- fonctions equivalentes.

f, g définies au voisinage d'un point a

$f \overset{a}{\sim} g \iff \boxed{f-g = o_a(g) \;(= o_a(f))}$

$\forall \varepsilon \; \exists \delta \; (U =]a-\delta, a+\delta[)$ tq.

$\forall x \in U \quad |(f-g)(x)| < \varepsilon \cdot |g(x)|$

$\qquad\qquad\quad \overset{\shortparallel}{|f(x)-g(x)|}$

$\boxed{\boxed{\lim_{x\to a} \frac{f}{g} = 1} \quad \text{alors } f \overset{a}{\sim} g}$

$f = o_a(g)$

$\boxed{\lim_{x\to a} \frac{f}{g} = 0}$

$f = o_a(g)$

$f(x) = x,\; g(x) = 1$

$\lim_{x\to 0} \frac{x}{1} = \lim x = 0$

$x = o_0(1)$

- $f(x)$
  - $\lim_{x\to a} f(x) \overset{=\ell}{}$ existe et n'est pas 0

    alors $\ell(x) \equiv \ell$ const. est une fonction équivalente à f en a.

    $\lim_{x\to a} \frac{f(x)}{\ell} = \frac{\lim f(x)}{\ell} = \frac{\ell}{\ell} = 1 \Rightarrow f \sim \ell$ ← limite

  - $\lim_{x\to a} f(x) = 0$ existe mais est 0.

    $\boxed{\boxed{f(x) \overset{a}{\sim} 0} \text{ est vrai ssi } f(x) \equiv 0}$

$\triangle$ $f(x) \equiv 0 \Leftrightarrow f(x) \overset{a}{\sim} 0$

$f(x) \not\equiv 0$ si seulement $\lim_{x\to a} f(x) = 0$

$\not\Rightarrow f(x) \overset{a}{\sim} 0$

$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$

$\lim_{x\to a} \frac{f(x)-f(a)}{f'(a)(x-a)} = \frac{f'(a)}{f'(a)} = 1$

$f(x)$ est dérivable en a $\quad \underbrace{f(x) = f(a)}_{} + f'(a)\cdot(x-a) + o_a(x-a)$

$f(x) - f(a) = \boxed{f'(a)} \cdot (x-a) + o_a(x-a)$ si $f(a) = 0$

$f(x) = f(x) - f(a)$

$\sim f'(a)(x-a)$

si $\lim_{x\to a} f(x) = f(a) = 0$ on conclut que

$f(x) = 0 + f'(a)\cdot(x-a) + o_a(x-a)$.

si $f'(a) \neq 0$ on a $f(x) = f'(a)\cdot(x-a) + o_a(x-a)$

$f(x) - \underbrace{f'(a)\,(x-a)}_{g(x)} = o_a(x-a) = o_a(\underbrace{f'(a)\cdot(x-a)}_{g(x)})$

$\Rightarrow f(x) \overset{a}{\sim} g(x) = f'(a)\,(x-a)$.

- $\lim_{x\to a} f(x) = 0 \Rightarrow f(a) = 0$

$\begin{cases} \bullet\; f(x) \equiv 0 \iff f(x) \overset{a}{\sim} 0 \\ \bullet\; f(x) \not\equiv 0 \not\Rightarrow f(x) \overset{a}{\sim} 0 \end{cases}$

- si f dérivable en a

  $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$

  $\begin{cases} \bullet\; f'(a) \neq 0 \Rightarrow \lim_{x\to a} \frac{f(x) - \boxed{f(a)}^{\,\shortparallel 0}}{f'(a)(x-a)} = 1 \Rightarrow f(x) \sim f'(a)(x-a) \\ \bullet\; f'(a) = 0 \end{cases}$

  $f(x) = \underbrace{a_0} + \underbrace{a_1\cdot x} + \underbrace{a_2\cdot x^2} + a_3 x^3 + \cdots + o(x^n)$

  $f(a) = f'(a) = f''(a) = 0$

$f(x) = \cancel{\boxed{a_0}} + a_1 x + o_0(x)$

$\lim_{x\to 0} f(x) = f(0)$ $\bullet$ $f(x) \sim f(0)$ si $f(0) \neq 0$

$\bullet$ sinon $f(0) = 0$ si $f'(0) \neq 0$

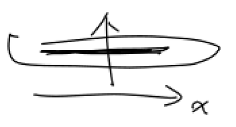
$f(x) \sim f'(0)\cdot x$

Ex 2 ① $f(x)$ $\quad \lim_{x\to 0} f(x) = 1$

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

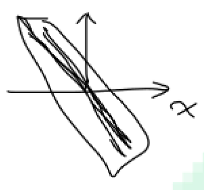
$\quad 1$

$f(x) \overset{0}{\sim} 1$

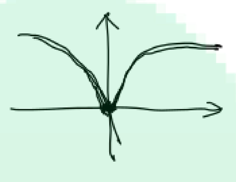


③ $\tan\left(\frac{\pi}{2x+1}\right)$ $\quad \lim_{x\to 0} f(x) = f(0) = \tan\pi = 0$ $\quad f(x) = \underset{0}{a_0} + a_1 x + \cdots$

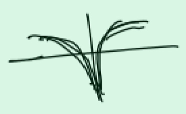
$f'(0) = -2\pi \Rightarrow f(x) \overset{0}{\sim} (-2\pi)\cdot x$



② $\sqrt{|2x - x^2|} \overset{0}{\sim} \sqrt{2|x|}$

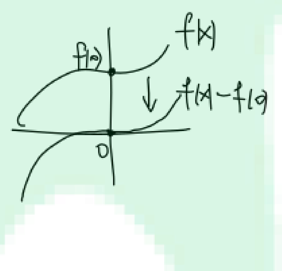


④ $\ln|\sin x| \overset{0}{\sim} \boxed{\ln|x|}$



$\sin x = x \cancel{- \frac{1}{6}x^3 + \frac{1}{120}x^5 \cdots}$

- DL en point 0 pour une fonction f.



