No mad families in Solovay's model

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Descriptive Set Theory in Paris, December 17, 2013

Mad families and Adrian Mathias, circa 1967

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2. *An a.d. family is maximal, or mad, if it is maximal among a.d. families under inclusion.*
3. *It is routine to show that AC implies that there are mad families.*
4. *An a.d. family $\mathcal{A}$ is analytic if it is analytic as a subset of $\mathcal{P}(\omega) \simeq 2^\omega$ (Cantor space).*

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Theorem (A.T., 2013)

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If $s \in \bigcup_{n\in\omega} \omega^n \times \omega^n$, we write s^0 and s^1 for the 1st and 2nd coordinate of s; i.e., $s = (s^0, s^1)$.

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- Continue defining the sets $F^{(s,t)}_{n,\alpha}$ by recursion on the rank of $\tilde{T}$, by removing the previously defined sets $F^{(s,t)}_{n,\beta}$ from $\mathcal{A}^2 \setminus \Delta$.

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Then every a.d. family admits a polarization.

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Claim

If $\mathcal{B} \subseteq \mathcal{A}$ is uncountable, then there is a diagonal sequence in $\mathcal{B}$ such that $B_0 = \mathrm{proj}_0(F^{(s,t)}_{n,j}) \cap \mathcal{B}$ for some (s,t),n and j.

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Repeat the argument on the set $\mathrm{proj}_1(F^{(s,t)}_{n,j}) \cap \mathcal{B}$ (i.e., using $\mathrm{proj}_1(F^{(s,t)}_{n,j}) \cap \mathcal{B}$ instead of $\mathcal{B}$, find new (s,t), n and j), continue building the diagonal sequence.

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The ordinal analysis will terminate at a countable ordinal α_0 because each step removes at least one triple $((s,t),n,j)$ from ever being considered again, and by the claim, as long as $\mathcal{A}_\alpha$ is uncountable, there *is* some triple $((s,t),n,j))$ which forms the first step of a diagonal sequence.

Construction

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Pick $m_i \geq k_0, n^{\alpha^*}_i, m_{i-1}$ which witness this fact.

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Then every a.d. family admits a polarization.

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There is a forcing extension in which this holds.

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Theorem (Feng)

This axiom holds in Solovay's model, it holds under AD, and it holds for X which are analytic subsets of Polish spaces.

Farah's variant of the Open Coloring Axiom

Ilijas Farah observed (in an unpublished note on his website) that a (standard) proof of the consistency of Todorcevic Open Coloring Axiom (OCA), which appears in a paper of Juris Steprans, actually proves something apparently stronger:

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- OCA_∞ *is the statement:* $\mathrm{OCA}_\infty(X)$ *holds for all separable metric spaces* X*.*

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Theorem (Essentially due to Feng)

AD implies OCA_∞*, and for any analytic subset* X *of a Polish metric spaces we have* $\mathrm{OCA}_\infty(X)$*.*

Polarizing an a.d. family using OCA_∞ and the PSP

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We let $K^0_n = [\mathcal{A}^2 \setminus \Delta]^2 \setminus K^1_n$.

The main claim

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Let $A_i = \mathrm{proj}_i[Y]$, which is still closed (compact). Fix trees $T^i \subseteq 2^{<\omega}$ such that $[T^i] = A_i$.

Main claim, II

Analogously with how we constructed a polarization of an a.d. analytic family, we can form a tree Γ on $(T^0)^{<\omega} \times (T^1)^{<\omega}$ of sequences (u, v) such that u is an increasing sequences in $(T^0)^{<\omega}$ and v in $(T^1)^{<\omega}$ with

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Then this is well-founded.

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Taking $(u, v) \in \Gamma$ terminal, and look at the set

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Then this is open; It can't be uncountable, because then there would be $(x, y), (z, w) \in U^{(u,v)}$ with $(x, y) \neq (z, w)$ and $\{(x, y), (z, w)\} \in K^0_{\mathrm{lh}(u^*)}$, contradicting that $U^{(u,v)}$ clearly is $K^1_{\mathrm{lh}(u^*}$-homogeneous.

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Taking $(u, v) \in \Gamma$ terminal, and look at the set

$$U^{(u,v)}\{(x, y) \in A_0 \times A_1 : x \supseteq u^*, y \supseteq v^*\}.$$

Then this is open; It can't be uncountable, because then there would be $(x, y), (z, w) \in U^{(u,v)}$ with $(x, y) \neq (z, w)$ and $\{(x, y), (z, w)\} \in K^0_{\mathrm{lh}(u^*)}$, contradicting that $U^{(u,v)}$ clearly is $K^1_{\mathrm{lh}(u^*}$-homogeneous.
By continuing in this way, removing countable open sets, we eventually must exhaust Y; but then Y would be countable, a contradiction.

□

Proof of the Lemma (B), continued

So, we can write

$$\mathcal{A}^2 \setminus \Delta = \bigcup F_n,$$

where each F_n is K^1_n homogeneous.

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Then

$$\mathcal{A}^2 \setminus \Delta = \bigcup_{n<\omega} \bigcup_{(s,t)\in I} F_n^{(s,t)}.$$

Proof of the Lemma (B), final claim.

Claim If $|F_n^{(s,t)}| > 1$, then for all $(x, y), (z, w) \in F_n^{(s,t)}$ we have $x \cap w \subseteq n$.

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Proof.

If $(x, y) \neq (z, w)$, it follows directly from K_n^1-homogeneity of F_n; if $(x, y) = (z, w)$, simply find some $(x', y') \in F_n^{(s,t)}$ such that $(x, y) \neq (x', y')$. Then $x \cap y \subseteq n$ by K_n^1-homogeneity. □

Proof of the Lemma (B), final claim.

Claim If $|F_n^{(s,t)}| > 1$, then for all $(x,y),(z,w) \in F_n^{(s,t)}$ we have $x \cap w \subseteq n$.

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This finishes the proof, as the claim shows that the decomposition

$$\mathcal{A}^2 \setminus \Delta = \bigcup_{n<\omega} \bigcup_{(s,t)\in I} F_n^{(s,t)}.$$

is a polarization of $\mathcal{A}$. □

Thank you
(time for drinks!)