# Geometry of hyperkähler manifolds 

 Géométries des variétés hyperkählériennesJieao Song<br>supervised by Olivier Debarre

Thesis defense
Soutenance de thèse

July 5, 2022

## Table of Contents

1 Generalities
Numerical aspects (joint with Thorsten Beckmann) Image of the period map

2 Geometry of Debarre-Voisin varieties
General results (joint with Vladimiro Benedetti) A special example

## Introduction

A hyperkähler manifold is a simply connected compact Kähler manifold $X$ such that

$$
H^{2,0}(X)=H^{0}\left(X, \Omega_{X}^{2}\right)=\mathbf{C} \sigma
$$

is generated by a nowhere vanishing holomorphic 2-form $\sigma$ (a holomorphic symplectic form).
(All known) examples

- In dimension 2: K3 surface (e.g., a quartic surface);
- K3 ${ }^{[n]}$ : Hilbert scheme of " $n$ points" on a K3 surface (and deformation);
- $\mathrm{Kum}_{n}$ : generalized Kummer variety (and deformation);
- Two examples found by O'Grady, in dimension 6 and 10.

Hyperkähler manifolds are very interesting because:
First, the cohomology ring $H^{*}(X, \mathbf{Q})$ has a lot of rich structure.

- The second cohomology group $H^{2}(X, \mathbf{Z})$ carries a natural quadratic form $q_{X}$ called the Beauville-Bogomolov-Fujiki form.
- The full cohomology $H^{*}(X, \mathbf{Q})$ admits an action of a Lie algebra (Looijenga-Lunts-Verbitsky algebra) and naturally decomposes into irreducible subrepresentations. One particularly important one is the Verbitsky component $\mathrm{SH}(X)$.

Second, a lot of information is encoded in the second cohomology group ( $\left.H^{2}(X, \mathbf{Z}), q_{X}\right)$ equipped with the BBF form, which provides a polarized Hodge structure. One form of the global Torelli theorem by Verbitsky says that one can (almost) recover $X$ from it.

The first part of this thesis studies some general properties of hyperkähler manifolds based on these features. We also emphasize on applying these results to the known examples.

From the point of view of algebraic geometry, it is natural to consider projective or polarized hyperkähler manifolds. We are particularly interested in locally complete families.
For K3 surfaces, the projective models are known in many small degrees:

- Degree 2: double cover of $\mathbf{P}^{2}$ ramified along a smooth sextic curve;
- Degree 4: quartic surface in $\mathbf{P}^{3}$;
- Degree 6: $(2,3)$-complete intersection in $\mathbf{P}^{4}$;
- Degree 8: $(2,2,2)$-complete intersection in $\mathbf{P}^{5}$;
- Degree $10,12, \ldots, 24,30,34,38$ : works of Mukai.

In higher dimensions, only very few examples are known.

- In dimension 4, the most well-studied example is the variety of lines of a cubic fourfold.
- Double EPW sextics, Debarre-Voisin varieties, VSP, ...

The main focus of the second part of this thesis is to prove many analogous results that are available for cubic fourfolds in the case of Debarre-Voisin varieties.

## Generalities

## J. Song

Let $X$ be a compact hyperkähler manifold of dimension $2 n$.

## Theorem (Beauville-Bogomolov-Fujiki form)

There exists a unique primitive integral quadratic form $q_{X}$ on $H^{2}(X, \mathbf{Z})$ of signature $\left(3, b_{2}-3\right)$ and a constant $C_{X} \in \mathbf{Q}$ satisfying

$$
\forall \beta \in H^{2}(X, \mathbf{Z}), \quad \int_{X} \beta^{2 n}=C_{X} \cdot q_{X}(\beta)^{n}
$$

More generally, let $\alpha \in H^{4 k}(X, \mathbf{Q})$ be a class that remains of type $(2 k, 2 k)$ on all small deformations of $X$ (e.g., any characteristic class), then there exists a constant $C(\alpha) \in \mathbf{Q}$ (generalized Fujiki constant of $\alpha$ ) such that

$$
\forall \beta \in H^{2}(X, \mathbf{Z}), \quad \int_{X} \alpha \cdot \beta^{2 n-2 k}=C(\alpha) \cdot q_{X}(\beta)^{n-k}
$$

