Generalized Perturbed Symmetry

Jean-Paul Allouche
CNRS, LRI
Bâtiment 490
F-91405 Orsay Cedex
France
allouche@lri.fr

Jeffrey Shallit*
Department of Computer Science
University of Waterloo
Waterloo, Ontario N2L 3G1
Canada
shallit@graceland.uwaterloo.ca

Abstract

Let w^R denote the reversal of the word w, and let $w^E = w$. We examine a generalization of Mendès France's "perturbed symmetry". Specifically, we study a map of the form

$$S(w) = w \prod_{1 \leq i \leq k-1} x_i \, w^{e_i},$$

where $e_i \in \{E, R\}$, and the infinite word arising from iteration of this map. In particular, if the x_i are fixed strings of identical length, the resulting infinite word is k-automatic, and we characterize precisely when such a word can be ultimately periodic. Finally, we apply our technique to prove a 1994 conjecture of Blanchard and Fabre.

1 Introduction

Mendès France [9] introduced a map on strings of symbols which he called a *perturbed symmetry*; also see [3, 7, 10, 14].

Let Σ be a finite alphabet and let $w, x \in \Sigma^*$. A perturbed symmetry is the map $S_x(w) = w\, x\, w^R$, where w^R denotes the symbols of w taken in reverse order. Since $S_x(w)$ starts with the symbols of w, we may iterate the map $S_x(w)$ to obtain the infinite word

$$S_x^\infty(w) = w\, x\, w^R\, x\, w\, x^R\, w^R \cdots .$$

As Mendès France pointed out [9], this infinite word has the property that it is periodic if and only if $x = x^R$. For a proof, see [14].

In this note, we generalize the notion of perturbed symmetry, and prove that the infinite word corresponding to this perturbed symmetry is generated by a finite automaton. Next, we consider the question of when the resulting infinite word is periodic. Finally, we apply our technique to prove a recent conjecture of Blanchard and Fabre.

*Research supported in part by a grant from NSERC.

2 Perturbed symmetry generates an automatic sequence

Let w^R denote the symbols of w taken in reverse order, and $w^E = w$, so that $R \circ R = E$, $E \circ E = E$, and $E \circ R = R \circ E = R$. Let k be a fixed integer ≥ 2, let $x_1, x_2, \ldots, x_{k-1}$ be fixed strings, each of the same length s, and let $e_1, e_2, \ldots, e_{k-1} \in \{E, R\}$. Then we define a *generalized perturbed symmetry* to be the map

$$S(w) = w \prod_{1 \leq i \leq k-1} x_i w^{e_i}.$$

Since $S(w)$ starts with w, we may iterate the map S to obtain an infinite word $S^\infty(w)$.

Recall that if a word is the image (under a coding) of the fixed point of a k-uniform homomorphism, it is said to be k-automatic.

Theorem 1 *The infinite word $S^\infty(w)$ is k-automatic.*

Proof. We first introduce an alphabet Δ of $4k-4$ new symbols,

$$\Delta = \{A_1, \ldots, A_{k-1}, B_1, \ldots, B_{k-1}, C_1, \ldots, C_{k-1}, D_1, \ldots, D_{k-1}\}.$$

We then define

$$X_j^{(i)} = \begin{cases} A_j & \text{if } e_i = E; \\ B_j, & \text{if } e_i = R, \end{cases}$$

$$Y_j^{(i)} = \begin{cases} C_j & \text{if } e_i = E; \\ D_j, & \text{if } e_i = R, \end{cases}$$

$$Z_j^{(i)} = \begin{cases} D_j & \text{if } e_i = E; \\ C_j, & \text{if } e_i = R. \end{cases}$$

Next we define the homomorphism σ on the alphabet Δ by

$$\begin{array}{rcl} \sigma(A_j) & = & A_1 \, X_2^{(1)} \, X_3^{(2)} \cdots X_{k-1}^{(k-2)} \, X_j^{(k-1)} \\ \sigma(B_j) & = & Z_{k-1}^{(k-1)} \cdots Z_1^{(1)} \, B_j \\ \sigma(C_j) & = & A_1 \, X_2^{(1)} \, X_3^{(2)} \cdots X_{k-1}^{(k-2)} \, Y_j^{(k-1)} \\ \sigma(D_j) & = & Z_{k-1}^{(k-1)} \cdots Z_1^{(1)} \, D_j \end{array}$$

for $1 \leq j \leq k-1$. Note that this homomorphism has constant length k.

