THE DESCRIPTIVE COMPLEXITY OF THE SET OF ARC-CONNECTED COMPACT SUBSETS OF THE PLANE

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ABSTRACT. We compute the exact descriptive class of the set of all compact arc-connected subsets of $\boldsymbol{R}^2$, which turns out to be strictly higher than the classical $\boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^1_1$ classes of analytic and coanalytic sets, but strictly lower than the class $\boldsymbol{\Pi}^1_2$ which is the exact descriptive class of the set of all compact arc-connected subsets of $\boldsymbol{R}^3$.

If X is any Polish space then it follows readily from the definitions that the set $\mathcal{KC}_{\mathrm{arc}}(X)$ of all compact arc-connected subsets of X, is a $\boldsymbol{\Pi}^1_2$ subset of the space $\mathcal{K}(X)$ of all compact subsets of X, endowed with the Vietoris topology. Moreover Ajtai and Becker showed independently (see [4], Theorem 37.11) that the set $\mathcal{KC}_{\mathrm{arc}}(\boldsymbol{R}^3)$ is actually $\boldsymbol{\Pi}^1_2$-complete.

The goal of the present work is to compute the exact descriptive complexity of the set $\mathcal{KC}_{\mathrm{arc}}(\boldsymbol{R}^2)$. More generally, given any space X we consider the set $\mathcal{C}_{\mathrm{arc}}(X)$ of all arc-connected closed subsets of X, viewed as a subset of the space $\mathcal{F}(X)$ of all closed subsets of X, endowed with the Effros Borel structure.

By a *planar Polish space* we mean a subspace of $\boldsymbol{R}^2$, on which the induced topology is Polish, that is a $\boldsymbol{G}_\delta$ subset of $\boldsymbol{R}^2$. Our first main result is the following:

Theorem A. *For any planar Polish space X the set $\mathcal{C}_{\mathrm{arc}}(X)$ is a $\check{\mathcal{A}}(\boldsymbol{\Pi}^1_1)$ set.*

where $\check{\mathcal{A}}(\boldsymbol{\Pi}^1_1)$ denotes the class of all complements of sets obtained from $\boldsymbol{\Pi}^1_1$ sets by Suslin operation $\mathcal{A}$. Let us recall that it was already known from Ajtai and Becker work that the set $\mathcal{KC}_{\mathrm{arc}}(\boldsymbol{R}^2)$ is not $\boldsymbol{\Pi}^1_1$ (see [4]).

The proof of Theorem A relies on recent results from [3]. Also as a by-product of this proof we obtain the following property which is specific of the plane topology, since the analog is no more true in $\boldsymbol{R}^3$.

Theorem B. *For any planar arc-connected Polish space X there exists a Borel mapping which to any pair (x, y) of distinct points in X assigns an arc $J \subset X$ with endpoints x and y.*

In fact the proof of Theorem A relies on a parametrized version of Theorem B in which the space X is replaced by a variable closed subset of X, given with some code in an auxiliary space. The precise statement of this latter result (Theorem 6.9) necessitates a number of preliminaries, and we refer the reader to Section 6 for more details.

The second main result is that if $X = \boldsymbol{R}^2$, or the unit square $\mathbb{I}^2$, then the complexity bound given by Theorem A is best possible. More precisely we prove:

Theorem C. *The set $\mathcal{C}_{\mathrm{arc}}(\mathbb{I}^2)$ is $\check{\mathcal{A}}(\boldsymbol{\Pi}^1_1)$-complete.*

We also give in Section 6 several cases in which the set $\mathcal{C}_{\mathrm{arc}}(X)$ is in a strictly smaller class than the class $\boldsymbol{\Pi}^1_2$.

2020 AMS Subject Classification. 54H05, 03E15.

Key words and phrases. Borel sets, arc-connected, continua, plane topology, Suslin operation.

The present text is a simplification of that submitted to (and published in) Fund.Math. and can be found on ArXiv. It avoids the use of Effective Descriptive Set Theory.

It is worth noting that the class $\check{\mathcal{A}}(\mathbf{\Pi}^1_1)$ appeared already in previous complexity computations in the hyperspace $\mathcal{F}(X)$. We mention the following two results from [2] (Theorems 6.3 and 6.4) where $\mathcal{C}_{\mathrm{loc}}(X)$ denotes the set of all locally connected closed subsets of X:

a) For any Polish space X the set $\mathcal{C}_{\mathrm{loc}}(X)$ is a $\check{\mathcal{A}}(\mathbf{\Pi}^1_1)$ set.

b) There exists a Polish space $X \subset \mathbb{I}^3$ for which the set $\mathcal{C}_{\mathrm{loc}}(X)$ is $\check{\mathcal{A}}(\mathbf{\Pi}^1_1)$-complete

Note however that unlike for $\mathcal{C}_{\mathrm{arc}}(X)$, if X is a compact space the set $\mathcal{C}_{\mathrm{loc}}(X)$ is Borel ([2] Proposition 6.1).

1. Some descriptive preliminaries

Throughout this work by a *"space"* we shall always mean a subset of some Polish space, though we shall introduce in some situations an additional (possibly non separable) metric or topology on the initial given space. However all descriptive notions we shall consider will always refer to the Borel structure inherited from the Polish topology.

We shall consider various classical, and less classical, descriptive classes. For such a class $\mathbf{\Gamma}$ we denote by $\check{\mathbf{\Gamma}}$ its dual class, that is the class of all complements of sets in $\mathbf{\Gamma}$, and by $\Delta(\mathbf{\Gamma}) = \Delta(\check{\mathbf{\Gamma}})$ the self-dual class $\mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$. Then given any space X, subspace of some Polish space $\tilde{X}$, we denote by $\mathbf{\Gamma}(X)$ the set of all subsets of X which are the trace on X of a subset of $\tilde{X}$ which is in $\mathbf{\Gamma}$. If the class $\mathbf{\Gamma}$ is closed under Borel isomorphim, which will always be the case in this work, then $\mathbf{\Gamma}(X)$ does not depend on the particular choice of the surrounding space $\tilde{X}$. Note that in general $\mathbf{\Gamma}(X)$ is a proper subset of $\mathcal{P}(X) \cap \mathbf{\Gamma}(\tilde{X})$, and $\Delta(\mathbf{\Gamma})(X)$ is a proper subset of $\Delta(\mathbf{\Gamma}(X))$.

For any class $\mathbf{\Gamma}$ we consider also the class $\mathcal{A}(\mathbf{\Gamma})$ obtained from $\mathbf{\Gamma}$ by Suslin operation $\mathcal{A}$ and denote by $\check{\mathcal{A}}(\mathbf{\Gamma})$ its dual class. Since countable unions and countable intersections are particular instances of operation $\mathcal{A}$, it follows from the idempotence of operation $\mathcal{A}$ that the classes $\mathcal{A}(\mathbf{\Gamma})$ and $\check{\mathcal{A}}(\mathbf{\Gamma})$ are closed under countable unions and intersections.

Following logician notation we denote by $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ the classical projective classes, and set $\mathbf{\Delta}^1_n = \Delta(\mathbf{\Sigma}^1_n) = \Delta(\mathbf{\Pi}^1_n)$. In particular $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ denote respectively the classes of analytic and coanalytic sets, $\mathbf{\Sigma}^1_2$ the class of projections of $\mathbf{\Pi}^1_1$ sets and $\mathbf{\Pi}^1_2$ its dual class.

We recall that a mapping $f : X \to Y$ is said to be $\mathbf{\Gamma}$-*measurable* if the inverse image of any open subset of Y is in $\mathbf{\Gamma}(X)$. If moreover $\mathbf{\Gamma}$ is closed under countable unions and intersections then the inverse image of any Borel subset of Y is in $\mathbf{\Gamma}$. It follows that for any class $\mathbf{\Gamma}$, the notions of $\mathcal{A}(\mathbf{\Gamma})$-measurability and $\check{\mathcal{A}}(\mathbf{\Gamma})$-measurability are the same. Also since $\mathbf{\Sigma}^1_1 = \mathcal{A}(\mathbf{\Delta}^1_1)$ and $\mathbf{\Pi}^1_1 = \check{\mathcal{A}}(\mathbf{\Delta}^1_1)$ we have the following property:

Proposition 1.1. *The inverse image of any $\mathbf{\Sigma}^1_1$ (respectively $\mathbf{\Pi}^1_1$) set by an $\mathcal{A}(\mathbf{\Gamma})$-measurable mapping is in $\mathcal{A}(\mathbf{\Gamma})$ (respectively $\check{\mathcal{A}}(\mathbf{\Gamma})$).*

In particular the right composition $f \circ g$ of an $\mathcal{A}(\mathbf{\Gamma})$-measurable mapping f with a Borel mapping g, is $\mathcal{A}(\mathbf{\Gamma})$-measurable; and if $\mathbf{\Gamma}$ is closed under Borel isomorphisms then the left composition $h \circ f$ of f with a Borel mapping h is $\mathcal{A}(\mathbf{\Gamma})$-measurable too.

Of particular interest for our study is the notion of bianalyticity that we recall.

Definition 1.2. *Given spaces X, Y:*

A subset A of X is said to be bianalytic *in X if A is in $\mathbf{\Sigma}^1_1(X) \cap \mathbf{\Pi}^1_1(X)$.*

A mapping $f : X \to Y$ is said to be bianalytic *if f is $\mathbf{\Sigma}^1_1$ (equivalently $\mathbf{\Pi}^1_1$)-measurable.*

So the set of bianalytic subsets of X is $\Delta(\mathbf{\Sigma}^1_1(X)) = \Delta(\mathbf{\Pi}^1_1(X))$ while the set of Borel subsets of X is $\Delta(\mathbf{\Sigma}^1_1)(X) = \Delta(\mathbf{\Pi}^1_1)(X)$. Note that by the separation Theorem of analytic sets, if X is analytic then the notions of bianalytic and Borel coincide. In fact the notion of bianalyticity is interesting mainly in the frame of non analytic, and more specifically coanalytic, spaces. Also in this latter context many properties of Borel sets (mappings) extend to bianalytic sets (mappings), and we state next two such properties which we will need.

Proposition 1.3. *Let X and Y be coanalytic spaces. Any partial bianalytic mapping $f : D \to Y$ with $D \subset X$, admits an extension $\tilde{f} : \tilde{D} \to Y$ to a bianalytic mapping with coanalytic domain $D \subset \tilde{D} \subset X$.*

The proof of Proposition 1.3 follows standard arguments. Note that if moreover the graph of f is a subset of some coanalytic set $Z \subset X \times Y$ then applying the previous result to the mapping $(Id_D, f) : D \to Z$ we can impose that the graph of the mapping $\tilde{f}$ is also a subset of Z.

The second property we will need is less elementary and is a bianalytic replica of the classical fact that the projection of a Borel set with compact sections is Borel (a particular case of Arsenin-Kunugui Theorem). The result is probably not new, but in the absence of known references we sketch the basic arguments of the proof.

Theorem 1.4. *For any space X, and any standard Borel space Y, if B is a bianalytic subset of $X \times Y$, with compact sections in Y then the projection P of B on X is bianalytic, the mapping $\Phi : x \mapsto B(x)$ from P to $\mathcal{K}(Y)$ is bianalytic and admits a bianalytic selection (i.e. there exists a bianalytic mapping $\varphi : P \to Y$ such that for all $x \in P$, $\varphi(x) \in B(x)$).*

Proof. We embed X into a Polish space $\tilde{X}$, and fix in $\tilde{X} \times Y$ a $\mathbf{\Sigma}^1_1$ subset A and a $\mathbf{\Pi}^1_1$ subset C such that $B = A \cap (X \times Y) = C \cap (X \times Y)$. We also fix on Y a bounded distance d and a basis $(U_s)_{s\in S}$ of the topology, where $S \subset 2^{<\omega}$ is a tree and for all $s \in S$, $\operatorname{diam}(U_s) < 2^{-|s|}$ and $\overline{U_s} \subset U_{s^*}$ if $s \neq \emptyset$ (where s^* is the restriction of s to $|s| - 1$); and we set $S_n = S \cap 2^n$.

Let $\pi : \tilde{X} \times Y \to \tilde{X}$ be the canonical projection. Then for all $s \in S$ the set $P_s = \pi(A \cap (\tilde{X} \times U_s))$ is $\mathbf{\Sigma}^1_1$, hence $A_n = \bigcup_{s\in S_n} P_s \times U_s$ is $\mathbf{\Sigma}^1_1$ too, and $\tilde{A} = \bigcap_{n\in\omega} \downarrow A_n$ is $\mathbf{\Sigma}^1_1$ too. Moreover

$$(x, y) \in \tilde{A} \iff \forall\, n,\ \exists\, s \in S_n,\ y_n \in U_s,\ (x, y_n) \in A$$

and since each section $B(x)$ of B is compact, then $B \subset \tilde{A}$.

Then by the reduction property of the class $\mathbf{\Pi}^1_1$ we can fix a sequence $(C_n)_{n\in\omega}$ of pairwise disjoint $\mathbf{\Pi}^1_1$ sets such that for all n, $C_n \subset (\tilde{X} \times Y) \setminus A_n$ and $\tilde{C} := \bigcup_{n\in\omega} C_n = \bigcup_{n\in\omega} (\tilde{X} \times Y) \setminus A_n = (\tilde{X} \times Y) \setminus \tilde{A}$. So $C'_n := \bigcup_{m\leq n} C_m \subset (\tilde{X} \times Y) \setminus A_n$ and $C''_n := \tilde{C} \setminus C'_n = \bigcup_{m>n} C_m$, hence C'_n is a bianalytic subset of $\tilde{C}$, and $A_n \subset \tilde{C}_n = C''_n \cup \tilde{C}$.

Note that by definition for all $s \in 2^n$:

$$P_s \subset \{x \in \tilde{X} :\ \{x\} \times U_s \subset A_n\} \subset \{x \in \tilde{X} :\ \{x\} \times U_s \subset \tilde{C}_n\} := R_s$$

and R_s is $\mathbf{\Pi}^1_1$. Then again by the reduction property of the class $\mathbf{\Pi}^1_1$ we can find a bianalytic set $P_s \subset Q_s \subset R_s$ such that $R_s \times U_s \subset \tilde{C}_n$.

Hence $B \subset \bigcap_n \bigcup_{s\in S_n} Q_s \times U_s$ and $P = \pi(B) \subset X \cap \bigcap_n \bigcup_{s\in S_n} Q_s$. Conversely if $x \in X \cap \bigcap_n \bigcup_{s\in S_n} Q_s$ then it follows from the compactness of $B(x)$ that $x \in P$, which proves that P is a bianaytic subset of X. Then for any open set V in Y, since $B \cap (X \times V)$ is bianalytic in $X \times Y$, the set $\pi(B \cap (X \times V)$ is a bianaytic subset of X. It follows that the mapping $\Phi : P \to \mathcal{K}(Y)$ is bianaytic, and if $\mathbf{c} : \mathcal{K}(Y) \to Y$ is any Borel choice mapping on $\mathcal{K}(Y)$ (*i.e.* $\mathbf{c}(K) \in K$ for all $K \in \mathcal{K}(Y)$) then $\varphi = \mathbf{c} \circ \Phi$ satisfies the conclusion of Theorem 1.4. □

1.5. *The class $\mathbf{\Sigma}$:* We denote by $\mathbf{\Sigma}$ the smallest class containing both classes $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$, and closed under countable unions and intersections; so $\mathbf{\Sigma} \subset \mathcal{A}(\mathbf{\Pi}^1_1) \cap \check{\mathcal{A}}(\mathbf{\Pi}^1_1)$. We recall the following classical result:

Theorem 1.6. *(*Yankov - von Neumann*) For any $\mathbf{\Sigma}^1_1$ set $A \subset X \times Y$ in a product space, if P is the projection of A on X then there exists a $\mathbf{\Sigma}$-measurable mapping $f : P \to Y$ with graph contained in A.*

We finish with the following two complements to Theorem 1.6.

Lemma 1.7. *Let X and Y be two Polish spaces and $\hat{Y} := Y \cup \{*\}$, where $* \notin Y$. Let D be a countable set and f be a $\mathbf{\Sigma}$-measurable function from X to $\hat{Y}^D$. If for each $x \in X$, the set $\{d \in D : f_d(x) \neq *\}$ is infinite, then there exists a $\mathbf{\Sigma}$-measurable function $\tilde{f} : X \to Y^\omega$ such that for all $x \in X$*

$$\{f_d(x) : d \in D \text{ and } f_d(x) \neq *\} = \{\tilde{f}_n(x) : n \in \omega\}$$

Proof. Enumerate D as a sequence $(d_k)_{k\in\omega}$ and for all $x \in X$ denote D_x the countable infinite set $\{d \in D : f_d(x) \neq *\} = \{d \in D : f_d(x) \in Y\}$. Then take for $\tilde{f}_n(x)$ the n^{th} term of the infinite sequence (with partial domain) $\big(f_{d_k}(x)\big)_{k\in D_x}$. This ensures that

$$Y \supset \{f_d(x) : d \in D \text{ and } f_d(x) \neq *\} = \{f_d(x) : d \in D_x\} = \{\tilde{f}_n(x) : n \in \omega\}$$

To prove that $\tilde{f}$ is $\mathbf{\Sigma}$-measurable we have to show that for all $n \in \omega$ and all open set $V \subset Y$ the set $\tilde{f}_n^{-1}(V)$ belongs to $\mathbf{\Sigma}$. For all $u \in \omega$ the set $L_u = \{x \in X : f_{d_u}(x) \neq *\}$ belongs to $\mathbf{\Sigma}$ and so does $L^V_u = \{x \in L_k : f_{d_u}(x) \in V\}$. Then the set $\{x : \tilde{f}_n(x) \in V\}$ is equal to

$$\bigcup\nolimits_{u_1<u_2<\cdots<u_n=k} \Big(L^V_k \cap \bigcap\nolimits_{i\leq n} L_{u_i} \cap \bigcap\nolimits_{i\in k\setminus\{u_1,u_2,\ldots,u_n\}} (X \setminus L_{u_i})\Big)$$

which belongs to $\mathbf{\Sigma}$. □

Lemma 1.8. *Let X and Y be Polish spaces, $* \notin Y$ and Z be an analytic subset of $X \times Y$. Then there exists a $\mathbf{\Sigma}$-measurable fonction $f = X \to \hat{Y} = Y \cup \{*\}$ such that $(x, f(x)) \in Z$ if x belongs to the projection $\pi(Z)$ of Z on $\mathscr{X}$ and $f(x) = * \iff x \notin \pi(Z)$.*

Proof. It follows from Theorem 1.6 that there is a $\mathbf{\Sigma}$-measurable fonction $f : \pi(Z) \to Y$ such that $f(x) \in \{y \in Y : (x, y) \in Z\}$ whenever x belongs to the analytic set $\pi(Z)$. Then extending f by $*$ on the coanalytic set $X \setminus \pi(Z)$ yields a $\mathbf{\Sigma}$-measurable function from X to $\hat{Y}$. □

2. The arc-connection relation

In this section we present briefly the main notions and known results, which we will use freely in the sequel, concerning the arc-connection relation, namely in the plane. For more details we refer the reader to [3].