## Theorem (Looijenga-Lunts-Verbitsky decomposition)

The Looijenga-Lunts-Verbitsky algebra $\mathfrak{g}(X)$ is the subalgebra of End $H^{*}(X, \mathbf{Q})$ generated by $\mathfrak{s l}_{2}$-triples $\left(L_{\alpha}, h, \Lambda_{\alpha}\right)$ for all $\alpha \in H^{2}(X, \mathbf{Q})$ satisfying the Lefschetz property.

- $\mathfrak{g}(X)$ is isomorphic to $\mathfrak{s o}\left(H^{2}(X, \mathbf{Q}) \oplus U\right)$;
- $H^{*}(X, \mathbf{Q})$ is naturally a $\mathfrak{g}(X)$-module and decomposes into irreducible $\mathfrak{g}$-submodules which respect the Hodge structures;
- The subalgebra $\mathrm{SH}(X, \mathbf{Q})$ of $H^{*}(X, \mathbf{Q})$ generated by $H^{2}(X, \mathbf{Q})$ is an irreducible $\mathfrak{g}$-submodule, known as the Verbitsky component.
- The decompositions for all known examples have been determined by Green-Kim-Laza-Robles.

An important object is $\mathfrak{q} \in \operatorname{Sym}^{2} H^{2}(X, \mathbf{Q}) \subset H^{4}(X, \mathbf{Q})$, the dual of the Beauville-Bogomolov-Fujiki form $q_{X}$.

Classes in $\mathrm{SH}(X)$ can be studied by pairing them with $H^{2}(X)$.


Figure: A "picture" of the LLV decomposition and the Verbitsky component

## Numerical aspects

Let $X$ be a hyperkähler manifold of dimension $2 n$ for $n \geq 2$.
Question. When does the second Chern class $c_{2}=c_{2}(X)$ lie in the Verbitsky component $\mathrm{SH}(X, \mathbf{Q})$ ? Equivalently, when is $c_{2}$ a multiple of $\mathfrak{q}$ ?

## Theorem

If $C\left(c_{2}^{2}\right)>2 C\left(c_{4}\right)$ or equivalently, $C\left(\mathrm{ch}_{4}\right)>0$, then we have the following bound for the second Betti number $b_{2}$

$$
b_{2}(X) \leq \frac{10}{\frac{C\left(c_{2}^{2}\right)}{C\left(c_{4}\right)}-2}-2 n+9
$$

with equality holds if and only if $c_{2}$ lies in the Verbitsky component. If $C\left(c_{2}^{2}\right) \leq 2 C\left(c_{4}\right)$, then $c_{2}$ is not contained in $\mathrm{SH}(X)$.

## Some remarks

Among known examples of compact hyperkähler manifolds of dimension $2 n \geq 4$, the second Chern class lies in the Verbitsky component only in the cases of $\mathrm{K} 3^{[2]}, \mathrm{K3}^{[3]}, \mathrm{Kum}_{2}, \mathrm{OG}_{6}$, and $\mathrm{OG}_{10}$.

The bound on $b_{2}$ can also be equivalently expressed in terms of the Riemann-Roch polynomial of $X$.

Curiously, one gets the same bound for $\mathrm{K}^{[5]}$ and $\mathrm{OG}_{10}\left(b_{2} \leq 24\right)$, as well as for $\mathrm{Kum}_{3}$ and $\mathrm{OG}_{6}\left(b_{2} \leq 8\right)$. In both cases, the bound is only attained by the O'Grady example.

In fact, for both O'Grady examples, all characteristic classes lie in the Verbitsky component.

## Cohomology class of a Lagrangian plane

Let $P \subset X$ be a Lagrangian $n$-plane and let $\ell \in H_{2}(X, \mathbf{Z})$ be the class of a line in $P$.

