Univoque numbers and automatic sequences

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Summary. A set of binary sequences related to the iteration of unimodal continuous functions of the interval $[0, 1]$ appears in a 1982–1983 work of Cosnard and the first author. An almost identical set of binary sequences occurs in a 1990 paper by Erdős, Joó, and Komornik; it consists of expansions of 1 in *univoque* bases β in $(1, 2)$ (the base β is univoque if 1 admits a unique β-expansion). We generalize a result of the second author and Niu by proving, using the 1982–1983 results, that a large class of Thue-Morse-like sequences belong to these sets of binary sequences. The case of alphabets of size larger than 2 yields similar results.

1 Introduction

A set of binary sequences related to the iteration of continuous unimodal functions of the interval $[0, 1]$ was introduced at the beginning of the 80's by M. Cosnard and the first author [4, 13, 1]. This set is obtained by looking at the kneading sequences of the point 1 under continous unimodal maps from $[0, 1]$ into itself, then replacing R, L in the kneading sequences by $0, 1$, and finally replacing each binary sequence $(a_n)_{n\geq 0}$ obtained in that way by the sequence $(\sum_{0\leq j\leq n} a_j \bmod 2)_{n\geq 0}$. This set is fractal (actually self-similar in some sense, see [4, 1]). An almost identical set was introduced independently in 1990 by Erdős, Joó, and Komornik [20] to characterize univoque real numbers in the interval $(1, 2)$. Recall that a real number $\beta > 1$ is called *univoque* if there is only one expansion of the number 1 as $1 = \sum_{n\geq 1} a_n/\beta^n$, with $a_n \in \{0, 1, \dots, \lceil\beta\rceil - 1\}$. The set studied in [20] is the set of binary sequences $(a_n)_{n\geq 0}$ such that the (unique) real number $\beta > 1$ satisfying $1 = \sum_{n\geq 1} a_n/\beta^n$ is univoque.

Those two sets of binary sequences are respectively the sets Γ and Γ_{strict} defined by

$$\Gamma := \{A = (a_n)_{n\geq 0} \in \{0, 1\}^{\mathbb{N}},\ \forall k \geq 0,\ \overline{A} \leq \sigma^{(k)}A \leq A\}$$

$$\Gamma_{strict} := \{A = (a_n)_{n\geq 0} \in \{0, 1\}^{\mathbb{N}},\ \forall k \geq 1,\ \overline{A} < \sigma^{(k)}A < A\}$$

where σ is the *shift* on sequences and the *bar* operation switches 0's and 1's, i.e., if $A = (a_n)_{n\geq 0}$, then $\sigma A := (a_{n+1})_{n\geq 0}$, and $\overline{A} := (1 - a_n)_{n\geq 0}$; the symbol $\leq$ denotes the lexicographical order on sequences induced by $0 < 1$, the notation $A < B$ meaning as usual that $A \leq B$ and $A \neq B$.

Note that the original set studied in [4, 13, 1] is equal, with our notation here, to $\Gamma \setminus \{(10)^\infty\}$. Also note that the original definition of the set that we call here Γ_{strict} is slightly different from but equivalent to the definition above, see [6, Remark 4, p. 328]. Finally note that the sets Γ and Γ_{strict} only differ by a set of (purely) periodic sequences.

In 1998, Komornik and Loreti [22] proved that there exists a smallest univoque number β_0, and that the expansion of 1 in base β_0 is the shifted Thue-Morse sequence: 1 1 0 1 0 0 1 1 .... (For more about the Thue-Morse sequence, see, e.g., [10].) J. Shallit indicated to the first author that this result was contained in the 1982–1983 work of Allouche-Cosnard: namely, the smallest nonperiodic element of Γ (see [4, 1]) is clearly the smallest element of Γ_{strict} (see [22], see also [6, 5]).

