Connecting Heterogeneous Structures in Banach Space Theory

[1] Department of Mathematics
National Technical University of Athens
Athens, Greece

2008 / Paris

The saturation method

- The saturation method, invented by B.S. Tsirelson, is one of the most important and fundamental discoveries in the recent Banach space theory. This method concerns the following class of problems:
- **Problem**: Assume that there exists a Banach space X containing a Schauder basic sequence $(x_n)_n$ satisfying a property (A).
  Does there exist another space Y such that every Schauder basic sequence of Y satisfies property (A)?

The saturation method

- The saturation method, invented by B.S. Tsirelson, is one of the most important and fundamental discoveries in the recent Banach space theory. This method concerns the following class of problems:

- **Problem**: Assume that there exists a Banach space X containing a Schauder basic sequence $(x_n)_n$ satisfying a property (A).
  Does there exist another space Y such that every Schauder basic sequence of Y satisfies property (A)?

From J. Schreier (1930) to B.S. Tsirelson (1971).

- **(J. Schreier).** There exists a Banach space X containing a weakly null sequence $(x_n)_{n\in\mathbb{N}}$ with no Cesaro summable subsequence.
- **(B.S. Tsirelson)** There exists a reflexive Banach space T such that every normalized basic sequence has no Cesaro summable subsequence.

From J. Schreier (1930) to B.S. Tsirelson (1971).

- **(J. Schreier).** There exists a Banach space X containing a weakly null sequence $(x_n)_{n\in\mathbb{N}}$ with no Cesaro summable subsequence.
- **(B.S. Tsirelson)** There exists a reflexive Banach space T such that every normalized basic sequence has no Cesaro summable subsequence.

The paradigms

From J. Schreier (1930) to B.S. Tsirelson (1971).

- **(J. Schreier).** There exists a Banach space X containing a weakly null sequence $(x_n)_{n\in\mathbb{N}}$ with no Cesaro summable subsequence.
- **(B.S. Tsirelson)** There exists a reflexive Banach space T such that every normalized basic sequence has no Cesaro summable subsequence.

From B. Maurey – H. Rosenthal (1977) to W.T. Gowers – B. Maurey

- **(B. Maurey - H. Rosenthal)** There exists a Banach space X containing a weakly null sequence with no unconditional subsequence.
- **(W.T. Gowers - B. Maurey)** There exists a reflexive Banach space X with no unconditional Schauder basic sequence.
- Gowers Maurey construction is based on Schlumprecht space, which is a Banach space with an unconditional basis similar to Tsirelson's space and which appeared (1990) as an ad-hoc construction to provide an example of an arbitrarily distortable Banach space.

From B. Maurey – H. Rosenthal (1977) to W.T. Gowers – B. Maurey

- **(B. Maurey - H. Rosenthal)** There exists a Banach space X containing a weakly null sequence with no unconditional subsequence.
- **(W.T. Gowers - B. Maurey)** There exists a reflexive Banach space X with no unconditional Schauder basic sequence.
- Gowers Maurey construction is based on Schlumprecht space, which is a Banach space with an unconditional basis similar to Tsirelson's space and which appeared (1990) as an ad-hoc construction to provide an example of an arbitrarily distortable Banach space.

From B. Maurey – H. Rosenthal (1977) to W.T. Gowers – B. Maurey

- **(B. Maurey - H. Rosenthal)** There exists a Banach space X containing a weakly null sequence with no unconditional subsequence.
- **(W.T. Gowers - B. Maurey)** There exists a reflexive Banach space X with no unconditional Schauder basic sequence.
- Gowers Maurey construction is based on Schlumprecht space, which is a Banach space with an unconditional basis similar to Tsirelson's space and which appeared (1990) as an ad-hoc construction to provide an example of an arbitrarily distortable Banach space.

From B. Maurey – H. Rosenthal (1977) to W.T. Gowers – B. Maurey

- **(B. Maurey - H. Rosenthal)** There exists a Banach space X containing a weakly null sequence with no unconditional subsequence.
- **(W.T. Gowers - B. Maurey)** There exists a reflexive Banach space X with no unconditional Schauder basic sequence.
- Gowers Maurey construction is based on Schlumprecht space, which is a Banach space with an unconditional basis similar to Tsirelson's space and which appeared (1990) as an ad-hoc construction to provide an example of an arbitrarily distortable Banach space.

Tsirelson and Mixed Tsirelson saturations

- Let $\mathcal{M}$ be an adequate and spreading compact family of finite subsets of $\mathbb{N}$, and $0 < \theta < 1$. The $(\mathcal{M}, \theta)$**-operation** on a subset D of $c_{00}(\mathbb{N})$ is the family of all $\phi \in c_{00}(\mathbb{N})$ with

$$\phi = \theta \sum_{i=1}^{d} \phi_i \qquad \text{where}$$

$\phi_1 < \phi_2 < \cdots < \phi_d$ and $\{\min \phi_i\}_{i=1}^{d} \in \mathcal{M}$.

Tsirelson and Mixed Tsirelson saturations

- Let $\mathcal{M}$ be an adequate and spreading compact family of finite subsets of $\mathbb{N}$, and $0 < \theta < 1$. The $(\mathcal{M}, \theta)$**-operation** on a subset D of $c_{00}(\mathbb{N})$ is the family of all $\phi \in c_{00}(\mathbb{N})$ with

$$\phi = \theta \sum_{i=1}^{d} \phi_i \qquad \text{where}$$

$\phi_1 < \phi_2 < \cdots < \phi_d$ and $\{\min \phi_i\}_{i=1}^{d} \in \mathcal{M}$.

Tsirelson saturations

- A subset D of $c_{00}(\mathbb{N})$ is $(\mathcal{M}, \theta)$ **saturated** if it is the minimal subset of $c_{00}(\mathbb{N})$ satisfying the following properties:
  1. $(\pm e_n^*)_n \subseteq D$
  2. D is closed for the $(\mathcal{M}, \theta)$ operation.

- Tsirelson's space is the completion of $c_{00}(\mathbb{N})$ with norm $\|\cdot\|_D$, where D is $(\mathcal{S}, 1/2)$ saturated.
  Here $\mathcal{S}$ is the Schreier family (i.e. the family used by Schreier in his original construction.)

Tsirelson saturations

- A subset D of $c_{00}(\mathbb{N})$ is $(\mathcal{M},\theta)$ **saturated** if it is the minimal subset of $c_{00}(\mathbb{N})$ satisfying the following properties:
  1. $(\pm e_n^*)_n \subseteq D$
  2. D is closed for the $(\mathcal{M},\theta)$ operation.
- Tsirelson's space is the completion of $c_{00}(\mathbb{N})$ with norm $\|\cdot\|_D$, where D is $(\mathcal{S}, 1/2)$ saturated.
  Here $\mathcal{S}$ is the Schreier family (i.e. the family used by Schreier in his original construction.)

Tsirelson saturations

- A subset D of $c_{00}(\mathbb{N})$ is $(\mathcal{M}, \theta)$ **saturated** if it is the minimal subset of $c_{00}(\mathbb{N})$ satisfying the following properties:
  1. $(\pm e_n^*)_n \subseteq D$
  2. D is closed for the $(\mathcal{M}, \theta)$ operation.

