Schrödinger operators with Rudin-Shapiro potentials are not palindromic

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Abstract We prove a conjecture of A. Hof, O. Knill and B. Simon by showing that the Rudin-Shapiro sequence is not *palindromic*, i.e., does not contain arbitrarily long palindromes. We prove actually this property for all paperfolding sequences and all Rudin-Shapiro sequences deduced from paperfolding sequences. As a consequence and as guessed by the above authors, their method cannot be used for establishing that discrete Schrödinger operators with Rudin-Shapiro potentials have a purely singular continuous spectrum.

I. Introduction

In [1] the authors study the spectrum of discrete Schrödinger operators with potentials taking values in a finite set of real numbers. They prove *inter alia* the following: *suppose that the sequence of potentials generates a subshift X that is strictly ergodic (i.e., both minimal and uniquely ergodic). Then, a sufficient condition ensuring the existence of a generic set in X for which the operator has a purely singular continuous spectrum, is that there exists a z in X that is* palindromic, *i.e. that contains arbitrarily long palindromes.*

This result covers many cases where X is generated by the fixed point of a primitive substitution (Fibonacci, Prouhet-Thue-Morse, period-doubling, binary and ternary non-Pisot) and also the case of sequences defined by circle maps. As stated by the authors, their results cover almost all, but not all, results in [2], and give also new results.

On the other hand the results in [2] do not cover the case of the Rudin-Shapiro substitutions: as the authors say in [2] "this sequence retains its mystery". The authors of [1] conjecture that the Rudin-Shapiro sequence does not contain arbitrarily long palindromes and hence that their method cannot be applied to this sequence.

The purpose of this note is to prove this conjecture: *if u is a Rudin-Shapiro sequence in the sense of [3], then it does not contain palindromes of length ≥ 15. If u is a paperfolding sequence, then it does not contain palindromes of length ≥ 14.*

II. Discrete Schrödinger operators

Given a sequence $(x_n)_n$ of potentials taking finitely many real values, one can define a (tight-binding) Schrödinger operator L on $\ell^2(\mathbb{Z})$ by

$$L(x) : (u_n)_n \longrightarrow ((L(x)u)_n)_n, \text{ with } (L(x)u)_n = u_{n+1} + u_{n-1} + x_n u_n.$$

Roughly speaking, properties of the sequence $(x_n)_n$ imply properties of the operator $L(x)$. An interesting case is when the sequence $(x_n)_n$ lies between order and disorder, in particular when $(x_n)_n$ is generated by a (primitive) substitution (also called morphism or inflation rule). A first example, i.e., the Fibonacci sequence, has been treated in [4] and [5]. Then the Thue-Morse sequence has been studied first in [6, 7, 8], then, as quoted in [1], in [9]. A nice and rather complete survey on these Schrödinger operators, with many references, in particular for the substitutive case, is [10].

Let us summarize quickly the results of [1]. The authors use essentially three steps. They first show that a minimal set, with only aperiodic elements, containing a palindromic sequence (therefore containing only palindromic sequences) contains uncontably many strongly palindromic sequences: the sequence x is called *palindromic* if it contains arbitrarily long palindromes; it is called *strongly palindromic* if, for any positive constant B, it contains infinitely many palindromes $w_1, w_2, \ldots$, such that w_i has length l_i, is centered at m_i, with $\lim_i m_i = +\infty$ and $\lim_i \frac{e^{Bm_i}}{l_i} = 0$. The authors then adapt a result of Jitomirskaya and Simon [11] to prove that $L(x)$ has no eigenvalue if x is a strongly palindromic sequence. Their last step uses a result of Kotani [12] to show that, if a compact shift-invariant set X of sequences is strictly ergodic (i.e., minimal and uniquely ergodic), and if there is an $x \in X$ such that $L(x)$ has no eigenvalue, then, there exists a generic subset Y of X such that, for every $y \in Y$, the spectrum of $L(y)$ is purely singular continuous.

The results apply to well-known substitutions like Fibonacci, Thue-Morse, as well as the period-doubling sequence and binary and ternary non-Pisot sequences. They also apply to circle maps corresponding to any irrational angle and any half-open interval.

