Lectures on Coxeter groups

Jean Michel

Beijing, April–May 2014

The Coxeter-Dynkin diagrams appear in many different classification contexts in mathematics: platonic solids, semi-simple Lie algebras, algebraic groups, finite simple groups, quivers with finitely many indecomposables, cluster algebras with finitely many seeds, singularities of hypersurfaces, and many others.

The most basic occurrence is the classification of finite Coxeter groups, or equivalently of finite groups generated by reflections in a real vector space. It is the purpose of the series of lectures to present this case. The plan is as follows

- Abstract characterisation of Coxeter groups.
- Finite real reflection groups are the finite Coxeter groups.
- All Coxeter groups are reflection groups.
- Classification of finite Coxeter groups.

Reflections, and real reflection groups

Let V be an $\mathbb{R}$-vector space. An element $s \in \mathrm{GL}(V)$ is a *reflection* if $H_s := \mathrm{Ker}(s - \mathrm{Id}_V)$ is a hyperplane and $s^2 = 1$. Thus s has a unique eigenvalue not 1, equal to -1.

A reflection takes the form $s(x) = x - \check{r}(x)r$ where $\check{r} \in V^*$ is a linear form of kernel H_s, and where r is an eigenvector for the eigenvalue -1 of s, provided that these data satisfy $\check{r}(r) = 2$. We will call r (resp. $\check{r}$) a root (resp. coroot) attached to the reflection s. These data are unique up to multiplying r by some scalar and $\check{r}$ by the inverse scalar.

Given a subgroup $W \subset \mathrm{GL}(V)$, we denote $\mathrm{Ref}(W)$ the set of reflections it contains. We say that W is a *real reflection group* if it is generated by $\mathrm{Ref}(W)$. It is clear that the set $\mathrm{Ref}(W)$ is stable by W-conjugacy.

Given a group W and morphism $\rho : W \to \mathrm{GL}(V)$ whose image is a reflection group, we say that ρ is a *reflection representation* of W.

A reflection group is called *irreducible* if the representation V is an irreducible representation of W, that is if V does not admit any proper W-invariant subspace. If W is finite, then V is a direct sum of irreducible representations, and W is the direct product of the corresponding subgroups.

4 Coxeter groups

Let W be a group generated by a set S of elements stable by taking inverses. Let $\{w_i, w'_i\}_{i\in I}$ be words in the elements of S (finite sequences of elements of S; the set of all words on S is denoted S^* and called the free monoid on S). We say that $\langle S \mid w_i = w'_i \text{ for } i \in I\rangle$ is a *presentation* of W is W is the "most general group" where the relation $w_i = w'_i$ holds. Formally, we take for W the quotient of S^* by the congruence relation on words generated by the relations $w_i = w'_i$.

Let $w \in W$ be the image of $s_1 \dots s_k \in S^*$. Then this word is called a *reduced expression* for $w \in W$ if it is a word of minimal length representing w; we then write $l(w) = k$.

We assume now the set S which generates W consists of involutions, that is each element of S is its own inverse. Notice that reversing words is then equivalent to taking inverses in W. For $s, s' \in S$ we will denote $\Delta^{(m)}_{s,s'}$ the word $\underbrace{ss'ss'\cdots}_{m \text{ terms}}$. If the product ss' has finite order m, we will just denote $\Delta_{s,s'}$ for $\Delta^{(m)}_{s,s'}$; then the relation $\Delta_{s,s'} = \Delta_{s',s}$ holds in W. Writing the relation $(ss')^m = 1$ this way has the advantage that transforming a word by the use of this relation does not change the length — this will be useful. This kind of relation is called a *braid relation* because it is the kind of relations which defines the braid groups, groups related to the Coxeter groups which have a topological definition.

Definition 4.1. *A pair (W, S) where S is a set of involutions generating the group W is a* Coxeter system *if*

$$\langle s \in S \mid s^2 = 1, \Delta_{s,s'} = \Delta_{s',s} \textit{ for pairs } s, s' \in S \textit{ such that } ss' \textit{ has finite order}\rangle$$

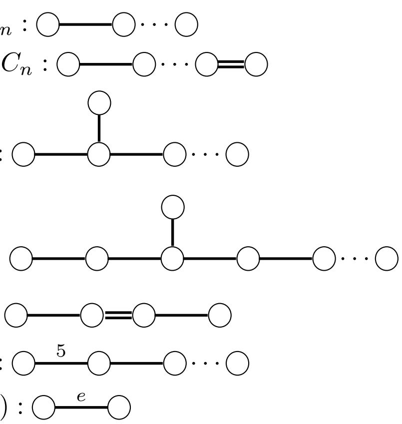
is a presentation of W.

We may ask if a presentation of the above kind defines always a Coxeter system. That is, given a presentation with relations $\Delta^{(m)}_{s,s'} = \Delta^{(m)}_{s',s}$, is m the order of ss' in the defined group? This is always the case, but it is not obvious. We shall prove it by constructing a faithful representation of the group defined by the above presentation where the image of elements of S are reflections and where ss' has the expected order.

If W contains a set S such that (W, S) is a Coxeter system we say that W is a *Coxeter group* and that S is a *Coxeter generating set*. Considering that W has a faithful reflection representation we will also some times call S the *generating reflections* of W, and the set R of W-conjugates of elements of S the *reflections* of W.

Coxeter groups are represented by their *Coxeter graph* which is a graph with vertices S and an edge between s and s' when the order m of ss' is > 2. This edge is labelled by the order of ss'. The label is ∞ if m is infinite. The label is omitted if $m = 3$, and instead of a label the edge is doubled when $m = 4$ and tripled when $m = 6$.

Here is a preview of the classification: the finite irreducible Coxeter groups are



A_n : (n vertices)
B_n/C_n : (n vertices, $n \geq 2$)
D_n : (n vertices, $n \geq 4$)
E_n : (n vertices, $n = 6, 7, 8$)
F_4 :
H_n : (n vertices, $n = 3, 4$)
$I_2(e)$: ($e = 5$ or $e \geq 7$)

A finite Coxeter group is called a *Weyl group* if it has a reflection representation over $\mathbb{Q}$. These groups are particularly important in mathematics; they are those which occur in the classifications mentioned at the beginning. The Weyl groups are the types A, B, D, E, F and $G_2 := I_2(6)$ — note that $I_2(2) = A_1 \times A_1$, $I_2(3) = A_2$ and $I_2(4) = B_2$ are also Weyl groups.

