# MINIMAX AND LIPSCHITZ OPERATORS

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Abstract. The aim of this short note is to show that the minimax equality proved in [3] cannot be extended to the case where linear operators are replaced by Lipschitz operators.

In his paper [3], extending the previous [2], the author established the following minimax theorem :

**Theorem.** *Let $E$ be a infinite-dimensional Banach space, $F$ be a Banach space, $X$ be a convex subset of $E$ whose interior is non-empty for the weak topology on bounded sets, $\Delta$ a finite-dimensional convex compact subset of $\mathcal{L}(E,F)$, $\varphi : F \to \mathbb{R}$ be a continuous convex coercive map, and $\psi : \Delta \to \mathbb{R}$ a convex continuous function. Assume moreover that $\Delta$ contains at most one compact operator. Then*

$$\sup_{x\in X} \inf_{T\in\Delta} \varphi(Tx) + \psi(T) = \inf_{T\in\Delta} \sup_{x\in X} \varphi(Tx) + \psi(T)$$

In particular, taking $\varphi(x) = \|x\|$ and $\psi = 0$, this gets the minimax equality :

$$\sup_{x\in X} \inf_{T\in\Delta} \|Tx\| = \inf_{T\in\Delta} \sup_{x\in X} \|Tx\| \ .$$

The aim of this note is to replace the finite-dimensional convex compact subset $\Delta$ of $\mathcal{L}(E,F)$ by a finite-dimensional convex compact set of Lipschitz mappings from $E$ to $F$, and more precisely of mappings of the type $x \mapsto \Phi(x) + \psi(x)$ where $\Phi$ is a fixed continuous surjective linear mapping from $E$ to $F$, and $\psi : E \to F$ is a Lipschitz mapping.

If $E$ and $F$ are Banach spaces and $\Phi : E \to F$ a continuous linear mapping with closed range (in particular if $\Phi$ is onto) we will denote by $\nu(\Phi)$ the quantity

$$\nu(\Phi) = \sup_{\substack{y\in\Phi(E)\\ \|y\|\le 1}} d\big(0, \Phi^{-1}(y)\big)\ .$$

Of course, if moreover $\Phi$ is one-to-one, $\nu(\Phi)$ is the norm of the continuous linear mapping $\Phi^{-1} : \Phi(F) \to E$. And if $\ker(\Phi) \neq \{0\}$, $\Phi$ factors through the quotient $\hat{E} = E/\ker(\Phi)$ and a one-to-one linear mapping $\hat{\Phi} = \hat{E} \to \Phi(E)$, and then $\nu(\Phi)$ is the norm of the linear mapping $\hat{\Phi}^{-1}$.

The following result which addresses the case where $\Delta$ is an interval, is due to B. Ricceri and follows from Theorem 2.1 in [1].

**Theorem.** *Let $X$ and $Y$ be two Banach spaces, with $\dim(Y) \geq 2$. Let $\Phi : X \to Y$ be a continuous surjective linear operator and let $\Psi_1, \Psi_2 : X \to Y$ be two $\beta$-Lipschitz operators, where $1/\beta = \nu(\Phi)$. Then, one has*

$$\sup_{x\in X} \inf_{\lambda\in[0,1]} \|\Phi(x) + \lambda\Psi_1(x) + (1-\lambda)\Psi_2(x)\| = \min\Big\{\sup_{x\in X} \|\Phi(x) + \Psi_1(x)\|\, , \sup_{x\in X} \|\Phi(x) + \Psi_2(x)\|\Big\}$$

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Consider now the case where the finite-dimensional compact convex set $\Delta$ has dimension at least 2 and its elements have small Lipschitz constants.