One might want to compare these results with those of [2]. The period-doubling and binary non-Pisot sequences are not obtained in [2]. Note also that all examples of substitutional sequences given in [2], but one, are covered by the methods of [1]. Recall that the method of Bovier and Ghez in [2] uses the *reduced trace map* and *semi-primitive* sequences. We will not describe their method here, but rather note that they need their sequences to contain *squares*, i.e., identical consecutive blocks. In other words a purely combinatorial property of the sequence of potentials, (to contain squares or to contain palindromes), implies results on the spectra.

The case of potentials given by the Rudin-Shapiro sequence is not covered by any of these two papers. It results from [1] that the corresponding spectrum has no absolutely continuous part, as the underlying substitution is primitive. But it is not known whether this spectrum is purely continuous. In [1] the authors conjecture that the Rudin-Shapiro sequence is not palindromic, hence that their method does not apply. We will prove here this conjecture. More precisely we prove that the Rudin-Shapiro sequence, but also the generalized Rudin-Shapiro sequences in the sense of [3], and the paperfolding sequences (see [13]), are not palindromic.

III. The Rudin-Shapiro sequence

Originally constructed by Shapiro [14] then by Rudin [15], the Rudin-Shapiro sequence can be defined by $v_n = (-1)^{a(n)}$, where $a(n)$ is the number of (possibly overlapping) blocks 11 in the binary expansion of n. This sequence has the unexpected property [14, 15]:

$$\sup_{\theta \in [0,1[} \left| \sum_{n=0}^{N-1} v_n e^{2i\pi n\theta} \right| \leq C\sqrt{N}.$$

Note also that the spectral measure of the Rudin-Shapiro sequence is the Lebesgue measure, although this sequence is deterministic. We recall that

- for any sequence $(c_n)_n$, one has:

$$\sqrt{N} = \left\|\sum_{n=0}^{N-1} c_n e^{2i\pi n\theta}\right\|_2 \leq \left\|\sum_{n=0}^{N-1} c_n e^{2i\pi n\theta}\right\|_\infty = \sup_{\theta\in[0,1[}\left|\sum_{n=0}^{N-1} c_n e^{2i\pi n\theta}\right| \leq N;$$

- if $(c_n)_n$ is a periodic (or even an almost-periodic) sequence, then

$$\sup_{\theta\in[0,1[}\left|\sum_{n=0}^{N-1} c_n e^{2i\pi n\theta}\right| \asymp N;$$

- for almost all (for the Lebesgue measure) sequences $(c_n)_n$, one has

$$\sup_{\theta\in[0,1[}\left|\sum_{n=0}^{N-1} c_n e^{2i\pi n\theta}\right| \leq \sqrt{N\log N}.$$

  Furthermore the spectral measure of almost all sequences is equal to the Lebesgue measure.

Another way of generating the Rudin-Shapiro sequence is the following. Consider the alphabet $\{A, B, C, D\}$ and the substitution σ defined on this alphabet by

$$\begin{array}{rcl} \sigma(A) & = & AB, \\ \sigma(B) & = & AC, \\ \sigma(C) & = & DB, \\ \sigma(D) & = & DC. \end{array}$$

Define then the map $\varphi : \{A, B, C, D\} \longrightarrow \{-1, +1\}$ by:

$$\begin{array}{rcl} \varphi(A) & = & +1, \\ \varphi(B) & = & +1, \\ \varphi(C) & = & -1, \\ \varphi(D) & = & -1. \end{array}$$

Then the Rudin-Shapiro sequence is the limit, as k goes to infinity, of $\varphi(\sigma^k(A))$. The sequence

$$\lim_{k\to\infty} \sigma^k(A) = A\ B\ A\ C\ A\ B\ D\ B\ A\ B\ A\ C\ D\ C\ \ldots$$

is sometimes called the Rudin-Shapiro sequence on four letters, whereas the sequence

$$\lim_{k\to\infty} \varphi(\sigma^k(A)) = +1\quad +1\quad +1\quad -1\quad +1\quad +1\quad -1\quad +1\quad +1\quad +1\quad +1\quad -1\quad -1\quad -1\quad \ldots$$

is simply called the Rudin-Shapiro sequence. In what follows we "change the names" of the letters by considering sequences with values in $\{0, 1\}$: we replace $+1$'s by 0's and -1's by 1's. Hence the Rudin-Shapiro sequence is the sequence

$$0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ \ldots$$

Of course this change of letters transforms palindromes into palindromes and conversely.

