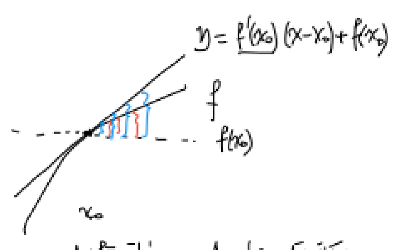
Test $\int_a^b f(x)\,dx = F(x)\Big|_a^b$

Ex 2 Q3 $x \longmapsto \tan\frac{\pi}{2x+1}$

Ex 3 f dérivable en x_0 $f'(x_0) \neq 0$

alors $\underline{f(x)-f(x_0)} \overset{x_0}{\sim} \boxed{f'(x_0)}(x-x_0)$

différence de valeur de f — droite tangente



Définition de la dérivée

$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$

$\lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)\right) = 0$

$\lim_{x\to x_0}\frac{f(x)-f(x_0)-f'(x_0)\cdot(x-x_0)}{x-x_0} = 0$

$\lim_{x\to a}\frac{f}{g} = 0$ alors $f = o_a(g)$

$\Rightarrow f(x)-f(x_0)-f'(x_0)\cdot(x-x_0) = o_{x_0}(x-x_0)$

$f(x)-f(x_0) = \boxed{f'(x_0)(x-x_0)} + o_{x_0}(x-x_0)$

$f \overset{a}{\sim} g$

$f = g + o_a(g)$

$(f-g = o_a(g))$

Donc on veut que

$o_{x_0}(x-x_0) = o_{x_0}(f'(x_0)\cdot(x-x_0))$

qui est immédiate si $f'(x_0) \neq 0$

Dans le cas $\Rightarrow f(x)-f(x_0) \overset{x_0}{\sim} f'(x_0)\cdot(x-x_0)$

si $f'(x_0)=0$ $f'(x_0)\cdot(x-x_0) = 0\cdot(x-x_0) = 0$

si $f(x)-f(x_0) \overset{x_0}{\sim} 0 \Rightarrow \underline{f(x)-f(x_0)=0} \Rightarrow \underline{f(x)=f(x_0)}$ constante au voisinage de x_0

qui n'est nécessairement vrai

$x_0=0$ $f(x)=x^2$ $f'(x)=2x$ $f'(0)=2\cdot 0=0$

$f(x)-f(x_0) = x^2-0^2 = x^2 \overset{0}{\not\sim} f(x_0)\cdot(x-x_0) = 0\cdot(x-0) = 0$

$\tan\frac{\pi}{2x+1}$

① $f(x) = \tan\boxed{\frac{\pi}{2x+1}}$

② $\boxed{g(u) = \tan u}$ g est dérivable en $u_0=\pi$ avec $g'(u_0) = \frac{1}{\cos^2 u_0} = 1$

$\underline{u_0=\pi}$ $\quad \underline{g(u)-g(u_0) = g'(u_0)\cdot(u-u_0) + o_{u_0}(u-u_0)}$ valable qu'au voisinage de $u_0=\pi$

$g(\pi) = \tan\pi = 0$ $\quad g(u) - \underset{0}{g(\pi)} = g'(\pi)(u-\pi) + o_\pi(u-\pi)$

$= 1\cdot(u-\pi) + o_\pi(u-\pi)$

$\underline{g(u) = u-\pi+o_\pi(u-\pi)}$ pour u au voisinage π

$\boxed{x\to 0}$, $\frac{\pi}{2x+1} \to \frac{\pi}{2\cdot 0+1} = \pi$ $\quad u\to\pi$ $\quad \boxed{g(u) \overset{\pi}{\sim} u-\pi}$

$\tan\frac{\pi}{2x+1} = g\left(\frac{\pi}{2x+1}\right) \overset{0}{\sim} \frac{\pi}{2x+1} - \pi$

$= \frac{\pi-\pi(2x+1)}{2x+1} = \frac{-2\pi x}{2x+1}$ quand $x\to 0$ $2x+1 \to 1$ $\quad \boxed{f \overset{a}{\sim} f\cdot g}$ avec $\underline{\lim_{x\to a} g = 1}$

$= \boxed{-2\pi x}\cdot\cancel{\boxed{\frac{1}{2x+1}}}$ $\quad \frac{1}{2x+1} \to 1$

