A relative of the Thue-Morse Sequence

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Abstract

We study a sequence, \( c \), which encodes the lengths of blocks in the Thue-Morse sequence. In particular, we show that the generating function for \( c \) is a simple product.
Consider the sequence
\[ c : c_0, c_1, c_2, c_3, \ldots = 1, 3, 4, 5, 7, 9, 11, 12, 13, \ldots \]
defined to be the lexicographically least sequence of positive integers satisfying
\[ n \in c \text{ implies } 2n \notin c. \]
In fact, the lexicographic minimality of \( c \) makes it possible to replace the previous “implies” with “if and only if.” Equivalently, \( c \) is defined inductively by \( c_0 = 1 \) and
\[
c_{k+1} = \begin{cases} 
  c_k + 1 & \text{if } (c_k + 1)/2 \notin c \\
  c_k + 2 & \text{otherwise}
\end{cases}
\]
for \( k \geq 0 \). This sequence was the focus of a problem of C. Kimberling in the *American Mathematical Monthly* [6]. (In fact, he looked at the sequence \( 4c_0, 4c_1, 4c_2, \ldots \)) The solution was given by D. Bloom [4]. Our Corollary 7 answers essentially the same question. Related results have recently been announced by J. Tamura [9].

At the 4ème Colloque Séries Formelles et Combinatoire Algébrique (Montréal, June 1992) S. Plouffe and P. Zimmermann [8] posed the following problem. Show that the generating function for \( c \) is
\[
\sum_{k \geq 0} c_k x^k = \frac{1}{1 - x} \prod_{j \geq 1} \frac{1 - x^{2e_j}}{1 - x^{e_j}} = \frac{1}{1 - x} \prod_{j \geq 1} (1 + x^{e_j})
\]
the sequence of exponents being
\[ e : e_1, e_2, e_3, e_4, \ldots = 1, 1, 3, 5, 11, 21, 43, \ldots \]
where \( e_1 = 1 \) and
\[
e_{j+1} = \begin{cases} 
  2e_j + 1 & \text{if } j \text{ is even} \\
  2e_j - 1 & \text{if } j \text{ is odd}
\end{cases}
\]
for \( j \geq 1 \). They found this conjecture by using a method that goes back to Euler. First they assumed that the generating function was of the form
\[
\prod_{j \geq 0} \frac{1 - x^{a_j}}{1 - x^{b_j}}
\]
for a certain pair of sequences \( a_j, b_j \). Then they took the logarithm to convert the product into a sum. Finally they used Möbius inversion to determine the candidate sequences. Details of this procedure can be found in the text of G. Andrews [2, Theorem 10.3].

The purpose of this note is to prove (2). Before doing this, however, we will show that \( c \) has a number of other interesting properties. Chief among these is the fact that \( c \) is closely related to the famous Thue-Morse sequence, \( t \). See the survey article of J. Berstel [3] for more information about \( t \).

First we need to have a characterization of the integers in the sequence \( c \).
Proposition 1 If $n$ is any positive integer then $n \in c$ if and only if $n = 2^{2i}(2j+1)$ for some nonnegative integers $i$ and $j$.

Proof. Every positive integer $n$ can be uniquely written in the form $n = 2^k(2j+1)$ where $k, j \geq 0$. We will proceed by induction on $k$.

If $k = 0$, then $n$ is odd. But then $n/2$ is not an integer, and so $n$ is in the sequence by definition (1).

Now assume that $k \geq 1$ and that the proposition holds for all powers less than $k$ of 2. If $k = 2i$ is even, then by induction we have $2^{2i-1}(2j+1) \not\in c$. So $n = 2^{2i}(2j+1) \in c$ by (1). On the other hand, if $k = 2i + 1$ is odd, then induction implies $2^{2i}(2j+1) \not\in c$. Thus $n = 2^{2i+1}(2j+1) \not\in c$ as desired. ■

Let $\chi$ be the characteristic function of $c$, i.e.,

$$
\chi(n) = \begin{cases} 
1 & \text{if } n \in c \\
0 & \text{otherwise.}
\end{cases}
$$

Restating the previous proposition in terms of $\chi$ yields the next result.

Lemma 2 The function $\chi$ is uniquely determined by the equations

$$
\begin{align*}
\chi(2n+1) &= 1 \\
\chi(4n+2) &= 0 \\
\chi(4n) &= \chi(n).
\end{align*}
$$

Another way of obtaining the sequence $\chi(n)$ for $n \geq 1$ is as follows. Starting from the sequence

$$
101 \bullet 101 \bullet 101 \bullet 101 \bullet \ldots
$$

defined on the alphabet \{0, 1, \bullet\}, fill in the successive holes with the successive terms of the sequence itself, obtaining:

$$
101110101101101 \bullet \ldots
$$

Iterating this process infinitely many times (by inserting the initial sequence into the holes at each step), one gets a “Toeplitz transform” which is nothing but our sequence $\chi$. The proof of this fact is easily obtained using Lemma 2. See the article of J.-P. Allouche and R. Bacher [1] for more information about Toeplitz transformations.