Characterizations of Coxeter groups

Theorem 4.2. *Let W be a group generated by the set S of involutions. Then the following are equivalent:*

(i) (W, S) is a Coxeter system.

(ii) There exists a map N from W to the set of subsets of R, the set of W-conjugates of S, such that $N(s) = \{s\}$ for $s \in S$ and for $x, y \in W$ we have $N(xy) = N(y) \dot{+} y^{-1}N(x)y$, where $\dot{+}$ denotes the symmetric difference of two sets (the sum $\pmod 2$ of the characteristic functions).

(iii) (Exchange condition) If $s_1 \ldots s_k$ is a reduced expression for $w \in W$ and $s \in S$ is such that $l(sw) \leq l(w)$, then there exists i such that $sw = s_1 \ldots \hat{s}_i \ldots s_k$.

(iv) W satisfies $l(sw) \neq l(w)$ for $s \in S$, $w \in W$, and (Matsumoto's lemma) two reduced expressions of the same word can be transformed one into the other by using just the braid relations. Formally, given any monoid M and any morphism $f : S^* \to M$ such that $f(\Delta_{s,s'}) = f(\Delta_{s',s})$ when ss' has finite order then f is constant on the reduced expressions of a given $w \in W$.

Note that (iii) could be called the "left exchange condition". By symmetry there is a right exchange condition where sw is replaced by ws.

Proof. We first show that (i)⇒(ii). The definition of N may look technical and mysterious, but the intuition is that W has a reflection representation where it acts on a set of roots stable under the action of W (there are two opposed roots attached to each reflection), that these roots are divided into positive and negative by a linear form which does not vanish on any root, and that $N(w)$ records the reflections whose roots change sign by the action of w.

Computing step by step $N(s_1 \dots s_k)$ by the two formulas of (ii), we find

$$N(s_1 \dots s_k) = \{s_k\} \dot{+} \{{}^{s_k} s_{k-1}\} \dot{+} \dots \dot{+} \{{}^{s_k s_{k-1} \dots s_2} s_1\}. \tag{1}$$

Let us show that the function thus defined on S^* factors through W which will show (ii). To do that we need that N is compatible with the relations defining W, that is $N(ss) = \emptyset$ and $N(\Delta_{s,s'}) = N(\Delta_{s',s})$. This is straightforward.

We now show (ii)⇒(iii). We will actually check the right exchange condition; by symmetry if (i) implies this condition it also implies the left condition. We first show that if $s_1 \dots s_k$ is a reduced expression for w, then $|N(w)| = k$, that is all the elements of R which appear on the RHS of (1) are distinct. Otherwise, there would exist $i < j$ such that $s_k \dots s_i \dots s_k = s_k \dots s_j \dots s_k$; then $s_i s_{i+1} \dots s_j = s_{i+1} s_{i+2} \dots s_{j-1}$ which contradicts that the expression is reduced.

We next observe that $l(ws) \le l(w)$ implies $l(ws) < l(w)$. Indeed $N(ws) = \{s\} \dot{+} s^{-1} N(w) s$ thus by the properties of $\dot{+}$ we have $l(ws) = l(w) \pm 1$. Also, if $l(ws) < l(w)$, we must have $s \in s^{-1} N(w) s$ or equivalently $s \in N(w)$. It follows that there exists i such that $s = s_k \dots s_i \dots s_k$, which multiplying on left by w gives $ws = s_1 \dots \hat{s}_i \dots s_k$ q.e.d.

We now show (iii)⇒(iv). The exchange condition implies $l(sw) \ne l(w)$ because if $l(sw) \le l(w)$ it gives $l(sw) < l(w)$. Given $f : S^* \to M$ as in (iv) we use induction on $l(w)$ to show that f is constant on reduced expressions. Otherwise, let $s_1 \dots s_k$ and $s'_1 \dots s'_k$ be two reduced expressions for the same element w whose image by f differ. By the exchange condition there exists i such that $s'_1 s_1 \dots s_k = s_1 \dots \hat{s}_i \dots s_k$ in W, thus $s'_1 s_1 \dots \hat{s}_i \dots s_k$ is another reduced expression for w. If $i \ne k$ we may apply induction to deduce that $f(s_1 \dots s_k) = f(s'_1 s_1 \dots \hat{s}_i \dots s_k)$ and similarly apply induction to deduce that $f(s'_1 \dots s'_k) = f(s'_1 s_1 \dots \hat{s}_i \dots s_k)$, a contradiction. Thus $i = k$ and $s'_1 s_1 \dots s_{k-1}$ is a reduced expression for w such that $f(s'_1 s_1 \dots s_{k-1}) \ne f(s_1 \dots s_k)$.

Arguing the same way, starting this time from the pair of expressions $s_1 \dots s_k$ and $s'_1 s_1 \dots s_{k-1}$, we get that $s_1 s'_1 s_1 \dots s_{k-2}$ is a reduced expression for w such that

$$f(s_1 s'_1 s_1 \dots s_{k-2}) \ne f(s'_1 s_1 \dots s_{k-1});$$

Going on this process will stop when we get two reduced expressions of the form $\Delta^{(m)}_{s_1,s'_1}$, $\Delta^{(m)}_{s'_1,s_1}$, such that $f(\Delta^{(m)}_{s_1,s'_1}) \ne f(\Delta^{(m)}_{s'_1,s_1})$. We cannot have m greater that the order of $s_1 s'_1$ since the expressions are reduced, nor less than that order, because the order would be smaller. And we cannot have m equal to the order of $s_1 s'_1$ because this contradicts the assumption.

We finally show (iv)⇒(i). (i) can be stated as: given any group G and a morphism of monoids $f : S^* \to G$ such that $f(s)^2 = 1$ and $f(\Delta_{s,s'}) = f(\Delta_{s',s})$ then f factors through a morphism $g : W \to G$. Let us define g by $g(w) =$

$f(s_1 \dots s_k)$ when $s_1 \dots s_k$ is a reduced expression for w. By (iv) the map g is well-defined. To see that g factors f we need to show that for any expression $w = s_1 \dots s_k$ we have $g(w) = f(s_1 \dots s_k)$. This will follow by induction on the length of the expression if we show that $f(s)g(w) = g(sw)$ for $s \in S, w \in W$. If $l(sw) > l(w)$ this equality is immediate from the definition of g. If $l(sw) < l(w)$ we use $f(s)^2 = 1$ to rewrite the equality $g(w) = f(s)g(sw)$ and we apply the reasoning of the first case. Finally $l(sw) = l(w)$ is excluded by assumption. □

Exercise 4.3. Show that $(\mathfrak{S}_n, \{(i, i+1)\}_{i=1\dots n-1})$ is a Coxeter system (of type A_{n-1}) by showing that $N(w) = \{(i,j) \mid i < j \text{ and } w(i) > w(j)\}$ satisfies the assumptions of 4.2(ii)).