Finally, we define the homomorphism τ from Δ to Σ^* by

$$\begin{array}{rcl} \tau(A_j) & = & w x_j \\ \tau(B_j) & = & w^R x_j \\ \tau(C_j) & = & w x_j^R \\ \tau(D_j) & = & w^R x_j^R \end{array}$$

for $1 \leq j \leq k-1$. These definitions immediately yield

$$\begin{array}{rcl} \tau(X_j^{(i)}) & = & w^{e_i} \, x_j; \\ \tau(Y_j^{(i)}) & = & w^{e_i} \, x_j^R; \\ \tau(Z_j^{(i)}) & = & (w^{e_i})^R \, x_j^R. \end{array}$$

Now an easy simultaneous induction shows that, for every $n \geq 1$, we have

$$\begin{array}{rcl}
\tau(\sigma^n(A_j)) & = & S^n(w)\, x_j \\
\tau(\sigma^n(B_j)) & = & (S^n(w))^R\, x_j \\
\tau(\sigma^n(C_j)) & = & S^n(w)\, x_j^R \\
\tau(\sigma^n(D_j)) & = & (S^n(w))^R\, x_j^R \\
\tau(\sigma^n(X_j^{(i)})) & = & (S^n(w))^{e_i}\, x_j \\
\tau(\sigma^n(Y_j^{(i)})) & = & (S^n(w))^{e_i}\, x_j^R \\
\tau(\sigma^n(Z_j^{(i)})) & = & (S^n(w)^{e_i})^R\, x_j^R.
\end{array}$$

We do one case of the induction proof, and leave the rest to the reader. We have

$$\begin{array}{rcl}
\tau(\sigma(A_j)) & = & \tau(A_1\, X_2^{(1)}\, X_3^{(2)} \cdots X_{k-1}^{(k-2)}\, X_j^{(k-1)}) \\
 & = & w\, x_1\, w^{e_1}\, x_2\, w^{e_2}\, x_3 \cdots w^{e_{k-2}}\, x_{k-1}\, w^{e_{k-1}}\, x_j \\
 & = & S(w)\, x_j,
\end{array}$$

and

$$\begin{array}{rcl}
\tau(\sigma^{n+1}(A_j)) & = & \tau(\sigma^n(\sigma(A_j))) \\
 & = & \tau(\sigma^n(A_1\, X_2^{(1)}\, X_3^{(2)} \cdots X_{k-1}^{(k-2)}\, X_j^{(k-1)})) \\
 & = & S^n(w)\, x_1\, (S^n(w))^{e_1}\, x_2\, (S^n(w))^{e_2}\, x_3 \cdots (S^n(w))^{e_{k-2}}\, x_{k-1}\, (S^n(w))^{e_{k-1}}\, x_j \\
 & = & S(S^n(w))\, x_j \\
 & = & S^{n+1}(w)\, x_j.
\end{array}$$

Hence letting $n \to \infty$, we obtain

$$S^\infty(w) = \tau(\sigma^\infty(A_1)).$$

Clearly $\sigma^\infty(A_1)$ is k-automatic, and since the k-automatic sequences are closed under q-block substitution [4, §5], we conclude that $S^\infty(w) = \tau(\sigma^\infty(A_1))$ is also k-automatic. ■

Example.

Let $n \geq 3$ be an integer and let $f(n) = \sum_{k \geq 0} n^{-2^k}$. Then it is shown in [3, 14] that the continued fraction expansion of $f(n)$ is given by

$$[0,\ n-1,\ S^\infty(v)]$$

where $v = (n+2,\ n)$ and $S(w) = (w,\ n,\ n-2,\ w^R)$. Hence we find

$$f(n) = [0,\ n-1,\ n+2,\ n,\ n,\ n-2,\ n,\ n+2,\ n,\ n-2,\ n+2,\ n,\ n-2,\ n,\ n,\ n+2,\ \ldots].$$

The map S is a generalized perturbed symmetry with $k = 2$, $x_1 = (n+2, n)$, and $e_1 = R$. It now follows from Theorem 1 that if $f(n) = [a_0, a_1, a_2, \ldots]$, then $(a_i)_{i \geq 0}$ is a 2-automatic sequence. In fact, it is generated by the automaton in Figure 1. This automaton takes as input the base-2 representation of j (starting with the *most* significant digit) and outputs a_j.

