TOPOLOGICAL GAMES AND OPTIMIZATION PROBLEMS

GABRIEL DEBS AND JEAN SAINT RAYMOND

Let X be a completely regular space, and $C_b(X)$ the space of all bounded continuous real valued functions on X equipped with the metric associated to the uniform norm. For $f \in C_b(X)$ and $\lambda \in \mathbb{R}$ we use the following standard notations: $\inf(f) = \inf_{x \in X} f(x)$ and $\{f < \lambda\} = \{x \in X : f(x) < \lambda\}$.

In this paper we investigate the following subsets of $C_b(X)$:

$\Gamma(X) = \{f \in C_b(X)$ *which attain their minimum at some point in* $X\}$;
$\Gamma_1(X) = \{f \in C_b(X)$ *which attain their minimum at a unique point in* $X\}$;
$\Gamma_0(X) = \{f \in C_b(X)$ *which attain their minimum at a unique point for which the sets* $\{f < \inf(f) + \delta\}$ *for* $\delta > 0$, *form a basic system of neighbourhoods*$\}$.

These families of functions were considered, with different notations, by Čoban, Kenderov, Revalski, Stegall in ([**2**], [**4**], [**6**]) where the authors investigate conditions under which one of these families form a residual subset of $C_b(X)$. These problems appear in the study of the differentiability of convex functions on Banach spaces.

In this work we show the existence of natural transformations which send any open subset W of $\Gamma(X)$, $\Gamma_1(X)$, $\Gamma_0(X)$ to an open subset V of X, $G_1(X)$, $G_0(X)$, where:

> $G_1(X)$ *denotes the set of all* $x \in X$ *such that* $\{x\}$ *is a* G_δ *subset of* X;
> $G_0(X)$ *denotes the set of all* $x \in X$ *which admit a countable basis of neighbourhoods.*

The fundamental fact is that each of these transformations $\Phi: W \mapsto V$ is monotone $(W' \subset W \Rightarrow \Phi(W') \subset \Phi(W))$ and so transforms any chain $W_0 \supset W_1 \supset W_1 \ldots \supset W_n$ of open sets in $\Gamma(X)$, $\Gamma_1(X)$, $\Gamma_0(X)$ into a chain of open sets in X, $G_1(X)$, $G_0(X)$. On the other hand, although Φ has no inverse $V \mapsto W$, any chain $V_0 \supset V_1 \supset V_1 \ldots \supset V_n$ of open sets in X, $G_1(X)$, $G_0(X)$ can be pulled back to a chain $W_0 \supset W_1 \supset W_1 \ldots \supset W_n$ of open sets in $\Gamma(X)$, $\Gamma_1(X)$, $\Gamma_0(X)$ satisfying $\Phi(W_i) \subset V_i$. Moreover we have the following compatibility property: The pull back of the chain $W_0 \supset W_1 \supset W_1 \ldots \supset W_n \supset W_{n+1}$ can be obtained by extending the pull back of the chain $W_0 \supset W_1 \supset W_1 \ldots \supset W_n$.

A simple example of such a transformation appears implicitly in the arguments of Kenderov and Revalski in [**4**]. This example will be detailed in Section 3. In Sections 5 and 6 we describe several transformations with very similar, but different, properties. It happens that these properties can be related very clearly in the language of games. And as we shall show, all these transformations can be viewed as different instances of the same notion of "game morphism", but relatively to different games. This notion is introduced in Section 4. All the topological games considered in this paper are classical or new variants of the Banach–Mazur game. These games are discussed in Section 2 and 6. The applications to the families $\Gamma(X)$, $\Gamma_1(X)$, $\Gamma_0(X)$ are given in Section 7.

This approach, which gives several new results, can also be used to obtain very simple and unified proofs for some other known results. Thus all the following statements can be proved by the same scheme of proof.

(1) (Kenderov–Revalski) $\Gamma(X)$ is residual in $C_b(X)$, if, and only if, X is α-favourable (in the Banach–Mazur game). (Theorem 7.1)

(2) $\Gamma_1(X)$ is residual in $C_b(X)$, if, and only if, X is α-favourable and contains a residual subset on which the topology is finer than the topology of some complete metric. (Theorem 7.2 and Theorem 2.6)

(3) (Čoban–Kenderov–Revalski) $\Gamma_0(X)$ is residual in $C_b(X)$, if, and only if, X is α-favourable and contains a residual subset on which the topology is defined by some complete metric. (Theorem 7.3 and Theorem 2.8)

(4) $\Gamma(X)$ is a Baire space, if, and only if, X is a Baire space. (Theorem 7.1)

(5) For a metrizable space X: Any closed subspace of $\Gamma_0(X)$ is a Baire space, if, and only if, any closed subspace of X is a Baire space.

(6) $\Gamma(X)$ is a $\mathbf{G}_\delta$ subset of $C_b(X)$, if, and only if, $\Gamma(X)=C_b(X)$. (Theorem 7.6)

(7) $\Gamma_0(X)$ is a $\mathbf{G}_\delta$ subset of $C_b(X)$, if, and only if, there exists a continuous and open mapping from some complete metric space onto the set of all points in X which have a countable basis of neighbourhoods. (Theorem 7.4)

(8) For a metrizable space X.

(a) $\Gamma_0(X)$ is a $\mathbf{G}_\delta$ subset of $C_b(X)$, if, and only if, X is Čech complete. (Theorem 7.5)

(b) $\Gamma_1(X)$ is a $\mathbf{G}_\delta$ subset of $C_b(X)$, if, and only if, X is compact. (Theorem 7.7)

§1. *Games.* All games considered here are with two players whom we shall denote by α and β. We shall also denote by γ one or the other player; then $\bar{\gamma}$ will denote the opponent player.

It will be convenient to define a game G by: a *domain* $D=\mathrm{Dom}\,(G)$, a *rule* $R\subset\bigcup_{n\in\omega}D^n$, and a *win condition* (C). A *play* in the game G is a sequence $(v_k)_{k\in\omega}$ in D such that all its finite subsequences $(v_k)_{k<n}$ are in R and constructed by induction by both players in the following way: Player β starts the play by choosing v_0 and then the players choose alternatively v_k. Notice that for an even index k Player β chooses v_k, and for an odd index k Player α chooses v_k. Any play is won by either α or β, and this is determined by the win condition (C). In the sequel we shall very often define (C) indirectly by defining all plays won by one of the players.

