On countable dense and strong n-homogeneity

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Descriptive set theory in Paris 2010.

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- The first result in this area is due to Cantor, who showed that the reals are CDH. Fréchet and Brouwer, independently, proved that the same is true for the n-dimensional Euclidean space $\mathbb{R}^n$.

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- There are many other CDH-spaces, as the following results show.

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- Bennett proved in 1972 that a *connected* CDH-space is homogeneous. (The converse is not true.)
- Hence for *connected* spaces, countable dense homogeneity can be thought of as a strong form of homogeneity.

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2. A space X is *strongly n-homogeneous* provided that for all n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of distinct points of X there is a homeomorphism f of X such that $f(x_i) = y_i$ for all $i \leq n$.

- Ungar proved:

Theorem

Let X be a locally compact separable metrizable space such that no finite set separates X. Then the following statements are equivalent:

(a) X *is* CDH.

(b) X *is n-homogeneous for every n.*

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- Observe that $(c) \Rightarrow (b)$ is trivial, and that $(c) \Rightarrow (a)$ seems trivial since the obvious approach is to use the standard back-and-forth method. The problem that one faces however is how to ensure that an inductively constructed sequence of homeomorphisms converges to a homeomorphism. At first glance it seems that $(a) \Rightarrow (c)$ is the most surprising implication.

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- The question whether one can prove a similar result with the assumption of local compactness relaxed to that of completeness is a natural one in this context.
- It turns out that the implication (a) $\Rightarrow$ (c) in Ungar's Theorem holds for all spaces, in essence even without connectivity assumptions.

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Theorem

If the group G makes the space X CDH *and no set of size $n-1$ separates X, then G makes X strongly n-homogeneous.*

Proposition

Let X be a space. Suppose that G is a subset of $\mathscr{H}(X)$ that makes X CDH. *If $F \subseteq X$ is finite and $D, E \subseteq X \setminus F$ are countable and dense in X, then there are elements $\alpha, \beta \in G$ such that $\alpha{\restriction}F = \beta{\restriction}F$ and $(\alpha^{-1} \circ \beta)(D) \subseteq E$.*

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- Let h_0 be an arbitrary element in G. Suppose $\{h_\beta : \beta < \alpha\} \subseteq G$ have been constructed for some $\alpha < \omega_1$. Now by CDH, pick $h_\alpha \in G$ such that

$$(\dagger) \qquad h_\alpha(F \cup E) = \bigcup_{\beta<\alpha} h_\beta(D).$$

For $1 \leq \alpha < \omega_1$, let T_α be a nonempty finite subset of $[1, \alpha)$ such that $h_\alpha(F) \subseteq \bigcup_{\beta \in T_\alpha} h_\beta(D)$. We claim that, for $T\colon [1, \omega_1) \to [\omega_1]^{<\omega}$ defined by $T(\alpha) = T_\alpha$, the fiber $T^{-1}(A)$ is uncountable for some $A \in [\omega_1]^{<\omega}$. If the latter were not true, there would exist a strictly increasing sequence $\{\alpha_n\}_n$ of countable ordinal numbers such that $T^{-1}(A) \subseteq \alpha_{n+1}$ for each $A \in T([1, \alpha_n])$. Then, letting $\alpha = \sup_n \alpha_n$ and $A = T(\alpha)$, one can find n such that $A \subseteq \alpha_n$; hence $A \in T([1, \alpha_n])$, which contradicts $\alpha \in T^{-1}(A) \subseteq \alpha_{n+1}$. (This is of course nothing but the standard argument in the proof of the Pressing Down Lemma.) So pick an $A \in [\omega_1]^{<\omega}$ for which $E = T^{-1}(A)$ is uncountable. Then $h_\alpha(F) \subseteq \bigcup_{\beta \in A} h_\beta(D)$ for every $\alpha \in E$. Since $\bigcup_{\beta \in A} h_\beta(D)$ is countable, and E is uncountable, we may consequently assume without loss of generality that $h_\alpha \restriction F = h_\beta \restriction F$ for all $\alpha, \beta \in E$. Hence if $\alpha, \beta \in E$ are such that $\beta < \alpha$, then $h_\alpha \restriction F = h_\beta \restriction F$ and by $(\dagger)$, $(h_\alpha^{-1} \circ h_\beta)(D) \subseteq E$.

Corollary

Let X be a space without isolated points. Assume that the group G makes X CDH*. Then for every finite subset $F \subseteq X$, every G_F-invariant subset of $X \setminus F$ is open.*

Proof.

For $x \in X \setminus F$, let $Y = G_F x$. To show that Y has nonempty interior, assume that it is not the case. Then we may pick a countable dense set D in X which is contained in $X \setminus (F \cup Y)$. By 9, there exists $h \in G_F$ such that $h(D \cup \{x\}) \subseteq D$, a contradiction because $h(x) \in G_F x = Y$ and $h(x) \in D \subseteq X \setminus Y$. So Y is open, being an orbit. $\square$

Proof of Theorem 8.

Assume that G makes the space X CDH and no set of size $n-1$ separates X. All we need to show is that for every $F \in [X]^{n-1}$ the group G_F acts transitively on $X \setminus F$. By 10 every orbit $G_F x$ for $x \in X \setminus F$ is open. Since orbits are disjoint, they are clopen. So we are done by connectivity. $\square$

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- *Is there for every homogeneous space X a nice topological group acting transitively on it?*
- Nice because $\mathcal{H}(X)$ with the discrete topology acts transitively if X is homogeneous.