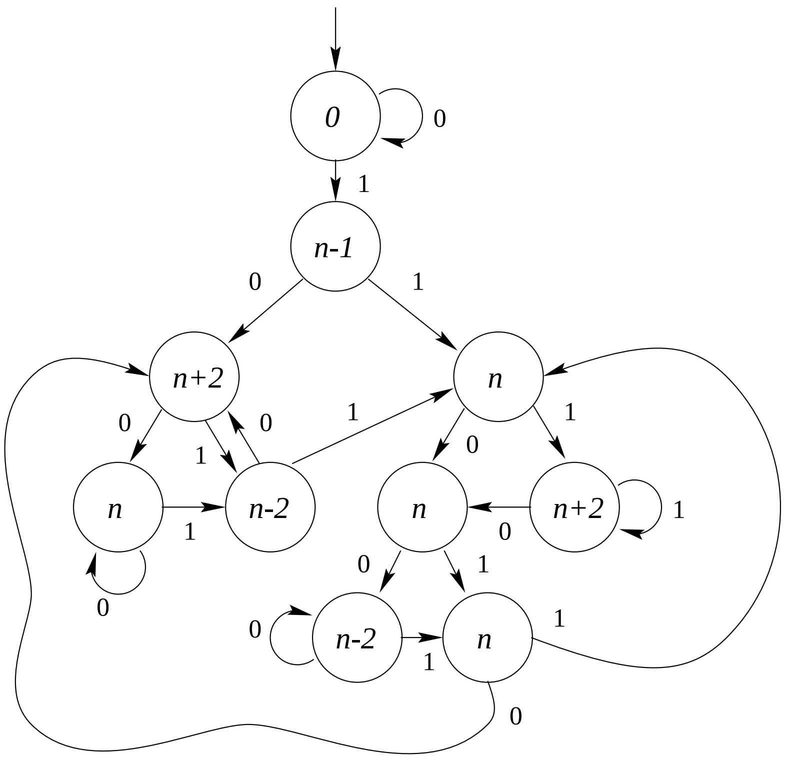

Figure 1: Automaton generating the continued fraction for $f(n)$

Remark.

The technique used above applies equally well to any involution in place of w^R. Hence our method also applies to the generalized paperfolding sequences studied by Razafy Andriamampianina [11, 12, 13], Bercoff [1], and Koskas [8]. Also see Wen and Wen [15]; Cohn [5].

3 Periodicity

We now address the question of the possible periodicity of a sequence obtained by generalized perturbed symmetry. A first remark is that $S^\infty(w) = S^\infty(S^t(w))$ for any $t \geq 1$, which easily implies that any word that occurs in the sequence $S^\infty(w)$ occurs infinitely often and with bounded gaps. In particular if the sequence $S^\infty(w)$ is ultimately periodic, then it is purely periodic.

Recall that a homomorphism is said to be *primitive* if there is an iterate of it having the property that the image of each letter contains, at least once, every letter of the alphabet on which the homomorphism is defined. Our main tool is the following result due to Dekking [6].

Theorem 2 ([6, p. 226]) *Let θ be a primitive homomorphism of constant length k on an alphabet $\mathcal{A}$, and let $W = (W_n)_{n \geq 0}$ be a sequence that is a fixed point of θ. Define $h(\theta)$, the* *height of θ, by*

$$h(\theta) = \max\{n \geq 1 : \ \gcd(n,k) = 1, \ n | \gcd\{a : \ W_a = W_0\}\}.$$

If θ is one-to-one, then W is periodic if and only if $h(\theta) = \#\mathcal{A}$.

To study the periodicity of the sequence $S^\infty(w)$, we first notice that the sequence $S^\infty(w) = \tau(\sigma^\infty(A_1))$ is of the form $w_1 z_1 w_2 z_2 \cdots$, where $w_i \in \{w, w^R\}$ and $z_i \in \bigcup_{1 \leq j \leq k-1} \{x_j, x_j^R\}$. We introduce two sequences U and V defined as follows:

(i) the sequence $U = w_1 w_2 \cdots$ is obtained from $S^\infty(w)$ by erasing the z_i's, i.e., is obtained by keeping only the w_i's;

(ii) the sequence $V = z_1 z_2 \cdots$ is obtained from $S^\infty(w)$ by erasing the w_i's, i.e., is obtained by keeping only the z_i's.

We define the homomorphisms λ and μ on the alphabet Δ as follows:

$$\begin{aligned} \lambda(A_j) &= w \\ \lambda(B_j) &= w^R \\ \lambda(C_j) &= w \\ \lambda(D_j) &= w^R, \end{aligned}$$

and

$$\begin{aligned} \mu(A_j) &= x_j \\ \mu(B_j) &= x_j \\ \mu(C_j) &= x_j^R \\ \mu(D_j) &= x_j^R, \end{aligned}$$

for $1 \leq j \leq k-1$. Then

$$U = \lambda(\sigma^\infty(A_1)) \quad \text{and} \quad V = \mu(\sigma^\infty(A_1)).$$

Note that, since $|w_i|$ is a constant independent of i, and similarly for $|z_i|$, the sequence $S^\infty(w)$ is periodic if and only if the sequences U and V are both periodic.