It is important to distinguish in the sequel the notion of "*strategy*" from the notion of "*winning strategy*". These notions that we shall not define formally, are to be understood in the sense of games with perfect information. In other words, if σ is a strategy for Player γ then the response given by σ at the n^{th} move (with n even if $\gamma=\beta$, and n odd if $\gamma=\alpha$) can depend on the sequence $(v_k)_{k<n}$ of all previous moves. Notice that the notion of "strategy" depends only on the rule R, whereas the notion of "winning strategy" depends also on the win condition (C). In particular, different games with same domain and same rule, have the same strategies. We recall that a game G is said to be γ-*favourable* if Player γ has a winning strategy in G. We shall say that two games G and G' are *equivalent* and we shall write $G\approx G'$ if for any $\gamma\in\{\alpha,\beta\}$ we have:

$$(G \text{ is } \gamma\text{-favourable}) \quad \Leftrightarrow \quad (G' \text{ is } \gamma\text{-favourable}).$$

In this work we shall use only basic concepts and facts about topological games, that we shall recall throughout the text. The reader who is interested in more information is referred to [7].

§2. *Some topological games.* By a topological game we mean a game the domain of which is a subset of the set $\mathcal{T}(X)$ of all open subsets of some topological space X.

§2.1. *The games $\mathcal{J}(X)$, $\mathcal{J}_1(X)$, $\mathcal{J}_0(X)$.* Given any non empty topological space X, we consider three topological games $\mathcal{J}(X)$, $\mathcal{J}_1(X)$, $\mathcal{J}_0(X)$ with same domain and same rule. The domain is the set $\mathcal{T}_+(X)$ of all non empty open subsets of X. The rule is the following: at the first move Player β chooses $V_0\in\mathcal{T}_+(X)$; at move $n>0$ the concerned player chooses $V_n\in\mathcal{T}_+(X)$ such that $V_n\subset V_{n-1}$. The win condition for each of these games is defined as follows: if $(V_n)_{n\in\omega}$ is a play in these games then:

(1) Player α wins the play in $\mathcal{J}(X)$, if, and only if, $\bigcap_{n\in\omega} V_n$ is non empty;
(2) Player α wins the play in $\mathcal{J}_1(X)$, if, and only if, $\bigcap_{n\in\omega} V_n$ is a singleton;
(3) Player α wins the play in $\mathcal{J}_0(X)$, if, and only if, $\bigcap_{n\in\omega} V_n$ is a singleton for which the family $(V_n)_{n\in\omega}$ form a basic system of neighbourhoods.

The game $\mathcal{J}(X)$ is the well known Banach–Mazur game, also called by some authors the Choquet game. The game $\mathcal{J}_1(X)$ is considered by Kenderov–Revalski in [4].

In fact eliminating the case $X=\varnothing$ is artificial and one can give natural definitions for the games $\mathcal{J}(X)$, $\mathcal{J}_1(X)$, $\mathcal{J}_0(X)$ which cover this case. With these extended definitions all three games happen to be α-favourable for $X=\varnothing$. But this would oblige us to use a non symmetric rule, forcing only Player β to play non empty sets. Unfortunately this minor modification in the definitions would necessitate very heavy modifications for the rest of this work. For these reasons we shall never consider in the sequel empty spaces, even when games are not involved.

§2.2. *The game $\mathcal{J}(X, M)$.* Let (M, d) be a complete metric space. We consider on the set $\mathcal{T}(M)$ of all open subsets of M the relation $\sqsubseteq$ defined by:

$$(V\sqsubseteq V') \quad \Leftrightarrow \quad (\bar{V}\subset V' \quad \text{and} \quad \operatorname{diam}(V)\leqslant \tfrac{1}{2}\operatorname{diam}(V')).$$

For any non emptyy subset X of M we shall define a game denoted by $\mathscr{J}(X, M)$. Its domain is the set $\mathscr{T}_+(X, M)$ of all open subsets of M which meet X. The rule is the following: at the first move Player β chooses $V_0 \in \mathscr{T}_+(X, M)$; at move $n > 0$ the concerned player chooses $V_n \in \mathscr{T}_+(X, M)$ such that $V_n \sqsubseteq V_{n-1}$. If $(V_n)_{n\in\omega}$ is a play in $\mathscr{J}(X, M)$ then $\bigcap_{n\in\omega} V_n$ is a singleton $\{x\}$; Player α wins the play, if, and only if, $x \in X$. It is clear that:

$$\mathscr{J}(X, M) \approx \mathscr{J}(X) \approx \mathscr{J}_1(X) \approx \mathscr{J}_0(X).$$

THEOREM 2.3. *The game $\mathscr{J}(X)$ is β-favourable, if, and only if, X is not a Baire space.*

For a proof see [5]. The next result is general folklore.

THEOREM 2.4. *For a metric space X, the game $\mathscr{J}(X)$ is α-favourable, if, and only if, X is a residual subset in its completion.*

Let X be a topological space. We recall that X is said *submetrizable* if its topology is finer than the topology defined by some metric on X. We shall say that X is *completely submetrizable* if its topology is finer than the topology defined by some complete metric on X, and *completely metrizable* if its topology can be defined by some complete metric.

THEOREM 2.5. *If X contains a residual subset which is submetrizable, then the games $\mathscr{J}(X)$ and $\mathscr{J}_1(X)$ are equivalent.*

Proof. Let $G = \bigcap_{n\in\omega} G_n$ be a residual subset of X with each G_n dense and open, and suppose that the topology on G is finer than the topology defined by some metric d.

Denote by J' one of the two games $\mathscr{J}(X)$ and $\mathscr{J}_1(X)$ and by J'' the other game. We recall that plays and strategies for J' and J'' are the same. Now given a winning strategy σ for Player γ in J', we can define a winning strategy τ for the same Player γ in the same game J' satisfying the following additional condition: If V_n is a response given by τ at the n-th move (with n even if $\gamma = \beta$, and n odd if $\gamma = \alpha$) then $V_n \subset G_n$ and diam $(G \cap V_n) < 2^{-n}$, where the diameter is relative to d. The existence of such a strategy τ is obvious.

If $(V_n)_{n\in\omega}$ is a play compatible with τ then $A = \bigcap_{n\in\omega} V_n$ satisfies: $A \subset G$ and diam $(A) = 0$, so that A is non-empty, if, and only if, A is a singleton. It then follows that τ is also a winning strategy in the other game J''.

THEOREM 2.6. *For any topological space X the following are equivalent.*

(i) *$\mathscr{J}_1(X)$ is α-favourable.*

(ii) *$\mathscr{J}(X)$ is α-favourable and X contains a submetrizable, residual subset.*

(iii) *$\mathscr{J}(X)$ is α-favourable and X contains a completely submetrizable, residual subset.*

Proof. (iii) $\Rightarrow$ (ii) is obvious and (ii) $\Rightarrow$ (i) follows from Theorem 2.5.

To prove (i)$\Rightarrow$(iii) consider σ a winning strategy for α in $\mathcal{J}_1(X)$. By standard arguments we can find a sequence $(\mathcal{W}_n)_{n\in\omega}$ satisfying the three conditions.

(1) $\mathcal{W}_n$ is a family of disjoint open subsets of X.

