Unitary representations of oligomorphic groups

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Unitary representations

Let G be a topological group.

Definition

A unitary representation of the group G on a Hilbert space $\mathcal{H}$ is a strongly (or, equivalently, weakly) continuous homomorphism $\pi: G \to U(\mathcal{H})$.

The importance of unitary representations stems partly from the fact that one can produce them from actions of the group on other objects, for example, measure spaces or some combinatorial objects.

Classically, the theory is usually restricted to locally compact groups (because of the Haar measure).

Irreducible representations

A closed subspace $\mathcal{K} \subseteq \mathcal{H}$ is invariant under π if $\pi(g)\mathcal{K} \subseteq \mathcal{K}$ for all $g \in G$.

Complete reducibility: If $\mathcal{K}$ is an invariant subspace, then $\mathcal{K}^\perp$ is also invariant, i.e., π splits as a direct sum of two representations, one on $\mathcal{K}$ and one on $\mathcal{K}^\perp$.

Definition

A unitary representation π is irreducible if it has no non-trivial invariant subspaces.

In general, it is not true that every representation is a sum of irreducibles (think of the left-regular representation $\mathbb{R} \curvearrowright L^2(\mathbb{R})$).

Traditionally, one tries to understand the irreducible representations of a given group and the way that some important representations are built out of irreducibles.

Permutation groups

Definition

A permutation group $G \curvearrowright \mathbf{X}$ is a topological group G acting continuously and faithfully on a countable set $\mathbf{X}$.

If we denote by $S(\mathbf{X})$ the group of all permutations of $\mathbf{X}$, then $S(\mathbf{X}) \curvearrowright \mathbf{X}$ is naturally a permutation group, where $S(\mathbf{X})$ is equipped with the pointwise convergence topology ($\mathbf{X}$ is taken to be discrete). This group is also known as S_∞ (if $\mathbf{X}$ is infinite).

The group S_∞ has a basis at 1 consisting of open subgroups (the stabilizers of finite sets).

A permutation group $G \curvearrowright \mathbf{X}$ is closed if G is a closed subgroup of $S(\mathbf{X})$. Every closed permutation group has a basis at 1 of open subgroups (the converse is also true).

Gelfand–Raikov

Theorem (Gelfand–Raikov)

Let G be a locally compact group. Then the irreducible unitary representations separate points of G (i.e. for every $1 \neq x \in G$, there exists an irreducible representation π with $\pi(x) \neq I$).

This theorem fails in general for non-locally compact groups. For example, $\mathrm{Homeo}^+(\mathbb{R})$ has no non-trivial representations whatsoever and $L^0(\mu, \mathbb{T})$ has a faithful representation but no irreducible representations.

Theorem

The conclusion of Gelfand–Raikov holds for closed subgroups of S_∞.

More general phenomenon: in some situations closed subgroups of S_∞ resemble locally compact groups (cf. the result of Glasner–Weiss about boolean actions or Hjorth's results on turbulence).

Classification of irreducible representations

Theorem (Peter–Weyl)

Let G be a compact group. Then the following hold:

- Every irreducible representation of G is finite-dimensional and every representation is a direct sum of irreducibles.
- The left-regular representation $G \curvearrowright L^2(G)$ contains as direct summands all irreducible representations.
- In particular, if G is metrizable, G has only countably many irreducible representations. If G is finite, there are only finitely many.

Theorem

If G is a countable, discrete group which is not abelian-by-finite, then unitary equivalence of irreducible representations of G

- (Thoma, 1968) is not smooth;
- (Hjorth, 1997) is not classifiable by countable structures.

Some non-locally compact groups

A complete classification of the unitary representations has been established for:

- S_∞ (Lieberman 1972; another proof by Olshanski 1985);
- The unitary group and related groups (infinite-dimensional orthogonal, symplectic) (Kirillov 1973; later many proofs by Olshanski);
- $\mathrm{GL}(\infty, \mathbf{F}_q)$ (Olshanski 1991).

The proofs of Olshanski use his *semigroup method*.

All of the proofs use essentially the fact that there is an inductive limit of compact subgroups dense in G. Olshanski's semigroup method also relies on certain sets of double cosets having a semigroup structure.

Uniformities on a group

Let G be a topological group.

A function $f\colon G \to \mathbb{C}$ is left uniformly continuous if for every $\epsilon > 0$, there exists U a neighborhood of 1 such that

$$x^{-1}y \in U \implies |f(x) - f(y)| < \epsilon.$$

f is right uniformly continuous if for every $\epsilon > 0$, there exists U a neighborhood of 1 such that

$$xy^{-1} \in U \implies |f(x) - f(y)| < \epsilon.$$

A function $f\colon G \to \mathbb{C}$ is uniformly continuous if it is both left and right uniformly continuous.