Lemma 3 *The sequence U is periodic if and only if at least one of the following conditions holds:*

(i) the word w is a palindrome (i.e., $w = w^R$);

(ii) all the e_i's are equal to E;

(iii) k is even, and either $(e_1, e_2, \ldots, e_{k-1}) = (ER)^{\frac{k-2}{2}} R$ or $(e_1, e_2, \ldots, e_{k-1}) = (RE)^{\frac{k-2}{2}} R$;

(iv) k is odd, and $(e_1, e_2, \ldots, e_{k-1}) = (RE)^{\frac{k-1}{2}}$.

Proof. It is quite clear that, if any one of the four conditions above holds, then the sequence U is purely periodic. More precisely U equals $www\cdots$ in cases (i) and (ii), $(w(ww^R)^{\frac{k-2}{2}} w^R)^\infty$ or $(ww^R)^\infty$ in case (iii), and $(ww^R)^\infty$ in case (iv).

Now suppose that the sequence U is periodic. Define the homomorphism σ_1 on $\{E, R\}$ by

$$\begin{aligned} \sigma_1(E) &= E e_1 \cdots e_{k-1} \\ \sigma_1(R) &= (R \circ e_{k-1})(R \circ e_{k-2}) \cdots (R \circ e_1) R. \end{aligned}$$

Note that this homomorphism has constant length k. Furthermore, defining the homomorphism φ on the alphabet Δ by

$$\begin{array}{rclc} \varphi(A_j) & = & \varphi(C_j) & = E \\ \varphi(B_j) & = & \varphi(D_j) & = R, \end{array}$$

for $1 \leq j \leq k-1$, we see that

$$\begin{array}{ll} \varphi(X_j^{(i)}) & = e_i \\ \varphi(Y_j^{(i)}) & = e_i \\ \varphi(Z_j^{(i)}) & = R \circ e_i \end{array}$$

for $1 \leq j \leq k-1$. Hence we easily obtain

$$\sigma_1 \varphi = \varphi \sigma.$$

On the other hand, we have that $\sigma^\infty(A_1)$ is a fixed point of σ, hence $Z := \varphi(\sigma^\infty(A_1))$ is a fixed point of σ_1, since

$$\sigma_1(\varphi(\sigma^\infty(A_1))) = \varphi(\sigma(\sigma^\infty(A_1))) = \varphi(\sigma^\infty(A_1)).$$

Now we look again at the homomorphism λ and the equality $U = \lambda(\sigma^\infty(A_1))$. We claim that if the sequence U is periodic, then either (a) $w = w^R$ or (b) $\varphi(\sigma^\infty(A_1))$ is periodic. For if $w \neq w^R$, then there exists an integer i such that the i'th letter of w and w^R differ. Now examine the sequence Q formed by taking every i'th letter of U, i.e., $Q = (U_{i+|w|n})_{n \geq 0}$. Up to renaming, Q is precisely the sequence obtained from U by replacing each w with E and each w^R with R. But this sequence is exactly $\varphi(\sigma^\infty(A_1))$, and by the above reasoning, $\varphi(\sigma^\infty(A_1))$ is a fixed point of σ_1.

To apply Theorem 2 we need the homomorphism σ_1 to be primitive and one-to-one. Primitivity is clearly achieved if and only if at least one of the e_i's is equal to R. Furthermore the homomorphism σ_1 is one-to-one if and only if the words

$$E e_1 e_2 \cdots e_{k-1}$$

and

$$(R \circ e_{k-1})(R \circ e_{k-2}) \cdots (R \circ e_1) R$$

are different. Hence, either all the e_i's are equal to E, or the words $E e_1 e_2 \cdots e_{k-1}$ and $(R \circ e_{k-1})(R \circ e_{k-2}) \cdots (R \circ e_1) R$ are equal (which is exactly to say that k is even, and either $(e_1, e_2, \ldots, e_{k-1}) = (ER)^{\frac{k-2}{2}} R$ or $(e_1, e_2, \ldots, e_{k-1}) = (RE)^{\frac{k-2}{2}} R$), or we can apply Theorem 2. In this last case, $h(\theta)$ has to be equal to 2, from Theorem 2. This implies on one hand that k is odd, and on the other hand, that $\gcd\{a : \ Z_a = Z_0 = E\} = 2$, which is equivalent to saying that $(e_1, e_2, \ldots, e_{k-1}) = (RE)^{\frac{k-1}{2}}$. ■