(2) $G_n = \bigcup \mathcal{W}_n$ is a dense open set in X.

(3) For any decreasing sequence $(W_n)_{n\in\omega}$ with $W_n \in \mathcal{W}_n$ there exists a play $(V_n)_{n\in\omega}$ in $\mathcal{J}_1(X)$ compatible with σ and such that:

$$W_{2n+1} = V_{2n+1} = \sigma(V_0, \ldots, V_{2n}).$$

Then $G = \bigcap_{n\in\omega} G_n$ is a residual subset of X and it is easy to see that the topology on G generated by the sets of the form $W \cap G$ with $W \in \bigcup_{n\in\omega} \mathcal{W}_n$ can be defined by a complete metric.

THEOREM 2.7. *Let X be a regular space. If X contains a residual subset which is metrizable then the games $\mathcal{J}(X)$ and $\mathcal{J}_0(X)$ are equivalent.*

Proof. We distinguish two cases. If X is not a Baire space then $\mathcal{J}(X)$ is β-favourable and *a fortiori* $\mathcal{J}_0(X)$ is also β-favourable, hence the result is obvious in this case.

Suppose now that X is a Baire space; then any residual subset G of X is dense in X. Notice that if $(V_n)_{n\in\omega}$ is a sequence of open subsets of X such that $(G \cap V_n)_{n\in\omega}$ form a basic system of neighbourhoods in G for some $x \in G$, then $(V_n)_{n\in\omega}$ is a basic system of neighbourhoods of x in X. To see this consider any neighbourhood W of x in X and fix V an open neighbourhood of x in X such that $\bar{V} \subset W$ and then fix $n \in \omega$ such that $(V_n \cap G) \subset (V \cap G)$; since G is dense we have: $V_n \subset \overline{V_n \cap G} \subset \bar{V} \subset W$. We then can argue as in the proof of Theorem 2.5 to show that the games $\mathcal{J}(X)$ and $\mathcal{J}_0(X)$ are equivalent.

The following result can be deduced from Theorem 2.7 in the same way that Theorem 2.6 was deduced from Theorem 2.5.

THEOREM 2.8. *For any regular space X the following conditions are equivalent.*

(i) *$\mathcal{J}_0(X)$ is α-favourable.*

(ii) *$\mathcal{J}(X)$ is α-favourable and X contains a metrizable, residual subset.*

(iii) *$\mathcal{J}(X)$ is α-favourable and X contains a completely metrizable, residual subset.*

§3. *An example of game morphism.* In this section we discuss a particular case of the notion of "game morphism" that we shall define in the next section.

Let X be a completely regular space. In [4] P. Kenderov and J. P. Revalski define for any subset W of $C_b(X)$

$$M(W) = \bigcup_{f\in W} \{f = \inf(f)\}.$$

If $W = \{f\}$ we write $M(f)$ for $M(\{f\})$. The main results of [4] are the following.

PROPOSITION 3.1. (*Kenderov–Revalski*)

(1) *If W is open in $C_b(X)$ then $M(W)$ is open in X.*

(2) *If W is a non empty open set in $C_b(X)$ and V a non empty open set in X such that $V \subset M(W)$, then there exists a non empty open set W' in $C_b(X)$ such that $M(W') \subset V$.*

(3) *If $(W_n)_{n\in\omega}$ is a decreasing sequence of open sets in $C_b(X)$ such that $\lim_{n\to\infty} \operatorname{diam}(W_n)=0$ and $\bigcap_{n\in\omega} W_n$ is a singleton $\{f\}$, then $\bigcap_{n\in\omega} M(W_n)=M(f)$.*

In particular,

$$f\in\Gamma(X) \quad \Leftrightarrow \quad \bigcap_{n\in\omega} M(W_n)\neq\varnothing.$$

THEOREM 3.2. (*Kenderov–Revalski*) *$\Gamma(X)$ is a residual subset of $C_b(X)$, if, and only if, $\mathscr{J}(X)$ is α-favourable.*

The proof of this Theorem in [4] uses for both implications the properties of the transformation M given in Proposition 3.1. For one implication, supposing $\Gamma(X)$ residual in $C_b(X)$, the proof constructs a winning strategy σ for α in $\mathscr{J}(X)$. In fact in this case α has a winning strategy τ in $\mathscr{J}(\Gamma(X))$ and the strategy σ can explicitly be derived from the strategy τ. For the other implication, assuming α has a winning strategy τ in $\mathscr{J}(X)$ and proving that $\Gamma(X)$ is residual in $C_b(X)$, the argument in [4] is more elaborated. However it is very easy to check that in a completely similar way one can derive from τ, a winning strategy σ for α. Then applying the general result of Theorem 2.4 one obtains a simple proof of the other implication of Theorem 3.2.

Moreover replacing the game $\mathscr{J}(\Gamma(X))$ by the equivalent game $\mathscr{J}(\Gamma(X), C_b(X))$, the arguments are also valid for Player β. In particular combining this with Theorem 2.3 one obtains:

THEOREM 3.3. *$\Gamma(X)$ is a Baire space, if, and only if, X is a Baire space.*

We shall not give here more details for these arguments since we shall give later in Theorem 7.1 another (complete) proof for these results among others.

§4. *Morphisms of transitive games.* We define in this section the notion of "morphism" which will be our main tool for proving the equivalence of some games. For this definition we have to restrict ourselves to some particular class of games that we introduce first.

§4.1. *Transitive games.* We shall say that a game G is transitive if it satisfies the following conditions (1) and (2).

(1) The domain of G is equipped with two transitive relations $<_\alpha$ and $<_\beta$, which determine the rule of G in the following way. At the first move Player β chooses any $u_0\in\operatorname{Dom}(G)$; at the n-th move, with $n>0$, we distinguish two cases,

   (i) if n is odd, Player α chooses $u_n <_\alpha u_{n-1}$,

   (ii) if n is even, Player β chooses $u_n <_\beta u_{n-1}$.

(2) The win result of some play $\bar{u}$ for Player γ depends only on the subsequence of $\bar{u}$ played by γ. This is equivalent to saying that if two plays $\bar{u}=(u_n)_{n\in\omega}$ and $\bar{v}=(v_n)_{n\in\omega}$ satisfy:

$$(u_{2n}=v_{2n}, \quad \forall n\in\omega) \qquad \text{or} \qquad (u_{2n+1}=v_{2n+1}, \quad \forall n\in\omega)$$

then $\bar{u}$ and $\bar{v}$ are both won by the same player.

The games $\mathscr{J}(X)$, $\mathscr{J}_1(X)$, $\mathscr{J}_0(X)$ are clearly transitive games with both relations $<_\alpha$ and $<_\beta$ equal to $\subset$. Also the game $J(X, M)$ is transitive with both relations $<_\alpha$ and $<_\beta$ equal to $\sqsubseteq$. In Section 6 we shall discuss interesting transitive games for which the relations $<_\alpha$ and $<_\beta$ are not equal.