IV. Paperfolding and generalized Rudin-Shapiro sequences

Paperfolding sequences are obtained by iterating an infinite number of times the operation of folding a piece of paper on itself. The interested reader can find in [13] and its bibliography more details on paperfolding, he can also read [16]. We give here a formal definition, (note that the indices of the paperfolding sequences will be taken in the set $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ although all other sequences in this paper have their indices in $\mathbb{N}$). *Furthermore, from now on, all the sequences we consider have their values in* $\{0,1\}$.

Definition *A sequence $(z_n)_{n\geq 1}$ with values in $\{0,1\}$ is called a paperfolding sequence if*

- $\exists i_0 \in \{0,1\}$, $\forall n \in \mathbb{N}$, $z_{2n+1} \equiv i_0 + n \pmod 2$.
- *the sequence $(z_{2n})_n$ is a paperfolding sequence.*

In other words a paperfolding sequence is characterized by its "folding instructions" $i_m = z_{2^m}$. Conversely, given a sequence $(i_m)_m \in \{0,1\}^{\mathbb{N}}$, one can define a paperfolding sequence $(z_n)_n$ by

$$\forall m \geq 0,\ \forall k \geq 0,\ z_{2^m(2k+1)} \equiv i_m + k \pmod 2.$$

To a paperfolding sequence $(z_m)_{m\geq 1}$ one can associate a "generalized" Rudin-Shapiro sequence $(v_n)_{n\geq 0}$ by

$$v_0 = 0,\quad \forall n \geq 0,\ v_n = \sum_{k=1}^{n} z_k \pmod 2.$$

One checks easily that, conversely,

$$\forall n \geq 1,\ z_n = v_n + v_{n-1} \pmod 2,$$

thus the correspondence between z and v is bijective.

These sequences $(v_n)_{n\geq 0}$ have been studied in [3] and they have the "Rudin-Shapiro property"

$$\sup_{\theta\in[0,1[} \left| \sum_{n=0}^{N-1} (-1)^{v_n} e^{2i\pi n\theta} \right| \leq C_v\sqrt{N}.$$

The spectral measure of a generalized Rudin-Shapiro sequence is also the Lebesgue measure. Furthermore the usual (two-letter) Rudin-Shapiro sequence corresponds to the sequence $(i_m)_m$ defined by:

$$i_0 = 0;\ \forall m \geq 1,\ i_m \equiv m+1 \pmod 2.$$

The relations between the paperfolding sequences and the generalized Rudin-Shapiro sequences permit to deduce properties of the factors (blocks) of the generalized Rudin-Shapiro sequences from properties of the factors of paperfolding sequences [17, 18, 19]. This remark will be used in the next paragraph.

V. The main theorem

We prove in this paragraph that, if a sequence is either a paperfolding sequence or a generalized Rudin-Shapiro sequence, then it does not contain arbitrarily long palindromes.

Theorem

A paperfolding sequence does not contain palindromes of length ≥ 14.

A generalized Rudin-Shapiro sequence does not contain palindromes of length ≥ 15.

Proof

The proof will consist of three steps: we first show that it suffices to prove the result for the paperfolding sequences. Second we study the palindromes of even length in a paperfolding sequence. The last step studies the palindromes of odd length in a paperfolding sequence.

- *First step*

Let $(v_n)_{n\geq 0}$ be a generalized Rudin-Shapiro sequence and let $(z_n)_{n\geq 1}$ be the associated paperfolding sequence ($z_n = v_n + v_{n-1}$ modulo 2). Suppose that the sequence v contains a palindrome of length > 1. If this palindrome has even length, say $2d$, with $d \geq 1$, then there exist an index $n \geq 0$ and letters $a_1, \ldots, a_d$ in $\{0, 1\}$ such that

$$v_n v_{n+1} \ldots v_{n+d-1} v_{n+d} \ldots v_{n+2d-1} = a_1 a_2 \ldots a_{d-1} a_d a_d a_{d-1} \ldots a_2 a_1.$$

Hence, using the relation between the sequences v and z, one has:

$$z_{n+1} \ldots z_{n+d-1} z_{n+d} z_{n+d+1} \ldots z_{n+2d-1} = a_2 + a_1 \ \ldots \ a_d + a_{d-1} \ 0 \ a_{d-1} + a_d \ \ldots \ a_1 + a_2,$$

which gives a palindrome of length $2d - 1$ for the sequence z.