Exercise 4.4. This time we look at the *hyperoctaedral* group B_n which is the group W of permutations of $\{-n, \dots, -1, 1, \dots, n\}$ which preserves the pairs $\{-i, i\}$. Show that (W, S) is a Coxeter system where

$$S = \{(-1,1), (1,2)(-1,-2), \dots, (n-1,n)(-n+1,-n)\}$$

by showing that

$$l(w) = 1/2|\{-n \le i < j \le n \mid w(i) > w(j)\}| + 1/2|\{1 \le i \le n \mid w(i) < 0\}|.$$

Exercise 4.5. The group D_n is the subgroup of B_n where elements have an even number of sign changes. This time the Coxeter generating set is

$$S = \{(-1,2)(1,-2), (1,2)(-1,-2), \dots, (n-1,n)(-n+1,-n)\}$$

and

$$l(w) = 1/2|\{-n \le i < j \le n \mid w(i) > w(j)\}| - 1/2|\{1 \le i \le n \mid w(i) < 0\}|.$$

Finite Coxeter groups: the longest element

Proposition 4.6. *Let (W,S) be a Coxeter system. Then the following properties are equivalent for an element $w_0 \in W$:*

(i) $l(w_0 s) < l(w_0)$ for all $s \in S$.

(ii) $l(w_0 w) = l(w_0) - l(w)$ for all $w \in W$.

(iii) w_0 has maximal length amongst elements of W.

If such an element exists, it is unique and it is an involution, and W is finite.

Proof. It is clear that (ii) implies (iii) and that (iii) implies (i).

To see that (i) implies (ii), we will show by induction on $l(w)$ that w_0 as in (i) has a reduced a expression ending by a reduced expression for w^{-1}. Write $w^{-1} = vs$ where $l(v) + l(s) = l(w)$. By induction we may write $w_0 = yv$ where $l(w_0) = l(y) + l(v)$. The (right) exchange condition, using that $l(w_0 s) < l(w_0)$

but vs is reduced, shows that $w_0 s = \hat{y}v$ where $\hat{y}$ represents y with a letter omitted. It follows that $\hat{y}vs$ is a reduced expression for w_0.

An element satisfying (ii) is an involution since $l(w_0^2) = l(w_0) - l(w_0) = 0$ an is unique since another w_1 has same length by (iii) and $l(w_0 w_1) = l(w_0) - l(w_1) = 0$ thus $w_1 = w_0^{-1} = w_0$.

If w_0 as in (i) exists then S is finite since $S \subset N(w_0)$ and W is then finite by (iii). □

Exercise 4.7. Let $(W, \{s, s'\})$ be a Coxeter system with $m = m_{s,s'} < \infty$ (type $I_2(m)$). Show that if $m \equiv 2 \pmod 4$ then $(W, \{s, w_0 s', w_0\})$ is also a Coxeter system.

5 Finite real reflection groups

In this section $W \subset \mathrm{GL}(V)$ is a finite reflection group on a finite-dimensional space $V = \mathbb{R}^n$. It is associated to the W-invariant hyperplane system $\mathcal{A}_W = \{H_s\}_{s \in \mathrm{Ref}(W)}$.

Lemma 5.1. *Given $H \in \mathcal{A}_W$, there is a unique reflection $s_H \in W$ such that $H_{s_H} = H$.*

Proof. A reflection of hyperplane H belongs to $C_W(H)$. Since $C_W(H)$ is finite, H has a $C_W(H)$-stable complement (Maschke's theorem), which is a line. The finite group $C_W(H)$ is determined by its action on this line, which is ± 1 since they are the only elements of finite order of $\mathbb{R}$. □

We will see that this lemma definitely fails when W is infinite.

The connected components of $V - \bigcup_{H \in \mathcal{A}_W} H$ are called *chambers* of the arrangement $\mathcal{A}_W$; given a chamber C the *walls* of C are the $H \in \mathcal{A}_W$ such that $H \cap \overline{C}$ contains a nonempty open set of H.

We show now W is a Coxeter group by using yet another characterization of Coxeter groups:

Lemma 5.2. *Let W be group generated by the set S of involutions and let $\{D_s\}_{s \in S}$ be a set of subsets of W such that:*

- $D_s \ni 1$.
- $D_s \cap sD_s = \emptyset$.
- *If for $s, s' \in S$ we have $w \in D_s, ws' \notin D_s$ then $ws' = sw$.*

Then (W, S) is a Coxeter system, and $D_s = \{w \in W \mid l(sw) > l(w)\}$.

Proof. We will show the exchange condition. Let $s_1 \dots s_k$ be a reduced expression for $w \notin D_s$ and let i be minimal such that $s_1 \dots s_i \notin D_s$; we have $i > 0$ since $1 \in D_s$. From $s_1 \dots s_{i-1} \in D_s$ and $s_1 \dots s_i \notin D_s$ we get $ss_1 \dots s_{i-1} = s_1 \dots s_i$, whence $sw = s_1 \dots \hat{s}_i \dots s_k$ thus $l(sw) < l(w)$ and we have checked the exchange condition in this case. If $w \in D_s$ then $sw \notin D_s$ by the first part $l(w) < l(sw)$ so we have nothing to check. □

Notice that V affords a W-invariant scalar product, by the

Lemma 5.3. *If $W \subset \mathrm{GL}(V)$ is a finite subgroup, there exists a symmetric definite positive bilinear form on V (a scalar product) which is W-invariant.*

Proof. Choose a form B which has the required properties excepted W-invariance. Then $\sum_{w\in W} B(wx, wy)$ is W-invariant and inherits the required properties from B. □

It follows that the reflections in W are orthogonal, since different eigenspaces are orthogonal for an invariant scalar product.