§4.2. *Morphisms.* Let G and H be two transitive games for which we denote by the same symbols $<_\alpha$ and $<_\beta$ the transitive relations defining their respective rules. A morphism from G onto H is a mapping

$$\varphi: \mathrm{Dom}\,(G) \to \mathrm{Dom}\,(H)$$

satisfying for any: $\gamma\in\{\alpha, \beta\}$, u and $u'\in\mathrm{Dom}\,(G)$, $v\in\mathrm{Dom}\,(H)$, the following conditions:

(1) $u'<_\gamma u \quad \Rightarrow \quad \varphi(u')<_\gamma \varphi(u)$;
(2) $v<_\gamma \varphi(u) \quad \Rightarrow \quad \exists u'\colon\ u'<_\gamma u$ and $\varphi(u')<_{\bar{\gamma}} v$;
(2′) $\forall v,\ \exists u'\colon\ \varphi(u')<_\alpha v$;
(3) Player γ wins the play $(u_u)_{n\in\omega}$ in G, if, and only if, Player γ wins the play $(\varphi(u_n))_{n\in\omega}$ in H.

§4.3. *Remarks.* (1) As we shall see, using a morphism, one can associate to any strategy in one of the games, a strategy for the same player in the other game. In this operation, conditions (2) and (2′) will play very similar roles: Condition (2′) will be used for defining the first move of the new strategy in the case where this strategy is intended for player α, whereas condition (2) will be used for the other moves. In fact (2′) is a consequence of (2) if the following condition is fulfilled:

$$\exists a\in\mathrm{Dom}\,(G)\colon \quad \forall v\in\mathrm{Dom}\,(H), \quad v<_\beta \varphi(a).$$

All the morphisms that we shall consider later will fulfil this supplementary condition.

(2) In spite of the terminology, if φ is a morphism from G onto H, the mapping φ: Dom $(G) \to$ Dom (H) is not necessarily onto. However condition (2) allows one to pull back some constructions in Dom (H) to similar constructions in Dom (G).

§4.4. *Example.* It follows from Proposition 3.1 that $W\mapsto M(W)$ is a morphism from $\mathscr{J}(\Gamma(X), C_b(X))$ onto $\mathscr{J}(X)$. Notice that in this example the condition of Remark 4.3.1 is fulfilled since $M(C_b(X))=X$.

THEOREM 4.5. *Let G and H be two transitive games. If there exists a morphism from G onto H then the games G and H are equivalent.*

Proof. Formally the proof should contain four parts, since we have to show that given any winning strategy for one of the two players in one of the two games we can find a winning strategy for the same player in the other game. However although the situation is non symmetric, the arguments are completely similar and we shall develop only one case.

Supposing that α has a winning strategy σ in G, we shall construct a winning strategy τ for α in H.

Let φ denote a morphism from G onto H. The strategy τ will consist in fact in associating to any play $\bar{v}=(v_n)_{n\in\omega}$ in H, a play $\bar{u}=(u_n)_{n\in\omega}$ in G compatible with σ and satisfying for all n:

$$\varphi(u_{2n+1})=v_{2n+1}.$$

We first prove that such a strategy τ is automatically winning for α in H. Notice that if $\bar{u}$ and $\bar{v}$ are as above then from condition (1) of (4.2) we can derive:

$$\begin{array}{ccccccccccccc} \ldots <_\beta & \varphi(u_{2n+1}) & <_\alpha \ldots <_\beta & \varphi(u_3) & <_\alpha & \varphi(u_2) & <_\beta & \varphi(u_1) & <_\alpha & \varphi(u_0) \\ & \| & & \| & & & & \| & & \\ \ldots <_\beta & v_{2n+1} & <_\alpha \ldots <_\beta & v_3 & <_\alpha & u_2 & <_\beta & v_1 & <_\alpha & v_0. \end{array}$$

Since σ is winning, Player σ wins $\bar{u}$ in G and hence by condition (3) of (4.2) Player α wins also $\varphi(\bar{u})$ in H. But then by the transitivity of the game H Player α wins also the play $\bar{v}$. This proves that τ is a winning strategy for α in H.

We now describe the construction of τ. If β chooses $v_0\in \mathrm{Dom}\,(H)$ then α fixes $u_0\in \mathrm{Dom}\,(G)$ given by (2′) such that $\varphi(u_0)<_\alpha v_0$ and sets

$$u_1=\sigma(u_0), \qquad \text{and} \qquad v_1=\tau(v_0)=\varphi(u_1).$$

Suppose that we have defined $(u_0, u_1, \ldots, u_{2n+1})$ compatible with σ and $(v_0, v_1, \ldots, v_{2n+1})$ compatible with the rule of H and satisfying $v_{2k+1}=\varphi(u_{2k+1})$ for all $k\leqslant n$. If in the next move β plays $v_{2n+2}<_\beta v_{2n+1}=\varphi(u_{2n+1})$ then by condition (2) we can find u_{2n+2} such that $u_{2n+2}<_\beta u_{2n+1}$ and $\varphi(u_{2n+2})<_\alpha v_{2n+2}$ and then define

$$\begin{cases} u_{2n+3}=\sigma(u_0, u_1, \ldots, u_{2n+1}, u_{2n+2}), \\ v_{2n+3}=\tau(v_0, v_1, \ldots, v_{2n+1}, v_{2n+2})=\varphi(u_{2n+3}). \end{cases}$$

Since $u_{2n+3}<_\alpha u_{2n+2}$ it follows from condition (1) that

$$\varphi(u_{2n+3})<_\alpha \varphi(u_{2n+2})<_\alpha v_{2n+2},$$

and this shows that the choice of v_{2n+2} respects the rule of H which finishes the construction of τ.

§5. *Some more game morphisms.* Throughout this section X denotes a completely regular space. For $f\in C_b(X)$, W an open subset of $C_b(X)$, and $\delta>0$

we let:

$$M_\delta(f)=\{f<\inf(f)+\delta\},$$

$$M_\delta(W)=\bigcup_{f\in W} M_\delta(f),$$

$$M_0(W)=\bigcup_{f\in W}\{f<\inf(f)+\operatorname{dist}(f, W^c)\},$$

where *dist* denotes the distance of an element to a subset, given by

$$\operatorname{dist}(f, W^c)=\inf\{\|f-g\|;\, g\in C_b(X)\setminus W\}.$$

For the definitions of $\Gamma(X)$, $\Gamma_1(X)$, $\Gamma_0(X)$ and $G_1(X)$, $G_0(X)$ see the introduction.