We treat $\ell$ as a class in $H^{2}(X, \mathbf{Q})$ by considering its dual class $L \in H^{2}(X, \mathbf{Q})$ using the Beauville-Bogomolov-Fujiki form and the Poincaré duality. In other words, $L$ satisfies

$$
\forall \alpha \in H^{2}(X, \mathbf{Q}) \quad q(L, \alpha)=\ell \cdot \alpha
$$

The square $q(\ell):=q(L)$ is conjectured by Hassett-Tschinkel to be a constant depending only on the deformation type of $X$.

For example, for K 3 surfaces, $\ell$ is the class of a smooth rational curve so we have $\ell^{2}=-2$.

For $\mathrm{K} 3^{[n]}$-type, it is shown that $q(\ell)=-\frac{n+3}{2}$. In lower dimensions, the proof was done by analysing the possible Hodge classes to determine the cohomology class of the plane $P$ (involving some hardcore Diophantine analysis).

- $n=2$ by Hassett-Tschinkel: $[P]=\frac{1}{2} L^{2}+\frac{1}{24} c_{2}$;
- $n=3$ by Harvey-Hassett-Tschinkel: $[P]=\frac{1}{6} L^{3}+\frac{1}{24} c_{2} L$;
- $n=4$ by Bakker-Jorza: $\left[\mathbb{P}^{4}\right]=\frac{1}{337920}\left(880 \rho^{4}+1760 \rho^{2} c_{2}(X)-3520 \theta^{2}+4928 \theta c_{2}(X)-1408 c_{2}(X)^{2}\right)$;
- any $n$ by Bakker, but no formula for $[P]$.

It would be interesting to determine the cohomology class $[P]$ in general.

Using the numerical perspective, it turns out that one can easily obtain a partial result: we can determine the orthogonal projection $\overline{[P]} \in \mathrm{SH}(X)$ to the Verbitsky component.

## Theorem

Let $P \subset X$ be a Lagrangian $n$-plane, we have

$$
\begin{aligned}
\overline{[P]} & =\overline{\left[\frac{\mu^{n}}{c_{X}} \exp (L / \mu) \operatorname{td}_{X}^{1 / 2}\right]_{n}} \\
& =\frac{\mu^{n}}{c_{X}}\left(\frac{1}{n!}\left(\frac{L}{\mu}\right)^{n}+\frac{1}{(n-2)!}\left(\frac{L}{\mu}\right)^{n-2} \overline{\operatorname{td}_{2}^{1 / 2}}+\cdots\right)
\end{aligned}
$$

where $\mu=\sqrt{\frac{-q(L)}{2 r_{X}}}$, and $[-]_{n}$ means the degree-n (cohomological degree-2n) part of a class. $c_{X}$ and $r_{X}$ are constants depending only on the deformation type of $X$.

One would then conjecture that $q(L)=-2 r_{X}$ so $\mu=1$ (which holds for $\mathrm{K} 3^{[n]}$-type, $\mathrm{Kum}_{2}$, and the two O'Grady examples).

In the $\mathrm{K} 3^{[n]}$-type case, we obtain the following simple formula

$$
\overline{[P]}=\overline{\left[\exp (L) \operatorname{td}_{X}^{1 / 2}\right]_{n}} .
$$

One could conjecture that this equality in fact holds before projection.

## Conjecture

For $X$ of $\mathrm{K3}^{[n]}$-type containing a Lagrangian plane $P$, we have

$$
[P]=\left[\exp (L) \operatorname{td}_{X}^{1 / 2}\right]_{n}
$$

Under the assumption that the line class $\ell$ is primitive in $H_{2}(X, \mathbf{Z})$, Bakker showed that all classes $[P]$ lie in the same monodromy orbit, so it suffices to verify the formula for one example.

In this case, I checked the conjecture for $n \leq 6$, and a full proof has recently been found by Oberdieck.

