Borel complexity and automorphism groups

David Kerr

Texas A&M University

December 2013

Automorphism groups

Automorphisms of a space or algebra A come naturally parametrized as elements of the group

$$\mathrm{Aut}(A)$$

where isomorphism is the relation of belonging to the same orbit under the action

$$\mathrm{Aut}(A) \curvearrowright \mathrm{Aut}(A)$$

by conjugation. In our cases of interest A carries a natural Polish topology under which this action is continuous.

Borel reducibility

Definition

Let E and F be equivalence relations on standard Borel spaces X and Y, respectively. We say that E is **Borel reducible** to F if there is a Borel map $\theta : X \to Y$ such that

$$x_1 E x_2 \Leftrightarrow \theta(x_1) F \theta(x_2)$$

for all $x_1, x_2 \in X$.

Smooth relations

Definition

An equivalence relation on a standard Borel space is **smooth** if it is Borel reducible to equality on $\mathbb{R}$.

This notion was introduced by Mackey, who conjectured:

Theorem (Glimm)

The classification of the irreducible unitary representations of a locally compact group is smooth iff the group is type I.

Note: for countable groups, type I $\Leftrightarrow$ Abelian-by-finite (Thoma).

Theorem (Ornstein)

Bernoulli shifts $\mathbb{Z} \curvearrowright (Y, \nu)^{\mathbb{Z}}$ are classified by their entropy.

Glimm-Effros dichotomy

Effros realized that Glimm's theorem could be derived as an application of a general theory that applies Baire category ideas to Polish transformation groups.

The Glimm-Effros dichotomy takes the following general form:

Theorem (Harrington-Kechris-Louveau)

A Borel equivalence relation on a Polish space is smooth iff the relation E_0 of tail equivalence on $\{0,1\}^{\mathbb{N}}$ cannot be Borel reduced to it.

Proper topological ergodicity

Key point relating Borel reducibility to topological dynamics:

- A Borel map $X \to Y$ between Polish spaces is continuous on a comeager set $X_0 \subseteq X$.

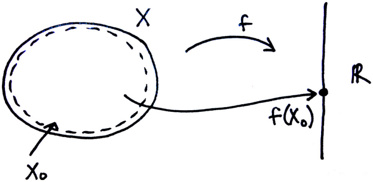
Proposition

Let $G \curvearrowright X$ be a Polish transformation group such that

(i) *there exists a dense orbit, and*

(ii) *every orbit is meager.*

Then the orbit equivalence relation is not smooth.



Classification by countable structures

Hjorth's idea:

- Require in addition that some orbit satisfy a *local* density condition, called **turbulence**, so as to obstruct classification by countable structures.

An example of classification of countable structures would be Borel reduction to the following parametrization of countable groups:

Give a countable group G, enumerate its elements as $g_1, g_2, \ldots$. The multiplication map $G \times G \to G$ determines the structure of G, and we can encode it as the map $\omega : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by

$$g_n g_m = g_{\omega(n,m)}.$$

Two such encodings represent isomorphic groups if they lie in the same orbit of S_∞ acting on $\mathbb{N}^{\mathbb{N} \times \mathbb{N}}$ by simultaneous permutation of the coordinates.

Example

Theorem (Halmos-von Neumann)

Discrete spectrum transformations of a probability space (X, μ) *are classified by the eigenvalues of the associated unitary operator on* $L^2(X, \mu)$ *counted with multiplicity.*

We thus have "reasonably concrete" classifications of measure-preserving dynamics at the two stochastic extremes:

- Bernoulli shifts (multiplicative structure)
- discrete spectrum transformations (additive/periodic structure, in the quantized unitary sense)

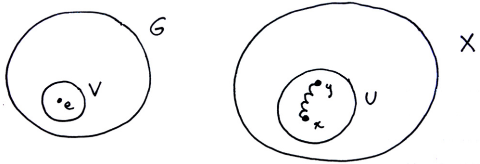
General measure-preserving transformations combine the phenomena at these extremes to produce a higher level of complexity, as an application of turbulence will demonstrate.