LEMMA 5.1. *If $W'\subset W$ then $M_0(W')\subset M_0(W)$.*

LEMMA 5.2. *If $f\in W$ then $M_0(W)\subset M_\delta(f)$ with $\delta=3\operatorname{diam}(W)$.*

Proof. Let $x\in M_0(W)$ and fix $g\in W$ such that

$$g(x)<\inf(g)+\operatorname{dist}(g, W^c).$$

Since the open ball of centre g and radius $r=\operatorname{dist}(g, W^c)$ is contained in W, we have $2r\leqslant\operatorname{diam}(W)$. So

$$f(x)<\inf(f)+2\|f-g\|+r\leqslant\inf(f)+3\operatorname{diam}(W).$$

LEMMA 5.3. *Let V be an open set in X, and $\delta>0$. If $x\in V\cap M_\delta(f)$ for some $f\in C_b(X)$ then we can find $f'\in C_b(X)$ and $\delta'>0$ satisfying:*
(1) $\|f'-f\|<\delta$;
(2) $M_{\delta'}(f')\subset V$; *and*
(3) $f'(x)=\inf(f')$.
If moreover $x\in G_i(X)$ for $i=0, 1$ then we can choose f' in $\Gamma_i(X)$.

Proof. Fix $0<\varepsilon<\delta$ such that $x\in M_\varepsilon(f)$ and let $\delta'=\frac{1}{2}(\delta-\varepsilon)$. Since $U=V\cap M_\varepsilon(f)$ is a neighbourhood of x, we can find $g\in\Gamma(X)$ satisfying

$$\inf(f)-2\delta'<g(x)=\inf(g)<\inf(f)-\delta', \qquad \text{and}$$

$$g=\sup(f) \qquad \text{on} \qquad X\setminus U.$$

Finally let $f'=\inf(f, g)$ which satisfies:

$$\begin{aligned}
&f'(x)=\inf(f')=\inf(g)=g(x);\\
&f'\leqslant f, &&\text{on}\quad X;\\
&f'=f, &&\text{on}\quad X\setminus U;\\
&0\leqslant f-f'<\delta, &&\text{on}\quad U.
\end{aligned}$$

To check the last relation notice that for $y \in U$ we have

$$\begin{aligned} 0 \leqslant f(y) - f'(y) &\leqslant (\inf(f) + \delta - 2\delta') - (\inf(f')) \\ &< (\inf(f) + \delta - 2\delta') - (\inf(f) - \delta')) \\ &< \delta - \delta'. \end{aligned}$$

It follows then that $\|f - f'\| < \delta$.

Moreover if $y \in M_{\delta'}(f')$ then

$$f'(y) < \inf(f') + \delta' < \inf(f) \leqslant f(y),$$

and so $y \in U$. This proves that $M_{\delta'}(f') \subset V$.

Finally it is clear that if $x \in G_i(X)$ for $i = 0, 1$ then we can choose $g \in \Gamma_i(X)$ and obtain $f' \in \Gamma_i(X)$.

LEMMA 5.4. *Let W be an open set in $C_b(X)$ and V an open set in X such that $V \cap M_0(W) \neq \varnothing$. Then we can find W' a non empty open set in $C_b(X)$ satisfying $W' \subset W$ and $M_0(W') \subset V$.*

If moreover $V \cap M_0(W) \cap G_i(X) \neq \varnothing$ for $i = 0, 1$ then we can choose W' such that $W' \cap \Gamma_i(X) \neq \varnothing$.

Proof. Since $V \cap M_0(W) \neq \varnothing$ we can find $f \in W$ such that for $\delta = \operatorname{dist}(f, W^c)$ the set $U = V \cap M_\delta(f)$ is not empty. Then pick any $x \in U$ and let (f', δ') obtained by applying Lemma 5.3 to (x, V, f, δ). Since $\|f - f'\| < \delta$ we have necessarily that $f' \in W$. So by Lemma 5.2 any open subset W' of W containing f' and of diameter $< \frac{1}{3}\delta'$, satisfies $M_0(W') \subset M_{\delta'}(f') \subset V$.

The last Γ_i variation of the conclusion follows from the similar variation in Lemma 5.3.

LEMMA 5.5. *If $(W_n)_{n \in \omega}$ is a decreasing sequence of open sets in $C_b(X)$ such that $\lim_{n \to \infty} \operatorname{diam}(W_n) = 0$ and $\bigcap_{n \in \omega} W_n$ is a singleton $\{f\}$, then $\bigcap_{n \in \omega} M_0(W_n) = M(f)$.*

Moreover:

(a) *$f \in \Gamma(X)$, if, and only if, $\bigcap_{n \in \omega} M_0(W_n)$ is non empty*;

(b) *$f \in \Gamma_1(X)$, if, and only if, $\bigcap_{n \in \omega} M_0(W_n)$ is a singleton*; *and*

(c) *$f \in \Gamma_0(X)$, if, and only if, $\bigcap_{n \in \omega} M_0(W_n)$ is a singleton for which the family $(M_0(W_n))_{n \in \omega}$ is a basic system of neighbourhoods.*

Proof. Let $\rho_n = \operatorname{diam}(W_n)$. If $x \in \bigcap_{n \in \omega} M_0(W_n)$ then for any n there exists $f_n \in W_n$ such that $f_n(x) < \inf(f_n) + \delta_n$ with $\delta_n = \operatorname{dist}(f_n, W_n^c) \leqslant \rho_n$, and since $f = \lim_{n \to \infty}(f_n)$ we have $x \in M(f)$. The other inclusion is obvious.

Then (a) and (b) follow easily from the last equality. Finally by Lemma 5.2 we have $M_0(W_n) \subset M_{3\rho_n}(f)$ from which we can derive (c).

The following is an immediate consequence of Lemma 5.3.

PROPOSITION 5.6. *Let $i = 0, 1$.*

(a) *$\Gamma(X)$ is dense in $C_b(X)$.*

(b) *$\Gamma_i(X)$ is dense in $C_b(X)$, if, and only if, $G_i(X)$ is dense in X.*

(c) *$\Gamma_i(X)$ is non empty, if, and only if, $G_i(X)$ is non empty.*

§5.7. *M_0-morphisms.* Notice first that for any $W \subset C_b(X)$ the set $M_0(W)$ is open in X, and is non empty if W is non empty. Let Φ denote the restriction of M_0 to $\mathscr{D} = \mathscr{T}_+(C_b(X))$. It follows from the complete regularity of X that $\Gamma(X)$ is dense in $C_b(X)$ and therefore that $\mathscr{D} = \text{Dom}\,(\mathscr{J}(\Gamma(X), C_b(X)))$. We claim that Φ is a game morphism from $\mathscr{J}(\Gamma(X), C_b(X))$ onto $\mathscr{J}(X)$. Since $\Phi(\Gamma(X)) = X$ it is sufficient, by Remark 4.3.1, to check conditions (1), (2), (3) of 4.2. But these conditions are clearly ensured by Lemma 5.1 for (1), by Lemma 5.4 for (2), and by Lemma 5.5 (a) for (3).

