BEHAVIOR OF THE SU(2)-REIDEMEISTER TORSION FORM BY MUTATION

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Abstract. In this paper, we prove that the Reidemeister torsion twisted by the adjoint representation, which is considered as a 1-form, on the SU(2)-character variety of a knot exterior is invariant under mutation along a Conway sphere.

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1. INTRODUCTION

If \( K \) is a hyperbolic knot in \( S^3 \), we know that each mutant of \( K \) is also hyperbolic and that their volumes are the same (see [11]). It is well-known that the Alexander polynomial (i.e. the abelian Reidemeister torsion) is invariant by mutation like most of the classical or quantum knot invariants. In [12, 13], S. Tillmann have studied the behavior of the character variety of a knot group by mutation, and proved that generically the character varieties of a knot group and one of its mutant are birationally equivalent.
In this paper, we study the behavior of the twisted Reidemeister torsion with coefficients in the adjoint representation associated to a generic SU(2)-representation, viewed as a 1-form on the character variety, under a mutation. In fact, we prove that this kind of twisted Reidemeister torsion, with sign, is invariant by positive mutation. Our technique is to use a “cut and past argument” which involves Mayer–Vietoris sequences and Turaev’s refined version of torsion.

In [7], P. Menal–Ferrer and J. Porti study the behavior by mutation of the Reidemeister torsion twisted by a representation into $\text{SL}_2(\mathbb{C})$ especially in the case of hyperbolic knots, and prove that it is invariant at the discrete and faithful representation corresponding to the complete structure. In [2], N. Dunfield, S. Friedl and N. Jackson make some computer computations to calculate the twisted Alexander invariant for some knots and their mutant. They observe that this invariant is not invariant by mutation. In [6], P. Kirk and C. Livingston have already observed that some special twisted Alexander polynomials are not invariant by mutation and which could be used to distinguished some pairs of mutant knots.

**Organization**

The paper is organized as follows. Section 2 deals with some reviews on the needed tools: twisted cohomology, Reidemeister torsion (with sign) and Mayer–Vietoris property for Reidemeister torsion. In Section 3, we explain in details the construction of the torsion form, which is a 1-form on the character variety of the knot exterior. In Section 4 we describe the concept of mutation, give a precise definition of the notion of positive mutation, and explain how to associate to any SU(2)-representation of the group of a knot $K$ a representation of the group of one of its mutant. Section 5 deals with the proof of the main theorem: the invariance of the torsion form by positive mutation. In the last Section 6 we discuss some open problems related to mutation and Reidemeister torsions theory.

**Acknowledgments**

The author would like to warmly thanks S. Friedl, J. Porti for helpful comments and encouragements related to the present work.

2. Preliminaries

2.1. Twisted cohomology and derivations. In this subsection we review a method to describe the first twisted cohomology group by using twisted derivations.

Let $G$ be a group of finite type, and consider a representation $\rho: G \to \text{SU}(2)$. The composition of a representation $\rho: G \to \text{SU}(2)$ with the adjoint action $\text{Ad}$ of $\text{SU}(2)$ on $\text{su}(2)$ gives us the following representation, called the adjoint representation associated to $\rho$:

$$\text{Ad} \circ \rho: G \to \text{Aut}(\text{su}(2)) = \text{SO}(3)$$

$$\gamma \mapsto (v \mapsto \rho(\gamma)v\rho(\gamma)^{-1})$$

A twisted derivation (twisted by $\text{Ad} \circ \rho$) is a mapping $d: G \to \text{su}(2)$ satisfying the following cocycle relation:

$$d(g_1g_2) = d(g_1) + \text{Ad}_{\rho(g_1)}d(g_2), \text{ for all } g_1, g_2 \in G.$$  

We let $\text{Der}_\rho(G)$ denote the set of twisted derivation of $G$. Among twisted derivations, we distinguish the inner ones. A map $\delta: G \to \text{su}(2)$ is an inner derivation, if there exists $a \in \text{su}(2)$ such that

$$\delta(g) = a - \text{Ad}_{\rho(g)}a, \text{ for all } g \in G.$$  

We let $\text{Inn}_\rho(G)$ denote the set of interior derivations of $G$ twisted by $\rho$. Observe that, if $\rho: G \to \text{SU}(2)$ is irreducible, then $\text{Inn}_\rho(G) \cong \text{su}(2)$. 

Let $W$ be a finite CW–complex and let $G = \pi_1(W)$ be its group. The Lie algebra $\mathfrak{su}(2)$ endows a structure of a (right) $\mathbb{Z}[G]$–module via the action $Ad \circ \rho$. In the sequel, $\mathfrak{su}(2)_\rho$ denote this structure. Let $\tilde{W}$ be the universal cover of $W$, it is well–known that the complex $C_\ast(W; \mathbb{Z})$ is also a (left) $\mathbb{Z}[G]$–module by using the action of $G = \pi_1(W)$ on $\tilde{W}$ by the covering transformations. The $(\mathfrak{su}(2)_\rho)$–twisted cochain complex of $W$ is

$$C^\ast(W; \mathfrak{su}(2)_\rho) = \text{Hom}_{\mathbb{Z}[G]} \left( C_\ast(\tilde{W}; \mathbb{Z}); \mathfrak{su}(2)_\rho \right).$$

This twisted complex $C^\ast(W; \mathfrak{su}(2)_\rho)$ computes the $(Ad \circ \rho)$–twisted cohomology $H^\ast_{\rho}(W) = H^\ast(W; \mathfrak{su}(2)_\rho)$, which is a finite dimensional real vector space. It is well–known that (see [5]):

$$Z^1_\ast(G) \cong \text{Der}_\rho(G), \quad B^1_\ast(G) \cong \text{Inn}_\rho(G), \quad H^1_\ast(G) \cong \text{Der}_\rho(G)/\text{Inn}_\rho(G),$$

and

$$H^0_\ast(G) = \mathfrak{su}(2)^{Ad \circ \rho(G)} = \{ v \in \mathfrak{su}(2) \mid v = Ad_{\rho(g)}v, \forall g \in G \}.$$ 

For each irreducible representation $\rho$ of $G$ we thus have the short exact sequence

$$0 \to \mathfrak{su}(2) \to \text{Der}_\rho(G) \to H^1_\rho(G) \to 0.$$ 

If $X$ is a $K(G,1)$–space (for example knot exteriors are $K(\pi_1,1)$–spaces), then

$$\text{Der}_\rho(X) = \text{Der}_\rho(G), \quad \text{Inn}_\rho(X) = \text{Inn}_\rho(G).$$

2.2. Reidemeister torsion. We review the basic notions and results about the sign–determined Reidemeister torsion introduced by Turaev which are needed in this paper. Details can be found in Milnor’s survey [9] and in Turaev’s monograph [15].

**Torsion of a chain complex.** Let $C_\ast = (0 \to C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to C_0 \to 0)$ be a chain complex of finite dimensional vector spaces over $\mathbb{R}$. Choose a basis $c^i$ for $C_i$ and a basis $h^i$ for the $i$-th homology group $H_i = H_i(C_\ast)$. The torsion of $C_\ast$ with respect to these choices of bases is defined as follows.

Let $b^i$ be a sequence of vectors in $C_i$ such that $d_i(b^i)$ is a basis of $B_{i-1} = \text{im}(d_i : C_i \to C_{i-1})$ and let $\tilde{h}^i$ denote a lift of $h^i$ in $Z_i = \ker(d_i : C_i \to C_{i-1})$. The set of vectors $d_{i+1}(b^{i+1})\tilde{h}^ib^i$ is a basis of $C_i$. Let $[d_{i+1}(b^{i+1})\tilde{h}^ib^i/c^i] \in C^\ast$ denote the determinant of the transition matrix between those bases (the entries of this matrix are coordinates of vectors in $d_{i+1}(b^{i+1})\tilde{h}^ib^i$ with respect to $c^i$). The **sign-determined Reidemeister torsion** of $C_\ast$ (with respect to the bases $c^i$ and $h^i$) is the following alternating product (see [14, Definition 3.1]):

$$\text{Tor}(C_\ast, c^i, h^i) = (-1)^{|C_\ast|} \prod_{i=0}^n \det [d_{i+1}(b^{i+1})\tilde{h}^ib^i/c^i]^{(-1)^{i+1}} \in \mathbb{R}^\ast.$$ 

Here

$$|C_\ast| = \sum_{k \geq 0} \alpha_k(C_\ast) \beta_k(C_\ast),$$

where $\alpha_k(C_\ast) = \sum_{k=0}^i \dim C_k$ and $\beta_i(C_\ast) = \sum_{k=0}^i \dim H_k$.