## Period map and period domain

Consider a fixed deformation type of hyperkähler manifolds so we get a fixed lattice

$$
\Lambda:=\left(H^{2}(X, \mathbf{Z}), q_{X}\right)
$$

Recall that a hyperkähler manifold $X$ has $H^{2,0}(X)=\mathbf{C} \sigma$, so the Hodge structure on $H^{2}$ is of type $\left(1, b_{2}-2,1\right)$.

The period map associates with each $X$ the point $\left[H^{2,0}(\underset{\sim}{X})\right]$ inside the projective space $\mathbf{P}\left(\Lambda_{\mathbf{C}}\right)$ (up to an isometry $\eta: H^{2}(X, \mathbf{Z}) \xrightarrow{\sim} \Lambda$ )

$$
[X] \longmapsto\left[H^{2,0}(X)\right] \in \mathbf{P}\left(\Lambda_{\mathbf{C}}\right) .
$$

A fundamental result is the global Torelli theorem by Verbitsky. We focus on the polarized version, obtained by Markman.

For a fixed polarization type $T$, i.e., an $\mathrm{O}(\Lambda)$-orbit in $\Lambda$ of a primitive element $h$ with $h^{2}>0$, there is a coarse moduli space $\mathcal{M}_{T}$ for polarized hyperkähler manifolds ( $X, H$ ) of type $T$ (there exists an isometry $\eta: H^{2}(X, \mathbf{Z}) \rightarrow \Lambda$ that maps $H$ to $\left.h\right)$.

The corresponding period domain $\mathcal{P}_{T}$ is defined as

$$
\mathcal{P}_{T}:=\left\{[x] \in \mathbf{P}\left(\Lambda_{\mathbf{C}}\right) \mid q(x)=q(x, h)=0, q(x, \bar{x})>0\right\} / \operatorname{Mon}(\Lambda, h),
$$

the set of all possible period points quotient by the action of the monodromy group.

The polarized version of the global Torelli theorem states that the period map

$$
\begin{aligned}
\mathfrak{p}: \quad \mathcal{M}_{T} & \longmapsto \\
& \longmapsto(X, H)]
\end{aligned} \mathcal{P}_{T} \longmapsto\left[H^{2,0}(X)\right]
$$

is an open immersion of algebraic varieties. The complement of the image is a union of Heegner divisors.

If we take an element $\kappa \in h^{\perp} \subset \Lambda$ with negative square, its orthogonal gives a hyperplane in $\mathbf{P}\left(\Lambda_{\mathbf{C}}\right)$ and the image in the period domain $\mathcal{P}_{T}$ is a divisor that we call a Heegner divisor. The Hodge structures parametrized by the elements of this divisor would then have $\kappa$ as a Hodge class.

Each Heegner divisor can be labeled using its discriminant, which is the discriminant of the sublattice orthogonal to both $h$ and $\kappa$.

For K3 ${ }^{[n]}$-type and $\mathrm{Kum}_{n}$-type,

- When $n$ is large, the moduli space $\mathcal{M}_{T}$ may not be connected. This was first studied by Apostolov in the K3 ${ }^{[n]}$-type case (and Onorati in the $\mathrm{Kum}_{n}$-type case).
- In such case, the period map is defined on each component, but different components can have different images.


## Geometry of Debarre-Voisin varieties

## Review: cubic fourfolds and their varieties of lines

## Theorem (Beauville-Donagi)

Let $X \subset \mathbf{P}^{5}=\mathbf{P}\left(V_{6}\right)$ be a smooth cubic fourfold. Let $F=F(X):=\left\{\ell \in \operatorname{Gr}\left(2, V_{6}\right) \mid \ell \subset X\right\}$ be its variety of lines. Consider also the incidence variety

$$
X \stackrel{p \quad I}{\longleftrightarrow}:=\{(x, \ell) \mid x \in \ell \subset X\} \xrightarrow{q}{ }^{\frac{p}{\longleftrightarrow}} F
$$