$\overset{0}{\sim} \underline{-2\pi x}$

① $f(x) = \tan\frac{\pi}{2x+1}$ $\quad f(0) = \tan\pi = 0$

$\boxed{f'(0) = -2\pi} = \underset{1}{\underline{\tan'(\pi)}}\cdot\underset{-2\pi}{\left(\frac{\pi}{2x+1}\right)'(0)}$

$\underline{x_0=0}$ $\quad f(x)-f(x_0) = f'(x_0)(x-x_0) + o_{x_0}(x-x_0)$ $\quad f(x) = -2\pi x + o_0(-2\pi x)$

$f(x) = \overset{f(0)}{0} + \overset{f'(0)}{(-2\pi)}(x-0) + o_0(x-0)$ $\quad \Rightarrow f(x) \overset{0}{\sim} -2\pi x$

$= -2\pi x + o_0(x) = \boxed{-2\pi x} + o_0(-2\pi x)$

$f(x) = \tan\frac{\pi}{2x+1}$ $\quad g(u) = \tan u$ $\quad u = h(x)\to\pi$ u au voisinage de π

$= g(h(x)) = g\left(\frac{\pi}{2x+1}\right)$ $\quad \boxed{h(x) = \frac{\pi}{2x+1}}$ $\quad x\to 0$ x au voisinage de 0

$f'(x) = \underline{g'(h(x))}\cdot\underline{h'(x)}$ composée $\quad \underline{f(x)-f(x_0)} = \underline{f(x)-f(0)} = \underline{f'(0)}$

$f'(0) = g'(h(0))\cdot h'(0) = \boxed{g'(\pi)}\,h'(0)$

Q4 $\underline{\ln|\sin x|}$ en $x=0$

$\ln\circ|\sin x|$ $\quad x\to 0$ $\sin x\to 0$ $|\sin x|\to 0$ $\quad$ $u\to 0$ $\ln(u)\to -\infty$ $\Big\}$ $\underline{\ln|\sin x| \to -\infty}$

~~$\ln|\sin x| \overset{0}{\sim} -\infty$~~

$\boxed{\sin x \overset{0}{\sim} x}$ $\left(\lim_{x\to 0}\frac{\sin x}{x} = 1 \;/\; \sin' 0 = \cos 0 = 1\right)$.

$\sin x = x + o_0(x)$

$\ln|\sin x| = \ln|x+o_0(x)| = \ln\left(|x|\cdot\left|\frac{x+o_0(x)}{x}\right|\right)$ $\quad \frac{o_0(x)}{x} = o_0(1)$ $\quad f = o_0(x)$, $f\cdot\frac{1}{x} = o_0(x\cdot\frac{1}{x}) = o_0(1)$

$= \ln(|x|\cdot|1+o_0(1)|)$ $\quad f = o_a(g) \Rightarrow f\cdot h = o_a(g\cdot h)$

$= \underline{\ln|x| + \boxed{\ln|1+o_0(1)|}}$ $\quad \ln|1+o_0(1)| = o_0(\ln|x|)$

$x\to 0$ $\quad o_0(1)\to 0 \Rightarrow \ln|1+o_0(1)| \to \ln|1| = \underline{\ln 1 = 0}$ $\quad \lim_{x\to 0}\boxed{\frac{\ln|1+o_0(1)|}{\ln|x|}} = 0$

$\ln|x| \to -\infty$ $\quad \ln|1+o_0(1)| = o_0(\ln|x|)$ $\quad \frac{0{,}0\cdots 1}{-1000\cdots 0}$ $\frac{0}{-\infty} = 0$

$\underline{\ln|x| + o_0(\ln|x|)} \Rightarrow \underline{\ln|\sin x| \overset{0}{\sim} \ln|x|}$

$\forall\varepsilon\ \exists\delta>0$ tq. $|x|<\delta$ $\quad |f(x)| < \varepsilon|g(x)|$ $\quad$ 0 $\quad |-\infty|$ $\quad +\infty$

Ex4 $f \overset{x_0}{\sim} \phi$ $\quad g \overset{x_0}{\sim} \psi$ $\quad \underline{\phi \text{ et } \psi \text{ sont de même signe au voisinage de } x_0}$

$\Rightarrow \boxed{f+g \overset{x_0}{\sim} \phi+\psi} \Leftrightarrow \underline{(f+g)-(\phi+\psi)} = o_{x_0}(\phi+\psi)$

$f \overset{x_0}{\sim} \phi \Rightarrow f-\phi = o_{x_0}(\phi)$ $\quad (f+g)-(\phi+\psi)$