The torsion $\text{Tor}(C_\ast, c^i, h^i)$ does not depend on the choices of $b^i$ and $\tilde{h}^i$. Note that if $C_\ast$ is acyclic (i.e. if $H_i = 0$ for all $i$), then $|C_\ast| = 0$. 


Torsion of a CW-complex. Let $W$ be a finite CW-complex; consider a representation $\rho: \pi_1(W) \to SU(2)$. We let $\{e^{(i)}_1, \ldots, e^{(i)}_{n_i}\}$ denote the set of $i$-dimensional cells of $W$. Choose a lift $\tilde{e}^{(i)}_j$ of the cell $e^{(i)}_j$ in the universal cover $\tilde{W}$ of $W$ and choose an arbitrary order and an arbitrary orientation for them. Thus, for each $i$, $e^i = \{e^{(i)}_1, \ldots, e^{(i)}_{n_i}\}$ is a $\mathbb{Z}[\pi_1(W)]$-basis of $C_i(\tilde{W}; \mathbb{Z})$ and we associate to it the corresponding “dual” basis over $\mathbb{R}$

$$e^i_{SU(2)} = \{\tilde{e}^{(i)}_1, \tilde{e}^{(i)}_j, \ldots, \tilde{e}^{(i)}_{n_i} : i, j, k\}$$

of $C_i(W; Ad \circ \rho) = \text{Hom}_{\pi_1(\chi)}(C_i(\tilde{W}; \mathbb{Z}), su(2))$. Here

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } k = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

If $h^*$ is a basis of $H^2_0(W)$ then $\text{Tor}(\ast^*(W; Ad \circ \rho), e^i_{SU(2)}, h^*) \in \mathbb{R}^*$ is well-defined.

The cells $\{\tilde{e}^{(i)}_j\}_{0 \leq i \leq \dim W; 1 \leq j \leq n_i}$ are in one-to-one correspondence with the cells of $W$ and their order and orientation induce an order and orientation for the cells $\{e^{(i)}_j\}_{0 \leq i \leq \dim W; 1 \leq j \leq n_i}$. We thus produce a basis over $\mathbb{R}$ for $C_*(W; \mathbb{R})$ which is denoted $e^*$.

Choose a homology orientation of $W$, i.e. an orientation of the real vector space $H_*(W; \mathbb{R}) = \bigoplus_{i \geq 0} H_i(W; \mathbb{R})$; let $\sigma$ denote such an orientation. Provide each vector space $H_i(W; \mathbb{R})$ with a reference basis $h^*$ such that the basis $h^* = \{h_0^0, \ldots, h_i^{\dim W}\}$ of $H_i(W; \mathbb{R})$ is positively oriented with respect to the cohomology orientation $\sigma$. Compute the sign-determined Reidemeister torsion $\text{Tor}(C_*(W; \mathbb{R}), c^*, h^*) \in \mathbb{R}^*$ of the resulting basis and cohomology based chain complex $C_*(W; \mathbb{R})$ and consider its sign $\tau_0 = \text{sgn}(\text{Tor}(C_*(W; \mathbb{R}), c^*, h^*)) \in \{\pm 1\}$. Further observe that $\tau_0$ does not depend on the choice of the positively oriented basis $h^*$.

The sign-determined Reidemeister torsion of the cohomology oriented CW-complex $W$ twisted by the representation $Ad \circ \rho$ is the product

$$\text{Tor}(W; Ad \circ \rho, h^*, \sigma) = \tau_0 \cdot \text{Tor}(\ast^*(W; Ad \circ \rho), e^i_{SU(2)}, h^*) \in \mathbb{R}^*.$$

The torsion $\text{Tor}(W; Ad \circ \rho, h^*, \sigma)$ is the $(Ad \circ \rho)$-twisted Reidemeister torsion of $W$. It is well-defined. It does not depend on the choice of the lifts $\tilde{e}^{(i)}_j$ nor on the order and orientation of the cells (because they appear twice). Finally, it just depends on the conjugacy class of $\rho$.

One can prove that $\tau_0$ is invariant under cellular subdivision, homeomorphism class and simple homotopy type. In fact, it is precisely the sign $(-1)^{|C^1|}$ in (2) which ensures all these properties of invariance (see Turaev’s monographs [14, 15]).

2.3. Mayer–Vietoris sequence. In this subsection we briefly review the so-called Mayer-Vietoris formula for Reidemeister torsions which will be used in the proof of the main results. This formula is based on the multiplicity property of Reidemeister torsions.

**Theorem 1** (Mayer-Vietoris formula). Let $W$ be a finite CW-complex, let $W_1$ and $W_2$ be two closed subcomplexes such that $W = W_1 \cup W_2$ and $V = W_1 \cap W_2$ is not void. Consider any representation $\rho: \pi_1(W) \to SU(2)$ and let us $\rho_i = \rho|_{\pi_1(W_i)}$ and $\rho_V = \rho|_{\pi_1(V)}$ denote its restrictions respectively to $\pi_1(W_i)$ and $\pi_1(V)$. If $\mathcal{H}$ denotes the Mayer–Vietoris sequence in twisted cohomology associated to the splitting $W = W_1 \cup V W_2$ and to the representation $\rho$, then one has the Mayer–Vietoris formula:

$$\text{Tor}(W_1; Ad \circ \rho_1) \cdot \text{Tor}(W_2; Ad \circ \rho_2) = (-1)^{\varepsilon + \alpha} \text{Tor}(W; Ad \circ \rho) \cdot \text{Tor}(V; Ad \circ \rho_V) \cdot \text{tor}(\mathcal{H}).$$

Here

$$\alpha = \alpha(\ast^*(W; Ad \circ \rho), \ast^*(V; Ad \circ \rho_V))$$

and

$$\varepsilon = \varepsilon(\ast^*(W; Ad \circ \rho), \ast^*(W_1; Ad \circ \rho_1) \oplus \ast^*(W_2; Ad \circ \rho_2), \ast^*(V; Ad \circ \rho_V)).$$
Proof. This formula follows from the Multiplicativity Lemma for the torsions (see [15]) applied to the following sequence of complexes:

\[ 0 \rightarrow C^* (W; Ad \circ \rho_1) \oplus C^* (W_2; Ad \circ \rho_2) \rightarrow C^* (V; Ad \circ \rho) \rightarrow 0, \]

which induces the Mayer–Vietoris long exact sequence $\mathcal{H}$. (see [10, Proposition 0.11] for details).

3. Review on the construction of the torsion form

In this section, we are interested with knots. So, let $K$ be a knot in $S^3$ and consider its exterior $M_K = S^3 \setminus V(K)$, where $V(K)$ is an open tubular neighborhood of $K$. Let $G_K = \pi_1 (M_K)$ be the fundamental group of $M_K$, we call it the group of $K$. Observe that $M_K$ is a compact connected three–dimensional manifold whose boundary consists in a single two–dimensional torus $\partial M_K = T^2$.

It is well–known (see for example [10] or [1]) that, for any representation $\rho: G_K \rightarrow \text{SU}(2)$, the $(Ad \circ \rho)$–twisted cohomology never vanishes, actually one can prove using Poincaré duality that:

\[ \dim \mathbb{R} H^1_\rho (M_K) \geq 1. \]

For an irreducible representation $\rho: G_K \rightarrow \text{SU}(2)$, we moreover know that $H^0_\rho (M_K) = 0$. As a consequence, using the fact that the Euler characteristic of $M_K$ vanishes, one has:

\[ \dim \mathbb{R} H^2_\rho (M_K) = \dim \mathbb{R} H^1_\rho (M_K), \]

for any irreducible representation $\rho: G_K \rightarrow \text{SU}(2)$.

3.1. The notion of regular representation. In this subsection, we review the notion of regularity for representations of a knot group in $\text{SU}(2)$.

**Definition 1.** A representation $\rho: G_K \rightarrow \text{SU}(2)$ is called regular, if $\dim \mathbb{R} H^1_\rho (M_K) = 1$.

Of course, if a representation $\rho: G_K \rightarrow \text{SU}(2)$ is regular, then every conjugate of $\rho$ is also regular. Thus the notion of regularity is well–defined not only for representations but for characters. Important properties concerning the set $\text{Reg}(K)$ of regular representations up to conjugation is summarized into the following result:

**Theorem 2** (see [4, 1]). The set $\text{Reg}(K)$ is a one–dimensional manifold, and if $\rho \in \text{Reg}(K)$ then the tangent space to the character variety $T_\rho X (M_K)$ is isomorphic to $H^3_\rho (M_K)$.

