Lebesgue density and exceptional points

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1. all open neighbourhoods of x have measure $+\infty$
2. some open neighbourhood of x has measure 0
3. $\mathcal{D}_A^-(x) < \mathcal{D}_A^+(x)$, where

$$\mathcal{D}_A^-(x) = \liminf_{\varepsilon \to 0^+} \frac{\mu(A \cap \mathcal{B}_\varepsilon(x))}{\mu(\mathcal{B}_\varepsilon(x))}$$

$$\mathcal{D}_A^+(x) = \limsup_{\varepsilon \to 0^+} \frac{\mu(A \cap \mathcal{B}_\varepsilon(x))}{\mu(\mathcal{B}_\varepsilon(x))}$$

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be the oscillation of A at x; and

$$\Phi(A) = \{x \in X \mid \mathcal{D}_A(x) = 1\}$$

be the set of points with density 1 in A, called the set of *density points* of A or the *density set* of A.

Lebesgue density

If A, B are measure equivalent, in symbols $A \equiv B$, then $\mathcal{D}_A^\pm = \mathcal{D}_B^\pm$, so $\mathcal{D}_A = \mathcal{D}_B$ and $\Phi(A) = \Phi(B)$.

Thus $\mathcal{D}_\bullet^\pm, \mathcal{D}_\bullet, \Phi$ induce functions on $MALG(X)$, the measure algebra of X.

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- $Exc(A) = Shrp(A) \cup Blr(A)$ (the *exceptional* points of A)

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Proof.

For $\mathbf{\Delta}^0_3$ degrees, define operations on sets that take density sets to density sets and allow to climb the Wadge hierarchy.
For the degree of true $\mathbf{\Pi}^0_3$ sets, prove that in fact that for comeagre many $[A] \in MALG(2^{\mathbb{N}})$, the set $\Phi(A)$ is $\mathbf{\Pi}^0_3$-complete.

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Moreover, there is a compact $K \subseteq 2^{\mathbb{N}}$ such that $\Phi(K) = Int(K)$ and $Shrp(K)$ is $\mathbf{\Pi}^0_3$-complete.

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Moreover, there is a compact $K \subseteq 2^{\mathbb{N}}$ such that $\Phi(K) = Int(K)$ and $Shrp(K)$ is $\mathbf{\Pi}^0_3$-complete.
Moreover, for any fixed $r \in]0,1[$, K can be chosen so that $\{x \in 2^{\mathbb{N}} \mid \mathcal{D}_K(x) = r\}$ is $\mathbf{\Pi}^0_3$-complete.

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Question. Why we could show the existence of lots of examples of sets whose blurry points form a $\mathbf{\Sigma}^0_3$-complete set, by only few instances of sets whose sharp points form a $\mathbf{\Pi}^0_3$-complete set?

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- Because for the generic $[A] \in MALG(2^{\mathbb{N}})$, is quasi-dualistic, i.e. function $\mathcal{D}_A$ assumes only values $0, 1$ or undefined.

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(the two options are not mutually exclusive)

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Theorem

Let ν on $2^{\mathbb{N}}$ be non-singular, and (X, d, μ) be a Polish measure space such that

$$\exists Y \in B(X)\ \nu(2^{\mathbb{N}}) < \mu(Y) < +\infty$$

Then there is $H : 2^{\mathbb{N}} \to X$ continuous, injective, measure preserving.

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Using the previous embedding theorem, one can in fact show

Theorem

If (X, d, μ) is a Heine-Borel space such that every compact set has finite measure and μ is non-singular, then

- the set of elements of $MALG(X)$ that contain a closed (a compact) member is $\mathbf{\Pi}^0_3$-complete

Euclidean spaces vs Cantor space

Given a measurable subset A of a measure metric space X, say that

- A is *quasi-dualistic* if, whenever $\mathcal{D}_A(x)$ is defined, $\mathcal{D}_A(x) \in \{0, 1\}$
- A is *dualistic* if $\forall x \in X\ \mathcal{D}_A(x) \in \{0, 1\}$
- A is *solid* if $dom\mathcal{D}_A = X$
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The situation is different for Euclidean spaces.

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- There are spongy sets in $\mathbb{R}$ (they can even be taken open, or closed)

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Remark. The estimates used in the proof of this theorem, adapted to the two dimensional case, should give the existence of spongy sets in $\mathbb{R}^2$. It is not clear that the same calculations would lead to a spongy set in $\mathbb{R}^n$, for bigger n. (However, the obvious conjecture is that spongy sets in $\mathbb{R}^n$ do exist.)

Quasi-Euclidean spaces

A Polish measure space (X, d, μ) is *quasi-Euclidean* if:

- X is locally compact, connected
- μ is fully supported, locally finite
- for all $x \in X$, the function $[0, +\infty[\to [0, +\infty], r \mapsto \mu(\mathcal{B}_r(x))$ is continuous
- (X, d, μ) satisfies the density point property

Example

All $\mathbb{R}^n$ with Lebesgue measure and the p-norm $(1 \leq p \leq \infty)$ are quasi-Euclidean.

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$$\forall x, y \in X \ \forall r \in \mathbb{R}^+ \ (d'(y, x) = r \Rightarrow \mathcal{D}_{\mathcal{B}'_r(x)}(y) = \rho)$$

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Then

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2. $\{x \mid \mathcal{D}_A(x) = \rho\}$ is a dense G_δ subset of $Fr_\mu A$
3. $\rho = \frac{1}{2}$

Solid sets in Euclidean spaces

Corollary

In $\mathbb{R}^n$ endowed with the p-norm ($1 \leq p \leq \infty$) and Lebesgue measure, for any solid, non-trivial A, the generic point of $Fr_\mu A$ has density $\frac{1}{2}$ w.r.t. A.

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At least for $n = 1$ there is another known way to get this last corollary.

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- $\delta(X) = \inf\{\delta_A \mid [A] \in MALG \setminus \{[\emptyset], [X]\}\}$

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- (Kurka, 2012) $\delta(\mathbb{R}) \simeq 0.268486$ is the unique real root of $8x^3 + 8x^2 + x - 1$

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It seems safe to conjecture that the study of $\delta(\mathbb{R}^n)$ should lead to the same kind of results. However it is not clear (at least to us) how to get there.