2.1. *Arcs:* By an arc I we mean as usual a compact space homeomorphic to the unit interval $\mathbb{I} = [0, 1]$. For an arc I we denote by $e(I)$ the set of its endpoints and set $\overset{\circ}{I} = I \setminus e(I)$. The set $\mathcal{J}(X)$ of all arcs in some space X, viewed as a subset of the space $\mathcal{K}(X)$, is Borel (in fact $\mathbf{\Pi}^0_3 = \boldsymbol{F}_{\sigma\delta}$) and the mapping $e : \mathcal{J}(X)$ to $\mathcal{K}(X)$ is Borel (see [1]).

For practical reasons, we shall consider a singleton $\{a\}$ as a *degenerated arc* with a as a unique endpoint. All notions and notations for arcs extend trivially to degenerated arcs. A set will said to be a *possibly degenerated arc*, and we shall write *p.d. arc*, if it is either an arc or a singleton. We denote by $\hat{\mathcal{J}}(X)$ the set of all p.d. arcs.

If I is an arc, for any $\{a, b\} \subset I$ we denote by $I^{\{a,b\}}$ the sub-arc of I with endpoints $\{a, b\}$. The mapping which to any arc I and any pair $\{a, b\} \subset I$ assigns the arc $I^{\{a,b\}}$, is Borel, since its graph: $\{(I, J) \in \mathcal{J}(X)^2 :\ J \subset I \text{ and } e(J) = \{a, b\}\}$ is Borel.

2.2. *Triods:* A *simple triod* in a space X is a compact subset $T = J_0 \cup J_1 \cup J_2$ of X which is the union of three arcs J_i such that for all $i \neq j$, $J_i \cap J_j = \{c\}$ is a singleton. The arcs J_i, which are uniquely determined up to a permutation, are called the *branches* of T, and c is called the *center* of T.

In this work we shall never consider the more general notion of *triod* introduced initially by Moore in [7], and in the sequel by a *"triod"* we shall always mean a *"simple triod"*.

Since the set $\mathcal{J}(X)$ of all arcs is a Borel subset of $\mathcal{K}(X)$ and the $\cup$ and $\cap$ operations on $\mathcal{K}(X)$ are Borel, it follows from the almost uniqueness of the decomposition of a simple triod, that if X is a Polish space then the set $\mathcal{T}(X)$ of all simple triods in X is a Borel subset of $\mathcal{K}(X)$ and the mapping $\mathbf{c} : \mathcal{T} \to X$, which assigns to any simple triod T its center, is Borel.

2.3. *Arc-components:* We denote by E_X the arc-connection equivalence relation on X. If X is a Polish space then E_X is analytic as the projection on X^2 of the Borel set

$$B = \{\big((x,y), J\big) \in X^2 \times \ \hat{\mathcal{J}}(X) : \ e(J) = \{x,y\}\}.$$

Any arc-component C in a space X is:

– either a singleton,

– or admits a one-to-one continuous parametrization $\varphi : I \to C$ where I is a (closed, open, half-open) interval in $\boldsymbol{R}$ or the unit circle, and we shall then say that C is a *curve*,

– or else contains a *triod* and we shall then say that C is a *triodic component*.

We denote by Θ^X the union of all triodic components of X, to which we will refer as the *triodic part*; and the space X is said to be *atriodic* if $\Theta^X = \emptyset$. If the space X is Polish then the equivalence relation E_X is analytic, hence each triodic component, as well as the triodic part, is analytic. Note that there exist in $\boldsymbol{R}^3$ compact subsets with non Borel arc-components (see [6]).

2.4. *Canonical arc-metrics and arc-topologies:* Given any metric space (X, d) we can consider the mapping $\delta : X \times X \to [0, +\infty]$ defined by

$$\delta(x,y) = \inf\{\operatorname{diam}(H) : \ H \text{ arc-connected s. t. } \{x,y\} \subset H \subset X\}$$

where $\inf \emptyset = \infty$. So if $x \neq y$ are in the same arc-component then

$$d(x,y) \leq \delta(x,y) = \inf\{\operatorname{diam}(J) : \ J \in \mathcal{J}(X) \text{ s.t. } e(J) = \{x,y\}\} < \infty$$

and if not then $\delta(x,y) = \infty$. Moreover setting by convention $\alpha + \infty = \infty$ for any $\alpha \in [0, \infty]$ we have for all $x, y, z \in X$

$$\delta(x,z) \leq \delta(x,y) + \delta(y,z)$$

Hence strictly speaking δ is not a distance on X, but δ induces a distance on any arc-component of X. We shall refer to δ as the *canonical arc-pseudo-metric defined by* (X, d), and to its restriction to any subset A of some arc-component of X as the *canonical arc-metric on A defined by d.*

The set of all δ-open balls with finite radius, constitutes a basis of a metrizable topology τ on X, finer than the initial topology t defined by d. Note that:

$$x = \lim_n x_n \text{ in } (X, \tau) \iff \begin{cases} \exists\, (J_n)_n \text{ in } \mathcal{J}(X,t) : \ \forall n, \ e(J_n) = \{x, x_n\} \\ \text{and } \{x\} = \lim_n J_n \text{ in } \mathcal{K}(X,t) \end{cases}$$

so the topology τ can in fact be defined directly from the topology t, and we shall say that τ is the *canonical arc-topology* defined by t.

We emphasize that even if the initial topology t is separable the topology τ is not in general. For example if we fix in the unit circle in $\boldsymbol{R}^2$ a copy C of the Cantor space then the union of all rays joining the center to an element of C, is an arc-connected compact space X. But if δ denotes the canonical arc-metric on X defined by the euclidean metric then for any two distinct element $a \neq b$ in C, $\delta(a,b) \geq 1$.

The canonical pseudo-metric δ was introduced in [3] for a Polish planar space X. But as the reader can easily check the following properties extracted from [3], that we state without proof, do not rely on this additional assumption.

Theorem 2.5. *Let δ be the canonical arc-pseudo-metric defined by the metric space (X, d), and let t and τ be respectively the d-topology and δ-topology on X.*

1. *If (X, d) is complete then (X, δ) is complete*
2. *$\mathcal{J}(X, t) = \mathcal{J}(X, \tau)$,*
3. *For any arc $J \subset X$, d-$\mathrm{diam}(J) = \delta$-$\mathrm{diam}(J)$*
4. *For any arc $J \subset X$, $(J, t) = (J, \tau)$*
5. *All open δ-balls with finite radius are arc-connected,*

Note that by property (2) the arc-connected subsets of (X, d) and (X, δ) are the same, so the reference to arcs and arc-connectedness in the following properties is non ambiguous.

The plane arc-connection relation: All specific properties of the arc-connection relation in the plane are due to the following fundamental property of the plane topology.

Theorem 2.6. (Moore) *Any family of pairwise disjoint triods in the plane is countable.*

In particular any planar set admits at most countably many triodic components.

Definition 2.7. *If τ is the canonical arc-topology of some space (X, t), the* triodic kernel *of X, that we denote by Σ^X, is the τ-closure of the set of all centers of triods in (X, t).*

The following theorem is a synthetic summary of the main results of [3].

Theorem 2.8. *Let (X, t) be a planar Polish space, and let τ be the corresponding arc-topology.*

a) Σ^X is τ-separable, hence (Σ^X, τ) is a Polish space, and Σ^X is a Borel subset of (X, t).

b) The set $B = \{(x, J) \in X \times \mathcal{J}(X) : \ J \subset X,\ e(J) = \{x, y\} \text{ and } J \cap \Sigma^X = \{y\}\}$ is Borel and for all $x \in X \setminus \Sigma^X$, $\mathrm{card}(B(x)) \leq 2$.

c) The equivalence relation E_X is Borel.

Remark 2.9. Note that the projection on X of the set B in property b) is the set $\Sigma' = \Theta^X \setminus \Sigma^X$. Since B is a Borel set with finite sections then it admits a Borel uniformization. Hence there exist Borel mappings $\Psi : \Sigma' \to \mathcal{J}(X)$ and $\psi : \Sigma' \to \Sigma^X$ such that for all $x \in \Sigma'$, $e(\Psi(x)) = \{x, \psi(x)\}$ and $\Psi(x) \cap \Sigma^X = \{\psi(x)\}$.

3. The partial operation $\bigvee$ on oriented arcs.

3.1. *Oriented arcs:* An oriented arc is a triple $\vec{I} = (I, a, b)$ where I (the *domain* of $\vec{I}$) is an arc and the two elements set $e(I) = \{a, b\}$ is ordered by the pair (a, b). We shall then say that I is an *arc joining a to b* and set:

$$dom\,(\vec{I}) = I \ ; \ e(\vec{I}) = (a, b) \ ; \ a = e_0(\vec{I}) \text{ and } b = e_1(\vec{I})$$

so $e(I)$ is a two elements subset of X, while $e(\vec{I}) \in X^2$. We will denote by $\mathfrak{f}$ the flip operation which assigns to any oriented arc $\vec{I} = (I, a, b)$ the arc $\mathfrak{f}(\vec{I}) = (I, b, a)$.

Given any oriented arc $\vec{I} = (I, a, b)$ the relation on I defined by:

$$x \leq_{\vec{I}} y \iff I^{\{a,x\}} \subset I^{\{a,y\}} \iff I^{\{y,b\}} \subset I^{\{x,b\}}$$

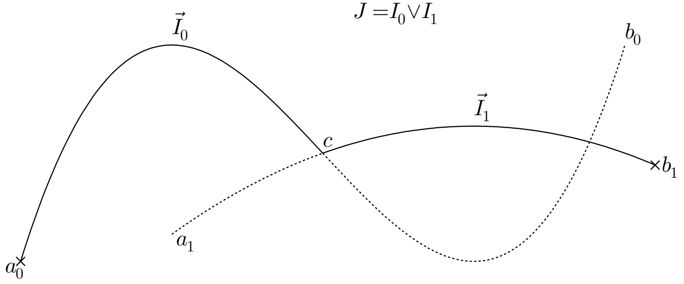
is a total ordering and we denote by $<_{\vec{I}}$ the corresponding strict ordering, to which we refer as *the* total order on I defined by $\vec{I}$. In the sequel the notation $\vec{I}$ will always suppose implicitly that $dom\,(\vec{I}) = I$. Also for simplicity, when there is no ambiguity on the orientation of I we shall set $e_0(I) = e_0(\vec{I})$, $e_1(I) = e_1(\vec{I})$, and denote by $<_I$ the total order on I defined by $\vec{I}$.

The set $\vec{\mathcal{J}}(X) = \{(I,a,b) \in \mathcal{J}(X) \times X^2 : \ e(I) = \{a,b\}\}$ of all oriented arcs is clearly Borel. Hence the set $\{(\vec{I},x,y) \in \vec{\mathcal{J}}(X) \times X^2 : \ x <_{\vec{I}} y\}$ is Borel too, and the flip operation $\mathfrak{f}$ is Borel.

Definition 3.2. *Given two oriented arcs* $\vec{I_0} = (I_0, a_0, b_0)$ *and* $\vec{I_1} = (I_1, a_1, b_1)$ *such that* $I_0 \cap I_1 \neq \emptyset$ *and* $a_0 \neq b_1$ *we define the oriented arc:*

$$\vec{I_0} \vee \vec{I_1} = (I, a_0, b_1) \text{ with } \begin{cases} I = I_0^{\{a_0,c\}} \cup I_1^{\{c,b_1\}} \\ \text{and} \\ c = \min_{<_{\vec{I_0}}}(I_0 \cap I_1) \end{cases}$$

Figure 1



It follows readily from the previous definition that if $\vec{I_0} \vee \vec{I_1}$ is defined then

$$\mathrm{dom}(\vec{I_0} \vee \vec{I_1}) \subset I_0 \cup I_1 \quad \text{and} \quad e(\vec{I_0} \vee \vec{I_1}) = (a_0, b_1)$$

Definition 3.3. *a) For any finite sequence* $(\vec{I_m})_{m \leq n}$ *of oriented arcs, we define inductively the oriented arcs:*

$$\vec{J_0} = \bigvee_0 \vec{I_0} = \vec{I_0} \quad \text{and} \quad \vec{J_n} = \bigvee_{m \leq n} \vec{I_m} = \vec{J}_{n-1} \vee \vec{I_n}.$$

b) If $(\vec{I_n})_{n \in \omega}$ *is an infinite sequence in* $\vec{\mathcal{J}}(X)$*, we shall say that* $\vec{J} = \bigvee_{n \in \omega} \vec{I_n}$ *is defined if for all* n*,* $\vec{J_n} = \bigvee_{m \leq n} \vec{I_m}$ *is defined, and* $\vec{J} = \lim \vec{J_n}$*.*

Remarks 3.4. a) We emphasize that $\bigvee$ is a *partial operation*, so $\vec{J_n} = \bigvee_{m \leq n} \vec{I_m}$ is defined only if all the terms in its definition are defined, that is if for all $0 < m \leq n$:

$$J_{m-1} \cap I_m \neq \emptyset \quad \text{and} \quad e_1(\vec{J_m}) \neq e_0(\vec{J_0})$$

and if so then:

$$J_n \subset \bigcup_{m \leq n} I_m \qquad \text{and} \quad e(\vec{J_n}) = (e_0(\vec{I_0}), e_1(\vec{I_n}))$$

b) If for all $0 < m \leq n$, $e_0(\vec{I_m}) = e_1(\vec{I}_{m-1}) \neq e_0(\vec{J_0})$ then $e_1(\vec{I}_{m-1}) \neq e_0(\vec{J_m}) = e_0(\vec{J_0})$ and $e_1(\vec{J}_{m-1}) = e_0(\vec{I_m}) \in J_{m-1} \cap I_m$, hence $\vec{J_n} = \bigvee_{m \leq n} \vec{I_m}$ is defined.

Lemma 3.5. *The set of all finite or infinite sequences* $(\vec{I_n})_{n<N}$ *in* $\vec{\mathcal{J}}(X)$ *such that* $\vec{J_N} = \bigvee_{n<N} \vec{I_n}$ *is defined, is Borel, and the mapping which assigns* $\vec{J_N}$ *to* $(\vec{I_n})_{n<N}$ *is Borel.*

Proof. It follows from its definition that the set $\mathcal{D}_2$ of all pairs $(\vec{I}_0, \vec{I}_1) \in \big(\vec{\mathcal{J}}(X)\big)^2$ such that $\vec{I}_0 \vee \vec{I}_1$ is defined, is Borel. Since the $\cap$ an $\cup$ operations in $\mathcal{K}(X)$, and the mapping $(I, a, b) \mapsto I^{\{a,b\}}$ are Borel, then the mapping $(\vec{I}_0, \vec{I}_1) \mapsto \vec{I}_0 \vee \vec{I}_1$ is Borel on $\mathcal{D}_2$. This proves the lemma for $N = 2$, and the general finite case follows by a straightforward induction; and the infinite case follows from the definition. □

3.6. *Subdivisions:* Let $\vec{J} = (J, a, b)$ be an oriented arc and let $<$ be the total order on J defined by $\vec{J}$. A *subdivison* of $\vec{J}$ is a finite sequence $(\vec{J}^k)_{0 \le k \le \ell}$ in $\vec{\mathcal{J}}(J)$ such that :

$$a = e_0(J^0) < e_1(J^0) = e_0(J^1) < \cdots < e_1(J^{k-1}) = e_0(J^k) < \cdots < e_1(J^{\ell-1}) = e_0(J^\ell) < e_1(J^\ell) = b$$

Theorem 3.7. *Let X be a space and $(\vec{I}_n)_{n\in\omega}$ be a sequence in $\vec{\mathcal{J}}(X)$ satisfying :*

a) for all n, $\vec{J}_n = \bigvee_{m\le n} \vec{I}_m$ is defined,

b) the sequence $(I_n)_{n\in\omega}$ converges in $\mathcal{K}(X)$ to a singleton $\{b\}$ with $b \ne a = e_0(I_0)$.

Then $\vec{J} = \bigvee_{n\in\omega} \vec{I}_n$ is defined and $e(\vec{J}) = (a, b)$.

Proof. Set for all n, $\vec{I}_n = (I_n, a_n, b_n)$ and $\vec{J}_n = (J_n, a, b_n)$, so $b_n \ne a$. For all $s, t \in \omega^{<\omega}$, we denote by $s \wedge t$ the largest initial segment of $s \cap t$, and if $s \ne \emptyset$ we set: $s^* = s_{||s|-1}$.