Moreover, for a regular representation $\rho: G_K \rightarrow \text{SU}(2)$, one has $\dim \mathbb{R} H^2_\rho (M_K) = 1$.

3.2. Construction of the torsion form. To construct a Reidemeister torsion, especially in a non acyclic context, we need to define some distinguished bases for homology groups. Let us review here Porti’s construction of a distinguished basis for $H^2_\rho (M_K)$ (see [10, 1]).

To fix the notation, we suppose that $S^3$ is oriented and that we have chosen an orientation for the knot $K$. The knot exterior $M_K$ inherits an orientation from that of $S^3$ and its boundary $\partial M_K$ is also oriented using the convention “the inward pointing normal vector in the last position”. As $K$ is oriented, the peripheral system $(\mu, \lambda)$ inherits an orientation as follows. First, we orient $\mu$ by the rule $ell_k (\mu, K) = +1$, and $\lambda$ is oriented using the intersection number defined by the orientation of $\partial M_K$: int$(\mu, \lambda) = +1$.

Let $\rho: G_K \rightarrow \text{SU}(2)$ be an irreducible representation. Observe that $\rho (\mu) \neq \pm 1$ (because, in a Wirtinger presentation of $G_K$ each generator is conjugate to $\mu$, and as $\rho$ is irreducible $\rho (G_K) \not\subset \{ \pm 1 \}$), as a consequence there exist only one couple $(\theta, P_\rho) \in (0, \pi) \times S^2$ such that

\[ \rho (\mu) = \cos (\theta) + \sin (\theta) P_\rho. \]
The vector $P_{\rho}$ generates the common axe of the rotations $Ad \circ \rho(\pi_1(\partial M_K))$, and thus $P_{\rho}$ generates $H^0_\rho(\partial M_K)$.

Let us denote by $\epsilon$ the generator of $H^2(\partial M_K; \mathbb{Z}) = \text{Hom}(H_2(\partial M_K; \mathbb{Z}), \mathbb{Z})$ corresponding to the fundamental class $[\partial M_K] \in H_2(\partial M_K; \mathbb{Z})$ induced by the orientation of $\partial M_K$.

With such notation, one has (see [10], Proposition 1.3.2):

**Lemma 3.** If $\rho: G_K \to SU(2)$ is regular, then the homomorphism $\iota^* : H^2_\rho(M_K) \to H^2(\partial M_K; \mathbb{R})$ induced by the inclusion $i : \partial M_K \hookrightarrow M_K$ is an isomorphism.

In this way, we have defined a volume form $\tau$ on the one-dimensional manifold $\text{Reg} \( K \)$.

Now, to fix the ambiguity of the sign in the torsion, and following Turaev’s construction of refined torsions [15], we need to define an homological orientation. The knot exterior $M_K$ is equipped with a distinguished homology orientation denoted $\circ$ (see [15]) given by $H_0(M_K; \mathbb{R}) = \mathbb{R}[\mu], H_1(M_K; \mathbb{R}) = \mathbb{R}[\mu], H_i(M_K; \mathbb{R}) = 0 \ (i \geq 2)$.

Here $[\mu]$ denotes the class of a point and $[\mu]$ denotes the class of the meridian $\mu$ of $K$.

Let $\rho$ be a regular representation of $G_K$, in that case $T_\rho X(M_K)$ and $H^1_\rho(M_K)$ are isomorphic (see Theorem 2), explicitly the isomorphism $\varphi_\rho : T_\rho X(M_K) \to H^1_\rho(M_K)$ is induced by (see [10, Paragraphe 3.1.3]):

$$
T_\rho R(G_K) \to Z^1_\rho(G_K), \quad \frac{d\rho_t}{dt} \bigg|_{t=0} \mapsto \left\{ \begin{array}{ll}
G_K \to \text{su}(2) & g \mapsto \varphi_\rho(g) \rho(g^{-1}) |_{t=0} \\
\end{array} \right.
$$

with $\rho_0 = \rho$.

The torsion form is the 1-form defined by:

$$
\tau^K_\rho : T_\rho X(M_K) \to \mathbb{C}, \quad \tau^K_\rho(v_\rho) = \begin{cases} 
\text{Tor}(M_K; \text{su}(2); \{\varphi_\rho(v_\rho), h_\rho^{(2)}\}; \circ) & \text{if } v_\rho \neq 0 \\
0 & \text{if } v_\rho = 0 
\end{cases}
$$

In this way, we have defined a volume form $\tau^K$ on the one-dimensional manifold $\text{Reg}(K)$.

Here is some remarks concerning the definition of the volume form $\tau^K$.

**Remark 2.** For a regular representation $\rho: G_K \to SU(2)$, there exists a unique $P_\rho \in S^2$ and a unique $\tilde{P}_\rho \in S^2$ such that:

$$
\rho(\mu) = \cos(\theta) + \sin(\theta)P_\rho \quad \text{and} \quad \rho(\mu) = \cos(2\pi - \theta) + \sin(2\pi - \theta)\tilde{P}_\rho
$$

with $\theta \in (0, \pi)$. Changing $P_\rho$ into $\tilde{P}_\rho$ in the construction has for consequence to change the volume form $\tau^K$ into $-\tau^K$.

**Remark 3.** The 1-form $\tau^K$ does not depend on the orientation of $K$.

**Remark 4.** In general $\text{Reg}(K)$ is not compact, the problem of integration on the character variety and with respect to the torsion form $\tau^K$ is not easy. It has been considered recently in [3].
4. Mutation and mutant representations

4.1. Review on mutation of knots. Let $K \subset S^3$ be a knot. We let $F$ be a 4-punctured 2-sphere which is incompressible in $M_K$ and whose closure in $S^3$ is an embedded sphere cutting $K$ transversally into four points. Such a surface is called a mutation sphere. We adopt the following notation. The 3-sphere splits along $S^2$ into two 3-balls: $S^3 = B_1 \cup_{S^2} B_2$. Let $M_i = B_i \cap M_K$ and write $K_i = B_i \cap K$ for $i = 1, 2$. We have $K = K_1 \cup K_2$ and $M_K = M_1 \cup_{id} M_2$ where $id: F \to F$ is the identity map. Note that each $K_i$ consists of two arcs.

The surface $F$ admits some orientation preserved involutions, we consider the three $\pi$-angle rotations of $S^2$ that leave the four points $K \cap S^2$ invariant. Let $\tau$ be such a rotation (see Fig. 1). The mutant knot $K^{\tau}$ is the knot $K_1 \cup_{\tau} K_2$ obtained by cutting $K$ along $F \cap K$ and gluing again after the application of $\tau$. On Fig. 2 one can find the example of the Kinoshita-Terasaka knot $K_{KT}$ and its mutant the Conway knot $K_C$.

The following diagram of natural inclusions is commutative:

\[
\begin{array}{ccc}
F & \hookrightarrow & M_1 \\
\downarrow_{i_1} & & \downarrow_{j_1}
\end{array}
\quad
\begin{array}{ccc}
M_1 & \to & M_K \\
\downarrow_{i_2} & & \downarrow_{j_2}
\end{array}
\quad
\begin{array}{ccc}
M_2 & \to & M_K \\
\downarrow_{i_2} & & \downarrow_{j_2}
\end{array}
\]

and the fundamental group of $F$, which is a free group of rank 3, admits the following presentation:

$\pi_1(F) = \langle a,b,c,d \mid abed = 1 \rangle \simeq F_3$.

Given a mutation sphere $(F, \tau)$, there is a fixed point of the rotation $\tau$. In what follows, we chose this fixed point as the base point of the fundamental groups: $\pi_1(F)$, $\pi_1(M_1)$, $\pi_1(M_2)$ and $G_K = \pi_1(M_K)$. Thus, using the Seifert–Van Kampen Theorem, we get a decomposition for the group of $K$:

$G_K \simeq \pi_1(M_1) \ast_{\pi_1(F)} \pi_1(M_2)$.

Of course we get a similar decomposition for the group of the mutant knot $K^{\tau}$:

$G_{K^{\tau}} \simeq \pi_1(M_1) \ast_{\pi_1(F)} \pi_1(M_2)$.