If the palindrome in v has odd length, say $2d + 1$ with $d \geq 1$, then there exist an index $n \geq 0$ and letters $a_1, \ldots, a_d, a_{d+1}$ in $\{0, 1\}$ such that

$$v_n v_{n+1} \ldots v_{n+d} v_{n+d+1} v_{n+d+2} \ldots v_{n+2d} = a_1 a_2 \ldots a_{d-1} a_d a_{d+1} a_d a_{d-1} \ldots a_2 a_1.$$

Hence, using the relation between the sequences v and z, one has:

$$z_{n+1} \ldots z_{n+d} z_{n+d+1} \ldots z_{n+2d} = a_2 + a_1 \ \ldots \ a_{d+1} + a_d \ a_d + a_{d+1} \ \ldots \ a_1 + a_2,$$

which gives a palindrome of length $2d$ for the sequence z.

This first step proves that it suffices to study the palindromes in the paperfolding sequences.

- *Second step*

Suppose that a paperfolding sequence, say $(z_n)_{n\geq 1}$, has a palindrome of even length ≥ 8. Then this sequence contains necessarily a palindrome of length 8 that is a "subpalindrome" of the previous one. Hence there exist an index $n \geq 1$ and letters a, b, c, d in $\{0, 1\}$ such that

$$z_n z_{n+1} \ldots z_{n+7} = a \ b \ c \ d \ d \ c \ b \ a.$$

But we know from the definition of a paperfolding sequence that either

$$a \ c \ d \ b = a \ \overline{a} \ a \ \overline{a},$$

if n is odd or

$$b\ d\ c\ a = b\ \overline{b}\ b\ \overline{b},$$

if n is even, where we define $\overline{0} = 1$ and $\overline{1} = 0$.

In the first case one has

$$a\ b\ c\ d\ d\ c\ b\ a = a\ \overline{a}\ \overline{a}\ a\ a\ \overline{a}\ \overline{a}\ a$$

and in the second case

$$a\ b\ c\ d\ d\ c\ b\ a = \overline{b}\ b\ b\ \overline{b}\ \overline{b}\ b\ b\ \overline{b}.$$

So in both cases we have a square of length 2×4 in a paperfolding sequence, which contradicts Corollary 5.2 of [19].

- *Third step*

Suppose now that the paperfolding sequence $(z_n)_{n\geq 1}$ contains a palindrome of odd length, say $2d+1$. Then there exist an index n and letters $a_1, \ldots, a_d, a_{d+1}$ in $\{0,1\}$ such that

$$z_n \ldots z_{n+d-1} z_{n+d} z_{n+d+1} \ldots z_{n+2d} = a_1 \ldots a_d a_{d+1} a_d \ldots a_1.$$

As $z_{n+d-1} = z_{n+d+1} = a_d$, the integer $n+d+1$ cannot be odd, hence it is even (from the definition of a paperfolding sequence the subsequence for the odd indices is an alternating sequence). So the word

$$\ldots z_{n+d-3} z_{n+d-1} z_{n+d+1} z_{n+d+3} \ldots = \ldots a_{d-2} a_d a_d a_{d+2} \ldots$$

is a factor of a paperfolding sequence (definition of a paperfolding sequence). This gives two cases according to the parity of d.

If d is odd, say $d = 2e+1$, then

$$\ldots z_{n+d-3} z_{n+d-1} z_{n+d+1} z_{n+d+3} \ldots = a_1 a_3 \ldots a_{2e+1} a_{2e+1} \ldots a_3 a_1$$

is a factor of a paperfolding sequence. But this is a palindrome of length $2e+2$. From Step 2 we know that $2e+2 \leq 6$, hence $2d+1 = 4e+3 \leq 11$.

If d is even, say $d = 2e$, then

$$\ldots z_{n+d-3} z_{n+d-1} z_{n+d+1} z_{n+d+3} \ldots = a_2 a_4 \ldots a_{2e} a_{2e} \ldots a_4 a_2$$

is a factor of a paperfolding sequence. But this is a palindrome of length $2e$. From Step 2 we know that $2e \leq 6$, hence $2d+1 = 4e+1 \leq 13$ and the theorem is proved.

Remark

What we proved actually is that the only possible lengths of palindromes for a paperfolding sequence are in the set: $\{1, 2, 3, 4, 5, 6, 7, 9, 11, 13\}$. Hence the only possible lengths for a generalized Rudin-Shapiro sequence are in the set: $\{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14\}$.

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