Then starting from $s_0 = \langle 0\rangle$ and $\hat{s}_0 = \langle J_0\rangle = \langle I_0\rangle$, we construct inductively two sequences $(s_n)_{n\in\omega}$ and $(\hat{s}_n)_{n\in\omega}$ in $\omega^{<\omega} \setminus \{\emptyset\}$ and $\mathcal{J}(X)$ respectively such that for all n, $|\hat{s}_n| = |s_n|$ and $\hat{s}_n = (J_n^j)_{j<|s_n|}$ is a subdivision of $\vec{J}_n$, and for all $m < n$:

(1) s_n is an increasing sequence of integers $\le n$ with $s_n(0) = 0$,
(2) for all $j < |s_n|$, $J_n^j \subset I_{s_n(j)}$,
(3) $s_n \preceq s_{n-1}$ or $s_n = s^\frown\langle n\rangle$ with $s \preceq s_{n-1}$,
(4) if $|s_m \wedge s_n| = k + 1$ then for all $j < k$, $J_n^j = J_m^j$, and $J_n^k \subset J_m^k$

Suppose that $(J_n^k)_{k<|s_n|}$ is already defined satisfying conditions (1) to (4). Then by definition $J_{n+1} = J_n^{\{a,c\}} \cup I_{n+1}^{\{c,b_{n+1}\}}$ with $c \in J_n \cap I_{n+1}$. Let

$$k = \min\{-1 \le j < |s_n| :\ c \le_{J_n} e_1(J_n^j)\} < |s_n|\}$$

with the convention $e_1(J_n^{-1}) = e_0(J_n^0) = a$, so $k = -1$ if $c = a$. We then distinguish two cases:

(i) if $c = b_{n+1}$ then $J_{n+1} = J_n^{\{a,b_{n+1}\}} = J_n^{\{a,a_n^k\}} \cup J_n^{\{a_n^k,b_{n+1}\}} = \big(\bigcup_{j<k} J_n^j\big) \cup J_n^{\{a_n^k,b_{n+1}\}}$

and we set: $s_{n+1} = s_{n|k+1}$ and $J_{n+1}^j = \begin{cases} J_n^j & \text{if } j < k \\ J_n^{\{a_n^k,b_{n+1}\}} & \text{if } j = k \end{cases}$

(ii) if $c \ne b_{n+1}$ then $J_{n+1} = J_n^{\{a,c\}} \cup I_{n+1}^{\{c,b_{n+1}\}} = \big(\bigcup_{j<k} J_n^j\big) \cup J_n^{\{a_n^k,c\}} \cup I_{n+1}^{\{c,b_{n+1}\}}$

and we set: $s_{n+1} = s_{n|k+1}{}^\frown\langle n+1\rangle$ and $J_{n+1}^j = \begin{cases} J_n^j & \text{if } j < k \\ J_n^{\{a_n^k,c\}} & \text{if } j = k \\ I_{n+1}^{\{c,b_{n+1}\}} & \text{if } j = k+1 \end{cases}$

In particular if $c = a$ then $k = -1$, so $s_{n|k+1} = \emptyset$ and $s_{n+1} = \langle n+1\rangle$ is of length 1, and $J_{n+1}^0 = I_{n+1}^{\{a,b_{n+1}\}}$ is the unique element of $\hat{s}_{n+1}$. This finishes the definition of s_{n+1} and $\hat{s}_{n+1}$ which clearly satisfy conditions (1), (2), (3); and we now prove condition (4).

So suppose that $m < n+1$ and $u = s_m \wedge s_{n+1}$, then by condition (3) $u \preceq s_n$ hence $u \preceq s_m \wedge s_n$. If $m < n$ then (4) follows from the induction hypothesis; and if $m = n$ then $u \preceq s_n$ and (4) follows from the definition of s_{n+1}.

This ends up the construction of the sequences $(s_n)_{n\in\omega}$ and $(\hat{s}_n)_{n\in\omega}$, and we set $s_{-1} = \hat{s}_{-1} = \emptyset$.

Lemma 3.8. *For all n:*

(5) *if* $\ell < m < n$ *and* $s_\ell \preceq s_n$ *then* $s_\ell \preceq s_m$
(6) *if* $\ell < n$ *and* $s_\ell \prec t \prec s_n$ *then there exists* m *such that* $\ell < m < n$ *and* $t = s_m$

Proof. The proof is by induction on n. For $n = -1$ the statements are trivial. So suppose they are true for n.

Suppose that $\ell < m < n+1$ and $s_\ell \preceq s_{n+1}$. Since $\ell < n+1$ then necessarily $s_\ell \preceq s^*_{n+1} \preceq s_n$. If $m < n$ then by the induction hypothesis $s_\ell \preceq s_m$; and if $m = n$ then $s_\ell \preceq s_n = s_m$. This proves (5) for $n+1$.

Suppose that $\ell < n+1$ and $s_\ell \prec t \prec s_{n+1}$. Since $t \neq s_{n+1}$ then $t \preceq s^*_{n+1} \preceq s_n$. Then either $t = s_n$ and we are done; or else $t \prec s_n$ and then by the induction hypothesis there exists m such that $\ell < m < n$ and $t = s_m$. This proves (6) for $n+1$. ☐

Lemma 3.9. *One the following two alternatives holds:*
(I) *There exists* $m \geq -1$ *such that the set* $N_m = \{n > m : \ s_n = s_m \text{ or } s^*_n = s_m\}$ *is infinite,*
(II) *There exists an increasing sequence* $(n_i)_{i \geq -1}$ *such that for all* i*, and all* $n \geq n_i$*,* $s_{n_i} \preceq s_n$.

Proof. Note that by condition (6) the set $S = \{s_n; \ n \geq -1\}$ is a tree, and consider the set $S' = \{s_m \in S : \ \forall n > m, \ s_n \succeq s_m\}$ which is nonempty since $\emptyset = s_{-1} \in S'$. It follows from condition (5) that if $\ell < m$ and s_ℓ, s_m are in S' then $s_\ell \preceq s_m$, hence S' is a $\preceq$-chain.

– If S' is finite, Let s be the $\preceq$-maximum of S' and $m = \min\{\ell : \ s_\ell = s\}$. Then for all $n > m$, $s_m \preceq s_n$, hence either $s_n = s_m$, or $s^*_n = s_m$, or $|s_n| > |s_m| + 1$ and then by condition (6) (applied to $t = s_n||s_m|+1$) there exists n' such that $m < n' < n$ and $t = s_{n'}$ hence $s^*_{n'} = s_m$; and it follows that alternative (I) holds.

– If S' is infinite then there exists a unique increasing sequence $(n_i)_{i\in\omega}$ in ω such that $S' \setminus \emptyset = \{s_{n_i}; \ i \in \omega\}$ and alternative (II) clearly holds. ☐

For the rest of the proof of Theorem 3.7 we distinguish two cases according to Lemma 3.9. Also for more clarity we shall split alternative (I) into two sub-alternatives.

Case (I.a): There exists m *such that the set* $N = \{n \in \omega : \ s_n = s_m\}$ *is infinite*

Set $|s_m| = k+1$; then $H = \bigcup_{j<k} J^j_m$ is an arc with $e(H) = \{a, b^{k-1}_m\}$ and for all $n \in N$, $J_n = H \cup J^k_n$, with $e(J^k_n) = \{a^k_n, b^k_n\} = \{b^{k-1}_m, b^k_n\}$ and $J^k_n \supset J^k_{n+1}$, hence $J' = \bigcap_{n\geq m} J^k_n$ is an arc with $e(J') = \{b^{k-1}_m, b\}$ and $J = \lim_{n\in N} J_n = H \cup J'$ exists and is an arc with $e(J) = \{a, b\}$.

Let $(n_i)_{i\in\omega}$ be the increasing enumeration of N. If $n_i \leq n < n_{i+1}$ then by definition of N we have $s_{n_i} \preceq s_n$; hence by condition (4) $J_{n_i} \subset J_n$ and by condition (2)

$$J_n \setminus J_{n_i} \subset \bigcup \{I_{s_n(j)} : \ |s_{n_i}| \leq j < |s_n|\}\,.$$

If $k_i = |s_{n_i}| - 1$ then $p_i = s_n(k_i) = s_{n_i}(k_i) \nearrow \infty$, and since $\lim I_n$ is a singleton then $\operatorname{diam}(J_n \setminus J_{n_i}) \leq \operatorname{diam}(\bigcup_{p\geq p_i} I_p) \searrow 0$. Moreover since $e_0(I_{p_i}) = e_0(I_{s_{n_i(k_i)}}) = e_0(J^{k_i}_{n_i})$ then $I_{p_i} \cap J_{n_i} \neq \emptyset$. Hence if d_H is the Hausdorff distance on $\mathcal{K}(X)$ associated to some compatible distance on X, then $d_H(J_n, J_{n_i})) = \operatorname{diam}(J_n \setminus J_{n_i}) \searrow 0$, which proves that $\lim_n J_n = \lim_i J_{n_i} = J$.

Case (I.b): There exists m *such that the set* $N^* = \{n \in \omega : \ s_n = s_m{}^\frown\langle n\rangle\}$ *is infinite*

The argument is essentially the same. Setting $|s_m| = k+1$ and $H = \bigcup_{j<k} J^j_m$ as in (I.a) observe that if $n \in N^*$, since $s_n = s_m{}^\frown\langle n\rangle$, then $|s_m| = k+2$ and $J_n = H \cup J^k_n \cup J^{k+1}_n$. By the same arguments as in (I.a), $J = \lim_{n\in N^*} H \cup J^k_n$ is an arc with $e(J) = \{a, b\}$. But by condition (2) the additional term J^{k+1}_n is a subset of $I_{s_n(k+1)}$ which by hypothesis b) of the theorem converges to the singleton $\{b\}$; and it follows that $J = \lim_{n\in N^*} J_n$. The rest of the argument is exactly the same as in (I.a)

Case II: S' *is infinite.*

Set $S' \setminus \{\emptyset\} = \{s_n; n \in M\}$ and for all $n \in M$, $|s_n| = k_n + 1$ and $H_n = \bigcup_{j<k_n} J_n^j \subset J_n$. By condition (4) $(H_n)_{n\in M}$ is an increasing sequence of arcs with $e(H_n) = \{a, c_n\}$ and $c_n = b_n^{k_n-1} = a_n^{k_n} \in I_{s_n(k_n)}$. Hence $\lim_{n\in M} c_n = b$ and $J = \overline{\bigcup_{n\in M} H_n} = \lim_{n\in M} H_n$ exists and is an arc with $e(J) = \{a, b\}$. Finally as in Case (I.a) if $(m_i)_{i\in\omega}$ is the increasing enumeration of M then it follows from condition (2) that for all $m = m_i \leq n < m_{i+1}$, if $p_m = s_m(|s_m| - 1)$ then $d_H(J_n, H_{m_i}) = \operatorname{diam}(J_n \setminus H_{m_i}) \searrow 0$ which proves that $\lim_n J_n = J$. □

Remark 3.10. Let $(I_n)_{n\in\omega}$ be an infinite sequence in $\mathcal{J}(X)$ such that :

a) for all n, $e(I_n) = \{a_n, a_{n+1}\}$ for some sequence $(a_n)_{n\in\omega}$ in X.

b) the sequence $(I_n)_{n\in\omega}$ converges in $\mathcal{K}(X)$ to a singleton $\{b\}$ with $b \neq a_0$.

Then by Remark 3.4.b) for all n, $\bigvee_{m\leq n} \vec{I}_m$ is defined, hence by Theorem 3.7 the oriented arc $\bigvee_{n\in\omega}(I_n, a_n, a_{n+1}) = (J, a_0, b)$ is defined; and we shall write $\bigvee_{n\in\omega} I_n = J$.

Moreover by Proposition 3.5 the mapping which to any such sequence $(I_n)_{n\in\omega}$ assigns the arc $\bigvee_{n\in\omega} I_n$ is Borel.

4. ARC-LIFTING

We recall that E_X denotes the arc-connection equivalence relation on a given space X.

Definition 4.1. *Given any subset S of some space X, an* arc-lifting of S in X *is a mapping $\psi : S^2 \cap E_X \to \hat{\mathcal{J}}(X)$ such that for all $(x, y) \in S^2 \cap E_X$, $e(\psi(x, y)) = \{x, y\}$. If $S = X$ we shall say that ψ is an* arc-lifting of X.

If X is Polish then the equivalence relation E_X is analytic, and it follows then from Theorem 1.6 that X admits a $\mathbf{\Sigma}$-measurable arc-lifting. But as we shall see next the existence of a Borel arc-lifting for a space is a very strong assumption.

Proposition 4.2. *If a Polish space X admits a Borel arc-lifting then the equivalence relation E_X is Borel.*

Proof. Suppose that $\psi : E_X \to \hat{\mathcal{J}}(X)$ is a Borel arc-lifting. Since E_X and $\hat{\mathcal{J}}(X)$ are Borel then by classical results ψ admits a Borel extension $\tilde{\psi} : \mathcal{D} \to \hat{\mathcal{J}}(X)$ to a Borel domain $E_X \subset \mathcal{D} \subset X^2$ such that for all $(x, y) \in \mathcal{D}$, $e(\tilde{\psi}(x, y)) = \{x, y\}$; hence necessarily $E_X = \mathcal{D}$, and so E_X is Borel. □

We recall (see Theorem 2.8) that if $X \subset \boldsymbol{R}^2$ then E_X is Borel, but there are compact spaces in $\boldsymbol{R}^3$ with non Borel arc-components, hence which do not admit a Borel arc-lifting.

Definition 4.3. *Given any subset S of some space X, and any element $a \in X$, a* $\star$-arc-lifting of S in X, with summit a, *is a mapping $\varphi : S \to \hat{\mathcal{J}}(X)$ such that for all $x \in S$, $e(\varphi(x)) = \{a, x\}$.*

Note that if S admits a $\star$-arc-lifting with summit a then a is not necessarily an element of S, but $S \cup \{a\}$ is a subset of some arc-component of X.

Remark 4.4. Let Θ denote the partial mapping which to any pair $(J_0, J_1) \in \mathcal{J}(X)$ such that $e(J_0) = \{a, x\}$ and $e(J_1) = \{a, y\}$ with $x \neq y$, associates the arc J defined by $(J, x, y) = (J_0, x, a) \vee (J_1, a, y)$. Then for any $\star$-arc-lifting $\varphi : S \to \hat{\mathcal{J}}(X)$ of S in X, with summit a, and $x \neq y$ in S, the arc

$$J = \psi(x, y) = \begin{cases} \Theta(\varphi(x), \varphi(y)) & \text{if } x \neq a \text{ and } y \neq a \\ \varphi(x) & \text{if } x = a \\ \varphi(y) & \text{if } y = a \end{cases}$$

is well defined and $e(J) = \{x, y\}$. The mapping $\psi : S^2 \to \mathcal{J}(X)$ thus defined is clearly an arc-lifting of S in X. Moreover it follows from Lemma 3.5 that if φ is Borel then ψ is Borel too. We shall refer to ψ as *the canonical arc-lifting* defined by φ.

Theorem 4.5. *Let (M, δ) be a complete, arc-connected and locally arc-connected, metric space. Then any separable subset S of M admits a Borel arc-lifting in some separable subspace $N \supset S$.*

Proof. Fixing an element $a \in S$ we shall construct a $\star$-arc-lifting of summit a, of S. Since the set of isolated points in S is countable we may suppose that S has no isolated points. Also replacing S by $\overline{S}$ we may suppose that S is closed.

Since S is separable we can fix a countable family $\mathcal{U}$ of open arc-connected subsets of M whose restriction to S form a basis for the topology of S. Then starting from $U_\emptyset = X$ we can construct a family $(U_s)_{s\in\omega^{<\omega}}$ in $\mathcal{U}$, satisfying for all $s \in \omega^{<\omega}$ and all $n \in \omega$:

(1) $S \cap U_s \neq \emptyset$, $\operatorname{diam}(U_s) < 2^{-|s|}$ and $\overline{U_{s\frown\langle n\rangle}} \subset U_s$,
(2) $S \cap U_s \subset \bigcup_{n\in\omega} U_{s\frown\langle n\rangle}$.

Then for all $\sigma \in \omega^{<\omega}$, $\bigcap_{n\in\omega} U_{\sigma|n}$ is a singleton $\{\varphi(\sigma)\} \subset S$ and the mapping $\psi : \omega^\omega \to S$ thus defined is onto, continuous and open, hence admits a Borel section $\varphi : S \to \omega^\omega$.

Set $a_\emptyset = a$. Since S has no isolated points, then for all $s \in T \setminus \{\emptyset\}$ we can fix an element $a_s \in S \cap U_s \setminus \{a_{s_{|k}};\ k < |s|\}$. So if $s^* = s_{||s|-1}$ then by construction $a_s \neq a_{s^*}$ and $\{a_{s^*}, a_s\} \subset U_s$; and since U_s is arc-connected we can fix an arc $I_s \subset U_s$ such that $e(I_s) = \{a_{s^*}, a_s\}$, hence $\operatorname{diam}(I_s) < 2^{-|s|}$. Then for any $\sigma \in \omega^{<\omega}$, $a_\sigma = \lim_k a_{\sigma_{|k}}$ exists, and if $a_\sigma \neq a$ then (see Remark 3.10) the arc $\bigvee_{k\in\omega} I_{\sigma_{|k}} = (J_\sigma, a, a_\sigma)$ is defined, and the mapping $\sigma \mapsto J_\sigma$ is Borel.

Hence the mapping $\Phi : S \to \hat{\mathcal{J}}(X)$ defined by $\Phi(x) = J_{\varphi(x)}$ if $x \neq a$ and $\Phi(a) = \{a\}$ is a Borel $\star$-arc-lifting with summit a, of S in the separable space $N = S \bigcup_{s\in\omega^{<\omega}} I_s$, so by Remark 4.4, S admits a Borel arc-lifting in N. in X. □

Corollary 4.6. *Any locally connected Polish space admits a Borel arc-lifting.*

Proof. It is well known that any locally connected Polish space is locally arc-connected. Hence X admits countably many arc-components $\{C_n;\ n \in \omega\}$ and each C_n is a clopen subset of X, so a Polish space. Hence by Theorem 4.5 each C_n admits a Borel arc-lifting Ψ_n in X, and then $\Psi = \bigcup_{n\in\omega} \Psi_n$ is clearly a Borel arc-lifting of X. □

Corollary 4.7. *Let (X, t) be a Polish space, and let τ be the arc-topology defined by (X, t). Then any τ-separable subset S of some arc-component C of X admits a Borel arc-lifting in X.*

Proof. Fix a complete distance d on X compatible with t. Then the topology τ can be defined by the pseudo-arc-metric δ defined by (X, d), and δ is a metric on C of X. Then (C, δ) is complete, arc-connected and locally arc-connected, metric space, hence by Theorem 4.5 there exists a τ separable space $N \supset S$ and a Borel arc-lifting mapping from (C, τ) to $\hat{\mathcal{J}}(N, \tau)$. Then since C is t-Borel and the identity mapping from $\hat{\mathcal{J}}(N, \tau)$ to $\hat{\mathcal{J}}(N, t)$ is continuous, then Φ induces a Borel arc-lifting of $S = (S, t)$ in X. □

Theorem 4.8. *Any planar Polish space X admits a Borel arc-lifting.*

Proof. Let d be a compatible complete distance on X. We denote by Y and Z respectively the atriodic part and the triodic part of X. Since X is a plane subset then Z admits at most countably many arc-components $(Z_n)_{n\in N}$, and since the equivalence relation E_X is Borel (Theorem 2.8.c)) then all the Z_n's as well as Y are Borel subsets of X. Note that

$$E_X = E_Y \cup \bigcup_{n\in N} E_{Z_n} = E_Y \cup \bigcup_{n\in N} Z_n^2$$

and we shall define Borel arc-liftings in X separately for Y and for each Z_n .

