Some examples of generic Banach spaces

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Main goal:

Study of the existence of uncountable “biorthogonal-like” sequences in a Banach space.

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Method of forcing, direct limits of direct systems of finite dimensional polyhedral spaces.

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Definition

Recall that a (normalized) sequence $(x_i)_{i\in\mathbb{N}}$ in a Banach space X is called a basic sequence if there is a constant $C \geq 1$ such that for every sequence of scalars $(a_i)_i$ and every integer m one has that

$$\|\sum_{i\leq m} a_i x_i\| \leq \|\sum_{i\in\mathbb{N}} a_i x_i\|.$$

Proposition (Mazur)

Every infinite dimensional Banach space has a basic sequence.

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A (normalized) sequence $(x_\alpha)_{\alpha<\kappa}$ in a Banach space X is called a Schauder basic sequence if there is a constant such that for every $\gamma \leq \kappa$ and every sequence of scalars $(a_\alpha)_{\alpha<\kappa}$ one has that

$$\|\sum_{\alpha<\gamma} a_\alpha x_\alpha\| \leq \|\sum_{\alpha<\kappa} a_\alpha x_\alpha\|.$$

Proposition

Let X be a Banach space, and suppose that κ is the weak*-density of X^*. Then X contains a Schauder basic sequence of length κ.

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Let X be a Banach space, and suppose that κ is the weak*-density of X^*. Then X contains a Schauder basic sequence of length κ.

Definition

Let X be a Banach space, and κ be a cardinal. A sequence $(x_\alpha, f_\alpha)_{\alpha<\kappa} \in (X \times X^*)^\kappa$ is called a *biorthogonal system* if $f_\alpha(x_\beta) = \delta_{\alpha,\beta}$.

The sequence $(x_\alpha, f_\alpha)_\alpha$ is called *fundamental* if $\overline{\langle x_\alpha \rangle_{\alpha<\kappa}} = X$.

The sequence $(x_\alpha, f_\alpha)_\alpha$ is called *total* if $\overline{\langle f_\alpha \rangle_\alpha}^{w^*} = X^*$.

A fundamental and total biorthogonal system is called a *Markushevich* basis of X (or simply M-basis).

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Every separable Banach space has a M-basis.

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A sequence $(x_\alpha, f_\alpha)_{\alpha<\kappa} \in (X \times X^*)^\kappa$ is called a ε-biorthogonal system ($\varepsilon > 0$) if $f_\alpha(x_\alpha) = 1$, $f_\alpha(x_\beta) = 0$ and $|f_\beta(x_\alpha)| \leq \varepsilon$ for $\alpha < \beta$.

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Theorem

For every $\varepsilon > 0$ there is a c.c.c.-forcing construction giving a Banach space $\mathfrak{X}$ of density ω_1 having an uncountable ε-biorthogonal sequence but no uncountable biorthogonal sequences. Moreover, the space have the following properties:

(I) *The space $\mathfrak{X}$ is a Gurarij space.*

(II) *There is an uncountable $D \subseteq S_{\mathfrak{X}}$ dense subset of $S_{\mathfrak{X}}$ such that every normalized basic sequence is finitely representable by blocks in any uncountable separated sequence of points in D.*

(III) *For every finite dimensional normed space F, every non-separable subspace X of $\mathfrak{X}_G$ and every $\delta > 0$ there is a subspace $\widetilde{F} \subseteq X$ such that the Banach-Mazur distance $d(F, \widetilde{F}) \leq 1 + \delta$.*

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(IV) Every operator between a subspace X of $\mathfrak{X}$ and $\mathfrak{X}$ is a multiple of the identity plus a separable range operator.

(V) The distance between two non-separable subspaces is zero.

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There is a c.c.c.-forcing construction giving a Banach space $\mathfrak{X}$ of density ω_1 having a biorthogonal sequence $(x_\alpha, f_\alpha)_{\alpha<\omega_1}$ which is total in $\overline{\langle x_\alpha \rangle}_{\alpha<\omega_1}$ but not having uncountable basic sequences.

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Theorem

There is a c.c.c.-forcing construction giving a Banach space $\mathfrak{X}$ of density ω_1 which does not have uncountable ε-biorthogonal sequences, for every $\varepsilon > 0$, and such that $(\mathfrak{X}, w)$ is not Hereditarily Lindelöf.