This unexpected occurrence of the Thue-Morse sequence $(t_n)_{n\geq 0}$ is not isolated. Other variations or avatars of this sequence also belong to the sets Γ and Γ_{strict} or to their generalizations to alphabets of size > 2: the q-mirror sequences (see [4, 1]), the sequences $(d+t_{n+1})_{n\geq 0}$ and $(d+t_{n+1}-t_n)_{n\geq 0}$ for a fixed integer d (see [23]), the fixed point beginning with 3 of the morphism $3 \to 31$, $2 \to 30$, $1 \to 03$, $0 \to 02$ that governs several sequences in (generalizations of) the set Γ_{strict}, including the sequences $(d+t_{n+1})_{n\geq 0}$ and $(d+t_{n+1}-t_n)_{n\geq 0}$ above (see [7]).

In [25] M. Niu and the second author exhibited a class of generalized Thue-Morse sequences $(\varepsilon_n)_{n\geq 1}$, called the "$m$-tuplings Morse sequences", which among other properties belong to the set Γ_{strict}. The "m-tuplings Morse sequence" $(\varepsilon_n)_{n\geq 1}$ will be called the *m-fold Morse sequence* here. It is defined as the fixed point beginning with 0 of the morphism $0 \to 01^{m-1}$, $1 \to 10^{m-1}$. The purpose of the present paper is to prove that a result more general than the one in [25] can be easily deduced from several lemmas proved by the first author in [1].

Remark 1. Note that m-fold Morse sequences are particular cases of the generalized Thue-Morse sequences defined by Doche in [19], and of symmetric D0L words defined by Frid in [21] (see also the paper of Astudillo [12]).

2 A class of sequences belonging to Γ_{strict}

Before stating the main theorem of this paper, we need to introduce some notation. Recall that the length of a (finite) word u is denoted by $|u|$.

Definition 1. *For any integer $r \geq 2$ we define the map Φ_r on periodic sequences of the form $(u0)^\infty$ with minimal period $|u|+1$ by*

$$\Phi_r((u0)^\infty) := (u1(\overline{u}1)^{r-2}\overline{u}0)^\infty.$$

Theorem 1. *Let u be a finite word on the alphabet $\{0,1\}$, such that the sequence $(u0)^\infty$ belongs to Γ and has minimal period $|u|+1$. Then*

- *the sequence $\Phi_r((u0)^\infty)$ belongs to Γ and has minimal period $(r|u|+r)$;*
- *the limit $\lim_{k\to\infty} \Phi_r^{(k)}((u0)^\infty)$ exists, and it belongs to Γ_{strict}.*

This theorem will be proved in the next section. We deduce from Theorem 1 the following corollaries. The first one is essentially the first part of [1, théorème fondamental de structure, p. 24] for $(|u|+1)$-*mirror sequences*, and the second one is Theorem 1 in [25].

Corollary 1 ([1]). *If the sequence $(u0)^\infty$ belongs to Γ and has minimal period $|u|+1$, then the sequence $\lim_{k\to\infty} \Phi_2^{(k)}((u0)^\infty)$ belongs to Γ_{strict}.*

Corollary 2 ([25]). *Let m be an integer ≥ 2, and let $(\varepsilon_n)_{n\geq 0}$ be the fixed point beginning with 0 of the morphism $0 \to 01^{m-1}$, $1 \to 10^{m-1}$. Then the sequence $(\varepsilon_n)_{n\geq 1}$ belongs to Γ_{strict}.*

Proof. The sequence $(1^{m-1}0)^\infty$ clearly belongs to Γ and it has minimal period m. Hence, applying Theorem 1 with $u := 1^{m-1}$ and $r = m$, we see that the sequence $\lim_{k\to\infty} \Phi_m^{(k)}((1^{m-1}0)^\infty)$ belongs to Γ_{strict}. It thus suffices to prove that the fixed point of the morphism λ_m defined by $\lambda_m(0) := 01^{m-1}$, $\lambda_m(1) := 10^{m-1}$, i.e., the sequence $\lambda_m^{(\infty)}(0)$, satisfies: $\lambda_m^{(\infty)}(0) = 0 \lim_{k\to\infty} \Phi_m^{(k)}((1^{m-1}0)^\infty)$.