Proposition 5.4. *Let W be a finite reflection group in a finite-dimensional vector space V. Then*

(i) Let C be a chamber, $\mathcal{M}$ the set of its walls, and let $S = \{s_H \mid H \in \mathcal{M}\}$. Then (W, S) is a Coxeter system, and $m_{s_H,s_{H'}} = |\{H'' \in \mathcal{A}_W \mid H'' \supset H \cap H'\}|$.

(ii) Let $x \in V$ and let C be a chamber such that $x \in \overline{C}$. Then the group $C_W(x)$ is generated by reflections with respect to the walls of C containing x, and $C_W(x) = C_W(F)$ where F is the intersection of the walls of C containing x.

Proof. Let W' be the subgroup of W generated by S. We first show that for any $x \in V$, there exists an element of the W'-orbit of x in $\overline{C}$. Choose $a \in C$ and let y an element of the W'-orbit of x at minimal distance of a. Then we claim $y \in \overline{C}$. Otherwise, there exists a wall H_s of C which separates a and y, hence $s_H(y)$ is closer to a than y (remember that the reflections are orthogonal).

It follows that any chamber is in the W'-orbit of C. Indeed, for a chamber C' we have seen there exists $w \in W'$ such that $w(C') \cap \overline{C} \neq \emptyset$, which implies $w(C') = C$. It follows also that W' contains all reflections of S. Indeed take any $s_H \in W$, let C' be a chamber which has H as a wall and let $w \in W'$ be such that $w(C') = C$. Then $w(H)$ is a wall of C, thus $ws_Hw^{-1} \in S$ which implies $s_H \in W'$. We get thus $W' = W$.

To show (i) we now apply lemma 5.2 by defining for $s \in S$ the set D_s to consist of the $w \in W$ such that C and $w(C)$ are on the same side of H_s. The first two items of 5.2 are trivial. It remains to show that if $w \in D_s$ and $ws' \notin D_s$, then $ws' = sw$. By assumption $ws'(C)$ and $w(C)$ are on different sides of H_s, thus $s'(C)$ and C are on different sides of $w^{-1}(H_s)$. But $H_{s'}$ is the only wall separating $s'(C)$ and C thus $H_{s'} = w^{-1}(H_s)$, *i.e.* $s = w^{-1}sw$ q.e.d.

Before showing the stated value for $m_{s_H,s_{H'}}$, let us show (ii). We consider the Coxeter system defined by C and show by induction on $l(w)$ (the length for this Coxeter system) that $w(x) = x$ implies that w belongs to the subgroup generated by the s_H where H is a wall of C containing x. If $w \neq 1$, there exists a wall H of C such that $l(s_Hw) < l(w)$. Since $w(x) = x$ we have $x \in \overline{C} \cap w(\overline{C})$; on the other hand, since $w \notin D_{s_H}$, the hyperplane H separates $w(C)$ and C, which implies $x \in H$. Finally by the conclusion of (iii) any element of $C_W(x)$ fixes F.

Finally, to show the stated value for $m_{s_H,s_{H'}}$ we can reduce to the case of rank 2: we replace V by the plane $V' = (H \cap H')^\perp$ and W by the subgroup $C_W(H \cap H')$ of $\mathrm{GL}(V')$. By (ii) this group is generated by s_H and $s_{H'}$. In the plane V' the product $s_H s_{H'}$ acts by a rotation of angle 2θ, if θ is the angle between H and H'; thus the order of $s_H s_{H'}$ is π/θ. A finite group of $\mathrm{GL}(\mathbb{R}^2)$ generated by reflections is dihedral, with m hyperplanes if π/m is the smallest angle between two hyperplanes, which is the case of θ since H and H' are walls of a chamber. □

Remark 5.5. The proof that we get a Coxeter group can be extended to the case of groups generated by affine reflections, but finite modulo the translations they contain. One gets this way in particular the *affine Weyl groups*.

It will follow from (ii) that $C_W(x)$ is a Coxeter groups: we will show in the next lecture that the subgroup generated by a subset of S is a Coxeter group.

From the above theorem, it follows that W is in bijection with the set of chambers: each chambers is uniquely of the form $w(C)$.

The chamber $-C$ is the unique chamber separated from C by all the elements of $\mathcal{M}$. It follows that $-C = w_0(C)$, where w_0 is the longest element of W introduced in 4.6.

Lemma 5.6. *Assume that $W \subset \mathrm{GL}(V)$ is an irreducible group which contains at least one reflection. Then the only elements of* $\mathrm{End}(V)$ *which commute with W are the scalars.*

Proof. Let $u \in \mathrm{End}(V)$ commute with W, thus in particular to a reflection s of W. Then u stabilizes the line $\mathrm{Ker}(1-s)$, thus acts by some scalar α on it. Then $u - \alpha\,\mathrm{Id}$ is still W-invariant and has a non-trivial kernel, which is stabilized by W. As W acts irreducibly on V this kernel must be the whole of V, thus u is a scalar. □

As a particular case, note that if a finite reflection group contains a non-trivial central element, this element is a scalar, thus equal to -1 since it is of finite order. And by the remark above lemma 5.6, it is w_0.

Lemma 5.7. *Let $W \subset \mathrm{GL}(V)$ be a finite subgroup such that the only elements of* $\mathrm{End}(V)$ *which commute with W are the scalars. Then there is a unique bilinear form invariant by W up to a scalar.*

Proof. First notice that W is irreducible, otherwise there is a W-invariant non-trivial subspace V' and there is a W-invariant projector to V'.

Then notice that a W-invariant bilinear form B is non-degenerate otherwise the orthogonal of V for B would be a proper W-invariant subspace. It follows that B is an isomorphism between V and V^*. As two such isomorphisms differ by an element of $\mathrm{GL}(V)$, another W-invariant bilinear form must be of the form $(x, y) \mapsto B(u(x), y)$ for some $u \in \mathrm{GL}(V)$. Now for $w \in W$ we have $B(u(x), y) = B(u(w(x)), w(y)) = B((w^{-1}uw)(x), y)$ from which it results that $w^{-1}uw = u$, thus u is a scalar. □

Proposition 5.8. *Let W be as in 5.4 and irreducible. Let B be a W-invariant scalar product and for $H \in \mathcal{M}$ let e_H be the unit ($B(e_H, e_H) = 1$) vector orthogonal to H pointing towards C. Then*

(i) The e_H are linearly independent.