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Remark

Recall that a space Z is hereditarily Lindelöf iff Z has no uncountable right separated sequences, i.e. for every uncountable sequence $(x_\alpha)_{\alpha<\omega_1}$ there is $\alpha < \omega_1$ which is in the closure of $\{x_\beta : \beta > \alpha\}$.
X has no uncountable ε-biorthogonal sequences iff for every uncountable sequence $(x_\alpha)_{\alpha<\omega_1}$ there is $\alpha < \omega_1$ such that $x_\alpha \in \overline{co\{x_\beta\}_{\beta>\alpha}}$.

It follows that if X is a Banach space which is w-HL, then it has no uncountable ε-biorthogonal sequences.

Remark

Under $\mathfrak{b} = \omega_1$ (Todorcevic) or CH (Kunen) there are compact spaces K such that $(C(K)^n, w^n)$ is HL for every n.

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The spaces are builded by the method of forcing. One can think on this as a direct limit of a directed system of finite dimensional Banach spaces and isometries (in our case, the inclusion mapping).

Definition (Basic forcing)

Conditions of $\mathbb{P}$ are triples $p = (D_p, A_p, F_p)$ with the following properties:

(C.0) $A_p \subseteq D_p \subseteq \omega_1$ and D_p is finite.

(C.1) $F_p \subseteq c_{00}(D_p, \mathbb{Q})$ is finite, symmetric and such that for every $\alpha \in D_p$ there is some $0 < \lambda \leq 1$ such that $\lambda u_\alpha \in F_p$ (NORMING).

(C.2) For every $\alpha \in A_p$ there is $f_\alpha \in F_p$ such that $u^*_\alpha(f_\alpha) = 1$ and $u^*_\beta(f_\alpha) = 0$ for every $\beta \in D_p \cap \alpha$ (LARGE DENSITY).

Observe that (C.1) implies that the formula for $x \in c_{00}(D_p)$ defined by

$$\|x\|_p = \max\{\langle f, x\rangle \,:\, f \in F_p\} \tag{1}$$

is a norm on the f.d. vector space $c_{00}(D_p)$. We denote this normed space by X_p.

Definition

The ordering $p \leq q$ is defined by:

(O.0) $D_q \subseteq D_p$ and $A_q \subseteq A_p$.

(O.1) For every $x \in c_{00}(D_q)$ one has that $\|x\|_p = \|x\|_q$.

The basic intuition behind the forcing notion $\mathbb{P}$ is that the desired Banach space is obtained by taking some sort of limits over a set finite approximations and for this we need the following notion.

Definition

A subset $\mathbb{F}$ of $\mathbb{P}$ is a *filter* if

(F.1) For all $p, q \in \mathbb{F}$, there is $r \in \mathbb{F}$ such that $r \leq p$ and $r \leq q$,

(F.2) For all $p, q \in \mathbb{P}$, if $p \leq q$ and $p \in \mathbb{F}$ then $q \in \mathbb{F}$.

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Note that any filter $\mathbb{F}$ of $\mathbb{P}$ naturally leads to a norm $\|.\|_{\mathbb{F}}$ on $c_{00}(\bigcup_{p\in\mathbb{F}} D_p)$ given by $\|x\|_{\mathbb{F}} = \|x\|_p$, where $p \in \mathbb{F}$ is such that $x \in x_p$. Let $X_{\mathbb{F}}$ denote its completion.

Definition

A subset $\mathbb{D}$ of $\mathbb{P}$ is *dense-open* if

(D.1) For all $p \in \mathbb{P}$ there is $q \in \mathbb{D}$ such that $q \leq p$,

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For example, the following is a natural family of dense-open subsets of $\mathbb{P}$.

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Proposition

For all $\alpha < \omega_1$, the set $\mathbb{D}_\alpha = \{p \in \mathbb{P} : \alpha \in D_p\}$ is a dense-open subset of $\mathbb{P}$.

Proof.

Fix $p \in \mathbb{P}$. If $\alpha \in D_p$, then we are done, so suppose that $\alpha \notin D_p$. Let $q = (\{\alpha\} \cup D_p, A_p, F_p \cup \{\pm u_\alpha\})$. Then $q \in D_\alpha$ and $q \leq p$. □ □

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Definition

A Banach space X is *generic* relative to the set $\mathbb{P}$ of finite approximations if $X = X_{\mathbb{F}}$ for some filter $\mathbb{F}$ of $\mathbb{P}$ that intersects *all* dense open subsets of $\mathbb{P}$.