- Tsirelson's space is the completion of $c_{00}(\mathbb{N})$ with norm $\| \cdot \|_D$, where D is $(\mathcal{S}, 1/2)$ saturated.
  Here $\mathcal{S}$ is the Schreier family (i.e. the family used by Schreier in his original construction.)

Mixed Tsirelson saturations.

- Let $(\mathcal{M}_n, \theta_n)_n$ be a sequence of pairs with $\mathcal{M}_n$ of increasing complexity and $\theta_n \searrow 0$.
  A subset D of $c_{00}(\mathbb{N})$ is said $(\mathcal{M}_n, \theta_n)_n$ saturated, if D is the minimal subset of $c_{00}(\mathbb{N})$ satisfying the following properties
  1. $(\pm e_n^*)_n \subseteq D$.
  2. It is closed in the $(\mathcal{M}_n, \theta_n)$ operation for all $n \in \mathbb{N}$.

- The norm in Schlumprecht space is defined by the sequence $(\mathcal{A}_n, 1/\log_2(n+1))_n$ where $\mathcal{A}_n$ is the family of subsets of $\mathbb{N}$ with cardinality less or equal to n.

Mixed Tsirelson saturations.

- Let $(\mathcal{M}_n, \theta_n)_n$ be a sequence of pairs with $\mathcal{M}_n$ of increasing complexity and $\theta_n \searrow 0$.
  A subset D of $c_{00}(\mathbb{N})$ is said $(\mathcal{M}_n, \theta_n)_n$ saturated, if D is the minimal subset of $c_{00}(\mathbb{N})$ satisfying the following properties
  1. $(\pm e_n^*)_n \subseteq D$.
  2. It is closed in the $(\mathcal{M}_n, \theta_n)$ operation for all $n \in \mathbb{N}$.

- The norm in Schlumprecht space is defined by the sequence $(\mathcal{A}_n, 1/\log_2(n+1))_n$ where $\mathcal{A}_n$ is the family of subsets of $\mathbb{N}$ with cardinality less or equal to n.

Mixed Tsirelson saturations.

- Let $(\mathcal{M}_n, \theta_n)_n$ be a sequence of pairs with $\mathcal{M}_n$ of increasing complexity and $\theta_n \searrow 0$.
  A subset D of $c_{00}(\mathbb{N})$ is said $(\mathcal{M}_n, \theta_n)_n$ saturated, if D is the minimal subset of $c_{00}(\mathbb{N})$ satisfying the following properties
  1. $(\pm e_n^*)_n \subseteq D$.
  2. It is closed in the $(\mathcal{M}_n, \theta_n)$ operation for all $n \in \mathbb{N}$.

- The norm in Schlumprecht space is defined by the sequence $(\mathcal{A}_n, 1/\log_2(n+1))_n$ where $\mathcal{A}_n$ is the family of subsets of $\mathbb{N}$ with cardinality less or equal to n.

Mixed Tsirelson saturations.

- Let $(\mathcal{M}_n, \theta_n)_n$ be a sequence of pairs with $\mathcal{M}_n$ of increasing complexity and $\theta_n \searrow 0$.
  A subset D of $c_{00}(\mathbb{N})$ is said $(\mathcal{M}_n, \theta_n)_n$ saturated, if D is the minimal subset of $c_{00}(\mathbb{N})$ satisfying the following properties
  1. $(\pm e_n^*)_n \subseteq D$.
  2. It is closed in the $(\mathcal{M}_n, \theta_n)$ operation for all $n \in \mathbb{N}$.
- The norm in Schlumprecht space is defined by the sequence $(\mathcal{A}_n, 1/\log_2(n+1))_n$ where $\mathcal{A}_n$ is the family of subsets of $\mathbb{N}$ with cardinality less or equal to n.

Mixed Tsirelson saturations.

- Let $(\mathcal{M}_n, \theta_n)_n$ be a sequence of pairs with $\mathcal{M}_n$ of increasing complexity and $\theta_n \searrow 0$.
  A subset D of $c_{00}(\mathbb{N})$ is said $(\mathcal{M}_n, \theta_n)_n$ saturated, if D is the minimal subset of $c_{00}(\mathbb{N})$ satisfying the following properties
  1. $(\pm e_n^*)_n \subseteq D$.
  2. It is closed in the $(\mathcal{M}_n, \theta_n)$ operation for all $n \in \mathbb{N}$.

- The norm in Schlumprecht space is defined by the sequence $(\mathcal{A}_n, 1/\log_2(n+1))_n$ where $\mathcal{A}_n$ is the family of subsets of $\mathbb{N}$ with cardinality less or equal to n.

H.I. saturations
(Gowers - Maurey spaces)

The norming set $D \subseteq c_{00}(\mathbb{N})$ in a Gowers- Maurey type spaces satisfies the following property:
There exists a sequence $(\mathcal{M}_n, \theta_n)_n$ such that

1. $(e_n^*)_n \subseteq D$.
2. D is closed in $(\mathcal{M}_{2n}, \theta_{2n})_n$ operations.
3. D is "partially" closed in $(M_{2n-1}, \theta_{2n-1})_n$ operations.

H.I. saturations
(Gowers - Maurey spaces)

The norming set $D \subseteq c_{00}(\mathbb{N})$ in a Gowers- Maurey type spaces satisfies the following property:
There exists a sequence $(\mathcal{M}_n, \theta_n)_n$ such that

1. $(e_n^*)_n \subseteq D$.
2. D is closed in $(\mathcal{M}_{2n}, \theta_{2n})_n$ operations.
3. D is “partially" closed in $(M_{2n-1}, \theta_{2n-1})_n$ operations.

H.I. saturations
(Gowers - Maurey spaces)

The norming set $D \subseteq c_{00}(\mathbb{N})$ in a Gowers- Maurey type spaces satisfies the following property:
There exists a sequence $(\mathcal{M}_n, \theta_n)_n$ such that

1. $(e_n^*)_n \subseteq D$.
2. D is closed in $(\mathcal{M}_{2n}, \theta_{2n})_n$ operations.
3. D is "partially" closed in $(M_{2n-1}, \theta_{2n-1})_n$ operations.

H.I. saturations
(Gowers - Maurey spaces)

The norming set $D \subseteq c_{00}(\mathbb{N})$ in a Gowers- Maurey type spaces satisfies the following property:
There exists a sequence $(\mathcal{M}_n, \theta_n)_n$ such that

1. $(e_n^*)_n \subseteq D$.
2. D is closed in $(\mathcal{M}_{2n}, \theta_{2n})_n$ operations.
3. D is “partially" closed in $(M_{2n-1}, \theta_{2n-1})_n$ operations.

Saturated extensions in Banach spaces

Saturation methods in Banach spaces share common metamathematical ideas with the forcing method of Set Theory. As is known, Tsirelson's initial approach, for the construction of his space was based on forcing method. In a work with A. Tolias, we have attempted to develop the concept of the saturated extensions, viewing each Banach space as a model of the theory.