For Y this is straightforward: Since the set

$$B = \{(x, y, J) \in Y^2 \times \hat{\mathcal{J}}(X) :\ J \subset X \text{ and } e(J) = \{x, y\}\}$$

is Borel and for all $(x,y) \in Y^2 \cap E_X$ the section $B(x,y) = \{J_{(x,y)}\}$ is a singleton, then the mapping $(x,y) \to J_{(x,y)}$ is a Borel arc-lifting of Y in X.

We now fix n, set $C = Z_n$ and $S = \Sigma^X \cap Z_n$, and fix some element $a \in S$. To construct an arc-lifting of C in X, it is enough to construct a $\star$-arc-lifting $\Phi : C \to \mathcal{J}(X)$ of C in X, and again we shall define Φ separately on S and $C \setminus S$.

By Theorem 2.8.a) the set S is τ-separable, then by Corollary 4.7 we get an arc-lifting $\Psi_S : S^2 \to \mathcal{J}(X)$ of S in X. In particular $\Phi_S = \Psi_S(a,\cdot) : S \to \mathcal{J}(X)$ is a $\star$-arc-lifting with summit a of S in X, and we now construct a $\star$-arc-lifting $\Phi_{S'}$ with summit a of $S' = C \setminus S$ in X.

By Remark 2.9 we can fix two Borel mappings $\Psi : S' \to \mathcal{J}(X)$ and $\psi : S' \to S$ such that for all $x \in S'$, if $J = \Psi(x)$ then $e(J) = \{x, \psi(x)\}$ and $J \cap S = \{\psi(x)\}$. Then for all $x \in S'$, $x \neq a$, and since $e(\Phi_S(\psi(x)) = \{a, \psi(x)\}$ then $\psi(x) \in \Psi(x) \cap \Phi_S(\psi(x))$, hence the oriented arc $(\Psi(x), x, \psi(x)) \vee (\Phi_S(\psi(x)), \psi(x), a)$ is defined, and is of the form $(\Phi_{S'}(x), x, a)$. Then by Lemma 3.5 the mapping $\Phi_{S'} : S' \to \mathcal{J}(X)$ thus defined is Borel and by construction $\Phi_{S'}$ is a $\star$-arc-lifting with summit a of S' in X. □

5. Uniform arc-liftings

Definition 5.1. *Given a space X and $\mathcal{G} \subset \mathcal{F}(X)$ let $E_{\mathcal{G}} = \{(F,x,y) \in \mathcal{G} \times X^2 : (x,y) \in E_F\}$. A* uniform arc-lifting *of $\mathcal{G}$ is a mapping $\Psi : E_{\mathcal{G}} \to \hat{\mathcal{J}}(X)$ such that for any $F \in \mathcal{G}$, the partial mapping $\Psi(F,\cdot) : E_F \to \hat{\mathcal{J}}(X)$ is an arc-lifting of F (in F).*

As we shall see the complexity of the set $\mathcal{C}_{\mathrm{arc}}(X)$ is intimately related to the complexity of a potential uniform arc-lifting of $\mathcal{C}_{\mathrm{arc}}(X)$. Note that if X is a Polish space and $\mathcal{G}$ is an analytic subset of $\mathcal{F}(X)$ then the set $E_{\mathcal{G}}$ is analytic, hence the set $\mathcal{B} = \{\big((F,x,y), J\big) \in E_{\mathcal{G}} \times \mathcal{J}(X) : J \subset F \text{ and } e(J) = \{x,y\}\}$ is analytic too. It follows then from Theorem 1.6 that $\mathcal{G}$ admits a $\mathbf{\Sigma}$-measurable uniform arc-lifting. If moreover $X \subset \boldsymbol{R}^2$ then by Theorem 4.8 any $F \in \mathcal{G}$, admits a Borel arc-lifting, and it is natural to ask whether in this context, $\mathcal{G}$ admits a Borel uniform arc-lifting. As we shall see if the set $\mathcal{G}$ is rich enough then even the existence of a bianalytic uniform arc-lifting is a very strong requirement.

Proposition 5.2. *For any Polish space X and all $n \geq 1$, the set*

$$\mathcal{C}^{[n]}_{\mathrm{arc}}(X) = \big\{F \in \mathcal{F}(X) : \ \forall\, x,y \in F, \ \exists^{\leq n} J \in \mathcal{J}(X) : \ J \subset F \text{ and } e(J) = \{x,y\}\big\}$$

admits a bianalytic uniform arc-lifting.

Proof. Note that the set $\mathcal{B} = \{(F,x,y,J) \in \mathcal{F}(X) \times X^2 \times \mathcal{J}(X) : \ J \subset F \text{ and } e(J) = \{x,y\}\}$ is Borel hence by Proposition 1.4 the set

$$\mathcal{C} = \{(F,x,y) \in \mathcal{F}(X) \times X^2 : \ \exists^{\leq n} J \in \mathcal{J}(X) : \ (F,x,y,J) \in \mathcal{B}\}$$

is $\mathbf{\Pi}^1_1$ and $\mathcal{B} \cap (\mathcal{C} \times \mathcal{J}(X))$ is the graph of a bianalytic mapping Φ on $\mathcal{C}$, which to any $F,x,y) \in \boldsymbol{C}$ assigns a nonempty finite set $\{J_1, \cdots, J_n\}$ in $\mathcal{J}(X)$ with graph contained in $\mathcal{B}$. Then any bianalytic section $\Psi : \mathcal{C} \to \mathcal{J}(X)$ of Φ, given by Proposition 1.4, is a uniform arc-lifting for $\mathcal{C}^{[n]}_{\mathrm{arc}}(X)$. □

Proposition 5.3. *Suppose that $\mathcal{G} \subset \mathcal{C}_{\mathrm{arc}}(X)$ is in $\mathbf{\Pi}^1_1(\mathcal{C}_{\mathrm{arc}}(X))$, that is $\mathcal{G} = \mathcal{H} \cap \mathcal{C}_{\mathrm{arc}}(X)$ where $\mathcal{H}$ is $\mathbf{\Pi}^1_1$. If $\mathcal{G}$ admits a bianalytic uniform arc-lifting then $\mathcal{G}$ is $\mathbf{\Pi}^1_1$.*

Proof. Let $\Psi : E_{\mathcal{G}} \to \hat{\mathcal{J}}(X)$ be a bianalytic uniform arc-lifting. Since the condition "$J \subset F$ and $e(J) = \{x,y\}$" is Borel, then by Proposition 1.3 Ψ admits an extension $\tilde{\Psi}$ to a bianalytic mapping with $\mathbf{\Pi}^1_1$ domain $\mathcal{D} \subset \mathcal{H} \times X^2$ and such that if $J = \tilde{\Psi}(F,x,y)$ then $J \subset F$ and $e(J) = \{x,y\}$. Then the set $\mathcal{C}_* = \{F \in \mathcal{H} : \ \forall\, x,y \in F, \ (F,x,y) \in \mathcal{D}\}$ is $\mathbf{\Pi}^1_1$. If $F \in \mathcal{G}$ then

$F \in \mathcal{H}$ and $E_F = F^2$ hence $\{F\} \times F^2 \subset E_{\mathcal{G}} \cap (\mathcal{H} \times X^2) \subset \mathcal{D}$; so $F \in \mathcal{C}_*$. Conversely if $F \in \mathcal{C}_*$ then then $F \in \mathcal{H}$ and the mapping $\tilde{\Psi}$ witnesses that $F \in \mathcal{C}_{\mathrm{arc}}(X)$, hence $F \in \mathcal{G}$. It follows that $\mathcal{G} = \mathcal{C}_*$ is $\mathbf{\Pi}^1_1$. ☐

Corollary 5.4. *For any Polish space X, if $\mathcal{C}_{\mathrm{arc}}(X)$ admits a bianalytic uniform arc-lifting then $\mathcal{C}_{\mathrm{arc}}(X)$ is $\mathbf{\Pi}^1_1$.*

Corollary 5.5. *For any Polish space X, each of the following subsets of $\mathcal{C}_{\mathrm{arc}}(X)$ is $\mathbf{\Pi}^1_1$ and admits a bianalytic uniform arc-lifting:*

a) the set $\mathcal{C}^{\check{\Theta}}_{\mathrm{arc}}(X)$ of all atriodic closed arc-connected subsets of X,

b) the set $\mathcal{C}^{0}_{\mathrm{arc}}(X)$ of all closed arc-connected subsets of X that contain no Jordan curve.

In particular if X is atriodic or contains no Jordan curve then $\mathcal{C}_{\mathrm{arc}}(X)$ is $\mathbf{\Pi}^1_1$.

Proof. a) If $F \in \mathcal{C}^{\check{\Theta}}_{\mathrm{arc}}(X)$ then any arc-component of F is a curve (see Section 2.3) hence $\mathcal{C}^{\check{\Theta}}_{\mathrm{arc}}(X) \subset \mathcal{C}^{[2]}_{\mathrm{arc}}(X)$ so by Proposition 5.2, $\mathcal{C}^{\check{\Theta}}_{\mathrm{arc}}(X)$ admits a bianalytic uniform arc-lifting. Moreover $\mathcal{C}^{\check{\Theta}}_{\mathrm{arc}}(X) = \mathcal{H} \cap \mathcal{C}_{\mathrm{arc}}(X)$ where $\mathcal{H}$ is the $\mathbf{\Pi}^1_1$ set of all atriodic closed subsets of X, hence by Proposition 5.3, $\mathcal{C}^{\check{\Theta}}_{\mathrm{arc}}(X)$is $\mathbf{\Pi}^1_1$.

b) The argument is similar: if $F \in \mathcal{C}^{0}_{\mathrm{arc}}(X)$ then any arc-component of F is the continuous image of subinterval of the real line, hence $\mathcal{C}^{0}_{\mathrm{arc}}(X) \subset \mathcal{C}^{[1]}_{\mathrm{arc}}(X)$ and by Proposition 5.2, $\mathcal{C}^{\check{\Theta}}_{\mathrm{arc}}(X)$ admits a bianalytic uniform arc-lifting. Also $\mathcal{C}^{0}_{\mathrm{arc}}(X) = \mathcal{H} \cap \mathcal{C}_{\mathrm{arc}}(X)$ where $\mathcal{H}$ is the $\mathbf{\Pi}^1_1$ set of all closed subsets of X not containing any Jordan curve, hence by Proposition 5.3, $\mathcal{C}^{\check{\Theta}}_{\mathrm{arc}}(X)$ is $\mathbf{\Pi}^1_1$. ☐

6. Uniform arc-liftings in the codes

As we mentioned before it was already known from the work of Becker that the set $\mathcal{C}_{\mathrm{arc}}(\mathbb{I}^2)$ is not $\mathbf{\Pi}^1_1$, hence by Proposition 5.3, $\mathcal{C}_{\mathrm{arc}}(\mathbb{I}^2)$ does not admit a bianalytic uniform arc-lifting. Nevertheless we shall prove that for any Polish $X \subset \boldsymbol{R}^2$ the set $\mathcal{C}_{\mathrm{arc}}(X)$ admits a weak form of bianalytic uniform arc-lifting. Notice that since by Corollary 5.5 the set $\mathcal{C}^{\check{\Theta}}_{\mathrm{arc}}(X)$ does admit a bianalytic uniform arc-lifting we can restrict the study to the set $\mathcal{C}^{\Theta}_{\mathrm{arc}}(X) = \mathcal{C}_{\mathrm{arc}}(X) \setminus \mathcal{C}^{\check{\Theta}}_{\mathrm{arc}}(X)$ of all triodic arc-connected closed subsets of X. To state the main result precisely we need to introduce some preliminary notions.

6.1. *Coding δ-separable subsets:* For all $F \in \mathcal{F}(X)$ let d_F denote the distance on F induced by d, and let δ^F be the arc-pseudo-metric defined on F by the metric structure (F, d_F). Our goal is to code pairs (S, F) where $F \in \mathcal{F}(X)$ and S is a δ^F-separable closed subset of F contained in some arc-component of F (so that δ^F induces a genuine distance on S). Since the space (S, δ^F) is separable, it is entirely determined by the restriction of δ^F to any countable dense subset D of S.

From now on we fix a Polish space X and set $\mathcal{F} = \mathcal{F}(X)$, $\mathcal{J} = \mathcal{J}(X)$ and $\boldsymbol{A} = X^\omega \times \mathcal{J}^\omega$.

Definition 6.2. *Let $\boldsymbol{C}$ be the set of all $(\alpha, F) \in \boldsymbol{A} \times \mathcal{F}$ with $\alpha = (\alpha_0, \alpha_1) = \big((a_n)_{n\in\omega}, (I_p)_{p\in\omega}\big)$ satisfying for all $m, n \in \omega$:*

$$\begin{cases} (1)\ a_n \in F \ \text{ and } \ I_n \subset F, \\ (2)\ \delta^F(a_m, a_n) = \inf\{\mathrm{diam}(I_p) :\ e(I_p) = \{a_m, a_n\}\} \ \textit{if}\ a_m \neq a_n. \end{cases}$$

We then set: $\alpha^* = a_0,\ D_\alpha = \{a_n;\ n \in \omega\},\ S_{(\alpha,F)} = \overline{D_\alpha}^{\delta^F}$ *and* $S'_{(\alpha,F)} = F \setminus S_{(\alpha,F)}$.

One should think to an element $(\alpha, F) \in \boldsymbol{C}$ as a code for the separable metric space $(S_{(\alpha,F)}, \delta^F)$.

Lemma 6.3. *The set $\boldsymbol{C}$ is $\boldsymbol{\Pi}^1_1$.*

Proof. Condition (1) is clearly Borel. If $\sigma(\alpha)$ denotes the righthand side of the equation in (2), then the mapping $\alpha \mapsto \sigma(\alpha)$ is Borel too; and it follows from the definition of δ^F that $\delta^F_\alpha \leq \sigma(\alpha)$, hence condition (2) is equivalent to:

$$\forall\, m, n \in \omega,\ \forall\, J \in \mathcal{J},\ \text{if } (J \subset F \text{ and } e(J) = \{a_m, a_n\}) \text{ then } \sigma(\alpha) \leq \operatorname{diam}(J)$$

which is clearly $\boldsymbol{\Pi}^1_1$. □

Set:

$\boldsymbol{R} = \{(\alpha, F, x) \in \boldsymbol{C} \times X :\ x \in F\}$, $\boldsymbol{S} = \{(\alpha, F, x) \in \boldsymbol{C} \times X :\ x \in S_{(\alpha,F)}\}$ and $\boldsymbol{S}' = \boldsymbol{R} \setminus \boldsymbol{S}$

so

$$\boldsymbol{R} = \boldsymbol{S} \cup \boldsymbol{S}'$$

and we also set:

$\boldsymbol{R}^{(2)} = \{(\alpha, F, x, y) \in \boldsymbol{C} \times X^2 :\ x, y \in F\}$ and $\boldsymbol{S}^{(2)} = \{(\alpha, F, x, y) \in \boldsymbol{C} \times X^2 :\ x, y \in S_{(\alpha,F)}\}$

Lemma 6.4. *For all $r > 0$:*

a) The set $\{(F, x, y) \in \mathcal{F} \times X^{(2)} :\ x, y \in F$ and $\delta^F(x, y) < r\}$ is $\boldsymbol{\Sigma}^1_1$.

b) The set $\{(\alpha, F, x, y) \in \boldsymbol{S}^{(2)} :\ \delta^F(x, y) < r\}$ is bianalytic in $\boldsymbol{S}^{(2)}$.

Similarly if we replace $<$ by $\leq$.

Proof. a) follows readily from the definition of δ^F. For b) observe that by the density of D_α in $S_{(\alpha,F)}$ and the triangle inequality, for $(\alpha, F, x, y) \in \boldsymbol{S}^{(2)}$:

$$\begin{aligned} \delta^F(x,y) \geq r \iff{} & \forall\, r' < r,\ \delta^F(x,y) > r' \\ \iff{} & \forall\, r' < r,\ \exists\, \varepsilon > 0, m, n \in \omega,\ \begin{cases} \delta^F(a_m, a_n) > r' + 2\varepsilon, \\ \delta^F(x, a_m) < \varepsilon \text{ and } \delta^F(y, a_n) < \varepsilon\,. \end{cases} \end{aligned}$$

□

Lemma 6.5. *a) The set $\boldsymbol{R}$ is a Borel subset of $\boldsymbol{C} \times X$, hence a $\boldsymbol{\Pi}^1_1$ set.*

b) The set $\boldsymbol{S}$ is in $\boldsymbol{\Sigma}^1_1(\boldsymbol{R})$, hence $\boldsymbol{S}'$ is a $\boldsymbol{\Pi}^1_1$ set.

Proof. Part a) is obvious, and part b) follows from Lemma 6.4 since for $(\alpha, F, x) \in \boldsymbol{R}$:

$$(\alpha, F, x) \in \boldsymbol{S} \iff \forall\, \varepsilon > 0,\ \exists\, n,\ \delta^F(a_n, x) < \varepsilon.$$

□

Theorem 6.6. *There exists a bianalytic mapping $\Phi = (\Phi^{(0)}, \Phi^{(1)}) : \boldsymbol{S} \to \omega^\omega \times \hat{\mathcal{J}}$ such that for all $(\alpha, F, x) \in \boldsymbol{S}$ if $\Phi(x) = (\sigma, J)$ then:*

1) $x = \delta^F\text{-}\lim_n \alpha_0(\sigma(n))$

2) $J \subset F$ and $e(J) = \{\alpha^*, x\}$

In particular the partial mapping $\Phi^{(1)}(\alpha, F, \cdot) : S_{(\alpha,F)} \to \hat{\mathcal{J}}$ is a Borel $\star$-arc-lifting for $S_{(\alpha,F)}$ in F, with summit α^*.