(ii) We have $B(e_H, e_{H'}) = -\cos(\pi/m_{s_H,s_{H'}})$.

Proof. As in the proof of 5.4 to see (ii) one can look at the situation in the 2-dimensional space $V' = (H \cap H')^\perp$.

To see (i), notice first that (ii) says that $H \neq H'$ implies $B(e_H, e_{H'}) \leq 0$. By contradiction, assume there was a dependence relation. By separating the positive and negative coefficients, this relation can be written $\sum_{H\in\mathcal{M}_1} c_H e_H = \sum_{H\in\mathcal{M}_2} c_H e_H$. Let v be the common sum on both sides. If $v = 0$ then we compute the scalar product with some element $w \in C$; the choice of e_H implies $B(w, e_H) > 0$ so since the c_H on each side are positive they have to be 0. If $v \neq 0$ then $0 < B(v, v) = \sum_{H\in\mathcal{M}_1, H'\in\mathcal{M}_2} c_H c_{H'} B(e_H, e_{H'})$, a contradiction since $B(e_H, e_{H'}) \leq 0$. □

We see in particular that $|S| = \dim V$ when W is irreducible.

Geometric representation of Coxeter groups

A Coxeter system W, S is defined by the *Coxeter matrix* $\{m_{s,s'}\}_{s,s'\in S}$ where $m_{s,s'}$ is the order of ss' (thus the entries are in $\mathbb{N} \cup \{\infty\}$).

The following proposition "implements" the remark after the definition 4.1:

Theorem 5.9. *Any symmetric matrix whose entries off-diagonal are in $\mathbb{N}_{\geq 2} \cup \{\infty\}$ and on the diagonal are 1 is the Coxeter matrix of some Coxeter group W.*

We will show this theorem by constructing W as a reflection group in a vector space with basis indexed by S.

Proof. The construction is suggested by the case of finite reflection groups. On $V = \mathbb{R}^S$, with basis $\{e_s\}_{s\in S}$, we define a bilinear form by $\langle e_s, e_{s'} \rangle = -\cos(\pi/m_{s,s'})$, where by convention $\pi/m_{s,s'} = 0$ if $m_{s,s'} = \infty$.

Lemma 5.10. *The map $s \mapsto (x \mapsto x - 2\langle x, e_s \rangle e_s)$ defines a reflection representation on V of $W = \langle s \in S \mid s^2 = 1, (ss)^{m_{s,s'}} = 1\rangle$ for which $\langle -, - \rangle$ is a W-invariant bilinear form.*

Proof. In the constructed representation it is clear that s acts by a reflection. Let us check that a reflection s preserves $\langle -, - \rangle$:

$$\begin{aligned}\langle sx, sy \rangle &= \langle x - 2\langle x, e_s \rangle e_s, y - 2\langle y, e_s \rangle e_s \rangle \\ &= \langle x, y \rangle - 2\langle x, e_s \rangle\langle e_s, y \rangle - 2\langle y, e_s \rangle\langle x, e_s \rangle + 4\langle x, e_s \rangle\langle y, e_s \rangle\langle e_s, e_s \rangle \\ &= \langle x, y \rangle\end{aligned}$$

where the last equality uses $\langle e_s, e_s \rangle = 1$.

Let us compute the order of ss'. Let $\lambda = \langle e_s, e_{s'} \rangle$. We get

$$ss'(e_s) = s(e_s - 2\lambda e'_s) = -e_s - 2\lambda(e_{s'} - 2\lambda e_s) = (4\lambda^2 - 1)e_s - 2\lambda e_{s'}$$

and $ss'(e_{s'}) = 2\lambda e_s - e_{s'}$. If $\lambda = -1$, then $ss'(e_s + e_{s'}) = e_s + e_{s'}$, whence, iterating the first formula which can be written $ss'(e_s) = 2(e_s + e_{s'}) + e_s$, we get $(ss')^m(e_s) = 2m(e_s + e_{s'}) + e_s$ thus ss' has infinite order.

When $\lambda \neq -1$, we do the computation in $\mathbb{C} \simeq \mathbb{R}^2$ with the usual scalar prodict: we identifying e_s to 1 and $e_{s'}$ to $-e^{-i\theta}$ where $\theta = \pi/m_{s,s'}$. We find

$$ss'(e_s) = (4\cos^2\theta - 1) - 2\cos\theta e^{-i\theta} = (e^{i\theta} + e^{-i\theta})^2 - 1 - (e^{i\theta} + e^{-i\theta})e^{-i\theta} = e^{2i\theta}$$

and $ss'(e_{s'}) = -2\cos\theta + e^{-i\theta} = -e^{i\theta} = e^{2i\theta}e_{s'}$, thus ss' acts by a rotation by $2\pi/m_{s,s'}$. Since ss' acts trivially on the $\langle -, - \rangle$-orthogonal of the subspace spanned by e_s and $e_{s'}$, its order is indeed $m_{s,s'}$. □

We have already seen the claim that our matrix is a Coxeter matrix since $m_{s,s'}$ is the order of ss' in the constructed group. But we have not yet seen the claim that W is a reflection group since we have not shown that our representation is injective.

For this, we will get the analogous result to 5.4, that W acts faithfully on the set of chambers, but for the contragredient representation on V^* (which is not isomorphic in general to the representation on V when W is infinite). For $f^* \in V^*$, the contragredient action is given by $(sf^*)(x) = f^*(sx)$. If $\{e^*_s\}_{s\in S}$ is the dual basis to $\{e_s\}_{s\in S}$, since for $s' \neq s$ we have $e^*_{s'}(sx) = e^*_{s'}(x - 2(x, e_s)e_s) = e^*_{s'}(x)$ we have that $se^*_{s'} = e^*_{s'}$ for $s' \neq s$, thus the reflecting hyperplane for the contragredient action on V^* of s is defined by the linear form e_s. For $I \subset S$ let $C_I = \{x^* \in V^* \mid x^*(e_s) > 0 \forall s \in I\}$, and let $C = C_S$, a chamber for the dual hyperplane system. The faithfulness of the representation will follow from the

Lemma 5.11. *(Tits) If $w \neq 1$, then $w(C) \cap C = \emptyset$.*

Proof. We start with a general lemma on *parabolic subgroups* of Coxeter groups.

