ON THE ARC-WISE CONNECTION RELATION IN THE PLANE

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Abstract. We prove that the arc-wise connection relation in a $\boldsymbol{G}_\delta$ subset of the plane is Borel.

Let X be a separable metrizable space. By an *arc* we mean as usual a compact space homeomorphic to the unit interval $\mathbb{I} = [0, 1]$. We recall that the set $\mathcal{J}(X)$ of all arcs in X is a Borel subset of the space $\mathcal{K}(X)$ of all compact subsets of X in the Vietoris topology (see [2]). In particular if X is a Polish space then $\mathcal{K}(X)$ is Polish too, hence $\mathcal{J}(X)$ is an absolute Borel space, and it follows that the arc-wise connectedness equivalence relation E_X in X is analytic.

In [3] Kunen and Starbird constructed a compact connected set $K \subset \mathbb{R}^3$ with an (analytic) non Borel arc component, hence with E_K non Borel. This result can be strengthened in various directions, for example one can impose that all components of K are non Borel ([4]) or that all components of K are Borel but E_K is non Borel ([1]). However in all these constructions working in a three dimensional space is fundamental, and in ([3], Problem 1) Kunen and Starbird asked:

Question: *Is there a compact connected set $K \subset \mathbb{R}^2$ with a non Borel arc-wise component?*

In fact this question is actually equivalent to ask whether the equivalence relation E_K itself is Borel. Indeed Becker and Pol showed ([2], Proposition 5.1) that for a $\boldsymbol{G}_\delta$ subset X of the plane if all arc components are Borel then the equivalence relation E_X is Borel. They also pointed out that no example of a $\boldsymbol{G}_\delta$ subset of the plane with a non Borel relation E_X, is known; and the main goal of this note is to prove:

Theorem 1. *If X is any $\boldsymbol{G}_\delta$ subset of the plane then the equivalence relation E_X is Borel.*

Let us first fix some notation and recall a few basic facts.

Arcs: For an arc J we denote by $e(J)$ the set of its endpoints and we set $\mathring{J} = J \setminus e(J)$. The mapping $e : J \mapsto e(J)$ from $\mathcal{J}(X)$ to $\mathcal{K}(X)$ is Borel, even of the first Baire class. Also if we endow X with some Borel total ordering $<$ (via any Borel embedding of X in 2^ω) and set $e_0(J) = \min(e(J))$ and $e_1(J) = \max(e(J))$ then the mappings $e_i : \mathcal{J}(X) \to X$ are also Borel. We also recall that given any *path* in some space X, that is a continuous, non necessarily one-to-one, mapping $\varphi : [0, 1] \to X$, there exists an arc $J \subset \varphi([0, 1])$ such that $e(J) = \varphi(\{0, 1\})$.

Triods: By a *simple triod* in a space X we will mean a compact subset $T = J_0 \cup J_1 \cup J_2$ which is the union of three arcs J_i such that:

$$\forall\, i \neq j,\ J_i \cap J_j = \{c_T\}.$$

The arcs J_i, which are uniquely determined up to a permutation, are called the *branches* of T; and c_T is called the *center* of T.

Notice that this notion is more restrictive than Moore's initial notion of *triod* introduced in [6] where the branches J_i are only assumed to be irreducible continua. In particular since the set $\mathcal{J}(X)$ of all arcs is a Borel subset of $\mathcal{K}(X)$ and the $\cup$ and $\cap$ operations on $\mathcal{K}(X)$ are Borel, it follows from the unicity of the decomposition of a simple triod, that if X is Polish then the set

2020 AMS Subject Classification. 54H05, 03E15.
Key words and phrases. Borel sets, Arc-connectedness, plane topology .

$\mathcal{T}(X)$ of all simple triods in X is a Borel subset of $\mathcal{K}(X)$ and the mapping $\mathbf{c} : \mathcal{T} \to X$, which assigns to any simple triod T its center c_T, is Borel. We also recall the fundamental property of planar triods (see [6]):

Lemma. (Moore) *Any family of pairwise disjoint triods in the plane is countable.*

Arc-wise components: If C is an arc-wise component in some separable metrizable space X then:

– either $C = \{c\}$ is a singleton and we shall then say that c is a *quasi-isolated* point in X,

– or C admits a one-to-one continuous parametrization $\varphi : I \to C$ where I is a (closed, open, half-open) interval in $\mathbb{R}$ or the unit circle, and we shall then say that C is a *curve component*,

– or else C contains a *simple triod* and we shall then say that C is a *triodic component*.

