an integer. A $d \times d$ morphism on A is a map that sends each letter of A to a $d \times d$ matrix on A and that is extended "morphically", i.e., the image of a square matrix is obtained by replacing each of its entries by the corresponding $d \times d$ matrix (see example 2 below).
- Let $d \geq 2$ be an integer. A two-dimensional sequence $(u_{m,n})_{m,n\geq 0}$ is said to be d-automatic if it is the pointwise image of a two-dimensional sequence that is a fixed point of a $d \times d$ morphism.

Example 2

- Let A be the alphabet $\{0, 1\}$. Define the 2×2 morphism σ on A by

$$\sigma(0) = \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \qquad \sigma(1) = \begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix}$$

Then taking images under σ, we obtain successively:

$$1 \longrightarrow \begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix} \longrightarrow \begin{matrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{matrix} \longrightarrow \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{matrix} \longrightarrow \cdots$$

The readers can easily convince themselves that continuing the process generates a two-dimensional sequence that is a fixed point of σ (generalized to infinite two-dimensional sequences as in the one-dimensional case). This sequence is nothing but the Pascal triangle reduced modulo 2 and completed with 0's to make an infinite two-dimensional sequence. Furthermore, up to "renormalization" and up to replacing 1's by black boxes and 0's by white boxes, the infinite pattern we thus obtain is the familiar *Sierpinski triangle* (see e.g., [123]).

- Let now be τ the 2×2 morphism defined on the alphabet $\{a, b, c, d\}$ by

$$\tau(a) = \begin{matrix} a & a \\ a & a \end{matrix} \qquad \tau(b) = \begin{matrix} b & c \\ b & d \end{matrix}$$

$$\tau(c) = \begin{matrix} a & a \\ d & c \end{matrix} \qquad \tau(d) = \begin{matrix} b & c \\ c & b \end{matrix}$$

The patient readers will show easily that the fixed point of the morphism τ is the two-dimensional sequence

$$\begin{matrix} b & c & a & a & a & a & a & a & \dots \\ b & d & d & c & a & a & a & a & \dots \\ b & c & b & c & b & c & a & a & \dots \\ b & d & c & b & c & b & d & c & \dots \\ b & c & a & a & b & c & a & a & \dots \\ b & d & d & c & b & d & d & c & \dots \\ b & c & b & c & a & a & b & c & \dots \\ b & d & c & b & d & c & b & d & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

and that the 2-automatic sequence obtained as the pointwise image of this fixed point under the map θ defined by $\theta(a) = 0$, $\theta(b) = 1$, $\theta(c) = 0$, $\theta(d) = 1$ is the infinite two-dimensional sequence

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & \dots \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \dots \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

that can be viewed as the time evolution of the one-dimensional cellular automaton associated with the polynomial $(1 + X + X^2)$ over $\mathbb{Z}/2\mathbb{Z}$ with the initial condition $\dots\ 0\ 0\ 1\ 0\ 0\ \dots$. In other words the initial condition as a 1 at index 0 and 0's elsewhere, and the neighbors of a cell are the cell itself and the two cells immediately on its left. The new value of a cell is obtained by adding modulo 2 the values of its neighbors.

Having seen the above examples the reader will not be too surprised by the following theorem [3].

Theorem 13 *Let $R(X)$ and $A(X)$ be two polynomials in $\mathbb{Z}[X]$, and let m be an integer ≥ 2. Suppose $A(X)$ is such that there does not exist any prime number p that divides m such that $A(X) = 0$ modulo p. Consider the additive cellular automaton defined on $\mathbb{Z}/m\mathbb{Z}$ by the polynomial $R(X)$ modulo m. Then the following three cases are the only possible ones.*

- *There exist two different prime numbers p and q dividing m, such that the two polynomials $R(X)$ modulo p and $R(X)$ modulo q are not monomials. Then, the two-dimensional sequence generated by the additive cellular automaton, with initial condition $A(X)$ modulo m, is not k-automatic, for any $k \geq 2$.*
- *There exists one prime number p dividing m for which $R(X)$ modulo p is not a monomial, and for every other prime divisor q of m (if any), the polynomial $R(X)$ modulo q is a monomial. Then, the two-dimensional sequence generated by the additive cellular automaton, with initial condition $A(X)$ modulo m, is p^a-automatic, for every $a \geq 1$, and this sequence is not k-automatic for any $k \notin \{p^a;\ a \geq 1\}$.*
- *For every prime number p dividing m, the polynomial $R(X)$ modulo p is a monomial. Then, the two-dimensional sequence generated by the additive cellular automaton, with initial condition $A(X)$ modulo m, is k-automatic for every $k \geq 2$.*

Remark 6 This theorem contains the main result of [110, 4] that can be essentially formulated as follows (taking the initial condition equal to 1): *the patterns generated by the Pascal cellular automaton modulo $d \geq 2$ form a k-automatic two-dimensional sequence ($k \geq 2$) if and only if d and k are powers of a same prime number.*

10 Block-complexity of patterns generated by one-dimensional additive cellular automata

The *block-complexity* of a two-dimensional sequence is the function $(m, n) \to P(m, n)$ that counts the number of rectangular blocks of size (m, n) that occur inside the sequence. This function measures somehow "how complicated" the sequence is. The following result can be found in [2].