Proof. The argument is a reminiscence of the proof of Theorem 4.5. We fix a Borel bijection $\rho : n \mapsto (\ell_n, r_n)$ from ω onto $\omega \times \boldsymbol{Q}^+$. Let $\tau : n \mapsto \ell_n$ denote the first coordinate of ρ and for $(\alpha, F) \in \boldsymbol{C}$ set for all n: $a_n = \alpha_0(n)$ and $c_n = \alpha_0(\ell_n) = \alpha_0(\tau(n))$.

We then define a tree $T = T(\alpha, F) \subset \omega^{<\omega}$ on ω as follows:

$$\begin{aligned} &(i) && T \cap \omega^1 = \{\langle m \rangle;\ m \in \omega\} \\ &(ii) && \begin{cases} \text{if } t = s^\frown\langle m \rangle \in T \text{ then:} \\ t^\frown\langle n \rangle \in T \iff r_n < \min\{r_m - \delta^F(c_n, c_m)\ ,\ \dfrac{r_m}{2}\ ,\ \{\delta^F(c_n, c_{t(j)});\ j < |t|\}\} \end{cases} \end{aligned}$$

For all n let $B_n = B_n(\alpha, F)$ and $\tilde{B}_n = \tilde{B}_n(\alpha, F)$ denote respectively the open δ^F-ball, and the closed δ^F-ball in $S_{(\alpha,F)}$, of center c_n and radius r_n. Then it follows from condition (ii) and the triangle inequality, that if $u = t^\frown\langle n\rangle = s^\frown\langle m\rangle^\frown\langle n\rangle \in T$ then $\tilde{B}_n \subset B_m$, $r_n < \dfrac{r_m}{2}$, and for all $j < |t|$, $c_n \neq c_{t(j)}$, since $r_n < \delta^F(c_n, c_{t(j)})$.

Hence if $s = \langle n_0, n_1, \cdots, n_k\rangle \in T$ with $k \geq 1$ then $B_{n_0} \supset \tilde{B}_{n_1} \supset B_{n_1} \cdots \supset \tilde{B}_{n_k} \supset B_{n_k}$ and $r_{n_k} < r_{n_0}\, 2^{-k}$. Then setting $S = S_{(\alpha,F)}$, for any $x \in S \cap B_{n_k}$ we can find $r \in \mathbb{Q}^+$ such that

$$r < \min\{r_{n_k} - \delta^F(x, \alpha_0(n_k))\ ,\ \frac{r_{n_k}}{2}\ ,\ \delta^F(x, \alpha_0(t(j))) \text{ for all } j < |t|\}.$$

Also by the density of D_α in S we can find some ℓ such that the same inequality holds when replacing x by a_p; and if $(\ell, r) = (\ell_n, r_n)$ then $s^\frown\langle n\rangle \in T$. Hence any $s \in T$ has a (strict) extension in T and $B_{n_k} \subset \bigcup\{B_n :\ s^\frown\langle n\rangle \in T\}$. Since by condition (i) $S = \bigcup\{B_n :\ \langle n\rangle \in T\}$ then by induction $S = \bigcup_{\sigma\in\lceil T\rceil}\bigcap_{j\in\omega} B_{\sigma(j)}$ with $\bigcap_{j\in\omega}\downarrow B_{\sigma(j)} = \{b_\sigma\}$ for all $\sigma \in \lceil T\rceil$.

So for any $x \in S$ there exists at least one infinite branch $\sigma \in \lceil T\rceil$ such that $x \in \bigcap_j B_{\sigma(j)}$, hence $x = b_\sigma = \delta^F\text{-}\lim_k c_{\sigma(k)} = \delta^F\text{-}\lim_k a_{(\ell_{\sigma(k)})}$. If $\Phi^{(0)}(\alpha, F, x) = \sigma$ is the lexicographical minimum of the set $\{\sigma' \in \lceil T\rceil :\ x = b_{\sigma'}\}$ then:

$$\begin{aligned}\sigma(k) = n \quad &\iff x \in \textstyle\bigcap_{j\leq k} B_{\sigma^x(j)} \setminus \bigcup\{B_m :\ m < n \text{ and } \sigma_{|k}{}^\frown\langle m\rangle \in T\}\\ &\iff \forall\, j < k,\ x \in B_{\sigma^x(j)}, \text{ and } (\forall\, m < n,\ \sigma_{|k}{}^\frown\langle m\rangle \notin T \text{ or } x \notin B_m).\end{aligned}$$

Since by condition (2) of Definition 6.2, for all m, n, the mapping $F \mapsto \delta^F(a_n, a_m)$ is Borel on $\boldsymbol{C}$, then for all $s \in \omega^{<\omega}$, the set $\{(\alpha, F) \in \boldsymbol{C} :\ s \in T(\alpha, F)\}$ is a Borel subset of $\boldsymbol{C}$, hence the mapping $(\alpha, F) \mapsto T(\alpha, F)$ is Borel on $\boldsymbol{C}$. Moreover for all n, since $a_n \in S_{(\alpha,F)}$, then by Lemma 6.4 the set $\{(\alpha, F, x) \in \boldsymbol{S} :\ x \in B_n\}$ is bianalytic in $\boldsymbol{S}$. Hence the mapping $\Phi^{(0)}$ is bianalytic on $\boldsymbol{S}$.

Then by condition (2) of Definition 6.2 for all $(m, n) \in \omega^2$ we can define on $\boldsymbol{C}$ a Borel mapping $\mu_{(m,n)} : \boldsymbol{C} \to \omega$ by $\mu_{(m,n)}(\alpha, F) = \min\{p :\ e(I_p) = \{a_m, a_n\} \text{ and } \operatorname{diam}(I_p) < 2\,\delta^F(a_m, a_n,)\}$. If $(\alpha, F, x) \in \boldsymbol{S}$ with $\varphi(\alpha, F, x) = \tau{\circ}\Phi^{(0)}(\alpha, F, x) = (\ell_{k_n})_n \in \omega^\omega$ and $p_n = \mu_{(\ell_{k_n}, \ell_{k_{n+1}})}(\alpha, F)$ we set $\psi(\alpha, F, x) = (p_n)_n$; then the mapping $\psi : \boldsymbol{S} \to \omega^\omega$ thus defined is bianalytic on $\boldsymbol{S}$. Moreover by the definition of μ, for all n, $e(I_{p_n}) = \{c_{k_n}, c_{k_{n+1}}\}$, $\operatorname{diam}(I_{p_n}) < 2\,\delta^F(c_{k_n}, c_{k_{n+1}})$ and $x = \delta^F\text{-}\lim_n c_{k_n}$, hence $x = d\text{-}\lim_n c_{k_n}$. If $x \neq a_0$ it follows from Remark 3.4. b) that the sequence $(I_{p_n})_n$ satisfies the hypothesis of Theorem 3.7, hence the arc $J = \Phi^{(1)}(\alpha, F, x) = \bigvee_{n\in\omega} I_{(p_n)}$ is well defined and satisfies $J \subset F$ and $e(J) = \{a_0, x\}$. Finally by Lemma 3.5, setting $\Phi^{(1)}(\alpha, F, a_0) = \{a_0\}$, the mapping $\Phi^{(1)}$ thus defined on $\boldsymbol{S}$ satisfies the conclusion of Theorem 6.9. □

Corollary 6.7. *The set* $\boldsymbol{S}$ *is* $\boldsymbol{\Pi}^1_1$, *hence* $\boldsymbol{S}$ *and* $\boldsymbol{S}'$ *are bianalytic subsets of* $\boldsymbol{R}$.

Proof. Let $\varphi = \Phi^{(0)} : \boldsymbol{S} \to \omega^\omega$ be the bianalytic mapping given by Theorem 6.6. Since $\boldsymbol{S} \subset \boldsymbol{R}$ and $\boldsymbol{R}$ is $\boldsymbol{\Pi}^1_1$, then φ admits an extension to a bianalytic mapping $\tilde{\varphi} : \boldsymbol{U} \to \omega^\omega$ with $\boldsymbol{\Pi}^1_1$ domain $\boldsymbol{S} \subset \boldsymbol{U} \subset \boldsymbol{R}$. Moreover observe that since δ^F is complete on F then for $(\alpha, F, x) \in \boldsymbol{R}$ and $\sigma \in \omega^\omega$:

$$x = \delta^F\text{-}\lim_n \alpha_0(\sigma(n)) \iff x = \lim_n \alpha_0(\sigma(n)) \text{ and } (\alpha_0(\sigma(n)))_n \text{ is } \delta^F\text{-Cauchy}$$

and it follows from Lemma 6.4 b) that the lefthand side of this equivalence is a bianalytic condition on (α, F, x). Hence the set $\tilde{\boldsymbol{S}} = \{(\alpha, F, x) \in \boldsymbol{U} :\ x = \delta^F\text{-}\lim_n \alpha_0(\tilde{\varphi}((\alpha, F, x))(n))\}$ is $\boldsymbol{\Pi}^1_1$ and $\boldsymbol{S} \subset \tilde{\boldsymbol{S}}$; and since the converse inclusion is obviously true then $\boldsymbol{S} = \tilde{\boldsymbol{S}}$, and so $\boldsymbol{S}$ is $\boldsymbol{\Pi}^1_1$. □

From now on we restrict our study to closed subsets F of the given space X, which are: *triodic, arc-connected, and* δ^F*-separable.* For this we consider

$$\boldsymbol{C}_\star = \{(\alpha, F) \in \boldsymbol{C} :\ F \in \mathcal{C}^\Theta_{\text{arc}} \quad\text{and}\quad S_{(\alpha,F)} \supset \Sigma^F\}$$

and for any set $\boldsymbol{Q} \subset \boldsymbol{C} \times Y$ defined previously we set $\boldsymbol{Q}_\star = \boldsymbol{Q} \cap (\boldsymbol{C}_\star \times Y)$.

Theorem 6.8. *There exists a bianalytic mapping* $\Phi' : \boldsymbol{S}'_\star \to X \times \hat{\mathcal{J}}$ *such that for all* $(\alpha, F, x) \in \boldsymbol{S}'_\star$*, if* $\Phi'(\alpha, F, x) = (y, J)$ *then* $y \in S_{(\alpha,F)}$*,* $J \subset F$ *and* $e(J) = \{x, y\}$*.*

Proof. Set $\tilde{\boldsymbol{C}}_\star = \boldsymbol{C}_\star \times X^2 \times \mathcal{J}$ and consider the sets:

$$\boldsymbol{B} = \{(\alpha, F, x, y, J) \in \tilde{\boldsymbol{C}}_\star : \ J \subset F, \ e(J) = \{x, y\}, \ x \in S'_{(\alpha,F)} \text{ and } y \in J \cap S_{(\alpha,F)}\}$$
$$\boldsymbol{B}_0 = \{(\alpha, F, x, y, J) \in \boldsymbol{B} : \ \{y\} = J \cap S_{(\alpha,F)}\}$$
$$\boldsymbol{B}'_\varepsilon = \{(\alpha, F, x, y, J, z) \in \boldsymbol{B} \times X : \ z \in J \cap S_{(\alpha,F)} \text{ and } d(y, z) \geq \varepsilon\}, \quad \text{for } \varepsilon > 0.$$

By Corollary 6.7 the sets $\boldsymbol{B}$ and $\boldsymbol{B}'_\varepsilon$ are bianalytic subsets of $\tilde{\boldsymbol{C}}_\star$ and $\tilde{\boldsymbol{C}}_\star \times X$ respectively. Moreover by Theorem 2.5 4) for all $(\alpha, F) \in \boldsymbol{C}$ and $J \in \mathcal{J}$ the set $J \cap S_{(\alpha,F)}$ is compact, hence for all $(\alpha, F, x, y, J) \in \boldsymbol{B}$ the set $\boldsymbol{B}'_\varepsilon(\alpha, F, x, y, J)$ is compact too. It follows then from Theorem 1.4 that $\pi(\boldsymbol{B}'_\varepsilon)$ – the projection of $\boldsymbol{B}'_\varepsilon$ on $\boldsymbol{B}$ – is bianalytic in $\boldsymbol{B}$, hence in $\tilde{\boldsymbol{C}}_\star$. So $\boldsymbol{B}_0 = \bigcap_\varepsilon \boldsymbol{B} \setminus \pi(\boldsymbol{B}'_\varepsilon)$ is bianalytic in $\boldsymbol{B}$.

Then for all $(\alpha, F, x) \in \boldsymbol{S}'_\star$, since $F \in \mathcal{C}_{\text{arc}}(X)$ and $\Sigma^F \neq \emptyset$, we can join x to Σ^F, hence to $S_{(\alpha,F)}$, by an arc J, and if J is minimal for $\subset$ and $e(J) = \{x, y\}$ then $(\alpha, F, x, y, J) \in \boldsymbol{B}$; hence $\boldsymbol{B}(\alpha, F, x)$ is nonempty. Observe also that if (y, J) and (y', J') are two distinct elements of $\boldsymbol{B}(\alpha, F, x)$ then necessarily $J \neq J'$. Moreover if $(\alpha, F, x) \in \boldsymbol{S}'_\star$ then $x \in S'_{(\alpha,F)}$, and since $(\alpha, F) \in \boldsymbol{C}_\star$ then $x \notin \Sigma^F$, in particular x is not the center of a triod. It follows that for all $(\alpha, F, x) \in \boldsymbol{S}'_\star$ the set $\boldsymbol{B}(\alpha, F, x)$ contains at least one element, and at most two elements, hence by Theorem 1.4 the set $\boldsymbol{B}$ admits a bianalytic selection Φ' which satisfies the conclusion of Theorem 6.8. $\square$

Recall that by definition

$$(\alpha, F, x, y) \in \boldsymbol{R}^{(2)}_\star \iff (\alpha, F) \in \boldsymbol{C}_\star(X) \text{ and } (x, y) \in F^2$$

and

$$(\alpha, F) \in \boldsymbol{C}_\star \iff \begin{cases} - F \text{ is a } \textit{triodic, arc-connected and } \delta^F\textit{-separable} \text{ subset of } X, \\ - (\alpha, F) \in \boldsymbol{C} \text{ is a code satisfying } \Sigma^F \subset S_{(\alpha,F)} \subset F. \end{cases}$$

Theorem 6.9. *For any Polish space* X*, there exists a bianalytic mapping* $\Psi : \boldsymbol{R}^{(2)}_\star \to \hat{\mathcal{J}}(X)$ *such that for all* $(\alpha, F) \in \boldsymbol{C}_\star$*, the partial mapping* $\Psi(\alpha, F, \cdot) : F^2 \to \hat{\mathcal{J}}$ *is a Borel arc-lifting for* F*.*

Proof. Let Φ and Φ' be as in Theorem 6.6 and Theorem 6.8, and let ϕ_0 denote the restriction of the mapping $\Phi^{(1)}$ to $\boldsymbol{S}_\star$. So for all $(\alpha, F) \in \boldsymbol{C}_\star$, the mapping $\phi_0(\alpha, F, \cdot) : S_{(\alpha,F)} \to \hat{\mathcal{J}}$ is a $\star$-arc-lifting of $S_{(\alpha,F)}$ in F, with summit α^* .

Also for all $(\alpha, F, x) \in \boldsymbol{S}'_\star$ if $y = \Phi'^{(0)}(\alpha, F, x)$ and $J = \phi'(\alpha, F, x)$ is the arc defined by $(J, x, \alpha^*) = \big(\Phi'^{(1)}(\alpha, F, x), x, y\big) \vee (\Phi^{(1)}(\alpha, F, y), y, \alpha^*)$, then the mapping $\phi_1 : \boldsymbol{S}'_\star \to \hat{\mathcal{J}}$ thus defined is bianalytic and for all $(\alpha, F) \in \boldsymbol{C}_\star$, the mapping $\phi_1(\alpha, F, \cdot) : S'_{(\alpha,F)} \to \hat{\mathcal{J}}$ is a $\star$-arc-lifting of $S'_{(\alpha,F)}$ in F, with summit α^* .

Since $\boldsymbol{S}_\star$ and $\boldsymbol{S}'_\star$ are bianalytic subsets of $\boldsymbol{R}_\star$, the mapping $\phi = \phi_0 \cup \phi_1$ on $\boldsymbol{R}_\star$ is bianalytic too, and for all $(\alpha, F) \in \boldsymbol{C}_\star$ the mapping $\phi(\alpha, F, \cdot) : F \to \hat{\mathcal{J}}$ is a $\star$-arc-lifting of F, with summit α^*; and since F is Polish then $\phi(\alpha, F, \cdot)$ is Borel. Then if $\Psi(\alpha, F, \cdot, \cdot)$ denotes the canonical arc-lifting on F defined by $\phi(\alpha, F, \cdot)$ (see Remark 4.4) the mapping $\Psi : \boldsymbol{R}_\star \to \hat{\mathcal{J}}(X)$ satisfies clearly the conclusion of Theorem 6.9. $\square$

Theorem 6.10. *Let* X *be Polish space and suppose that for any* $F \in \mathcal{C}^\Theta_{\text{arc}}(X)$ *the set* Σ^F *is* δ^F*-separable, and let* $\boldsymbol{\Gamma} \supset \boldsymbol{\Pi}^1_1$ *be a given class.*

a) If there exists a $\boldsymbol{\Gamma}$*-measurable mapping* $\boldsymbol{\sigma} : \mathcal{C}^\Theta_{\text{arc}}(X) \to \boldsymbol{A}$ *such that for all* $F \in \mathcal{C}^\Theta_{\text{arc}}(X)$*,* $(\boldsymbol{\sigma}(F), F) \in \boldsymbol{C}_\star$ *then the set* $\mathcal{C}_{\text{arc}}(X)$ *is in* $\check{\mathcal{A}}(\boldsymbol{\Gamma})$*.*

b) If there exists a $\boldsymbol{\Gamma}$-measurable mapping $\boldsymbol{\sigma}_0 : \mathcal{C}^{\Theta}_{\mathrm{arc}}(X) \to X^\omega$ such that for all $F \in \mathcal{C}^{\Theta}_{\mathrm{arc}}(X)$, the δ^F-closure of the range of the sequence $\boldsymbol{\sigma}_0(F)$ contains the set Σ^F, then the set $\mathcal{C}_{\mathrm{arc}}(X)$ is in $\check{\mathcal{A}}(\check{\mathcal{A}}(\boldsymbol{\Gamma}))$.