Lemma-Definition 5.12. *Let (W, S) be a Coxeter system, let I be a subset of S, and let W_I be the subgroup of W generated by I. Then (W_I, I) is a Coxeter system. An element $w \in W$ is said* I-reduced *if it satisfies one of the equivalent conditions:*

(i) For any $v \in W_I$, we have $l(vw) = l(v) + l(w)$.

(ii) For any $s \in I$, we have $l(sw) > l(w)$.

(iii) w is of minimal length in the coset $W_I w$.
There is a unique I-reduced element in $W_I w$.

Proof. It is clear that (W_I, I) satisfies the exchange condition (a reduced expression in W_I is reduced in W by the exchange condition, and then satisfies the exchange condition in W_I) thus is a Coxeter system.

It is clear that (iii)⇒(ii) since (iii) implies $l(sw) \geq l(w)$ when $s \in I$. Let us show that not (iii)⇒ not (ii). If w' does not have minimal length in $W_I w'$, then $w' = vw$ with $v \in W_I$ and $l(w) < l(w')$; adding one by one the terms of a reduced expression for v to w, applying at each stage the exchange condition, we find that w' has a reduced expression of the shape $\hat{v}\hat{w}$ where $\hat{v}$ (resp. $\hat{w}$) denotes a subsequence of the chosen reduced expression. As $l(\hat{w}) \leq l(w) < l(w')$, we have $l(\hat{v}) > 0$, thus w' has a reduced expression starting by an element of I, thus w' does not satisfy (ii).

(i)$\Rightarrow$(iii) is clear. Let us show not (i)$\Rightarrow$ not (iii). If $l(vw) < l(v) + l(w)$ then a reduced expression for vw has the shape $\hat{v}\hat{w}$ where $l(\hat{w}) < l(w)$. Then $\hat{w} \in W_I w$ and has a length smaller than that of w.

Finally, an element satisfying (i) is clearly unique in $W_I w$. □

For $I \subset S$ and $w \in W$, let $h_I(w) \in W_I$ be the unique element such that $h_I(w)^{-1}w$ is I-reduced. We will show by induction on $l(w)$ that

$$w(C) \subset h_I(w)C_I \text{ for all } w \in W \text{ and all } I \subset S, |I| \leq 2. \tag{$*$}$$

Tit's lemma will follow since for $w \neq 1$, there exists $s \in S$ such that $h_{\{s\}}(w) = s$, whence $w(C) \subset sC_{\{s\}} = -C_{\{s\}}$ and $-C_{\{s\}} \cap C = \emptyset$.

The start of the induction is for $w = 1$ where (*) reduces to $C \subset C_I$, which is clear.

Assume now that $l(w) > 0$ and (*) holds for all $w' \in W$ such that $l(w') < l(w)$. We first show (*) for $I = \{s\}$. If $h_{\{s\}}(w) = s$ then $w = sw'$ with $l(w') < l(w)$ and $h_{\{s\}}(w') = 1$ whence $w(C) = sw'(C) \subset sC_{\{s\}}$ by induction q.e.d.

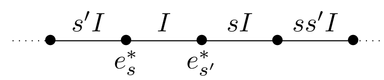
If $h_{\{s\}}(w) = 1$ let $s' \in S$ be such that $h_{\{s'\}}(w) = s'$ and write $w = h_{\{s,s'\}}(w)w'$. Since $l(w') < l(w)$ induction gives $w(C) = h_{\{s,s'\}}(w)w'(C) \subset h_{\{s,s'\}}(w)(C_{\{s,s'\}})$. It is thus sufficient to solve the question for the dihedral group $W_{\{s,s'\}}$, that is to show that if $w' = h_{\{s,s'\}}(w) \in W_{\{s,s'\}}$ satisfies $h_{\{s\}}(w') = 1$ then $w'C_{\{s,s'\}} \subset C_{\{s\}}$. Further, we can work in the quotient V'^* of V^* by the $e^*_{s''}$ for $s'' \notin \{s, s'\}$ (which is dual to the subspace $V' \subset V$ generated by e_s and $e_{s'}$) since $C_{\{s,s'\}}$ and $C_{\{s\}}$ are preimages of the analogous sets C' and $C'_{\{s\}}$ in this quotient.

It is easy to compute explicitly the contragredient action in V'^*: as before we have $s(e^*_{s'}) = e^*_{s'}$ and

$$(se^*_s)(e_s) = e^*_s(se_s) = -1$$
$$(se^*_s)(e_{s'} = e^*_s(se_{s'}) = e^*_s(e_{s'} - 2(e_{s'}, e_s)e_s) = -2(e_{s'}, e_s)$$

thus $s(e^*_s) = -e^*_s - 2(e_{s'}, e_s)e^*_{s'}$; and we have a symmetric formula for the action of s'.

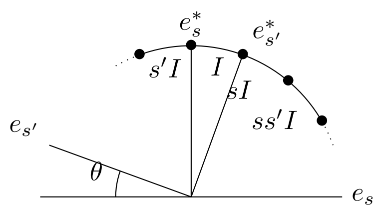
When $m_{s,s'} = \infty$ we have $s(e^*_s) = -e^*_s + 2e^*_{s'}$ and both reflections preserve the affine line $\lambda e^*_s + (1 - \lambda)e^*_{s'}$ through e^*_s and $e^*_{s'}$. On this line s (resp. s') acts as as a reflection with respect to $e^*_{s'}$ (resp e^*_s). The intersection of C' with this affine line is the segment I between e^*_s and $e^*_{s'}$. The chamber system is described by its intersection with this affine line. The picture looks like



from which it can readily be seen that if w' has a reduced expression starting with s, the image of I by w' is on the right side of $e_{s'}^*$, and this right side is the intersection of the line with $sC'_{\{s\}}$.

Remark 5.13. In the subspace V', the reflection hyperplanes of s and s' are both spanned by $e_s + e_{s'}$, and the fundamental chamber is fixed by ss'. This shows the need to go to the dual space.

If $m_{s,s'} < \infty$ we can make a similar picture intersecting this time with the unit circle. The intersection I of C' with the unit circle is the arc between e_s^* and $e_{s'}^*$; the transforms sI,$s'I$,etc... are arcs as above, with $\Delta_{s,s'}I = -I$.