Now let $i = 0, 1$. It is clear that if $W \cap \Gamma_i(X) \neq \varnothing$ then $M_0(W) \cap G_i(X) \neq \varnothing$, so that $\Phi_i(W) = M_0(W) \cap G_i(X)$ defines a mapping from $\text{Dom}\,(\mathscr{J}(\Gamma_i(X), C_b(X))$ into $\text{Dom}\,(\mathscr{J}_i(G_i(X))$. Then using the Γ_i variation of the previous lemmas one checks easily that Φ_i is also a game morphism from $\mathscr{J}(\Gamma_i(X), C_b(X))$ onto $\mathscr{J}_i(G_i(X))$.

From these observations and from Theorem 4.4 we obtain

THEOREM 5.8.
(a) *The games $\mathscr{J}(\Gamma(X))$ and $\mathscr{J}(X)$ are equivalent.*
(b) *The games $\mathscr{J}(\Gamma_1(X))$ and $\mathscr{J}_1(G_1(X))$ are equivalent.*
(c) *The games $\mathscr{J}(\Gamma_0(X))$ and $\mathscr{J}_0(G_0(X))$ are equivalent.*

§6. *Some $*$-topological games.* By a $*$ game on a topological space X we mean a game the domain of which is a subset of

$$\mathscr{E}(X) = \{(x, V) \in X \times \mathscr{T}(X) : x \in V\}.$$

§6.1. *The games $\mathscr{J}^*(X)$ and $\mathscr{J}_0^*(X)$.* These are transitive games with same domain and same rule. The domain is the set $\mathscr{E}(X)$. The rule is defined by the following transitive relations:

$$\begin{cases} (x, V) <_\alpha (x', V') & \Leftrightarrow \quad V \subset V' \text{ and } x = x'; \\ (x, V) <_\beta (x', V') & \Leftrightarrow \quad V \subset V'. \end{cases}$$

If $(x_n, V_n)_{n\in\omega}$ is a play in these games then:
(1) Player α wins in $\mathscr{J}^*(X)$, if, and only if, $\bigcap_{n\in\omega} V_n$ is non empty; and
(2) Player α wins in $\mathscr{J}_0^*(X)$, if, and only if, $\bigcap_{n\in\omega} V_n$ is a singleton for which the family $(V_n)_{n\in\omega}$ is a basic system of neighbourhoods.

$\mathscr{J}^*(X)$ was introduced by G. Choquet who called this game "the strong game" by comparison with $\mathscr{J}(X)$. Notice that if X is metrizable then $\mathscr{J}^*(X) \approx \mathscr{J}_0^*(X)$.

§6.2. *The game $\mathscr{J}^*(X, M)$.* This is a $(*)$ version of the game $\mathscr{J}(X, M)$. So M is a metric space and X is a subspace of M. The domain of $\mathscr{J}^*(X, M)$ is the set

$$\mathscr{E}(X, M) = \{(x, V) \in X \times \mathscr{T}(M) : x \in V\}.$$

The rule of $\mathscr{J}^*(X, M)$ is obtained from the rule of $\mathscr{J}^*(X)$ by replacing $\subset$ the inclusion relation by the relation $\sqsubset$ defined in 2.2. If $(x_n, V_n)_{n\in\omega}$ is a play in

$\mathscr{J}^*(X, M)$ then $\bigcap_{n\in\omega} W_n$ is a singleton $\{x\}$ and Player α wins this play if $x\in X$. It is clear that

$$\mathscr{J}^*(X, M)\approx\mathscr{J}^*(X)\approx\mathscr{J}_0^*(X).$$

THEOREM 6.3. *Let X be a metrizable space.*
(a) *$\mathscr{J}^*(X)$ is α-favourable, if, and only if, X is completely metrizable.*
(b) *$\mathscr{J}^*(X)$ is β-favourable, if, and only if, X contains a closed subset which is not a Baire space.*

For (a) see [1] *and for (b) see* [3].

THEOREM 6.4. *$\mathscr{J}_0^*(X)$ is α-favourable, if, and only if, X is the image of some complete metric space by an open and continuous mapping.*

Proof. Suppose that $\varphi: S\to X$ is an open and continuous mapping from the complete metric space S onto X. Let $T=\mathscr{T}^+(S)$ and $Y=\mathscr{T}^+(X)$ the sets of all non empty open subsets of S and X. We define Φ: $S\times T\to X\times Y$ by

$$\Phi(s, U)=(\varphi(s), \varphi(U)).$$

Then a simple checking of the conditions of 4.2 shows that Φ is a game morphism from $\mathscr{J}_0^*(S)$ onto $\mathscr{J}_0^*(X)$. Since by Theorem 6.3, $\mathscr{J}_0^*(S)$ is α-favourable, then it follows from Theorem 4.5 that $\mathscr{J}_0^*(X)$ is α-favourable.

Conversely suppose that $\mathscr{J}_0^*(X)$ is α-favourable, and fix σ some winning strategy for α in this game. Denote by Σ the set of all plays in $\mathscr{J}_0^*(X)$ which are compatible with σ. So Σ appears as a subset of $(X\times Y)^\omega$. If we endow $X\times Y$ with the discrete topology then Σ appears as a closed subset of $(X\times Y)^\omega$. We consider from now on Σ as endowed by its subspace topology which is completely metrizable. Let $\psi:\Sigma\to X$ which assigns to any play $(x_n, V_n)_{n\in\omega}$ compatible with σ the unique point of the singleton $\bigcap_{n\in\omega} W_n$. We shall prove that ψ is open, continuous and onto.

For any finite sequence

$$p=\langle(x_0, V_0), (x_1, V_1), \ldots, (x_{2n+1}, V_{2n+1})\rangle$$

of even length and compatible with σ, denote by Σ_p the set of all (infinite) plays compatible with σ and extending p. It is clear that the family of all sets of the form Σ_p is a basis for the topology of Σ.

To see that ψ is onto fix any $x\in X$, then it is clear that there exists a play $\pi=(x_n, V_n)_{n\in\omega}$ compatible with σ and such that $x_n=x$ for all n, so that $\psi(\pi)=x$. The same argument shows that for p as above $\psi(\Sigma_p)=V_{2n+1}$ and therefore that ψ is open. Finally it follows from the win condition of $\mathscr{J}_0^*(X)$ that given any $\pi\in\Sigma$ the sets $\psi(\Sigma_p)$, for p any finite beginning of π with even length, form a basic system of neighbourhoods of $\psi(\pi)$ and this ensures the continuity of ψ.

§6.5 *More M_0-morphisms.* In the rest of this section X denotes a completely regular space. Using M_0 we shall define a new game morphism Φ^* from the game $\mathscr{J}^*(\Gamma_0(X), C_b(X))$ onto the game $\mathscr{J}_0^*(G_0(X))$.