Ground sets

- A subset G of $c_{00}(\mathbb{N})$ is a **ground set** if
  1. The set G is countable, symmetric and for all $\phi \in G$, $\|\phi\|_\infty \leq 1$.
  2. $\{\pm e_n^*\} \subseteq G$.
  3. For every $\phi \in G$ and every interval E of $\mathbb{N}$ we have that $E^\phi \in G$.
- For every Banach space X with a Schauder basis $(x_n)_{n\in\mathbb{N}}$ there exists a ground set G such that $\|\cdot\|_G$ induced by G on $c_{00}(\mathbb{N})$ makes the basis $(e_n)_n$ of $c_{00}(\mathbb{N})$ equivalent to $(x_n)_n$.

Ground sets

- A subset G of $c_{00}(\mathbb{N})$ is a **ground set** if
  1. The set G is countable, symmetric and for all $\phi \in G$, $\|\phi\|_\infty \leq 1$.
  2. $\{\pm e_n^*\} \subseteq G$.
  3. For every $\phi \in G$ and every interval E of $\mathbb{N}$ we have that $E\phi \in G$.
- For every Banach space X with a Schauder basis $(x_n)_{n\in\mathbb{N}}$ there exists a ground set G such that $\|\cdot\|_G$ induced by G on $c_{00}(\mathbb{N})$ makes the basis $(e_n)_n$ of $c_{00}(\mathbb{N})$ equivalent to $(x_n)_n$.

Ground sets

- A subset G of $c_{00}(\mathbb{N})$ is a **ground set** if
  1. The set G is countable, symmetric and for all $\phi \in G$, $\|\phi\|_\infty \leq 1$.
  2. $\{\pm e_n^*\} \subseteq G$.
  3. For every $\phi \in G$ and every interval E of $\mathbb{N}$ we have that $E\phi \in G$.
- For every Banach space X with a Schauder basis $(x_n)_{n\in\mathbb{N}}$ there exists a ground set G such that $\|\cdot\|_G$ induced by G on $c_{00}(\mathbb{N})$ makes the basis $(e_n)_n$ of $c_{00}(\mathbb{N})$ equivalent to $(x_n)_n$.

Ground sets

- A subset G of $c_{00}(\mathbb{N})$ is a **ground set** if
  1. The set G is countable, symmetric and for all $\phi \in G$, $\|\phi\|_\infty \leq 1$.
  2. $\{\pm e_n^*\} \subseteq G$.
  3. For every $\phi \in G$ and every interval E of $\mathbb{N}$ we have that $E^\phi \in G$.
- For every Banach space X with a Schauder basis $(x_n)_{n\in\mathbb{N}}$ there exists a ground set G such that $\|\cdot\|_G$ induced by G on $c_{00}(\mathbb{N})$ makes the basis $(e_n)_n$ of $c_{00}(\mathbb{N})$ equivalent to $(x_n)_n$.

Ground sets

- A subset G of $c_{00}(\mathbb{N})$ is a **ground set** if
  1. The set G is countable, symmetric and for all $\phi \in G$, $\|\phi\|_\infty \leq 1$.
  2. $\{\pm e_n^*\} \subseteq G$.
  3. For every $\phi \in G$ and every interval E of $\mathbb{N}$ we have that $E^\phi \in G$.
- For every Banach space X with a Schauder basis $(x_n)_{n\in\mathbb{N}}$ there exists a ground set G such that $\|\cdot\|_G$ induced by G on $c_{00}(\mathbb{N})$ makes the basis $(e_n)_n$ of $c_{00}(\mathbb{N})$ equivalent to $(x_n)_n$.

- Mixed Tsirelson extensions
- For a ground set G and a sequence $(\mathcal{M}_n, \theta_n)_n$, we define the set D_G as the minimal subset of $c_{00}(\mathbb{N})$ containing G and closed in the $(\mathcal{M}_n, \theta_n)$ operations
- HI extensions
- Similar to the above is defined the HI extension of a ground set G as the minimal set D_G containing G, closed in the $(\mathcal{M}_{2n}, \theta_{2n})$ operations and partially closed in the $(\mathcal{M}_{2n-1}, \theta_{2n-1})$ operations.

- Mixed Tsirelson extensions

- For a ground set G and a sequence $(\mathcal{M}_n, \theta_n)_n$, we define the set D_G as the minimal subset of $c_{00}(\mathbb{N})$ containing G and closed in the $(\mathcal{M}_n, \theta_n)$ operations

- HI extensions

- Similar to the above is defined the HI extension of a ground set G as the minimal set D_G containing G, closed in the $(\mathcal{M}_{2n}, \theta_{2n})$ operations and partially closed in the $(\mathcal{M}_{2n-1}, \theta_{2n-1})$ operations.

- Mixed Tsirelson extensions

- For a ground set G and a sequence $(\mathcal{M}_n, \theta_n)_n$, we define the set D_G as the minimal subset of $c_{00}(\mathbb{N})$ containing G and closed in the $(\mathcal{M}_n, \theta_n)$ operations

- HI extensions

- Similar to the above is defined the HI extension of a ground set G as the minimal set D_G containing G, closed in the $(\mathcal{M}_{2n}, \theta_{2n})$ operations and partially closed in the $(\mathcal{M}_{2n-1}, \theta_{2n-1})$ operations.

- Mixed Tsirelson extensions

- For a ground set G and a sequence $(\mathcal{M}_n, \theta_n)_n$, we define the set D_G as the minimal subset of $c_{00}(\mathbb{N})$ containing G and closed in the $(\mathcal{M}_n, \theta_n)$ operations

- HI extensions

- Similar to the above is defined the HI extension of a ground set G as the minimal set D_G containing G, closed in the $(\mathcal{M}_{2n}, \theta_{2n})$ operations and partially closed in the $(\mathcal{M}_{2n-1}, \theta_{2n-1})$ operations.

- Observe that mixed Tsirelson and HI extensions are referred to the ground set G rather than to the space X_G. ($X_G = \overline{< c_{00}(\mathbb{N}), \| \cdot \|_G >}$). In particular the identity operator

$$I : (c_{00}(\mathbb{N}), \| \cdot \|_{D_G}) \to (c_{00}(\mathbb{N}), \| \cdot \|_G)$$

is a norm one operator.

- In the work with A. Tolias (Memoirs AMS, 2004) we have applied the saturated extension to the study of quotients of HI spaces.

- Observe that mixed Tsirelson and HI extensions are referred to the ground set G rather than to the space X_G. ($X_G = \overline{< c_{00}(\mathbb{N}), \|\cdot\|_G >}$). In particular the identity operator

$$I : (c_{00}(\mathbb{N}), \|\cdot\|_{D_G}) \to (c_{00}(\mathbb{N}), \|\cdot\|_G)$$

is a norm one operator.

- In the work with A. Tolias (Memoirs AMS, 2004) we have applied the saturated extension to the study of quotients of HI spaces.

- **Theorem** *(Argyros – Tolias)* Let G be a ground set such that $\ell^1 \not\hookrightarrow X_G$. Then there exists a sequence $(\mathcal{M}_n, \theta_n)_n$ such that the HI extension X_{D_G} is a HI space.
- **Theorem** *(Argyros – Tolias)* Every separable Banach space X not containing ℓ_1 is a quotient of a HI space.

- **Theorem** *(Argyros – Tolias)* Let G be a ground set such that $\ell^1 \not\hookrightarrow X_G$. Then there exists a sequence $(\mathcal{M}_n, \theta_n)_n$ such that the HI extension X_{D_G} is a HI space.
- **Theorem** *(Argyros – Tolias)* Every separable Banach space X not containing ℓ_1 is a quotient of a HI space.