Theorem 14 *The block-complexity $P(m, n)$ of the Pascal triangle modulo $d \geq 2$ satisfies*

$$\exists A > 0,\ \exists B > 0,\ \forall m, n \geq 1,\ A \sup(m, n)^{2\omega(d)} \leq P(m, n) \leq B \sup(m, n)^{2\omega(d)},$$

where $\omega(d)$ is the number of different prime divisors of the integer d.

This theorem was generalized to all one-dimensional additive cellular automata in [20].

11 Examples of cellular automata in physics, biology and chemistry

One of the main interesting features of cellular automata in physical modelling is to display the relation between the two possible levels of description: the microscopic level based on simple interactions between units and the average or "ensemble" behavior of the whole system. For this reason, cellular automata are used to model some non-equilibrium macroscopic processes like diffusion, reaction-diffusion, pattern formation, hydrodynamic flows, phase transitions etc. Practitioners of computational cellular automta assert that their simulations of these processes are more physically enlightening than the ones based on non-linear partial differential equations. Several cellular-automata-machines have been developed for this purpose, giving real time visualization of processes. In 1981, T. Toffoli and N. Margolus created the first cellular automata machine [145, 146] for very fast and efficient computation of reversible mechanical processes. Later on, in 1987, A. Clouqueur and D. d'Humières created a second machine suitable for fluid mechanics [38]. Since then, several machines with parallel processing computation were proposed. We give below some of the main applications of cellular automata in natural sciences.

11.1 Hydrodynamic flows

Around 1985, cellular automata became a new tool for fluid simulations. They provide better understanding of microscopic and mesoscopic mechanisms of hydrodynamic flows. The fluid is represented as an ensemble of particles moving along the links of a lattice. The "Lattice Gas" model was first proposed by J. Hardy, O. de Pazzis and Y. Pomeau in 1976 [89], and was further studied in many papers. Let us briefly describe it. (For a survey see also [26].)

A system of particles of equal masses is moving, only one step at a time, along a discrete space represented by the square lattice $\mathbb{Z}^2$. Each site is either occupied or not. If the site is occupied, a particle has a unit velocity represented by a directed link toward one of the four possible directions $(1, 2, 3, 4)$ issued from the occupied site. We suppose there is no more than one particle per site for a given direction. The state of the automaton at each site is given by the 4-vector (x_1, x_2, x_3, x_4). Each component, taking the value 0 or 1, represents the absence or presence of a particle in the corresponding direction. The updating consists of moving each particle to the nearest neighbor along the links (propagation) and, if two particles are hitting on the same node, they are deflected by 90 degrees, this corresponds to a collision which satisfies the conservation of the total momentum.

The model provides in the macroscopic limit the Navier-Stokes equation. However, the stress-tensor in this model is not isotropic. For this purpose, U. Frisch, B. Hasslacher and Y. Pomeau [64, 65] used a triangular lattice with hexagonal symmetry. The new model allowed a larger family of collision rules making possible many hydrodynamical simulations (transport coefficients, decay of temporal velocity auto-correlations, long-time tails, etc.) [140]. It was also used to simulate the complicated interface between two different fluids in the Rayleigh-Taylor instability where a dense fluids invades a light one creating vortices which are not easy to observe in other simulations using partial differential equations [127]. The pattern of the interface between two fluids having different viscosities shows fingering which occurs when the dense viscous fluid fills a porous material.

Moreover, J.-P. Boon, D. Dab, R. Kapral and A. Lawniczak [27] extended lattice gas models to include chemical-like reactive processes between several species of particles and to simulate spirals wave-patterns in non-equilibrium reacting systems of the type of Belousov-Zaikin-Zhabotinsky.

More details on lattice gas cellular automata can be found in [128, 126]. For dimensions ≥ 3 the reader can look at [66].

11.2 Turbulence and space-time chaos

Based on the systematic numerical computations of patterns generated by cellular automata starting from random configurations, S. Wolfram advocated a cellular automata model of the turbulent behavior of a fluid [156]. The fluid is represented as an ensemble of macroscopic particles each of which is in turbulent or in laminar state. Using rules from (Wolfram)-class 3, one observes the propagation of turbulent regions in the laminar ones and the formation of complex self-similar geometry reminiscent of coherent structures of turbulent flows. In the same spirit, H. Chaté and P.