In particular, one can think of the representation space $R(G_K; SU(2))$ as a subspace in $R(M_1; SU(2)) \times R(M_2; SU(2))$ and the inclusion is simply given by the restrictions to $\pi_1(M_1)$ and $\pi_1(M_2)$ (see [12]).
4.2. Positive and negative mutations. Fix an orientation of the knot $K$. The orientation of $K$ induces an orientation for its two parts $K_i = B_i \cap K$ ($i = 1, 2$). Moreover, the orientation of $K$ induces an orientation for its mutant $K^\tau = K_1 \cup_\tau K_2$ defined using the orientation of the unchanged part $K_2$ of the knot. Each meridian $\mu$ of $K$ is oriented by the rule: $\ell_k(\mu, K) = +1$. All the curves $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ respectively corresponding to the generators $a, b, c, d$ of $\pi_1(F)$ are oriented using the same rule: $\ell_k(\gamma, K) = +1$ where $\gamma \in \{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\}$. Moreover observe that necessarily the curves $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ are coupled in two pairs where the curves in the same pair belongs to the same component of $K_i$ ($i = 1, 2$).

We assign to each curve $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ in $F$ a sign $\pm$ as follows. Let $\gamma \in \{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\}$, when passing through $\gamma$ along the oriented knot $K$ if we go from $M_1$ to $M_2$, then we assign $+$ to the curve $\gamma$, if not we assign $-$. Of course this convention depends on the orientation of the knot, if we reversed the orientation of $K$, then all signs change. Moreover, two of the four curves are assign with $+$ and the two other with $-$. Observe that if we consider a pair of curves which lie on the same component of $K_i$, then necessarily one is assign with $+$ and the other with $-$. A mutation $\tau$ sends the set of sign–oriented curves $\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\}$ to itself (but the sign of the curves could be changed in the mutation). We say that the mutation $\tau$ is positive if $\tau$ preserves signs, which means that for all $\gamma \in \{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\}$, the curves $\gamma$ and $\tau(\gamma)$ in $F$ are assigned with the same sign. If not, we say that the mutation is negative.

Observe that there exist only one positive mutation among the three possible ones and that this notion does not depend on the orientation of $K$. We say that $K^\tau$ is a positive mutant (resp. negative mutant) of $K$, if the mutation $\tau$ is positive (resp. negative). As an example, the Conway knot $K_C$ is a positive mutant of the Kinoshita–Terasaka knot $K_{KT}$ (see Fig. 2).

In what follows, we choose a common meridian for $K$ and $K^\tau$. The meridian $\mu$ of $K$ is chosen to be a circle $\partial D^2 \times \{pt\}$ in $\partial M_2 = F \cup \partial D^2 \times I \cup \partial D^2 \times I$. The meridian of $K^\tau$ is chosen as same as $K$, and denoted by $\mu^\tau$. Moreover we endow $M_K$ (resp. $M_K^\tau$) with the usual homological orientations defined by the meridian $\mu$ (resp. $\mu^\tau$).

![Figure 2. The Kinoshita-Terasaka knot $K_{KT}$ and its mutant the Conway knot $K_C$.](image)

4.3. Some homology computations. The aim of this paragraph is to give some observations on the (twisted and non–twisted) homology groups of the mutation sphere $F$ and on the manifolds $M_i$ ($i = 1, 2$).

**Proposition 5.** The homology groups with real coefficients of the punctured 2-sphere $F$ are described as follows:

$$H_j(F; \mathbb{R}) \simeq \begin{cases} \mathbb{R} & \text{if } j = 0, \\ \mathbb{R}^3 & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$
Proposition 6. The homology groups with real coefficients of the manifold $M_i$ are described as follows:

$$H_j(M_i; \mathbb{R}) \simeq \begin{cases} \mathbb{R} & j = 0, \\ \mathbb{R}^2 & j = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. It follows from the Mayer–Vietoris sequence for the decomposition of 3-ball $B^3_i = M_i \cup (D^2 \times I \cup D^2 \times I)$. □

Remark 6. The knot $K$ is cut as four arcs and each 3-ball $B_i$ contains two arcs denoted by $K_i$. We can choose a pair of meridians for two arcs in $B_i$ as a basis of $H_1(M_i; \mathbb{R})$ and denote by $\xi_i$ and $\eta_i$ the homology classes.

The Mayer–Vietoris sequence $\mathcal{V}_\mathbb{R}$ with real coefficients associated to the splitting $M_K = M_1 \cup_{a_1} M_2$ is:

$$\begin{array}{cccccccc}
0 & \longrightarrow & H_1(F; \mathbb{R}) & \bigoplus & H_1(M_1; \mathbb{R}) & \bigoplus & H_1(M_2; \mathbb{R}) & \longrightarrow & H_0(M_K; \mathbb{R}) & \longrightarrow & 0.
\end{array}$$

By counting dimensions, the connecting homomorphism $\delta$ is zero. The first homology group $H_1(M_K; \mathbb{R})$ is generated by the meridian $\mu$. Moreover, one has the following properties on the connecting maps into $\mathcal{V}_\mathbb{R}$.

Lemma 7. Let $\xi_i$ and $\eta_i$ denote meridians of $K_i$ such that they give a basis of $H_1(M_i; \mathbb{R})$ for $i = 1, 2$. In the Mayer–Vietoris sequence $\mathcal{V}_\mathbb{R}$, the homomorphisms $j_1^*$ and $j_2^*$ satisfy the following identities:

$$j_1^*(\xi_1) = j_1^*(\eta_1) = j_2^*(\xi_2) = j_2^*(\eta_2) = [\mu] \text{ in } H_1(M_K; \mathbb{R}),$$

where $\mu$ denotes an oriented meridian of $K$.

Proof of the Lemma. The boundary of $M_1$ consists of a four punctured sphere $F$ and two annuli. An annulus in $M_1$ is connected with two annuli in $M_2$ in $M_K$, thus joining these four annuli alternately, we obtain the boundary torus of $M_K$. When an annulus in $\partial M_1$ has the boundary $a \cup (-b)$ on $F$ and contains the meridian $\xi_1$, the map $(i_1^*, i_2^*)$ sends $a$ and $b$ to $[\xi_1, \xi_2]$ and $[\xi_1, \eta_2]$. Such elements are contained in the kernel $j_1^* - j_2^*$. Hence we have $j_1^*(\xi_1) = j_2^*(\xi_2) = j_1^*(\eta_2) = j_2^*(\eta_1)$. It follows from the surjectivity of $j_1^* - j_2^*$ that all $\xi_i$ and $\eta_i$ are send to the meridian $\mu$. □

Remark 7. For a positive mutation $\tau$, Lemma 7 also holds for the Mayer–Vietoris sequence associated to the splitting $M_{K^\tau} = M_1 \cup_{\tau} M_2$. For a negative mutation $\tau$, it holds that $j_1^k(\xi_1) = j_2^k(\eta_1)$ for each $k = 1, 2$. But $j_1^k(\xi_1)$ has a different sign than the one of $j_2^k(\xi_2)$.

4.4. Mutant representation. For any representation $\rho: G_K \to \text{SU}(2)$, its restriction $\rho_F: \pi_1(F) \to \text{SU}(2)$ to $\pi_1(F)$ is such that $\chi_{\rho_F}(a) = \chi_{\rho_F}(b) = \chi_{\rho_F}(c) = \chi_{\rho_F}(d)$ (see Fig. 1). We say that $\rho$ is $F$-irreducible if its restriction $\rho_F$ is irreducible.

The following result computes the twisted homology groups of $F$ and $M_i$. 
Lemma 8. If \( \rho : G_K \to \text{SU}(2) \) is an F-irreducible representation of \( G_K \), then we have:
\[
\dim \mathbb{C} H^j_{\rho_F}(F) = \begin{cases} 
6 & \text{if } j = 1, \\
0 & \text{otherwise} 
\end{cases}
\]
and
\[
\dim \mathbb{C} H^1_{\rho_i}(M_i) \geq 3.
\]

Proof.

(1) The punctured sphere \( F \) has the same homotopy type as a bouquet of 3 circles, thus it has the same homotopy type as a 1-dimensional CW-complex and its Euler characteristic is \(-2\). Hence using the irreducibility of \( \rho_F \), we conclude that all its twisted homology groups vanish except in degree 1 for which:
\[
\dim \mathbb{C} H^1_{\rho_F}(F) = -\chi(F) \cdot \dim \mathbb{C} \text{su}(2) = 6.
\]

(2) The boundary of the three-dimensional manifold \( M_i, i = 1, 2 \), is a surface of genus two, thus \( \chi(M_i) = -1 \). Moreover, as the representation \( \varphi = (\rho_i)_{\pi_i(F)} \) is irreducible, we observe that \( \rho_i : \pi_1(M_i) \to \text{SU}(2) \) are also irreducible for \( i = 1, 2 \). The long exact sequence in twisted cohomology with coefficients in \( \text{Ad} \circ \rho_i \) associated to the pair \( (M_i, \partial M_i) \) reduces to:
\[
\begin{array}{ccccccc}
0 & \rightarrow & H^1_{\rho_i}(M_i, \partial M_i) & \rightarrow & H^1_{\rho_i}(M_i) & \rightarrow & i^* \rightarrow & H^1_{\rho_i}(\partial M_i) \\
& & \rightarrow & \rightarrow & \rightarrow & \rightarrow & 0.
\end{array}
\]

Observe that \( \dim H^1_{\rho_i}(M_i) = 3 + \dim H^2_{\rho_i}(M_i) \) and using Poincaré duality, we obtain \( \text{rk} i^* = 3 \), which implies that \( \dim H^1_{\rho_i}(M_i) \geq 3 \).