In particular any non triodic arc-wise component is σ-compact. For more details we refer the reader to [2].

Proof of Theorem 1: By ([2], Proposition 5.1) we only need to prove that any triodic arc-wise component of X is Borel.

Since X is a Polish space we can fix a complete distance d compatible with the topology of X, and define $\delta : X \times X \to [0, \infty]$ by:

$$\delta(x, y) = \inf\{\mathrm{diam}(H) : \ H \text{ arc-wise connected s. t. } \{x, y\} \subset H \subset X\}$$

where $\inf \emptyset = \infty$. So if $x \neq y$ are in the same arc-wise component then

$$\delta(x, y) = \inf\{\mathrm{diam}(J) : \ J \in \mathcal{J}(X) \text{ s.t. } e(J) = \{x, y\}\} < \infty$$

and if not then $\delta(x, y) = \infty$. Moreover setting by convention $\alpha + \infty = \infty$ for any $\alpha \in [0, \infty]$ we have for all $x, y, z \in X$

$$\delta(x, z) \leq \delta(x, y) + \delta(y, z)$$

Hence δ induces a metric on each arc-wise component, and defines a topology τ on X; and since $\delta \geq d$ then τ is finer than the initial topology t induced by $\mathbb{R}^2$. But unless stated otherwise all topological notions are to be understood relatively to t. Notice that for an arc connected subset $H \subset X$ the d-diameter $\mathrm{diam}(H)$ and the δ-diameter are equal since for any $x, y \in H$:

$$d(x, y) \leq \delta(x, y) \leq \mathrm{diam}(H).$$

Lemma 2. *Each arc-wise component C of X is a clopen subset of (X, τ) and the metric space (C, δ) is complete.*

Proof. If $x_0 \in C$ then $C = \{x \in X : \ \delta(x, x_0) < \infty\}$ is an open subset of (X, τ), and the same holds for any other components. So all components are open, hence closed.

Suppose that (x_n) is a δ-Cauchy sequence. Since $d \leq \delta$ then (x_n) is a d-Cauchy sequence hence converges to some $x \in X$. We can also extract from (x_n) a subsequence (x'_n) satisfying for all $n > 0$, $\delta(x'_n, x'_{n+1}) < 2^{-n}$ and we can fix then an arc J_n with $\mathrm{diam}(J_n) < 2^{-n}$ and a continuous function $\varphi_n : [\frac{1}{n+1}, \frac{1}{n}] \to J_n$ such that $\varphi_n(\{\frac{1}{n+1}, \frac{1}{n}\}) = \{x'_{n+1}, x'_n\} = e(J_n)$. Then for all $n > 0$, $\hat{\varphi}_n = \bigcup_{m \geq n} \varphi_m$ defines a continuous function from $]0, \frac{1}{n}]$ onto $\bigcup_{m \geq n} J_m$; and since $\lim_n J_n = \{x\}$ then setting $\hat{\varphi}_n(0) = x$ we get a continuous path from $[0, \frac{1}{n}]$ onto $\{x\} \cup \bigcup_{m \geq n} J_m$ such that $\left(\hat{\varphi}_n(0), \hat{\varphi}_n(\frac{1}{n})\right) = (x, x_n)$, hence there exists an arc $\hat{J}_n \subset \{x\} \cup \bigcup_{m \geq n} J_m$ with $e(\hat{J}_n) = \{x'_n, x\}$. Then $\mathrm{diam}(\hat{J}_n) \leq \sum_{m \geq n} \mathrm{diam}(J_m) < 2^{-n+1}$ so $\delta(x'_n, x) < 2^{-n+1}$ and x is a δ-cluster value of (x_n), hence (x_n) δ-converges to x, and since C is δ-closed then $x \in C$. $\square$

Lemma 3. *$\mathcal{J}(X,\tau)=\mathcal{J}(X)$ and for any $J\in\mathcal{J}(X), t_{|J}=\tau_{|J}$.*

Proof. If $J\in\mathcal{J}(X,\tau)$ and $\varphi:[0,1]\to X$ is a τ-continuous parametrization of J then φ is t-continuous; hence $\mathcal{J}(X,\tau)\subset\mathcal{J}(X)$.