Theorem 4 *The sequence $S^\infty(w)$ is periodic if and only if at least one of the following conditions holds:*

(i) all the e_i's are equal to E, and there exists a word x such that $x_i = x$ for $1 \le i \le k-1$;

(ii) the integer k is even, $(e_1, \dots, e_{k-1}) = (ER)^{\frac{k-2}{2}}R$, and there exists a palindrome x such that $x_i = x$ for $1 \le i \le k-1$;

(iii) the integer k is even, $(e_1, \dots, e_{k-1}) = (RE)^{\frac{k-2}{2}}R$, and there exists a palindrome x such that $x_i = x$ for $1 \le i \le k-1$;

(iv) the integer k is odd, $(e_1, \dots, e_{k-1}) = (RE)^{\frac{k-1}{2}}$, and there exist two palindromes x and y such that $x_1 = x_3 = \cdots = x_{k-2} = x$ and $x_2 = x_4 = \cdots = x_{k-1} = y$;

(v) the word w is a palindrome, and there exists a palindrome x such that $x_i = x$ for $1 \le i \le k-1$.

Proof. ($\Leftarrow$): If the word w, the e_i's, and the x_i's satisfy one of the above conditions, then the sequence $S^\infty(w)$ is a periodic sequence, respectively equal to:

(i) $(wx)^\infty$;

(ii) $(wx(wxw^Rx)^{\frac{k-2}{2}}w^Rx)^\infty$;

(iii) $(wxw^Rx)^\infty$;

(iv) $(wxw^Ry)^\infty$;

(v) $(wx)^\infty$.

($\Rightarrow$): Suppose now that the sequence $S^\infty(w)$ is periodic. Mimicking the proof of Theorem 1, let $w' \in \Sigma^*$ and define a morphism $\tau' : \Delta \to \Sigma^*$ by

$$\begin{aligned} \tau'(A_j) &= w'x_j \\ \tau'(B_j) &= (w')^R x_j \\ \tau'(C_j) &= w'x_j^R \\ \tau'(D_j) &= (w')^R x_j^R, \end{aligned}$$

for $1 \le j \le k-1$. Then we have

$$S^\infty(w') = \tau'(\sigma^\infty A_1).$$

In particular, if we take $w' = x_1 \cdots x_{k-1}$, we check immediately that $\tau' = \mu\sigma$ and hence

$$S^\infty(x_1 \cdots x_{k-1}) = \mu(\sigma(\sigma^\infty A_1)) = \mu(\sigma^\infty(A_1)) = V.$$

Hence, if the sequence $S^\infty(w)$ is periodic, then the sequence V is periodic, and applying Lemma 3 again we see that one of the following conditions holds:

(a) the word w' is a palindrome (i.e., $x_1 \cdots x_{k-1} = x_{k-1}^R \cdots x_1^R$);

(b) all the e_i's are equal to E;

(c) k is even, and either $(e_1, e_2, \dots, e_{k-1}) = (ER)^{\frac{k-2}{2}}R$ or $(e_1, e_2, \dots, e_{k-1}) = (RE)^{\frac{k-2}{2}}R$;

(d) k is odd, and $(e_1, e_2, \dots, e_{k-1}) = (RE)^{\frac{k-1}{2}}$.

As for the sequence $S^\infty(w)$, the sequence $V = S^\infty(x_1 \cdots x_{k-1})$ can be written

$$S^\infty(x_1 \cdots x_{k-1}) = w_1'z_1'w_2'z_2' \cdots$$

where $w'_j \in \{(x_1x_2\cdots x_{k-1}), (x_1x_2\cdots x_{k-1})^R\}$, and $z'_j \in \bigcup_{1\le j\le k-1}\{x_j, x_j^R\}$. We will distinguish five cases according to the pattern $(e_1, \ldots, e_{k-1})$.

Case 1: all the e_i's are equal to E.

In this case, it is clear that the sequence V is equal to

$$x_1x_2\cdots x_{k-1}x_1x_1x_2\cdots x_{k-1}x_2\cdots$$

In other words, writing $V = z'_1z'_2\cdots$, where $z_i \in \bigcup_{1\le j\le k-1}\{x_j\}$, we have

$$\begin{array}{ll} z'_{kn+i} & = z'_i, \ \forall i \ge 1, \ i \not\equiv 0 \pmod{k}, \\ z'_{kn} & = z'_n. \end{array}$$

Since the sequence V is periodic, the sequence of the z'_i's is also periodic. Let T be its smallest period. We claim that $\gcd(T, k) = 1$. Namely if $p \ge 2$ is a number that divides both k and T, we have

$$z'_{n+\frac{T}{p}} = z'_{k(n+\frac{T}{p})} = z'_{kn+\frac{k}{p}T} = z'_{kn} = z'_n,$$

and $T/p < T$, a contradiction.