**Theorem 1.** *Let $X$ and $Y$ be Banach spaces, $\Phi$ be a linear continuous operator from $X$ onto $Y$, $\beta = 1/\nu(\Phi)$ and $\Delta$ be a finite-dimensional compact convex set of Lipschitz mappings from $X$ to $Y$. If each element of $\Delta$ has a Lipschitz constant $< \beta$, then*

$$\inf_{\psi\in\Delta}\sup_{x\in X}\|\Phi(x)+\psi(x)\| = \sup_{x\in X}\inf_{\psi\in\Delta}\|\Phi(x)+\psi(x)\| = +\infty$$

*Proof.* On the compact space $\Delta$, the function which assigns to each $\psi$ its Lipschitz constant $L_\psi = \sup_{\substack{x,z\in X\\ x\neq z}} \dfrac{\|\psi(x)-\psi(z)\|}{\|x-z\|}$ is continuous and attains its supremum $\lambda < \beta$. Also by compactness $R = \sup_{\psi\in\Delta}\|\psi(0)\| < +\infty$.

For $\varepsilon > 0$ let $X_\varepsilon = \big\{x \in X : \|\Phi(x)\| \geq \beta(1-\varepsilon)\,\|x\|\big\}$. By definition of $\beta$, we have $\Phi(X_\varepsilon) = \Phi(X) = Y$, hence $\sup_{x\in X_\varepsilon}\|x\| = +\infty$. Indeed if $X_\varepsilon$ was bounded so would be $Y = \Phi(X_\varepsilon)$. Thus for each $\psi \in \Delta$ and each $x \in X_\varepsilon$ we have

$$\begin{aligned}\|\Phi(x)+\psi(x)\| &\geq \|\Phi(x)\| - \|\psi(x)\| \geq \|\Phi(x)\| - \Big(\|\psi(0)\| + \|\psi(x)-\psi(0)\|\Big)\\ &\geq \beta(1-\varepsilon)\,\|x\| - \Big(R + \lambda\,\|x\|\Big) = \big(\beta(1-\varepsilon)-\lambda\big)\,\|x\| - R\,.\end{aligned}$$

Choosing $\varepsilon < \dfrac{1}{2}(1-\dfrac{\lambda}{\beta})$, we get $\inf_{\psi\in\Delta}\|\Phi(x)+\psi(x)\| \geq \dfrac{\beta-\lambda}{2}\,\|x\| - R$ for $x \in X_\varepsilon$, hence

$$\sup_{x\in X}\inf_{\psi\in\Delta}\|\Phi(x)+\psi(x)\| \geq \sup_{x\in X_\varepsilon}\inf_{\psi\in\Delta}\|\Phi(x)+\psi(x)\| \geq \frac{\beta-\lambda}{2}\cdot\Big(\sup_{x\in X_\varepsilon}\|x\|\Big) - R = +\infty$$

and

$$\inf_{\psi\in\Delta}\sup_{x\in X}\|\Phi(x)+\psi(x)\| \geq \sup_{x\in X}\inf_{\psi\in\Delta}\|\Phi(x)+\psi(x)\| = +\infty\,.$$

So the proof is complete. □

We now look at the case where $\Delta$ has dimension $> 1$ and each element of $\Delta$ is $\beta$-Lipschitz, with $\beta = 1/\nu(\Phi)$ and want to prove that the previous minimax equality does no longer hold by constructing a counterexample.

**Theorem 2.** *There exist $X$ and $Y$ Banach spaces, $\Phi : X \to Y$ linear and onto, and $(\Psi_j)_{1\leq j\leq 3} : X \to Y$, $\beta$-Lipschitz, where $\beta = 1/\nu(\Phi)$ such that*

$$\sup_{x\in X}\inf_{\alpha\in\Delta}\Big\|\Phi(x) + \sum\alpha_j\Psi_j(x)\Big\| < \inf_{\alpha\in\Delta}\sup_{x\in X}\Big\|\Phi(x) + \sum\alpha_j\Psi_j(x)\Big\|$$

*where $\Delta$ is the canonical 2-dimensional simplex : $\{\alpha \in \mathbb{R}^3_+ : \alpha_1+\alpha_2+\alpha_3 = 1\}$.*

The following result is probably well known.

**Lemma 3.** *Let $H$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H$. Denote by $p$ the orthogonal projection on $C$ and by $\varpi : x \mapsto p(x) - x$ the projecting line. For all $x$ and $y$ in $H$ we have*

$$\|p(x)-p(y)\|^2 + \|\varpi(x)-\varpi(y)\|^2 \leq \|x-y\|^2$$

*In particular $p$ and $\varpi$ are 1-Lipschitz.*

*Proof.* In the case where $C$ is a closed linear subspace this inequality is in fact an equality and corresponds to Pythagoras' theorem.