Turbulence

Let $G \curvearrowright X$ be a Polish transformation group. For an $x \in X$ and neighbourhoods $U \ni x$ and $V \ni e$ we define the local orbit $\mathcal{O}(x, U, V)$ as the set of all $y \in U$ such that there exist $g_1, \ldots, g_n \in V$ satisfying

(i) $g_n g_{n-1} \cdots g_1 x = y$,

(ii) $g_k g_{k-1} \cdots g_1 x \in U$ for all $k = 1, \ldots, n$.



An orbit is **turbulent** if, for any (equivalently, all) of its points, every local orbit is somewhere dense.

Turbulence

Definition

The action $G \curvearrowright X$ is **turbulent** if

(i) every orbit is dense and turbulent, and

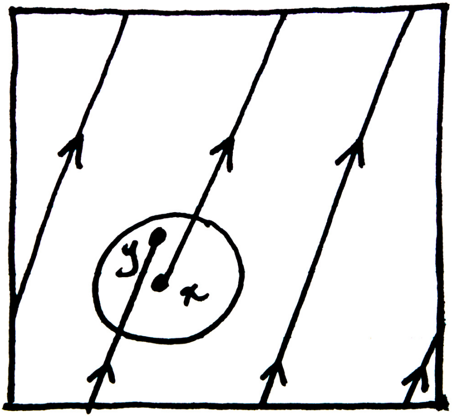
(ii) every orbit is meager.

It is **generically turbulent** if “every” is replaced by “some” in (i).

Theorem (Hjorth)

The orbit equivalence relation of a generically turbulent action is not classifiable by countable structures.

Examples



Kronecker flow on the torus

- Actions of locally compact groups are never turbulent.

Examples

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space and $\mathcal{U}(\mathcal{H})$ its unitary group.

Theorem (Hjorth)

For a non-type-I countable discrete group G, the conjugation action $\mathcal{U}(\mathcal{H}) \curvearrowright \mathsf{Irr}(G, \mathcal{H})$ on irreducible representations is generically turbulent.

Theorem (Kechris-Sofronidis)

The conjugation action $\mathcal{U}(\mathcal{H}) \curvearrowright \mathcal{U}(\mathcal{H})$ is generically turbulent.

Measurable dynamics

Theorem (Foreman-Weiss)

For an atomless standard probability space (X, μ), the conjugation action $\mathrm{Aut}(X, \mu) \curvearrowright \mathrm{Aut}(X, \mu)$ is generically turbulent.

If a conjugation action $G \curvearrowright G$ has good periodic approximation then one can use the following result of Rosendal to establish the meagerness of orbits:

- If for every infinite set $I \subseteq \mathbb{N}$ and neighbourhood V of e in G the set $\{g \in G : g^n \in V \text{ for some } n \in I\}$ is dense, then the action $G \curvearrowright G$ has meager orbits.

This applies to $\mathrm{Aut}(X, \mu)$ as a consequence of the Rokhlin lemma.

Rosendal property

Proposition

Let G be a Polish group with the Rosendal property such that the relation of conjugacy in G is generically $E^X_{S_\infty}$-ergodic for every Polish S_∞-space X. Let H be a Polish group and $\varphi\colon G \to H$ a continuous homomorphism such that $\varphi(G) \neq \{1_H\}$. Let F be the equivalence relation on $\varphi(G)$ such that xFy if there is an $h \in H$ for which $y = hxh^{-1}$. Then F is not classifiable by countable structures.

Dynamics on the Cantor set

Homeomorphisms of the Cantor set are classifiable by countable structures, as they correspond by Stone duality with the automorphisms of the Boolean algebra of clopen subsets.

Moreover, there is a generic homeomorphism of the Cantor set (Kechris-Rosendal).