Moreover, for restrictions to \( \pi_1(F) \) of F-irreducible representations we have the following lemma (see [11, Theorem 2.2] and [12, Lemma 2.1.1]).

Lemma 9. If \( \psi : \pi_1(F) \to \text{SU}(2) \) is an irreducible representation such that \( \chi_{\psi}(a) = \chi_{\psi}(b) = \chi_{\psi}(c) = \chi_{\psi}(d) \), then there is an element \( x \in \text{SU}(2) \) such that:
\[
(4) \quad \psi \circ \tau_x = \text{Ad}_x \circ \psi.
\]

Remark 8. The element \( x \) in Equation (4) is not unique in general. Actually, by Schur’s lemma, \( x \) is defined up to sign.

The rest of this section consists in the construction of the so-called mutant representation associated to a representation of \( G_K \). It is an SU(2)-representation of \( G_K \) corresponding to a representation of \( G_K \) obtained by twisting its restrictions to \( R(M_1; \text{SU}(2)) \) and \( R(M_2; \text{SU}(2)) \), using the pull–back of \( \tau_x \). Note that the pull–back of \( \tau_x \) is defined only on \( R(F; \text{SU}(2)) \). However, Lemma 9 says that the pull–back of \( \tau_x \) is expressed as the adjoint action, so we can use this adjoint action as the twisting on \( R(M_1; \text{SU}(2)) \) instead of the pull–back of \( \tau_x \).

Let \( \rho : G_K \to \text{SU}(2) \) be an F-irreducible representation. The mutant representation \( \rho^\tau : G_K^\tau \to \text{SU}(2) \) associated to \( x \) as in Equation (4) is defined by (see [12, Section 2.2]):
\[
\rho^\tau_1 = \rho_{\tau x}(M_1) = \text{Ad}_{\rho^{-1}} \circ \rho_{\pi_1(M_1)}, \quad \rho^\tau_2 = \rho_{\pi_1(M_2)}.
\]
One can observe that this definition is consistent because both parts agree on the amalgamating subgroup.

It is easy to see that \( \rho : G_K \to \text{SU}(2) \) is an irreducible representation if and only if \( \rho^\tau : G_K^\tau \to \text{SU}(2) \) is as well (see for example [11] or [12]).
Remark 9 (A digression on $\text{SL}_2(\mathbb{C})$-character variety and hyperbolic knots). If $K$ is a hyperbolic knot, then there is a discrete and faithful representation of $G_K$ into $\text{PSL}_2(\mathbb{C})$ which lifts to a representation $\rho_0: G_K \to \text{SL}_2(\mathbb{C})$. Such a representation is irreducible. D. Ruberman proved [11] the following result about the discrete and faithful representation (see also [13, Corollaries 3 and 4]).

Proposition 10. Let $K$ be a hyperbolic knot and consider a mutation sphere $(F, \tau)$. Then the mutant knot $K^\tau$ is hyperbolic (with the same volume as $K$), and the discrete and faithful representation $\rho_0: G_K \to \text{SL}_2(\mathbb{C})$ is $F$-irreducible, moreover the corresponding mutant representation $\tilde{\rho}_0: G_{K^\tau} \to \text{SL}_2(\mathbb{C})$ is the discrete and faithful representation of the hyperbolic structure of $K^\tau$.

Sketch of the proof. The part on hyperbolicity and volume is [11, Corollary 1.4]. Let $\psi_0$ be the restriction of $\rho_0$ to $\pi_1(F)$. One can observe that as a restriction, $\psi_0$ is discrete and faithful, so it is in particular irreducible thus $\rho_0$ is $F$-irreducible. □

S. Tillmann also proved [13] that the geometric components of the character varieties of $G_K$ and $G_{K^\tau}$ are birationally equivalent.

4.5. Regularity property of mutant representations. In this subsection, we construct an isomorphism

\[ \tau^z : H^1_{\rho}(M_K) \to H^1_{\rho^z}(M_{K^\tau}) \]

and introduce a notion of regularity in a sense “compatible” with the splittings $M_K = M_1 \cup M_2$ and $M_{K^\tau} = M_1 \cup M_2$. Again, $\rho_i = \rho|_{\pi_1(M_i)}, i = 1, 2$, and $\rho_F = \rho|_{\pi_1(F)}$ denotes the restrictions of $\rho$.

Notation. In the sequel, we use the following notation:

\[ \tau^z = Ad_x \circ \tau, \text{ for } x \in \text{SU}(2) \text{ and } \rho \text{ a representation}. \]

The construction of the isomorphism of Equation (5) is based on the following technical result.

Lemma 11. If $z \in \text{Der}_{\rho_F}(F)$, then $z \circ \tau \in \text{Der}_{\rho_F}(F)$ is such that:

\[ z \circ \tau = \tau^z + \delta \]

with some $\delta \in \text{Inn}_{\rho_F}(F)$, where $\tau^z = Ad_x \circ z$.

Proof. The proof of Equation (6) essentially consists in writing down the derivative of the equality $\rho_F \circ \tau = \tau^z \circ \rho_F$ (see Lemma 9).

It is easy to observe that $z \circ \tau \in \text{Der}_{\rho_F}(F)$. Let $\varphi_t : \pi_1(F) \to \text{SU}(2)$ be a germ at origin such that $\varphi_0 = \rho_F$ and satisfying, for all $g \in \pi_1(F)$, the following identity:

\[ z(g) = \frac{d}{dt} \varphi_t(g) \rho_F(g)^{-1} \bigg|_{t=0}. \]

For all $t$ in a neighborhood of 0, there exist $x_t \in \text{SU}(2)$ such that $\varphi_t \circ \tau_s = x_t \varphi_t = Ad_{x_t} \circ \varphi_t$.

Set $X = \frac{dx_t}{dt}|_{t=0}$ and take the derivative of the preceding equality with respect to $t$, one has, for all $g \in \pi_1(F)$,

\[ \frac{d\varphi_t(\tau_s(g))}{dt} \bigg|_{t=0} = X \rho_F(g)x^{-1} + x \frac{d\varphi_t(g)}{dt} \bigg|_{t=0} x^{-1} - x \rho_F(g)x^{-1}X x^{-1}. \]

Moreover, $\rho_F(\tau_s(g)) = x \rho_F(g)x^{-1}$, thus

\[ \frac{d\varphi_t(\tau_s(g))}{dt} \bigg|_{t=0} \rho_F(\tau_s(g))^{-1} = X x^{-1} - x \rho_F(g)x^{-1}(X x^{-1})x \rho_F(g)x^{-1} + x \frac{d\varphi_t(g)}{dt} \bigg|_{t=0} \rho_F(g)^{-1}x^{-1}. \]
Finally, for all \( g \in \pi_1(F) \),
\[
z \circ \tau_*(g) = \tau_*(g) + (1 - Ad_{\rho F}(g))Xx^{-1},
\]
with \( Xx^{-1} \in \mathfrak{su}(2) \).
\( \square \)

From this technical lemma, we deduce:

**Corollary 12.** If \( h \in H^1_{\rho F}(F) \), then \( h \circ \tau_* \in H^1_{\rho F}(F) \) and \( h \circ \tau_* = \tau h = Ad_x \circ h \).

Observe that the twisted cohomology groups \( H^1_{\rho F}(M_1) \) and \( H^1_{\rho F^{-1}}(M_1) \) are isomorphic. Moreover, the isomorphism is induced by \( \phi_x : z \mapsto Ad_{\rho^{-1}}z \) and will be denoted \( \phi_x \) in the sequel. Using Corollary 12 it is easy to deduce the following result.