Quasi – reflexive HI spaces.

- **Theorem** We set $G = \{\pm\chi_E : E \subseteq \mathbb{N},\ \text{interval}\}$. Then for every $(\mathcal{M}_n, \theta_n)_n$ such that the HI extension of the set G is a HI space, yields that the space X_{D_G} is quasi – reflexive.
- This answered a problem posed by G. Godefroy.

Quasi – reflexive HI spaces.

- **Theorem** We set $G = \{\pm\chi_E : E \subseteq \mathbb{N},\ \text{interval}\}$. Then for every $(\mathcal{M}_n, \theta_n)_n$ such that the HI extension of the set G is a HI space, yields that the space X_{D_G} is quasi – reflexive.
- This answered a problem posed by G. Godefroy.

Quasi – reflexive HI spaces.

- **Theorem** We set $G = \{\pm\chi_E : E \subseteq \mathbb{N}, \text{ interval}\}$. Then for every $(\mathcal{M}_n, \theta_n)_n$ such that the HI extension of the set G is a HI space, yields that the space X_{D_G} is quasi – reflexive.
- This answered a problem posed by G. Godefroy.

The rest of the present lecture is divided into the following two parts:

- Connecting heterogeneous structures.
- A $\mathcal{L}_\infty$ Hereditarily Indecomposable Banach space solving the $\lambda I + K$ problem.

  (Joint work with Richard Haydon.)

The rest of the present lecture is divided into the following two parts:

- Connecting heterogeneous structures.
- A $\mathcal{L}_\infty$ Hereditarily Indecomposable Banach space solving the $\lambda I + K$ problem.

  (Joint work with Richard Haydon.)

The rest of the present lecture is divided into the following two parts:

- Connecting heterogeneous structures.
- A $\mathcal{L}_\infty$ Hereditarily Indecomposable Banach space solving the $\lambda I + K$ problem.

(Joint work with Richard Haydon.)

Two antipodal classes of Banach spaces

- Banach spaces with an unconditional basis.
- Indecomposable Banach spaces.

Two antipodal classes of Banach spaces

- Banach spaces with an unconditional basis.
- Indecomposable Banach spaces.

Two antipodal classes of Banach spaces

- Banach spaces with an unconditional basis.
- Indecomposable Banach spaces.

Connecting Heterogeneous Structures

- A Banach space X has an unconditional basis if there exists a sequence $(x_n)_n$ such that each $x \in X$ is expressed as $\sum_{n=1}^{\infty} a_n x_n$ and $\sum_{n=1}^{\infty} a_n x_n$ converges iff $\sum_{n=1}^{\infty} \epsilon_n a_n x_n$ converges for every choice of signs $(\epsilon_n)_{n\in\mathbb{N}}$.

- Most of the classical spaces ($\ell^p(\mathbb{N})$, $L^p(\lambda)$ $1 < p < \infty$, $c_0(\mathbb{N})$) have an unconditional basis.

Connecting Heterogeneous Structures

- A Banach space X has an unconditional basis if there exists a sequence $(x_n)_n$ such that each $x \in X$ is expressed as $\sum_{n=1}^{\infty} a_n x_n$ and $\sum_{n=1}^{\infty} a_n x_n$ converges iff $\sum_{n=1}^{\infty} \epsilon_n a_n x_n$ converges for every choice of signs $(\epsilon_n)_{n \in \mathbb{N}}$.

- Most of the classical spaces ($\ell^p(\mathbb{N})$, $L^p(\lambda)$ $1 < p < \infty$, $c_0(\mathbb{N})$) have an unconditional basis.

Connecting Heterogeneous Structures

- The most extreme example are Hilbert spaces.

- If $(x_n)_{n\in\mathbb{N}}$ is an unconditional basis for X then for every L subset of $\mathbb{N}$ the operator

$$\sum_{n=1}^{\infty} a_n x_n \to \sum_{n\in L} a_n x_n$$

is a bounded projection.

Connecting Heterogeneous Structures

- The most extreme example are Hilbert spaces.
- If $(x_n)_{n\in\mathbb{N}}$ is an unconditional basis for X then for every L subset of $\mathbb{N}$ the operator

$$\sum_{n=1}^{\infty} a_n x_n \to \sum_{n\in L} a_n x_n$$

is a bounded projection.

Approaching the contradiction

- A Banach space X is said to be **Indecomposable** if it admits no (nontrivial) projection.

- The space X is **Hereditarily Indecomposable (HI)** if every infinite dimensional closed subspace of X is indecomposable.

Approaching the contradiction

- A Banach space X is said to be **Indecomposable** if it admits no (nontrivial) projection.

- The space X is **Hereditarily Indecomposable (HI)** if every infinite dimensional closed subspace of X is indecomposable.

Connecting Heterogeneous Structures

- The first example of a HI space is due to W.T. Gowers and B. Maurey (JAMS, 1993) which also was the first Indecomposable Space.

- **Theorem** (W.T. Gowers, GAFA 1994, Annals of Math. 2004) Let X be a Banach space. Then one of the following two holds:
    - There exists a HI Y subspace of X.
    - Every Y subspace of X contains a further subspace Z with an unconditional basis. (i.e. X is unconditionally saturated).

Connecting Heterogeneous Structures

- The first example of a HI space is due to W.T. Gowers and B. Maurey (JAMS, 1993) which also was the first Indecomposable Space.

- **Theorem** (W.T. Gowers, GAFA 1994, Annals of Math. 2004) Let X be a Banach space. Then one of the following two holds:
  - There exists a HI Y subspace of X.
  - Every Y subspace of X contains a further subspace Z with an unconditional basis. (i.e. X is unconditionally saturated).

- The first example of a HI space is due to W.T. Gowers and B. Maurey (JAMS, 1993) which also was the first Indecomposable Space.

- **Theorem** (W.T. Gowers, GAFA 1994, Annals of Math. 2004) Let X be a Banach space. Then one of the following two holds:
  - There exists a HI Y subspace of X.
  - Every Y subspace of X contains a further subspace Z with an unconditional basis. (i.e. X is unconditionally saturated).

Connecting Heterogeneous Structures

- The first example of a HI space is due to W.T. Gowers and B. Maurey (JAMS, 1993) which also was the first Indecomposable Space.

- **Theorem** (W.T. Gowers, GAFA 1994, Annals of Math. 2004) Let X be a Banach space. Then one of the following two holds:
  - There exists a HI Y subspace of X.
  - Every Y subspace of X contains a further subspace Z with an unconditional basis. (i.e. X is unconditionally saturated).

Results connecting Conditional and unconditional structures

- **Theorem** (S. Argyros and V. Felouzis, JAMS 2000) There exists a reflexive HI Banach space X which has as quotient a separable Hilbert space.

- As consequence the space $\ell^2(\mathbb{N})$ is isomorphic to a subspace of a reflexive indecomposable Banach space. (In particular the dual X^* of a HI space X is not necessarily HI).

Results connecting Conditional and unconditional structures

- **Theorem** (S. Argyros and V. Felouzis, JAMS 2000) There exists a reflexive HI Banach space X which has as quotient a separable Hilbert space.