Since for each letter x in $\{0,1\}$ we have $\lambda_m(\overline{x}) = \overline{\lambda_m(x)}$, we have for any word w on $\{0,1\}$ the equality $\lambda_m(\overline{w}) = \overline{\lambda_m(w)}$. Now define x_k by $\lambda_m^{(k)}(0) = 0x_k$. We thus have $\lambda_m^{(k)}(1) = 1\overline{x_k}$. Furthermore

$$0x_{k+1} = \lambda_m^{(k+1)}(0) = \lambda_m^{(k)}(\lambda_m(0)) = \lambda_m^{(k)}(01^{m-1}) = \lambda_m^{(k)}(0)(\lambda_m^{(k)}(1))^{m-1}.$$

Hence $0x_{k+1} = 0x_k(1\overline{x_k})^{m-1}$, which shows that

$$x_{k+1} = x_k(1\overline{x_k})^{m-1}.$$

Now, since the sequence $\lambda_m^{(\infty)}(0)$ is the limit of the sequence of words $\lambda_m^{(k)}(0)$ when k goes to infinity, it is also the limit of the sequence of periodic sequences $(\lambda_m^{(k)}(0))^\infty = (0x_k)^\infty = 0(x_k0)^\infty$ when k goes to infinity. Hence $\lambda_m^{(\infty)}(0) = 0 \lim_{k\to\infty}(x_k0)^\infty = 0 \lim_{k\to\infty}(x_{k+1}0)^\infty$. But

$$\Phi_m((x_k0)^\infty) = (x_k1(\overline{x_k}1)^{m-2}\overline{x_k}0)^\infty = (x_k(1\overline{x_k})^{m-1}0)^\infty = (x_{k+1}0)^\infty.$$

An immediate induction on k implies that

$$\Phi_m^{(k)}((1^{m-1}0)^\infty) = \Phi_m^{(k)}((x_1 0)^\infty) = (x_{k+1}0)^\infty.$$

Hence

$$\lambda_m^{(\infty)}(0) = 0 \lim_{k\to\infty}(x_{k+1}0)^\infty = 0 \lim_{k\to\infty}\Phi_m^{(k)}((1^{m-1}0)^\infty). \quad \square$$

3 Proof of Theorem 1

We first note that the limit in the second assertion of the theorem exists: namely, $\Phi_r^{(k)}((u0)^\infty)$ and $\Phi_r^{(k+1)}((u0)^\infty)$ coincide on their prefixes of length $r^k|u|+r^k-1$, and this quantity tends to infinity. Since a limit of sequences belonging to Γ clearly belongs to Γ, what we have to prove is the first assertion, and the fact that $\lim_{k\to\infty}\Phi_r^{(k)}((u0)^\infty)$ is not periodic.

3.1 Proof of the first assertion

We will make use of two results proved in [1] and that we recall below (up to notation).

Lemma 1 ([1]). *If a binary sequence belongs to Γ and begins with $u\overline{u}$ for some word u, then it is equal to $(u\overline{u})^\infty$.*

Lemma 2 ([1]). *Let $(u0)^\infty$ be a sequence in Γ, of smallest period $|u|+1$. Suppose $u = xy$ where x, y are two binary words, and x is not empty. Then $\overline{x}\ \overline{y}\ 0 < y\ 1\ \overline{x} < x\ y\ 1$, and $\overline{x}\ \overline{y}\ 0 < \overline{y}\ 1\ \overline{x} < x\ y\ 1$.*

Remark 2. Lemma 1 is [1, Lemme 2, p. 26-27]. The first double inequality in Lemma 2 is [1, Lemme 5, p. 30], while the second can be easily obtained by combining the second assertion of [1, Proposition 2, p. 34] and [1, Lemme 3, p. 27].

Now suppose that $(u0)^\infty$ is a sequence in Γ, of smallest period $|u|+1$. We want to prove that $\Phi_r((u0)^\infty)$ belongs to Γ and has minimal period $r|u|+r$, i.e., that we have

$$\begin{cases} (\overline{u}0(u0)^{r-2}u0)^\infty \le (u1(\overline{u}1)^{r-2}\overline{u}0)^\infty \\ (\overline{u}0(u0)^{r-2}u0)^\infty \le \sigma^k((u1(\overline{u}1)^{r-2}\overline{u}0)^\infty) < (u1(\overline{u}1)^{r-2}\overline{u}0)^\infty, \quad \forall k \ge 1 \end{cases} \qquad (*)$$

Note that, if u is reduced to 1, or if u begins with 10, then, from Lemma 1, the sequence $(u0)^\infty$ must be equal to $(10)^\infty$, hence the sequence $(u1(\overline{u}1)^{r-2}\overline{u}0)^\infty$ must be equal to $(11(01)^{r-2}00)^\infty$ and inequalities $(*)$ clearly hold. We thus may suppose that u begins with 11.