**Claim 13.** Let \( \iota_\ell : M_\ell \hookrightarrow M_K, \ell = 1, 2, \) be the usual inclusions. The following diagram is commutative:

\[
\begin{array}{ccc}
H^1_{\rho_1}(M_1) \oplus H^1_{\rho_2}(M_2) & \xrightarrow{i_1^\vee + \cdot \cdot \cdot + i_\ell^\vee} & H^1_{\rho F}(F) \\
\phi_x \oplus \text{Id} & \cong & i_1^\vee + \cdot \cdot \cdot + i_\ell^\vee \\
H^1_{\rho^{-1}}(M_1) \oplus H^1_{\rho_2}(M_2) & & \\
\end{array}
\]

Let \( j_\ell : M_\ell \hookrightarrow M_K \) and \( j'_\ell : M_\ell \hookrightarrow M_{K^\tau}, \ell = 1, 2, \) be the usual inclusions. Write down the Mayer-Vietoris sequences in cohomology respectively associated to the splittings \( M_K = M_1 \cup_{\text{Id}} M_2 \) and \( M_{K^\tau} = M_1 \cup_{\tau} M_2 \) and twisted by \( \rho \) and \( \rho^\tau \). We obtain:

\[
\begin{array}{ccc}
0 & \xrightarrow{i_1^\vee + \cdot \cdot \cdot + i_\ell^\vee} & H^1_{\rho_1}(M_1) \oplus H^1_{\rho_2}(M_2) \xrightarrow{i_1^\vee + \cdot \cdot \cdot + i_\ell^\vee} H^1_{\rho F}(F) \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{i_1^\vee + \cdot \cdot \cdot + i_\ell^\vee} & H^1_{\rho_1}(M_1) \oplus H^1_{\rho_2}(M_2) \xrightarrow{i_1^\vee + \cdot \cdot \cdot + i_\ell^\vee} H^1_{\rho F}(F) \\
\end{array}
\]

where \( \rho_1^\tau = \tau^{-1} \rho_1, \rho_2^\tau = \rho_2 \) and \( \rho_{F}^\tau = \rho_{F} \).

Combine these two exact sequences by using Diagram (7), one has the following commutative diagram:

\[
\begin{array}{ccc}
0 & \xrightarrow{i_1^\vee + \cdot \cdot \cdot + i_\ell^\vee} & H^1_{\rho_1}(M_1) \oplus H^1_{\rho_2}(M_2) \xrightarrow{i_1^\vee + \cdot \cdot \cdot + i_\ell^\vee} H^1_{\rho F}(F) \\
\phi_x \oplus \text{Id} & \cong & i_1^\vee + \cdot \cdot \cdot + i_\ell^\vee \\
0 & \xrightarrow{i_1^\vee + \cdot \cdot \cdot + i_\ell^\vee} & H^1_{\rho_1}(M_1) \oplus H^1_{\rho_2}(M_2) \xrightarrow{i_1^\vee + \cdot \cdot \cdot + i_\ell^\vee} H^1_{\rho F}(F) \\
\end{array}
\]

Thus we can restrict the isomorphism \( \phi_x \oplus \text{Id} : H^1_{\rho_1}(M_1) \oplus H^1_{\rho_2}(M_2) \to H^1_{\rho_1^\tau}(M_1) \oplus H^1_{\rho_2^\tau}(M_2) \) to an isomorphism

\[
\tau^\tau : H^1_{\rho}(M_K) \to H^1_{\rho^\tau}(M_{K^\tau}).
\]

An immediate consequence is the following: a representation \( \rho : G_K \to \text{SU}(2) \) is regular if, and only if, \( \rho^\tau : G_{K^\tau} \to \text{SU}(2) \) is also regular.

To compare the torsion form of \( K \) and the one of one of its mutant \( K^\tau \) we make the following technical hypothesis:

**Hypothesis.** Fix a (positive) mutation \( (F, \tau) \) and suppose that \( \rho : G_K \to \text{SU}(2) \) is \( \tau \)-regular, which means that:

1. \( \rho \) is regular;
2. and \( H^2_{\rho_i}(M_i) = 0 \), for \( i = 1, 2 \).

We have the following remarks.

**Remark 10.** It is easy to prove that for a regular representation \( \rho : G_K \to \text{SU}(2) \), the following assertions are equivalent:
(1) $\rho: G_K \to SU(2)$ is $\tau$-regular (i.e. $H^2_{\rho_i}(M_i) = 0$, for $i = 1, 2$),
(2) $\dim H^1_{\rho_i}(M_i) = 3$, for $i = 1, 2$,
(3) the homomorphism $H^1_{\rho_i}(M_i, \partial M_i) \to H^1_{\rho_i}(M_i)$ is 0, for $i = 1, 2$.

Remark 11. If $\rho: G_K \to SU(2)$ is $\tau$-regular, then $\rho$ is $F$-irreducible. Thus, all its restrictions $\rho_F = \rho_{\pi_1(F)}$ and $\rho_i = \rho_{\pi_1(M_i)}$ ($i = 1, 2$) are irreducible.

5. Behavior of the torsion form by positive mutation

In this section, we prove that the torsion form is invariant by positive mutation.

Theorem 14. If $\rho: G_K \to SU(2)$ is a $\tau$-regular representation, then

$$\tau^K_{\rho^*} \circ \tau^\tau = \tau^K_{\rho}.$$ 

Here $\rho^*: G_{K^\tau} \to SU(2)$ denotes the mutant representation associated to the representation $\rho: G_K \to SU(2)$ and $\tau^\tau: H^1_{\rho}(M_K) \to H^1_{\rho^*}(M_{K^\tau})$ is the isomorphism of Equation (10).

The rest of this section is devoted to the proof of Theorem 14. The proof is divided into two parts: in the first one we are interested in the “twisted part” of the torsion, and in the second one in its “sign part”.

We compute the torsions in the geometric bases described as follows. Fix a presentation of the group $G_K$:

$$\Gamma_K = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_{n-1} \rangle.$$

It is known, using a result due to Waldhausen [16], that $M_K$ has the same homotopy type as the two-dimensional CW-complex $W_K$ constructed as follows. The 0-skeleton of $W_K$ consists in a single 0-cell, its 1-skeleton is a wedge of $n$ oriented circles corresponding to the generators $x_1, \ldots, x_n$ and the 2-skeleton consists in $(n - 1)$ 2-cells $D_1, \ldots, D_{n-1}$ where the attaching maps are given by the relations $r_1, \ldots, r_{n-1}$. Let us write

$$T^K_{\rho}(v) = \text{Tor}(C_*(W_K; Ad \circ \rho); \{\varphi_{\rho}(v), h_{\rho}^{(2)}\}) = \text{Tor}(C_*(M_K; Ad \circ \rho); \{\varphi_{\rho}(v), h_{\rho}^{(2)}\}),$$

and

$$\varepsilon^K = \text{sgn}(\text{Tor}(C_*(W_K; \mathbb{R}; \circ)); \text{sgn}(\text{Tor}(C_*(M_K; \mathbb{R}; \circ)).$$

One has

$$\tau^K_{\rho}(v) = \varepsilon^K \cdot T^K_{\rho}(v).$$

5.1. Computation of the twisted part of the torsion. Here we compute the twisted part of the torsions of $M_K$ and $M_{K^\tau}$, using the Mayer–Vietoris formula respectively associated to the splittings $M_K = M_1 \cup_{id} M_2$ and $M_{K^\tau} = M_1 \cup_{\tau} M_2$.

Remark 12. In the case of a positive mutation, using Lemma 7, observe that the meridian of $K$ and the one of $K^\tau$ can be defined by the same loop in $M_2$ (more precisely in the boundary of the mutation sphere). As a consequence, we can choose the same $(Ad \circ \rho)$-invariant vector $P_{\rho} \in H^0_{\rho}(T^2)$ for both $K$ and its positive mutant $K^\tau$. In the sequel, we will do that.

Proof of Theorem 14, “twisted part”. Let $v \in H^1_{\rho}(M_K)$ be a non zero vector. The Mayer–Vietoris sequence for twisted cohomology associated to the splitting $M_K = M_1 \cup_{id} M_2$ is the following exact sequence denoted $\mathcal{H}^*$:

$$\mathcal{H}^* : 0 \longrightarrow H^1_{\rho}(M_K) \xrightarrow{f} H^1_{\rho_1}(M_1) \oplus H^1_{\rho_2}(M_2) \xrightarrow{\iota^* + \iota_2^*} H^1_{\rho}(F) \xrightarrow{\partial} H^2_{\rho}(M_K) \longrightarrow 0.$$ 

Applying the Mayer–Vietoris formula for the torsions gives us:

$$T^K_{\rho}(v) \cdot \text{Tor}(F; Ad_{\rho_F}h_F) \cdot \text{Tor}(\mathcal{H}^*) = (-1)^n \text{Tor}(M_1; Ad_{\rho_1}h_{M_1}) \cdot \text{Tor}(M_2; Ad_{\rho_2}h_{M_2}),$$

for
where \( h_{M_i} \) is a basis of \( H^i_{\rho_i}(M_i) \), \( i = 1, 2 \), and \( \rho_{RF} \) is a basis of \( H^2_{\rho}(F) \).