$g \overset{x_0}{\sim} \psi \Rightarrow g-\psi = o_{x_0}(\psi)$ $\quad = (f-\phi)+(g-\psi)$

il suffit que $o_{x_0}(\phi) = o_{x_0}(\phi+\psi)$ $\quad$ il est important que $\underline{\phi \text{ et } \psi \text{ soient de même signe}}$

et $o_{x_0}(\psi) = o_{x_0}(\phi+\psi)$

Sinon $\quad f \overset{x_0}{\sim} \phi$, $g \overset{x_0}{\sim} \psi$, $\not\Downarrow$, $f+g \overset{x_0}{\sim} \phi+\psi$ $\quad x_0=0$ $\quad 1+x \overset{0}{\sim} 1$, $+$, $-1 \overset{0}{\sim} -1$, $\Downarrow$, $x \overset{0}{\sim} 0$

Comme ϕ et ψ sont de même signe au voisinage de x_0

on a $|\phi+\psi| = |\phi|+|\psi|$

$\forall\varepsilon>0, \exists U$ tq. $x\in U$ $\quad |f-\phi| < \varepsilon|\phi| \leq \varepsilon|\phi+\psi|$

$\Rightarrow f-\phi = o_{x_0}(\phi+\psi)$ pareil pour $g-\psi$

Ex 6 $\sqrt{x+1} - x = o_{+\infty}(x^2)$

$\sqrt{x+1} = o_{+\infty}(x^2)$ parce que $\lim_{x\to+\infty}\frac{\sqrt{x+1}}{x^2} = \lim\sqrt{\frac{x+1}{x^4}} = \lim\sqrt{\frac{1}{x^3}+\frac{1}{x^4}} \to 0$

$x = o_{+\infty}(x^2)$ $\quad \lim_{x\to+\infty}\frac{x}{x^2} = \lim\frac{1}{x} \to 0$

Ex 8 ① $\text{ch}\,x = \boxed{\frac{e^x+e^{-x}}{2}} \overset{-\infty}{\sim} \frac{e^{-x}}{2}$ $\quad \Rightarrow \boxed{\text{sh}\,x \overset{+\infty}{\sim} \text{ch}\,x}$

$\text{sh}\,x = \frac{e^x-e^{-x}}{2} \overset{+\infty}{\sim} \frac{e^x}{2}$

$\text{ch}\,x = \frac{e^x}{2} + \frac{e^{-x}}{2} \overset{-\infty}{\sim} \frac{e^{-x}}{2}$

$x\to-\infty$ $\quad e^x\to 0$ $\quad e^{-x}\to+\infty$ $\quad \frac{e^x}{2} = o_{-\infty}\left(\frac{e^{-x}}{2}\right)$

$x\to+\infty$ $\quad e^x\to+\infty$ $\quad e^{-x}\to 0$ $\quad \frac{e^{-x}}{2} = o_{+\infty}\left(\frac{e^x}{2}\right) \Rightarrow \underline{\text{ch}\,x \overset{+\infty}{\sim} \frac{e^x}{2}}$

$\underline{\text{sh}\,x = \frac{e^x}{2} - \frac{e^{-x}}{2} \overset{+\infty}{\sim} \frac{e^x}{2}}$

$\text{sh}\,x \overset{+\infty}{\sim} \frac{e^x}{2} \overset{+\infty}{\sim} \text{ch}\,x$ par la transitivité

② en 0 $\quad \text{sh}'x = \text{ch}\,x$ $\quad \text{ch}'x = \text{sh}\,x$ $\Rightarrow$ dérivable en 0

$\underline{\text{sh}'0 = \text{ch}\,0 = 1 \quad \text{ch}'0 = \text{sh}\,0 = 0}$ $\quad \text{ch}\,0 = 1$

$\text{sh}\,x = \text{sh}\,0 + \text{sh}'0\cdot(x-0) + o_0(x-0)$ $\quad \underline{\text{ch}\,x \overset{0}{\sim} 1}$

$= 0 + 1\cdot x + o_0(x) = x + o_0(x) \Rightarrow \text{sh}\,x \overset{0}{\sim} x$ $\quad (\cos x \overset{0}{\sim} 1)$

$(\sin x \overset{0}{\sim} x)$

$\cos x \overset{0}{\sim} 1$ $\quad \underline{\cos x - 1} = o_0(\cos x) = o_0(1)$

