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ON A QUESTION ABOUT NON-UNIQUENESS OF GLOBAL MINIMA

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Let X be a topological space, I and J be sequentially l.s.c. real functions on X such that J is non-negative with sequentially compact sublevels, and I/J is not bounded from below outside the sublevel $J^{-1}([0,c[)$ of J for some $c > 0$. Is it true that for any large enough λ and any increasing l.s.c. real function φ the function $x \mapsto I(x) + \lambda J(x) + \mu\varphi\circ J(x)$ has at least two global minima for some positive μ ? We give a positive answer to this question assuming that $I + \lambda J + \mu\circ\varphi$ has sequentially compact sublevels for some λ and all $\mu > 0$.

During my stay in Catania in May 2022, Prof. B. Ricceri asked me the above stated question related to his paper [1]. Theorem 6 of the present paper answers positively to this question. This improves Theorem 1.2 of [1] and gives a positive answer to Problem 1 of the same paper.

Let X be a topological space, I and J be two real sequentially l.s.c. functions on X satisfying the following three:

(i) The sublevels $S_\alpha := J^{-1}(]-\infty,\alpha])$ of J are sequentially compact and $\inf_{x\in X} J(x) = 0$,

(ii) There exists $c > 0$ such that $\inf_{J(x)>c} \dfrac{I(x)}{J(x)} = -\infty$,

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(iii) There exists a l.s.c. increasing function $\varphi : J(X) \to \mathbb{R}^+$ such that for all $\mu > 0$, the function

$$\psi_\mu : x \mapsto I(x) + J(x) + \mu\varphi\circ J(x)$$

admits a unique global minimum[1] x_μ,

Under conditions (i) and (ii), X cannot be compact : the function I would be bounded from below on X, thus also the quotient I/J on $X \setminus S_c$.

Since J attains its minimum, there exists x^* such that $J(x^*) = 0$. And we can assume that I attains at x^* its minimum on the compact set $S_0 \neq \emptyset$. Up to replacing φ by the increasing l.s.c. function : $t \mapsto \varphi(t) - \varphi(0)$, in what case ψ_μ is replaced by $\psi_\mu - \mu\varphi(0)$ which has the same minima, we will always assume $\varphi(0) = 0$.

Notice that the function φ in (iii) has to be unbounded : indeed condition (ii) implies that $I + \lambda J$ cannot be bounded from below but so is $I + \lambda J + \mu\varphi\circ J$ which attains a finite global minimum.

Lemma 1. There exist an $\varepsilon > 0$, an integer n_0 and some $\lambda^* > 0$ such that for $\lambda \geq \lambda^*$, we never have simultaneously for any $x \in X$:

$$\lambda J(x) \leq n_0 \quad \text{and} \quad I(x) \leq I(x^*) - n_0\,.$$

Proof. Take $\varepsilon > 0$ in $J(X)$. Then the set S_ε is sequentially compact and the set $F_n = \{x \in X : I(x) \leq I(x^*) - n\}$ is sequentially closed. Moreover $\bigcap_{n\in\mathbb{N}} F_n$ is empty, and it follows that the non-increasing sequence $(S_\varepsilon \cap F_n)_n$ of sequentially compact sets has empty intersection : thus there exists some integer $n_0 > 0$ such that $F_{n_0} \cap S_\varepsilon = \emptyset$. Take $\lambda^* = \dfrac{n_0}{\varepsilon}$. Then if $x \in F_{n_0}$ and $\lambda \geq \lambda^*$, we must have $x \notin S_\varepsilon$, hence

$$\lambda J(x) > \lambda\varepsilon \;\geq \lambda^*\varepsilon = n_0\,,$$

the wanted result. □

Up to taking $\lambda \geq \lambda^*$ and replacing J by $J' = \lambda J$ and φ by $\varphi' : t \mapsto \varphi(t/\lambda)$, hence

$$I(x) + J'(x) + \mu\varphi'\circ J'(x) = I(x) + \lambda J(x) + \mu\varphi\circ J(x)\,,$$

[1] In fact we can assume φ l.s.c., increasing on $J(X)$ and defined on $[0,+\infty[$ By hypothesis J is unbounded, and if $t \in \mathbb{R}^+ \setminus J(X)$, there exists $\alpha, \beta \in J(X)$ with $\alpha < t < \beta$. We then put $\alpha^* = \sup J(X) \cap [0,t[$ and $\beta^* = \inf J(X) \cap]t,\infty[$. Si $\alpha^* = t = \beta^*$, we define $\varphi(t) = \sup\{\varphi(\alpha) : \alpha \in J(X),\ \alpha < t\}$, what ensures the semi-continuity at t.

In contrary if $\alpha^* < \beta^*$, we put $\varphi(\alpha^*) = \sup\{\varphi(\alpha) : \alpha \in J(X),\ \alpha \leq \alpha^*\}$, and for $\alpha^* < t \leq \beta^*$, $\varphi(t) = \inf\{\varphi(\beta) : \beta \in J(X),\ \beta \geq \beta^*\}$. It is easily checked that this extends φ to an increasing l.s.c. function. It follows that, so defined on $\mathbb{R}^*$, φ is increasing on $J(X)$.

we will later on assume that

$$(iv) \qquad J(x) \le n_0 \Longrightarrow I(x) > I(x^*) - n_0 .$$

Theorem 2. The function $\rho : \mu \mapsto x_\mu$ is continuous from $\mathbb{R}_+^*$ to X.