- As consequence the space $\ell^2(\mathbb{N})$ is isomorphic to a subspace of a reflexive indecomposable Banach space. (In particular the dual X^* of a HI space X is not necessarily HI).

Results connecting Conditional and unconditional structures

- **Theorem** (S. Argyros and V. Felouzis, JAMS 2000) There exists a reflexive HI Banach space X which has as quotient a separable Hilbert space.

- As consequence the space $\ell^2(\mathbb{N})$ is isomorphic to a subspace of a reflexive indecomposable Banach space. (In particular the dual X^* of a HI space X is not necessarily HI).

The structure of the subspaces of indecomposable spaces

- In the late 60's J. Lindenstrauss asked if there exists an indecomposable Banach space. This is one more problem answered by Gowers – Maurey HI construction.
- **Theorem** (S. Argyros and H. Raikoftsalis (to appear)) Every reflexive space is isomorphic to a subspace of a reflexive indecomposable space.

The structure of the subspaces of indecomposable spaces

- In the late 60's J. Lindenstrauss asked if there exists an indecomposable Banach space. This is one more problem answered by Gowers – Maurey HI construction.

- **Theorem** (S. Argyros and H. Raikoftsalis (to appear)) Every reflexive space is isomorphic to a subspace of a reflexive indecomposable space.

The structure of the subspaces of indecomposable spaces

- The following is the ultimate problem concerning the structures of the subspaces of indecomposable Banach spaces:
- Problem: Let X be a separable Banach space satisfying the following property: Whenever X is isomorphic to a subspace of a separable Banach space Y, the space Y is indecomposable. Does this imply that $c_0 \hookrightarrow X$?
- Of particular interest is the space L^1.
- Jointly with R. Haydon and H. Raicftsalis have shown that ℓ^1 embeds into an indecompsable space.

The structure of the subspaces of indecomposable spaces

- The following is the ultimate problem concerning the structures of the subspaces of indecomposable Banach spaces:
- Problem: Let X be a separable Banach space satisfying the following property: Whenever X is isomorphic to a subspace of a separable Banach space Y, the space Y is indecomposable. Does this imply that $c_0 \hookrightarrow X$?
- Of particular interest is the space L^1.
- Jointly with R. Haydon and H. Raicftsalis have shown that ℓ^1 embeds into an indecompsable space.

The structure of the subspaces of indecomposable spaces

- The following is the ultimate problem concerning the structures of the subspaces of indecomposable Banach spaces:
- Problem: Let X be a separable Banach space satisfying the following property: Whenever X is isomorphic to a subspace of a separable Banach space Y, the space Y is indecomposable. Does this imply that $c_0 \hookrightarrow X$?
- Of particular interest is the space L^1.
- Jointly with R. Haydon and H. Raicftsalis have shown that ℓ^1 embeds into an indecompsable space.

The structure of the subspaces of indecomposable spaces

- The following is the ultimate problem concerning the structures of the subspaces of indecomposable Banach spaces:
- Problem: Let X be a separable Banach space satisfying the following property: Whenever X is isomorphic to a subspace of a separable Banach space Y, the space Y is indecomposable. Does this imply that $c_0 \hookrightarrow X$?
- Of particular interest is the space L^1.
- Jointly with R. Haydon and H. Raicftsalis have shown that ℓ^1 embeds into an indecompsable space.

The structure of the subspaces of indecomposable spaces

- The following results answered a problem posed by W.T. Gowers and H.P. Rosenthal.
- **Theorem** (S. Argyros and A. Manoussakis, Studia 2003) There exists a reflexive unconditionally saturated indecomposable space.
- Problem. Let $1 \leq p < \infty$. Does there exist an indecomposable Banach space X which is ℓ^p saturated?

The structure of the subspaces of indecomposable spaces

- The following results answered a problem posed by W.T. Gowers and H.P. Rosenthal.
- **Theorem** (S. Argyros and A. Manoussakis, Studia 2003) There exists a reflexive unconditionally saturated indecomposable space.
- Problem. Let $1 \leq p < \infty$. Does there exist an indecomposable Banach space X which is ℓ^p saturated?

The structure of the subspaces of indecomposable spaces

- The following results answered a problem posed by W.T. Gowers and H.P. Rosenthal.
- **Theorem** (S. Argyros and A. Manoussakis, Studia 2003) There exists a reflexive unconditionally saturated indecomposable space.
- Problem. Let $1 \leq p < \infty$. Does there exist an indecomposable Banach space X which is ℓ^p saturated?

Indecomposability and unconditionality in duality

The following results show the divergence of the structure between X and its dual X^*.

- **Theorem** (S. Argyros and A. Tolias, GAFA 2004) There exists a reflexive HI X such that X^* is unconditionally saturated.
- **Theorem** (Argyros - A. Arvanitakis - A. Tolias) There exists a HI Banach space X not containing reflexive subspace, such that X^* is separable and saturated by reflexive spaces with an unconditional basis.

Indecomposability and unconditionality in duality

The following results show the divergence of the structure between X and its dual X^*.

- **Theorem** (S. Argyros and A. Tolias, GAFA 2004) There exists a reflexive HI X such that X^* is unconditionally saturated.
- **Theorem** (Argyros - A. Arvanitakis - A. Tolias) There exists a HI Banach space X not containing reflexive subspace, such that X^* is separable and saturated by reflexive spaces with an unconditional basis.

Indecomposability and unconditionality in duality

The following results show the divergence of the structure between X and its dual X^*.

- **Theorem** (S. Argyros and A. Tolias, GAFA 2004) There exists a reflexive HI X such that X^* is unconditionally saturated.
- **Theorem** (Argyros - A. Arvanitakis - A. Tolias) There exists a HI Banach space X not containing reflexive subspace, such that X^* is separable and saturated by reflexive spaces with an unconditional basis.

The Solution of the "Scalar plus Compact" Problem.

- **Problem** Does there exist a HI Banach space X with X^* having an unconditional basis?

- As we have seen, the dual of a HI Banach space is not necessarily HI. (it could be unconditionally saturated!)

- Within the class of reflexive Banach spaces, the above problem has a negative answer.

The Solution of the "Scalar plus Compact" Problem.

- **Problem** Does there exist a HI Banach space X with X^* having an unconditional basis?

- As we have seen, the dual of a HI Banach space is not necessarily HI. (it could be unconditionally saturated!)

- Within the class of reflexive Banach spaces, the above problem has a negative answer.

The Solution of the "Scalar plus Compact" Problem.

- **Problem** Does there exist a HI Banach space X with X^* having an unconditional basis?

- As we have seen, the dual of a HI Banach space is not necessarily HI. (it could be unconditionally saturated!)

- Within the class of reflexive Banach spaces, the above problem has a negative answer.

- J. Bourgain-F. Delbaen $\mathcal{L}_\infty$ spaces
- In the late 70s J. Bourgain and F. Delbaen invented a general method for constructing non classical $\mathcal{L}_\infty$ spaces.
- A separable Banach space X is a λ-$\mathcal{L}_\infty$ space if there exists a strictly increasing sequence $(F_n)_n$ of finite dimensional subspaces of X such that $X = \overline{\cup_n F_n}$ and $d(F_n, \ell^\infty_{dimF_n}) \leq \lambda$.