$f(x) = a_0 + a_1\cdot x + a_2\cdot x^2 + \cdots + a_n\cdot x^n + o_0(x^n)$

$[n=0]$ $\quad f(x) = a_0 + o_0(x^0) = a_0 + o_0(1)$ $\quad \lim_{x\to 0} \overset{f(0)}{f(x)} = a_0$

$\left(o_0(1) = g(x) \Rightarrow \lim_{x\to 0} g(x) = 0 \right)$

$[n=1]$ $\quad f(x) = a_0 + a_1\cdot x + o_0(x^1)$ $\quad \boxed{f(x) = f(0) + f'(0)\cdot x + o_0(x)}$

$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x-0}$

$0 = \lim_{x\to 0} \frac{f(x) - f(0) - f'(0)\cdot(x-0)}{x-0} \Rightarrow f(x) - f(0) - f'(0)\cdot x = o_0(x)$

En general $\quad$ Taylor-Young

$\Rightarrow a_n = \frac{f^{(n)}(0)}{n!}$ $\quad \left(a_0 = \frac{f(0)}{0!} = f(0) \quad a_1 = \frac{f'(0)}{1!} = f'(0) \right)$

$n! = n\times(n-1)\cdots\times 2\times 1$

$f(x) = \sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots$

$\quad\quad\quad\quad\quad\quad\quad\uparrow 3! \quad \uparrow 5!$

$f(0)$	$f'(0)$	$f''(0)$	$f'''(0)$	$f^{(4)}(0)$	
0	1	0	-1	0	1 0 -1 …
a_0	a_1	a_2	a_3	$f^{(4)}(x) = \sin x = f(x)$	
$\cancel{\frac{0}{0!}}$	$\frac{1}{1!}$	$\cancel{\frac{0}{2!}}$	$\frac{-1}{3!}$	$\cancel{\frac{0}{4!}}$	$\frac{1}{5!}$ $\cancel{\frac{0}{6!}}$ $\frac{-1}{7!}$

$g(x) = \cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots$

$\quad\quad\quad\quad\quad\quad\quad\uparrow 2! \quad \uparrow 4!$

1	0	-1	0	…

$h(x) = e^x \quad h'(x) = e^x$

1	1	1	1

$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \cdots$

$\frac{1}{1-x}$ $\quad 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \frac{1}{1-x}$ $\quad \lim_{n\to\infty} \left(\sum_{k=0}^{n} x^k \right) = \frac{1}{1-x}$

$-1 < x < 1$

$DL_{0,n}\left(\frac{1}{1-x}\right) = 1 + x + x^2 + \cdots + x^n + o_0(x^n)$

$DL_n(0)$

$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots$

$= 1 - x + x^2 - x^3 + x^4 \cdots$

$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^5)}{1 - \underbrace{\frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^5)}}$ $\quad = \frac{\boxed{A}}{1 - \boxed{B}} = \boxed{A} \cdot \frac{1}{1-\boxed{B}}$

$x \to 0$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad = \boxed{A} \cdot 1 + B + B^2 + B^3 + \cdots$

$u = \frac{1}{2}x^2 - \frac{1}{24}x^4 + o(x^5)$ $\quad \begin{matrix} x\to 0 \\ u\to 0 \end{matrix}$

$u = \frac{1}{2}x^2 + o(x^2)$ $\quad u^2 = \frac{1}{4}x^4 + o(x^4)$

$u^3 = \frac{1}{8}x^6 + o(x^6)$

$= \left(x - \frac{1}{6}x^3 + \cdots\right) \cdot \frac{1}{1-u}$

$= \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^5)\right)\left(1 + u + u^2 + o(x^5)\right)$

$= \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^5)\right)\left(1 + \left(\frac{1}{2}x^2 - \frac{1}{24}x^4 + \cancel{o(x^5)}\right)\right.$

$(A+B)^2 = A^2 + 2AB + B^2$

$+ \left(\frac{1}{2}x^2 - \frac{1}{24}x^4 + o(x^5)\right)^2 = \left(\frac{1}{2}x^2\right)^2 + \cancel{2\left(\frac{1}{2}x^2\right)\left(-\frac{1}{24}x^4 + o(x^5)\right)}$

$+ \cancel{\left(-\frac{1}{24}x^4 + o(x^5)\right)^2}$

$\left. + o(x^5) \right)$

$\frac{-1+6}{24} = \frac{5}{24}x^4$

$= \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^5)\right)\left(1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{4}x^4 + o(x^5)\right)$

$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{2}x^3 - \frac{1}{12}x^5 + \cancel{\frac{1}{240}x^7} \quad \frac{5}{24}x^5 - \cancel{\frac{5}{144}x^7} \quad \cancel{x^9} + o(x^5)$

$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^5)$