The size of conjugacy classes of automorphism groups

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Fraïssé limits

Let $\mathcal{F} = \langle A, (R^{\mathcal{F}}_{i,n_i})_{i \in I}, (f^{\mathcal{F}}_{j,n_j})_{j \in J} \rangle$ be a Fraïssé limit of a class $\mathcal{K}$.

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Simple examples

- $\mathcal{K} = \{$finite sets$\} \leftrightarrow$ countably infinite set
- $\mathcal{K} = \{$finite linear orders$\} \leftrightarrow \langle \mathbb{Q}, < \rangle$
- $\mathcal{K} = \{$finite graphs$\} \leftrightarrow \mathcal{R}$

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We are interested in groups of type $Aut(\mathcal{F})$.

Automorphism groups

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Theorem. Let G be a Polish group. TFAE:

- $G \cong Aut(\mathcal{F})$ for a Fraïssé limit $\mathcal{F}$
- $G < S_\infty$ and G is closed
- $e \in G$ has a neighbourhood basis of open subgroups

Generic elements

Definition. If $f, g \in Aut(\mathcal{F})$ we say that f and g are *conjugate*, if there exists an $h \in Aut(\mathcal{F})$ such that $h^{-1}fh = g$.
Note: if $f, g \in Aut(\mathcal{F})$ then

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Definition. An automorphism is called *generic* if its conjugacy class is co-meagre.

Conjugacy classes

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Kechris, Rosendal: Characterisation of the existence of generic elements for a limit of a Fraïssé class $\mathcal{K}$, in terms of properties of the class $\mathcal{K}_p$, that is,

$$\{\langle \mathcal{A}, \Psi \rangle | \mathcal{A} \in \mathcal{K}, \Psi : \mathcal{B} \to \mathcal{C} \text{ isomorphism and } \mathcal{B}, \mathcal{C} < \mathcal{A}\}.$$

Measure

Definition. Let $(G,\cdot)$ be a Polish topological group and μ is a Borel measure on G. We say that λ is a *left Haar measure* on G if

- for every $g \in G$ and Borel set $B \subset G$
$$\lambda(B) = \lambda(gB),$$
- for every B Borel and V open set
$$\lambda(B) = \inf\{\lambda(U) : B \subset U,\ U \text{ open}\}$$
$$\lambda(V) = \sup\{\lambda(K) : K \subset V,\ K \text{ compact}\},$$
- for every K compact set $\lambda(K) < \infty$ and $\lambda(G) > 0$.

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Theorem. (Haar, Weil) Let $(G, \cdot)$ be a Polish topological group. There exists a left Haar measure on G if and only if G is locally compact.

Measure

Definition. (Christensen) Let $(G, \cdot)$ be a Polish group and $B \subset G$ Borel. We say that B is *Haar null* if there exists Borel probability measure μ on G such that for every $g, h \in G$ we have $\mu(gBh) = 0$.

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- (Christensen) Haar null sets coincide with measure zero sets w. r. t. left (and right) Haar measures in locally compact groups.
- (Solecki) In non-locally compact groups the ideal of Haar null sets is not ccc.
- If for every compact set K there exist g, h with $gKh \subset B$ then B is not Haar null.

Measure in S_∞

Theorem. (Dougherty, Mycielski) Almost all elements of S_∞ have infinitely many infinite orbits and only finitely many finite orbits.

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Therefore, almost all permutations included in the union of countably many conjugacy classes.
Theorem. (Dougherty, Mycielski) All of these classes are Haar positive.

Measure and Fraïssé limits

Definition. Let $\mathcal{F} = \langle A, \ldots \rangle$ be a structure, $a \in A$ and $X \subset A$. a *is algebraic over* X if $|\{f(a) : f \in Aut(\mathcal{F}), f|_X = id|_X\}| < \infty$.
Definition. The structure $\mathcal{F}$ *has no algebraicity* if for every $a \in A$ and finite $X \subset A \setminus \{a\}$ we have that a is not algebraic over X.