The Solution of the "Scalar plus Compact" Problem.

- J. Bourgain-F. Delbaen $\mathcal{L}_\infty$ spaces
- In the late 70s J. Bourgain and F. Delbaen invented a general method for constructing non classical $\mathcal{L}_\infty$ spaces.
- A separable Banach space X is a λ-$\mathcal{L}_\infty$ space if there exists a strictly increasing sequence $(F_n)_n$ of finite dimensional subspaces of X such that $X = \overline{\cup_n F_n}$ and $d(F_n, \ell^\infty_{dimF_n}) \leq \lambda$.

The Solution of the "Scalar plus Compact" Problem.

- J. Bourgain-F. Delbaen $\mathcal{L}_\infty$ spaces
- In the late 70s J. Bourgain and F. Delbaen invented a general method for constructing non classical $\mathcal{L}_\infty$ spaces.
- A separable Banach space X is a λ-$\mathcal{L}_\infty$ space if there exists a strictly increasing sequence $(F_n)_n$ of finite dimensional subspaces of X such that $X = \overline{\cup_n F_n}$ and $d(F_n, \ell^\infty_{dimF_n}) \leq \lambda$.

The Solution of the "Scalar plus Compact" Problem.

- The easy examples of $\mathcal{L}_\infty$ spaces are $C(K)$ spaces which include the space $c_0(\mathbb{N})$. More generally all ℓ_1 preduals are $\mathcal{L}_\infty$ spaces.
- The BD constructions concern $\mathcal{L}_\infty$ with peculiar subspaces' structure. In particular for every $1 < p < \infty$ there exists a $\mathcal{L}_\infty$ space $\mathfrak{X}_p$ which is ℓ_p-saturated and $\mathfrak{X}_p^* \cong \ell^1(\mathbb{N})$.
- R. Haydon, in an unpublished work, had constructed a BD $\mathcal{L}_\infty$ space saturated by Tsirelson type spaces.

The Solution of the "Scalar plus Compact" Problem.

- The easy examples of $\mathcal{L}_\infty$ spaces are $C(K)$ spaces which include the space $c_0(\mathbb{N})$. More generally all ℓ_1 preduals are $\mathcal{L}_\infty$ spaces.
- The BD constructions concern $\mathcal{L}_\infty$ with peculiar subspaces' structure. In particular for every $1 < p < \infty$ there exists a $\mathcal{L}_\infty$ space $\mathfrak{X}_p$ which is ℓ_p-saturated and $\mathfrak{X}_p^* \cong \ell^1(\mathbb{N})$.
- R. Haydon, in an unpublished work, had constructed a BD $\mathcal{L}_\infty$ space saturated by Tsirelson type spaces.

The Solution of the "Scalar plus Compact" Problem.

- The easy examples of $\mathcal{L}_\infty$ spaces are $C(K)$ spaces which include the space $c_0(\mathbb{N})$. More generally all ℓ_1 preduals are $\mathcal{L}_\infty$ spaces.
- The BD constructions concern $\mathcal{L}_\infty$ with peculiar subspaces' structure. In particular for every $1 < p < \infty$ there exists a $\mathcal{L}_\infty$ space $\mathfrak{X}_p$ which is ℓ_p-saturated and $\mathfrak{X}_p^* \cong \ell^1(\mathbb{N})$.
- R. Haydon, in an unpublished work, had constructed a BD $\mathcal{L}_\infty$ space saturated by Tsirelson type spaces.

The Solution of the "Scalar plus Compact" Problem.

- Putting together BD method with HI constructions we arrived to the following:
- **Theorem** (S. Argyros and R. Haydon) There exists a $\mathcal{L}_\infty$ space $\mathfrak{X}_K$, subspace of $\ell^\infty(\mathbb{N})$, such that
  1. The space $\mathfrak{X}_K$ is HI.
  2. If $Q : \ell^1(\mathbb{N}) \to \mathfrak{X}_K^*$ is the restriction operator, then Q is an onto isomorphism.
- Hence the problem of the duality between HI spaces and spaces with an unconditional basis has a positive solution.

The Solution of the "Scalar plus Compact" Problem.

- Putting together BD method with HI constructions we arrived to the following:
- **Theorem** (S. Argyros and R. Haydon) There exists a $\mathcal{L}_\infty$ space $\mathfrak{X}_K$, subspace of $\ell^\infty(\mathbb{N})$, such that
  1. The space $\mathfrak{X}_K$ is HI.
  2. If $Q : \ell^1(\mathbb{N}) \to \mathfrak{X}_K^*$ is the restriction operator, then Q is an onto isomorphism.
- Hence the problem of the duality between HI spaces and spaces with an unconditional basis has a positive solution.

The Solution of the "Scalar plus Compact" Problem.

- Putting together BD method with HI constructions we arrived to the following:

- **Theorem** (S. Argyros and R. Haydon) There exists a $\mathcal{L}_\infty$ space $\mathfrak{X}_K$, subspace of $\ell^\infty(\mathbb{N})$, such that
  1. The space $\mathfrak{X}_K$ is HI.
  2. If $Q : \ell^1(\mathbb{N}) \to \mathfrak{X}_K^*$ is the restriction operator, then Q is an onto isomorphism.

- Hence the problem of the duality between HI spaces and spaces with an unconditional basis has a positive solution.

The Solution of the "Scalar plus Compact" Problem.

- Putting together BD method with HI constructions we arrived to the following:

- **Theorem** (S. Argyros and R. Haydon) There exists a $\mathcal{L}_\infty$ space $\mathfrak{X}_K$, subspace of $\ell^\infty(\mathbb{N})$, such that
  1. The space $\mathfrak{X}_K$ is HI.
  2. If $Q : \ell^1(\mathbb{N}) \to \mathfrak{X}_K^*$ is the restriction operator, then Q is an onto isomorphism.

- Hence the problem of the duality between HI spaces and spaces with an unconditional basis has a positive solution.

The Solution of the "Scalar plus Compact" Problem.

- Putting together BD method with HI constructions we arrived to the following:
- **Theorem** (S. Argyros and R. Haydon) There exists a $\mathcal{L}_\infty$ space $\mathfrak{X}_K$, subspace of $\ell^\infty(\mathbb{N})$, such that
  1. The space $\mathfrak{X}_K$ is HI.
  2. If $Q : \ell^1(\mathbb{N}) \to \mathfrak{X}_K^*$ is the restriction operator, then Q is an onto isomorphism.
- Hence the problem of the duality between HI spaces and spaces with an unconditional basis has a positive solution.

Problems concerning $\mathcal{L}(X)$

As is known for a given Banach space X by $\mathcal{L}(X)$ (or $\mathcal{B}(X)$) we denote the Banach space of its bounded linear operators, $T : X \to X$ endowed with the operator norm.

- **Problem** Let X be a Banach space.
  1. How many bounded linear operators $T : X \to X$ do there exist?
  2. What is the structure of $\mathcal{L}(X)$ as a Banach space (or a Banach algebra)?

Problems concerning $\mathcal{L}(X)$

As is known for a given Banach space X by $\mathcal{L}(X)$ (or $\mathcal{B}(X)$) we denote the Banach space of its bounded linear operators, $T : X \to X$ endowed with the operator norm.

