MINIMAL OSCILLATION AND VANISHING OF SMOOTH FUNCTIONS

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Abstract. In his paper [2], B. Ricceri considers, for X bounded convex subset of the real Hilbert space H, the quantity

$$\delta_X = \inf_{\varphi \in \Gamma_X} \Big(\sup_{x \in X} \left(\|x\|^2 + \varphi(x) \right) - \inf_{x \in X} \left(\|x\|^2 + \varphi(x) \right) \Big)$$

where Γ_X denotes the set of real convex functions on X, and shows that $\delta_X > 0$ for X non singleton without giving any quantitative estimation of this quantity. And he asks whether δ_X can be controlled by a function of the diameter of X.

In this paper we show that δ_X is exactly the square of the Chebyshev radius of X, hence is at least $\dfrac{\mathrm{diam}(X)^2}{4}$. We deduce from the main result of [2] a quantitative statement on the zeros of a $\mathcal{C}^1$-operator on H with Lipschitz derivative, and show that this statement is optimal.

Let H be a real Hilbert space and X be a convex bounded subset of H. If φ is a real convex function on X, we will denote by $\theta(\varphi, X)$ the oscillation on X of the convex function $x \mapsto \|x\|^2 + \varphi(x)$, so

$$\theta(\varphi, X) = \sup_{x \in X} \left(\|x\|^2 + \varphi(x) \right) - \inf_{x \in X} \left(\|x\|^2 + \varphi(x) \right),$$

hence $\delta_X = \inf_{\varphi \in \Gamma_X} \theta(\varphi, X)$. B. Ricceri shows in [2] in a rather indirect way that $\delta_X > 0$ for any convex subset of H containing more than one point. He notices also that C. Zalinescu gave him in a private communication another indirect (and not quantitative) proof of this result based on quite hard arguments of convex analysis.

Recall that if C is a bounded subset of some normed space E, the *Chebyshev radius* (for short *radius*) of C is the infimum ρ of all $r > 0$ such that C is contained in some ball $B(x, r)$ for $x \in E$. So

$$\rho = \inf_{a \in E} \sup_{x \in C} \|x - a\| .$$

And a *Chebyshev center* (for short *center*) of C is any point $a \in E$ such that $C \subset B(a, \rho)$. Of course there are examples where such a center does not exist or is not unique. One can find in [1] what is well known about radius and center of a bounded set and Jung inequalities. In particular we will use in the sequel the following three theorems.

Theorem 1. *If the Banach space E is uniformly convex (in particular if it is a Hilbert space), any nonempty bounded subset has a unique center.*

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Theorem 2. *Let H be a Hilbert space and X be a bounded subset of H of center a and radius ρ. Then for any $\varepsilon > 0$, a belongs to the closed convex hull of the set of those points of X such that $\|x-a\| \geq \rho - \varepsilon$. In particular, $a \in \overline{\mathrm{conv}}(X)$.*

Theorem 3. *Let H be a Hilbert space and X be a bounded subset of H of center a and radius ρ. Then $\rho\sqrt{2} \leq \mathrm{diam}(X) \leq 2\rho$. Moreover if H has finite dimension d, one has even $\rho \leq \mathrm{diam}(X)\sqrt{\dfrac{d}{2(d+1)}}$.*

We now prove our main theorem.

Theorem 4. *If X is a bounded convex subset of H having radius ρ, δ_X is equal to ρ^2. In particular $\delta_X \geq \dfrac{\mathrm{diam}(X)^2}{4}$.*

Proof. For the proof of this statement we need the next lemmas.

Lemma 5. *Let X be a convex bounded subset of H. One has $\delta_X = \delta_{X+a}$ and $\delta_{\lambda X} = \lambda^2 \delta_X$. Moreover if $X \subset Y$, one has $\delta_Y \geq \delta_X$.*

Proof. If φ is convex on X the function $\varphi_a : x \mapsto \varphi(x-a)$ is convex on $X+a$, and so is the function $\varphi'_a = x \mapsto \varphi_a(x) + \|a\|^2 - 2\langle a, x\rangle$. Then for $y = x + a \in X + a$:

$$\begin{aligned}\|y\|^2 + \varphi'_a(y) &= \|x+a\|^2 + \varphi_a(y) - \|a\|^2 + 2\langle a, y\rangle \\ &= \|x\|^2 + \|a\|^2 + 2\langle a, x\rangle + \varphi(x) + \|a\|^2 - 2\langle a, x+a\rangle = \varphi(x)\end{aligned}$$

whence we deduce that $\delta_{X+a} \geq \delta_X$, and $\delta_{X+a} \leq \delta_X$ by replacing a by $-a$.