Let $\mathscr{D} = \operatorname{Dom}(\mathscr{J}^*(\Gamma_0(X), C_b(X)))$. Given any $f \in \Gamma_0(X)$ we denote by $m(f)$ the unique point of X where f attains its minimum, so $M(f) = \{m(f)\}$. For any W a non empty open subset of $C_b(X)$ and any $f \in W \cap \Gamma_0(X)$ with $(f, W) \in \mathscr{D}$, we define

$$\Phi^*(f, W) = (m(f), M_0(W) \cap G_0(X)).$$

Since $\Phi^*(f, C_b(X)) = (m(f), G_0(x))$ we can again use Remark 4.3.1 so that we only need to check conditions (1), (2), (3) of 4.2, that we shall now write in this particular situation. Notice that because of the non equality of the relations $<_\alpha$ and $<_\beta$, each of conditions (1) and (2) splits in two subconditions.

(1) Let $(x, V) = \Phi^*(f, W)$ and $(x', V') = \Phi^*(f', W')$.

(1a) If $W' \sqsubseteq W$ and $f' = f$ and $V' \subset V$ and $x' = x$.

(1b) If $W' \sqsubseteq W$ then $V' \subset V$.

(2) Let $(f, W) \in \mathscr{D}$ and $(x, V) \in \operatorname{Dom}(\mathscr{J}_0^*(G_0(X)))$.

(2a) If $V \subset M_0(W)$, and $x = m(f)$ then there exists $(f', W') \in \mathscr{D}$ such that

$$f' = f, \qquad W' \sqsubseteq W, \qquad M_0(W') \subset V.$$

(2b) If $V \subset M_0(W)$ and $x \in V \cap G_0(X)$ then there exists $(f', W') \in \mathscr{D}$ such that

$$W' \sqsubseteq W, \qquad M_0(W') \subset V, \qquad x = m(f').$$

(3) Player α wins the play $(f_n, W_n)_{n \in \omega}$ in $\mathscr{J}^*(\Gamma_0(X), C_b(X))$, if, and only if, Player α wins the play $(m(f_n), M_0(W_n) \cap G_0(X))_{n \in \omega}$ in the game $\mathscr{J}_0^*(G_0(X))$.

Condition (1) is obvious; condition (2b) follows from Lemma 5.3 and Lemma 5.2; and condition (3) follows from Lemma 5.5. Finally for (2a) notice that since $f \in \Gamma_0(X)$ and $x = m(f)$, we can find $\delta > 0$ such that $M_\delta(f) \subset V$ then it follows from Lemma 5.2 that (2a) is satisfied by any open set $W' \sqsubseteq W$ containing f and such that $\operatorname{diam}(W') < \frac{1}{3}\delta$.

Then applying Theorem 4.5 we obtain

THEOREM 6.6. *The games $\mathscr{J}^*(\Gamma_0(X), C_b(X))$ and $\mathscr{J}_0^*(G_0(X))$ are equivalent.*

§7. *Applications.* In all this section X denotes a completely regular space.

We recall that a topological space X is said to be α-favourable, if, and only if, the Banach–Mazur game $\mathscr{J}(X)$ is α-favourable.

THEOREM 7.1.

(a) *$\Gamma(X)$ is α-favourable, if, and only if, X is α-favourable.*

(*b*) *$\Gamma(X)$ is a Baire space, if, and only if, X is a Baire space.*

Proof. Apply Theorem 5.8 and Theorem 2.3.

THEOREM 7.2.
(a) *$\Gamma_1(X)$ is α-favourable, if, and only if, $G_1(X)$ is α-favourable and contains a submetrizable residual subset.*
(b) *$\Gamma_1(X)$ is a residual subset of $C_b(X)$, if, and only if, X is α-favourable and contains a submetrizable residual subset.*
(c) *Suppose X submetrizable. Then $\Gamma_1(X)$ is a Baire space, if, and only if, X is a Baire space.*

Proof.
(a) Apply Theorem 2.6 and Theorem 5.8.
(b) If $\Gamma_1(X)$ is residual in $C_b(X)$ then by (a) and Proposition 5.6 the space X which contains a dense α-favourable subspace is α-favourable. Moreover any residual subset of $G_1(X)$ is then automatically residual in X. Conversely if X is α-favourable and contains a submetrizable residual subset R then necessarily $R \subset G_1(X)$, so by (a) and Proposition 5.6 the space $\Gamma_1(X)$ is α-favourable and dense in the complete metric space $C_b(X)$ and therefore $\Gamma_1(X)$ is a residual subset of $C_b(X)$.
(c) Notice that since X is submetrizable we have $G_1(X)=X$ and apply Theorem 2.3, Theorem 2.5 and Theorem 5.8.

The following result is proved with completely similar arguments.

THEOREM 7.3.
(a) *$\Gamma_0(X)$ is α-favourable, if, and only if, $G_0(X)$ is α-favourable and contains a metrizable residual subset.*
(b) *$\Gamma_0(X)$ is a residual subset of $C_b(X)$, if, and only if, X is α-favourable and contains a metrizable residual subset.*
(c) *Suppose X metrizable. Then $\Gamma_0(X)$ is a Baire space, if, and only if, X is a Baire space.*

THEOREM 7.4. *$\Gamma_0(X)$ is a G_δ subset in $C_b(X)$, if, and only if, $G_0(X)$ is the image of some complete metric space by an open and continuous mapping.*

Proof. Apply Theorem 6.3(a), Theorem 6.4, and Theorem 6.6.

THEOREM 7.5. *Let X be a metrizable space.*
(a) *Any closed subspace of $\Gamma_0(X)$ is a Baire space, if, and only if, any closed subspace of X is a Baire space.*
(b) *$\Gamma_0(X)$ is a G_δ subset of $C_b(X)$, if, and only if, X is completely metrizable.*

Proof.
(a) Apply Theorem 6.3(b) and Theorem 6.6.
(b) Since X is metrizable we have $G_0(X)=X$, and it follows from Theorem 6.3 and Theorem 6.4 that $\mathcal{J}_0^*(X)$ is α-favourable, if, and only if, X is completely metrizable. Then apply Theorem 7.4.

The proof of the next result is independent from the other sections and will not use games arguments. We recall that $\Gamma(X)$ is always dense in $C_b(X)$ and that $\Gamma_i(X)$ is dense in $C_b(X)$, if, and only if, $G_i(X)$ is dense in X.

THEOREM 7.6. *Consider the following conditions.*
(i) $\Gamma_1(X)$ *is a dense* G_δ *subset in* $C_b(X)$.
(ii) $\Gamma_1(X)=\Gamma_0(X)\neq\emptyset$.
(iii) $\Gamma(X)$ *is a* G_δ *subset in* $C_b(X)$.
(iv) $\Gamma(X)=C_b(X)$.
(v) $\Gamma_1(X)=\Gamma_0(X)$.
Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v).