## Then

- $F$ is a hyperkähler fourfold of $\mathrm{K} 3^{[2]}$-type. The Plücker ample class $H$ has $q(H)=6$ and is of divisibility 2 .
- There is a Hodge isometry

$$
q_{*} p^{*}: H^{4}(X, \mathbf{Z})_{\mathrm{van}} \longrightarrow H^{2}(F, \mathbf{Z})_{\text {prim }}(-1) .
$$



$$
H^{*}(X)
$$


$H^{*}(F)$

Figure: Hodge diamonds of a cubic fourfold and its variety of lines

## Review: cubic fourfolds and their varieties of lines

## Theorem (Voisin, Hassett, Laza, Looijenga, ...)

We have the diagram

$$
\begin{gathered}
\mathcal{M}_{\text {cubic }} \xrightarrow{\mathfrak{m}} \mathcal{M}_{6}^{(2)} \xrightarrow{\mathfrak{p}} \mathcal{P}_{6}^{(2)} \\
{[X] \longmapsto[(F, H)] \longmapsto\left[H^{2,0}(F)\right]}
\end{gathered}
$$

- The moduli space of cubics $\mathcal{M}_{\text {cubic }}$ is the GIT quotient $\left(\mathbf{P}\left(\mathrm{Sym}^{3} V_{6}^{\vee}\right) \backslash \Delta\right) / / \operatorname{SL}\left(V_{6}\right)$.
- The period map $\mathfrak{p}$ misses the Heegner divisor $\mathcal{D}_{6}$, while the composition $\mathfrak{p} \circ \mathfrak{m}$ misses $\mathcal{D}_{2} \cup \mathcal{D}_{6}$.
- Heegner divisors $\mathcal{D}_{d}$ correspond to special cubic fourfolds $X$ (and $F$ ) with extra geometry.


## Review: cubic fourfolds and their varieties of lines

For example, for a general member in ...

- $\mathcal{D}_{6}: X$ has an ordinary double point, $F$ is birational to $S_{6}^{[2]}$;
- $\mathcal{D}_{8}: X$ contains a plane, $F$ is isomorphic to a moduli of sheaves on $\left(S_{2}, \beta\right)$;
- $\mathcal{D}_{14}: X$ is Pfaffian, $F$ is isomorphic to $S_{14}^{[2]}$.


## Generalities on Debarre-Voisin varieties

Let $V_{10}$ be a 10-dimensional complex vector space and let $\sigma \in \Lambda^{3} V_{10}^{\vee}$ be an alternating 3 -form.

We consider the following subvariety in $\mathrm{Gr}\left(6, V_{10}\right)$

$$
X_{6}:=\left\{\left[V_{6}\right] \in \operatorname{Gr}\left(6, V_{10}\right)|\sigma|_{V_{6}}=0\right\} .
$$

For a general $\sigma$, Debarre and Voisin showed that $X_{6}$ is a smooth 4-dimensional hyperkähler manifold of $\mathrm{K}{ }^{[2]}$-type. We have $q(H)=22$ and $H$ is of divisibility 2 .

There are two Fano varieties that one can associate with the trivector $\sigma$.

## The Fano 20-fold $X_{3}$

$X_{3} \subset \mathrm{Gr}\left(3, V_{10}\right)$ is the 20-dimensional Plücker hyperplane section defined by $\sigma$. In other words,

$$
X_{3}=\left\{\left[V_{3}\right] \in \operatorname{Gr}\left(3, V_{10}\right)|\sigma|_{V_{3}}=0\right\} .
$$

We define the vanishing cohomology of $X_{3}$ as

$$
H^{20}\left(X_{3}, R\right)_{\text {van }}:=\operatorname{ker}\left[j_{*}: H^{20}\left(X_{3}, R\right) \longrightarrow H^{22}\left(\operatorname{Gr}\left(3, V_{10}\right), R\right)\right],
$$

where $j: X_{3} \hookrightarrow \operatorname{Gr}\left(3, V_{10}\right)$ is the natural embedding and $R$ is $\mathbf{Z}$ or $\mathbf{Q}$. It is of type $(1,20,1)$, and is equipped with the intersection product.