- First case: $k \equiv 0 \bmod (|u|+1)$.
  If $k \equiv 0 \bmod (r|u|+r)$ inequalities $(*)$ are clearly true (note that u must begin with 1 since $(u0)^\infty$ belongs to Γ hence $(\overline{u}1)^\infty \le (u0)^\infty$).

If $k \equiv j(|u|+1) \bmod (r|u|+r)$, with $j \in [1, r-2]$, the sequence $\sigma^k((u1(\overline{u}1)^{r-2}\overline{u}0)^\infty)$ begins with $\overline{u}1$, and the inequalities are clear.
If $k \equiv (r-1)(|u|+1) \bmod (r|u|+r)$, then the sequence $\sigma^k((u1(\overline{u}1)^{r-2}\overline{u}0)^\infty)$ begins with $\overline{u}0u1$, and the inequalities are clear again.

- Second case: $k \equiv j(|u|+1)-1 \bmod (r|u|+r)$, with $j \in [1, r]$, then the sequence $\sigma^k((u1(\overline{u}1)^{r-2}\overline{u}0)^\infty)$ begins with $1\overline{u}$ or $0u$. It thus suffices to check that $\overline{u}0 < 1\overline{u} < u1$ and that $\overline{u}0 < 0u < u1$, which are easy consequences of the fact that u begins with 11.
- Third case: $k \not\equiv 0, -1 \bmod (|u|+1)$. There exist two words x and y with $x \neq \emptyset$ and $u = xy$ such that the sequence $\sigma^k((u1(\overline{u}1)^{r-2}\overline{u}0)^\infty)$ begins either with $y1\overline{x}$, or with $\overline{y}1\overline{x}$, or with $\overline{y}0x$, and Lemma 2 permits to conclude.

3.2 The sequence $\lim_{k\to\infty} \Phi_r^{(k)}((u0)^\infty)$ is not periodic

As will be proved in Theorem 2 below, if the sequence $(u0)^\infty$ is periodic with smallest period $|u|+1$, and if we set $(x_n)_{n\geq 0} := \lim_{k\to\infty} \Phi_r^{(k)}((u0)^\infty)$, then the sequence $(x_{qn})_{n\geq 1}$ is the shifted sequence of the r-fold Morse sequence $(\varepsilon_n)_{n\geq 0}$. It thus suffices to prove that the r-fold Morse sequence is not eventually periodic. Being the fixed point of the morphism λ_r, $0 \to 01^{r-1}$, $1 \to 10^{r-1}$, the sequence $(\varepsilon_n)_{n\geq 0}$ satisfies

$$\begin{aligned} \varepsilon_{rn} &= \varepsilon_n && \text{for all } n \geq 0, \\ \varepsilon_{rn+j} &= 1 - \varepsilon_n && \text{for all } n \geq 0, \text{ for all } j \in [1, r-1]. \end{aligned}$$

Now suppose that $(\varepsilon_n)_{n\geq 0}$ is eventually periodic with smallest period T. We will show that this is impossible by looking at T modulo r. If $T = r\ell$ for some integer ℓ, then, for n large enough, we have $\varepsilon_{n+\ell} = \varepsilon_{rn+r\ell} = \varepsilon_{rn+T} = \varepsilon_{rn} = \varepsilon_n$, which would imply that $(\varepsilon_n)_{n\geq 0}$ is eventually periodic with period $\ell < T$. If $T = r\ell + j$ for some integer ℓ, and some integer $j \in [1, r-1]$, then, for n large enough, we have $\varepsilon_{n+\ell} = 1 - \varepsilon_{rn+r\ell+j} = 1 - \varepsilon_{rn+T} = 1 - \varepsilon_{rn} = 1 - \varepsilon_n$. This implies that $\varepsilon_{n+2\ell} = 1 - \varepsilon_{n+\ell} = \varepsilon_n$ for n large enough. Hence the sequence $(\varepsilon_n)_{n\geq 0}$ is eventually periodic with period 2ℓ. But $2\ell \leq r\ell < r\ell + j = T$.