The three corresponding uniformities on G are called the left, right and lower (or Roelcke) uniformity, respectively.

Roelcke precompact groups

A topological group G is precompact if its left uniformity is precompact; equivalently, for every $1 \ni U$ open, there exists a finite $F \subseteq G$ such that $FU = G$.

Definition

A topological group G is called Roelcke precompact if its lower uniformity is precompact. Equivalently, G is Roelcke precompact iff for every neighborhood U of 1, there exists a finite set F such that $G = UFU$.

Basic observations:

- If G is Roelcke precompact, then every uniformly continuous function on G is bounded.
- A locally compact group is Roelcke precompact iff it is compact. Indeed, if U is a compact neighborhood of e, $G = UFU$ is compact.
- An abelian (or, more generally, a SIN) group is Roelcke precompact iff it is precompact.

Examples of Roelcke precompact groups

The following groups are Roelcke precompact:

- $\mathrm{Homeo}^+(\mathbb{R})$ with the compact-open topology (Roelcke–Dierolf);
- the unitary group $U(\mathcal{H})$ of a separable Hilbert space $\mathcal{H}$ with the strong operator topology (Uspenski);
- $\mathrm{Aut}(X,\mu)$, the group of measure-preserving automorphisms of a standard probability space (X,μ) (Glasner);
- $\mathrm{Iso}(\mathbf{U}_1)$, the isometry group of the Urysohn metric space of diameter 1 (Uspenski);
- oligomorphic permutation groups.

Closure properties:

- Roelcke precompact groups are stable under open subgroups, products, inverse limits, group extensions, and homomorphisms with dense image.
- They are not stable under taking closed subgroups.

Some easy consequences

Definition (Rosendal)

A topological group G has property (OB) if every time G acts by isometries on a metric space X so that for each $x \in X$, the map $G \to X, g \mapsto g \cdot x$ is continuous, then every orbit is bounded.

Proposition

Every Roelcke precompact group has property (OB).

Proof.

If x_0 is any point, the function $g \mapsto d(x_0, g \cdot x_0)$ is uniformly continuous. $\square$

The property of Roelcke precompactness is likely to be important every time uniformly continuous functions are involved, for example, matrix coefficients of unitary representations:

$$G \to \mathbb{C}, \quad g \mapsto \langle \pi(g)\xi, \eta \rangle.$$

Oligomorphic groups

Definition

A closed permutation group $G \curvearrowright \mathbf{X}$ is called oligomorphic if the induced action $G \curvearrowright \mathbf{X}^n$ has finitely many orbits for each n. G is called oligomorphic if it can be realized as an oligomorphic permutation group.

A closed subgroup of S_∞ is Roelcke precompact if for every open subgroup $V \leq G$, the set of double cosets $\{VxV : x \in G\}$ is finite.

Theorem

For a closed subgroup $G \leq S_\infty$, the following are equivalent:

- G is Roelcke precompact;
- for every continuous transitive action $G \curvearrowright \mathbf{X}$ on a countable set $\mathbf{X}$, the permutation group $G \curvearrowright \mathbf{X}$ is oligomorphic;
- G is an inverse limit of oligomorphic groups.

Oligomorphic groups in model theory

- Closed permutation groups are studied in model theory as automorphism groups of countable structures.
- Oligomorphic groups are especially important because there is a direct correspondence between properties of the permutation group and model-theoretic properties of the structure and its theory.

Theorem (Engeler, Ryll-Nardzewski, Svenonius)

Let G be the automorphism group of a countable structure $\mathbf{X}$. Then the following are equivalent:

- the permutation group $G \curvearrowright \mathbf{X}$ is oligomorphic;
- the structure $\mathbf{X}$ is ω-categorical ($\mathbf{X}$ is the unique countable model of the first-order theory of $\mathbf{X}$).

ω-categorical structures via Fraïssé's construction

A countable structure $\mathbf{X}$ is called homogeneous if every isomorphism between finite substructures of $\mathbf{X}$ extends to a full automorphism of $\mathbf{X}$.

Every homogeneous structure in a finite relational language is ω-categorical.

Fraïssé's construction:

class of finite structures $\longrightarrow$ a homogeneous infinite structure.

Examples:

- $\{$finite sets$\} \longrightarrow$ (countably infinite set);
- $\{$finite linear orders$\} \longrightarrow (\mathbf{Q}, <)$;
- $\{$finite graphs$\} \longrightarrow$ (the random graph);
- $\{$finite vector spaces over $\mathbf{F}_q\} \longrightarrow$ (infinite-dimensional vector space over $\mathbf{F}_q$);
- $\{$finite boolean algebras$\} \longrightarrow (\mathrm{Clopen}(2^{\mathbb{N}}))$.