Let $x$ and $y$ be two points of $H$, $\xi = p(x)$ and $\eta = p(y)$ be their projections on $C$. We have

$$\langle x-\xi, \eta-\xi\rangle \leq 0 \text{ and } \langle y-\eta, \xi-\eta\rangle \leq 0\,,$$

thus

$$\begin{aligned}\langle x-y, \xi-\eta\rangle &= \big\langle (x-\xi)+(\xi-\eta)+(\eta-y), \xi-\eta\big\rangle \\ &= \langle x-\xi, \xi-\eta\rangle + \langle \xi-\eta, \xi-\eta\rangle + \langle \eta-y, \xi-\eta\rangle \\ &= \|\xi-\eta\|^2 - \langle x-\xi, \eta-\xi\rangle - \langle y-\eta, \xi-\eta\rangle \\ &\geq \|\xi-\eta\|^2\,.\end{aligned}$$

It follows that

$$\begin{aligned}\|p(x)-p(y)\|^2 + \|\varpi(x)-\varpi(y)\|^2 &= \|\xi-\eta\|^2 + \|(\xi-x)-(\eta-y)\|^2 \\ &= \|\xi-\eta\|^2 + \|(\xi-\eta)-(x-y)\|^2 \\ &= 2\,\|\xi-\eta\|^2 + \|x-y\|^2 - 2\langle x-y, \xi-\eta\rangle \\ &\leq \|x-y\|^2 + 2\,\|\xi-\eta\|^2 - 2\,\|\xi-\eta\|^2 = \|x-y\|^2\,,\end{aligned}$$

whence the statement follows. □

**Notations**

Consider a triangle $T$ in the euclidean plane $\mathbb{R}^2$, with edges $a_1, a_2, a_3$, and suppose all of its angles are acute. Let $\Gamma$ be the circumscribed circle to $T$, $\omega$ be the center and $\rho$ the radius of $\Gamma$. Then $\omega$ is interior to $T$.

To simplify the notations, for $j \in \mathbb{Z}$, $a_j$ will denote the point $a_i$ with $i \in \{1,2,3\}$ such that $i = j \pmod 3$. Denote $m_j = \dfrac{a_{j+1}+a_{j+2}}{2}$ the midpoint of the side $J_j = [a_{j+1}, a_{j+2}]$ opposite to $a_j$ ; in particular $\omega - m_j$ is orthogonal to $J_j$ . We have

$$\begin{aligned}(a_j - a_{j+1}) \wedge (a_{j+1} - a_{j+2}) &= (a_j - a_{j+2}) \wedge (a_{j+1} - a_{j+2}) - (a_{j+1} - a_{j+2}) \wedge (a_{j+1} - a_{j+2}) \\ &= (a_j - a_{j+2}) \wedge (a_{j+1} - a_{j+2}) = (a_{j+3} - a_{j+2}) \wedge (a_{j+1} - a_{j+2}) \\ &= -(a_{j+2} - a_{j+3}) \wedge (a_{j+1} - a_{j+2}) \\ &= (a_{j+1} - a_{j+2}) \wedge (a_{j+2} - a_{j+3})\end{aligned}$$