Remark 3. Since the morphism λ_r is primitive, the sequence $(\varepsilon_n)_{n\geq 0}$ is minimal. Hence it cannot be eventually periodic without being periodic. A general criterion for periodicity of fixed points of constant length morphisms is given in [16, II.9 (iii), p. 226].

4 Automatic sequences and the sets Γ and Γ_{strict}

The sequences in Γ_{strict} given by Theorem 1 are generalizations of the m-fold Morse sequences of [25]. We will prove that they can be obtained by shuffling m-fold Morse sequences. In particular they are automatic (for more about automatic sequences, see, e.g., [11]). We start with a definition and a lemma.

Definition 2. *If r is an integer ≥ 2, we define the map Ψ_r on binary words by*

$$\Psi_r(w) := w(\overline{w})^{r-1}.$$

Lemma 3. *Let u be a binary word such that the sequence $(u0)^\infty$ has minimal period $|u|+1$. Then for all $k \geq 0$*

$$0\ \Phi_r^{(k)}((u0)^\infty) = (\Psi_r^{(k)}(0u))^\infty \quad \text{and} \quad 0 \lim_{k\to\infty} \Phi_r^{(k)}((u0)^\infty) = \lim_{k\to\infty} \Psi_r^{(k)}(0u)$$

Proof. The first assertion is trivial for $k = 0$, and it is an immediate consequence of the definitions of Φ_r and Ψ_r for $k = 1$. Suppose it is true for some k, then, using the induction hypothesis for k, and the case $k = 1$ of the induction hypothesis for the sequence $(u1(\overline{u}1)^{r-2}\overline{u}0)^\infty$, we have

$$\begin{aligned} 0\ \Phi_r^{(k+1)}((u0)^\infty) &= 0\ \Phi_r^{(k)}(\Phi_r((u0)^\infty) = 0\ \Phi_r^{(k)}(u1(\overline{u}1)^{r-2}\overline{u}0)^\infty) \\ &= (\Psi_r^{(k)}(0u1(\overline{u}1)^{r-2}\overline{u}))^\infty = (\Psi_r^{(k)}(0u(1\overline{u})^{r-1}))^\infty \\ &= (\Psi_r^{(k)}(\Psi_r(u0)))^\infty = (\Psi_r^{(k+1)}(0u))^\infty. \end{aligned}$$

The second assertion is a consequence of the first one and of the two remarks that $\Psi_r^{(k+1)}(w)$ begins with $\Psi_r^{(k)}(w)$ for all binary words w, and that $|\Psi_r^{(k)}(w)|$ tends to infinity with k. □

Theorem 2. *Let u be a finite word on the alphabet $\{0, 1\}$, such that the sequence $(u0)^\infty$ belongs to Γ and has minimal period $q := |u| + 1$. Let $(x_n)_{n\geq 1}$ be the binary sequence $\lim_{k\to\infty} \Phi_r^{(k)}((u0)^\infty)$. Then each of the sequences $(x_{qn+j})_{n\geq 0}$ for $j = 1, 2, \ldots, q-1$ is either the r-fold Morse sequence $(\varepsilon_n)_{n\geq 0}$ or the sequence $\overline{(\varepsilon_n)}_{n\geq 0}$. Furthermore the sequence $(x_{qn})_{n\geq 1}$ is the shifted sequence of the r-fold Morse sequence. In other words, we have*

$$\begin{aligned} x_{qn+j} &= \varepsilon_n + x_j \quad \text{for all } n \geq 0 \text{ and for all } j = 1, 2, \ldots, q-1, \\ x_{qn} &= \varepsilon_n \quad \text{for all } n \geq 1. \end{aligned}$$

In particular the sequence $\lim_{k\to\infty} \Phi_r^{(k)}((u0)^\infty)$ is r-automatic.