## The Fano sixfold $X_{1}$

$X_{1} \subset \mathbf{P}\left(V_{10}\right)$ is a certain degeneracy locus, known as the Peskine variety

$$
X_{1}:=\left\{\left[V_{1}\right] \in \mathbf{P}\left(V_{10}\right) \mid \operatorname{rank} \sigma\left(V_{1},-,-\right) \leq 6\right\} .
$$

It is Fano of (expected) dimension 6.
In fact, $X_{1}$ more naturally embeds into the flag variety $\operatorname{Flag}\left(1,4, V_{10}\right)$.
And we similarly define the vanishing cohomologies of $X_{1}$ for $k=4,6,8$ as

$$
H^{k}\left(X_{1}, R\right)_{\text {van }}:=\operatorname{ker}\left[j_{*}: H^{k}\left(X_{1}, R\right) \longrightarrow H^{k+42}\left(\operatorname{Flag}\left(1,4, V_{10}\right), R\right)\right]
$$

where $j: X_{1} \hookrightarrow \operatorname{Flag}\left(1,4, V_{10}\right)$ is the embedding and $R=\mathbf{Z}$ or $\mathbf{Q}$. All three pieces are of type $(1,20,1)$. The middle piece $H^{6}\left(X_{1}, R\right)_{\text {van }}$ is equipped with the intersection product.


Figure: Hodge diamonds of a Debarre-Voisin variety and related Fano varieties

## Analogue of Beauville-Donagi

## Theorem

There exist Hodge isometries

$$
H^{20}\left(X_{3}, \mathbf{Z}\right)_{\mathrm{van}} \xrightarrow{\sim} H^{2}\left(X_{6}, \mathbf{Z}\right)_{\text {prim }}(-1) \stackrel{\sim}{\sim} H^{6}\left(X_{1}, \mathbf{Z}\right)_{\mathrm{van}}
$$

given by algebraic correspondences $X_{3} \leftarrow I_{3,6} \rightarrow X_{6}$ and $X_{1} \leftarrow I_{1,6} \rightarrow X_{6}$, whenever they are smooth of expected dimension.

## Corollary

The integral Hodge conjecture holds for the two Fano varieties $X_{1}$ and $X_{3}$.
The corollary follows from the IHC of curves on $\mathrm{K3}^{[n]}$ by Mongardi-Ottem, where they used this same idea to reprove the IHC for cubic fourfolds.

## Global picture

We have the following picture of the various moduli spaces

where $\mathcal{M}:=\mathbf{P}\left(\bigwedge^{3} V_{10}^{\vee}\right) / / \mathrm{SL}\left(V_{10}\right)$ is the GIT quotient, and $\mathcal{M}_{22}^{(2)}$ and $\mathcal{P}_{22}^{(2)}$ are the moduli space for polarized hyperkähler manifolds and the period domain respectively.

We have $\mathrm{SL}\left(V_{10}\right)$-invariant hypersurfaces in $\mathbf{P}\left(\bigwedge^{3} V_{10}^{\vee}\right)$ given by certain $\mathrm{SL}\left(V_{10}\right)$-invariant conditions on the trivector $\sigma$, as well as Heegner divisors in the period domain giving rise to extra Hodge classes on $X_{6}$. One could then try to relate the two.

## Three divisors

We consider the following three $\mathrm{SL}\left(V_{10}\right)$-invariant conditions on the trivector $\sigma$.
$\sigma$ satisfying the condition $(3,3,10)$

$$
\exists V_{3} \quad \sigma\left(V_{3}, V_{3},-\right)=0 .
$$

We obtain the discriminant hypersurface $\Delta=\Delta^{3,3,10} \subset \mathbf{P}\left(\bigwedge^{3} V_{10}^{\vee}\right)$.

## Proposition

$X_{3}$ and $X_{6}$ are smooth of expected dimension if and only if $\sigma$ does not lie in $\Delta^{3,3,10}$.

For a general $[\sigma] \in \Delta^{3,3,10}$, the hyperplane section $X_{3}$ acquires an ordinary double point at $\left[V_{3}\right] ; X_{6}$ is not smooth but birational to $S_{22}^{[2]}$.