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Definition. The structure $\mathcal{F}$ *has no algebraicity* if for every $a \in A$ and finite $X \subset A \setminus \{a\}$ we have that a is not algebraic over X.
Theorem. Suppose that $\mathcal{F}$ is a Fraïssé limit with no algebraicity. Then almost all elements of $Aut(\mathcal{F})$ have finitely many orbits and infinitely many infinite ones.
Remark. For relational Fraïssé limits no algebraicity is equivalent to the *strong amalgamation property* of the corresponding Fräissé class.

Measure and $Aut(\mathbb{Q})$

$f \in Aut(\mathbb{Q})$ extends to a $\bar{f} \in Homeo^+(\mathbb{R})$.
Definition. A $+$ *orbital* ($-$ *orbital*) of f is a maximal interval $I \subset \mathbb{R}$ such that for every $x \in I$ we have $\bar{f}(x) > x$ ($\bar{f}(x) < x$).
Let $Fix(\bar{f}) = \{x \in \mathbb{R} : \bar{f}(x) = x\}$.

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Let $Fix(\bar{f}) = \{x \in \mathbb{R} : \bar{f}(x) = x\}$.
Proposition. $f, g \in Aut(\mathbb{Q})$ are conjugate if and only if there exists an order and rationality preserving isomorphism between $Fix(\bar{f})$ and $Fix(\bar{g})$ so that the corresponding orbitals have the same sign.

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Theorem. For almost every element of $Aut(\mathbb{Q})$

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Theorem. There are $\aleph_0$ many Haar positive conjugacy classes in $Homeo^+([0,1])$ and their union is almost everything.

Measure and graphs

Theorem. There are $\mathfrak{c}$ many Haar positive conjugacy classes in $Aut(\mathcal{R})$ and in Aut(random K_n free graph), Aut(random tournament).

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Theorem. There are $\aleph_0$ many Haar positive conjugacy classes and their union is co-Haar null in

- Aut(e. r. with $\aleph_0$ many $\aleph_0$ sized classes),
- Aut(e. r. with n many $\aleph_0$ sized classes),
- Aut(e. r. with $\aleph_0$ many n sized classes).

Questions

1. How many Haar positive conjugacy classes are there?
2. Is the union of the Haar null conjugacy classes is Haar null?

Examples

	$\bigcup$ of Haar null classes is Haar null		
	C	LC $\setminus$ C	NLC
0			
n			
$\aleph_0$			
$\mathfrak{c}$			
	$\bigcup$ of Haar null classes is not Haar null		
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	$\bigcup$ of Haar null classes is Haar null		
	C	LC \ C	NLC
0	–	–	–
n	$\mathbb{Z}_n$	HNN	???
$\aleph_0$	???	$\mathbb{Z}$	S_∞
$\mathfrak{c}$	–	–	$Aut(\mathbb{Q})$; $Aut(\mathcal{R})$?
	$\bigcup$ of Haar null classes is not Haar null		
	C	LC \ C	NLC
0	2^ω	$\mathbb{Z} \times 2^\omega$	$\mathbb{Z}^\omega$
n	$\mathbb{Z}_n \times (\mathbb{Z}_2 \ltimes \mathbb{Z}_3^\omega)$	HNN $\times (\mathbb{Z}_2 \ltimes \mathbb{Z}_3^\omega)$	$\mathbb{Z}_n \times (\mathbb{Z}_2 \ltimes \mathbb{Q}_d^\omega)$
$\aleph_0$	???	$\mathbb{Z} \times (\mathbb{Z}_2 \ltimes \mathbb{Z}_3^\omega)$	$S_\infty \times (\mathbb{Z}_2 \ltimes \mathbb{Z}_3^\omega)$
$\mathfrak{c}$	–	–	$Aut(\mathbb{Q}) \times (\mathbb{Z}_2 \ltimes \mathbb{Z}_3^\omega)$

Open problems

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Question. Does there exist a Polish group such that it consistently has κ many Haar positive conjugacy classes with $\aleph_0 < \kappa < \mathfrak{c}$?
Problem. Formulate necessary and sufficient model theoretic conditions which characterise the measure theoretic behaviour of the conjugacy classes!

Thank you for your attention!