Let a and b be positive integers such that $bk - aT = 1$. Then, for any $i \not\equiv 0 \pmod{k}$,

$$z'_{i+1} = z'_{i+bk-aT} = z'_{i+bk} = z'_i.$$

But, for $i = 1, 2, \ldots, k-1$, we have $z'_i = x_i$, hence

$$x_1 = x_2 = \cdots = x_{k-1}.$$

Case 2: k is even, and $(e_1, e_2, \ldots, e_{k-1}) = (ER)^{\frac{k-2}{2}}R$.

As in Lemma 3, the sequence $w'_1w'_2\cdots$ is equal to $(w'(w'(w')^R)^{\frac{k-2}{2}}(w')^R)^\infty$. Let

$$\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \beta_1, \beta_2, \ldots, \beta_{k-1}\}$$

be a new alphabet with $2k-2$ letters. Define a "projection" π as follows: $\pi(V) = \pi(S^\infty(x_1\cdots x_{k-1}))$ is obtained from $S^\infty(x_1\cdots x_{k-1}) = w'_1z'_1w'_2z'_2\cdots$ where w' is rewritten as $x_1\cdots x_{k-1}$ and $(w')^R$ as $(x_{k-1})^R\cdots(x_1)^R$, by replacing each x_i by α_i and each $(x_i)^R$ by β_i. The property of the pattern $(e_1, e_2, \ldots, e_{k-1})$ gives

$$\begin{array}{ll} (\pi V)_{i+nk^2} & = (\pi V)_i, \ \forall i \ge 1, \ i \not\equiv 0 \pmod{k}, \\ (\pi V)_{kn} & = (\pi V)_n. \end{array}$$

Now let T be the smallest period of (πV). We claim, as above, that $\gcd(T, k) = 1$. Namely if $p \ge 2$ is a number that divides both k and T, we have

$$(\pi V)_{n+\frac{T}{p}} = (\pi V)_{k(n+\frac{T}{p})} = (\pi V)_{kn+\frac{k}{p}T} = (\pi V)_{kn} = (\pi V)_n,$$

and $T/p < T$, a contradiction.

Hence $\gcd(k^2, T) = 1$. Let a and b be positive integers such that $bk^2 - aT = 1$. Then, for any $i \not\equiv 0 \pmod{k}$,

$$(\pi V)_{i+1} = (\pi V)_{i+bk^2-aT} = (\pi V)_{i+bk^2} = (\pi V)_i.$$

Hence, in particular

$$(\pi V)_1 = (\pi V)_2 = \cdots = (\pi V)_{k-1}.$$

Furthermore

$$(\pi V)_{2k^2+1} = (\pi V)_1.$$

But, for $i = 1, 2, \ldots, k-1$, the letter $(\pi V)_i$ is the image of x_i by π, and $(\pi V)_{2k^2+1}$ is the image of $(x_{k-1})^R$ by π. Hence, by taking for π the map which associates to each x_i and each $(x_i)^R$ its j-th letter, we obtain that all the x_i's are equal, and we see that $x_{k-1}^R = x_1$. Finally there exists a palindrome x such that $x_i = x$, for all $i = 1, 2, \ldots, k-1$.

Case 3: k is even, and $(e_1, e_2, \ldots, e_{k-1}) = (RE)^{\frac{k-2}{2}} R$.

As in Lemma 3, the sequence $w_1' w_2' \cdots$ is equal to $(w'(w')^R)^\infty$. Taking a projection π as above, we have

$$\begin{aligned} (\pi V)_{i+2nk} &= (\pi V)_i, \ \forall i \geq 1, \ i \not\equiv 0 \pmod{k}, \\ (\pi V)_{kn} &= (\pi V)_n. \end{aligned}$$

As above, the smallest period T of the sequence V has to be prime to k. But k is even, hence 2 does not divide T. Hence $\gcd(2k, T) = 1$, which implies the existence of two positive integers a and b such that $2bk - aT = 1$. Then, for $i \not\equiv 0 \pmod{k}$,

$$(\pi V)_{i+1} = (\pi V)_{i+2bk-aT} = (\pi V)_{i+2bk} = (\pi V)_i.$$

Hence

$$(\pi V)_1 = (\pi V)_2 = \cdots = (\pi V)_{k-1},$$

and

$$(\pi V)_{2k-1} = (\pi V)_{2k} = (\pi V)_2.$$

But, for $i = 1, 2, \ldots, k-1$, the letter $(\pi V)_i$ is the image of x_i by π, and the letter $(\pi V)_{2k-1}$ is the image of $(x_1)^R$ by π. Hence, by taking for π the map which associates to each x_i and each $(x_i)^R$ its j-th letter, we obtain

$$x_1 = x_2 = \cdots = x_{k-1} = (x_1)^R,$$

Hence, there exists a palindrome x such that $x_i = x$, for all $i = 1, 2, \ldots, k-1$.