- **Problem** Let X be a Banach space.
  1. How many bounded linear operators $T : X \to X$ do there exist?
  2. What is the structure of $\mathcal{L}(X)$ as a Banach space (or a Banach algebra)?

Problems concerning $\mathcal{L}(X)$

As is known for a given Banach space X by $\mathcal{L}(X)$ (or $\mathcal{B}(X)$) we denote the Banach space of its bounded linear operators, $T : X \to X$ endowed with the operator norm.

- **Problem** Let X be a Banach space.
  1. How many bounded linear operators $T : X \to X$ do there exist?
  2. What is the structure of $\mathcal{L}(X)$ as a Banach space (or a Banach algebra)?

Problems concerning $\mathcal{L}(X)$

As is known for a given Banach space X by $\mathcal{L}(X)$ (or $\mathcal{B}(X)$) we denote the Banach space of its bounded linear operators, $T : X \to X$ endowed with the operator norm.

- **Problem** Let X be a Banach space.
  1. How many bounded linear operators $T : X \to X$ do there exist?
  2. What is the structure of $\mathcal{L}(X)$ as a Banach space (or a Banach algebra)?

- A well known problem in the theory of Banach spaces formulated by J. Lindenstrauss is the "scalar plus compact" problem.
- **Problem 1** Does there exist a Banach space X such that every $T \in \mathcal{L}(X)$ is of the form $\lambda I + K$ with λ scalar K a compact operator.
  It is convenient to formulate that problem as

$$dim \mathcal{L}(X)/\mathcal{K}(X) = 1$$

The Solution of the "Scalar plus Compact" Problem.

- A well known problem in the theory of Banach spaces formulated by J. Lindenstrauss is the "scalar plus compact" problem.
- **Problem 1** Does there exist a Banach space X such that every $T \in \mathcal{L}(X)$ is of the form $\lambda I + K$ with λ scalar K a compact operator.
  It is convenient to formulate that problem as

$$dim\mathcal{L}(X)/\mathcal{K}(X) = 1$$

- **Problem 2** Does there exist a Banach space X such that every operator $T : X \to X$ has a non trivial invariant subspace.
- N. Aronszajn and K. Smith well known theorem shows that a positive answer to Problem 1 yields also a positive answer to Problem 2.

- **Problem 2** Does there exist a Banach space X such that every operator $T : X \to X$ has a non trivial invariant subspace.
- N. Aronszajn and K. Smith well known theorem shows that a positive answer to Problem 1 yields also a positive answer to Problem 2.

The Solution of the "Scalar plus Compact" Problem.

- We denote by $\mathcal{S}(X)$ the ideal of strictly singular operators (those operators that restricted on every subspace of X are not an isomorphism). Note that in general $\mathcal{K}(X)$ is a proper subspace of $\mathcal{S}(X)$. In some cases like ℓ_p the two ideals coincide.
- Among the seminal properties of HI spaces is that they have small spaces of bounded linear operators as the following describes
- **Theorem** (W.T. Gowers and B. Maurey, V. Ferenczi)
  1. For every complex HI space X $dim\mathcal{L}(X)/\mathcal{S}(X) = 1$
  2. For every real Banach X one of the following holds

$$\mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{R}, \quad \mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{C}, \quad \mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{H}$$

The Solution of the "Scalar plus Compact" Problem.

- We denote by $\mathcal{S}(X)$ the ideal of strictly singular operators (those operators that restricted on every subspace of X are not an isomorphism). Note that in general $\mathcal{K}(X)$ is a proper subspace of $\mathcal{S}(X)$. In some cases like ℓ_p the two ideals coincide.
- Among the seminal properties of HI spaces is that they have small spaces of bounded linear operators as the following describes
- **Theorem** (W.T. Gowers and B. Maurey, V. Ferenczi)
  1. For every complex HI space X $dim\mathcal{L}(X)/\mathcal{S}(X) = 1$
  2. For every real Banach X one of the following holds

$$\mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{R}, \quad \mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{C}, \quad \mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{H}$$

The Solution of the "Scalar plus Compact" Problem.

- We denote by $\mathcal{S}(X)$ the ideal of strictly singular operators (those operators that restricted on every subspace of X are not an isomorphism). Note that in general $\mathcal{K}(X)$ is a proper subspace of $\mathcal{S}(X)$. In some cases like ℓ_p the two ideals coincide.
- Among the seminal properties of HI spaces is that they have small spaces of bounded linear operators as the following describes
- **Theorem** (W.T. Gowers and B. Maurey, V. Ferenczi)
  1. For every complex HI space X $dim \mathcal{L}(X)/\mathcal{S}(X) = 1$
  2. For every real Banach X one of the following holds

$$\mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{R}, \quad \mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{C}, \quad \mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{H}$$

The Solution of the "Scalar plus Compact" Problem.

- We denote by $\mathcal{S}(X)$ the ideal of strictly singular operators (those operators that restricted on every subspace of X are not an isomorphism). Note that in general $\mathcal{K}(X)$ is a proper subspace of $\mathcal{S}(X)$. In some cases like ℓ_p the two ideals coincide.
- Among the seminal properties of HI spaces is that they have small spaces of bounded linear operators as the following describes
- **Theorem** (W.T. Gowers and B. Maurey, V. Ferenczi)
  1. For every complex HI space X $dim\mathcal{L}(X)/\mathcal{S}(X) = 1$
  2. For every real Banach X one of the following holds

$$\mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{R}, \quad \mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{C}, \quad \mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{H}$$

The Solution of the "Scalar plus Compact" Problem.

- We denote by $\mathcal{S}(X)$ the ideal of strictly singular operators (those operators that restricted on every subspace of X are not an isomorphism). Note that in general $\mathcal{K}(X)$ is a proper subspace of $\mathcal{S}(X)$. In some cases like ℓ_p the two ideals coincide.
- Among the seminal properties of HI spaces is that they have small spaces of bounded linear operators as the following describes
- **Theorem** (W.T. Gowers and B. Maurey, V. Ferenczi)
  1. For every complex HI space X $dim \mathcal{L}(X)/\mathcal{S}(X) = 1$
  2. For every real Banach X one of the following holds

$$\mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{R}, \quad \mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{C}, \quad \mathcal{L}(X)/\mathcal{S}(X) \cong \mathbb{H}$$

The Solution of the "Scalar plus Compact" Problem.

After the discovery of the space $\mathfrak{X}_K$ we studied the structure of the space $\mathcal{L}(\mathfrak{X}_K)$ showing the following:

- **Theorem** (S.Argyros and R. Haydon) Every bounded linear operator $T : \mathfrak{X}_K \to \mathfrak{X}_K$ is of the form $T = \lambda I + K$ with $K : \mathfrak{X}_K \to \mathfrak{X}_K$ a compact operator.
- This also yields the following:
- **Theorem** For the space $\mathfrak{X}_K$ the following hold:
  - Every bounded linear operator $T : \mathfrak{X}_K \to \mathfrak{X}_K$ has a non trivial invariant subspace.
  - The space $\mathcal{L}(\mathfrak{X}_K)$ is separable
  - The space $\mathcal{K}(\mathfrak{X}_K)$ is complemented in $\mathcal{L}(\mathfrak{X}_K)$
  - c_0 does not embed into $\mathcal{L}(\mathfrak{X}_K)$.