We finally show (*) for $I = \{s, s'\}$. If $h_{\{s,s'\}}(w) = 1$ then $h_{\{s\}}(w) = h_{\{s'\}}(w) = 1$ and by the previous case $w(C) \subset C_{\{s\}} \cap C_{\{s'\}} = C_{\{s,s'\}}$ q.e.d. Otherwise $w = h_{\{s,s'\}}(w)w'$ where $w'(C_{\{s,s'\}}) = C_{\{s,s'\}}$ whence the result. □

□

Classification of finite Coxeter groups

Proposition 5.14. *Let Γ be a Coxeter graph, W the corresponding Coxeter group, V the geometric representation of W defined in 5.9 and $B(\Gamma)$ the corresponding W-invariant bilinear form. Then*

(i) V is irreducible if and only if Γ is connected.

(ii) W is finite if and only if $B(\Gamma)$ is definite positive.

Proof. For (i), it is clear that V is the direct sum of representations corresponding to different connected components of Γ. Conversely, assume that $U \subset V$ is a W-stable subspace. For any $s \in S$, either $e_s \in U$ or $B(e_s, U) = 0$: if U is s-stable and is not a subspace of H_s then $U \ni e_s$ (since if $x \in U, x \notin H_s$, then $s(x) - x \in U$ and is a multiple of e_s); and if $U \subset H_s$ then $B(e_s, U) = 0$ by definition. Thus U defines a partition $S = S_1 \coprod S_2$ where $S_1 = \{s \mid e_s \in U\}$ and $S_2 = \{s \mid B(e_s, U) = 0\}$ such that if $s \in S_1$ and $S' \in S_2$ then $\langle e_s, e_{s'} \rangle = 0$, *i.e.* a partition of Γ into two connected components.

For (ii), we may assume V irreducible since $B(\Gamma)$ is definite positive (resp. $W(\Gamma)$ is finite) if and only if this holds for each connected component of Γ. If W is finite then any invariant bilinear form is definite positive by 5.3 and 5.7(ii).

Conversely, if $B(\Gamma)$ is definite positive its orthogonal group is compact and W is a discrete subgroup of this orthogonal group, thus finite.

Here discrete means that there is an open neighbourhood of 0 in $\mathrm{GL}(V*)$ meeting only one element of W: take, for $x \in C$, the set $\{g \in \mathrm{GL}(V^*) \mid g(x) \in C\}$.

A discrete subgroup of a compact group is finite otherwise it would contain a convergent sequence $\{w_n\}_n$. Then $w_n^{-1} w_{n+1}$ would converge to 0 which contradicts discreteness. □

Theorem 5.15. *The only Coxeter graphs giving rise to discrete groups are the graphs of type A, B, D, E, F, G, H, I.*

Proof. The proof has two parts. The first proves that these graphs actually define finite groups. The second proves that only these graphs are possible.

Bothe parts will need the values of $-\cos \pi/m$ and of its square for small values of m, which are in the following table

m	2	3	4	5	6
$-\cos \pi/m$	0	$-1/2$	$-1/\sqrt{2}$	$-(1+\sqrt{5})/4$	$-\sqrt{3}/2$
$(\cos \pi/m)^2$	0	$1/4$	$1/2$	$(3+\sqrt{5})/8$	$3/4$

Let us do the first part of the proof: the diagrams A–I give positive definite forms. It sufficient to prove that the determinants of $B(\Gamma)$ are positive, since subgraphs of graphs as in the theorem are unions of graphs of the same type, which proves the positivity of principal minors, which is sufficient to have a positive definite bilinear form.

To compute the determinant, it will be convenient to:

- multiply by 2 the matrix of $B(\Gamma)$ (this corresponds to replacing e_s by $\sqrt{2}e_s$).
- conjugate by a diagonal matrix, which does not change the principal minors; this will get rid of the irrational entries in types B, F, G.

These operations will bring the matrix $B(\Gamma)$ to the "Cartan matrix" $C(\Gamma)$ (attached to a "root system"); note that we obtain an integral matrix for types A–G, which proves that the corresponding groups are defined over $\mathbb{Q}$.

If the graph with n vertices Γ_n ends with a subgraph of type A_2, we have the pattern $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ on the bottom right corner. If Γ_{n-1} (resp. Γ_{n-2}) is obtained by removing the last vertex (resp. the last 2 vertices) developing with respect to the last line gives the induction formula: $\det C(\Gamma_n) = 2 \det C(\Gamma_{n-1}) - \det C(\Gamma_{n-2})$.

- Starting from: $\det C(A_1) = 2$, $\det C(A_2) = 3$ this gives $\det C(A_n) = n+1$.
- Starting from: $\det C(B_1) = \det C(A_1) = 2$, $\det C(B_2) = 2$ we get $\det C(B_n) = 2$.
- Starting from: $\det C(D_3) = \det C(A_3) = 4$, $\det C(D_2) = \det C(A_1 \times A_1) = (\det C(A_1))^2 = 4$ we get $\det C(D_n) = 4$.

- Starting from: $\det C(D_5) = 4$, $\det C(A_4) = 5$ we get $\det C(E_6) = 3$, $\det C(E_7) = 2$, $\det C(E_8) = 1$.
- Starting from: $\det C(B_3) = 2$, $\det C(A_2) = 3$ we get $\det C(F_4) = 1$.
- We have $\det C(I_2(m)) = 4(1 - cos^2(\pi/m))$.
- Finally starting from: $\det C(A_1) = 2$, $\det C(I_2(5)) = 4(1 - (3+\sqrt{5})/8) = (5-\sqrt{5})/2$ we get $\det C(H_3) = 3 - \sqrt{5}$ and $\det C(H_4) = (7 - 3\sqrt{5})/2$.

□

Note that the values for types A–G are the *connexion index* of the corresponding root system, which is the order of the fundamental group of the corresponding algebraic group.

We now do the second part of the proof: we assume $(-,-)$ is a scalar product and see the conditions this imposes on Γ. We will call spherical such a graph.

From the above table of cosines it follows that if there is an edge between i and j then $(e_i, e_j) \leq -1/2$.

We now observe the following properties of a connected spherical graph Γ:

(i) Any subgraph defined by all the edges delimited by a subset of the vertices is spherical (since it defines a parabolic subgroup).