Case 4: k is odd, $e_1 = e_3 = \cdots = R$, and $e_2 = e_4 = \cdots = E$.

As in Lemma 3, the sequence $w_1' w_2' \cdots$ is equal to $(w'(w')^R)^\infty$. Taking a projection π as above, we have

$$\begin{aligned} (\pi V)_{i+2nk} &= (\pi V)_i, \ \forall i \geq 1, \ i \not\equiv 0 \pmod{k}, \\ (\pi V)_{kn} &= (\pi V)_n. \end{aligned}$$

As above, the smallest period T of the sequence V has to be prime to k. Hence there exist two positive integers a and b such that $bk - aT = 1$. Then, for $i \not\equiv 0 \pmod{k}$,

$$(\pi V)_{i+2} = (\pi V)_{i+2bk-2aT} = (\pi V)_{i+2bk} = (\pi V)_i.$$

Hence

$$(\pi V)_1 = (\pi V)_3 = \cdots = (\pi V)_{k-2},$$

$$(\pi V)_2 = (\pi V)_4 = \cdots = (\pi V)_{k-1} = (\pi V)_{k+1},$$

and

$$(\pi V)_{2k-1} = (\pi V)_{2k+1}.$$

But, for $i = 1, 2, \ldots, k-1$, the letter $(\pi V)_i$ is the image of x_i by π, the letter $(\pi V)_{k+1}$ is the image of $(x_{k-1})^R$ by π, the letter $(\pi V)_{2k-1}$ is the image of x_1^R by π, and the letter $(\pi V)_{2k+1}$ is the image of x_1 by π. Hence, by taking for π the map which associates to each x_i and each $(x_i)^R$ its j-th letter, we obtain

$$x_1 = x_3 = \cdots = x_{k-2},$$

$$x_2 = x_4 = \cdots x_{k-1} = x_{k-1}^R,$$

and

$$x_1^R = x_1.$$

Hence, there exist two palindromes x and y such that $x_i = x$, for all $i = 1, 3, \ldots, k-2$, and $x_i = y$ for all $i = 2, 4, \ldots, k-1$.

Case 5: the pattern of the e_i's is not one of those given in the first four cases.

Using Lemma 3, we know that w is a palindrome. Since $V = S^\infty(x_1 x_2 \cdots x_{k-1})$, the same lemma and the periodicity of V show that $x_1 x_2 \cdots x_{k-1}$ must also be a palindrome. This implies easily that, for each palindrome z, the word $S(z)$ is also a palindrome. Furthermore, defining the map S_1 by

$$S_1(z) = z \prod_{1 \le i \le k-1} x_i z,$$

it is clear that S and S_1 coincide on palindromes. Hence the sequence $S^\infty(w)$ is equal to the sequence $S_1^\infty(w)$, and its periodicity implies, using the first case, that all the x_i's are equal. Hence, finally, there exists a palindrome x such that $x_i = x$, for all $i = 1, 2, \ldots, k-1$. ∎

Remark. The condition of periodicity contains the one noticed by Mendès France [9] in the case of (usual) perturbed symmetry.

4 Application to a conjecture of Blanchard and Fabre

In a 1994 paper [2], F. Blanchard and S. Fabre studied some transformations on words and the infinite sequences generated by iterating these transformations. One of the transformations they considered is defined as follows. Let $t \geq 1$ be an integer. The map d_t is defined on the words z of length $\geq 2t$ by $d_t(z) = zz'$, where z' is obtained from z^R by removing the first t symbols. Hence, starting from a word $z = avb$, where $|a| = |b| = t$, we compute

$$\begin{aligned} d_t(z) &= avbv^Ra^R \\ d_t^2(z) &= avbv^Ra^Rvb^Rv^Ra^R \\ &\vdots \end{aligned}$$

Let $D_t(z)$ be the limit $D_t(z) = d_t^\infty(z) = \lim_{n\to\infty} d_t^n(z)$. The authors of [2] give a necessary and sufficient condition for $D_t(z)$ to be periodic, state without proof that this sequence can be generated by a generalized tag-machine, and conjecture this sequence is automatic (in the terminology they use, this means pointwise image of a homomorphism, not necessarily of constant length). We prove here the more precise result that this sequence is actually 2-automatic.