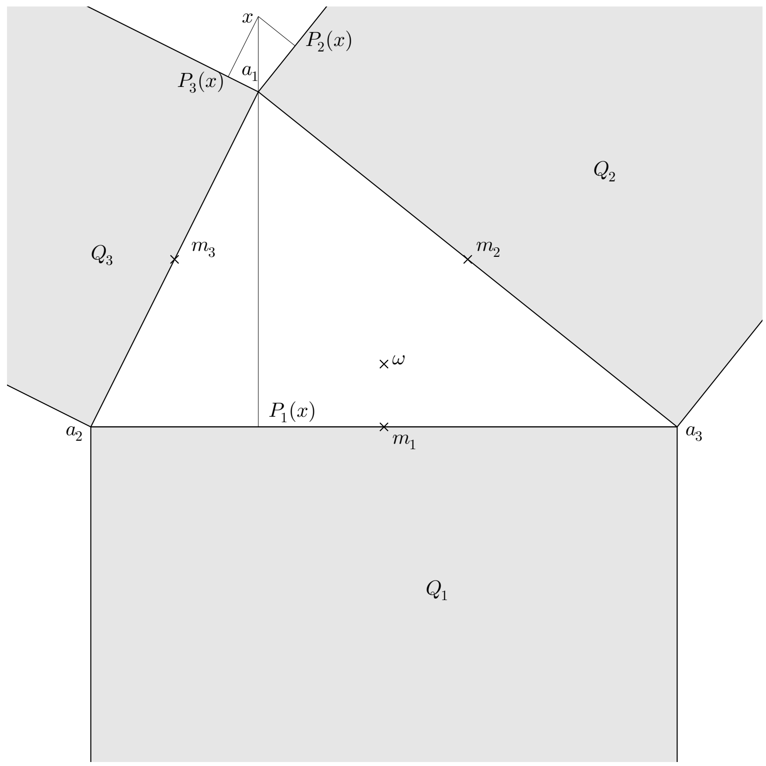
so this outer product is independent from $j$. Up to swapping $a_2$ and $a_3$ if necessary, we will assume that $(a_j - a_{j+1}) \wedge (a_{j+1} - a_{j+2}) > 0$ for all $j$. The linear functionals $u \mapsto \langle \omega - m_j, u\rangle$ and $u \mapsto u \wedge (a_{j+1} - a_{j+2})$ have same kernel hence are proportional; so there exists $\gamma_j \in \mathbb{R}$ such that $u \mapsto u \wedge (a_{j+1} - a_{j+2}) = \gamma_j \langle \omega - m_j, u\rangle$. Applying to $u = a_j - a_{j+1}$ we check that $\gamma_j \langle \omega - m_j, a_j - a_{j+1}\rangle = \gamma_j \langle \omega - m_j, a_j - m_j\rangle > 0$, hence that $\gamma_j > 0$.

Denote by $Q_j$ the half strip:

$$Q_j = \left\{x \in \mathbb{R}^2 : (x - m_j) \wedge (a_{j+1} - a_{j+2}) \leq 0 \leq |\langle x - m_j, a_{j+1} - a_{j+2}\rangle| \leq \frac{1}{2}\,\|a_{j+1} - a_{j+2}\|^2\right\},$$

whose boundary contains the side $J_j$, by $P_j$ the orthogonal projection on the closed convex set $Q_j$ and by $\varpi_j : x \mapsto P_j(x) - x$ the corresponding projecting line. It is easily seen that the intersection of $Q_j$ and the line $(a_{j+1}, a_{j+2}) = \{x : (x - m_j) \wedge (a_{j+2} - a_{j+1}) = 0\}$ is the segment $J_j$.

**Lemma 4.** *If $(x - m_j) \wedge (a_{j+1} - a_{j+2}) \geq 0$, then $P_j(x)$ belongs to $J_j$.*




*Proof.* If not the point $y = P_j(x)$ should satisfy $(y-m_j)\wedge(a_{j+1}-a_{j+2}) < 0$ since $P_j(x) \in Q_j$ and the inequality $\langle y - m_j, a_{j+1} - a_{j+2}\rangle \le 0$ holds. Then $y' = y + \varepsilon(\omega - m_j)$ would belong to $Q_j$ for $\varepsilon > 0$ small enough, and we would have $y - x = \lambda(a_{j+2} - a_{j+1}) - \mu(\omega - m_j)$ for some convenient $\lambda$ an $\mu$, with $\mu \ge 0$ since $(\omega - m_j) \wedge (a_{j+1} - a_{j+2}) > 0$ and

$$\mu(\omega - m_j) \wedge (a_{j+1} - a_{j+2}) = (x - m_j) \wedge (a_{j+1} - a_{j+2}) - (y - m_j) \wedge (a_{j+1} - a_{j+2}) \ge 0,$$