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- X is 1-homogeneous, but not 2-homogeneous.
- What does this advertising of X have to do with the present talk?

The counterexample

Theorem (Ungar)

Let X be a locally compact separable metrizable space such that no finite set separates X. Then $(a) \Leftrightarrow (b) \Leftrightarrow (c)$, where (a) X is CDH, *(b) X is n-homogeneous for every n, (c) X is strongly n-homogeneous for every n.*

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Example

There are a Polish space $\mathcal{X}$ and a (separable metrizable) topological group (G, τ) such that

1. (G, τ) acts on $\mathcal{X}$ by a continuous action, and makes $\mathcal{X}$ strongly n-homogeneous for every n,
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- But there is a significant difference. The components of $\mathcal{X}$ are points, so they are not wildly distributed. The pathology must be different.
- $\mathcal{X}$ is a tricky subspace of the product $\{0,1\}^\infty \times \mathfrak{E}_c$, where $\mathfrak{E}_c$ is the *complete Erdős space*.

- In 1940 Erdős proved that the 'rational Hilbert space' space $\mathfrak{E}$, which consists of all vectors in the real Hilbert space ℓ^2 that have only rational coordinates, has dimension one, is totally disconnected, and is homeomorphic to its own square. This answered a question of Hurewicz who proved that for every compact space X and every 1-dimensional space Y we have that $\dim(X \times Y) = \dim X + 1$.

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- It is not difficult to prove that $\mathfrak{E}$ has dimension at most 1. Erdős proved the surprising fact that every nonempty clopen subset of $\mathfrak{E}$ is unbounded, and hence that for no $x \in \mathfrak{E}$ and no $t > 0$ the open ball $\{y \in \mathfrak{E} : \|x - y\| < t\}$ contains a nonempty clopen subset of $\mathfrak{E}$. This implies among other things that $\mathfrak{E}$ is nowhere zero-dimensional.

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- It is not difficult to prove that $\mathfrak{E}$ has dimension at most 1. Erdős proved the surprising fact that every nonempty clopen subset of $\mathfrak{E}$ is unbounded, and hence that for no $x \in \mathfrak{E}$ and no $t > 0$ the open ball $\{y \in \mathfrak{E} : \|x - y\| < t\}$ contains a nonempty clopen subset of $\mathfrak{E}$. This implies among other things that $\mathfrak{E}$ is nowhere zero-dimensional.
- This is the crucial property that makes the Erdős spaces so interesting.

- Erdős also proved that the closed subspace $\mathfrak{E}_{\mathrm{c}}$ of ℓ^2 consisting of all vectors such that every coordinate is in the convergent sequence $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ has the same property. The space $\mathfrak{E}_{\mathrm{c}}$ is called *complete Erdős space* and was shown by Dijkstra to be homeomorphic to the 'irrational' Hilbert space, which consists of all vectors in the real Hilbert space ℓ^2 that have only irrational coordinates. All nonempty clopen subsets of $\mathfrak{E}_{\mathrm{c}}$ are unbounded just as the nonempty clopen subsets of $\mathfrak{E}$ are.

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- It was shown by Dijkstra that the complete Erdős space is homeomorphic to the so-called *harmonic Erdős space*. It is the subspace of all vectors x in the real Hilbert space ℓ^1 such that $x_n \in \{0, 1/n\}$ for all n. This space is a particularly elegant model of $\mathfrak{E}_{\mathrm{c}}$.

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- Let $C = \{0,1\}^{\mathbb{N}}$ be the Cantor set, and define $\varphi\colon C \to [0,\infty]$ by $\varphi(x) = \sum_{n=1}^{\infty} x_n/n$. The harmonic Erdős space is topologically the graph of the function $x \mapsto \varphi(x)$ on $\{x \in C : \varphi(x) < \infty\}$. So it is a subspace of $C \times [0,\infty)$.

- The space $\mathfrak{E}_{\mathrm{c}}$ surfaces at many places. For example, as the set of endpoints of certain dendroids (among them, the Lelek fan), the set of endpoints of the Julia set of the exponential map, the set of endpoints of the separable universal $\mathbb{R}$-tree, line-free groups in Banach spaces and Polishable ideals on $\mathbb{N}$.

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- $\mathfrak{E}_{\mathrm{c}}$ has the following property: every bounded closed subspace is somewhere zero-dimensional. (This property implies that $\mathfrak{E}_{\mathrm{c}} \not\approx \mathfrak{E}_{\mathrm{c}}^{\infty}$.) $\mathcal{X}$ contains arbitrarily small closed copies of $\mathfrak{E}_{\mathrm{c}}$, which are all nowhere zero-dimensional.

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- Metric and topological characterizations of $\mathfrak{E}_c$ were proved by Kawamura, Tymchatyn, Oversteegen, Dijkstra and van Mill.
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- $\mathfrak{E}_c$ has the following property: every bounded closed subspace is somewhere zero-dimensional. (This property implies that $\mathfrak{E}_c \not\approx \mathfrak{E}_c^\infty$.) $\mathcal{X}$ contains arbitrarily small closed copies of $\mathfrak{E}_c$, which are all nowhere zero-dimensional.
- This last property of $\mathcal{X}$ and a variation of Erős' original argument from 1940, give us that $\mathcal{X}$ is not CDH.

- So the pathology of $\mathcal{X}$ is not based upon connectivity, but upon the pathology present in the complete Erdős space.