Proof. It is enough to prove the continuity of ρ on each interval $[\alpha,\beta]$ where $0<\alpha\le\beta<\infty$. Recall that $x_\mu \in X$ is the unique global minimum of the function $\psi_\mu : x \mapsto I(x)+J(x)+\mu\varphi\circ J(x)$.

We prove first that there exists some $\theta>0$ such that $\rho([\alpha,\beta])\subset S_\theta$.

If $\alpha\le\nu<\mu\le\beta$, we have $\psi_\mu(x_\mu)\le\psi_\mu(x_\nu)$ and $\psi_\nu(x_\mu)\ge\psi_\nu(x_\nu)$, hence

$$(\mu-\nu)\varphi\circ J(x_\mu) = \psi_\mu(x_\mu)-\psi_\nu(x_\mu) \le \psi_\mu(x_\nu)-\psi_\nu(x_\nu) = (\mu-\nu)\varphi\circ J(x_\nu)$$

whence $\varphi\circ J(x_\mu)\le\varphi\circ J(x_\nu)$, and since φ^{-1} is increasing [2] : $J(x_\mu)\le J(x_\nu)$. And if we put $\theta = J(x_\alpha)$, we get $\rho(\mu)=x_\mu\in S_\theta$.

So we have $\rho([\alpha,\beta])\subset S_\theta$. If V is any open neighborhood of $y=\rho(\mu)$ for $\mu\in[\alpha,\beta]$ and if $\rho^{-1}(V)$ is not a neighborhood of μ, there exists a sequence (μ_n) in $[\alpha,\beta]\setminus\rho^{-1}(V)$ which converges to μ . And by sequential compactness of S_θ, the sequence $y_n = x_{\mu_n}$ possesses a cluster value $y^*\in S_\theta\setminus V$, and in particular $y^*\ne y$. Since $(\mu,x)\mapsto\psi_\mu(x)$ is sequentially l.s.c., and since $\psi_{\mu_n}(y_n)\le\psi_{\mu_n}(y)$, we have

$$\begin{aligned}\psi_\mu(y^*) \le \liminf_n \psi_{\mu_n}(y_n) \le \liminf_n \psi_{\mu_n}(y) &= \liminf_n \Big(I(y)+J(y)+\mu_n\varphi\circ J(y)\Big)\\ &= \psi_\mu(y)\end{aligned}$$

from what follows that y^* is a global minimum of ψ_μ, hence that $y^*=y$, a contradiction. This proves the continuity of ρ. □

Lemma 3. As μ tends to $+\infty$, $\rho(\mu)$ tends to $J^{-1}(0)$, in the sense that $\rho([1,+\infty[)$ is conditionnally compact in X and that the cluster values of ρ at $+\infty$ belong to S_θ, where $\theta = \inf(J(X)\setminus\{0\}$. More precisely, if $\varepsilon\in J(X)\setminus\{0\}$, we have $\rho(\mu)\in S_\varepsilon$ for all large enough μ.

Proof. Let $\varepsilon\in J(X)\setminus\{0\}$ and $\delta=\varphi(\varepsilon)$. We have $\delta>0$ since φ is increasing. If $x_\mu=\rho(\mu)\notin S_\varepsilon$, fix some $\nu<\mu$; we have

$$\begin{aligned}I(x^*) &= I(x^*)+J(x^*)+\varphi\circ J(x^*) = \psi_\mu(x^*)\\ &\ge \psi_\mu(x_\mu) = I(x_\mu)+J(x_\mu)+\mu\varphi\circ J(x_\mu)\\ &\ge I(x_\mu)+J(x_\mu)+\nu\varphi\circ J(x_\mu)+(\mu-\nu)\varphi\circ J(x_\mu)\\ &\ge \psi_\nu(x_\nu)+(\mu-\nu)\varphi\circ J(x_\mu)\\ &> \psi_\nu(x_\nu)+(\mu-\nu)\delta\end{aligned}$$

[2]if we assume only φ non-decreasing, we can find θ such that $\varphi(\theta)>\varphi\circ J(x_\alpha)$ since φ is unbounded, whence $\varphi\circ J(x_\mu)\le\varphi\circ J(x_\alpha)<\varphi(\theta)$ and $J(x_\mu)<\theta$ for all $\mu\in[\alpha,\beta]$.

whence $\mu < \nu + \frac{1}{\delta}\Big(I(x^*) - \psi_\nu(x_\nu)\Big)$, which is contradictory for large μ. Thus for all $\varepsilon > 0$ we have $\rho(\mu) \in S_\varepsilon$ for large enough μ. Since the set S_ε is compact, we see that $\rho([1,+\infty[)$ is contained in $S_\varepsilon \cup \rho([1,\mu])$ for a convenient μ, whence we get the conditional compactness. And if x is a cluster value of ρ at $+\infty$, we get[3] $x \in S_\varepsilon$ for all $\varepsilon > 0$, hence $x \in \bigcap_{\varepsilon>0,\ \varepsilon\in J(X)} S_\varepsilon = S_\theta$.

