Measurable colorings of graphs

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Part I

Ancient history

Credit

This is joint work with Alekos Kechris and, towards the end, with Ben Miller.

Axioms

We always work in ZFC!

Parenthesis

(It is possible to formulate everything without Choice, but the language becomes needlessly cumbersome.)

Definition

A *graph* G on a set X is a symmetric, irreflexive subset of X^2. We refer to points of X as the *vertices* of G, and pairs $(x, y) \in G$ as *edges* of G (and such x and y are called *adjacent*).

Definition

For $A \subseteq X$ the *restriction* of G to A, written $G|A$, is simply $G \cap A^2$.

Definition

We say that $A \subseteq X$ is *G-independent* (or *G-discrete*) if $G|A = \emptyset$.

Definition

A (*proper*) *coloring* of G is a function $c : X \to Y$ so that $c^{-1}(\{y\})$ is G-independent for all $y \in Y$. Equivalently, $xGy \Rightarrow c(x) \neq c(y)$.

Definition

The *chromatic number* of G, written $\chi(G)$, is the least cardinality of a set Y so that there exists some coloring $c : X \to Y$ of G.

Definition

A (G-)*path* is an injective sequence $x_0, \ldots, x_n \in X$ with $x_i G x_{i+1}$ for all $i < n$; we call n its *length*. A (G-)*cycle* of length $n \geq 3$ is an injective sequence $x_0, \ldots, x_{n-1} \in X$ with $x_i G x_{i+1} \pmod{n}$.

Remark

A nonempty graph G satisfies $\chi(G) = 2$ iff G has no cycles of odd length.

Definition

A graph G is *connected* if any two distinct points of X are the endpoints of some G-path.

Definition
The (G-)*degree* of a point $x \in X$, denoted $\deg(x)$, is the cardinality of $G_x = \{y \in X : xGy\}$.

Definition
The *degree* of a graph G, written $\Delta(G)$, equals $\sup_{x \in X} \deg(x)$.

Theorem (Brooks)
Almost every connected graph G satisfies $\chi(G) \leq \Delta(G)$.

The only counterexamples are odd cycles and finite complete graphs (those of the form $xGy \Leftrightarrow x \neq y$).

Remark
Of course, Brooks' theorem also holds for graphs which are not connected, but you must allow for the counterexamples to be mixed and matched a bit.

Remark
Brooks' theorem, like almost every theorem in classical graph theory, is first established for finite graphs by induction. A straightforward compactness argument then handles the general case.

Remark
This approach, of course, uses lots of Choice!

Kechris-Solecki-Todorcevic (1999) investigates what happens when definability restrictions are placed on the graph and the coloring function.

Definition

A graph G on a Polish space X is *Borel* when it is Borel as a subset of X^2.

Definition

The *Borel chromatic number* of G, denoted $\chi_B(G)$, is the least cardinality of a Polish space Y such that there is a Borel coloring $c : X \to Y$ of G.

Remark

Thus, if G is nonempty, we have $\chi_B(G) \in \{2, 3, \ldots, \aleph_0, 2^{\aleph_0}\}$.

Question

Is there an analog of Brooks' theorem in the context of Borel chromatic numbers?

Definition

A graph G on X is *locally finite* (resp., *locally countable*) if for all $x \in X$, $\deg(x) < \aleph_0$ (resp., $\deg(x) \leq \aleph_0$).

Theorem (KST)

If G is a locally finite, Borel graph on X, then $\chi_B(G) \leq \aleph_0$.

Proof

Color each point by the first basic open set separating it from its G-neighbors. $\square$

In the general locally countable context, things go terribly awry.

Definition

We denote by E_0 the equivalence relation on 2^ω of eventual agreement, i.e.,

$$xE_0y \Leftrightarrow |\{n \in \omega : x(n) \neq y(n)\}| < \aleph_0.$$

Example

The graph G on 2^ω given by $xGy \Leftrightarrow x \neq y$ and xE_0y is locally countable, but $\chi_B(G) = 2^{\aleph_0}$.

Proof

If there were a Borel coloring by countably many colors, we could obtain a Borel transversal of E_0 by taking the set of points of minimal color in their E_0-classes. $\square$

When the degree of a graph is bounded, KST comes closer to Brooks' theorem.

Theorem (KST)

If G is a Borel graph on X and $\Delta(G) < \aleph_0$, then $\chi_B(G) \leq \Delta(G) + 1$.