Proof. (iii) $\Leftrightarrow$ (iv). One implication is obvious. Suppose that there exists $f\in C_b(X)$ which does not attain the value $a=\inf(f)$ and let $(a_n)_{n\in\omega}$ a sequence of real numbers decreasing to a and such that all the open sets $U_n=\{a_{n+1}<f<a_n\}$ are non empty. For each n, fix x_n and V_n such that $x_n\in V_n$ and $\bar{V}_n\subset U_n$. Notice that since

$$\bigcap_{n\in\omega}\overline{\bigcup_{k\geqslant n}V_k}\subset\bigcap_{n\in\omega}\{f\leqslant a_n\}=\emptyset$$

the family $(V_n)_{n\in\omega}$ is locally finite in X. It follows that if $(g_n)_{n\in\omega}$ is a sequence of continuous functions on X such that for each n the support of g_n is contained in V_n, then the function g is defined by:

$$g(x)=\sum_{n=0}^{\infty}g_n(x)=\begin{cases}g_n(x), & \text{if } x\in V_n,\\ 0, & \text{if } x\in X\setminus\bigcup_{n\in\omega}V_n.\end{cases}$$

is continuous on X.

From now on, $(g_n)_{n\in\omega}$ denotes a fixed sequence in $C_b(X)$ as above, and satisfying the supplementary conditions

$$g_n(x_n)=1 \qquad \text{and} \qquad 0\leqslant g_n\leqslant 1.$$

Fix any enumeration $(r_n)_{n\in\omega}$ of the set R all rationals in $I=[0,1]$ and consider the mapping $\varphi\colon t\mapsto h_t$ from I into $C_b(X)$ defined by:

$$1-h_t=\sum_{n\in\omega}^{\infty}(1-|t-r_n|)g_n.$$

Then for any $t\in I$ we have

$$\begin{cases}|t-r_n|\leqslant h_t\leqslant 1, & \text{on } V_n,\\ h_t=1, & \text{on } X\setminus\bigcup_{n\in\omega}V_n.\end{cases}$$

It is clear then that $\inf(h_t)=0$ for any $t\in I$, and that $h_t\in\Gamma(X)$, if, and only if, $t\in R$. Since φ is continuous, if $\Gamma(X)$ were a $\mathbf{G}_\delta$ subset of $C_b(X)$ then $R=\varphi^{-1}(\Gamma(X))$ would also be a $\mathbf{G}_\delta$ subset of I which gives the contradiction.

(ii) $\Rightarrow$ (iv). Fix $f\in\Gamma_1(X)$ and let x be the unique point such that $f(x)=\inf(f)$. Suppose now that there exists $g\in C_b(X)$ which does not attain its minimum; we can moreover suppose that $\inf(g)=\inf(f)=0$, then $h=\inf(f,g)$ attains its minimum only at point x. Now consider any sequence

$(x_n)_{n\in\omega}$ in X such that $\lim_{n\to\infty} g(x_n)=0$. Since $g\notin\Gamma(X)$, such a sequence has no cluster point, however $\lim_{n\to\infty} h(x_n)=0$ and this proves that $h\notin\Gamma_0(X)$.

(iv) $\Rightarrow$ (v). Suppose that there exists $f\in\Gamma_1(X)\setminus\Gamma_0(X)$ and let x be the unique point such that $f(x)=\inf(f)=m$. Since $f\notin\Gamma_0(X)$ we can find a neighbourhood V of x such that $\inf_{y\in X\setminus V} f(y)=m$. Let g be any positive function in $C_b(X)$ with support in V satisfying $g(x)>0$. Then $h=f+g$ is not in $\Gamma(X)$ which contradicts (iv).

(i) $\Rightarrow$ (ii). Since (iv) $\Rightarrow$ (v) it is enough to prove (i) $\Rightarrow$ (iv). This is proved in the same way as (iii) $\Rightarrow$ (iv). Notice that the density condition on $\Gamma_1(X)$ allows us now to choose the functions g_n in $\Gamma_1(X)$, from which we could again derive that $R=\varphi^{-1}(\Gamma_1(X))$ is a $\mathbf{G}_\delta$ subset of I.

THEOREM 7.7. *For a metrizable space X the following conditions are equivalent.*

(i) *$\Gamma_1(X)$ is a $\mathbf{G}_\delta$ subset in $C_b(X)$.*
(ii) *$\Gamma(X)$ is a $\mathbf{G}_\delta$ subset in $C_b(X)$.*
(iii) *X is compact.*

Proof. (i) $\Rightarrow$ (ii). Since X is metrizable $\Gamma_1(X)$ is dense in $C_b(X)$ and the conclusion follows from Theorem 7.6.

(ii) $\Rightarrow$ (iii). By Theorem 7.6 we have that (ii) is equivalent to the condition $(\Gamma(X)=C_b(X))$ and it is well known that for metric spaces this condition is equivalent to the compactness of X.

(iii) $\Rightarrow$ (i). Since X is compact $\Gamma_1(X)=\Gamma_0(X)$ and the conclusion follows from Theorem 7.5(b).

References

1. G. Choquet. *Lectures on Analysis* (Benjamin, New York, 1969).
2. M. M. Čoban, P. S. Kenderov and J. P. Revalski. Generic well-posedness of optimization problems in topological spaces. *Mathematika*, 36 (1989), 301–304.
3. G. Debs. Espaces héréditairement de Baire. *Fund. Math.*, 129 (1988), 196–206.
4. P. S. Kenderov and J. P. Revalski. The Banach–Mazur game and generic existence of solutions to optimization problems. *Proc. Amer. Math. Soc.*, 118 (1993), 911–917.
5. J. Saint Raymond. Jeux topologiques et espaces de Namioka. *Proc. Amer. Math. Soc.*, 87 (1983), 499–504.
6. Ch. Stegall. Topological spaces with dense subspaces that are homeomorphic to complete metric space and the classification of $C(K)$ Banach spaces. *Mathematika*, 34 (1987), 101–107.
7. R. Telgarski. Topological games: On the 50-th anniversary of the Banach–Mazur game. *Rocky Mount. J. Math.*, 17 (1987), 227–276.

Professor G. Debs,
Equipe d'Analyse,
Université Paris 6,
4, Place Jussieu,
75252 Paris Cedex 05,
France.

Professor J. Saint Raymond,
Equipe d'Analyse,
Université Paris 6,
4, Place Jussieu,
75252 Paris Cedex 05,
France.

46E15: *FUNCTIONAL ANALYSIS: Linear function spaces and their duals; Banach spaces of continuous functions.*

Received on the 10*th of March*, 1993.