Proof. Recall that the r-fold Morse sequence $(\varepsilon_n)_{n\geq 0}$ is the fixed point beginning with 0 of the morphism λ_r defined by $\lambda_r(0) = 01^{r-1}$ and $\lambda_r(1) = 10^{r-1}$. We first note that the sequence $(\varepsilon_n)_{n\geq 0}$ (resp. $(\overline{\varepsilon_n})_{n\geq 0}$) satisfies $(\varepsilon_n)_{n\geq 0} = \lim_{k\to\infty} \Psi_r^k(0)$ (resp. $(\overline{\varepsilon_n})_{n\geq 0} = \lim_{k\to\infty} \Psi_r^k(1)$). Namely, it was stated during the proof of Corollary 2, that $\lambda_r^{(\infty)}(0) = 0 \lim_{k\to\infty} \Phi_r^k((1^{r-1}0)^\infty)$. Hence, using Lemma 3, we have $\lambda_r^{(\infty)}(0) = \lim_{k\to\infty} \Psi_r^{(k)}(01^{r-1})$. Since $|\Psi_r^{(k)}(0)|$ tends to infinity with k, we have $\lim_{k\to\infty} \Psi_r^{(k)}(01^{r-1}) = \lim_{k\to\infty} \Psi_r^{(k)}(0)$. Hence $(\varepsilon_n)_{n\geq 0} = \lambda_r^{(\infty)}(0) = \lim_{k\to\infty} \Psi_r^{(k)}(0)$, and $(\overline{\varepsilon_n})_{n\geq 0} = \lambda_r^{(\infty)}(1) = \lim_{k\to\infty} \Psi_r^{(k)}(1)$.

Now an induction on k shows that

$$\Psi_r^{(k)}(0u) = 0v_1b_1v_2b_2\dots b_{r^k-1}v_{r^k}$$

where $v_1 = u$, the v_j's are equal to u or $\overline{u}$ and b_j is 0 or 1. Furthermore

$$0v_1b_1v_2b_2\dots b_{r^{k+1}-1}v_{r^{k+1}} = 0v_1b_1v_2b_2\dots b_{r^k-1}v_{r^k}(\overline{0v_1b_1v_2b_2\dots b_{r^k-1}v_{r^k}})^{r-1}$$

which implies that for all $k \geq 0$

- the words $B_k := 0b_1b_2\dots b_{r^k-1}$ satisfy the relation $B_{k+1} = B_k(\overline{B_k})^{r-1} = \Psi_r(B_k)$,
- the words $V_k := v_1v_2\dots v_{r^k}$ satisfy the relation $V_{k+1} = V_k(\overline{V_k})^{r-1} = \Psi_r(V_k)$,
- if $c_{j,i}$ is the jth letter of v_i (with $j \in [1, |u|]$), then the words $C_{j,k} := c_{j,1}c_{j,2}\dots c_{j,r^k}$ satisfy the relation $C_{j,k+1} = C_{j,k}(\overline{C_{j,k}})^{r-1} = \Psi_r(C_{j,k})$.

All these relations imply that the sequence $(b_n)_{n\geq 1}$ is the shifted sequence of the r-fold Morse sequence $(\varepsilon_n)_{n\geq 0}$, and that each of the sequences $(c_{j,n})_{n\geq 0}$ for $j \in [1, |u|]$ is either the sequence $(\varepsilon_n)_{n\geq 0}$ or the sequence $(\overline{\varepsilon_n})_{n\geq 0}$.

The sequence $\lim_{k\to\infty} \Phi_r^{(k)}((u0)^\infty)$ is thus r-automatic: this is a classical result, given that all subsequences on arithmetic progressions of length q are r-automatic (see, e.g. [11]).

Remark 4.

- Generalizing the definition introduced in [1], we will call the sequences $\lim_{k\to\infty} \Phi_r^{(k)}((u0)^\infty)$ above "(q,r)-mirror sequences" (where $q = |u| + 1$). The q-mirror sequences in [1] are thus $(q,2)$-mirror sequences.
- It was proved in [25] that the r-fold Morse sequence $(\varepsilon_n)_{n\geq 0}$ has the property that $\varepsilon_n = 0$ if and only if the base r expansion of n has an even number of nonzero digits. The penultimate and last assertions of Theorem 2 are thus generalizations to (q,r)-mirror sequences of properties of q-mirror sequences (see [1, Propriétés, p. 21]).
- Relations like the ones satisfied by the sequences of words B_k, V_k, and $C_{j,k}$ above are called *locally catenative formulas*. For a systematic study of locally catenative formulas and morphic sequences, the reader can look at [26].