hence also $y' - x = \lambda(a_{j+2} - a_{j+1}) - (\mu - \varepsilon)(\omega - m_j)$ thus

$$\begin{aligned}\|y' - x\|^2 &= \lambda^2 \|a_{j+1} - a_{j+2}\|^2 + (\mu - \varepsilon)^2 \|\omega - m_j\|^2 \\ &< \lambda^2 \|a_{j+1} - a_{j+2}\|^2 + \mu^2 \|\omega - m_j\|^2 = \|x - y\|^2 ,\end{aligned}$$

so $d(x, Q_j) \le \|x - y'\| < \|x - y\| = d(x, Q_j)$, a contradiction. □

**Lemma 5.** *For all $x \in \mathbb{R}^2$ at least one $P_j(x)$ belongs to $T$.*

*Proof.* Indeed

$$\begin{aligned}\sum_{j=1}^{3}(x - m_j) \wedge (a_{j+1} - a_{j+2}) &= (x - \omega) \wedge \sum_{j=1}^{3}(a_{j+1} - a_{j+2}) + \sum_{j=1}^{3}(\omega - m_j) \wedge (a_{j+1} - a_{j+2}) \\ &= (x - \omega) \wedge 0 + \sum_{j=1}^{3}(\omega - m_j) \wedge (a_{j+1} - a_{j+2}) = \sum_{j=1}^{3} \gamma_j \langle \omega - m_j, \omega - m_j\rangle \\ &= \sum_{j} \gamma_j \|\omega - m_j\|^2 > 0 .\end{aligned}$$

Then at least one of the terms $(x-m_j)\wedge(a_{j+1}-a_{j+2})$ is positive and the conclusion follows from lemma 4. □

Define $X=Y=\mathbb{R}^2$, $\Phi(x)=x$ and $\Psi_j(x)=\varpi_j(x)-\omega$, hence $\Phi(x)+\Psi_j(x)=P_j(x)-\omega$. It is clear from what precedes that the mappings $P_j$ and $\Psi_j$ are 1-Lipschitz, and that $\Phi^{-1}$ is the identity mapping, so $\nu(\Phi)=1$.

Recall that $\Delta$ is the simplex $\{(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{R}^3_+ : \alpha_1+\alpha_2+\alpha_3=1\}$.

**Lemma 6.** *For whichever* $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in\Delta$ *the function* $\varphi_\alpha=\sum_{j=1}^3\alpha_j\big(P_j(x)-\omega\big)$ *satisfies*

$$\sup_{x\in\mathbb{R}^2}\|\varphi_\alpha(x)\|=+\infty\,.$$

*Thus* $\inf_{\alpha\in\Delta}\sup_{x\in\mathbb{R}^2}\|\varphi_\alpha(x)\|=+\infty$.

*Proof.* Consider first the case $\alpha_j=1$, $\alpha_{j+1}=\alpha_{j+2}=0$, hence $\varphi_\alpha(x)=P_j(x)-\omega$. Then, for $t\geq 0$, consider the point $x_t=m_j-t.(\omega-m_j)$ which belongs to the strip $Q_j$. We have

$$\|\varphi_\alpha(x_t)\|=\|x_t-\omega\|=\|m_j-\omega-t.(\omega-m_j)\|=(1+t)\,\|\omega-m_j\|$$

and $\sup_{x\in\mathbb{R}^2}\|\varphi_\alpha(x)\|\ \geq\sup_{t\geq 0}\|\varphi_\alpha(x_t)\|=+\infty$ since $m_j\neq\omega$.