In the same way if $\lambda \in \mathbb{R}^*$, and if φ is convex on X, the function $\varphi_\lambda : y \mapsto \lambda^2 \varphi(\dfrac{y}{\lambda})$ is convex on λX and for $y = \lambda x \in \lambda X$, we get $\|y\|^2 + \varphi_\lambda(y) = \lambda^2 \|x\| + \lambda^2 \varphi(x)$, whence $\delta_{\lambda X} \geq \lambda^2 \delta_X$, and the equality $\delta_{\lambda X} = \lambda^2 \delta_X$ by replacing λ by $1/\lambda$.

If $Y \supset X$ and if φ is convex on Y, the function $\psi = \varphi_{|X}$ is convex on X, and we get

$$\sup_{y\in Y} \|y\|^2 + \varphi(y) \geq \sup_X \|x\|^2 + \psi(x) \text{ and } \inf_{y \in Y}\big(\|y\|^2 + \varphi(y)\big) \leq \inf_X \|x\|^2 + \psi(x)\,.$$

Thus $\delta_X \leq \theta(\psi, X) \leq \theta(\varphi, Y)$, whence we deduce that $\delta_Y \geq \delta_X$. □

Lemma 6. *Let X be a convex bounded subset of H. If ρ is the radius of X one has $\delta_X \leq \rho^2$.*

Proof. If a is the center of X and φ_a the affine function $x \mapsto \|a\|^2 - 2\langle a, x\rangle$, one has $\rho = \sup_{x\in X} \|x - a\|$ and $\|x\|^2 + \varphi_a(x) = \|x-a\|^2$, hence $\sup_x\big(\|x\|^2 + \varphi_a(x)\big) = \rho^2$ and $\inf_x\big(\|x\|^2 + \varphi_a(x)\big) = \inf_{x\in X}\|x-a\|^2 = 0$, since $a \in \bar{X}$ by Theorem 2. Thus $\delta_X \leq \theta(\varphi_a, X) = \rho^2$. □

Lemma 7. *Let X be a bounded convex subset of H. If φ belongs to $\Gamma(X)$, there exists a convex lower semi-continuous function $\bar\varphi$ on $\overline{X}$ such that $\theta(\bar\varphi, \overline{X}) \leq \theta(\varphi, X)$.*

Proof. If φ is not bounded from below on X, one has $\theta(\varphi, X) = +\infty$. Then it is enough to take $\bar\varphi = 0$ for getting $\theta(\bar\varphi, \overline{X}) \leq \sup_{x\in X}\|x\|^2 < \theta(\varphi, X) = +\infty$.

On the contrary, if φ is bounded from below take $\bar\varphi : \overline{X} \to \mathbb{R}$ the lower semi-continuous envelope of φ, that is

$$\bar\varphi(x) = \liminf_{y\in X, y\to x} \varphi(y) = \sup\{\ell(x) : \ell \text{ affine continuous on } H,\ \ell \leq \varphi \text{ on } X\}$$

which is convex on $\overline{X}$. We clearly have $\bar{\varphi} \leq \varphi$ on X, hence

$$\sup_{x\in X}\left(\|x\|^2+\varphi(x)\right) \geq \sup_{x\in X}\left(\|x\|^2+\bar{\varphi}(x)\right) = \sup_{x\in \bar{X}}\left(\|x\|^2+\bar{\varphi}(x)\right).$$

And letting $\mu = \inf_{x\in X}\left(\|x\|^2+\varphi(x)\right)$, we have for all $y \in X$: $\varphi(y) \geq \mu - \|y\|^2$ hence

$$\bar{\varphi}(x) = \liminf_{y\to x} \varphi(y) \geq \liminf_{y\to x} \mu - \|y\|^2 = \mu - \|x\|^2$$

and $\mu = \inf_{x\in X}\left(\|x\|^2+\varphi(x)\right) \geq \inf_{x\in \bar{X}}\left(\|x\|^2+\bar{\varphi}(x)\right) \geq \mu$. We conclude that $\theta(\bar{\varphi}, X) \leq \theta(\varphi, X)$. □

The convex sets X and $\overline{X}$ have same center and same radius. We have already proved that $\delta_X \leq \rho^2$. We can reduce by translation the proof to the case where the center a of X is 0. Let φ be a convex function on X. Following lemma 7, there exists a l.s.c. convex function $\bar{\varphi}$ on $\overline{X}$ such that $\theta(\bar{\varphi}, \overline{X}) \leq \theta(\varphi, X)$. For proving that $\theta(\varphi, X) \geq \rho^2$, we can thus assume moreover that X is closed and that φ is l.s.c.