Proof

- We know that $\chi_B(G) \leq \aleph_0$.
- From this, we can build a Borel set A which is maximal G-independent.
- Then the graph $G|(X \setminus A)$ has degree strictly less than $\Delta(G)$.
- The result then follows by induction. $\square$

Example

Consider the action of $\mathbb{Z}$ on $2^{\mathbb{Z}}$ by shifts:

$$z + A = \{z + a : a \in A\}.$$

Restrict this action to its free part (i.e., those A for which $z + A = A \implies z = 0$).

We then define a graph G by setting A adjacent to $\pm 1 + A$.

Every vertex of this graph has degree 2, and in fact each connected component looks like a $\mathbb{Z}$ line. However, $\chi_B(G) = 3$.

Part II

Measurable colorings

We now shift our attention to graphs on standard probability spaces.

Definition

For a graph G on a standard probability space (X, μ), we define its (μ-)*measurable* chromatic number, $\chi_\mu(G)$, to be the least cardinality of a Polish space Y such that there is a (μ-)measurable coloring $c : X \to Y$ of G.

Remark

Certainly, $\chi_\mu(G) \leq \chi_B(G)$.

Remark

Since the shift action of $\mathbb{Z}$ on $2^{\mathbb{Z}}$ is free on a conull subset of $2^{\mathbb{Z}}$, the graph on the previous slide in fact satisfies $\chi_\mu(G) = 3$.

So, if we want Brooks' theorem to go through, we must further tweak our notion of chromatic number.

Definition

For a graph G on a standard probability space (X, μ), we define its *approximate* (μ-)*measurable* chromatic number, $\chi_\mu^{\mathrm{ap}}(G)$, to be the least cardinality of a Polish space Y such that for all $\varepsilon > 0$ there is a Borel set $A \subseteq X$ with $\mu(X \setminus A) < \varepsilon$ and a (μ-)measurable coloring $c : A \to Y$ of $G|A$.

Remark

Clearly $\chi_\mu^{\mathrm{ap}}(G) \leq \chi_\mu(G) \leq \chi_B(G)$.

Definition

A graph G on X generates an equivalence relation, which we denote by E_G. If G is locally countable, E_G is countable.

Definition

We say G is *hyperfinite* if its corresponding equivalence relation is hyperfinite, that is, E_G can be written as the increasing union $E_1 \subseteq E_2 \subseteq \cdots$ of Borel equivalence relations, each of which has finite classes.

Definition

When G is on a probability space (X, μ), we say G is *μ-hyperfinite* if it is hyperfinite on a conull, E_G-invariant set.

Theorem

Suppose that G is a locally finite, μ-hyperfinite, Borel graph on (X, μ). Then $\chi_\mu^{\mathrm{ap}}(G) \leq \chi(G)$.

Proof

- Without loss, assume G is in fact hyperfinite. Fix $\varepsilon > 0$ and witnesses $E_1 \subseteq E_2 \subseteq \cdots$ to the hyperfiniteness of E_G.
- Set $X_n = \{x \in X : \forall y \ (xGy \implies xE_n y)\}$. Intuitively, $x \in X_n$ iff E_n captures all of the neighbors of x.
- Since G is locally finite, X is the increasing union of $X_1 \subseteq X_2 \subseteq \cdots$.
- Pick n with $\mu(X_n) > 1 - \varepsilon$. Since $G|X_n$ has finite connected components, $\chi_B(G|X_n) \leq \chi(G|X_n) \leq \chi(G)$. $\square$

Question

Does the bound $\chi_\mu^{\mathrm{ap}}(G) \leq \chi(G)$ remain true for locally *countable*, μ-hyperfinite, Borel graphs?

Answer

No!

Example

Consider the graph G on 2^ω given by

$$xGy \Leftrightarrow \exists! n \ (x(n) \neq y(n)).$$

- We have $E_G = E_0$, which is certainly hyperfinite.
- It's not hard to see $\chi(G) = 2$, since G contains no odd cycles.
- However, $\chi_\mu^{\mathrm{ap}}(G) = 2^{\aleph_0}$!

There is a salvage when the graph contains no cycles.

Theorem

Suppose that G is a locally countable, μ-hyperfinite, acyclic, Borel graph on (X, μ). Then $\chi_\mu^{\mathrm{ap}}(G) \leq 2$.