Proof. Again since by Proposition 5.2 the set $\mathcal{C}^{\check{\Theta}}_{\mathrm{arc}}(X)$ is $\boldsymbol{\Pi}^1_1$, to prove that $\mathcal{C}_{\mathrm{arc}}(X)$ is in some class $\boldsymbol{\Lambda} \supset \boldsymbol{\Pi}^1_1$, we only need to prove that the set $\mathcal{C}^{\Theta}_{\mathrm{arc}}(X)$ is in $\boldsymbol{\Lambda}$.

a) Let Ψ the mapping given by Theorem 6.9. Since the domain $\boldsymbol{R}^{(2)}_\star$ of the mapping Ψ is a subset of the $\boldsymbol{\Pi}^1_1$ set $\boldsymbol{R}^{(2)}$, and the graph of Ψ is a subset of the $\boldsymbol{\Pi}^1_1$ set

$$\boldsymbol{U} = \{(\alpha, F, x, y, J) \in \boldsymbol{R}^{(2)} :\ \Sigma^F \subset S_{(\alpha,F)},\ J \subset F \text{ and } e(J) = \{x, y\}\}$$

we can extend Ψ to a bianalytic mapping $\tilde{\Psi} : \boldsymbol{Q} \to \hat{\mathcal{J}}$ with $\boldsymbol{\Pi}^1_1$ domain $\boldsymbol{R}^{(2)}_\star \subset \boldsymbol{Q} \subset \boldsymbol{R}^{(2)}$ and graph also contained in $\boldsymbol{U}$. It follows that the set

$$\boldsymbol{D} = \{(\alpha, F) \in \boldsymbol{C} :\ \forall\, (x, y) \in F^2,\ (\alpha, F, x, y) \in \boldsymbol{Q}\}$$

is $\boldsymbol{\Pi}^1_1$.

For all $F \in \mathcal{C}^{\Theta}_{\mathrm{arc}}(X)$, since $(\boldsymbol{\sigma}(F), F) \in \boldsymbol{C}_\star$, then by Theorem 6.9 for all $x, y \in F$, $(\alpha, F, x, y) \in \boldsymbol{R}^{(2)}_\star \subset \boldsymbol{Q}$; hence $(\boldsymbol{\sigma}(F), F) \in \boldsymbol{D}$. Conversely if $(\boldsymbol{\sigma}(F), F) \in \boldsymbol{D}$ then by the choice of $\tilde{\Psi}$, for all $x, y \in F$, $\tilde{\Psi}(\alpha, F, x, y)$ is a sub-arc of F with endpoints $\{x, y\}$; hence $F \in \mathcal{C}^{\Theta}_{\mathrm{arc}}(X)$. This proves that $\mathcal{C}^{\Theta}_{\mathrm{arc}}(X) = (\boldsymbol{\sigma}, Id)^{-1}(\boldsymbol{D})$ and the conclusion follows from Proposition 1.1.

b) Fix a Borel choice mapping $\mathbf{c} : \mathcal{F} \to X$ such that for all $F \in \mathcal{F}$, $\mathbf{c}(F) \in F$. Since for all $r > 0$ the set

$$\boldsymbol{B} = \{(F, a, b, r, J) \in \mathcal{F} \times X^2 \times \mathbb{Q}^+ \times \mathcal{J} :\ J \subset F,\ e(J) = \{a, b\} \text{ and } \operatorname{diam}(J) < r\}$$

is Borel there exists a $\boldsymbol{\Sigma}$-measurable mapping $\boldsymbol{\tau} : \mathcal{F} \times X^2 \times \mathbb{Q}^+ \to \hat{\mathcal{J}}$ such that for all $(F, a, b, r) \in \mathcal{F} \times X^2 \times \mathbb{Q}^+$ if $J = \boldsymbol{\tau}(F, a, b, r)$ then:

$$\begin{cases} (F, a, b, r, J) \in \boldsymbol{B} & \text{if } 0 < \delta^F(a, b) < r \\ J = \{\mathbf{c}(F)\} & \text{if not} \end{cases}$$

So if we fix an enumeration $(m_p, n_p, r_p)_{p \in \omega}$ of the set $\omega^2 \times \mathbb{Q}^+$ and set for all $F \in \mathcal{C}^{\Theta}_{\mathrm{arc}}(X)$:

$$\boldsymbol{\sigma}_1(F)(p) = \boldsymbol{\tau}(F, \boldsymbol{\sigma}_0(m_p), \boldsymbol{\sigma}_0(n_p), r_p)$$

then the mapping $\boldsymbol{\sigma}_1 : \mathcal{C}^{\Theta}_{\mathrm{arc}}(X) \to \hat{\mathcal{J}}^\omega$ satisfies condition 2) of Definition 6.2. It also follows from the hypothesis that the mapping $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_0, \boldsymbol{\sigma}_1) : \mathcal{C}^{\Theta}_{\mathrm{arc}}(X) \to \boldsymbol{A}$ is bianalytic and for all $F \in \mathcal{C}^{\Theta}_{\mathrm{arc}}(X)$, $(\boldsymbol{\sigma}(F), F) \in \boldsymbol{C}_\star$. Since $\boldsymbol{\sigma}_0$ is $\boldsymbol{\Gamma}$-measurable and $\boldsymbol{\tau}$ is $\boldsymbol{\Sigma}$-measurable one easily checks that $\boldsymbol{\sigma}_1$, as well as $\boldsymbol{\sigma}$, is measurable relatively to the σ-algebra class generated by $\mathcal{A}(\boldsymbol{\Gamma})$ and $\check{\mathcal{A}}(\boldsymbol{\Gamma}))$, hence $\boldsymbol{\sigma}$ is $\mathcal{A}(\check{\mathcal{A}}(\boldsymbol{\Gamma}))$-measurable and $\check{\mathcal{A}}(\mathcal{A}(\boldsymbol{\Gamma}))$-measurable. It follows then from part a) that $\mathcal{C}^{\Theta}_{\mathrm{arc}}(X)$ is in $\check{\mathcal{A}}(\mathcal{A}(\check{\mathcal{A}}(\boldsymbol{\Gamma}))) = \big(\mathcal{A}(\mathcal{A}(\check{\mathcal{A}}(\boldsymbol{\Gamma})))\big)\check{} = \check{\mathcal{A}}(\check{\mathcal{A}}(\boldsymbol{\Gamma}))$. □

Corollary 6.11. *Let X be a Polish space. If X contains at most countably many centers of triods then the set $\mathcal{C}_{\mathrm{arc}}(X)$ is in $\check{\mathcal{A}}(\check{\mathcal{A}}(\boldsymbol{\Pi}^1_1))$.*

Proof. Fix an enumeration $(c_n)_{n \in \omega}$ of the set S^X of all centers of triods in X. Then by definition for all $F \in \mathcal{C}^{\Theta}_{\mathrm{arc}}(X)$, the set S^F is a non empty subset of S^X, hence $\Sigma^F \subset \overline{(\Sigma^X \cap F)}^{\delta^F}$.

Consider then the Borel mapping $\boldsymbol{\sigma}_0 : \mathcal{C}^{\Theta}_{\mathrm{arc}}(X) \to \omega^\omega$ defined by

$$\boldsymbol{\sigma}_0(F)(k) = \begin{cases} \min\{n :\ c_n \in F \text{ and } \forall\, j < k,\ c_n \neq \sigma_0(F)(j),\}\} & \text{if this set is nonempty} \\ \boldsymbol{\sigma}_0(F)(k-1) & \text{if not} \end{cases}$$

Note that if $F \in \mathcal{C}^{\Theta}_{\mathrm{arc}}(X)$ then $S^F \neq \emptyset$ and so there exists n such that $c_n \in F$, so $\boldsymbol{\sigma}_0(F)(0)$ is defined; and it follows by induction that $\boldsymbol{\sigma}_0(F)(k)$ is defined for all k. Consequently $\boldsymbol{\sigma}_0$ is well defined and for all $F \in \mathcal{C}^{\Theta}_{\mathrm{arc}}(X)$, if $\boldsymbol{\sigma}_0(F) = \sigma$ then $S^F \subset S \cap S^X = \{c_{\sigma(k)};\ k \in \omega\}$, hence $\Sigma^F \subset \overline{\{c_{\sigma(k)};\ k \in \omega\}}^{\delta^F}$, and $\boldsymbol{\sigma}_0$ satisfies the hypothesis of part b) of Theorem 6.10 with $\boldsymbol{\Gamma} = \boldsymbol{\Pi}^1_1$, so the set $\mathcal{C}_{\mathrm{arc}}(X)$ is in $\check{\mathcal{A}}(\check{\mathcal{A}}(\boldsymbol{\Pi}^1_1))$. □

Corollary 6.12. *For any Polish space X, if for all $F \in \mathcal{C}^{\Theta}_{\mathrm{arc}}(X)$ the set Σ^F is δ^F-separable, then the set $\mathcal{C}^{\Theta}_{\mathrm{arc}}(X)$ is $\boldsymbol{\Delta}^1_2$.*

Proof. Note that the set $\boldsymbol{C}^\star = \{(F, \alpha) \in \mathcal{F} \times \boldsymbol{A} :\ S_{(\alpha,F)} \supset \Sigma^F\}$ is a $\boldsymbol{\Pi}^1_1$ subset of $\mathcal{F} \times \boldsymbol{A}$, and it follows from the hypothesis that $\pi(\boldsymbol{C}^\star)$ – the projection of $\boldsymbol{C}^\star$ onto the first factor – contains $\mathcal{C}^{\Theta}_{\mathrm{arc}}(X)$. By Kondo uniformization Theorem there exists a mapping $\boldsymbol{\sigma}_0 : \pi(\boldsymbol{C}^\star) \to \boldsymbol{A}$ with $\boldsymbol{\Pi}^1_1$ graph contained in $\boldsymbol{C}^\star$, hence satisfying for all $F \in \mathcal{C}^{\Theta}_{\mathrm{arc}}(X)$, $(F, \boldsymbol{\sigma}_0(F)) \in \boldsymbol{C}^\star$, so $(\boldsymbol{\sigma}_0(F), F) \in \boldsymbol{C}^\star$. Since the graph of $\boldsymbol{\sigma}_0$ is $\boldsymbol{\Pi}^1_1$ then the mapping $\boldsymbol{\sigma}_0$ is $\boldsymbol{\Sigma}^1_2$(hence $\boldsymbol{\Delta}^1_2$)-measurable on $\mathcal{C}^{\Theta}_{\mathrm{arc}}(X)$. Then by Theorem 6.10 b), $\mathcal{C}^{\Theta}_{\mathrm{arc}}(X)$ is in $\check{\mathcal{A}}(\check{\mathcal{A}}(\boldsymbol{\Delta}^1_2) = \boldsymbol{\Delta}^1_2$ by the invariance of projective classes by operation $\mathcal{A}$ (see [5] 38, Theorem 4). □

7. BACK TO PLANAR POLISH SPACES

In all this section X will be a *planar* Polish space. Note that it follows from Corollary 6.12 and Theorem 2.8 that the set $\mathcal{C}_{\mathrm{arc}}(X)$ is $\boldsymbol{\Delta}^1_2$, and our goal in this section is to prove the following more precise results.

Theorem 7.1. *For any Polish space $X \subset \boldsymbol{R}^2$ there exists a $\boldsymbol{\Sigma}$-measurable mapping $\boldsymbol{\sigma} : \mathcal{F}(X) \to \boldsymbol{A}$ such that for all $F \in \mathcal{C}^{\Theta}_{\mathrm{arc}}(X)$, $(\boldsymbol{\sigma}(F), F) \in \boldsymbol{C}_\star$.*

It follows then from Theorem 6.10

Corollary 7.2. *For any Polish space $X \subset \boldsymbol{R}^2$ the set $\mathcal{C}_{\mathrm{arc}}(X)$ is in $\check{\mathcal{A}}(\boldsymbol{\Pi}^1_1)$.*

The proof of Theorem 7.1 relies strongly on the following notion.

7.3. *Triods trap:* A *triod trap* is a quadruple $p = (W, I_0, I_1, I_2)$ where W is the bounded connected component of some Jordan curve ∂W and I_0, I_1, I_2 are three pairwise disjoint sub-arcs of ∂W; W will be called the *domain* of p, and we set $W = \mathrm{dom}(p)$ and $\mathrm{diam}(p) = \mathrm{diam}(W)$.

A triod T will said to be *compatible* with the trap (W, I_0, I_1, I_2) if $T = J_0 \cup J_1 \cup J_2 \subset \overline{W}$ and for all k, $J_k \cap \partial W = e(J_k) \setminus \{\mathbf{c}(T)\} \subset I_k$.

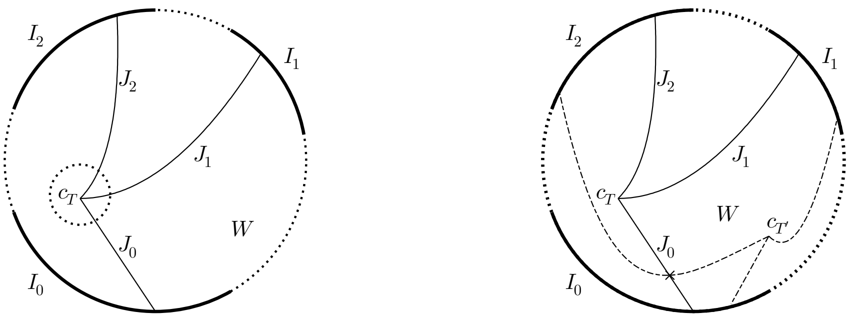
In fact the heart of Moore's proof of Theorem 2.6 ([7], Lemma 1) can be restated as follows:

Lemma 7.4. (MOORE). *Two triods compatible with the same trap have nonempty intersection.*

To derive Theorem 2.6 from Lemma 7.4 observe that there exists a countable set $\mathbb{P}$ of triod traps in the plane such that any triod in the plane is compatible with some $p \in \mathbb{P}$. For example one can take for $\mathbb{P}$ the countable set of all *rational circular triod traps* i.e. the set of all traps of the form $p = (W, I_0, I_1, I_2)$ where W is an open disc with rational radius and rational coordinates center, and, for each $i \in 3$, I_i is a sub-arc of ∂W with rational endpoints.

Observe that if $T = J_0 \cup J_1 \cup J_2$ is compatible with the trap $p = (W, I_0, I_1, I_2)$, W' is the bounded connected component of some Jordan curve $\partial W'$, $\overline{W'} \subset W$ and $\mathbf{c}(T) \in W'$ each J_k meets $\partial W'$. It follows that there is a unique triod $\tilde{T} = \tilde{J}_0 \cup \tilde{J}_1 \cup \tilde{J}_2$ contained in T with $\tilde{J}_k \subset J_k$, hence $\mathbf{c}(\tilde{T}) = \mathbf{c}(T)$, and such that $\tilde{J}_k \cap \partial W'$ is a singleton; and we shall then write $\tilde{T} = T_{|W'}$. For fixed W and W' the set of such pairs $(T, \tilde{T})$ is Borel, and the mapping $T \mapsto \tilde{T}$ is Borel. If moreover (I'_0, I'_1, I'_2) are pairwise disjoint sub-arcs of $\partial W'$ and $\tilde{T}$ is compatible with $p' = (W', I'_0, I'_1, I'_2)$ we shall say that T is *weakly compatible* with the trap p'. It follows that for all trap p' the set of triods T which are weakly compatible with p' is Borel.

FIGURE 2



<u>*Proof of Theorem 7.1*</u>:

Let $\mathbb{P}$ denote the set of all rational circular triod traps introduced in 7.3. For each triod trap p the set $\mathscr{T}_p(X)$ of all triods in X compatible with p is Borel, and the set $\mathcal{M}_p(X)$ of closed subsets of X which match p, i.e which contain a triod compatible with p, is analytic. For all $p \in \mathbb{P}$ and all $\varepsilon > 0$ we denote by $\mathbb{P}_{p,\varepsilon}$ the set of all triod traps $q \in \mathbb{P}$ such that $\overline{\mathrm{dom}(q)} \subset \mathrm{dom}(p)$ and $\mathrm{diam}(q) < \varepsilon$.

Lemma 7.5. *For all $p \in \mathbb{P}$ there exists a Borel function γ_p which assigns to each pair of triods $(T, T') \in \mathcal{T}_p(X)^2$ compatible with p an arc joining c_T to $c_{T'}$ inside $T \cup T'$.*

Proof. By Lemma 7.4, $T \cap T' \neq \emptyset$. It is well known that there exists a Borel choice function ρ assigning to each nonempty closed subset F of a Polish space $\mathscr{X}$ an element of F. So the function $(T, T') \mapsto \rho(T \cap T')$ is Borel. Moreover there is a unique arc $\vec{J}$ contained in T and joining c_T to $\rho(T \cap T')$ and similarly a unique arc $\vec{J}'$ contained in T' and joining $c_{T'}$ to $\rho(T \cap T')$. The graphs of these two mappings are Borel ; so the functions $(T, T') \mapsto \vec{J}$ and $(T, T') \mapsto \vec{J}'$ are Borel, and so is the function $(T, T') \mapsto \vec{J} \vee \mathfrak{f}(\vec{J}')$. And this latter arc joins c_T to $c_{T'}$ inside $T \cup T'$. ☐

Lemma 7.6. *For each $p \in \mathbb{P}$ there exists a $\mathbf{\Sigma}$-meaurable function φ_p defined on the set $\mathcal{F}^*$ of all closed nonempty subsets of X such that $\varphi_p(F) = (T, a)$ where $T \in \mathcal{T}_p(X)$, $T \subset F$ and $a = c_T$ if F matches p and $\varphi_p(F) = *$ if not.*

Proof. The set $\tilde{E}_p = \{(F, T, a) \in \mathcal{F}^* \times \mathcal{T}_p \times X : T \subset F \text{ and } a = c_T \text{ and } F \text{ matches } T\}$ is Borel. Its projection $\mathcal{M}_p$ on $\mathcal{F}^*$ is analytic and by Theorem 1.6 there exists a $\mathbf{\Sigma}$-measurable selection of $\tilde{E}_p$ on $\mathcal{M}_p$. Extending this selection by $*$ on $\mathcal{F}^* \setminus \mathcal{M}_p$ we define a $\mathbf{\Sigma}$-measurable function φ_p. For $F \in \mathcal{M}_p$ we will set $\varphi_p^1(F) = T$ and $\varphi_p^2(F) = a$ if $\varphi_p(F) = (T, a)$. ☐

Lemma 7.7. *Let $p \in \mathbb{P}$, $\varepsilon > 0$ and the $\mathbf{\Sigma}$-measurable function φ_p be as in Lemma 7.6. Then there exists a $\mathbf{\Sigma}$-measurable function $\lambda_{p,\varepsilon} : \mathcal{F}^* \to \mathbb{P}_{p,\varepsilon}$ such that for all $F \in \mathcal{F}^*$, if $\varphi_p(F) \neq *$, the triod $\varphi_p^1(F)$ is weakly compatible with the trap $\lambda_{p,\varepsilon}(F)$.*

Proof. The set $\{F : \varphi_p(F) \neq *\}$ is analytic hence belongs to $\mathbf{\Sigma}$.

