Embeddings of spaces of the form $C(K)$

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Corollary

$$\mathrm{univ}((\mathfrak{B}_\kappa, \leq_i)) \leq \mathrm{univ}((\mathfrak{B}_\kappa, \leq_m)) \leq \mathrm{univ}((\mathcal{A}_\kappa, \leq_e)).$$

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Now of course we want to iterate in length ω_2, every Boolean algebra of size $\aleph_1$ is in an intermediate model, bla *STOP* even if $\mathfrak{A}$ is in an intermediate universe, we cannot say that $C(\mathrm{St}(\mathfrak{A}))$ is, since new reals keep being added and hence so do new elements of $C(\mathrm{St}(\mathfrak{A}))$.

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Proof.

For (2), use the scenario attempted above, since no new reals are added. Iterability needs to be checked: not well met but good enough.
(1) For simplicity, prove that these is no universal. Suppose it is $C(\mathrm{St}(\mathfrak{A}))$, say in the ground model. Let p^* force $\dot{T}$ to be an isometric embedding from the first generic algebra to $C(\mathrm{St}(\mathfrak{A}))$. Let $p_i \geq p^*$ force that h_i is a simple rational function with rational coefficient with $||\dot{T}(\chi_{[a_i]}) - h_i|| < 1/2$ (so $h_i \in V$). Do several Δ-system arguments, take "clean" $i \neq j$. If $||h_i - h_j|| < 1$, show that there cannot be an extension of $p_i \cup p_j$ forcing a_i and a_j to be disjoint, contradiction. Similarly with $||h_i - h_j|| \geq 1$. $\square$

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Why isometries? Because they know how to model them with a model theoretic structure.

We would like a simple model theoretic class $(\mathcal{K}, \leq)$ such that $\mathrm{univ}((\mathcal{K}_\kappa, \leq)) \leq \mathrm{univ}((\mathfrak{B}_\kappa, \leq_i))$ for all relevant κ.

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which goes to 0.

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Hence the Conjecture isomorphism–isometry is proved in this case.