In particular, inside $\mathcal{M}:=\mathbf{P}\left(\bigwedge^{3} V_{10}^{\vee}\right) / / \operatorname{SL}\left(V_{10}\right)$ we have an irreducible divisor $\mathcal{D}^{3,3,10}$ given by the image of $\Delta^{3,3,10}$ and we define $\mathcal{M}^{\text {smooth }}$ to be its complement, which is precisely the open locus where the corresponding $X_{3}$ is smooth.

Since in this case the corresponding $X_{6}$ is also smooth, we get a morphism

$$
\mathfrak{m}: \mathcal{M}^{\text {smooth }} \longrightarrow \mathcal{M}_{22}^{(2)}
$$

## Theorem (Debarre-Voisin and O'Grady)

$\mathfrak{m}$ is an open immersion.
The period map can be extended to the divisor $\mathcal{D}^{3,3,10}$ which corresponds to the boundary divisor $\mathcal{D}_{22}$ in the period domain.
$\sigma$ satisfying the condition $(1,6,10)$

$$
\exists V_{1} \subset V_{6} \quad \sigma\left(V_{1}, V_{6},-\right)=0
$$

In this case, $X_{1}$ is singular at $\left[V_{1}\right]$ since it further degenerates.

## Proposition

$X_{1}$ is smooth of expected dimension 6 if and only if $\sigma$ is not in the union $\Delta^{3,3,10} \cup \Delta^{1,6,10}$.
$X_{6}$ contains a uniruled divisor that is a $\mathbf{P}^{1}$-fibration over a K3 surface $S_{6}$ and provides a natural Brauer class $\beta$. A general $X_{6}$ in the family is isomorphic to a moduli space of sheaves on $\left(S_{6}, \beta\right)$.

This case corresponds to the Heegner divisor $\mathcal{D}_{24}$.
$\sigma$ satisfying the condition $(4,7,7)$

$$
\exists V_{4} \subset V_{7} \quad \sigma\left(V_{4}, V_{7}, V_{7}\right)=0
$$

$X_{6}$ contains a Lagrangian plane $\operatorname{Gr}\left(2, V_{7} / V_{4}\right)=\mathbf{P}\left(\left(V_{7} / V_{4}\right)^{\vee}\right)$.

## Proposition

A (linearly embedded) $\mathbf{P}^{2}$ contained in a Debarre-Voisin fourfold $X_{6}$ is always of the form $\mathbf{P}\left(\left(V_{7} / V_{4}\right)^{\vee}\right)$ for a such flag $V_{4} \subset V_{7}$.

This case corresponds to the Heegner divisor $\mathcal{D}_{28}$.
The Hodge isometries are proved by specializing to this family.

In conclusion, for Debarre-Voisin varieties, we have the following picture

as well as the following divisors


## A special example

Using the representation theory of the simple group $\mathbf{G}:=\operatorname{PSL}\left(2, \mathbf{F}_{11}\right)$, one can construct a (unique) trivector $\sigma_{0}$ that is $\mathbf{G}$-invariant. We consider the corresponding Debarre-Voisin variety $X_{6}$.

Among the three divisorial conditions that we have obtained,

- Condition $(3,3,10)$ is not satisfied, so $X_{6}$ is a smooth hyperkähler fourfold;
- Condition $(1,6,10)$ is satisfied: there are 55 pairs $V_{1} \subset V_{6}$ and we get 55 distinct divisors on $X_{6}$. They generate the Picard lattice which is of (maximal) rank 21.
- Condition $(4,7,7)$ is satisfied: there are 220 pairs $V_{4} \subset V_{7}$ so there are 220 distinct planes contained in $X_{6}$.

With the explicit description of the Picard lattice, we can show that the (symplectic) automorphism group $\mathrm{Aut}_{H}^{s}\left(X_{6}\right)$ is in fact isomorphic to $\mathbf{G}$.

## A special example

## Thank you!