For each $q \in \mathbb{P}_{p,\varepsilon}$ the set $\{T \in \mathcal{T}(X) : T \textit{ is weakly compatible with } q\}$ is Borel. Thus the set $\mathscr{Z}_q = \{F : \varphi_p(F) \textit{ is weakly compatible with } q\}$ belongs to $\mathbf{\Sigma}$. And it is easy to check that

$$\mathcal{M}_p = \{F : \varphi_p(F) \neq *\} \subset \bigcup\nolimits_{q \in \mathbb{P}_{p,\varepsilon}} \mathscr{Z}_q$$

since for all $F \in \mathcal{M}_p$ the center of the triod $\varphi^1_p(F)$ belongs to the domain W' of some trap $q = (W', I'_0, I'_1, I'_2) \in \mathbb{P}_{p,\varepsilon}$. Thus it is possible to find a countable partition $(\mathscr{X}'_q)_{q\in\mathbb{P}_{p,\varepsilon}}$ of $\mathcal{M}_p$ such that $\mathscr{X}'_q \in \mathbf{\Sigma}$ and $\mathscr{X}'_q \subset \mathscr{X}_q$. Setting

$$\lambda_{p,\varepsilon}(F) = q \iff F \in \mathscr{X}'_q$$

completes the proof. □

Lemma 7.8. *For all* $(p, q) \in \mathbb{P}^2$*, all* $r \in \mathbb{Q}^+$ *and all* $k \in \omega$ *there exists a* $\mathbf{\Sigma}$*-measurable function* $\psi_{p,q,r,k}$ *on* $\mathcal{F}^*$ *such that* $\psi_{p,q,r,k}(F)$ *is an oriented arc* J *joining* $\varphi^2_p(F)$ *to* $\varphi^2_q(F)$ *inside* F *with* $\mathrm{diam}(J \cup \mathrm{dom}(p') \cup \mathrm{dom}(q')) < r$ *and* $\max(\mathrm{diam}(p'), \mathrm{diam}(q')) < 2^{-k}$ *if such a* J *exists, and* $\psi_{p,q,r,k}(F) = *$ *if not.*

Proof. As above the set

$$\hat{E}_{p,q,r} = \Big\{(F, T, T', J) \in \mathcal{F}^* \times \mathcal{T}_p(X) \times \mathcal{T}_q(X) \times \mathcal{J}(X) : J, T, T' \subset F$$
$$\text{and } e(J) = (c_T, c_{T'}) \text{ and } \mathrm{diam}(J \cup p \cup q) < r\Big\}$$

is Borel and has a $\mathbf{\Sigma}$-measurable selection $\chi_{p,q,r}$ on its projection $E^*_{p,q,r} = \pi(\hat{E}_{p,q,r})$ on $\mathcal{F}^*$ that we extend by $*$ on the coanalytic set $\mathcal{F}^* \setminus E^*_{p,q,r}$. If $\chi_{p,q,r}(F) = (T, T', J)$ we denote $\chi^1_{p,q,r}(F) = T$, $\chi^2_{p,q,r}(F) = T'$ and $\chi^3_{p,q,r}(F) = J$. Then for $F \in E^*_{p,q,r}$, we necessarily have $F \in \mathcal{M}_p \cap \mathcal{M}_q$.

Clearly, if $F \in E^*_{p,q,r}$, we also have $F \in E^*_{p,q,r'}$ for all $r' > r$; so the set $\{(p, q, r) : F \in E^*_{p,q,r}\}$ is either empty or infinite.

For $p, q \in \mathbb{P}$, $r \in \mathbb{Q}^+$ and $k \in \omega$, we consider for all $F \in \mathcal{M}_p \cap \mathcal{M}_q$: $p' = \lambda_{p,2^{-k}}(F)$, $W' = \mathrm{dom}(p')$, $q' = \lambda_{q,2^{-k}}(F)$ and $V' = \mathrm{dom}(q')$.

By definition of $\lambda_{p,2^{-k}}(F)$ the triod $T = \varphi^1_p(F)$ is weakly compatible with p'. So $T_{|W'}$ is compatible with p', and $\mathbf{c}(T) = \mathbf{c}(T_{|W'})$. Similarly the triod $S = \varphi_q(F)$ is weakly compatible with q'. Thus using the functions $\gamma_{p'}$ and $\gamma_{q'}$ from Lemma 7.5, we can consider the oriented arcs

$$\vec{I}_{p,k,r,F} = \gamma_{p'}\big(\varphi^1_{p'}(F), \chi^1_{p',q',r}(F)\big)$$
$$\vec{J}_{q,k,r,F} = \gamma_{q'}\big(\varphi^1_{q'}(F), \chi^2_{p',q',r}(F)\big)$$

respectively contained in $W' = \mathrm{dom}(p')$ and $V' = \mathrm{dom}(q')$, and joining respectively $\mathbf{c}(T)$ to $e_0(\chi^3_{p',q',r}(F))$ and $\mathbf{c}(S)$ to $e_1(\chi^3_{p',q',r}(F))$.

Assume moreover that $F \in E^*_{p,q,r}$. Since the functions γ_p are Borel, it is clear that the functions $F \mapsto \vec{I}_{p,k,r,F}$ and $F \mapsto \vec{J}_{q,k,r,F}$ are $\mathbf{\Sigma}$-measurable, and so is the concatenation

$$\psi_{p,q,r,k}(F) = \vec{I}_{p,k,r,F} \vee \chi^3_{p',q',r}(F) \vee \mathfrak{f}(\vec{J}_{q,k,r,F})$$

which is an oriented arc joining $\varphi^2_p(F)$ to $\varphi^2_q(F)$. Moreover $\psi_{p,q,r,k}(F) \subset \chi^3_{p',q',r}(F) \cup W' \cup V'$, hence $\mathrm{diam}(\psi_{p,q,r,k}(F)) < r$. □

Applying Lemma 1.8 to the family $\big(\varphi^2_p\big)_{p\in\mathbb{P}}$ and to the family $\big(\psi^3_{p,q,r}\big)_{(p,q,r)\in\mathcal{T}(X)^2\times\mathbb{Q}^+}$ we get $\mathbf{\Sigma}$-measurable functions $\tilde{\varphi} : \mathcal{F}^* \to X^\omega$ and $\tilde{\psi} : \mathcal{F}^* \to \vec{\mathcal{J}}(X)^\omega$ such that for every $F \in \mathcal{F}^*$

$$\{\tilde{\varphi}_n(F) : n \in \omega\} = \{\varphi^2_p(F) : F \in E_p,\ p \in \mathbb{P}\}$$
$$\{\tilde{\psi}_n(F) : n \in \omega\} = \{\psi^3_{p,q,r,k}(F) : p, q \in \mathbb{P},\ r \in \mathbb{Q}^*,\ k \in \omega\}$$

Then the sequence $(a_n) = \big(\tilde{\varphi}_n(F)\big)_n$ is τ-dense in Σ^F since for every trap p matched by F there is some n such that a_n is the center of some triod compatible with p.

Lemma 7.9. *If* $(a_m, a_n) = (\tilde{\varphi}_m(F), \tilde{\varphi}_n(F)) \in E_F$ *then*

$$\delta^F(a_m, a_n) = \inf\{\operatorname{diam}(J) : \exists k \in \omega \ J = \tilde{\psi}_k(F) \text{ and } e(J) = (a_m, a_n)\} .$$

Proof. If $(a_m, a_n) \in E_F$ then $\delta^F(a_n, a_m) < \infty$, and for all $r \in \mathbb{Q}^+$ such that $r > \delta^F(a_n, a_m)$ there exists an arc J_0 with endpoints a_n and a_m such that $\delta^F(a_n, a_m) \leq \operatorname{diam}(J_0) < r$ and an integer k such that $\operatorname{diam}(J_0) + 2^{1-k} < r$. There exist p and q in $\mathbb{P}$ such that $a_n = \varphi^2_p(F)$ and $a_m = \varphi^2_q(F)$, and we denote $p' = \lambda_{p,2^{-k}}(F)$ and $q' = \lambda_{p,2^{-k}}(F)$. We then have $\operatorname{diam}(\vec{I}_{p,k,r,F}) < 2^{-k}$ and $\operatorname{diam}(\vec{I}_{q,k,r,F}) < 2^{-k}$, hence

$$\operatorname{diam}\big(\vec{I}_{p,k,r,F} \vee J_0 \vee \mathfrak{f}(\vec{I}_{q,k,r,F})\big) < 2^{-k} + \operatorname{diam}(J_0) + 2^{1-k} = \operatorname{diam}(J_0) + 2^{1-k} < r .$$

It follows that $F \in E^*_{p',q',r}$ and that $J_1 = \psi_{p,q,r,F}$ satisfies $e(J_1) = (a_m, a_n)$ and $\operatorname{diam}(J_1) < r$. Then there exists $k \in \omega$ such that $J_1 = \tilde{\psi}_k(F)$ and we are done. ☐

To finish the proof of Theorem 7.1 define $\boldsymbol{\sigma} : \mathcal{F}(X) \to \boldsymbol{C}$ by $\boldsymbol{\sigma}(F) = \big(\tilde{\varphi}(F), \tilde{\psi}(F), F\big)$ and observe that if F is arc-connected then any two elements $\tilde{\varphi}_m(F), \tilde{\varphi}_n(F)$ of $\tilde{\varphi}(F)$ are E_F-equivalent and apply Lemma 7.9. ☐

8. The exact complexity of $\mathcal{C}_{\text{arc}}(\boldsymbol{R}^2)$

By Corollary 7.2 the set $\mathcal{C}_{\text{arc}}(\boldsymbol{R}^2)$ of arc-connected compact subsets of the plane $\boldsymbol{R}^2$ is a $\check{\mathcal{A}}(\boldsymbol{\Pi}^1_1)$ subset of $\mathcal{K}(\boldsymbol{R}^2)$, and we now prove that this upper bound complexity is optimal.

Theorem 8.1. *The set* $\mathcal{C}_{\text{arc}}(\boldsymbol{R}^2)$ *is* $\check{\mathcal{A}}(\boldsymbol{\Pi}^1_1)$*-complete.*

Proof. We start by some preliminary constructions.

The Cantor space $\boldsymbol{B}$: We first define inductively for all $s \in \omega^{<\omega}$ reals a_s and b_s in $\mathbb{I} = [0,1]$ as follows:

Set $a_\emptyset = 0$, $b_\emptyset = 1$, and fix two increasing sequences $(a_n)_{n\in\omega}$, $(b_n)_{n\in\omega}$ such that:

$$a_\emptyset < a_0 < b_0 < a_1 < b_1 < \cdots a_n < b_n < a_{n+1} < \cdots < b_\emptyset \quad \text{and} \quad b_\emptyset = \sup_n a_n = \sup_n b_n$$

then for all $s \neq \emptyset$ if $h_s : \xi \mapsto (1-\xi)a_s + \xi\, b_s$ is the affine function such that $h_s(\mathbb{I}) = [a_s, b_s]$ define $a_{s\frown\langle n\rangle} = h_s(a_n)$ and $b_{s\frown\langle n\rangle} = h_s(b_n)$.

Then $\rho = \sup_{n\in\omega}(b_n - a_n) < 1$ and for all $s \in \mathbb{S}$, $b_s - a_s \leq \rho^{|s|}$. Hence for all $\sigma \in \omega^\omega$ the real $b_\sigma = \inf_{s\prec\sigma} b_s = \sup_{s\prec\sigma} a_s$, is well defined, and we set:

$$\boldsymbol{B}_0 = \{b_s : s \in \omega^{<\omega}\} \quad ; \quad \boldsymbol{B}_1 = \{b_\sigma : \sigma \in \omega^\omega\} \quad \text{and} \quad \boldsymbol{B} = \boldsymbol{B}_0 \cup \boldsymbol{B}_1$$

The set $\boldsymbol{B}$ is clearly a perfect compact subset of $\mathbb{I}$ with empty interior.

The next construction is essentially due to Becker (see [4], 33.17 and 37.11). Let $\mathscr{T} \subset 2^{\omega^{<\omega}}$ denote the set of all trees on ω.

Lemma 8.2. *There is a continuous function* $B : \mathscr{T} \to \mathcal{K}(\boldsymbol{R}^2)$ *which assigns to any* $T \in \mathscr{T}$ *a connected compact subset* $B(T)$ *of the unit square* $\mathbb{I}^2$ *such that :*

a) $(0,1) \in B(T)$ *and* $(\mathbb{I} \times \{0\}) \cup (\{1\} \times \mathbb{I}) \subset B(T)$*,*

b) $B(T)$ *has at most two arc-components,*

c) T *is ill-founded* $\iff$ $B(T)$ *is arc-connected*

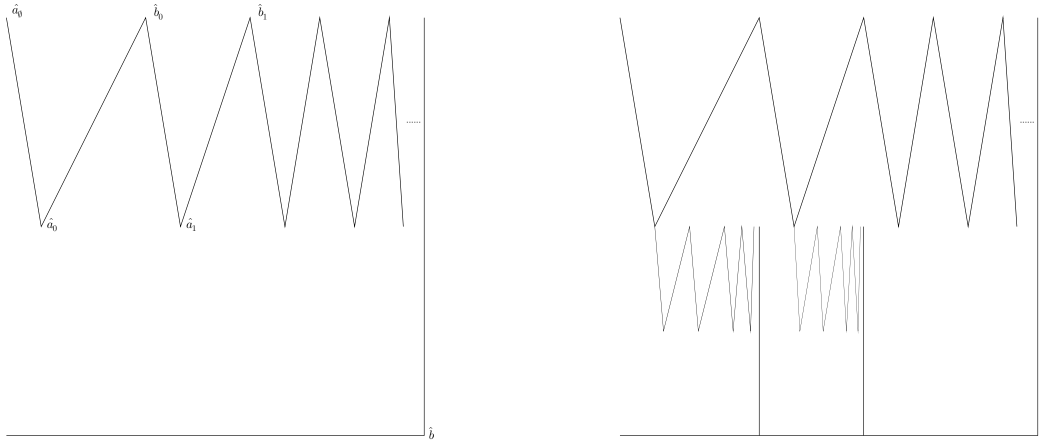
$\iff$ *there is an arc in* $B(T)$ *connecting* $(0,1)$ *and* $(1,0)$.

Proof. For any $u, v \in \boldsymbol{R}^2$ we denote by $[u, v]$ the line segment joining u to v. For all $s \in \omega^{<\omega}$ let $\hat{a}_s, \hat{b}_s$ the elements of $\boldsymbol{R}^2$ defined by $\hat{a}_s = (a_s, 2^{-|s|})$ and $\hat{b}_s = (b_s, 2^{1-|s|})$, and consider the family $(R_s)_{s\in\omega^{<\omega}}$ of compact subsets of $\mathbb{I}^2$ defined as follows:

$$R_\emptyset = (\mathbb{I} \times \{0\}) \cup (\{1\} \times \mathbb{I}) \cup [\hat{a}_\emptyset, \hat{a}_0] \cup \bigcup_{n\in\omega} \big([\hat{a}_n, \hat{b}_n] \cup [\hat{b}_n, \hat{a}_{n+1}]\big)$$

(see Fig. 3); and for all $s \in \omega^{<\omega}$, R_s is the image of $R_\emptyset$ under the affine mapping sending $\hat{a}_\emptyset$ to $\hat{a}_s$ and $\{b_\emptyset\} \times \mathbb{I}$ to $\{b_s\} \times [0, 2^{-|s|}]$. Finally for all $T \in \mathscr{T}$ let $B(T) = \bigcup_{s\in T} R_s$.

FIGURE 3



It is clear that each R_s is a connected, but not arc-connected, compact set. Since R_s and $R_{s\frown\langle n\rangle}$ have $\hat{a}_{s\frown\langle n\rangle}$ and $(b_s, 0)$ in common, then any point in $B(T)$ is connected by an arc either to $\hat{a}_\emptyset$ or to $\hat{b} = (1, 0)$. Hence $B(T)$ has at most two arc-components.

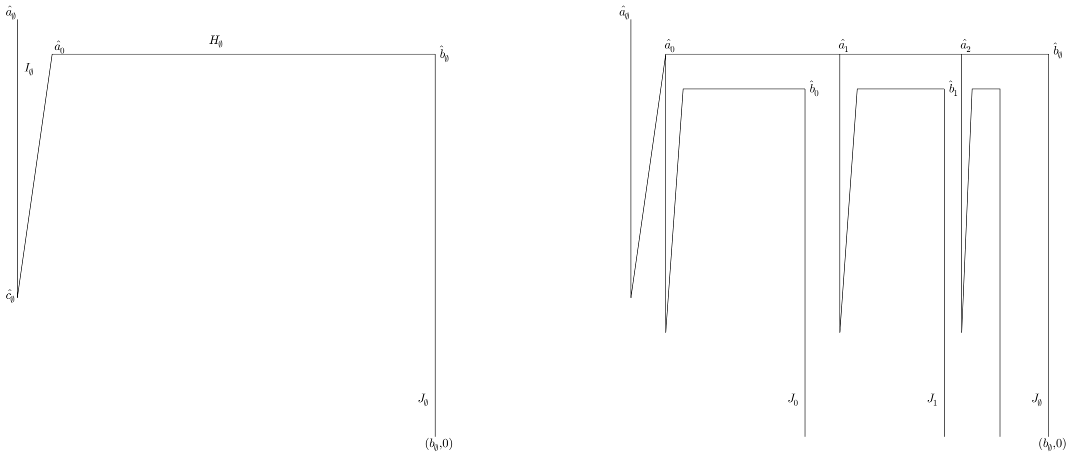
The set $B(T)$ is compact : indeed, if (z_i) is a sequence in $B(T)$ which converges to $z \in \mathbb{I}^2$, there exists a sequence $s^{(i)} \in T$ such that $z_i \in R_{s^{(i)}}$ and we can extract a subsequence such that
– either $s^{(i)}$ is a constant s, and then $z \in R_s$ since R_s is closed,
– or there exists $s \in T$ and a sequence (n_i) converging to ∞ such that $s\frown\langle n_i\rangle \preceq s^{(i)}$, and then $z \in \{b_s\} \times [0, 2^{-|s|}] \subset R_s \subset B(T)$,
– or else there exists $\sigma \in \omega^\omega$ such that $\sigma_{|i} \preceq s^{(i)}$, and $z_i \to (b_\sigma, 0) \in R_\emptyset \subset B(T)$.