Dynamics on the Cantor set

To construct a dense orbit in the homeomorphism group of the Cantor set X, use the product of a dense sequence $\alpha_1, \alpha_2, \ldots$ of automorphisms of $C(X)$:

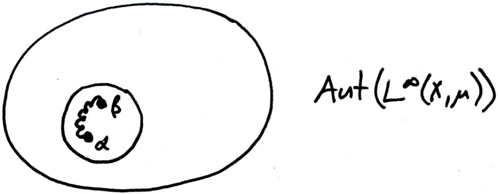
$$\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_m \otimes \ldots \otimes \alpha_n \otimes \ldots$$

$$C(X) \cong \underbrace{C(X) \otimes C(X) \otimes \ldots \otimes C(X)}_{} \otimes \ldots \otimes \underbrace{C(X)}_{} \otimes \ldots$$

proximity to a given automorphism α determined here

interchange these, where $\alpha_n \approx \alpha$

Turbulence in Aut(X, μ)



$$\alpha = \alpha_1 \otimes \dots \otimes \alpha_\ell \otimes \dots \otimes \alpha_m \otimes \dots \otimes \alpha_n \otimes \cdots$$

$$L^\infty(X,\mu) \cong \underbrace{\underbrace{L^\infty(X,\mu) \otimes \dots \otimes L^\infty(X,\mu)}_{\text{proximity to } \alpha \text{ determined here}} \otimes \dots \otimes L^\infty(X,\mu)}_{\text{proximity to } \beta \text{ determined here}} \otimes \dots \otimes \underbrace{L^\infty(X,\mu)} \otimes \dots$$

interchange these in small steps by cutting and swapping, with $\alpha_n \approx \beta$

Dynamics on the CAR algebra

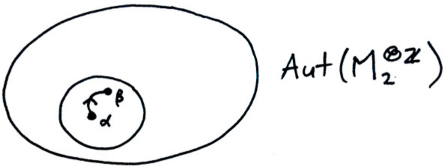
Consider now the CAR algebra

$$M_2^{\otimes \mathbb{N}} = M_2 \otimes M_2 \otimes M_2 \otimes \cdots$$

To show the existence of a turbulent point in $\mathrm{Aut}(M_2^{\otimes \mathbb{N}})$, we use the shift automorphism γ of $M_2^{\otimes \mathbb{Z}}$, which, in contrast to the case of the Cantor set,

- has dense conjugacy class
- has a malleability property

Dynamics on the CAR algebra



$$\alpha = \gamma\otimes\gamma\otimes\cdots\otimes\gamma\otimes\cdots\otimes\gamma\otimes\gamma\otimes\cdots\otimes\gamma\otimes\cdots$$

$$M_2^{\otimes\mathbb{Z}} \cong \underbrace{\underbrace{M_2^{\otimes\mathbb{Z}}\otimes M_2^{\otimes\mathbb{Z}}\otimes\cdots\otimes M_2^{\otimes\mathbb{Z}}}_{\text{proximity to }\alpha\text{ determined here}}\otimes\cdots\otimes M_2^{\otimes\mathbb{Z}}}_{\text{proximity to }\beta\text{ determined here}}\otimes\underbrace{M_2^{\otimes\mathbb{Z}}\otimes\cdots\otimes M_2^{\otimes\mathbb{Z}}}\otimes\cdots$$

continuously interchange these by a path that commutes with the shift

The Jiang-Su algebra

The Jiang-Su algebra $\mathcal{Z}$ is constructed as an inductive limit of algebras of the form

$$\left\{ f \in C([0,1], M_p \otimes M_q) : f(0) \in M_p \otimes 1 \text{ and } f(1) \in 1 \otimes M_q \right\}.$$

Theorem (K.-Lupini-Phillips-Winter)

The conjugation action $\mathrm{Aut}(\mathcal{Z}) \curvearrowright \mathrm{Aut}(\mathcal{Z})$ *is generically turbulent.*

The proof uses the following facts:

- the shift on $\mathcal{Z}^{\otimes \mathbb{Z}}$ has dense conjugacy class (Sato)
- the flip $a \otimes 1 \mapsto 1 \otimes a$ on $\mathcal{Z} \otimes \mathcal{Z}$ is asymptotically unitarily equivalent to the identity automorphism (Dadarlat-Winter)

Theorem

The conjugacy relation on automorphisms of a separable $\mathcal{Z}$-stable C^*-algebra is not classifiable by countable structures.

Theorem

Let M be a separable II_1 factor which is either McDuff or a free product of II_1 factors. Then the orbit equivalence relation of the conjugation action $\mathrm{Aut}(M) \curvearrowright \mathrm{Aut}(M)$ is not classifiable by countable structures.