(ii) Γ is a tree. Indeed, if $s_1, \ldots, s_r$ is a circuit (we may assume there is no bond between the s_i excepted between s_i and s_{i+1}, shortening if need be the circuit) and $v = e_{s_1} + \ldots + e_{s_r}$, then $\langle v, v \rangle = r + 2\sum_{i=1}^{r-1} \langle e_{s_i}, e_{s_{i+1}} \rangle + 2\langle e_{s_r}, e_{s_1} \rangle \leq 0$ since if e_i and e_j are connnected then $\langle e_i, e_j \rangle \leq -1/2$.

(iii) Let s^* be the set of neighbours of $s \in S$ in Γ; then $\sum_{j \in s^*} \langle e_s, e_j \rangle^2 < 1$. Indeed this inequality expresses that e_s is strictly longer that its orthogonal projection to the subspace generated by the e_j for $j \in s^*$, of which the e_j are an orthonormal basis by (ii).

As a consequence of (iii) the possibilities for s^* are:

- $|s^*| = 1$,
- $|s^*| = 2$ with an edge of label 3 and the other of label ≤ 5,
- $|s^*| = 3$ with 3 edges of label 3.

(iv) The graph Γ' obtained by removing an edge of label 3 and gluing the delimited vertices is still spherical. Indeed let $B' = B(\Gamma')$; we have to show that $B'(w, w) > 0$ for any w. If (s, s') is the removed edge, and e is the basis vector for $s = s'$ in Γ', then w is of te form $v + \lambda e$ where v is in the span of $e_{s''}$ for $s'' \neq s, s'' \neq s'$. We have

$$\begin{aligned} B'(v + \lambda e, v + \lambda e) &= B'(v, v) + 2B'(v, \lambda e) + \lambda^2 \\ &= \langle v, v \rangle + 2\lambda \langle v, e_s + e_{s'} \rangle + \lambda^2 \\ &= \langle v + \lambda(e_s + e_{s'}), v + \lambda(e_s + e_{s'}) \rangle - \lambda^2(1 + 2\langle e_s, e_{s'} \rangle) \\ &= \langle v + \lambda(e_s + e_{s'}), v + \lambda(e_s + e_{s'}) \rangle \end{aligned}$$

where the second line uses that $v = v_1 + v_2$ where v_1 is on the e_s-side of Γ and v_2 on the $e_{s'}$-side, so that $B'(v_1, e) = (v_1, e_s) = (v_1, e_s + e_{s'})$ and similarly $B'(v_2, e) = (v_2, e_{s'}) = (v_2, e_s + e_{s'})$.

(v) Γ has at most one edge of label > 3. Otherwise using (iv) we may move the edges of label > 3 together and get a configuration excluded by (iii). By a similar reasoning Γ if gamma has an edge with label > 3 it is a chain; and Γ has a most one order 3 vertex (otherwise similarly these 2 vertices could be moved together to make an order ≥ 4 vertex).

(vi) Given an oriented chain $C = s_1, \ldots, s_i$, define $e(C) := e_{s_1} + 2e_{s_2} \ldots + ie_{s_i}$. Notice that $(e(C), e(C)) = \sum_{k=1}^{i} k^2 - \sum_{k=1}^{i-1} k(k+1) = i^2 - i(i-1)/2 = i(i+1)/2$. Assume now that Γ is a chain with one edge (s, s') of label $m > 3$. The complement is the union of two chains C, C' such that (say) $C \ni s$ and $C' \ni s'$. Orient C (resp. C' so that its last vertex is s (resp. s'). Let i (resp. j) be the length of C (resp C') and assume $i \leq j$. We get $\langle e(C), e(C) \rangle = i(i+1)/2$, $\langle e(C'), e(C') \rangle = j(j+1)/2$ and $\langle e(C), e(C') \rangle = -ij \cos \pi/m$. The inequality $\langle e(C), e(C') \rangle^2 < \langle e(C), e(C) \rangle \langle e(C'), e(C') \rangle$ gives $(i+1)(j+1) > 4ij \cos^2 \pi/m$. Since $2ij \leq 4ij \cos^2 \pi/m$, we have $(i-1)(j-1) < 2$ which, since $i \leq j$ leaves $(1, j)$ and $(2, 2)$ as possibilities for (i, j). Feeding back these values in $(i+1)(j+1) > 4ij \cos^2 \pi/m$, we find for $(2, 2)$ that $\cos^2 \pi/m < \frac{9}{16}$ which implies $m = 4$. For $(1, j)$ we find $\frac{1}{2} + \frac{1}{2j} > \cos^2 \pi/m$ which for $j = 1$ leaves any value of m allowed, for $j = 2, 3$ leaves $m \leq 5$ and for greater j leaves only $m = 4$.

(vii) We finally consider the case where Γ has only edges of label 3 and there exists $s \in \Gamma$ such that $\Gamma - \{s\}$ is the union of 3 chains C,C',C'' of lengths p, q, r respectively. Orient C, C', C'' so their last vertex is a neighbour of s. Let $u = e(C), v = e(C'), w = e(C'')$. Notice that u, v, w are orthogonal to each other. Writing that e_s is longer than its projection on the subspace generated by u, v, w we get

$$\langle e_s, e_s \rangle^2 = 1 > \langle e_s, u \rangle^2 / \langle u, u \rangle + \langle e_s, v \rangle^2 / \langle v, v \rangle + \langle e_s, w \rangle^2 / \langle w, w \rangle,$$

which, taking in account $\langle e_s, u \rangle = -p/2$, $\langle u, u \rangle = p(p+1)/2$, and similarly for v, w can be written $1/(p+1) + 1/(q+1) + 1/(r+1) > 1$, which describes exactly the tri-chains of the theorem.

Exercise 5.16. Let V be the geometric representation of a Coxeter group W as in 5.9. We assume that $\dim V < \infty$ and that the representation is defined over $\mathbb{Z}$, that is there exists a W-invariant lattice in V. Then

- All $m_{s,s'}$ are in $\{2, 3, 4, 6, \infty\}$. Indeed, complete $\{e_s, e_{s'}\}$ by vectors orthogonal to the plane they span to make a basis. Then by the formulae in the proof of 5.10 we get $\text{Trace}(ss') = \dim V + 4(\cos^2 \pi/m - 1)$ which is integral only for the stated values.

- Conversely, if all $m_{s,s'}$ are in $\{2, 3, 4, 6\}$ and we can rescale the bilinear form to get an integral Cartan matrix, the group is defined over $\mathbb{Z}$.