Otherwise, if both $\alpha_{j+1}$ and $\alpha_{j+2}$ are not 0 (e.g. if $\sup_i\alpha_i<1$ and $\alpha_j=\min(\alpha_1,\alpha_2,\alpha_3)$), consider for $t\geq 0$, the point $x_t=a_j+t(\omega-m_j)$. We have $P_j(x_t)=P_j(x_0)\in J_j$ for all $t$. And for $k\neq j$, ($k=j+1$ or $k=j+2$), we have $P_k(x_t)=a_j+\theta(m_k-\omega)$ for some $\theta\geq 0$. Then since $x_t-P_k(x_t)$ is orthogonal to $m_k-\omega$, we get :

$$\begin{aligned}0&=\langle x_t-P_k(x_t),m_k-\omega\rangle=\big\langle\big(a_j+t(\omega-m_j)\big)-\big(a_j+\theta(m_k-\omega)\big),m_k-\omega\big\rangle\\&=t\langle\omega-m_j,m_k-\omega\rangle-\theta\langle m_k-\omega,m_k-\omega\rangle\end{aligned}$$

hence $\theta=t\dfrac{\langle\omega-m_j,m_k-\omega\rangle}{\|\omega-m_k\|^2}$, and

$$\langle\omega-m_j,P_k(x_t)-\omega\rangle=\langle\omega-m_j,a_j-\omega\rangle+t\frac{\big(\langle\omega-m_j,\omega-m_k\rangle\big)^2}{\|\omega-m_k\|^2}\,.$$

Finally we get

$$\begin{aligned}\langle\omega-m_j,\varphi_\alpha(x_t)\rangle&=\alpha_j\langle\omega-m_j,P_j(x_0)-\omega\rangle+\alpha_{j+1}\langle\omega-m_j,a_j-\omega\rangle+\alpha_{j+1}t\frac{\big(\langle\omega-m_j,\omega-m_{j+1}\rangle\big)^2}{\|\omega-m_{j+1}\|^2}\\&\qquad+\alpha_{j+2}\langle\omega-m_j,a_j-\omega\rangle+\alpha_{j+2}t\frac{\big(\langle\omega-m_j,\omega-m_{j+2}\rangle\big)^2}{\|\omega-m_{j+2}\|^2}\\&=A+Bt\end{aligned}$$

where $B=\alpha_{j+1}\dfrac{\big(\langle\omega-m_j,\omega-m_{j+1}\rangle\big)^2}{\|\omega-m_{j+1}\|^2}+\alpha_{j+2}\dfrac{\big(\langle\omega-m_j,\omega-m_{j+2}\rangle\big)^2}{\|\omega-m_{j+2}\|^2}>0$, since by the hypothesis on the angles of $T$ : $\langle\omega-m_j,\omega-m_k\rangle\neq 0$ for $j\neq k$.

So we get

$$\|\varphi_\alpha(x_t)\|\geq\frac{1}{\|\omega-m_j\|}\,|\langle\omega-m_j,\varphi_\alpha(x_t)\rangle|\geq\frac{Bt-|A|}{\|\omega-m_j\|}$$

whence $\lim_{t\to+\infty}\|\varphi_\alpha(x_t)\|=+\infty$ and $\sup_{x\in\mathbb{R}^2}\|\varphi_\alpha(t)\|=+\infty$. □

**Lemma 7.** *For whichever* $x \in \mathbb{R}^2$ *there exists some* $\alpha \in \Delta$ *such that* $\|\varphi_\alpha(x)\| \le \rho$. *In particular*

$$\sup_{x\in\mathbb{R}^2} \inf_{\alpha\in\Delta} \|\varphi_\alpha(x)\| \ \le \rho < +\infty\,.$$

*Proof.* In virtue of lemma 5, for all $x \in \mathbb{R}^2$ there is at least one $P_j(x)$ in $T$ and for such a $j$ we have

$$\inf_{\alpha\in\Delta} \|\varphi_\alpha(x)\| \le \|P_j(x) - \omega\| \le \rho$$

since $T$ is included in the disk of center $\omega$ and radius $\rho$. □

*Proof of theorem 2.* Lemmas 6 and 7 show that the previous construction yields the desired counterexample. □

## References


1. Ricceri, B., *Miscellaneous Applications of Certain Minimax Theorems II*, Acta Mathematica Vietnamica (2020) 45: 515–524
2. Saint Raymond J., *A minimax theorem for linear operators,* Minimax Theory and its Applications, **1** (2016), No 2 , 291–305
3. Saint Raymond J., *A new minimax theorem for linear operators,* Minimax Theory and its Applications, **3** (2018) , n° 1, 131–160



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