The Solution of the "Scalar plus Compact" Problem.

After the discovery of the space $\mathfrak{X}_K$ we studied the structure of the space $\mathcal{L}(\mathfrak{X}_K)$ showing the following:

- **Theorem** (S.Argyros and R. Haydon) Every bounded linear operator $T : \mathfrak{X}_K \to \mathfrak{X}_K$ is of the form $T = \lambda I + K$ with $K : \mathfrak{X}_K \to \mathfrak{X}_K$ a compact operator.
- This also yields the following:
- **Theorem** For the space $\mathfrak{X}_K$ the following hold:
  - Every bounded linear operator $T : \mathfrak{X}_K \to \mathfrak{X}_K$ has a non trivial invariant subspace.
  - The space $\mathcal{L}(\mathfrak{X}_K)$ is separable
  - The space $\mathcal{K}(\mathfrak{X}_K)$ is complemented in $\mathcal{L}(\mathfrak{X}_K)$
  - c_0 does not embed into $\mathcal{L}(\mathfrak{X}_K)$.

The Solution of the "Scalar plus Compact" Problem.

After the discovery of the space $\mathfrak{X}_K$ we studied the structure of the space $\mathcal{L}(\mathfrak{X}_K)$ showing the following:

- **Theorem** (S.Argyros and R. Haydon) Every bounded linear operator $T : \mathfrak{X}_K \to \mathfrak{X}_K$ is of the form $T = \lambda I + K$ with $K : \mathfrak{X}_K \to \mathfrak{X}_K$ a compact operator.
- This also yields the following:
- **Theorem** For the space $\mathfrak{X}_K$ the following hold:
  - Every bounded linear operator $T : \mathfrak{X}_K \to \mathfrak{X}_K$ has a non trivial invariant subspace.
  - The space $\mathcal{L}(\mathfrak{X}_K)$ is separable
  - The space $\mathcal{K}(\mathfrak{X}_K)$ is complemented in $\mathcal{L}(\mathfrak{X}_K)$
  - c_0 does not embed into $\mathcal{L}(\mathfrak{X}_K)$.

The Solution of the "Scalar plus Compact" Problem.

After the discovery of the space $\mathfrak{X}_K$ we studied the structure of the space $\mathcal{L}(\mathfrak{X}_K)$ showing the following:

- **Theorem** (S.Argyros and R. Haydon) Every bounded linear operator $T : \mathfrak{X}_K \to \mathfrak{X}_K$ is of the form $T = \lambda I + K$ with $K : \mathfrak{X}_K \to \mathfrak{X}_K$ a compact operator.
- This also yields the following:
- **Theorem** For the space $\mathfrak{X}_K$ the following hold:
  - Every bounded linear operator $T : \mathfrak{X}_K \to \mathfrak{X}_K$ has a non trivial invariant subspace.
  - The space $\mathcal{L}(\mathfrak{X}_K)$ is separable
  - The space $\mathcal{K}(\mathfrak{X}_K)$ is complemented in $\mathcal{L}(\mathfrak{X}_K)$
  - c_0 does not embed into $\mathcal{L}(\mathfrak{X}_K)$.

The Solution of the "Scalar plus Compact" Problem.

After the discovery of the space $\mathfrak{X}_K$ we studied the structure of the space $\mathcal{L}(\mathfrak{X}_K)$ showing the following:

- **Theorem** (S.Argyros and R. Haydon) Every bounded linear operator $T : \mathfrak{X}_K \to \mathfrak{X}_K$ is of the form $T = \lambda I + K$ with $K : \mathfrak{X}_K \to \mathfrak{X}_K$ a compact operator.
- This also yields the following:
- **Theorem** For the space $\mathfrak{X}_K$ the following hold:
  - Every bounded linear operator $T : \mathfrak{X}_K \to \mathfrak{X}_K$ has a non trivial invariant subspace.
  - The space $\mathcal{L}(\mathfrak{X}_K)$ is separable
  - The space $\mathcal{K}(\mathfrak{X}_K)$ is complemented in $\mathcal{L}(\mathfrak{X}_K)$
  - c_0 does not embed into $\mathcal{L}(\mathfrak{X}_K)$.

The Solution of the "Scalar plus Compact" Problem.

After the discovery of the space $\mathfrak{X}_K$ we studied the structure of the space $\mathcal{L}(\mathfrak{X}_K)$ showing the following:

- **Theorem** (S.Argyros and R. Haydon) Every bounded linear operator $T : \mathfrak{X}_K \to \mathfrak{X}_K$ is of the form $T = \lambda I + K$ with $K : \mathfrak{X}_K \to \mathfrak{X}_K$ a compact operator.
- This also yields the following:
- **Theorem** For the space $\mathfrak{X}_K$ the following hold:
  - Every bounded linear operator $T : \mathfrak{X}_K \to \mathfrak{X}_K$ has a non trivial invariant subspace.
  - The space $\mathcal{L}(\mathfrak{X}_K)$ is separable
  - The space $\mathcal{K}(\mathfrak{X}_K)$ is complemented in $\mathcal{L}(\mathfrak{X}_K)$
  - c_0 does not embed into $\mathcal{L}(\mathfrak{X}_K)$.

The Solution of the "Scalar plus Compact" Problem.

After the discovery of the space $\mathfrak{X}_K$ we studied the structure of the space $\mathcal{L}(\mathfrak{X}_K)$ showing the following:

- **Theorem** (S.Argyros and R. Haydon) Every bounded linear operator $T : \mathfrak{X}_K \to \mathfrak{X}_K$ is of the form $T = \lambda I + K$ with $K : \mathfrak{X}_K \to \mathfrak{X}_K$ a compact operator.
- This also yields the following:
- **Theorem** For the space $\mathfrak{X}_K$ the following hold:
  - Every bounded linear operator $T : \mathfrak{X}_K \to \mathfrak{X}_K$ has a non trivial invariant subspace.
  - The space $\mathcal{L}(\mathfrak{X}_K)$ is separable
  - The space $\mathcal{K}(\mathfrak{X}_K)$ is complemented in $\mathcal{L}(\mathfrak{X}_K)$
  - c_0 does not embed into $\mathcal{L}(\mathfrak{X}_K)$.

The Solution of the "Scalar plus Compact" Problem.

After the discovery of the space $\mathfrak{X}_K$ we studied the structure of the space $\mathcal{L}(\mathfrak{X}_K)$ showing the following:

- **Theorem** (S.Argyros and R. Haydon) Every bounded linear operator $T : \mathfrak{X}_K \to \mathfrak{X}_K$ is of the form $T = \lambda I + K$ with $K : \mathfrak{X}_K \to \mathfrak{X}_K$ a compact operator.
- This also yields the following:
- **Theorem** For the space $\mathfrak{X}_K$ the following hold:
  - Every bounded linear operator $T : \mathfrak{X}_K \to \mathfrak{X}_K$ has a non trivial invariant subspace.
  - The space $\mathcal{L}(\mathfrak{X}_K)$ is separable
  - The space $\mathcal{K}(\mathfrak{X}_K)$ is complemented in $\mathcal{L}(\mathfrak{X}_K)$
  - c_0 does not embed into $\mathcal{L}(\mathfrak{X}_K)$.