For $\varepsilon > 0$, define $X_\varepsilon = \{x \in X : \|x\| \geq \rho - \varepsilon\}$ and $K_\varepsilon = \overline{\text{conv}(X_\varepsilon)}$ then put $\mu_\varepsilon = \sup_{x\in X_\varepsilon} \varphi(x)$. For all $x \in X_\varepsilon$ one has :

$$\|x\|^2 + \varphi(x) \geq (\rho-\varepsilon)^2 + \varphi(x)\,,$$

hence

$$\sup_{x\in X_\varepsilon}\left(\|x\|^2+\varphi(x)\right) \geq (\rho-\varepsilon)^2 + \sup_{x\in X_\varepsilon}\varphi(x) = (\rho-\varepsilon)^2+\mu_\varepsilon\,,$$

and

$$\sup_{x\in X}\left(\|x\|^2+\varphi(x)\right) \geq \sup_{x\in X_\varepsilon}\left(\|x\|^2+\varphi(x)\right) \geq (\rho-\varepsilon)^2+\mu_\varepsilon\,.$$

Since the set C of those x in X such that $\varphi(x) \leq \mu_\varepsilon$ is a closed convex set containing X_ε, hence containing K_ε, it follows from Theorem 2 that $a = 0 \in C$, thus that $\varphi(0) \leq \mu_\varepsilon$. Thus we get

$$\theta(\varphi, X) \geq \left((\rho-\varepsilon)^2+\mu_\varepsilon\right) - \varphi(0) \geq \left((\rho-\varepsilon)^2+\mu_\varepsilon\right) - \mu_\varepsilon = (\rho-\varepsilon)^2\,.$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $\theta(\varphi, X) \geq \rho^2$, hence that

$$\delta_X = \inf_{\varphi\in\Gamma_X} \theta(\varphi, X) \geq \rho^2 \geq \delta_X\,.$$

By Therorem 3 we deduce that we necessarily have $\text{diam}(X) \leq 2\rho$ and thus $\delta_X = \rho^2 \geq \dfrac{\text{diam}(X)^2}{4}$. And this completes the proof of Theorem 4. □

Using Theorem 1.2 from [2] and Theorem 4 above allows us to state the following:

Theorem. *Let H be a real Hilbert space, $\Omega \subset H$ a convex open set, $\Phi : \Omega \to H$ an operator of class $\mathcal{C}^1$, with L-Lipschitz derivative and $V \subseteq \Omega$ a set such that $\eta := \inf_{x\in V} \|\Phi(x)\| > 0$.*

Then, for all bounded convex set $X \subseteq V$ with radius smaller than $2\eta.L$, we have $0 \notin \text{conv}(\Phi(X))$.

We can also rewrite this result in the following form, denoting by $\mathcal{A}_L$ the set of $\mathcal{C}^1$ functions with L-Lipschitz derivative from X into a Hilbert space and ρ the radius of the bounded convex open set X :

$$\sup\left\{\sqrt{\frac{2d(0,\Phi(X))}{L}} : \Phi \in \mathcal{A}_L,\ 0 \in \overline{\text{conv}}\big(\Phi(X)\big)\right\} \leq \rho$$

We now prove that this result is optimal.

Theorem 8. *Let X be a bounded convex nonempty open subset of the Hilbert space H and ρ be its radius. Then there exists a function Φ of class $\mathcal{C}^1$ with L-Lipschitz derivative from X to the Hilbert space $H \oplus \mathbb{R}$ satisfying $\rho^2 = \dfrac{2d(0, \Phi(X))}{L}$ and $0 \in \overline{\text{conv}}\big(\Phi(X)\big)$.*

Proof. Denote by a the center of X. Define $\lambda = \dfrac{1}{\rho\sqrt{2}}$, then the continuous functions $f : X \to \mathbb{R}$ and $\Phi : X \to H \times \mathbb{R}$ by

$$\begin{cases} f(x) &= \lambda\big(\rho^2 - \|x - a\|^2\big)\,, \\ \Phi(x) &= \big(x - a, f(x)\big)\,. \end{cases}$$

Since $\|x - a\| < \rho$ for all $x \in X$, we have $f > 0$ on X.