Conversely if $J\in\mathcal{J}(X)$ then J is t-compact hence τ-closed, so δ-complete. Moreover if $\varphi:[0,1]\to X$ is a t-continuous parametrization of J, then by the uniform continuity of φ, for any $\varepsilon>0$ we can find a covering of J by a finite family $(J_i)_{i\leq k}$ of arcs of d-diameter, hence of δ-diameter, $<\varepsilon$. So J is δ-complete and δ-precompact, hence τ-compact. It follows that $J\in\mathcal{J}(X,\tau)$ and $\tau_{|J}=t_{|J}$. □

For any set $A\subset X$ and any $\varepsilon>0$ let $\mathcal{T}_\varepsilon(A)=\{T\in\mathcal{T}(A):\ \operatorname{diam}(T)<\varepsilon\}$. We recall that for an arc-wise connected set, in particular for a simple triod, the d-diameter and the δ-diameter are equal. If $\mathcal{S}\subset\mathcal{T}(A)$ we shall say that:

– $\mathcal{S}$ is an *ε-total subset* of $\mathcal{T}(A)$ if for all $T\in\mathcal{T}_\varepsilon(A)$ there exists $S\in\mathcal{S}\cap\mathcal{T}_\varepsilon(A)$ s. t. $S\cap T\neq\emptyset$

– $\mathcal{S}$ is a *total subset* of $\mathcal{T}(A)$ if $\mathcal{S}$ is ε-total for any $\varepsilon>0$.

Lemma 4. *$\mathcal{T}(A)$ contains a countable total subset.*

Proof. For any $\varepsilon>0$ fix a maximal family $\mathcal{S}_\varepsilon$ of pairwise disjoint simple triods in $\mathcal{T}_\varepsilon(A)$. Then each $\mathcal{S}_\varepsilon$ is an ε-total subset of $\mathcal{T}(A)$, so by Moore's Lemma $\mathcal{S}_\varepsilon$ is countable. Hence $\mathcal{S}=\bigcup_n\mathcal{S}_{\frac{1}{n}}$ is a total countable subset of $\mathcal{T}(A)$. □

Lemma 5. *If $\mathcal{S}$ is a total subset of $\mathcal{T}(A)$ then the set $\mathbf{c}(\mathcal{S})$ is τ-dense in $\mathbf{c}(\mathcal{T}(A))$.*

Proof. Let U be a δ-open ball of center $x_0\in A$ and radius r, such that $U\cap\mathbf{c}(\mathcal{T}(A))\neq\emptyset$. Fix $T\in\mathcal{T}(A)$ and $n>0$ such that $\delta(x_0,c_T)+\dfrac{2}{n}<r$. Replacing T by a subtriod (necessarily with the same center) we may suppose that $\operatorname{diam}T<\dfrac{1}{n}$. Since $\mathcal{S}$ is a total subset of $\mathcal{T}(A)$ there exists some $S\in\mathcal{S}$ with $\operatorname{diam}(S)<\dfrac{1}{n}$ such that $S\cap T\neq\emptyset$. Then for any $x\in S\cap T$ we have $\delta(x_0,c_S)\leq\delta(x_0,c_T)+\delta(c_T,x)+\delta(x,c_S)<r$ hence $U\cap\mathbf{c}(\mathcal{S})\neq\emptyset$. □

We now fix a triodic arc-wise component C in X. Let $R=\mathbf{c}(\mathcal{T}(C))$ be the set of all centers of simple triods in C, and $S=\overline{R}^\tau$ be the τ-closure of R. Since C is τ-closed in X then $R\subset S\subset C$, and since C is an arc-wise component then

$$\mathcal{T}(C)=\{T\in\mathcal{T}(X):\ T\subset C\}=\{T\in\mathcal{T}(X):\ T\cap C\neq\emptyset\}$$

We recall that C is analytic, so since $\mathcal{T}(X)$ is a Borel subset of $\mathcal{K}(X)$, it follows from the previous equality that $\mathcal{T}(C)$ is analytic, and since the mapping $\mathbf{c}$ is Borel then $R=\mathbf{c}(\mathcal{T}(C))$ is an analytic subset of X, and one can easily derive from this that S is an analytic subset of X too. However we have the following:

Lemma 6. *(S,τ) is a Polish space, and S is a Borel subset of X.*

Proof. (S,δ) is a complete metric space and it follows from Lemmas 4 and 5 that the space (R,δ) is separable, hence (S,τ) is a Polish space. So (S,t) is an injective continuous image of (S,τ), hence (S,t) is an absolute Borel space, in particular S is a Borel subset of X. □

Lemma 7. *If J,J' are two arcs such that $J\not\subset J'$, $J\cap J'\neq\emptyset$ and $J\cap J'\not\subset e(J)$ then $\overset{\circ}{J}\cap J'$ contains the center of some simple triod.*