Commensurators

Two subgroups $H_1, H_2 \leq G$ are commensurate if $H_1 \cap H_2$ has finite index in both H_1 and H_2. The commensurator of H in G is

$$\mathrm{Comm}_G(H) = \{g \in G : H \text{ and } H^g \text{ are commensurate}\}.$$

If G is oligomorphic and $V \leq G$ is open, then

- V has finite index in $\mathrm{Comm}_G(V)$;
- $\mathrm{Comm}_G(\mathrm{Comm}_G(V)) = \mathrm{Comm}_G(V)$.

Call $H \leq G$ a commensurator if it is open and $H = \mathrm{Comm}_G(H)$. Commensurators are exactly the open subgroups that have no supergroups in which they are of finite index.

Induced representations

Let G be a group, H an open subgroup, and σ a representation of H.

The induced representation $\mathrm{Ind}_H^G(\sigma)$ is defined as follows. Let T be a complete system of left coset representatives of H in G. Let M be the space of all functions $f: G \to \mathcal{H}(\sigma)$ for which

$$f(gh) = \sigma(h^{-1}) f(g) \quad \text{for all } g \in G, h \in H.$$

For $f \in M$, define

$$\|f\| = \Big(\sum_{g \in T} \|f(g)\|^2 \Big)^{1/2}$$

Let

$$\mathcal{H} = \{ f \in M : \|f\| < \infty \} \cong \ell^2(G/H, \mathcal{H}(\sigma)).$$

The representation $\mathrm{Ind}_H^G(\sigma)$ on $\mathcal{H}$ is defined by

$$\big(\mathrm{Ind}_H^G(\sigma)(g) \cdot f \big)(x) = f(g^{-1}x).$$

Some representations of permutation groups

Natural representations of closed subgroups of S_∞ are the quasi-regular representations:

$$G \curvearrowright \ell^2(G/V).$$

for $V \leq G$ an open subgroup.
If $V \trianglelefteq H \leq G$, then

$$\ell^2(G/V) \cong \mathrm{Ind}_V^G(1_V) \cong \mathrm{Ind}_H^G\left(\mathrm{Ind}_V^H(1_V)\right) \cong \mathrm{Ind}_H^G(\lambda_{H/V}) \cong \bigoplus_\sigma \mathrm{Ind}_H^G(\sigma).$$

Proposition

Let G be an oligomorphic group. Then the following hold:

- If H is a commensurator, $V \trianglelefteq H$, and σ is a representation of H/V, then $\mathrm{Ind}_H^G(\sigma)$ is irreducible iff σ is.
- If H_1, H_2 are commensurators, $V_1 \trianglelefteq H_1$, $V_2 \trianglelefteq H_2$, and σ_1, σ_2 are irreducible representations of H_1/V_1, H_2/V_2, respectively, then $\mathrm{Ind}_{H_1}^G(\sigma_1) \cong \mathrm{Ind}_{H_2}^G(\sigma_2)$ iff there exists $g \in G$ such that $H_2 = H_1^g$ and $\sigma_2 \cong \sigma_1^g$.

The representations of oligomorphic groups

Theorem

Suppose that G is oligomorphic. Then every unitary representation of G is a sum of irreducible representations of the form $\mathrm{Ind}_H^G(\sigma)$, where H is a commensurator and σ is an irreducible representation of H that factors through a finite factor of H.

Fact: Every oligomorphic group has only countably many open subgroups.

Corollary

If G is oligomorphic, then it has only countably many irreducible representations and it is of type I.

Imaginaries

Definition

An imaginary element of a structure $\mathbf{X}$ is an equivalence class of a definable equivalence relation on a definable subset of $\mathbf{X}^n$.

Fact: the open subgroups of $\mathrm{Aut}(\mathbf{X})$ are exactly the stabilizers of imaginaries.

Definition

An ω-categorical structure $\mathbf{X}$ admits weak elimination of imaginaries if for every imaginary α, there exists a formula $\phi(\bar{x}, \bar{y})$ such that the set

$$\left\{\bar{c} \in \mathbf{X}^n : \alpha = \left\{\bar{x} \in \mathbf{X}^m : \phi(\bar{x}, \bar{c})\right\}\right\}$$

is finite and non-empty.