Lemma 9. *One has $\|\Phi(x)\| \geq \|\Phi(a)\| = \dfrac{\rho}{\sqrt{2}}$ for all $x \in X$.*

Proof. Indeed :

$$\begin{aligned} \|\Phi(x)\|^2 &= \|x - a\|^2 + |f(x)|^2 = \|x - a\|^2 + \lambda^2\big(\rho^2 - \|x - a\|^2\big)^2 \\ &= \lambda^2\rho^4 + \lambda^2 \|x - a\|^4 + (1 - 2\lambda^2\rho^2)\, \|x - a\|^2 = \lambda^2\rho^4 + \lambda^2 \|x - a\|^4 \\ &\geq \lambda^2\rho^4 = \frac{\rho^2}{2}\,, \end{aligned}$$

thus $\|\Phi(x)\| \ \geq \dfrac{\rho}{\sqrt{2}} = \|\Phi(a)\|$ for all $x \in X$. □

Lemma 10. *The function Φ is of class $\mathcal{C}^1$ with L-Lipschitz derivative, where L is equal to 2λ.*

Proof. For all $x \in X$ and $h \in H$ we have :

$$\Phi'(x).h = \big(h, f'(x).h\big) = \big(h, -2\lambda\langle x - a, h\rangle\big)$$

what shows that Φ is of class $\mathcal{C}^1$ (and even of class $\mathcal{C}^\infty$). Moreover if x and y belong to X, we have

$$\|\Phi'(x).h - \Phi'(y).h\| = \big\|\big(0, 2\lambda\langle y - x, h\rangle\big)\big\| \leq 2\lambda\, \|x - y\|\,.\, \|h\|$$

whence $\|\Phi'(x) - \Phi'(y)\| \leq 2\lambda\, \|x - y\|$ and the fact that Φ' is 2λ-Lipschitz. □

Lemma 11. *The origin of $H \oplus \mathbb{R}$ belongs to the closed convex hull of $\Phi(X)$*

Proof. It follows from Theorem 2 that for all $\varepsilon > 0$ there exist points $(x_i)_{i\leq n}$ of $X \setminus B(a, \rho - \varepsilon)$ and non-negative real numbers $(\alpha_i)_{i\leq n}$ with sum 1 such that $\|a - \sum_i \alpha_i x_i\| \ \leq \varepsilon$. Then we have

$$z = \sum_i \alpha_i \Phi(x_i) = \Big(\sum_i \alpha_i x_i - a\,,\ \sum_i \alpha_i f(x_i)\Big)$$

Since $\|x_i - a\| \ > \rho - \varepsilon$, one has $0 < f(x_i) < \lambda\big(\rho^2 - (\rho - \varepsilon)^2\big) < 2\lambda\rho\varepsilon = \varepsilon\sqrt{2}$, hence $0 \leq \sum_i \alpha_i f(x_i) < \varepsilon\sqrt{2}$ and finally

$$\|z\|^2 = \Big\|\sum\nolimits_i \alpha_i x_i - a\Big\|^2 + \Big|\sum\nolimits_i \alpha_i f(x_i)\Big|^2 \leq \varepsilon^2 + 2\varepsilon^2 = 3\varepsilon^2\,,$$

whence we deduce that $z \in \operatorname{conv}(\Phi(X)) \cap B\big((0,0), \varepsilon\sqrt{3}\big)$, thus that every neighborhood of $(0,0)$ in $H \oplus \mathbb{R}$ meets the convex hull of $\Phi(x)$ and finally that the origin $(0,0)$ of $H \oplus \mathbb{R}$ belongs to $\overline{\operatorname{conv}}\big(\Phi(X)\big)$. □

This completes the proof of Theorem 8. □

Corollary 12. *Let X be a bounded convex nonempty open subset of the Hilbert space H and ρ be its radius. Then*

$$\sup\Big\{\sqrt{\frac{2d(0,\Phi(X))}{L}} : \Phi \in \mathcal{A}_L,\ 0 \in \overline{\operatorname{conv}}\big(\Phi(X)\big)\Big\} = \rho.$$

Proof. This follows immediately from Theorems 4 and 8. □

References

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