The connection with the Thue-Morse sequence can now be obtained. This sequence is

$$
t : t_0, t_1, t_2, t_3, \ldots = 0, 1, 1, 0, 1, 0, 1, 0, 1, \ldots
$$
defined by the conditions
\[ t_0 = 0, \quad t_{2n+1} \equiv t_n + 1 \pmod{2}, \quad t_{2n} = t_n. \]

We will need a lemma relating \( t \) and \( \chi \). All congruences in this and any future results will be modulo 2.

**Lemma 3** *For every positive integer, \( n \), we have*
\[ \chi(n) \equiv t_n + t_{n-1}. \]

**Proof.** This is a three case induction based on Lemma 2 and the definitions of \( \chi \) and \( t \). We will only do one of the cases as the others are similar.

\[ t_{4n} + t_{4n-1} \equiv t_{2n} + t_{2n-1} + 1 \]
\[ \equiv t_n + t_{n-1} + 2 \]
\[ \equiv \chi(n) \]
\[ = \chi(4n). \]

Define \( d_k \) to be the first difference sequence of \( c_k \), i.e., \( d_k = c_k - c_{k-1} \), for \( k \geq 0 \) \((c_{-1} = 0)\). So \( d \) is the sequence
\[
d_0, d_1, d_2, d_3, d_4, \ldots = 1, 2, 1, 2, 2, 2, 1, 1, 2, 1, \ldots
\]

Note that from the definition of \( c \) in (1), the value of \( d_k \) is either 1 or 2. Write the Thue-Morse sequence in term of its blocks
\[ t = 011010011 \ldots = 0^{d_0}1^{d_1}0^{d_2}1^{d_3} \ldots \]

defining a sequence \( d'_k \). It is this sequence that is related to our original one via the difference operator.

**Theorem 4** *For all \( k \geq 0 \) we have \( d_k = d'_k \).*

**Proof.** Since both sequences consist of 1’s and 2’s, we need only verify that the 1’s appear in the same places in both. It will be convenient to let \( c'_k = \sum_{i \leq k} d'_i \). We now proceed by induction on \( k \), assuming that \( d_i = d'_i \) for \( i \leq k \). Then, from the definitions,
\[ d_{k+1} = 1 \Leftrightarrow \chi(c_k + 1). \]
But by the induction hypothesis, \( c_k = \sum_{i \leq k} d_i = \sum_{i \leq k} d'_i = c'_k \). So, from equation (4),
\[
d_{k+1} = 1 \iff \chi(c'_k + 1) = 1
\]
\[
\iff t_{c'_k+1} + t_{c'_k} \equiv 1 \quad \text{(Lemma 3)}
\]
\[
\iff t_{c'_k+1} \neq t_{c'_k}
\]
\[
\iff d'_{k+1} = 1 \quad \text{(definitions)}. \]

S. Brlek [5] used the sequence \( d \) in calculating the number of factors of \( t \) of given length. The paper of A. de Luca and S. Varricchio [7] attacks the same problem in a different way.

Now if \( n \in c \) then we will consider its \textit{rank}, \( r(n) \), which is the function satisfying \( c_{r(n)} = n \). Note that \( r(n) \) is not defined for all positive integers \( n \). In order to obtain a formula for \( r(n) \), we will need a definition. Let the base 2 expansion of \( n \) be
\[
n = \sum_{i \geq 0} \epsilon_i 2^i
\]
with the \( \epsilon_i \in \{0, 1\} \) for all \( i \). Define a function \( s \) by
\[
s(n) = \sum_{i \geq 0} (-1)^i \epsilon_i.
\]
In other words, \( s(n) \) is the alternating sum of the binary digits of \( n \).

\textbf{Theorem 5} \textit{If} \( n \in c \) \textit{then}
\[
r(n) = (2n + s(n))/3 - 1. \quad (5)
\]
\textbf{Proof.} The proof will be by induction. From Proposition 1, \( n \in c \) if and only if \( n \) is odd or \( n = 2^i(2j + 1) \) where \( i > 0 \) and \( j \geq 0 \). To facilitate the induction, it will be convenient to split the odd numbers into two groups depending upon whether the highest power of 2 dividing \( n + 1 \) is even or odd. So there will be three cases

1. \( n = 2^i(2j + 1) \)
2. \( n = 2^i(2j + 1) - 1 \)
3. \( n = 2^{i-1}(2j + 1) - 1 \)

where \( i > 0 \) and \( j \geq 0 \). The arguments are similar, so we will only do the first case.
So suppose $n$ is even (remember that $i > 0$). Thus $n + 1$ is odd and, by Proposition 1, we have $n + 1 \in \mathbf{c}$. Since both $n$ and $n + 1$ are in $\mathbf{c}$, the left side of equation (5) satisfies $r(n + 1) = r(n) + 1$. So, by induction, it suffices to show that $r'(n + 1) = r'(n) + 1$ where $r'(n)$ is the right side of this equation. Moreover, $n$ is a multiple of 4, hence $s(n + 1) = s(n) + 1$ (write down their binary expansions). Thus

$$r'(n + 1) = \frac{2n + 2 + s(n + 1)}{3} - 1$$

$$= \frac{2n + 2 + s(n) + 1}{3} - 1$$

$$= \frac{2n + s(n)}{3}$$

$$= r'(n) + 1. \quad \blacksquare$$

As straightforward corollaries we have the next two results.