Proof

A slight variation of the techniques in:

Theorem (B.D. Miller)

Suppose that G is a locally countable, μ-hyperfinite, acyclic, Borel graph on (X, μ). Then $\chi_\mu(G) \leq 3$.

Definition

A graph G is *aperiodic* if the corresponding equivalence relation E_G has no finite classes.

Warning

This has nothing to do with G being acyclic!

Theorem

Suppose that G is an aperiodic, Borel graph on (X, μ) with $\Delta(G) < \aleph_0$. Then $\chi_\mu^{\mathrm{ap}}(G) \leq \Delta(G)$.

Proof

- Fix $k \in \mathbb{N}$ and a marker set A with $\mu(A) < \varepsilon$ so that every point of X is connected to a point of A via a G-path of length at most k.
- Color the points furthest from A by at most $\Delta(G)$ colors.
- Extend this to a $\Delta(G)$ coloring of the next furthest points from A.
- Keep going until you've colored everything but A with $\Delta(G)$ colors. $\square$

Part III

Group actions

We now narrow our focus to graphs arising from free, measure-preserving actions of finitely generated groups.

Definition

Let $\mathrm{FR}(\Gamma, X, \mu)$ denote the space of (μ-a.e.) free, measure-preserving actions of a countably infinite group Γ on a standard probability space (X, μ).

Definition

For S a finite generating set of Γ and $a \in \mathrm{FR}(\Gamma, X, \mu)$, define the graph $G(a, S)$ on X to equal $\{(x, s^a(x)) : x \in X \text{ and } s \in S^{\pm 1}\}$.

Remark

Each connected component of $G(a, S)$ is isomorphic to the Cayley graph $\mathrm{Cay}(\Gamma, S)$ of Γ relative to S.

Remark

The approximate version of Brooks' theorem tells us that $\chi_\mu^{\mathrm{ap}}(G(a,S)) \leq d$, where $d = \Delta(G(a,S)) = |S^{\pm 1}|$.

Remark

It can in fact be shown that any action $a \in \mathrm{FR}(\Gamma, X, \mu)$ is weakly equivalent to some action $b \in \mathrm{FR}(\Gamma, X, \mu)$ with

$$\chi_\mu(G(b,S)) = \chi_\mu^{\mathrm{ap}}(G(b,S)) = \chi_\mu^{\mathrm{ap}}(G(a,S)).$$

So, modulo weak equivalence, Brooks' theorem holds for measurable chromatic numbers as well.

Question

But what can we say about the Borel chromatic numbers of such graphs?

Definition

Given a group Γ with finite generating set S, the number of *ends* of $\mathrm{Cay}(\Gamma, S)$ is the supremum of the number of infinite connected components which remain upon deleting any finite set of vertices.

Remark

This value is independent of the choice of finite generating set S, so we denote it by $e(\Gamma)$.

Remark

For any such Γ, $e(\Gamma) \in \{1, 2, \infty\}$.

Remark

All cost 1 groups, all property (T) groups, and all groups not containing $\mathbb{F}_2$ have finitely many ends.

Theorem (Stallings)

The groups with infinitely many ends are precisely amalgamated products over finite subgroups and nontrivial HNN extensions.

Remark

The groups $\mathbb{Z}$ and $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ have two ends.

We are finally in a position to state an analog of Brooks' theorem for Borel chromatic numbers.

Theorem

Suppose that Γ is a group with finite generating set S. Suppose further that Γ has finitely many ends, and that Γ is neither $\mathbb{Z}$ nor $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. Then for any free Borel action a of Γ on a Polish space, we have $\chi_B(G(a, S)) \leq d$.

Remark

Thus, at least among groups with finitely many ends, our initial counterexample is essentially the only one.

We mimic the proof of the approximate Brooks' theorem.

Proof

- Tile the space by large, finite rooted trees contained in the graph $G(a, S)$ so that each root is incident with two nonadjacent monovalent vertices (which we'll now call *leaves*).
- Color those special leaves by color 0.
- Now color the remaining leaves of the trees using at most d colors.
- Extend this to a d coloring of the non-root leaves which result from pruning the original leaves.
- Keep going until you've colored everything but the roots with d colors.
- Since each root is adjacent to two vertices of color 0, there has to be a color left over for each root. □

Part IV

Large chromatic number

Question

What about groups with infinitely many ends? For example, what about the free groups?