Proof. By hypothesis $F = J \cap J'$ is a proper closed subset of J which meets $\overset{\circ}{J}$. So there exists some $c \in \overset{\circ}{J}$ which is the endpoint of some connected component of $J \setminus F$. Then c is the center of a simple triod of the form $J \cup J''$ with $J'' \subset J'$ and $c \in e(J'')$. □

Lemma 8. *Let* $\mathcal{B}_S = \big\{(x, J) \in X \times \mathcal{J}(X);\ e(J) = \{x, y\} \text{ and } J \cap S = \{y\}\big\}$. *Then*
a) $\mathcal{B}_S$ *is Borel.*
b) $\forall\, x \in X$, *card* $\big(\mathcal{B}_S(x)\big) \leq 2$.
In particular the projection $\pi(\mathcal{B}_S)$ *of* $\mathcal{B}_S$ *on* X *is Borel.*

Proof. a) Observe that $\mathcal{B}_S$ is the complement in $X \times \mathcal{J}(X)$ of the projection of the set:

$$\mathcal{A} = \Big\{(x, J, z) \in X \times \mathcal{J}(X) \times X :\ \exists\, i \in \{0, 1\}\ ,\ e_i(J) = x\ ,\ e_{1-i}(J) \in S \text{ and } z \in J \cap S \setminus e(J)\Big\}$$

which is clearly Borel and all its sections $\mathcal{A}(x, J) = \overset{\circ}{J} \cap S$ are σ-compact, hence by the classical Arsenin-Kunugui Theorem $\mathcal{B}_S$ is Borel.

b) Suppose that $(x, J) \in \mathcal{B}_S$: since $J \cap S \neq \emptyset$ and $S \subset C$ then $J \subset C$; and since $\overset{\circ}{J} \cap S = \emptyset$ then $\overset{\circ}{J}$ does not contain the center of any simple triod in C. So if (x, J) and $(x, J') \in \mathcal{B}_S$ with $J \neq J'$ then by Lemma 7 we necessarily have $J \cap J' = \{x\}$, or else $J \cap J' = e(J) = e(J') = \{x, y\}$. It follows that $\mathcal{B}_S(x) = \{J, J'\}$ for otherwise x would be the center of a simple triod contained in C, which is impossible since $x \notin S$ and a fortiori $x \notin R$.

The last part of the conclusion follows then from part b) and again the Arsenin-Kunugui Theorem. □

Lemma 9. $C = S \cup \pi(\mathcal{B}_S)$.

Proof. If $x \in S \cup \pi(\mathcal{B}_S)$ then $x \in S$ or x can be joined to an element of S by an arc, so since $S \subset C$ then $x \in C$.

Conversely fix any element $x \in C$; if $x \notin S$ there exists an arc I with $e(I) = \{x, x_0\}$, hence $x \in I \setminus S$ and $x_0 \in I \cap S$. Since $S \cap I$ is a closed subset of I, there exists a sub-arc J of I such that $e(J) = \{x, y\}$ with $y \in S$, and $(J \setminus \{y\}) \cap S = \emptyset$, hence $\overset{\circ}{J} \cap S = \emptyset$; so $x \in \pi(\mathcal{B}_S)$. □

It follows from Lemmas 6, 8, 9 that any triodic arc-wise component of X is Borel, which finishes the proof of Theorem 1. □

Note that given any separable metrizable space X, if we set:

$$X = X^{(0)} \cup X^{(1)} \cup X^{(2)} \quad \text{where} \quad \begin{cases} X^{(0)} \text{ is the set of all quasi-isolated points,} \\ X^{(1)} \text{ is the union of all curve arc-wise components,} \\ X^{(2)} \text{ is the union of all triodic arc-wise components.} \end{cases}$$

then for any $x \in X$ we have:

$$\begin{aligned} x \in X^{(0)} &\iff \forall\, y \in X,\ x = y \text{ or } (x, y) \notin E_X \\ x \in X^{(2)} &\iff \exists\, T \in \mathcal{T}(X),\ x \in T \end{aligned}$$

So if X is Polish, since the space $\mathcal{T}(X)$ is Borel and the equivalence relation E_X is analytic, then $X^{(0)}$ is coanalytic and $X^{(2)}$ is analytic, and in this general setting the set $X^{(1)}$ appears as the difference of two analytic sets.