$\sin x \overset{0}{\sim} x$

Taylor-Young

$f(x)-f(x_0) = f'(x_0)(x-x_0) + \underbrace{o_{x_0}(x-x_0)}$ à l'ordre 1 $\quad \frac{x^2 = o_0(x)}{x = o_0(1)}$

$\boxed{(x-x_0)^2 = o_{x_0}(x-x_0)}$

$f(x) = f(x_0) + f'(x_0)\cdot(x-x_0) + \frac{f^{(2)}(x_0)}{2}(x-x_0)^2 + \underbrace{o_{x_0}((x-x_0)^2)}$ TY à l'ordre 2

DL en $x_0=0$

$\boxed{f(x) = \underline{a_0} + \underline{a_1}\cdot x + \underline{a_2}\cdot x^2 + \underline{a_3}\cdot x^3 + \cdots + \underline{a_n}\cdot x^n + o_0(x^n)}$

en $x_0=0$ on a une suite de fonctions $\quad x\to 0$ $\quad x = o_0(1)$

$1, x, x^2, x^3, x^4, \ldots$ $\quad x^2 = o_0(x) = o_0(1)$

$x^n = o_0(x^{n-1})$ $\quad x^3 = o_0(x^2) = o_0(x) = o_0(1)$ $\quad x^4$

$f(x) - f(0) = o_0(1)$ $\quad f(x)-f(0)-a_1 x = o_0(x) = o_0(1)$

$f(x)-f(0)-a_1 x - a_2 x^2 = o_0(x^2)$ $\quad n! = n\cdot(n-1)(n-2)\cdots 1$

$3! = 3\times2\times1 = 6$

$\boxed{\text{Taylor-Young}} \Rightarrow$ $\quad a_0 = f(0)$ $\quad a_3 = f^{(3)}(0)/3! = f^{(3)}(0)/6$

$a_1 = f'(0)$ $\quad a_4 = f^{(4)}(0)/4! = f^{(4)}(0)/24$

$a_2 = f''(0)/2$ $\quad \ldots$ $\quad a_n = f^{(n)}(0)/n!$

$\sin x \overset{0}{\sim} x$ $\quad x\in\mathbb{R}$

$\underline{\sin x = \overset{a_0}{0} + \overset{a_1}{1}\cdot x + o_0(x)}$

0 $\sin x$

1 $\sin' x = \cos x$

2 $\sin'' x = (\cos)' x = -\sin x$

3 $\sin^{(3)} x = (-\sin x)' = -\cos x$

4 $\sin^{(4)} x = (-\cos x)' = -(-\sin x) = \sin x$

5 $\sin^{5} = (\sin^{(4)})' = \sin' = \cos$

$\sin 0 = 0$ $\quad 0\ 1\ 0\ {-1}\ \ 0\ 1\ 0\ {-1}\ \ldots$

$\sin' 0 = \cos 0 = 1$

$\sin^{(2)}(0) = -\sin 0 = 0$

$\sin^{(3)}(0) = -\cos 0 = -1$

$0\ 1\ 0\ {-1}$

$f(x) = \underline{\sin x} = \underset{0}{\underbrace{f(0)}} + \underset{1}{f'(0)}\cdot x + \frac{\overset{0}{f''(0)}}{2!=2}x^2 + \frac{\overset{-1}{f^{(3)}(0)}}{3!}x^3 + \frac{\overset{0}{f^{(4)}(0)}}{4!}x^4 + \cdots + o_0(x^n)$

$= \cancel{0} + 1\cdot x + \cancel{0\cdot\frac{1}{2}x^2} + (-1)\cdot\frac{1}{6}x^3 + \cancel{0\cdot\frac{1}{24}x^4} + 1\cdot\frac{1}{120}x^5 + \underbrace{\cancel{0\cdot\frac{1}{720}x^6}} + \cdots$

$= \underline{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o_0(x^6)}$

$\cos x$ $\quad 1\ 0\ {-1}\ 0\ \big|\ 1\ 0\ {-1}\ 0\ \big|\ \ldots$

$\underline{\cos x} = 1 + \cancel{0\cdot x} + (-1)\cdot\frac{1}{2}x^2 + \cancel{0\cdot\frac{1}{6}x^3} + 1\cdot\frac{1}{24}x^4 + \cancel{0\cdot\frac{1}{120}x^5} + o_0(x^5)$

$= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o_0(x^5)$