Corollary 3. *For any $r \geq 2$ the set Γ_{strict} contains infinitely many r-automatic sequences.*

Proof. The sequences $(1^\ell 0)^\infty$ (with $\ell \geq 1$) clearly belong to Γ. They have minimal period $\ell + 1$. Using Theorems 1 and 2 we deduce that the (distinct) sequences $\lim_{k\to\infty} \Phi_r^{(k)}((1^\ell 0)^\infty)$ belong to Γ_{strict} and are r-automatic. □

Remark 5. Another proof of Corollary 3 can be deduced from a result in [9]. Namely, if B is any nonperiodic minimal r-automatic sequence, then the sequence $A := \max\{\sup\{\sigma^k B,\ k \geq 0\},\ \sup\{\sigma^k \overline{B},\ k \geq 0\}\}$ is r-automatic (see [9]), and it clearly belongs to Γ_{strict} (it belongs *a priori* to Γ and cannot be periodic since B is minimal and nonperiodic). Now the sequences $1^\ell A$ (with $\ell \geq 1$) belong to Γ_{strict} and they are r-automatic.

5 Alphabets with more than two letters

All the results above extend to (finite) alphabets with more than two letters. As noted in [1], the set Γ is the set of binary expansions of real numbers $x \in [0,1]$ such that for all $k \geq 0$, we have $1 - x \leq \{2^k x\} \leq x$, where $\{y\}$ is the fractional part of the real number y. It is natural to ask about the real numbers for which the inequalities $1 - x \leq \{b^k x\} \leq x$ hold (where b is some fixed integer ≥ 3). The base b expansions of these real numbers form the generalized Γ set

$$\Gamma := \{A = (a_n)_{n \geq 0} \in [0, b-1]^{\mathbb{N}},\ \forall k \geq 0,\ \overline{A} \leq \sigma^{(k)} A \leq A\}$$

whose combinatorial properties were studied in [1, Troisième partie, p. 63–90]. (Here σ is again the *shift* on sequences and the *bar* operation replaces $x \in [0, b-1]$ by $b - 1 - x$, i.e., if $A = (a_n)_{n \geq 0}$, then $\sigma A := (a_{n+1})_{n \geq 0}$, and $\overline{A} := (b - 1 - a_n)_{n \geq 0}$; the symbol $\leq$ denotes the lexicographical order on sequences induced by $0 < 1$, the notation $A < B$ meaning as usual that $A \leq B$ and $A \neq B$.) The corresponding set

$$\Gamma_{strict} := \{A = (a_n)_{n \geq 0} \in [0, b-1]^{\mathbb{N}},\ \forall k \geq 0,\ \overline{A} < \sigma^{(k)} A < A\}$$

is the set of *admissible sequences* introduced in [23]. (Note that the definition given in [23] is slightly different, but it can be proved equivalent, see, e.g., [7, Proposition 1].)

Results quite similar to the case of a binary alphabet, in particular generalizations of Theorems 1 and Theorem 2 above can be proved, by using the results of [1, Troisième partie]. We will not enter details for brevity, but no really serious difficulty occurs.

6 Conclusion

Combinatorial properties of the sets Γ and Γ_{strict} are crucial in the study of univoque numbers but also in the study of iterations of unimodal functions. An attempt to explain why these almost identical sets appear in seemingly disjoint fields can be found in [3]. Other papers on univoque numbers make use of combinatorial properties of Γ and Γ_{strict} (to cite a few, see [8, 14, 15, 24, 17, 18]). One may also ask whether the sets Γ and Γ_{strict} contain other "classical" infinite sequences from combinatorics on words. For example, it is not hard to prove that, defining the binary morphism τ by $\tau(1) := 10$, $\tau(0) := 1$ and denoting by F the Fibonacci sequence $\tau^{(\infty)}(1) = 10110101101\ldots$, then the sequence $1F = 110110101101\ldots$ belongs to Γ_{strict} (see, e.g., [2]). This provides us with an open question: which morphic sequences belong to Γ or Γ_{strict}?

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