ADDITIVE FAMILIES OF LOW BOREL CLASSES

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(JOINT WORK WITH JIŘÍ SPURNÝ)

First let us recall the following notions.

Definition 1. Let X be a topological space and $\mathcal{A}$ be a family of subset of X.

(i) The family $\mathcal{A}$ is called *σ-discretely decomposable* if for each $A \in \mathcal{A}$ there exist sets $A(n)$, $n \in \omega$, such that $A = \bigcup_{n\in\omega} A(n)$ and $\{A(n);\ A \in \mathcal{A}\}$ is a discrete family for each $n \in \omega$.

(ii) The family $\mathcal{A}$ is said to admit a *σ-discrete refinement* if there exists a refinement of $\mathcal{A}$ that is σ-discrete.

(iii) Let $\mathcal{B}$ be a family of sets in X. The family $\mathcal{A}$ is called *$\mathcal{B}$-additive* if $\bigcup \mathcal{A}' \in \mathcal{B}$ for every $\mathcal{A}' \subset \mathcal{A}$.

The theory of non-separable metric spaces, as developed by A. H. Stone, R. W. Hansell, and others (see, e.g., [11] or [4]), very often relies upon the possibility of decomposing a given family of sets into countably many discrete pieces. If this is the case, standard methods of descriptive set theory of separable spaces can be applied to get results analogous to the separable ones.

R. W. Hansell showed in [3, Theorem 2] that any Suslin-additive disjoint cover of an absolute Suslin metric space is σ-discretely decomposable. By an improvement due to J. Kaniewski and R. Pol (see [7, Theorem 1]), every point-finite Suslin-additive cover of an absolute Suslin space is σ-discretely decomposable.

If the assumption of point-finiteness is weakened to point-countability, a more appropriate notion of decomposability is the one of σ-discrete refinement. By [5, Theorem 3.1(b)], under suitable set theoretical assumptions there exists a point-countable Suslin-additive family in a Polish space that is not σ-discretely refinable. On the other hand, R. Pol showed in [8, Theorem 1.3] that any point-countable Borel-additive family of sets in an arbitrary metrizable space admits a σ-discrete refinement provided each member of the family is of weight at most $\aleph_1$. W. G. Fleissner showed in [1] (see also [2, Theorem 3N]) that under an additional axiom of set theory every point-countable Suslin-additive family is σ-discretely refinable.

Nevertheless, the central problem we are trying to solve is still wide open.

Question. Is it provable in ZFC that every point-countable Borel-additive cover of a complete metric space has a σ-discrete refinement?

R. W. Hansell answered Question affirmatively if the family is $\mathbf{\Sigma}^0_2$-additive (see [6, Theorem 3.3]). He also showed in [6, Example] that a $\mathbf{\Sigma}^0_2$-additive cover of a complete space need not admit a σ-discrete refinement if the assumption of point-countability is omitted. Using a method different to the one used in [6] J. Spurný obtained in [9, Theorem 6] that a $\mathbf{\Pi}^0_2$-additive cover of a complete metric space has a σ-discrete refinement. The method of the proof was an application of a Hurewicz-like construction. Our aim is to show that this construction can be refined to yield the following result further supporting the belief that the answer to Question is affirmative.

Theorem 2 (J. S.–M. Z.)**.** *Let $\mathcal{A}$ be an $\mathbf{\Pi}^0_3$-additive family in an absolute Suslin space. If $\mathcal{A}$ is point-countable, then $\mathcal{A}$ is σ-discretely refinable.*

The next theorem is one of the main tools used in our proof.

Theorem 3 (D. Fremlin ([2]))**.** *Let $\mathcal{A}$ be a cover of a completely metrizable space X such that $\mathcal{A}$ is a Borel-additive family and point-countable. Then there exists $A \in \mathcal{A}$ which is nonmeager.*

Further we proceed by contradiction. We assume that we have a $\mathbf{\Pi}_3^0$-additive cover $\mathcal{A}$ of a complete metric space X which is point-countable but does not admit σ-discrete refinement. Then we construct a homeomorphism φ of the Cantor space C to X such that a true $\mathbf{\Sigma}_3^0$ subset of $\varphi(C)$ is equal to $\varphi(C) \cap \bigcup \mathcal{A}'$ for some subfamily $\mathcal{A}' \subset \mathcal{A}$. This is a contradiction since $\bigcup \mathcal{A}'$ is $\mathbf{\Pi}_3^0$. Finally we generalize this result from complete metric spaces to absolute Suslin spaces (see [10] for details).

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