On the other hand, the Mayer–Vietoris sequence for twisted cohomology associated to the splitting \( M_{K'} = M_1 \cup_{\tau} M_2 \) is the exact sequence denoted \( \mathcal{H}^*_\tau \):

\[
\cdots \rightarrow H^1_{\rho_i}(M_1) \oplus H^2_{\rho_2}(M_2) \xrightarrow{\rho_{RF}} H^1_{\rho_2}(M_1) \oplus H^2_{\rho_1}(M_2) \xrightarrow{\rho_{RF}} H^2_{\rho}(F) \xrightarrow{\partial} H^2_{\rho}(M_K) \rightarrow 0
\]

Here, observe that \( \rho^2_1 = x^{-1}\rho_1 \) et \( \rho^2_2 = \rho_2 \). Another application of the Mayer–Vietoris formula for the torsions gives:

\[
T^H_{\rho}(\tau^f(v)) = \frac{\text{Tor}(\mathcal{H})}{\text{Tor}(\mathcal{H}_\tau)}.
\]

Now we have to compare the torsions \( \text{Tor}(\mathcal{H}) \) and \( \text{Tor}(\mathcal{H}_\tau) \).

**Claim 15.** For a positive mutation \( \tau \), one has:

\[
T^H_{\rho}(\tau^f(v)) = \frac{\text{Tor}(\mathcal{H})}{\text{Tor}(\mathcal{H}_\tau)} = 1.
\]

**Proof of the claim.** Let us compute in parallel the two Reidemeister torsions \( \text{Tor}(\mathcal{H}) \) and \( \text{Tor}(\mathcal{H}_\tau) \):

1. Let \( b \) be a basis of \( \text{im}(i^*_1 + i^*_2) \), \( \tilde{h}^{(2)}_{\rho} \) be a lift of \( h^{(2)}_{\rho} \) by \( \partial \) and \( c = f(v) \) a generator of \( \text{im } f \) (see Sequence (11)). The torsion \( \text{Tor}(\mathcal{H}) \) is equal to:

\[
\text{Tor}(\mathcal{H}) = [\tilde{b}\tilde{h}^{(2)}_{\rho}/h_{RF}] \cdot [c\tilde{b}/h_{M_1}, h_{M_2}]^{-1}.
\]

2. In the same way, let \( b' \) be a basis of \( \text{im}(\tau^*i^*_1 + i^*_2) \), \( \tilde{h}^{(2)}_{\rho_\tau} \) be a lift of \( h^{(2)}_{\rho_\tau} \) by \( \partial^* \) and \( c' = f^* \circ \tau^f(v) \) be a generator of \( \text{im } f^* \) (see Sequence (13)). The torsion \( \text{tor}(\mathcal{H}_\tau) \) is equal to:

\[
\text{tor}(\mathcal{H}_\tau) = [b\tilde{h}^{(2)}_{\rho_\tau}/h_{RF}] \cdot [c'b'/h_{M_1}, h_{M_2}]^{-1}.
\]

Further observe that \( c' = \tilde{c} \oplus \text{Id}(c) \), thus \( [c\tilde{b}/h_{M_1}, h_{M_2}] = [c'\tilde{b}'/h_{M_1}, h_{M_2}] \). As a result we obtain:

\[
T^H_{\rho}(\tau^f(v)) = [b\tilde{h}^{(2)}_{\rho}/b'] \cdot [\tilde{b}\tilde{h}^{(2)}_{\rho_\tau}/h_{RF}]^{-1}.
\]

It remains to compute the following bases change determinant \( [b\tilde{h}^{(2)}_{\rho}/b'] \cdot [\tilde{b}\tilde{h}^{(2)}_{\rho_\tau}/h_{RF}]^{-1} \). It is easy to observe that

\[
[b\tilde{h}^{(2)}_{\rho}/b\tilde{h}^{(2)}_{\rho_\tau}] = [b'/b] \cdot [\tilde{h}^{(2)}_{\rho}/\tilde{h}^{(2)}_{\rho_\tau}]
\]

and the computation of this bases change determinant uses the following commutative diagram (see Claim 13):

\[
\begin{array}{ccc}
\cdots & \xrightarrow{\phi_{\tau} \oplus \text{Id}} & H^1_{\rho_i}(M_1) \oplus H^2_{\rho_2}(M_2) \\
\phi_{\tau} & & \phi_{\tau} \oplus \text{Id} \\
\cdots & \xrightarrow{=} & H^1_{\rho_2}(M_1) \oplus H^2_{\rho_1}(M_2) \\
& & \xrightarrow{\partial^*} H^2_{\rho}(M_K) \\
\end{array}
\]

The computation is divided into two parts:
(1) Computation of $[b'/b]$. One can observe that

$$[b'/b] = \det((\tau^* \iota_1^* \tau + \iota_2^* \tau) \circ (\iota_1^* \tau + \iota_2^* \tau)^{-1})$$

where

$$\iota_1^* \tau + \iota_2^* \tau : H^1_{\rho_1}(M_1) \oplus H^1_{\rho_2}(M_2) / \ker(\iota_1^* \tau + \iota_2^* \tau) \mapsto \text{im}(\iota_1^* \tau + \iota_2^* \tau)$$

and

$$\tau^* \iota_1^* \tau + \iota_2^* \tau : H^1_{\rho_1}(M_1) \oplus H^1_{\rho_2}(M_2) / \ker(\tau^* \iota_1^* \tau + \iota_2^* \tau) \mapsto \text{im}(\tau^* \iota_1^* \tau + \iota_2^* \tau),$$

are respectively induced by $\iota_1^* \tau + \iota_2^* \tau$ and $\tau^* \iota_1^* \tau + \iota_2^* \tau$. The action of $\tau$ on the character variety of $\pi_1(F)$ is up to conjugation trivial (see Lemma 9). As a consequence $\tau^* : H^1_{\rho_1}(F) \to H^1_{\rho_1}(F)$ is the identity, which gives us $[b'/b] = 1$.

(2) Computation of $[\tilde{h}_p^{(2)}/\tilde{h}_p^{(2)}]$. In this part, we prove that $[\tilde{h}_p^{(2)}/\tilde{h}_p^{(2)}] = 1$ for a positive mutation. Actually, we prove that $\partial^* (\tilde{h}_p^{(2)})$ is exactly the reference generator $h_p^{(2)}$ of $H^2_p(M_{K^r})$.

Let us recall the precise definition of the reference generators $h_p^{(2)}$ and $h_p^{(2)}$. In the case of a positive mutation, the meridian $\mu$ of $K$ and the meridian $\mu^r$ of $K^r$ are represented by the same circle in the boundary of the mutation sphere $F$. Let $i : \partial M_K \hookrightarrow M_K$ and $i^r : \partial M_{K^r} \hookrightarrow M_{K^r}$ be the usual inclusions. Consider $c \in H^2(\partial M_K; \mathbb{R}) = \text{Hom}(H_2(\partial M_K; \mathbb{Z}), \mathbb{R})$ and $c^r \in H^2(\partial M_{K^r}; \mathbb{R}) = \text{Hom}(H_2(\partial M_{K^r}; \mathbb{Z}), \mathbb{R})$ the fundamental classes in $H_2(\partial M_K; \mathbb{Z})$ and $H_2(\partial M_{K^r}; \mathbb{Z})$ respectively. By the definition, one has:

$$P_{\rho} \sim i^*(h_p^{(2)}) = c \quad \text{and} \quad P_{\rho^r} \sim i^r*(h^{(2)}_p) = c^r.$$

Further observe that the orientation of $S^3$ induces the same orientation on $M_K$ and $M_{K^r}$, because we use the invariant part $M_2$ to define it.