Manneville [35] developed a new class of models in D-dimensional lattices with $D \geq 3$ which shows up space-time chaos having regular statistical behavior: some macroscopic observables (like spatial averages) have deterministic temporal periodicity. They called it non-trivial collective behavior. Moreover, they also showed that periods become longer as the system runs through a class of rules, and when this periodicity operates sudden changes the system has scaling laws which are reminiscent of critical behavior of phase transitions.

The characterization of space-time chaotic evolutions is still a challenging question. It was shown [45, 46, 47] that the 1-dimensional permutative cellular automata exhibit an abundant family of unstable travelling waves, densely distributed in the space of configurations, and having arbitrarily large velocity of propagation and wavelength.

11.3 Ising spin dynamics and non-equilibrium processes

In 1984, G. Vichniac first studied cellular-automata-dynamics of the Ising model [148] where he observed that the model did not correctly thermalize. In 1986, M. Creutz [48] proposed deterministic cellular automata which simulate more correctly the Ising model. The procedure allows to study non-equilibrium phenomena, heat flow, mixing and time correlations. O. Parodi and H. Ottavi showed that the Creutz model freezes at low temperature and they proposed improved versions [122]. On the other hand, growth processes of the domain of one phase into another phase in quenched spin systems was successfully described by cellular automata [52, 58].

Cellular automata algorithms which simulate the nonlinear diffusion of many particles have been developed by B. Chopard and M. Droz [36] who also studied the fractal properties of the diffusion front. These simulations are related to the family of lattice gases. From a different point of view, S. Takesue studied simple models of reversible cellular automata in which he derived a Boltzmann-type equation and heat conduction [144].

A famous toy model of non-equilibrium statistical mechanics, the ring Kac model [106], is in fact a cellular automaton. It was introduced by M. Kac in order to understand the relations between deterministic microscopic descriptions like the Liouville equation and stochastic descriptions like the Master equation. This model was extended and modified to be adapted to less simple microscopic interactions by several authors (see [44] and the references therein). Although too simple in comparison with hard spheres and other kinetic systems, it can explain some ingredients of the transition from reversible to irreversible processes.

11.4 Solitons and integrability

Cellular automata exhibiting properties of soliton systems, i.e., coherent particle-like structures (solitary waves) were found by J. K. Park, K. Steiglitz, and W. P. Thurston in 1986 (see [121], see also [63, 24]). Some of these structures propagate with fixed velocity and retain their identities after a collision. There were also many works by M. Bruschi, A. S. Fokas, C. R. Gilson, B. Grammaticos, A. Ramani and others on integrable cellular automata, i.e., automata with many constants of motion exhibiting some coherent structures like localized particles in space, wave-like patterns or solitons (see [63, 28] and references therein).

11.5 Pattern formation

It is well known that systems submitted to non-equilibrium constraints, like constant energy or matter flows, generate macroscopic spatial structures (e.g., spiral waves) in steady or oscillatory states. These ubiquitous systems are called dissipative structures or excitable media. They are the subject of experimental works and are generally described by nonlinear partial differential equations. As for fluids, cellular-automata-models for the formation of patterns of dissipative structures are widely used in order to generate patterns similar to the ones that are experimentally observed [82, 70, 135, 93, 115, 101, 150, 151].

11.6 Biology

There are so many applications of cellular automata in biology and ecology that we can only mention some examples. M. P. Hassell, H. N. Comins, and R. M. May [90, 40] modelled host-parasitoid discrete systems by 2-dimensional cellular automata in which each individual has one of 3 possible states characterizing the local density. The system has stable patterns with various complex textures of three types: stationary patterns, spiral waves and spatial disorder. The effect of ecological interactions and habitat patterns were also modelled by cellular automata (see [31, 57]). Several authors studied epidemic propagation cellular automata models [6, 131, 137, 129]. Finally, cellular automata were also used to model the activity of neural networks [95, 96].

12 Conclusion

It was of course impossible in this short article to address all results about cellular automata neither to describe all possible applications. Having to make choices, we omitted many interesting papers or books. We did not mention for example stochastic cellular automata (see for example [51]), threshold cellular automata (see for example [78]), sandpile models (see [13, 12], see also [73, 74, 43]), applications of cellular automata to modeling traffic (see for example [67, 25], see also [37]), ...

We end this paper by quoting the following papers, books or collective books where the reader will find much more material and more references: [23, 126, 34, 36, 53, 60, 86, 92, 113, 128, 156]. Many issues of the journals *Physica D* and *Complex Systems* also contain papers on cellular automata.

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