Ranks on the Baire class ξ functions

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Joint work with Márton Elekes and Zoltán Vidnyánszky.

Definition and motivation

Let us denote the class of Baire-ξ functions on some Polish space X by $\mathcal{B}_\xi$. By a rank on $\mathcal{B}_\xi$ we mean a map $\mathcal{B}_\xi \to \omega_1$.

Kechris and Louveau investigated the case of Baire class 1 functions. Our aim is to generalize their results to Baire class ξ functions.

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Are there ranks on the class of Baire-ξ functions with nice properties?

Definition

A rank $\mathrm{rk} : \mathcal{B}_\xi \to \omega_1$ is **nice** if

- it is unbounded in ω_1
- "linear" i.e. $\mathrm{rk}(cf + g) \leq \max\{\mathrm{rk}(f), \mathrm{rk}(g)\} \cdot \omega$
- translation invariant (if the space is a Polish group)
- $\mathrm{rk}(\chi_F \cdot f) \leq \mathrm{rk}(f) \cdot \omega$, if F is a closed set

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Ranks on the Baire class 1 functions

It is well-known that for every $H \in \mathbf{\Delta}^0_2$ there exists a decreasing continuous sequence of closed sets $\{F_\eta\}_{\eta<\xi}$ such that

$$H = \bigcup_{\substack{\eta<\xi \\ \eta \text{ is even}}} F_\eta \setminus F_{\eta+1}.$$

Define $\mathrm{rk}_{\mathbf{\Delta}^0_2}(H)$ as the minimal length of such a sequence.

For $A, B \subseteq X$ disjoint G_δ sets define

$$\alpha(A, B) = \min\{\mathrm{rk}_{\mathbf{\Delta}^0_2}(H) : A \subseteq H, B \subseteq H^c, H \in \mathbf{\Delta}^0_2\}.$$

Finally,

Definition

The **separation rank** of a Baire class 1 function $f : X \to \mathbb{R}$ is

$$\alpha(f) = \sup_{\substack{p,q\in\mathbb{Q} \\ p<q}} \alpha\left((f \le p), (f \ge q)\right).$$

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Ranks on the Baire class 1 functions

The other two ranks are the **oscillation rank** ($\beta(f)$) and the **convergence rank** ($\gamma(f)$), but we will not need the exact form of the somewhat technical definitions, so we omit them.

However, it is interesting to note that these three ranks are defined according to the three classical characterizations of the Baire class 1 functions.

Baire-1 ranks

separation rank (α) $\longleftrightarrow$ level sets of the function
oscillation rank (β) $\longleftrightarrow$ continuity point restricted to every nonempty closed set
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The most obvious thing to do is to generalize one of the ranks on Baire-1 functions. We can only do that with the separation rank.

Replace $\mathbf{\Delta}^0_2$, closed and G_δ sets by $\mathbf{\Delta}^0_{\xi+1}$, $\mathbf{\Pi}^0_\xi$ and $\mathbf{\Pi}^0_{\xi+1}$ sets in the definition of the separation rank.

Let us denote this rank by

$$\alpha_\xi.$$

It turns out that this rank has the same properties as the α rank on the Baire class 1 functions, it is unbounded in ω_1, but is not linear, since

$$\sup\{\alpha_\xi(f+g) : f, g \in \mathcal{B}_\xi, \alpha_\xi(f) = \alpha_\xi(g) = 2\} = \omega_1.$$

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Ranks on Baire class ξ functions

The natural remedy is the following.

Definition

For $f \in \mathcal{B}_\xi$ let

$$\overline{\alpha_\xi}(f) = \min\{\max\{\alpha_\xi(f_1), \dots, \alpha_\xi(f_n)\} : n \in \omega, f_1, \dots, f_n \in \mathcal{B}_\xi, \\ f = f_1 + \dots + f_n\}.$$

This time we only have a partial result.

Theorem (Elekes-K-Vidnyánszky)

*$\overline{\alpha_\xi}$ is bounded on the **characteristic** Baire class ξ functions.*

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Limit ranks

Another way to construct $\mathcal{B}_\xi$ ranks is the following.

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For a rank ρ on $\mathcal{B}_1$ and $f \in \mathcal{B}_2$ let

$$\rho_2'(f) = \min_{\substack{f_n \to f \\ f_n \in \mathcal{B}_1}} \sup_n \rho(f_n).$$

However, to our great surprise, this does not work.

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Topology refinement

A bit less natural approach involves topology refinements.

For an $f \in \mathcal{B}_\xi$ let

$$T_{f,\xi} = \{\tau' : \tau' \supseteq \tau \text{ Polish}, f \in \mathcal{B}_1(\tau'), \tau' \subset \mathbf{\Sigma}^0_\xi(\tau)\}.$$

And now the definition of the rank is the following.

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For a rank ρ on $\mathcal{B}_1$ and $f \in \mathcal{B}_\xi$ let

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Topology refinement

This turns out to be good for our purposes.

Theorem (Elekes-K-Vidnyánszky)

The ranks β^*_ξ and γ^*_ξ are nice.

All ranks are equal on bounded functions

Kechris and Louveau proved that for functions on a compact metric space, the following holds.

Theorem (Kechris–Louveau)

If $1 \leq \lambda < \omega_1$ is an ordinal and f is a bounded Baire class 1 function then $\alpha(f) \leq \omega^\lambda$ ($\Leftrightarrow \beta(f) \leq \omega^\lambda \Leftrightarrow \gamma(f) \leq \omega^\lambda$) $\Leftrightarrow$

$$f = c + \sum_{\eta<\omega^\lambda}^{*} (-1)^\eta f_\eta,$$

where $c \in \mathbb{R}$, (f_η) is monotone decreasing sequence of non-negative usc functions and $f_\eta \to 0$ as $\eta \to \omega^\lambda$.

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This yields the following.

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An application to the solvability of systems of difference equations

Definition

$D : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ is said to be a **difference operator**, if there are real number a_i, b_i $(i = 1, \dots, n)$ such that for every $f \in \mathbb{R}^{\mathbb{R}}$

$$(Df)(x) = \sum_{i=1}^{n} a_i f(x + b_i).$$

Definition

A system of difference equations is

$$D_i(f) = g_i \quad (i \in I),$$

where I is an arbitrary set of indices D_i is a difference operator and g_i is a given function for every $i \in I$.

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Elekes and Laczkovich investigated the solvability cardinal of classes of functions.

Definition (Elekes-Laczkovich)

Let $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ be a class of real functions. The solvability cardinal of $\mathcal{F}$ is the minimal cardinal $\mathrm{sc}(\mathcal{F})$ with the property that if every subsystem of size less than $\mathrm{sc}(\mathcal{F})$ of a system of difference equations has a solution in $\mathcal{F}$, then the whole system has a solution in $\mathcal{F}$.

Corollary (Elekes-K-Vidnyánszky)

If $\xi \geq 2$, then $sc(\mathcal{B}_\xi) \geq \omega_2$.

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Elekes and Laczkovich investigated the solvability cardinal of classes of functions.

Definition (Elekes-Laczkovich)

Let $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ be a class of real functions. The solvability cardinal of $\mathcal{F}$ is the minimal cardinal $\mathrm{sc}(\mathcal{F})$ with the property that if every subsystem of size less than $\mathrm{sc}(\mathcal{F})$ of a system of difference equations has a solution in $\mathcal{F}$, then the whole system has a solution in $\mathcal{F}$.

Corollary (Elekes-K-Vidnyánszky)

If $\xi \geq 2$, then $sc(\mathcal{B}_\xi) \geq \omega_2$.