The boundary of the mutation sphere $F$ is the disjoint union of four circles denoted $S^1_1, \ldots, S^1_4$ (cf. Fig. 1). The Mayer-Vietoris sequence combines with the restriction homomorphism onto the boundary to give us the following commutative diagrams:

$$\begin{array}{ccc}
H^2_p(M_K) & \overset{\partial}{\longrightarrow} & H^2_p(\partial M_K) \overset{P_{\rho}}{\longrightarrow} H^2(\partial M_K; \mathbb{R}) \\
\downarrow && \downarrow \\
H^1_{\rho_p}(F) \overset{\{\iota_1^*, \ldots, \iota_4^*\}}{\longrightarrow} \bigoplus_{i=1}^4 H^1_{\rho_p}(S^1_i) \overset{4}{\longrightarrow} \bigoplus_{i=1}^4 H^1(S^1_i; \mathbb{R})
\end{array}$$

and

$$\begin{array}{ccc}
H^2_{\rho_p}(M_{K^r}) & \overset{\partial^r}{\longrightarrow} & H^2_{\rho_p}(\partial M_{K^r}) \overset{P_{\rho^r}}{\longrightarrow} H^2(\partial M_{K^r}; \mathbb{R}) \\
\downarrow && \downarrow \\
H^1_{\rho_p}(F) \overset{\{\iota_1^*, \ldots, \iota_4^*\}}{\longrightarrow} \bigoplus_{i=1}^4 H^1_{\rho_p}(S^1_i) \overset{4}{\longrightarrow} \bigoplus_{i=1}^4 H^1(S^1_i; \mathbb{R})
\end{array}$$

Observe that $\delta(t) = c$ if and only if $\delta^r(t) = c^r$. Thus $\partial(\tilde{h}_p^{(2)}) = h_p^{(2)}$ and $\partial^r(\tilde{h}_p^{(2)}) = h_p^{(2)}$, as a conclusion $[\tilde{h}_p^{(2)}/\tilde{h}_p^{(2)}] = 1$. □

□
5.2. Computation of the sign part of the torsion. We are interested in the sign of the torsions of $M_K$ and $M_{K^r}$. To this purpose we compute the torsions of $C_*(M_K; \mathbb{R})$ and $C_*(M_{K^r}; \mathbb{R})$ by using the Mayer–Vietoris sequences with real coefficients associated to the splittings $M_K = M_1 \cup \text{id} M_2$ and $M_{K^r} = M_1 \cup \tau M_2$.

Remark 13. From our assumption that $\tau$ is positive and Lemma 7 and the following Remark 7, we can choose the same bases for both Mayer–Vietoris sequences with real coefficients of $M_K = M_1 \cup \text{id} M_2$ and $M_{K^r} = M_1 \cup \tau M_2$.

A consequence of the preceding Propositions 5 & 6 is that the Mayer–Vietoris sequence $\mathcal{V}_R$ splits into two short exact sequences:

$$\mathcal{V}_1 = 0 \rightarrow H_1(F; \mathbb{R}) \xrightarrow{(\iota_{(1)}^0, \iota_{(1)}^0)} H_1(M_1; \mathbb{R}) \oplus H_1(M_2; \mathbb{R}) \xrightarrow{j_{(1)}^1 - j_{(1)}^0} H_1(M_K; \mathbb{R}) \rightarrow 0$$

and

$$\mathcal{V}_0 = 0 \rightarrow H_0(F; \mathbb{R}) \xrightarrow{(\iota_{(0)}^0, \iota_{(0)}^0)} H_0(M_1; \mathbb{R}) \oplus H_0(M_2; \mathbb{R}) \xrightarrow{j_{(0)}^1 - j_{(0)}^0} H_0(M_K; \mathbb{R}) \rightarrow 0.$$ 

The Reidemeister torsion of $\mathcal{V}_R$ (here with real coefficients) is thus:

$$\text{Tor}(\mathcal{V}_R, h^*_v, \emptyset) = \text{Tor}(\mathcal{V}_0, h^*_v, \emptyset) \cdot \text{Tor}(\mathcal{V}_1, h^*_v, \emptyset)^{-1}$$

where $h^*_v$, $h^*_v$ and $h^*_v$ denote bases of homology groups in the exact sequences.

Corresponding to the splitting $M_{K^r} = M_1 \cup \tau M_2$, the Mayer–Vietoris with real coefficients $\mathcal{V}_R$ splits into short exact sequences:

$$\mathcal{V}_1^r = 0 \rightarrow H_1(F) \xrightarrow{(\iota_{(1)}^0, \iota_{(1)}^0)} H_1(M_1) \oplus H_1(M_2) \xrightarrow{j_{(1)}^1 - j_{(1)}^0} H_1(M_{K^r}) \rightarrow 0$$

and

$$\mathcal{V}_0^r = 0 \rightarrow H_0(F) \xrightarrow{(\iota_{(0)}^0, \iota_{(0)}^0)} H_0(M_1) \oplus H_0(M_2) \xrightarrow{j_{(0)}^1 - j_{(0)}^0} H_0(M_{K^r}) \rightarrow 0.$$ 

Moreover, we can see the generator $\mu^r$ of $H_1(M_{K^r}; \mathbb{R})$ is the image $j_{(1)}^2(\eta_2)$ since $\eta_2 \in H_1(M_2; \mathbb{R})$ is a loop which bounds a disk in $N(K_2)$ (see Lemma 7). The Reidemeister torsion of $\mathcal{V}_R$ (here with real coefficients) is thus:

$$\text{Tor}(\mathcal{V}_R^r, h^*_v, \emptyset) = \text{Tor}(\mathcal{V}_0^r, h^*_v, \emptyset) \cdot \text{Tor}(\mathcal{V}_1^r, h^*_v, \emptyset)^{-1}$$

Here we omit the bases of homology groups in the Mayer–Vietoris sequence $\mathcal{V}$ in the torsions for simplicity.

We obtain:

$$\frac{\text{Tor}(C_*(M_{K^r}; \mathbb{R}), c^*_v, h^*_v)}{\text{Tor}(C_*(M_K; \mathbb{R}), c^*_v, h^*_v)} = \frac{\text{Tor}(\mathcal{V}_R^r, h^*_v, \emptyset)}{\text{Tor}(\mathcal{V}_R, h^*_v, \emptyset)} = \text{det} \tau(1) \cdot \text{det}(\mu/[\mu^r]) \cdot \text{det}(\tau(0))^{-1}.$$

where $C_*(M_K; \mathbb{R})$ and $C_*(M_{K^r}; \mathbb{R})$ are endowed with the homology orientation $[\mu^r]$, $[\mu]$ and $[\mu^r]$, $[\mu]$.

Next we compute each terms in the right hand side of the preceding equality.

Claim 16. The sign of $\text{det} \tau(0)$ is positive.

Proof. As $\tau(0)$ is just the identity map, it maps the class of the point to itself. \hfill $\square$

Claim 17. One has $\text{det}(\mu/[\mu^r]) = +1$.

Proof. This equality comes from our convention. Both meridians $\mu$ and $\mu^r$ are given by the same loop in $M_2$ and does not have any influence from a mutation. \hfill $\square$

Claim 18. The sign of $\text{det} \tau(1)$ is positive.
Proof. The homology group $H_1(F;\mathbb{Z})$ is isomorphic to the following quotient

$$H_1(F) \simeq \mathbb{R}a \oplus \mathbb{R}b \oplus \mathbb{R}c \oplus \mathbb{R}d/(a + b + c + d = 0).$$

The isomorphism of $H_1(F)$ induced by a positive mutation $\tau$ is one of the following:

$$\tau_{(1)}: a \mapsto b, \ b \mapsto a, \ c \mapsto d, \ d \mapsto c,$$
$$\tau_{(1)}: a \mapsto c, \ b \mapsto d, \ c \mapsto a, \ d \mapsto b,$$
$$\tau_{(1)}: a \mapsto d, \ b \mapsto c, \ c \mapsto b, \ d \mapsto a.$$

In each case it is easy to see that $\det \tau_{(1)}$ is $+1$.

Using the three previous claims, we conclude that:

**Claim 19.** We have:

$$\text{sgn}(\text{Tor}(V^*, h^*_{\rho_k}, \emptyset)) = \text{sgn}(\text{Tor}(V, h^*_{\rho_k}, \emptyset)).$$

6. Conclusion and open questions

Among the Reidemeister torsion form, another important invariant in Reidemeister torsions theory is the so-called twisted Alexander invariant, which can be understand as a non abelian version of the well–known Alexander polynomial. J. Milnor [8] proved that torsions theory is the so–called twisted Alexander invariant $\pm (\text{Alexander polynomial})$ up to a constant $\alpha$.

It is easy to see that $\det \tau_{(1)}$ is $+1$.

Numerical computations made by N. Dunfield, S. Friedl and N. Jackson [2] show that the twisted Alexander invariant is not invariant by mutation, but the torsion form is. An interesting question will be to understand and to characterize the "default" between $\Delta_{K}^{Ad\rho_p}(t)$ and $\Delta_{K}^{Ad\rho_p}(t)$ and especially at the discrete and faithful representation corresponding to the complete structure for hyperbolic knots.

**References**


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