Clearly such a filter can only be an imaginary which will not prevent us using it in getting the corresponding generic Banach space $X_{\mathbb{G}}$ and study its properties. In fact there is a whole new universe of sets, *the forcing extension*, that $\mathbb{G}$ generated using the sets from our *ground* universe V.

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Let $\mathbb{G}$ be a generic filter. We define

$$D_{\mathbb{G}} = \bigcup_{p \in \mathbb{G}} D_p = \omega_1$$

$$A_{\mathbb{G}} = \bigcup_{p \in \mathbb{G}} A_p, \text{ and}$$

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The Generic space $X_{\mathbb{G}}$ is the completion of the normed space $(c_{00}(\omega_1), \|\cdot\|_{\mathbb{G}})$, where

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We have to guarantee that the space $X_{\mathbb{G}}$ is non-separable. In particular, we have to prove that $\mathbb{P}$ does not collapse ω_1. This is done by the following:

Proposition

The forcing notion $\mathbb{P}$ satisfies the countable chain condition.

Given an uncountable sequence $(p_\alpha)_{\alpha<\omega_1}$ of conditions of $\mathbb{P}$ we have to prove that at least two of them are compatible, , i.e. there are $\alpha < \beta$ and some condition $p \in \mathbb{P}$ such that $p \leq p_\alpha, p_\beta$; in other words, we can produce an amalgamation p of p_α and p_β.

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Given an uncountable sequence $(p_\alpha)_{\alpha<\omega_1}$ of conditions of $\mathbb{P}$ we have to prove that at least two of them are compatible, , i.e. there are $\alpha < \beta$ and some condition $p \in \mathbb{P}$ such that $p \leq p_\alpha, p_\beta$; in other words, we can produce an amalgamation p of p_α and p_β.

We have to guarantee that the space $X_{\mathbb{G}}$ is non-separable. In particular, we have to prove that $\mathbb{P}$ does not collapse ω_1. This is done by the following:

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Of course, for arbitrary pair $\alpha < \beta$ this is, in general, not possible.

However, since each condition $p = (D_p, A_p, F_p)$ are such that $A_p \subseteq D_p \subseteq \omega_1$ is finite, and $F_p \subseteq c_{00}(\omega_1, \mathbb{Q})$ is also finite, we can alway pass to an uncountable subsequence $(p_\alpha)_{\alpha \in \Gamma}$ such that

1. $(D_{p_\alpha})_{\alpha \in \Gamma}$ forms a Δ-system with root R,
2. for every $\alpha < \beta$ in Γ, one has that $|D_{p_\alpha}| = |D_{p_\beta}|$ and
3. if $\varpi_{\alpha,\beta} : D_{p_\alpha} \to D_{p_\beta}$, is the order-preserving bijection, then indeed $\varpi_{\alpha,\beta}$ is an *isomorphism* between the structures $p_\alpha = (D_{p_\alpha}, A_{p_\alpha}, F_{p_\alpha})$ and $p_\beta = (D_{p_\beta}, A_{p_\beta}, F_{p_\beta})$ in the obvious way.

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In general, we can put in F_p any $f \odot g$ for $f \in F_{p_\alpha}$ and $g \in F_{p_\beta}$ provided that they coincide in the root R

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The set A_p is the responsible of making the generic space non-separable: observe that for $\alpha < \beta$ in $A_{\mathbb{G}} = \bigcup_{p \in \mathbb{G}} A_p$ one has that

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Proof.

For each $\alpha < \omega_1$, we define the dense set

$$D_\alpha = \{p \in \mathbb{P} : \max A_p > \alpha\}.$$

□ □

In other cases $A_{\mathbb{G}}$ will give a biorthogonal sequece, a ε-biorthogonal sequence... by a modification of the requirement $(C.2)$.

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Definition

A sequence $(x_\alpha)_{\alpha<\kappa}$ in X is called ω-independent if for every sequence increasing $(\alpha_n)_n$ and every sequence $(a_n)_n$ of scalars if $(\sum_{n=1}^{m} a_n x_n)_n$ converges in norm to 0, then $a_n = 0$ for every n.

Remark

If $(x_\alpha, f_\alpha)_\alpha$ is a biorthogonal sequence then $(x_\alpha)_\alpha$ is an ω-independent family.

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A separable Banach space does not have uncountable independent families (Fremlin and Sersouri).

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