If 0 belongs to the closure of $J(X) \setminus \{0\}$, we have $\theta = 0$ and $S_\theta = J^{-1}(0)$. □

Lemma 4. We have $\liminf_{\mu\to 0} I{\circ}\rho(\mu) = -\infty$.

Proof. Let $\eta > 0$. There exists $c > 0$ such that $\inf_{x\notin S_c} \frac{I(x)}{J(x)} = -\infty$. Thus, for each integer $q > 1$, we can find some $z_q \in X$ such that $I(z_q) < -(q+2)J(z_q)$ and $J(z_q) \geq c$. We the have

$$I(z_q) + J(z_q) < -(1+q)J(z_q) \leq -(1+q)c$$

and one can find some $\mu \in\,]0,\eta[$ such that $\mu\varphi{\circ}J(z_q) < c$, and by definition of $x_\mu = \rho(\mu)$,

$$\begin{aligned} I{\circ}\rho(\mu) = I(x_\mu) &\leq I(x_\mu) + J(x_\mu) + \mu\varphi{\circ}J(x_\mu) \\ &\leq I(z_q) + J(z_q) + \mu\varphi{\circ}J(z_q) \\ &< -(1+q)J(z_q) + \mu\varphi{\circ}J(z_q) \\ &< -(1+q)c + c = -qc\,, \end{aligned}$$

and this completes the proof since qc est arbitrarily large. □

Theorem 5. Hypotheses (i), (ii), (iii) and (iv) cannot hold simultaneously.

Proof. Consider the two sets $H_1 = \rho^{-1}\big(\{x \in X : I(x) \leq I(x^*) - n_0\}\big)$ and $H_2 = \rho^{-1}\big(S_{n_0}\big)$. Since ρ is continuous both sets are sequentially closed, hence closed in $\mathbb{R}_+^*$. Il follows from condition (iv) that $H_1 \cap H_2 = \emptyset$, from lemma 3 that $H_2 \neq \emptyset$ and from lemma 4 that $H_1 \neq \emptyset$. By connectedness of $\mathbb{R}_+^*$, we cannot have $H_1 \cup H_2 = \mathbb{R}_+^*$. Thus there exists μ^* such that $\mu^* \notin H_1 \cup H_2$. Then we have, for $z^* = \rho(\mu^*)$, $J(z^*) > n_0$ and $I(z^*) > I(x^*) - n_0$, thus

$$\begin{aligned} I(x^*) < I(z^*) + J(z^*) &\leq I(z^*) + J(z^*) + \mu^*\varphi{\circ}J(z^*) \\ &\leq I(x^*) + J(x^*) + \mu^*\varphi{\circ}J(x^*) = I(x^*)\,, \end{aligned}$$

since $J(x^*) = 0$ and $\varphi(0) = 0$, a contradiction. □

[3]if φ is only assumed to be non-decreasing, we can find $\eta > 0$ such that $\delta = \varphi(\eta) > 0$. We will then have $\rho(\mu) \in S_\eta$ for all large μ, and the cluster values of ρ at $+\infty$ will belong to S_ζ, where $\zeta = \inf\{\varepsilon \in J(X) : \varphi(\varepsilon) > 0\}$.

Theorem 6. Let X be a topological space, I and J two real sequentially l.s.c. functions on X satisfying (i) and (ii). Then there exists some $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$ and all increasing l.s.c. function φ from $J(X)$ to $\mathbb{R}^+$ the function $\psi_\mu : x \mapsto I(x) + \lambda J(x) + \mu\varphi\circ J(x)$ has necessarily at least two global minima on X for some $\mu > 0$ if it has sequentially compact sublevels.

Proof. Notice that if the l.s.c. function $I + \lambda J + \mu\varphi\circ J$ has sequentially compact sublevels for some λ, μ, then for $\lambda' \geq \lambda$ and $\mu' \geq \mu$,

$$S'_\alpha = \{x : \big(I + \lambda' J + \mu'\varphi\circ J\big)(x) \geq \alpha\} \subset S_\alpha = \{x : \big(I + \lambda J + \mu\varphi\circ J\big)(x) \geq \alpha\}$$

hence S'_α which is closed in the sequentially compact set S_α is itself sequentially compact. For example this is the case if $\varepsilon I + \varphi\circ J$ is bounded from below for every $\varepsilon > 0$: S_α is contained in some sublevel of J.

It follows from what precedes that one can find λ^* such that for every $\lambda \geq \lambda^*$ the extra condition (iv) be satisfied for I and λJ. Then the result follows immediately from theorem 5 : the condition (iii) cannot hold ; for every increasing sequentially l.s.c. function φ there are some $\mu > 0$ for which ψ_μ does not have a unique global minimum. But since this function ψ_μ has sequentially compact sublevels, the global minimum is attained, necessarily at several points: ψ_μ has several global minima. □

REFERENCES

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