**Corollary 6** If $n \in \mathbf{c}$ then

$$r(n) = \frac{2n}{3} + O(\log n)$$

and $r(n)$ takes the value $2n/3$ infinitely often. \(\blacksquare\)

**Corollary 7** For any nonnegative integer $k$

$$c_k = 3k/2 + O(\log k)$$

and $c_k = 3k/2$ infinitely often. \(\blacksquare\)

We shall now prove the identity (2). First we note a property of the exponents $e_j$ which is a simple consequence of their definition (3).

**Lemma 8** For $k \geq 2$, let $f_k = \sum_{2 \leq j \leq k} e_j$. Then

$$f_k = \begin{cases} 
  e_{k+1} - 2 & \text{if } k \text{ is even} \\
  e_{k+1} - 1 & \text{if } k \text{ is odd.} \quad \blacksquare
\end{cases}$$

Finally, we come to the proof. We restate the generating function here for easy reference.

**Theorem 9** The generating function for $\mathbf{c}$ is

$$\sum_{k \geq 0} c_k x^k = \frac{1}{1 - x} \prod_{j \geq 1} (1 + x^{e_j}).$$
**Proof.** It suffices to show that if \( k \geq 2 \) then
\[
g_k(x) = \frac{1}{1-x} (1 + x^1)(1 + x^1)(1 + x^3) \cdots (1 + x^{e_k})
\]
is the generating function for the sequence
\[
1, 3, 4, 5, 7, \ldots, c_f, 2^k, 2^k, 2^k, \ldots
\]
with \( c_f = 2^k - 1 \). The proof is an induction, breaking up into two parts depending on the parity of \( k \). We will do the case where \( k \) is odd. (Even \( k \) is similar.) Now, by Lemma 8, \( g_k(x)(1 + x^{e_{k+1}}) \) is the generating function for the sequence
\[
1, 3, \ldots, c_f, 2^k + 1, 2^k + 3, \ldots, 2^k + c_f, 2^{k+1}, 2^{k+1}, \ldots
\]
Using Proposition 1 and the fact that \( k \) is odd, we see that \( 2^k + 1 = c_{f+1} \) and \( 2^k + c_f = 2^{k+1} - 1 = c_{f+1} \). So we want to show that
\[
c_{f+1}, c_{f+2}, \ldots, c_{f+1} = 2^k + c_0, 2^k + c_1, \ldots, 2^k + c_f.
\]
But if \( n < 2^k \), then the highest power of 2 dividing \( n \) is equal to the highest power dividing \( 2^k + n \). Thus, by Proposition 1 again, \( n \in c \) if and only if \( 2^k + n \in c \). This gives us the desired equality of the two sequences. ■

One possible generalization of \( c \) is the sequence \( c(\alpha) \) defined by \( n \in c(\alpha) \) if and only if \( \alpha n \notin c(\alpha) \). Thus \( c \) is the special case \( \alpha = 2 \).

The following observation is a direct consequence of our definitions.

**Proposition 10** If \( \chi^{(\alpha)}(n) \) is the characteristic function of \( c^{(\alpha)} \), then the sequence \( (\chi^{(\alpha)}(n)) \) is the unique fixed point of the morphism
\[
\begin{align*}
1 & \rightarrow 1^{\alpha-1}0 \\
0 & \rightarrow 1^{\alpha-1}1
\end{align*}
\]
which begins with 1. ■

One can also see that \( c^{(\alpha)} \) satisfies analogs of many of our previous theorems. For example, if one defines \( e_1^{(\alpha)} = 1 \) and
\[
e_{j+1}^{(\alpha)} = \begin{cases} 
\alpha e_j^{(\alpha)} + 1 & \text{if } j \text{ is even} \\
\alpha e_j^{(\alpha)} - 1 & \text{if } j \text{ is odd}
\end{cases}
\]
for \( j \geq 1 \), then the following result is a generalization of Theorem 9 and has an analogous proof.

**Theorem 11** The generating function for \( c^{(\alpha)} \) is
\[
\frac{1}{1-x} \prod_{j \geq 1} \frac{1 - x^{\alpha e_j^{(\alpha)}}}{1 - x^{e_j^{(\alpha)}}}.
\]
References


[9] J. Tamura, Some problems and results having their origin in the power series $\sum_{n=1}^{+\infty} z^{[\alpha n]}$ sum from 1 to infinity of $z$ to the integral part of $\alpha$ times $n$, in “Reports of the Meeting on Analytic Theory of Numbers and Related Topics,” Gakushuin Univ., 1992, 190–212.