Theorem 5 *The sequence $D_t(z)$ is 2-automatic.*

Proof. Define a sequence of words $z_1, z_2, \ldots$ by

$$d_t^n(z) = az_na^R$$

for all $n \geq 1$. This sequence is well defined (easy proof by induction on n). Furthermore, we have

$$az_{n+1}a^R = d_t^{n+1}(z) = az_na^Rz_n^Ra^R,$$

hence

$$z_{n+1} = z_na^Rz_n^R.$$

In other words, defining the generalized perturbed symmetry S by $S(w) = wa^Rw^R$, we have $z_{n+1} = S(z_n)$, which gives, for $n \geq 1$, the equality $z_n = S^n(z_1)$. Hence

$$D_t(z) = aS^\infty(z_1).$$

Note that this generalized perturbed symmetry is not necessarily a perturbed symmetry in the sense of [3], as the length of a can be greater than 1. As a consequence of Theorem 1, the sequence $S^\infty(z_1)$ is 2-automatic, and so is the sequence $D_t(z)$. ■

Remark. Note that the periodicity condition for $D_t(z)$ given in [2] is an easy consequence of what precedes: if $D_t(z) = aS^\infty(z_1)$ is periodic, then so is the sequence $S^\infty(z_1)$. Hence $a = a^R$ (from our Theorem 4 but also from the remark of Mendès France for the (usual) perturbed symmetry). Conversely, if $a = a^R$, then $S^\infty(z_1) = (z_1\, a\, z_1^R\, a)^\infty$, hence $D_t(z) = (a\, z_1\, a\, z_1^R)^\infty$.

5 Concluding remarks

Theorem 1 was obtained by the second author in July 1987 using a different proof, but never published.

Part of this work was done while the first author visited the University of Waterloo in August 1997.

References

[1] C. Bercoff. A family of tag systems for paperfolding sequences. In E. W. Mayr and C. Puech, editors, *STACS 95, Proc. 12th Symp. Theoretical Aspects of Comp. Sci.*, Vol. 900 of *Lecture Notes in Computer Science*, pages 303–312. Springer-Verlag, 1995.

[2] F. Blanchard and S. Fabre. Quelques procédés engendrant des suites infinies. *Theoret. Comput. Sci.* **123** (1994), 55–60.

[3] A. Blanchard and M. Mendès France. Symétrie et transcendance. *Bull. Sci. Math.* **106** (1982), 325–335.

[4] A. Cobham. Uniform tag sequences. *Math. Systems Theory* **6** (1972), 164–192.

[5] H. Cohn. Symmetry and specializability in continued fractions. *Acta Arith.* **75** (1996), 297–320.

[6] F. M. Dekking. The spectrum of dynamical systems arising from substitutions of constant length. *Z. Wahrscheinlichkeitstheorie* **41** (1978), 221–239.

[7] F. M. Dekking, M. Mendès France, and A. J. van der Poorten. Folds! *Math. Intelligencer* **4** (1982), 130–138, 173–181, 190–195.

[8] M. Koskas. About the p-paperfolding words. *Theoret. Comput. Sci.* **158** (1996), 35–51.

[9] M. Mendès France. Principe de la symétrie perturbée. In M.-J. Bertin, editor, *Séminaire de Théorie des Nombres, Paris 1979–80*, pages 77–98. Birkhäuser, 1981.

[10] M. Mendès France and J. Shallit. Wire bending. *J. Combin. Theory. Ser. A* **50** (1989), 1–23.

[11] D. Razafy Andriamampianina. Le p-pliage de papier. *Ann. Fac. Sci. Toulouse* **10** (1989), 401–414.

[12] D. Razafy Andriamampianina. Le p-pliage de papier et les polynômes. *C. R. Acad. Sci. Paris* **314** (1992), 875–878.

[13] D. Razafy Andriamampianina. Suites de Toeplitz, p-pliage, suites automatiques et polynômes. *Acta Math. Hung.* **73** (1996), 179–190.

[14] J. O. Shallit. Explicit descriptions of some continued fractions. *Fibonacci Quart.* **20** (1982), 77–81.

[15] Z.-X. Wen and Z.-Y. Wen. Some studies on the (p, q)-type sequences. *Theoret. Comput. Sci.* **94** (1992), 373–393.