Remark

Despite being acyclic, it turns out that graphs arising from free actions of free groups (with the standard free generating set) can be hard to color.

Definition

For each $n \geq 2$, let $G_{\mathbb{F}_n}$ denote the graph $G(a_{s,n}, S_n)$ where $a_{s,n}$ is the free part of the shift action of $\mathbb{F}_n$ on $2^{\mathbb{F}_n}$ (with the standard product measure) and S_n is a free set of n generators.

Theorem (indep. Lyons-Nazarov)

$\chi_B(G_{\mathbb{F}_n}) \geq \chi_\mu(G_{\mathbb{F}_n}) \geq \chi_\mu^{\mathrm{ap}}(G_{\mathbb{F}_n}) \geq \sqrt{n/2}$.

Remark

The argument is an infinite-dimensional generalization of the connection made by Hoffman between chromatic numbers of finite graphs and the spectra of their corresponding adjacency matrices.

Remark

Lyons-Nazarov in fact improve the asymptotic lower bound to the order of $n/\log n$ by using properties of random graphs (due to Bollobás and Frieze-Luczak).

Definition

A *homomorphism* from G on X to H on Y is a function $\varphi : X \to Y$ such that $x_0 G x_1 \implies \varphi(x_0) H \varphi(x_1)$.

Remark

In KST is defined an acyclic graph G_0 which is minimal (under Borel homomorphism) among Borel graphs of uncountable Borel chromatic number.

Question

It is asked therein whether such a minimal graph exists for graphs of infinite Borel chromatic number.

Definition

With any function $f : X \to X$ we may associate a graph G_f on X by setting $x_0 G_f x_1$ when $x_0 \neq x_1$ and either $f(x_0) = x_1$ or vice versa.

Hope

A proposed candidate for a minimal graph among those of infinite Borel chromatic number is G_s, where $s : [\mathbb{N}]^{\aleph_0} \to [\mathbb{N}]^{\aleph_0}$ is given by $s(a) = a \setminus \min(a)$. Note that G_s is acyclic and locally finite.

Remark

By considering, for example, the disjoint union of arbitrarily large finite complete graphs, it is clear that an additional assumption is necessary for this minimality to be plausible.

Definition

A graph G on X has *indecomposably infinite Borel chromatic number* if whenever $\{X_i : i \in \mathbb{N}\}$ is a Borel partition of X into G-invariant pieces, there is some $i \in \mathbb{N}$ with $\chi_B(G|X_i)$ infinite.

Remark

The graph G_s (and indeed any graph G_f with infinite Borel chromatic number) has indecomposably infinite Borel chromatic number.

Remark

There is a way of "sewing together" the graphs $G_{\mathbb{F}_n}$ using an ergodic automorphism T, creating a new graph $G_{\mathbb{F}}$. Provided T is chosen wisely, $G_{\mathbb{F}}$ is locally finite, acyclic, and the ergodicity of T ensures that it has indecomposably infinite Borel (and in fact measurable) chromatic number.

Despite these superficial similarities, the graph $G_{\mathbb{F}}$ is very unlike any graph induced by a function.

Theorem

Suppose that $f : X \to X$ is a Borel function. Then there is no Borel homomorphism from $G_{\mathbb{F}}$ to G_f. If, moreover, $\chi_B(G_f)$ is infinite, there is no Borel homomorphism from G_f to $G_{\mathbb{F}}$.

Proof

- That there is no Borel homomorphism from $G_{\mathbb{F}}$ to G_f follows from function graphs admitting measurable 3-colorings with respect to any Borel probability measure.
- That there is no Borel homomorphism from G_f to $G_{\mathbb{F}}$ uses the natural distinction between edges in $G_{\mathbb{F}}$ arising from T and those arising from some $G_{\mathbb{F}_n}$. $\square$

Dashed hope

Thus, G_s cannot be minimal even among locally finite, acyclic graphs of indecomposably infinite Borel chromatic number.

Question

Is there a graph of indecomposably infinite Borel chromatic number which is below both G_s and $G_{\mathbb{F}}$ with respect to Borel homomorphism?

Corollary

If we instead look at *injective* Borel homomorphism, there can be no minimal graph among locally finite, acyclic graphs of indecomposably infinite Borel chromatic number.

Part V

Thanks!