Proposition 10. *Suppose that* X *is Polish. If* $X^{(2)}$ *is Borel then* $X^{(0)}$ *and* $X^{(1)}$ *are Borel.*

Proof. By assumption $Y = X \setminus X^{(2)} = X^{(0)} \cup X^{(1)}$ is Borel hence the set $\mathcal{H} = \{(x, J) \in Y \times \mathcal{J}(X) : x \in J\}$ is Borel too, and $X^{(1)}$ is the projection of $\mathcal{H}$ on the first factor. Moreover if $x \in X^{(1)}$ then $E(x) = \bigcup_{n\in\omega} \uparrow H_n$ is the increasing union of a countable family of arcs H_n, hence the section $\mathcal{H}(x) = \bigcup_{n\in\omega} \uparrow \mathcal{J}(H_n)$ is σ-compact. It follows then from Arsenin-Kunugui Theorem that the set $X^{(1)}$ is Borel, so $X^{(0)} = X \setminus (X^{(1)} \cup X^{(2)})$ is Borel too. □

It follows then from Theorem 1:

Corollary 11. *If X is a $\boldsymbol{G}_\delta$ subset of the plane then the three sets $X^{(0)}$, $X^{(1)}$, $X^{(2)}$ are Borel.*

Remarks 12. (1) The proof of Theorem 1 does not give any bound on the Borel rank of E_X, which is most likely unbounded. However the situation is totally unclear concerning triodic arc-wise components, even in the compact case (we recall that non-triodic arc-wise components are all σ-compact).

(2) One can derive from Williams' work in [7] the construction of planar continua with an arc-wise component which is an $\boldsymbol{F}_{\sigma\delta} \setminus \boldsymbol{G}_{\delta\sigma}$ set. The reader can also find in [5] an explicit geometrical example, by Malicki, of such a continuum.

(3) The same boundedness questions can be considered also for the Borel decomposition $X = X^{(0)} \cup X^{(1)} \cup X^{(2)}$ of a planar $\boldsymbol{G}_\delta$ (or compact) set X. Note that by Moore's Lemma a uniform bound for the Borel rank of arc-wise components provides a uniform bound for rank of the set $X^{(2)}$, but has no impact on the rank of the sets $X^{(0)}$ and $X^{(1)}$.

(4) One can also derive from [7] the construction of a planar continuum K such that $K^{(0)}$ (which by Corollary 11 is Borel) is not an $\boldsymbol{F}_{\sigma\delta}$ set.

(5) Fix an enumeration $(q_n)_{n\in\omega}$ for the set $\mathbb{Q}$ of all rational numbers in $[0,1]$, let $\mathbb{P} = [0,1] \setminus \mathbb{Q}$ and consider the Sierpinski function $f : \mathbb{P} \to [-1,1]$ defined by $f(x) = \sum_{n\geq 0} 2^{-n-1} \sin \dfrac{1}{q_n - x}$. Then f is clearly continuous and the closure in $[0,1] \times [-1,1]$ of the graph of f is a compact set K with no triodic arc-wise component and $K^{(0)} = \{(\alpha, f(\alpha));\ \alpha \in \mathbb{P}\} \approx \mathbb{P}$ is a $\boldsymbol{G}_\delta \setminus \boldsymbol{K}_\sigma$ set, hence $K^{(1)} = K \setminus K^{(0)}$ is a $\boldsymbol{K}_\sigma \setminus \boldsymbol{G}_\delta$ set.

(6) Since the submission of the present paper we were able to improve Theorem 1 by proving that under the same hypothesis there exists a Borel function $\Phi : E_X \to \mathcal{J}$ wich assigns to any pair (x, y) of distinct elements in E_X, an arc J with endpoints $\{x, y\}$

Acknowledgment We would like to thank Roman Pol for pointing to us Williams' paper [7] and Malicki Thesis [5] mentioned above.

References

1. H. Becker, *The number of path-components of a compact space*, in: J.A. Makowsky, E.V. Ravve (Eds) Logic Colloquium' 95 (1995), 1–15.
2. H. Becker and R. Pol, *Note on path-components in complete spaces*, Top. and Appl. **114** (2001), 107–114.
3. K. Kunen and M.Starbird, *Arc components in metric continua*, Top. Appl. **14** (1982)167–170.
4. A. Le Donne, *Arc components in metric continua*, Top. Appl. **22** (1986) 83–84.
5. M. Malicki *Master Thesis* University of Warsaw (1999).
6. R.L. Moore, *Concerning triods in the plane and the junction points of plane continua*, Proc. Nat. Acad. Sci. **14** (1928) 85–88.
7. J. Williams, *Regarding arc-wise accessibility in the plane*, Fund.Math. **85** (1974), 134-136,