If $\mathbf{A} \subseteq \mathbf{X}$ is finite, let

$$G_{\mathbf{A}} = \{g \in G : g \cdot \mathbf{A} = \mathbf{A}\}.$$

There is a natural homomorphism $G_{\mathbf{A}} \to \mathrm{Aut}(\mathbf{A})$ given by restriction. It is surjective if $\mathbf{X}$ is homogeneous.

Theorem

Let $\mathbf{X}$ be an ω-categorical, homogeneous structure that admits weak elimination of imaginaries and $G = \mathrm{Aut}(\mathbf{X})$. Then the following is a complete list of the irreducible representations of G:

$$\{\mathrm{Ind}_{G_{\mathbf{A}}}^{G}(\sigma) : \mathbf{A} \subseteq \mathbf{X} \text{ is finite, algebraically closed and } \sigma \text{ is an irreducible representation of } \mathrm{Aut}(\mathbf{A})\}.$$

Moreover, this list is without repetitions (only one substructure appears of each isomorphism type).

The case of $\mathrm{Aut}(\mathbf{Q})$

- The automorphism groups of finite substructures of $\mathbf{Q}$ are trivial.
- Hence,
$$\{\mathrm{Aut}(\mathbf{Q}) \curvearrowright \ell^2(\mathbf{Q}^{[n]}) : n \in \mathbb{N}\}$$
is a complete list of the irreducible representations of G. ($\mathbf{Q}^{[n]}$ denotes the set of n-element subsets of $\mathbf{Q}$).
- $\mathrm{Aut}(\mathbf{Q})$ embeds as a dense subgroup of $\mathrm{Homeo}^+(\mathbb{R})$.
- Direct sums of the representations above clearly do not extend to representations of $\mathrm{Homeo}^+(\mathbb{R})$.
- Thus, we recover a partial version of a result of Megrelishvili: $\mathrm{Homeo}^+(\mathbb{R})$ has no non-trivial unitary representations.

Property (T)

Definition

Let G be a group, $Q \subseteq G$, $\epsilon > 0$. If π is a unitary representation of G, we say that a unit vector $\xi \in \mathcal{H}(\pi)$ is (Q, ϵ)-almost invariant if for all $x \in Q$, $\|\pi(x)\xi - \xi\| < \epsilon$. The topological group G is said to have Kazhdan's property (T) if there exist a compact $Q \subseteq G$ and $\epsilon > 0$ such that every representation π of G that has a (Q, ϵ)-almost invariant vector, actually has an invariant vector. G has the strong property (T) if Q can be chosen to be finite.

Property (T) (cont.)

Using the classification of the representations of the unitary group by Kirillov and Olshanski, Bekka showed that $U(\mathcal{H})$ has property (T) and exhibited an explicit Kazhdan pair.

A structure $\mathbf{X}$ has no algebraicity if the algebraic closure of every substructure $\mathbf{A}$ is $\mathbf{A}$ itself (for homogeneous structures, this is the same as SAP).

Theorem

Let $\mathbf{X}$ be an ω-categorical, relational, homogeneous structure with no algebraicity that admits weak elimination of imaginaries. Then $\mathrm{Aut}(\mathbf{X})$ has property (T).

In many concrete examples, one can find a finite Kazhdan set.

Question

Does every oligomorphic group have property (T)? More generally, does every Roelcke precompact Polish group have property (T)?

An example of Cherlin and Hrushovski

- Let E_n be a relation symbol of arity $2n$.
- Let $\mathcal{K}$ be the class of all finite structures $\mathbf{A}$, where each E_n is interpreted as an equivalence relation on n-element subsets of $\mathbf{A}$ with at most 2 equivalence classes.
- This is a Fraïssé class. Let $\mathbf{X}$ be the limit and G the automorphism group (which is oligomorphic).
- Then G surjects onto $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$.
- Bekka showed that if a compact group is amenable as a discrete group, then it does not have the strong property (T).
- Conclusion: G does not have the strong property (T).

Automatic continuity

Many oligomorphic groups G satisfy the following automatic continuity property:

Automatic continuity

Every homomorphism from G into a separable group is continuous.

The following groups are known to satisfy this property:

- S_∞, $\mathrm{GL}(\infty, \mathbf{F}_q)$, or, more generally, the automorphism group of any ω-stable, ω-categorical structure (Hodges–Hodkinson–Lascar–Shelah, Kechris–Rosendal);
- the automorphism group of the random graph (ibid.);
- $\mathrm{Aut}(\mathbf{Q})$, $\mathrm{Homeo}(2^{\mathbb{N}})$ (Rosendal–Solecki).

Corollary

For any G from the list above, the theorem holds for any representation of $(G, \mathrm{discrete})$ on a separable Hilbert space.