It is not difficult to see that $T \mapsto B(T)$ is continuous from $\mathscr{T}$ to $\mathcal{K}(\boldsymbol{R}^2)$. Moreover if σ is a branch of T, then for all $s \prec \sigma$ there exists an arc $J_n \subset R_s \subset B(T)$ with endpoints $\hat{a}_{\sigma_{|n}}$ and $\hat{a}_{\sigma_{|n+1}}$. And the concatenation of the J_n's yields an arc connecting $\hat{a}_\emptyset$ to $(b_\sigma, 0)$. Thus in this case $B(T)$ is arc-connected.

Conversely, if $B(T)$ is arc-connected let $\gamma : \mathbb{I} \to B(T)$ be a continuous mapping with $\gamma(t) = \big(\gamma_1(t), \gamma_2(t)\big)$, $\gamma(0) = \hat{a}_\emptyset$ and $\gamma(1) \in \mathbb{I} \times \{0\}$, and consider $\theta = \inf\{t : \gamma_2(t) = 0\} > 0$ and $\theta_k = \sup\{t < \theta : \gamma_2(t) \geq 2^{-k}\}$. Then for $\theta_k < t < \theta$, $\gamma_2(t) < 2^{-k}$ and there exists some $s^{(k)} \in \omega^k$ such that $\gamma(t) \in \bigcup\{R_s;\ s \in T,\ \text{and } s \succeq s^{(k)}\}$; so $s^{(k)} \in T$ and $s^{(k)} \prec s^{(k+1)}$. Hence there exists $\sigma \in \omega^\omega$ such that $s^{(k)} \prec \sigma$ for all k. Thus σ is a branch of T, and T is ill-founded. □

A connected compact set with uncountably many arc-components

Figure 4



Let $\boldsymbol{B}$ the Cantor set constructed above with the elements a_s, b_s and b_σ for $s \in \omega^{<\omega}$ and $\sigma \in \omega^\omega$, and fix a decreasing sequence $(\alpha_n) \in \mathbb{I}$ such that $\alpha_0 = 1$ and $\lim_n \alpha_n = \theta = \dfrac{2}{3}$.

For all $s \in \omega^{<\omega}$ with $|s| = k$, let P_s be the union of the following four segments of the unit square (see Fig. 4):

– the vertical segment I_s with endpoints $\hat{a}_s = (a_s, \alpha_k)$ and $\hat{c}_s = (a_s, \alpha_k - \theta)$ (of length θ),

– the segment with endpoints $\hat{c}_s$ and $\hat{a}_{s^\frown\langle 0\rangle}$,

– the horizontal segment H_s with endpoints $\hat{a}_{s^\frown\langle 0\rangle}$ and $\hat{b}_s = (b_s, \alpha_{k+1})$,

– the vertical segment J_s with endpoints $\hat{b}_s$ and $(b_s, 0)$ which has length α_{k+1}.

It is clear that each P_s is arc-connected and has the point $\hat{a}_{s^\frown\langle n\rangle}$ in common with $P_{s^\frown\langle n\rangle}$, hence $P_\infty := \bigcup_{s\in\omega^{<\omega}} P_s$ is arc-connected, and the compact set $P = \overline{P_\infty}$ is connected.

Lemma 8.3. *The set P is the union of P_∞ and the vertical segments $J_\sigma = \{b_\sigma\} \times [0, \theta]$. Moreover there is no arc in P connecting $\hat{a}_\emptyset$ to any J_σ.*

Proof. Let (z_i) be a sequence of points of P_∞ converging to $z = (x, y) \in P$. There exists a $s^{(i)} \in \omega^{<\omega}$ such that $z_i \in P_{s^{(i)}}$, and again one can extract a subsequence such that :

– either $s^{(i)}$ is a constant s and then $z \in P_s \subset P_\infty$ since P_s is closed,

– or there exists $s \in \omega^{<\omega}$ and (n_i) in ω converging to ∞ such that $s^\frown\langle n_i\rangle \preceq s^{(i)}$, and then $z \in J_s \subset P_s \subset P_\infty$,

– or else there exists $\sigma \in \omega^\omega$ such that $\sigma_{|i} \preceq s^{(i)}$, and then $y \leq \theta$ and $x = b_\sigma$, hence $z \in J_\sigma$.
Conversely, if $z = (b_\sigma, y) \in J_\sigma$, the points $(a_s, y + \alpha_k - \theta)$ for $s = \sigma_{|k}$ belong to P_∞ and converge to z; hence $J_\sigma \subset P$.

If there were an arc J connecting $\hat{a}_\emptyset$ and J_σ, then J should go through points in H_s for every $s \prec \sigma$ and J should contain I_s for every $s \prec \sigma$, which is impossible. So each J_σ for $\sigma \in \omega^\omega$ is an arc-component of P. □

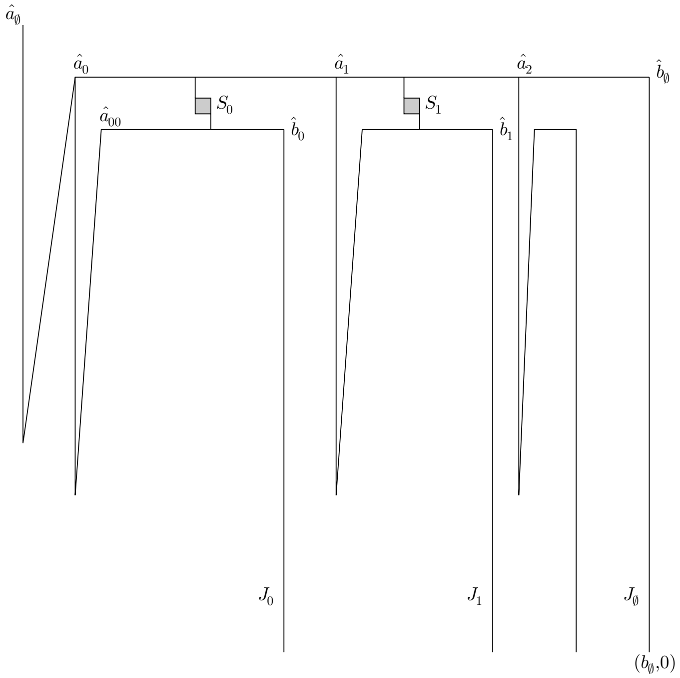
Construction of a compact connected subset of $\mathbb{I}^2$

We recall that for any $s \in \omega^{<\omega}$ with length $k = |s| > 0$ we denote by s^* the sequence of length $k - 1$ such that $s^* \prec s$.

We now modify the above constructed compact set P by adding shortcuts between H_s and H_{s^*} : choose for every $s \in \omega^{<\omega}$ of length $k \geq 1$ some small square Q_s inside the open rectangle

$]a_{s^\frown\langle 0\rangle}, b_s[\times]\alpha_{k+1}, \alpha_k[$ and consider the positive homothety h_s transforming the unit square $\mathbb{I}^2$ into Q_s. For a given tree $T_s \in \mathscr{T}$ the shortcut S_s will be the union of $h_s(B(T_s))$, where $B(T_s)$ is the compact set defined in Lemma 8.2, with two vertical segments connecting respectively $h_s(0,1)$ to H_{s^*} and $h_s(1,0)$ to H_s (see Fig. 5).

FIGURE 5



Then for any family $\tilde{T} = (T_s)_{s\in\omega^{<\omega}} \in \mathscr{T}^{(\omega^{<\omega})}$, let

$$\Psi(\tilde{T}) = P \cup \bigcup\nolimits_{k\geq 1} \bigcup\nolimits_{s\in\omega^k} S_s.$$

Lemma 8.4. *$\Psi(\tilde{T})$ is compact and connected. Moreover the function Ψ is continuous from the compact space $\mathscr{T}^{(\omega^{<\omega})}$ to $\mathcal{K}(\boldsymbol{R}^2)$.*

Proof. Since every point of $B(T_s)$ is connected to $(0,1)$ or to $(1,0)$ (both if T is ill-founded), every point of S_s is connected to P_∞. So $\Psi(\tilde{T})$ is the union of the compact set P and the arc-connected set $P_\infty \cup \bigcup_s S_s$, hence is connected.

To see that $\Psi(\tilde{T})$ is compact, it is enough to prove that if a sequence (z_i) in $\bigcup_s S_s$ converges to $z = (x,y) \in \mathbb{I}^2$, then $z \in \Psi(\tilde{T})$. Then there exists a sequence $(s^{(i)})$ in $\omega^{<\omega}$ such that $z_i \in S_{s_i}$. Again, if s_i is constant equal to s, S_s is closed and $z \in S_s \subset \Psi(\tilde{T})$. If there exist $s \in \omega^{<\omega}$ and (n_i) tending to ∞ such that $s^\frown\langle n_i\rangle \preceq s^{(i)}$, then $x = b_s$ and $z \in J_s \subset P \subset \Psi(\tilde{T})$. And finally, if there exists $\sigma \in \omega^\omega$ such that $\sigma_{|i} \preceq s^{(i)}$, then $x = b_\sigma$ and $y = \theta$, hence $z \in J_\sigma \subset \Psi(\tilde{T})$.

Observe that for every $\varepsilon > 0$ there are only finitely many $s \in \omega^{<\omega}$ such that $d(S_s, P) > \varepsilon$. It follows that if we fix an enumeration $(s^{(n)})_n$ of $\omega^{<\omega}$, then Ψ is the uniform limit of a sequence of continuous functions ψ_n from $\mathscr{T}^{(\omega^{<\omega})}$ to $\mathcal{K}(\boldsymbol{R}^2)$ associating to any $\tilde{T}$ the set $\psi_n(\tilde{T}) = P \cup \bigcup_{j=0}^n S_{s^{(j)}}$; hence Ψ is continuous. □

It results from what precedes that $\Psi(\tilde{T})\setminus\bigcup_{\sigma\in\omega^\omega} J_\sigma$ is arc-connected and that for each $\sigma\in\omega^\omega$, either J_σ is an arc-component of $\Psi(\tilde{T})$, or J_σ is connected to P_∞ by an arc in $\Psi(\tilde{T})$.

Lemma 8.5. *For any $\sigma\in\omega^\omega$, J_σ is connected to P_∞ by an arc in $\Psi(\tilde{T})$ if and only if the set $\{s\prec\sigma : T_s$ is well-founded$\}$ is finite.*

Proof. Suppose first that there exists $u\prec\sigma$ such that T_s is ill-founded for each s with $u\prec s\prec\sigma$. Let $s^{(k)}\in\omega^{<\omega}$ the beginning of σ with length $|u|+k$. Then since $S_{s^{(k)}}$ is arc-connected there exists for each $k\geq 1$ a continuous function $\gamma_k : [1-2^{1-k},1-2^{-k}]\to H_{s^{(k-1)}}\cup H_{s^{(k)}}\cup S_{s^{(k)}}$ such that $\gamma_k(1-2^{1-k})=\hat{a}_{s^{(k-1)}\frown\langle 0\rangle}$ and $\gamma_k(1-2^{-k})=\hat{a}_{s^{(k)}\frown\langle 0\rangle}$. It follows that there exists a continuous function $\gamma : [0,1[\to P_\infty\cup\bigcup_s S_s$ which extends all γ_k, that $\gamma(0)\in P_\infty$ and that $\gamma(\xi)\to(b_\sigma,\theta)\in J_\sigma$ when $\xi\to 1$. This shows that J_σ is connected to P_∞ by an arc in $\Psi(\tilde{T})$.

Conversely, suppose that there exists a continuous path $\gamma:\mathbb{I}\to\Psi(\tilde{T})$ connecting $\hat{a}_\emptyset$ to J_σ. Then $\xi^*=\inf\{\xi : \gamma(\xi)\in J_\sigma\}>0$. It is easily seen that for any $s\in\omega^{<\omega}$, $\Psi(\tilde{T})\setminus H_s$ is not connected and that H_s separates $\hat{a}_s$ from J_σ whenever $s\prec\sigma$. So $\xi_k=\sup\{\xi\leq\xi^* : \gamma(\xi)\in H_{\sigma_{|k}}\}$ is well defined for all $k\geq 0$, and $\xi_k<\xi_{k+1}<\xi^*$. Then if $\xi^{**}=\sup\xi_k\leq\xi^*$ we necessarily have $\gamma(\xi^{**})\in\limsup_k H_{\sigma_{|k}}=\{(b_\sigma,\theta)\}\subset J_\sigma$, hence $\xi^{**}=\xi^*$. Moreover by continuity of γ at ξ^* there exists k_0 such that for $\xi_{k_0}<\xi<\xi^* : d(\gamma(\xi),\gamma(\xi^*))<1-\theta$. It follows that no point $\hat{c}_{\sigma_{|k}}$ for $k>k_0$ can belong to $\gamma([\xi_{k_0},\xi^*])$. Thus γ has to go through all shortcuts $S_{\sigma_{|k}}$ for $k>k_0$. This implies that the corresponding trees $T_{\sigma_{|k}}$ are ill-founded for all $k>k_0$ and the set $\{s\prec\sigma : T_s$ is well-founded$\}$ is finite. □

Lemma 8.6. *The compact set $\Psi(\tilde{T})$ is arc-connected if and only if for all $\sigma\in\omega^\omega$ there are only finitely many T_s with $s\prec\sigma$ which are well-founded.*

Proof. The compact set $\Psi(\tilde{T})$ is arc-connected if and only if each J_σ for $\sigma\in\omega^\omega$ is connected to $\hat{a}_\emptyset$ by an arc; and by the previous lemma this happens if and only if only finitely many T_s with $s\prec\sigma$ are well-founded. □

To finish the proof of Theorem 8.1 let $Z\subset 2^\omega$ be any $\check{\mathcal{A}}(\mathbf{\Pi}^1_1)$-set. We want to construct a continuous function $\Phi : 2^\omega\to\mathcal{K}(\boldsymbol{R}^2)$ such that $\Phi(\zeta)$ is arc-connected if and only if $\zeta\in Z$. By definition of $\check{\mathcal{A}}(\mathbf{\Pi}^1_1)$, there is a Souslin scheme $(\Gamma_s)_{s\in\omega^{<\omega}}$ of $\mathbf{\Pi}^1_1$-sets, which can be assumed to be regular, such that

$$2^\omega\setminus Z=\bigcup_{\sigma\in\omega^\omega}\bigcap_{s\prec\sigma}\Gamma_s$$

Since the set WF of well-founded trees is $\mathbf{\Pi}^1_1$-complete in $\mathscr{T}$, we can find for each $s\in\omega^{<\omega}$ a continuous function $T_s : \zeta\mapsto T_s(\zeta)$ such that $T_s(\zeta)\in WF\iff\zeta\in\Gamma_s$. Then the function $\tilde{T} : 2^\omega\to\mathscr{T}^{(\omega^{<\omega})}$ defined by $\tilde{T}(\zeta)=(T_s(\zeta))_{s\in\omega^{<\omega}}$ is continuous too and so is $\Phi : \zeta\mapsto\Psi(\tilde{T}(\zeta))$.

If $\zeta\in Z$, then for all $\sigma\in\omega^\omega$, $\zeta\notin\bigcap_{s\prec\sigma}\Gamma_s$, thus there exists $s_0\prec\sigma$ such that $\zeta\notin\Gamma_{s_0}$ and since the Souslin scheme is regular we have also for $s_0\preceq s\prec\sigma : \zeta\notin\Gamma_s$ and the tree $T_s(\zeta)$ is ill-founded; it follows that J_σ is connected to $\hat{a}_\emptyset$ by an arc. Since this happens for all J_σ, the compact set $\Phi(\zeta)$ is arc-connected.

Conversely, if $\zeta\notin Z$ there is a $\sigma\in\omega^\omega$ such that for all $s\prec\sigma : \zeta\in\Gamma_s$ and $T_s(\zeta)\in WF$; it follows that J_σ is connected to $\hat{a}_\emptyset$ by none arc, and $\Phi(\zeta)$ is not arc-connected.

Thus $Z=\Phi^{-1}(\mathcal{C}_{\mathrm{arc}}(\boldsymbol{R}^2))$ and the proof is complete. □

References

1. H. Becker and R. Pol, *Note on path-components in complete spaces*, Top. and Appl. **114** (2001), 107–114.
2. G. Debs and J. Saint Raymond, *The descriptive complexity of connectedness in Polish spaces*. Fund. Math. **249** (2020) , n 3, 261286.

3. G. Debs and J. Saint Raymond, *On the arc-wise connection relation in the plane.* Topology and its Appl., doi :10.1016/j.topol.2024.108975, To appear .
4. A. Kechris, *Classical descriptive set theory.* Graduate Texts in Mathematics, **156**. Springer-Verlag, New York, 1995.
5. K. Kuratowski, *Topology Vol. I.* New edition, revised and augmented. Translated from the French by J. Jaworowski Academic Press, New York-London; Państwowe Wydawnictwo Naukowe, Warsaw (1966).
6. K. Kunen and M.Starbird, *Arc components in metric continua*, Top. Appl. **14** (1982)167–170.
7. R.L. Moore, *Concerning triods in the plane and the junction points of plane continua*, Proc. Nat. Acad. Sci. **14** (1928) 85–88.

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